mathtutor
geometry, vectors

The geometry of a circle

In this unit we find the equation of a circle, when we are told its centre and its radius. There are two different forms of the equation, and you should be able to recognise both of them. We also look at some problems involving tangents to circles.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- find the equation of a circle, given its centre and radius;
- find the centre and radius of a circle, given its equation in standard form;
- find the equation of the tangent to a circle through a given point on its circumference;
- decide whether a given line is tangent to a given circle.

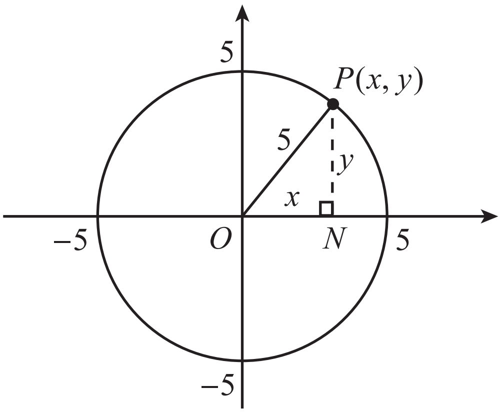
Contents

1. Introduction

The circle is a familiar shape and it has a host of geometric properties that can be proved using the traditional Euclidean format. But it is sometimes useful to work in co-ordinates and this requires us to know the standard equation of a circle, how to interpret that equation and how to find the equation of a tangent to a circle. This video will explore these particular facets of a circle, using co-ordinate geometry.

2. The equation of a circle centred at the origin

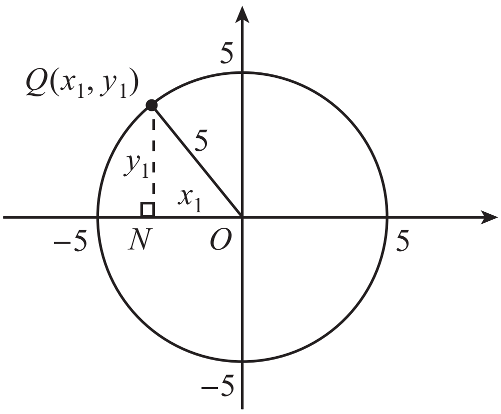
The simplest case is that of a circle whose centre is at the origin. Let us take an example. What will be the equation centred on the origin with radius 5 units?



If we take any point $P(x, y)$ on the circle, then $OP = 5$ is the radius of the circle. But OP is also the hypotenuse of the right-angled triangle OPN, formed when we drop a perpendicular from P to the x-axis. Now in the right-angled triangle, $ON = x$ and $NP = y$. Thus, using the theorem of Pythagoras,

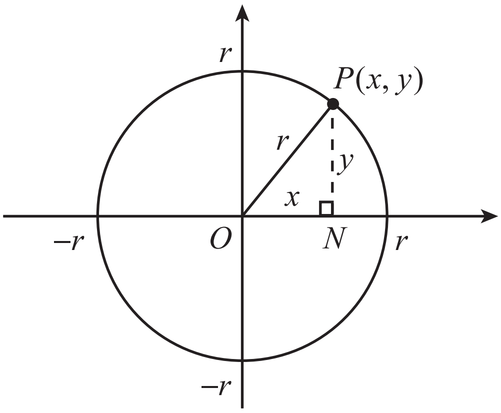
$$x^2 + y^2 = 5^2 = 25\,.$$

And this equation is true for any point on the circle. For instance, we could take a point $Q(x_1, y_1)$ in a different quadrant.



Once again, we can drop a perpendicular from Q to the x-axis. And now we can use the right-angled triangle OQN to see that $x_1^2 + y_1^2 = 5^2$. So the co-ordinates (x_1, y_1) of the point Q also satisfy the equation $x^2 + y^2 = 25$.

We shall now take the radius of the circle to be r.



