mathtutor∞

integration

Integration as the reverse of differentiation

By now you will be familiar with differentiating common functions and will have had the opportunity to practice many techniques of differentiation. In this unit we carry out the process of differentiation in reverse. That is, we start with a given function, $f(x)$ say, and ask what function or functions, $F(x)$, would have $f(x)$ as their derivative. This leads us to the concepts of an antiderivative and integration.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- understand what is meant by the antiderivative, $F(x)$, of a function $f(x)$

- calculate some antiderivatives by using a table of derivatives

- calculate some antiderivatives by using a table of antiderivatives

- understand the link between antiderivatives and definite integrals

- calculate simple definite integrals using antiderivatives

- calculate simple indefinite integrals.

Contents

1. Introduction

By now you will be familiar with differentiating common functions and will have had the opportunity to practice many techniques of differentiation. In this unit we carry out the process of differentiation in reverse. That is, we start with a given function, $f(x)$ say, and ask what function or functions, $F(x)$, would have $f(x)$ as their derivative. This leads us to the concepts of an antiderivative and integration.

An earlier unit, *Integration as summation*, explained integration as the process of adding rectangular areas. To evaluate integrals defined in this way it was necessary to calculate the limit of a sum — a process which is cumbersome and impractical. In this unit we will see how integrals can be found by reversing the process of differentiation — that is by finding antiderivatives.

2. Antiderivatives — differentiation in reverse.

Consider the function $F(x) = 3x^2 + 7x - 2$. Suppose we write its derivative as $f(x)$, that is $f(x) = \dfrac{\mathrm{d}F}{\mathrm{d}x}$. You already know how to find this derivative by differentiating term by term to obtain $f(x) = \dfrac{\mathrm{d}F}{\mathrm{d}x} = 6x + 7$. This process is illustrated in Figure 1.

differentiate

$F(x) = 3x^2 + 7x - 2$ $f(x) = 6x + 7$
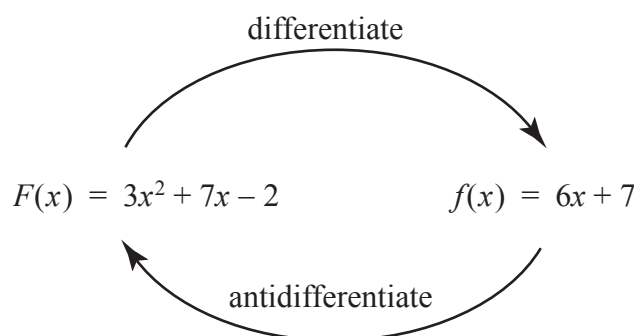
antidifferentiate

Figure 1. The function $F(x)$ is an antiderivative of $f(x)$

Suppose now that we work back to front and ask ourselves which function or functions could possibly have $6x + 7$ as a derivative. Clearly, one answer to this question is the function $3x^2 + 7x - 2$. We say that $F(x) = 3x^2 + 7x - 2$ is an **antiderivative** of $f(x) = 6x + 7$.

There are however other functions which have derivative $6x + 7$. Some of these are

$$3x^2 + 7x + 3, \qquad 3x^2 + 7x, \qquad 3x^2 + 7x - 11$$

The reason why all of these functions have the same derivative is that the constant term disappears during differentiation. So, all of these are antiderivatives of $6x+7$. Given any antiderivative of $f(x)$, all others can be obtained by simply adding a different constant. In other words, if $F(x)$ is an antiderivative of $f(x)$, then so too is $F(x) + C$ for any constant C.

Example

(a) Differentiate $F(x) = 4x^3 - 7x^2 + 12x - 4$ to find $f(x) = \dfrac{\mathrm{d}F}{\mathrm{d}x}$.

(b) Write down several antiderivatives of $f(x) = 12x^2 - 14x + 12$.

Solution

(a) Differentiating $F(x) = 4x^3 - 7x^2 + 12x - 4$ we find

$$f(x) = \frac{\mathrm{d}F}{\mathrm{d}x} = 12x^2 - 14x + 12.$$

We can deduce from this that an antiderivative of $12x^2 - 14x + 12$ is $4x^3 - 7x^2 + 12x - 4$.

(b) All other antiderivatives of $f(x)$ will take the form $F(x) + C$ where C is a constant. So, the following are all antiderivatives of $f(x)$:

$$4x^3 - 7x^2 + 12x - 4, \qquad 4x^3 - 7x^2 + 12x - 10, \qquad 4x^3 - 7x^2 + 12x, \qquad 4x^3 - 7x^2 + 12x + 3\,.$$

From these examples we deduce the following important observation.

 Key Point

A function $F(x)$ is an antiderivative of $f(x)$ if $\dfrac{\mathrm{d}F}{\mathrm{d}x} = f(x)$.

If $F(x)$ is an antiderivative of $f(x)$ then so too is $F(x) + C$ for any constant C.

