

# Polar co-ordinates

The  $(x, y)$  co-ordinates of a point in the plane are called its Cartesian co-ordinates. But there is another way to specify the position of a point, and that is to use polar co-ordinates  $(r, \theta)$ . In this unit we explain how to convert from Cartesian co-ordinates to polar co-ordinates, and back again.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- plot a point on a diagram when given its polar co-ordinates;
- convert the polar co-ordinates of a point into Cartesian co-ordinates;
- convert the Cartesian co-ordinates of a point into polar co-ordinates;
- convert the polar co-ordinate equation of a curve into a Cartesian co-ordinate equation;
- identify new portions of a curve which satisfy the Cartesian co-ordinate equation but which do not satisfy the polar co-ordinate equation.

## Contents

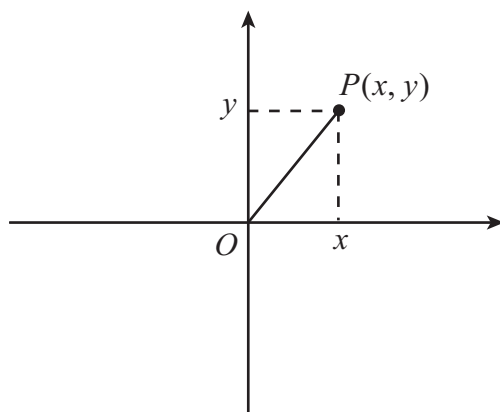
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# 1. Introduction

The position of a point in a plane can be described using Cartesian, or rectangular, co-ordinates.

In the Cartesian system we fix an origin, a point  $O$ , and then take two lines which are perpendicular to each other passing through  $O$ . These two lines are the axes. The  $x$ -axis is horizontal, and the  $y$ -axis is vertical. The position of any point  $P$  in the plane is then described by noting its displacement from  $O$  in the direction of the  $x$ -axis, the  $x$  co-ordinate, and also its displacement from  $O$  in the direction of the  $y$ -axis, the  $y$  co-ordinate.

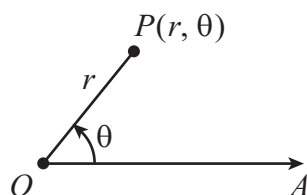


Notice that we talk about the displacement, and not the distance. This is because, in this system, we need to use the directions indicated by the arrows to help us to be able to specify a point uniquely. Thus moving up from the origin is a positive displacement, whereas moving down from the origin is a negative displacement. Similarly, moving to the right from the origin gives a positive displacement, and moving to the left gives a negative displacement.

The polar co-ordinate system is another system for specifying a point in the plane uniquely.

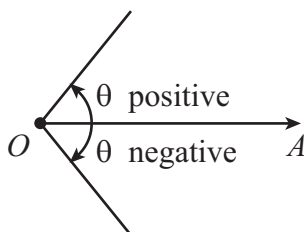
## 2. The polar co-ordinate system

For polar co-ordinates we take an origin (or pole)  $O$ , and a fixed initial line  $OA$ . A point  $P$  is then described by specifying a distance  $r$ , the distance  $O$  to  $P$  along the radius direction, and the angle  $\theta$  we have had to turn from the initial line to be looking along the radius direction. The point then has polar co-ordinates  $(r, \theta)$ .



There are similar conventions about directions with polar co-ordinates as there are with Cartesian co-ordinates. The conventions are:

1.  $\theta$  is measured in radians;
2.  $\theta$  is measured from the initial line, and is measured as positive in an anticlockwise direction from the initial line, or negative in a clockwise direction from the initial line;



3. the values of  $\theta$  are restricted to  $-\pi < \theta \leq \pi$ ;
4. the values of  $r$  are restricted to  $r > 0$  (strictly speaking, the origin  $O$  itself is excluded from the plane when the points are described using polar co-ordinates, although we will sometimes disregard this when drawing graphs).