If we take any point $P(x, y)$ on the circle, then $OP = r$ is the radius of the circle. But OP is also the hypotenuse of the right-angled triangle OPN, formed when we drop a perpendicular from P to the x-axis. In the right-angled triangle, $ON = x$ and $NP = y$. Thus, using the theorem of Pythagoras,

$$x^2 + y^2 = r^2\,,$$

and this is the equation of a circle of radius r whose centre is the origin $O(0, 0)$.

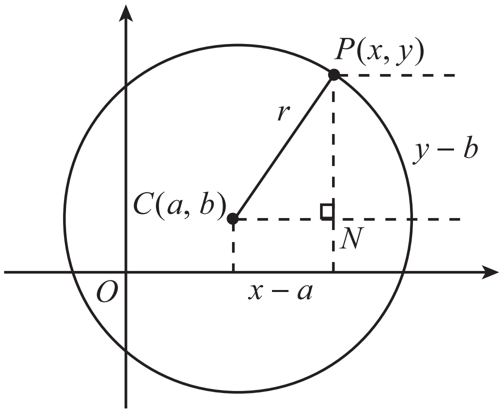
Key Point

The equation of a circle of radius r and centre the origin is

$$x^2 + y^2 = r^2\,.$$

3. The general equation of a circle

What is the equation of a circle of radius r, centred at the point $C(a, b)$?



We shall take a horizontal line through the centre C and drop a perpendicular from P to meet this horizontal line at N. Then again we have a right-angled triangle CPN, where $CP = r$ is the hypotenuse, and where we have $CN = x - a$ and $PN = y - b$. Thus using Pythagoras again we have

$$CN^2 + PN^2 = CP^2 \,,$$

so that

$$(x - a)^2 + (y - b)^2 = r^2 \,.$$

Expanding the brackets gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2 \,,$$

and if we bring r^2 to the left-hand side and rearrange we get

$$x^2 - 2ax + y^2 - 2by + a^2 + b^2 - r^2 = 0 \,.$$

It is a convention, at this point, to replace $-a$ by g and $-b$ by f. This gives

$$x^2 + 2gx + y^2 + 2fy + g^2 + f^2 - r^2 = 0 \,.$$

Now look at the last three terms on the left-hand side, $g^2 + f^2 - r^2$. These do not involve x or y at all, so together they just represent a single number that we can call c. Substituting this into the equation finally gives us

$$x^2 + 2gx + y^2 + 2fy + c = 0 \,.$$

This is the general equation of a circle. We can recognise it, because it is quadratic in both x and y, and it has two additional properties. First, there is no term in xy. And secondly, the coefficient of x^2 is the same as the coefficient of y^2. (In fact, our equation has both coefficients equal to 1, but you can always multiply an equation by a non-zero constant to obtain another valid equation, and so we must allow for this possibility.) The centre of the circle is then at $(a, b) = (-g, -f)$ and, since $c = g^2 + f^2 - r^2$, we have

$$r^2 = g^2 + f^2 - c \,,$$

so that the radius of the circle is given by

$$r = \sqrt{g^2 + f^2 - c} \,.$$

Key Point

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0\,,$$

where the centre is given by $(-g, -f)$ and the radius by $r = \sqrt{g^2 + f^2 - c}$. The equation can be recognised because it is given by a quadratic expression in both x and y with no xy term, and where the coefficients of x^2 and y^2 are equal.

Example

Find the centre and radius of the circle

$$x^2 + y^2 - 6x + 4y - 12 = 0\,.$$

Solution

First, we can check that the expression on the left-hand side is quadratic, that there is no term involving xy, and that the coefficients of x^2 and y^2 are equal. So this is the equation of a circle. If we compare this equation with the standard equation

$$x^2 + y^2 + 2gx + 2fy + c = 0\,,$$

we see that $g = -3$ and $f = 2$, so that the centre is $(-g, -f) = (3, -2)$. We also see that $c = -12$, so we can find the radius by calculating

$$\begin{aligned} r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(-3)^2 + 2^2 - (-12)} \\ &= \sqrt{9 + 4 + 12} \\ &= \sqrt{25} \\ &= 5\,. \end{aligned}$$

