Introduction to vectors

A vector is a quantity that has both a magnitude (or size) and a direction. Both of these properties must be given in order to specify a vector completely. In this unit we describe how to write down vectors, how to add and subtract them, and how to use them in geometry.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

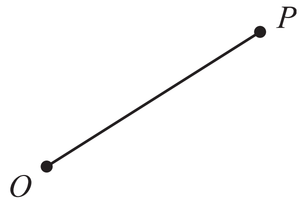
- distinguish between a vector and a scalar;
- understand how to add and subtract vectors;
- know when one vector is a multiple of another;
- use vectors to solve simple problems in geometry.

Contents

1. Introduction

Vector quantities are extremely useful in physics. The important characteristic of a vector quantity is that it has both a magnitude (or size) and a direction. Both of these properties must be given in order to specify a vector completely.

An example of a vector quantity is a displacement. This tell us how far away we are from a fixed point, and it also tells us our direction relative to that point.



Another example of a vector quantity is velocity. This is speed, in a particular direction. An example of velocity might be 60 mph due north.

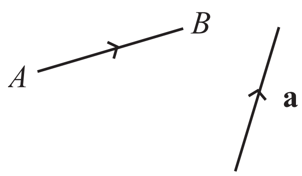
A quantity with magnitude alone, but no direction, is not a vector. It is called a *scalar* instead. One example of a scalar is distance. This tells us how far we are from a fixed point, but does not give us any information about the direction. Another example of a scalar quantity is the mass of an object.

Key Point

A vector has both magnitude and direction, and both these properties must be given in order to specify it. A quantity with magnitude but no direction is called a scalar.

2. Representing vector quantities

We can represent a vector by a line segment. This diagram shows two vectors.

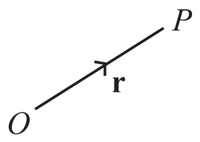


We have used a small arrow to indicate that the first vector is pointing from A to B. A vector pointing from B to A would be going in the opposite direction.

Sometimes we represent a vector with a small letter such as $\mathbf{a}$, in a bold typeface. This is common in textbooks, but it is inconvenient in handwriting. In writing, we normally put a bar underneath, or sometimes on top of, the letter: $\underline{a}$ or $\overline{a}$. In speech, we call this the vector "a-bar".

3. Position vectors

Sometimes vectors are referred to a fixed point, an origin. Such a vector is called a position vector. So we might refer to the position vector of a point P with respect to an origin O. In writing, might put $\overline{OP}$ for this vector. Alternatively, we could write it as $\mathbf{r}$. These two expressions refer to the same vector.



4. Some notation for vectors

What does it mean if, for two vectors, $\mathbf{a} = \mathbf{b}$? This means first that the length of $\mathbf{a}$ equals the length of $\mathbf{b}$, so that the two vectors have the same magnitude. But it also means that $\mathbf{a}$ and $\mathbf{b}$ are in the same direction. How can we write this down more succinctly?

If two vectors are "in the same direction", then they are parallel. We write this down as $\mathbf{a}//\mathbf{b}$.

For length, if we have a vector $\overline{AB}$, we can write its length as AB without the bar. Alternatively, we can write it as $|\overline{AB}|$. The two vertical lines give us the modulus, or size of, the vector. If we have a vector written as $\mathbf{a}$, we can write its length as either $|\mathbf{a}|$ with two vertical lines, or as a in ordinary type (or without the bar). This is why it is very important to keep to the convention that has been adopted in order to distinguish between a vector and its length.

Key Point

The length of a vector $\overline{AB}$ is written as

$$AB \text{ or } |\overline{AB}|,$$

and the length of a vector $\mathbf{a}$ is written as

$$a \text{ (in ordinary type, or without the bar) or as } |\mathbf{a}|.$$

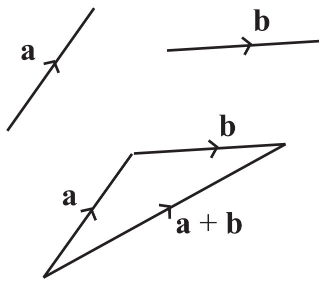
If two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, we write

$$\mathbf{a}//\mathbf{b}$$

5. Adding two vectors

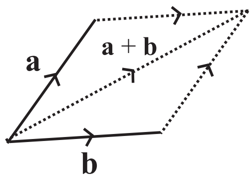
One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on.