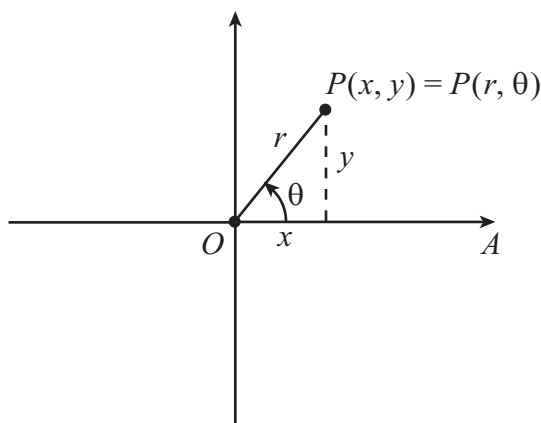


### Key Point

The polar co-ordinates of a point  $P$  with respect to a pole  $O$  and initial line  $OA$  are the numbers  $(r, \theta)$ , where  $r$  is the length of the line  $OP$  and  $\theta$  is the angle between  $OA$  and  $OP$ . We always take  $r > 0$ , and we always take  $-\pi < \theta \leq \pi$ .

## 3. Conversion between the two systems

Consider a point  $P$  with Cartesian co-ordinates  $(x, y)$  and polar co-ordinates  $(r, \theta)$ . We shall identify the pole with the origin of Cartesian co-ordinates, and the initial line with the  $x$ -axis.



From the diagram, we see that there are two pairs of equations we can use to help us to move from one system to another:

$$x = r \cos \theta, \quad y = r \sin \theta$$

to convert from polar co-ordinates to Cartesian co-ordinates, and

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}$$

to convert from Cartesian co-ordinates to polar co-ordinates. Both these sets of equations will be useful, but they must be used with care, as it is easy to obtain an incorrect value of  $\theta$  when the point does not lie in the first quadrant. So it is always a good idea to plot the points first.

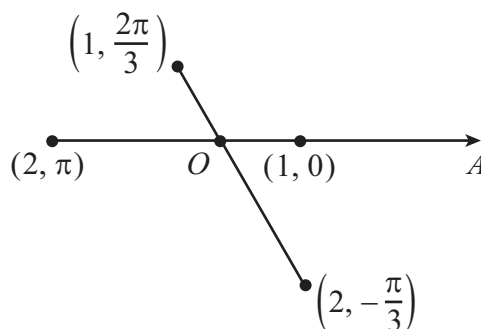
### Example

Plot the points with polar co-ordinates

- (a)  $(2, \pi)$ , (b)  $(1, 0)$ , (c)  $(2, -\pi/3)$ , (d)  $(1, 2\pi/3)$ .

### Solution

For each point, we rotate through the angle  $\theta$  given (the second co-ordinate), and then move out in the direction obtained by a distance equal to  $r$  (the first co-ordinate)



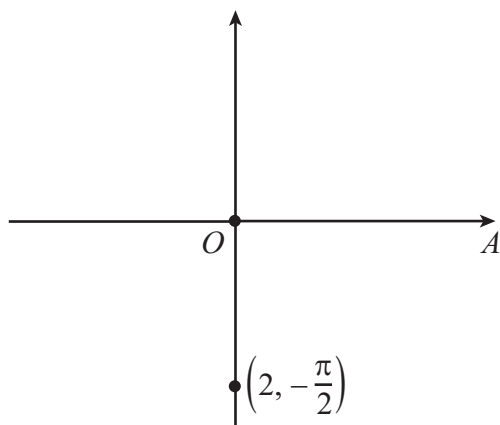
### Example

Assuming a common origin and the  $x$ -axis as the initial line, find the Cartesian co-ordinates of the points with the following polar co-ordinates:

- (a)  $(2, -\pi/2)$ , (b)  $(1, 3\pi/4)$ , (c)  $(2, -\pi/3)$ .