An alternative method is to attempt to reconstruct the equation of the circle in the form

$$(x - a)^2 + (y - b)^2 = r^2$$

by completing the square. We start by collecting together the terms in x, and the terms in y. So we rewrite our equation

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

as

$$x^2 - 6x + y^2 + 4y - 12 = 0\,.$$

Now the terms in x must come from $(x - a)^2$, a complete square, so we complete the square for the x terms, and similarly for the y terms, to get

$$(x - 3)^2 - 9 + (y + 2)^2 - 4 - 12 = 0\,.$$

So we have

$$(x-3)^2+(y+2)^2-25=0\,,$$

which we rewrite as

$$(x-3)^2+(y+2)^2=25=5^2\,.$$

We can now see that the centre of the circle is $(3,-2)$ and the radius is 5.

Example

Find the centre and radius of the circle

$$2x^2+2y^2-8x-7y=0\,.$$

Solution

Notice that this is the equation of a circle, even though the coefficients of x^2 and of y^2 are not equal to 1. But we can divide throughout by 2, and we get

$$x^2+y^2-4x-\tfrac{7}{2}y=0\,.$$

If we compare this with the standard equation

$$x^2+y^2+2gx+2fy+c=0$$

we see that $g=-2$ and $f=-\frac{7}{4}$, so the centre of the circle is $(-g,-f)=(2,\frac{7}{4})$. We also see that $c=0$, so we find the radius by calculating

$$\begin{aligned} r &= \sqrt{g^2+f^2-c} \\ &= \sqrt{(-2)^2+\left(-\tfrac{7}{4}\right)^2} \\ &= \sqrt{4+\tfrac{49}{16}} \\ &= \sqrt{\tfrac{113}{16}} \\ &= \tfrac{1}{4}\sqrt{113}\,. \end{aligned}$$

Alternatively, we could try completing the square to regain the form $(x-a)^2+(y-b)^2=r^2$. So we start with our equation

$$2x^2+2y^2-8x-7y=0\,,$$

and again we divide by 2 to get

$$x^2+y^2-4x-\tfrac{7}{2}y=0\,.$$

Collecting the x terms together and the y terms together, we get

$$x^2-4x+y^2-\tfrac{7}{2}y=0\,,$$

and then completing the square gives us

$$(x-2)^2-4+\left(y-\tfrac{7}{4}\right)^2-\tfrac{49}{16}=0$$

so that

$$(x-2)^2+\left(y-\tfrac{7}{4}\right)^2=\tfrac{113}{16}\,.$$

We can now see that the centre of the circle is $(2,\frac{7}{4})$ and the radius is $\frac{1}{4}\sqrt{113}$.

Example

Find the centre and radius of the circle

$$x^2 + y^2 + 8x + 7 = 0\,.$$

Solution

Notice that, in this example, there is no y term. If we compare our equation with the standard equation

$$x^2 + y^2 + 2gx + 2fy + c = 0\,,$$

we see that $g = 4$ and $f = 0$. So the centre of the circle is $(-g, -f) = (-4, 0)$. We also see that $c = 7$, so we find the radius by calculating

$$\begin{aligned} r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{4^2 + 0^2 - 7} \\ &= \sqrt{16 - 7} \\ &= \sqrt{9} \\ &= 3\,. \end{aligned}$$

Exercises

1. Find the equation of the circle with given centre and radius:

(a) centre $(3, 5)$, radius 3; (b) centre $(-2, 3)$, radius 1; (c) centre $(-1, -3)$, radius 2;
(d) centre $(2, -2)$, radius 5; (e) centre $(0, 5)$ radius 4.

2. Identify the centre and radius of the following circles:

(a) $x^2 + y^2 - 2x - 4y - 20 = 0$, (b) $x^2 + y^2 - 4x + 6y + 4 = 0$,
(c) $x^2 + y^2 + 2x - 3 = 0$, (d) $x^2 + y^2 + 6x + 7y - 14\frac{3}{4} = 0$,
(e) $3x^2 + 3y^2 - 6x + 9y + 5 = 0$.

