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algebra

Pascal's triangle and the binomial theorem

A binomial expression is the sum, or difference, of two terms. For example,

$$x + 1, \quad 3x + 2y, \qquad a - b$$

are all binomial expressions. If we want to raise a binomial expression to a power higher than 2 (for example if we want to find $(x + 1)^7$) it is very cumbersome to do this by repeatedly multiplying $x + 1$ by itself. In this unit you will learn how a triangular pattern of numbers, known as Pascal's triangle, can be used to obtain the required result very quickly.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- generate Pascal's triangle

- expand a binomial expression using Pascal's triangle

- use the binomial theorem to expand a binomial expression

Contents

1. Introduction

A **binomial expression** is the sum, or difference, of two terms. For example,

$$x + 1, \quad 3x + 2y, \qquad a - b$$

are all binomial expressions.

You will be familiar already with the need to expand brackets when squaring such quantities. You will know, for example, that

$$
\begin{aligned}
(x + 1)^2 &= (x + 1)(x + 1) \\
&= x^2 + x + x + 1 \\
&= x^2 + 2x + 1
\end{aligned}
$$

If we want to raise a binomial expression to a power higher than 2 (for example if we want to find $(x + 1)^7$) it is very cumbersome to do this by repeatedly multiplying $x + 1$ by itself. In this unit you will learn how a triangular pattern of numbers, known as **Pascal's triangle**, can be used to obtain the required result very quickly.

2. Pascal's triangle

We start to generate Pascal's triangle by writing down the number 1. Then we write a new row with the number 1 twice:

$$1$$

$$1 \qquad 1$$

We then generate new rows to build a triangle of numbers. Each new row must begin and end with a 1:

$$1$$

$$1 \qquad 1$$

$$1 \qquad * \qquad 1$$

$$1 \qquad * \qquad * \qquad 1$$

The remaining numbers in each row are calculated by adding together the two numbers in the row above which lie above-left and above-right.

So, adding the two 1's in the second row gives 2, and this number goes in the vacant space in the third row:

$$1$$

$$1 \qquad\qquad 1$$

$$1 \qquad 2 \qquad 1$$

$$1 \qquad * \qquad * \qquad 1$$

The two vacant spaces in the fourth row are each found by adding together the two numbers in the third row which lie above-left and above-right: $1 + 2 = 3$, and $2 + 1 = 3$. This gives:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & & & \searrow & \swarrow & & \\
 & 1 & & 2 & & 1 & \\
 & & \searrow & \swarrow & \searrow & \swarrow & \\
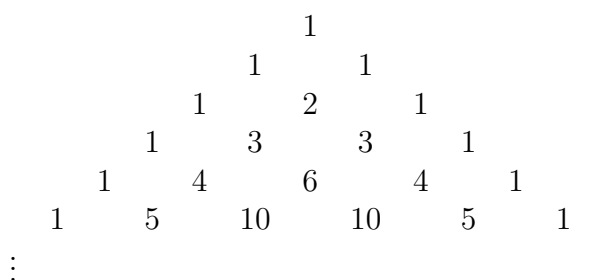1 & & 3 & & 3 & & 1
\end{array}
$$
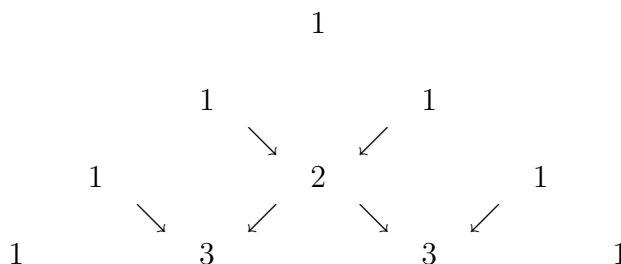
We can continue to build up the triangle in this way to write down as many rows as we wish. The Key Point below shows the first six rows of Pascal's triangle.