### Solution

- (a)

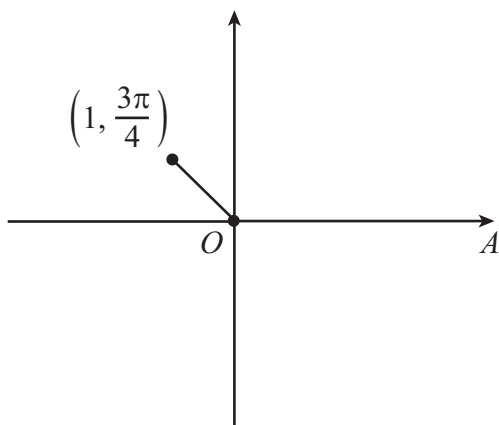


$$x = 2 \times \cos \left( -\frac{\pi}{2} \right) = 0$$

$$y = 2 \times \sin \left( -\frac{\pi}{2} \right) = -2$$

so that the Cartesian co-ordinates are  $(0, -2)$ .

(b)



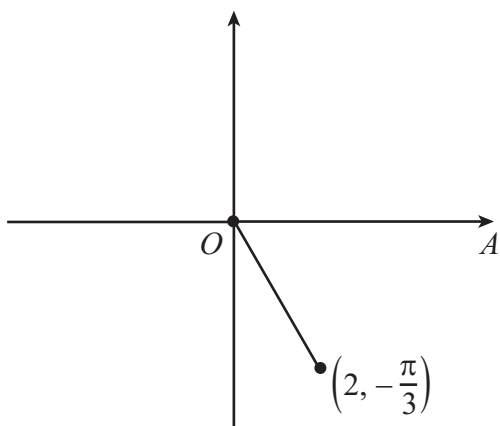
$$x = 1 \times \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$y = 1 \times \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

so that the Cartesian co-ordinates are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

(c)



$$x = 2 \times \cos\left(-\frac{\pi}{3}\right) = 1$$

$$y = 2 \times \sin\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

so that the Cartesian co-ordinates are  $(1, -\sqrt{3})$ .

Notice that each point may be seen to be in the correct quadrant of the diagram.

### Example

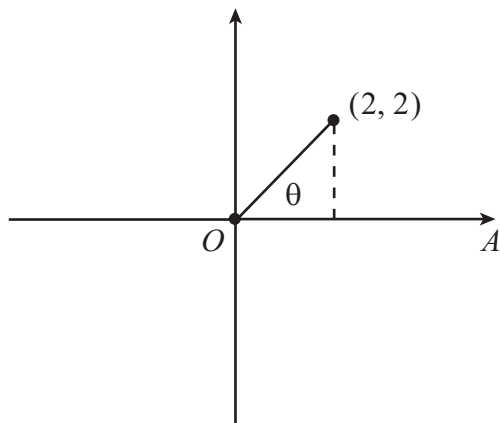
Assuming a common origin and the  $x$ -axis as the initial line, find the polar co-ordinates of the points with the following Cartesian co-ordinates:

(a)  $(2, 2)$ , (b)  $(-3, 4)$ , (c)  $(-2, -2\sqrt{3})$ , (d)  $(1, -1)$ .

### Solution

It is important to plot the points before calculation in order to be certain where they are and hence get the corresponding values of  $r$  and  $\theta$  correct.

(a)



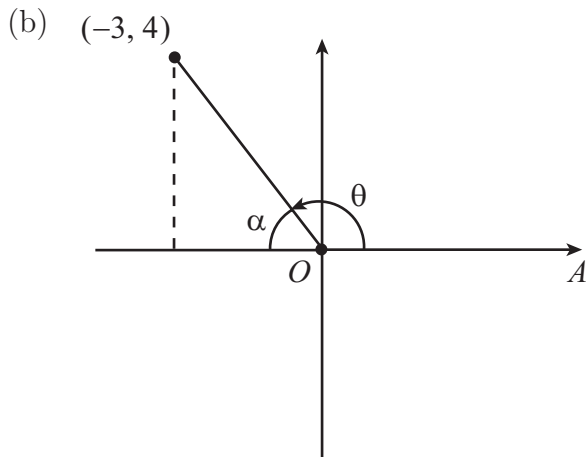
$$r^2 = 2^2 + 2^2 = 8$$

$$\text{so that } r = \sqrt{8} = 2\sqrt{2}$$

$$\tan \theta = 1$$

$$\text{so that } \theta = \pi/4$$

and thus the polar co-ordinates are  $(2\sqrt{2}, \pi/4)$ .