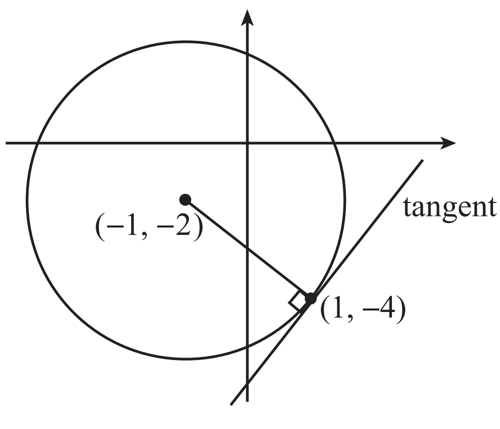
4. The equation of a tangent to a circle at a given point

What is the equation of the tangent to the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ at the point $(1, -4)$ on the circle?

For a question like this, we should check first that the given point does indeed lie on the circle. If we substitute $x = 1$ and $y = -4$ into the equation, we obtain

$$1^2 + (-4)^2 + 2 \times 1 + 4 \times (-4) - 3 \;=\; 1 + 16 + 2 + (-16) - 3 \;=\; 0\,,$$

and so the equation is satisfied. In fact we can see this from a diagram.



We have also marked the centre of the circle on the diagram. To find the centre, we note that $g = 1$ and $f = 2$, so that the centre is at $(-1, -2)$.

A tangent is a straight line that just touches the circle. To find the equation of a straight line, we need to know either two points on it, or one point on it together with its gradient. In this example, we know one point on the line, the point $(1, -4)$ where it is to touch the circle. But we do not know another point. Nor do we know the gradient. So what should we do?

One fact we do know is that the tangent to a circle is perpendicular to the radius at the point of contact. In this case, we know the point of contact $(1, -4)$, and we also know the centre $(-1, -2)$. We can therefore calculate the gradient of the radius from the centre to the point of contact, and hence the gradient of the tangent.

Now the gradient m of a straight line joining the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} .$$

So if we take $(x_1, y_1) = (1, -4)$ and $(x_2, y_2) = (-1, -2)$, the gradient m_1 of the radius is

$$m_1 = \frac{(-2) - (-4)}{(-1) - 1} = \frac{2}{-2} = -1 .$$

We now use the result that, if two lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$. Here, the gradient of the radius is $m_1 = -1$, and so the gradient of the tangent must be $m_2 = 1$.

Now we have enough information to find the equation of the tangent. We know that the equation of a straight line with a given gradient $m = 1$ and containing a given point $(x_1, y_1) = (1, -4)$ can be found from the formula

$$y - y_1 = m(x - x_1) ,$$

and so the equation of the tangent is given by

$$\begin{aligned} y - (-4) &= 1 \times (x - 1) \\ y + 4 &= x - 1 \\ y &= x - 5 . \end{aligned}$$

Key Point

To find the equation of the tangent to a circle through a given point of contact, you should first find the centre of the circle and then calculate the gradient m_1 of the line joining the centre to the point of contact.

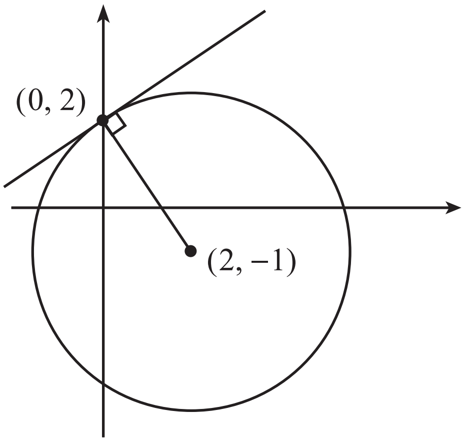
Having done this, you should find the gradient m_2 of the tangent, using the formula $m_1 m_2 = -1$. As you now know the gradient and one point on the tangent, you can find the equation of the tangent.

Example

Find the equation of the tangent at the point $(0, 2)$ to the circle $x^2 + y^2 - 4x + 2y - 8 = 0$.