In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So the second displacement must start where the first one finishes.



The sum of the vectors, $\mathbf{a} + \mathbf{b}$ (or the *resultant*, as it is sometimes called) is what we get when we join up the triangle. This is called the *triangle law* for adding vectors.

There is another way of adding two vectors. Instead of making the second vector start where the first one finishes, we make them both start at the same place, and complete a parallelogram. This is called the *parallelogram law* for adding vectors. It gives the same result as the triangle law, because one of the properties of a parallelogram is that opposite sides are equal and in the same direction, so that $\mathbf{b}$ is repeated at the top of the parallelogram.

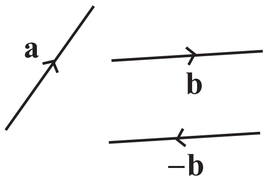


Key Point

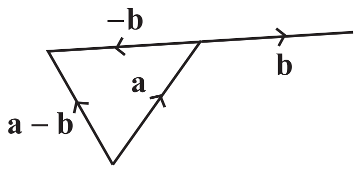
We can add two vectors $\mathbf{a}$ and $\mathbf{b}$ by making $\mathbf{b}$ start where $\mathbf{a}$ finishes, and completing the triangle. Alternatively, we can make $\mathbf{a}$ and $\mathbf{b}$ start at the same place, and take the diagonal of the parallelogram.

6. Subtracting two vectors

What is $\mathbf{a} - \mathbf{b}$? We think of this as $\mathbf{a} + (-\mathbf{b})$, and then we ask what $-\mathbf{b}$ might mean. This will be a vector equal in magnitude to $\mathbf{b}$, but in the reverse direction.



Now we can subtract two vectors. Subtracting $\mathbf{b}$ from $\mathbf{a}$ will be the same as *adding* $-\mathbf{b}$ to $\mathbf{a}$.

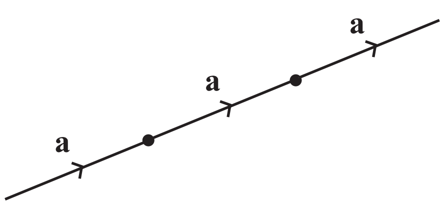


Key Point

$\mathbf{a} - \mathbf{b}$ means $\mathbf{a} + (-\mathbf{b})$

7. Adding a vector to itself

What happens when you add a vector to itself, perhaps several times? We write, for example, $\mathbf{a} + \mathbf{a} + \mathbf{a} = 3\mathbf{a}$.



In the same way, we would write

$$n\mathbf{a} = \underbrace{\mathbf{a} + \ldots + \mathbf{a}}_{n \text{ copies}}.$$

Key Point

A vector $n\mathbf{a}$ is in the same direction as the vector $\mathbf{a}$, but n times as long.

8. Vectors of unit length

There is one more piece of notation we shall use when writing vectors. If $\mathbf{a}$ is any vector, we shall write $\hat{\mathbf{a}}$ to represent a unit vector in the direction of $\mathbf{a}$. A unt vector is a vector whose length is 1, so that

$$|\hat{\mathbf{a}}| = 1.$$

This notation gives us another way of writing the vector $\mathbf{a}$: we can write it as $a\,\hat{\mathbf{a}}$, so that it is the length a multiplied by the unit vector $\hat{\mathbf{a}}$.

Key Point

A unit vector in the direction of the vector $\mathbf{a}$ is written as $\hat{\mathbf{a}}$, so that

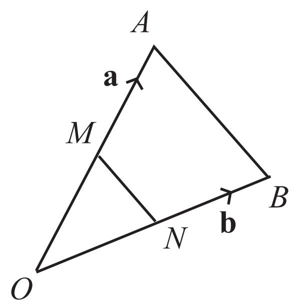
$$\mathbf{a} = a\,\hat{\mathbf{a}}.$$

9. Using vectors in geometry

Example

There is a useful theorem in geometry called the *mid-point theorem*. In this theorem, we take two points A and B, defined with respect to an origin O. Let us write $\mathbf{a}$ for the position vector of A, and $\mathbf{b}$ for the position vector of B. We can join A and B with a line, to give a triangle.