$$r^2 = (-3)^2 + 4^2 = 25$$

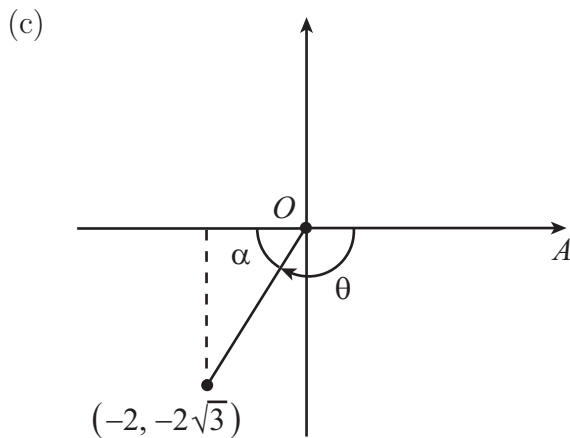
$$\text{so that } r = \sqrt{25} = 5$$

$$\tan \alpha = \frac{4}{3}$$

$$\text{so that } \alpha = 0.93 \text{ radians}$$

$$\text{giving } \theta = \pi - \alpha = 2.21 \text{ radians}$$

and thus the polar co-ordinates are  $(5, 2.21)$ .



$$r^2 = (-2)^2 + (-2\sqrt{3})^2 = 16$$

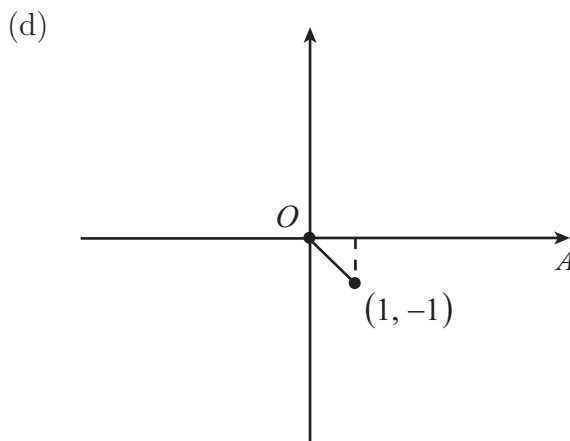
$$\text{so that } r = \sqrt{16} = 4$$

$$\tan \alpha = \sqrt{3}$$

$$\text{so that } \alpha = \pi/3$$

$$\text{giving } \theta = -(\pi - \pi/3) = -2\pi/3$$

and thus the polar co-ordinates are  $(4, -2\pi/3)$ .



$$r^2 = 1^2 + (-1)^2 = 2$$

$$\text{so that } r = \sqrt{2}$$

$$\tan \alpha = 1$$

$$\text{so that } \alpha = \pi/4$$

$$\text{giving } \theta = -\pi/4$$

and thus the polar co-ordinates are  $(\sqrt{2}, -\pi/4)$ .