Solution

We start by finding the centre of the circle. From the equation, we see that $g = -2$ and $f = 1$, so the centre of the circle is at $(2, -1)$.



Let us take $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (2, -1)$. Then the gradient m_1 of the radius joining these two points is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - 2}{2 - 0} = -\frac{3}{2}\,.$$

If the tangent has gradient m_2 then we must have $m_1 m_2 = -1$ as the tangent and the radius are perpendicular, and so $m_2 = \frac{2}{3}$.

Now we can find the equation of the tangent. We know the gradient $m_2 = \frac{2}{3}$, and we know a point $(x_1, y_1) = (0, 2)$. So the tangent is given by

$$\begin{aligned} y - y_1 &= m_2(x - x_1) \\ y - 2 &= \tfrac{2}{3}(x - 0) \\ y &= \tfrac{2}{3}x + 2\,. \end{aligned}$$

Note that the y-intercept of this line is 2, as we would expect from the fact that it passes through the given point $(0, 2)$.

Exercises

3. Find the equation of the tangent to each circle at the point specified:

(a) circle $x^2 + y^2 - 2x - 4y - 20 = 0$, point $(4, -2)$;
(b) circle $x^2 + y^2 + 4x + 2y - 20 = 0$, point $(1, 3)$;
(c) circle $x^2 + y^2 - 6x + 4y - 87 = 0$, point $(-3, -10)$;
(d) circle $x^2 + y^2 + 18x - 88 = 0$, point $(3, 5)$;
(e) circle $x^2 + y^2 - 6y - 160 = 0$, point $(12, 8)$.

4. Find the points of intersection of the line $y = 2x + 1$ and the circle $x^2 + y^2 - 2y + 4 = 0$. Show that the line $y = 2x + 1$ is a diameter of the circle. Find the equation of the tangent to the circle at one of the points of intersection.

5. Find the points of intersection of the line $y = x - 3$ and the circle $x^2 + y^2 - 2x + 2y + 1 = 0$. What are the tangents at the points of intersection? Where do they intersect?

6. Find the points where the circle $x^2 + y^2 - 10x - 10y + 40 = 0$ and the line $y + 2x = 10$ intersect. Find the equation of the tangent to the circle at each of the points of intersection. Find the point of intersection of these two tangents.

7. Show that the equation of the tangent at the point (x_1, y_1) on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0\,.$$

Answers

1.

(a)	$x^2 + y^2 - 6x - 10y + 25 = 0$	(b)	$x^2 + y^2 + 4x - 6y + 12 = 0$
(c)	$x^2 + y^2 + 2x + 6y + 6 = 0$	(d)	$x^2 + y^2 - 4x + 4y - 17 = 0$
(e)	$x^2 + y^2 - 10y + 9 = 0$		

2.

(a)	centre $(1, 2)$, radius 5	(b)	centre $(2, -3)$, radius 3	(c)	centre $(-1, 0)$, radius 2
(d)	centre $(-3, -\frac{7}{2})$, radius 6	(e)	centre $(1, -\frac{3}{2})$, radius $\sqrt{\frac{19}{12}}$		

3.

(a)	$4y = 3x - 20$	(b)	$4y + 3x = 15$	(c)	$4y + 3x + 49 = 0$
(d)	$5y + 12x = 61$	(e)	$5y + 12x = 184$		

4. The points of intersection are $(1, 3)$ and $(-1, -1)$. The mid-point of these is $(0, 1)$ which is the centre of the circle, and hence $y = 2x + 1$ is a diameter. The tangents are $2y + x = 7$ and $2y + x = 3$ respectively.

5. The points of intersection are $(1, -2)$ and $(2, -1)$. The tangents are $y = -2$ and $x = 2$ respectively. They intersect at the point $(2, -2)$.

6. The points of intersection are $(4, 2)$ and $(2, 6)$. The tangents are $3y + x = 10$ and $y = 3x$ respectively. They intersect at the point $(1, 3)$.