We can deduce many antiderivatives that we will need simply by reading a table of derivatives from right to left, instead of from left to right. However, in practice tables of antiderivatives are commonly available such as that shown in Table 1.

$f(x)$	$F(x)$, an antiderivative of $f(x)$
k constant	$kx + C$
x	$\dfrac{x^2}{2} + C$
x^2	$\dfrac{x^3}{3} + C$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$\sin mx$	$-\dfrac{1}{m}\cos mx + C$
$\cos mx$	$\dfrac{1}{m}\sin mx + C$
e^{mx}	$\dfrac{1}{m}\mathrm{e}^{mx} + C$

Table 1. Table of some common antiderivatives

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Exercises 1.

Use the Table of antiderivatives to find $F(x)$ given $f(x)$:

1. $f(x) = 5$ 2. $f(x) = x^8$ 3. $f(x) = \dfrac{1}{x^2}$

4. $f(x) = e^{3x}$ 5. $f(x) = e^{-2x}$ 6. $\cos\dfrac{1}{2}x$.

3. The link with integration as a summation

Consider that portion of the graph of $y = f(x)$ to the right of the y-axis as illustrated in Figure 2. Suppose $f(x)$ lies entirely above the x-axis. Clearly the area bounded by the graph and the x-axis will depend upon how far to the right we move. That is the area, A, will depend upon the value of x, $A = A(x)$.
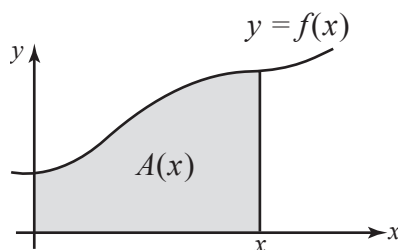

Figure 2. The area bounded by the graph will depend upon
how far to the right we move, that is $A = A(x)$.

Suppose we consider the additional contribution made to this area by moving a little further to the right as shown in Figure 3. Let this additional contribution be denoted by δA. Then δA is the change in area produced by increasing x by δx. Note then that $\delta A = A(x + \delta x) - A(x)$.
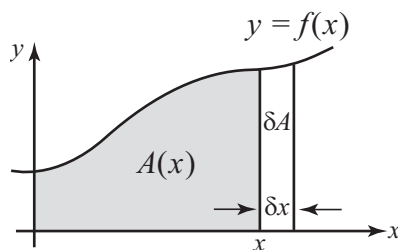

Figure 3. The additional contribution to the area is $\delta A \approx f(x)\,\delta x$.

This additional contribution, δA, can be approximated by assuming it has the form of a rectangle of height $y = f(x)$ and width δx, that is

$$\delta A \approx f(x)\delta x\,,$$

that is

$$\frac{\delta A}{\delta x} \approx f(x)\,.$$

In the limit as δx tends to zero we have

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x)\,.$$

We can use the previous section to interpret this result.

Given that $f(x)$ is the derivative of $A(x)$ then, $A(x)$ must be an antiderivative of $f(x)$, i.e.

$$A(x) = F(x) + C \qquad (1)$$

In Section 2 we wrote $F(x) + C$ for a whole family of functions all of which are antiderivatives when $F(x)$ is. Now, we are looking for a specific antiderivative by choosing a specific value for C. This is obtained by noting that when $x = 0$ the area bounded is zero, that is $0 = F(0) + C$, and this gives us a value for C, that is $C = -F(0)$.

We can use this result as follows: given a function $f(x)$, then the formula in Equation 1 can tell us the area under the graph provided we can calculate an antiderivative $F(x)$.

Suppose now we want to find the area under the graph between $x = a$ and $x = b$ as shown in Figure 4.

The total area up to $x = b$ is given, using Equation 1, by

$$A(b) = F(b) + C \,.$$

The total area up to $x = a$ is given, using Equation 1, by

$$A(a) = F(a) + C \,.$$

And so, subtracting, the area between a and b is

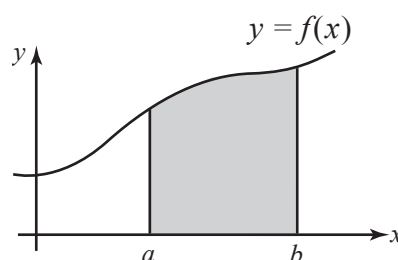$$A(b) - A(a) = F(b) - F(a) \,.$$

Note how the C's cancel out.

Figure 4. The area between a and b is $A(b) - A(a)$.

Key Point

When $f(x)$ lies entirely above the x axis between a and b we can find the area under $y = f(x)$ between a and b by finding an antiderivative $F(x)$, and evaluating this at $x = a$ and at $x = b$. The area is then

$$A = F(b) - F(a)$$

In the unit entitled **Integration as a summation** we looked at how the area under a curve could be found by adding the areas of rectangular elements. It was shown that the area under $y = f(x)$ between $x = a$ and $x = b$ is found from

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x\,.$$

This limiting process is denoted by

$$A = \int_a^b f(x)\,\mathrm{d}x$$

and this defines what we mean by the **definite integral** of $f(x)$ between the limits a and b.