Notice that, where the value of either  $x$  or  $y$  is negative, we have introduced another angle  $\alpha$  by drawing a right-angled triangle. This new angle  $\alpha$  will always be between 0 and  $\pi/2$ . Then we can find the value of  $\theta$  by subtraction, bearing in mind that we want  $\theta$  to have the correct value for the quadrant containing the point:

quadrant	range	value
1st ( $x > 0, y > 0$ )	$0 < \theta < \pi/2$	$\theta = \alpha$
2nd ( $x < 0, y > 0$ )	$\pi/2 < \theta < \pi$	$\theta = \pi - \alpha$
3rd ( $x < 0, y < 0$ )	$-\pi < \theta < -\pi/2$	$\theta = -(\pi - \alpha)$
4th ( $x > 0, y < 0$ )	$-\pi/2 < \theta < 0$	$\theta = -\alpha$



### Key Point

To convert from polar co-ordinates to Cartesian co-ordinates, use the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

To convert from Cartesian co-ordinates to polar co-ordinates, use the equations

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

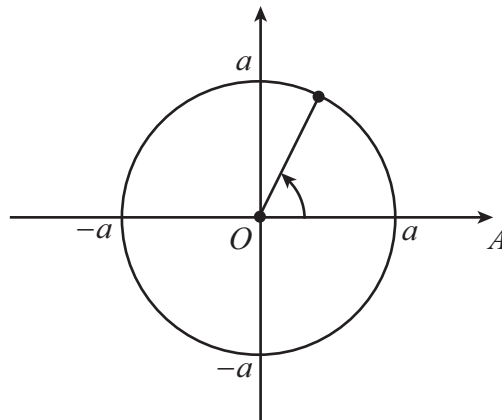
If the point is not in the first quadrant then you should find an acute angle  $\alpha$  using a right-angled triangle, and then find  $\theta$  by subtraction so that it has the correct value for the quadrant containing the point.

## 4. Some equations of curves in polar co-ordinates

The equation

$$x^2 + y^2 = a^2$$

is the equation of a circle, centred at the origin and with radius  $a$ , given in Cartesian co-ordinates.



As we look at the graph of the curve we can observe that, no matter in which direction we look out from the origin, the distance of the curve from the origin is always the same. Thus in polar co-ordinates the equation of the circle centre the origin is

$$r = a .$$

We can see this if we use the equations  $x = r \cos \theta$  and  $y = r \sin \theta$  to transform  $x^2 + y^2 = a^2$  into polar co-ordinates. We get

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = a^2$$

so that

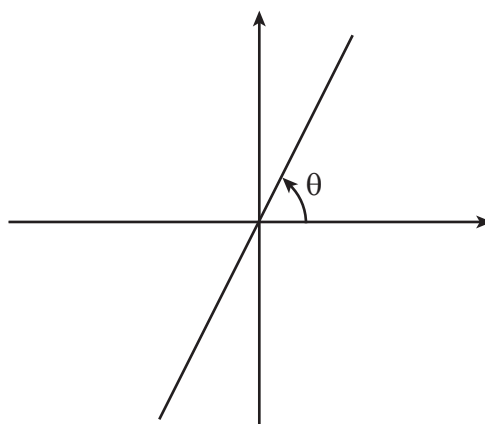
$$r^2(\cos^2 \theta + \sin^2 \theta) = a^2 .$$

But since  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$r^2 = a^2$$

so that  $r = a$ .

Similarly, the equation  $y = mx$  in Cartesian co-ordinates represents a straight line through the origin, where  $m$  is the gradient. Here,  $m = \tan \theta$  where  $\theta$  is the angle that the line makes with the positive direction of the  $x$ -axis.



Now we see that, no matter how far along the line we go, the gradient is always the same. We can also see this from the equation. From

$$r \sin \theta = m(r \cos \theta)$$

we obtain

$$\frac{\sin \theta}{\cos \theta} = m$$

so that

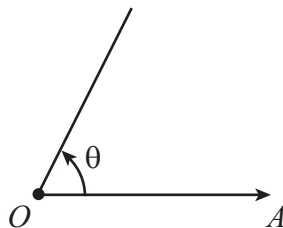
$$\tan \theta = m$$

and therefore

$$\theta = \text{constant}.$$



Note that this is not the full line  $y = mx$ , but only the half-line along the direction of  $\theta$ . This is because the value of  $r$  must be positive.