Key Point

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & & & & & & & \vdots &
\end{array}
$$

Exercise 1

1. Generate the seventh, eighth, and ninth rows of Pascal's triangle.

3. Using Pascal's triangle to expand a binomial expression

We will now see how useful the triangle can be when we want to expand a binomial expression. Consider the binomial expression $a + b$, and suppose we wish to find $(a + b)^2$.
We know that

$$
\begin{aligned}
(a + b)^2 &= (a + b)(a + b) \\
&= a^2 + ab + ba + b^2 \\
&= a^2 + 2ab + b^2
\end{aligned}
$$

That is,

$$
(a + b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2
$$

Observe the following in the final result:

1. As we move through each term from left to right, the power of a decreases from 2 down to zero.

2. The power of b increases from zero up to 2.

3. The coefficients of each term, (1, 2, 1), are the numbers which appear in the row of Pascal's triangle beginning 1,2.

4. The term $2ab$ arises from contributions of $1ab$ and $1ba$, i.e. $1ab + 1ba = 2ab$. This is the link with the way the 2 in Pascal's triangle is generated; i.e. by adding 1 and 1 in the previous row.

If we want to expand $(a + b)^3$ we select the coefficients from the row of the triangle beginning 1,3: these are 1,3,3,1. We can immediately write down the expansion by remembering that for each new term we decrease the power of a, this time starting with 3, and increase the power of b. So

$$(a + b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3$$

which we would normally write as just

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Thinking of $(a + b)^3$ as

$$(a + b)(a^2 + 2ab + b^2) = a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

we note that the term $3ab^2$, for example, arises from the two terms ab^2 and $2ab^2$; again this is the link with the way 3 is generated in Pascal's triangle - by adding the 1 and 2 in the previous row.

Example

Suppose we wish to find $(a + b)^4$.

To find this we use the row beginning 1,4, and can immediately write down the expansion.

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

We can apply the same procedure to expand any binomial expression, even when the quantities a and b are more complicated. Consider the following examples.

Example

Suppose we want to expand $(2x + y)^3$.

We pick the coefficients in the expansion from the relevant row of Pascal's triangle: (1,3,3,1). As we move through the terms in the expansion from left to right we remember to decrease the power of $2x$ and increase the power of y. So,

$$\begin{aligned} (2x + y)^3 &= 1(2x)^3 + 3(2x)^2y + 3(2x)^1y^2 + 1y^3 \\ &= 8x^3 + 12x^2y + 6xy^2 + y^3 \end{aligned}$$

Example

Suppose we want to expand $(1 + p)^4$.

We pick the coefficients in the expansion from the row of the triangle beginning 1,4; that is (1,4,6,4,1). As we move through the terms in the expansion from left to right we remember to increase the power of p. This example is simpler than the previous one because the first term in brackets is 1, and 1 to any power is still 1. So,

$$
\begin{aligned}
(1+p)^4 &= 1(1)^4 + 4(1)^3 p + 6(1)^2 p^2 + 4(1)p^3 + 1p^4 \\
&= 1 + 4p + 6p^2 + 4p^3 + p^4.
\end{aligned}
$$

Either or both of the terms in the binomial expression can be negative. When raising a negative number to an even power the result is positive. When raising a negative number to an odd power the result is negative. Consider the following example.

Example

Expand $(3a - 2b)^5$.

We pick the coefficients in the expansion from the row of Pascal's triangle beginning 1,5; that is 1,5,10,10,5,1. Powers of $3a$ decrease from 5 as we move left to right. Powers of $-2b$ increase.

$$
\begin{aligned}
(3a - 2b)^5 &= 1(3a)^5 + 5(3a)^4(-2b) + 10(3a)^3(-2b)^2 + 10(3a)^2(-2b)^3 + 5(3a)(-2b)^4 + 1(-2b)^5 \\
&= 243a^5 - 810a^4 b + 1080a^3 b^2 - 720a^2 b^3 + 240ab^4 - 32b^5
\end{aligned}
$$

Either or both of the terms could be fractions.

Example

Expand $\left(1 + \dfrac{2}{x}\right)^3$.

We pick the coefficients in the expansion from the row of Pascal's triangle (1,3,3,1). Powers of $\dfrac{2}{x}$ increase as we move left to right. Any power of 1 is still 1.

$$
\begin{aligned}
\left(1 + \frac{2}{x}\right)^3 &= 1(1)^3 + 3(1)^2 \left(\frac{2}{x}\right) + 3(1)^1 \left(\frac{2}{x}\right)^2 + 1 \left(\frac{2}{x}\right)^3 \\
&= 1 + \frac{6}{x} + \frac{12}{x^2} + \frac{8}{x^3}
\end{aligned}
$$

Exercises 2

Use Pascal's triangle to expand the following binomial expressions:

1. $(1 + 3x)^2$ 2. $(2 + x)^3$ 3. $(1 - x)^3$ 4. $(1 - 5x)^5$

5. $(x + 6)^3$ 6. $(a - b)^7$ 7. $\left(1 + \dfrac{3}{a}\right)^4$ 8. $\left(x - \dfrac{1}{x}\right)^6$.