Now take the mid-point M of the line OA, and the mid-point N of the line OB, and join M to N with a line. Can we say anything about the relationship between the line MN and the line AB?



We can answer this very easily with vectors. We can write the vector for the line segment $\overline{AB}$ as $\overline{AO} + \overline{OB}$. Now $\overline{AO}$ is the reverse of the vector $\mathbf{a}$, so it is $-\mathbf{a}$. And $\overline{OB}$ is the same as the vector $\mathbf{b}$. Therefore

$$\begin{aligned}\overline{AB} &= \overline{AO} + \overline{OB}\\ &= (-\mathbf{a}) + \mathbf{b}\\ &= \mathbf{b} - \mathbf{a}.\end{aligned}$$

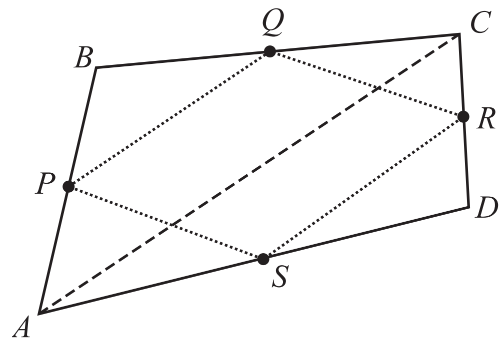
What about $\overline{MN}$? Following the same reasoning, this is $\overline{MO} + \overline{ON}$. But what is $\overline{MO}$? This is a vector half the length of $\overline{AO}$, and in the same direction, so it must be $\frac{1}{2}(-\mathbf{a})$. In the same way, $\overline{ON}$ is in the same direction as $\overline{OB}$, but is half the length, so it must be $\frac{1}{2}\mathbf{b}$. Therefore

$$\begin{aligned}\overline{MN} &= \overline{MO} + \overline{ON}\\ &= \tfrac{1}{2}(-\mathbf{a}) + \tfrac{1}{2}\mathbf{b}\\ &= \tfrac{1}{2}(\mathbf{b} - \mathbf{a}).\end{aligned}$$

Now we can compare $\overline{AB}$ and $\overline{MN}$. From our calculation, we can see that $\overline{MN}$ is $\frac{1}{2}\overline{AB}$. So, as this is a vector equation, it tells us two things. First, it tells us about magnitude, so that $MN = \frac{1}{2}AB$. Also, it tells us that MN and AB must be in the same direction, so that $MN//AB$. This is called the mid-point theorem for a triangle. It states that if you join the mid-points of two sides of a triangle then the resulting line is equal to half of the third side of the triangle, and is parallel to it.

Example

We can apply the mid-point theorem to a quadrilateral, or indeed to any four points in space, to give an interesting geometrical result. We shall call the four points A, B, C and D. We shall also give labels to the mid-points of the four sides: we shall call the mid-points P, Q, R and S. Now let us join the four mid-points, to make a new shape $PQRS$. What kind of shape is this?



We can identify the shape by joining the points A and C.

If we apply the mid-point theorem to triangle ABC, we see that

$$\overline{PQ} = \tfrac{1}{2}\overline{AC}.$$

But if we apply the mid-point theorem to the triangle ADC, we also see that

$$\overline{RS} = \tfrac{1}{2}\overline{AC}.$$

If we combine these two equations, we then obtain

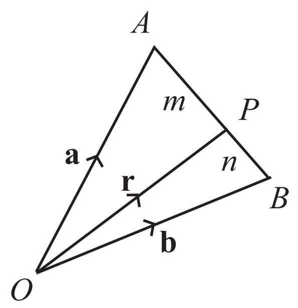
$$\overline{PQ} = \overline{RS}.$$

Now this is a vector equation, and so it tells us two things. First, it tells us that the length of PQ is the same as the length of RS. And secondly, it tells us that the direction of PQ is the same as the direction of RS, so that PQ and RS are parallel. But having two parallel sides of equal length is a property which defines a parallelogram, and so the shape $PQRS$ must be a parallelogram.