### Example

Find the polar equation of  $x^2 + y^2 = 9$ .

### Solution

We use the relationships

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Substituting these back into the equation, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 9.$$

Taking out the common factor of  $r^2$  we have

$$r^2(\cos^2 \theta + \sin^2 \theta) = 9,$$

and as  $\cos^2 \theta + \sin^2 \theta = 1$  we see that  $r^2 = 9$ , so that  $r = 3$ . So this is a circle of radius 3 units, centred on the origin.

### Example

Find the polar equation of  $xy = 4$ .

### Solution

This is the equation of a rectangular hyperbola, in Cartesian co-ordinates, so let us see what it looks like in polar co-ordinates. Again we substitute  $x = r \cos \theta$  and  $y = r \sin \theta$ , and we get

$$(r \cos \theta)(r \sin \theta) = 4$$

so that

$$r^2 \sin \theta \cos \theta = 4.$$

But  $\sin 2\theta = 2 \cos \theta \sin \theta$ , and so we have

$$r^2 \sin 2\theta = 8$$

as the equation of the hyperbola in polar co-ordinates. As you can see, a knowledge of the trigonometric identities is useful in order to simplify equations given in polar co-ordinates.

### Example

Find the Cartesian equation of the curve

$$\frac{2}{r} = 1 + \cos \theta.$$

### Solution

If we first multiply by  $r$ , we obtain

$$2 = r + r \cos \theta.$$

Now we can replace  $r$  by  $\sqrt{x^2 + y^2}$  and  $r \cos \theta$  by  $x$ , to obtain

$$2 = \sqrt{x^2 + y^2} + x.$$

So

$$\sqrt{x^2 + y^2} = 2 - x,$$

and squaring both sides to get rid of the square root sign we obtain

$$x^2 + y^2 = 4 - 4x + x^2.$$

This reduces to

$$y^2 = 4 - 4x,$$

which is the equation of a parabola

### Exercises

1. Plot on a diagram the following points with polar co-ordinates

(a)  $(2, 0)$ , (b)  $(1, \pi/2)$ , (c)  $(1, 5\pi/6)$ , (d)  $(2, \pi/3)$ , (e)  $(1, \pi/6)$ , (f)  $(1, -3\pi/4)$ .

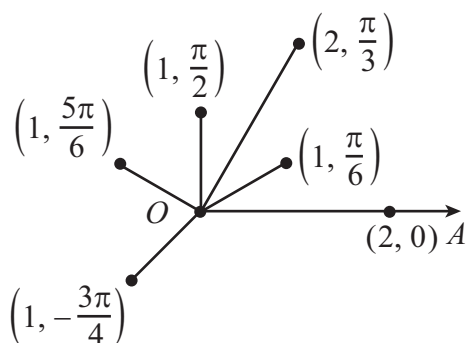
2. Calculate the Cartesian co-ordinates of each point in Exercise 1, assuming a common origin and the  $x$ -axis as the initial line.

3. Assuming a common origin and the  $x$ -axis as the initial line, find the polar co-ordinates of the points with the following Cartesian co-ordinates:

(a)  $(\sqrt{3}, 1)$ , (b)  $(3\sqrt{3}/2, -3\sqrt{3}/2)$ , (c)  $(0, 2)$ , (d)  $(-\sqrt{3}/2, -\frac{1}{2})$ .

### Answers

1.



2.

(a)  $(2, 0)$ , (b)  $(0, 1)$ , (c)  $(-\sqrt{3}/2, \frac{1}{2})$ , (d)  $(1, \sqrt{3})$ ,  
(e)  $(\sqrt{3}/2, \frac{1}{2})$ , (f)  $(-\sqrt{2}/2, -\sqrt{2}/2)$ .

3.

(a)  $(2, \pi/6)$ , (b)  $(3, -\pi/4)$ , (c)  $(2, \pi/2)$ , (d)  $(1, -5\pi/6)$ .