So we now have two ways of finding the area under a curve, one involving taking the limit of a sum, that is finding a definite integral, and one involving find an antiderivative. Generally, finding the limit of a sum is difficult, whereas with some practice, finding an antiderivative is much simpler. So, in future we can avoid the more difficult way and evaluate integrals using antiderivatives.

Key Point

If $F(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)\,.$$

Example

Suppose we wish to evaluate $\displaystyle\lim_{\delta x \to 0} \sum_{x=0}^{x=1} x^2\,\delta x$. This limit of a sum arises when we use the sum of small rectangles to find the area under $y = x^2$ between $x = 0$ and $x = 1$. This limit defines the definite integral

$$\int_0^1 x^2\,\mathrm{d}x\,.$$

Using the Key Point above we can evaluate this integral by finding an antiderivative of x^2. This is $F(x) = \dfrac{x^3}{3} + C$. We evaluate this at $x = 1$ and $x = 0$ and find the difference between the two results:

$$\int_0^1 x^2\,\mathrm{d}x = \left(\frac{1^3}{3} + C\right) - \left(\frac{0^3}{3} + C\right)$$
$$= \frac{1}{3}$$

Note that the C's cancel — this will always happen, and so in future when dealing with definite integrals we will omit the C altogether. It is conventional to set the previous calculation out as follows:

$$\begin{aligned} \int_0^1 x^2 \,\mathrm{d}x &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right) \\ &= \frac{1}{3}. \end{aligned}$$

More generally we write

$$\begin{aligned} \int_a^b f(x)\mathrm{d}x &= [F(x)]_a^b \\ &= F(b) - F(a). \end{aligned}$$

Exercises 2

Evaluate the following limits of sums using antiderivatives.

1. Find $\displaystyle\lim_{\delta x \to 0} \sum_{x=0}^{x=1} x^3 \,\delta x.$

2. Find $\displaystyle\lim_{\delta x \to 0} \sum_{x=1}^{x=2} x^4 \,\delta x.$

3. Find $\displaystyle\lim_{\delta x \to 0} \sum_{x=0}^{x=1} \mathrm{e}^{2x} \,\delta x.$

4. Indefinite integrals

As we have seen definite integration is closely associated with finding antiderivatives. Because of this it is common to think of antidifferentiation in general as integration. So when asked to reverse the process of differentiation this is commonly referred to as integration.

In the Example on page 2 we found antiderivatives of $f(x) = 12x^2 - 14x + 12$. Using integration notation we would commonly write

$$\int (12x^2 - 14x + 12)\mathrm{d}x = 4x^3 - 7x^2 + 12x + C$$

and refer to this as the **indefinite integral** of $12x^2 - 14x + 12$ with respect to x. The constant C is called the **constant of integration**. Tables of antiderivatives such as Table 1 are often called Tables of Integrals.

Key Point

The **indefinite integral** of the function $f(x)$, $\int f(x)\,\mathrm{d}x$, is another function. It is given by $F(x) + C$ where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

The **definite integral** of the function $f(x)$ between the limits a and b, $\int_a^b f(x)\,\mathrm{d}x$, is a number. It may be obtained from the formula $F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

In this unit we have described how to carry out differentiation in reverse, leading to the concept of an antiderivative. We have also seen how we can use antiderivatives to find integrals. In subsequent units you can learn how to integrate a much wider range of functions that are met in engineering and the physical sciences.

Exercises 3

Use a Table to find the following integrals:

1. $\displaystyle\int e^{2x}\,\mathrm{d}x$ 2. $\displaystyle\int \sin 3x\,\mathrm{d}x$ 3. $\displaystyle\int \cos 7x\,\mathrm{d}x$

4. $\displaystyle\int e^{-3x}\,\mathrm{d}x$ 5. $\displaystyle\int \frac{1}{x^3}\,\mathrm{d}x$ 6. $\displaystyle\int x^{1/2}\,\mathrm{d}x$

Answers

Exercises 1.

1. $5x + C$ 2. $\dfrac{x^9}{9} + C$ 3. $-\dfrac{1}{x} + C$

4. $\dfrac{1}{3}e^{3x} + C$ 5. $-\dfrac{1}{2}e^{-2x} + C$ 6. $2\sin\dfrac{1}{2}x + C$.

Exercises 2.

1. $\dfrac{1}{4}$ 2. $\dfrac{31}{5}$ 3. $\dfrac{1}{2}(e^2 - 1)$.

Exercises 3.

1. $\dfrac{1}{2}e^{2x} + C$ 2. $-\dfrac{1}{3}\cos 3x + C$ 3. $\dfrac{1}{7}\sin 7x + C$

4. $-\dfrac{1}{3}e^{-3x} + C$ 5. $-\dfrac{1}{2x^2} + C$ 6. $\dfrac{2x^{3/2}}{3} + C$.