Example

We shall now use vectors to prove one more theorem.

Take two points A and B, having position vectors $\mathbf{a}$, $\mathbf{b}$ with respect to an origin O. Draw the line AB, and take a point P on that line which divides it in the ratio of m to n. What is the position vector of P with respect to O?.



We can use the same method that we used before. We know that

$$\overline{OP} = \overline{OA} + \overline{AP}, \tag{1}$$

and we also know that $\overline{OA} = \mathbf{a}$. But what is $\overline{AP}$?

Now $\overline{AP}$ is in the same direction as $\overline{AB}$, and their lengths are in the ratio of m to $m+n$. So

$$\overline{AP} = \frac{m}{m+n}\overline{AB}. \tag{2}$$

We also know that

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= \mathbf{b} - \mathbf{a}. \end{aligned}$$

Now we can put these three statements together, replacing $\overline{AP}$ in equation (1) by using equation (2), and replacing $\overline{AB}$ in equation (2) by using the equation (3), so that everything will be written in terms of **a** and **b**. This gives us

$$\overline{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}).$$

Putting all this over a common denominator then gives

$$\overline{OP} = \frac{(m+n)\mathbf{a} + m(\mathbf{b} - \mathbf{a})}{m+n}.$$

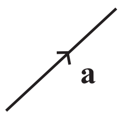
If we expand the brackets, the term $m\mathbf{a}$ will cancel with the term $m(-\mathbf{a})$, and so finally we have

$$\overline{OP} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}.$$

This formula gives us a way of calculating the position vector of the point P. For instance, if m and n were both 1 then P would be the mid-point of AB. The position vector of the midpoint would be $(\mathbf{a} + \mathbf{b})/2$. As another example, if $m = 2$ and $n = 1$, so that P was two-thirds of the way along the line, then the position vector of P would be $(\mathbf{a} + 2\mathbf{b})/3$.

Exercises

1. The vector **a** is shown below.



Sketch the vectors $2\mathbf{a}$, $3\mathbf{a}$, $\frac{1}{2}\mathbf{a}$ and $-2\mathbf{a}$.

2. In ΔOAB, $\overline{OA} = \mathbf{a}$ and $\overline{OB} = \mathbf{b}$. In terms of **a** and **b**,
   (a) What is $\overline{AB}$?
   (b) What is $\overline{BA}$?
   (c) What is $\overline{OP}$, where P is the midpoint of AB?
   (d) What is $\overline{AP}$?
   (e) What is $\overline{BP}$?
   (f) What is $\overline{OQ}$, where Q divides AB in the ratio 2:3?

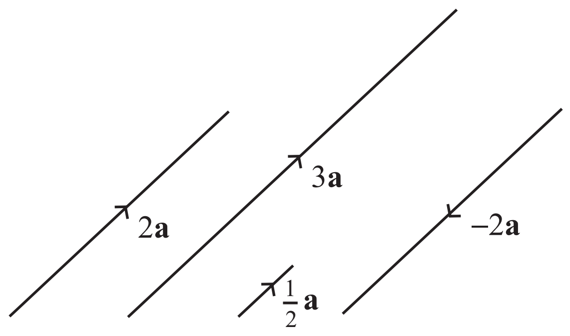
3. What is meant by a unit vector?

4. If **e** is a unit vector, what is the length of $3\mathbf{e}$?

5. In ΔABC, $AB = \mathbf{a}$, $BC = \mathbf{b}$, $CA = \mathbf{c}$. What is $\mathbf{a} + \mathbf{b} + \mathbf{c}$?

Answers

1.



2.

(a) $\mathbf{b} - \mathbf{a}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ (d) $\frac{1}{2}(\mathbf{b} - \mathbf{a})$

(e) $\frac{1}{2}(\mathbf{a} - \mathbf{b})$ (f) $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

3. A vector with length 1

4. 3

5. 0