4. The binomial theorem

If we wanted to expand a binomial expression with a large power, e.g. $(1+x)^{32}$, use of Pascal's triangle would not be recommended because of the need to generate a large number of rows of the triangle. An alternative method is to use the **binomial theorem**. The theorem enables us to expand $(a+b)^n$ in increasing powers of b and decreasing powers of a. We will look at expanding expressions of the form $(a+b)^2$, $(a+b)^3$, ..., $(a+b)^{32}$,..., that is when the power is a positive whole number. Under certain conditions the theorem can be used when n is negative or fractional and this is useful in more advanced applications, but these conditions will not be studied here.

 Key Point

The binomial theorem:

When n is a positive whole number

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \ldots + b^n$$

Note that this is a **finite** series (that is, it stops after a finite number of terms) and the last term is b^n.

A simpler form of the theorem is often quoted by taking the special case in which $a = 1$ and $b = x$. It is straightforward to verify that the theorem becomes:

 Key Point

The binomial theorem:

When n is a positive whole number

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots + x^n$$

Example

We shall apply the binomial theorem to expand $(1+x)^2$.

We use the theorem with $n = 2$ and stop when we have written down the term in x^2.

$$\begin{aligned}(1+x)^2 &= 1 + 2x + \frac{(2)(2-1)}{2!}x^2 \\ &= 1 + 2x + x^2\end{aligned}$$

which is the familiar and well-known result.

Example

We now apply the binomial theorem to expand $(1+x)^3$.

We use the theorem with $n = 3$.

$$\begin{aligned}(1+x)^3 &= 1 + 3x + \frac{(3)(3-1)}{2!}x^2 + \frac{(3)(3-1)(3-2)}{3!}x^3 \\ &= 1 + 3x + 3x^2 + x^3\end{aligned}$$

Example

Suppose we wish to apply the binomial theorem to find the first three terms in ascending powers of x of $(1+x)^{32}$.

We use the theorem with $n = 32$ and just write down the first three terms.

$$\begin{aligned}(1+x)^{32} &= 1 + 32x + \frac{(32)(32-1)}{2!}x^2 \\ &= 1 + 32x + 496x^2 + \ldots\end{aligned}$$

With some ingenuity we can use the theorem to expand other binomial expressions.

Example

Suppose we wish to find the first four terms in the expansion of $(1 + \frac{1}{3}y)^{10}$.

We use the theorem, replacing x with $\dfrac{y}{3}$ and letting $n = 10$. This gives

$$\begin{aligned}(1+\frac{1}{3}y)^{10} &= 1 + 10\left(\frac{y}{3}\right) + \frac{(10)(10-1)}{2!}\left(\frac{y}{3}\right)^2 + \frac{(10)(10-1)(10-2)}{3!}\left(\frac{y}{3}\right)^3 + \ldots \\ &= 1 + \frac{10}{3}y + 5y^2 + \frac{40}{9}y^3 + \ldots\end{aligned}$$

Example

Suppose we wish to find the first three terms in the expansion of $(3 - 5z)^{14}$.

We shall apply the binomial theorem in the original form given on page 6 with $a = 3$, $b = -5z$ and $n = 14$.

$$\begin{aligned}(3-5z)^{14} &= 3^{14} + 14(3^{13})(-5z) + \frac{(14)(13)}{2!}(3^{12})(-5z)^2 \\ &= 3^{14} - (3^{13})70z + (3^{12})2275z^2 \\ &= 3^{14}\left(1 - \frac{70z}{3} + \frac{2275}{9}z^2 - \ldots\right)\end{aligned}$$

Exercises 3

1. Use the binomial theorem to expand (a) $(1 + x)^4$ and (b) $(1 + x)^5$.
2. Use the binomial theorem to expand $(1 + 2x)^3$.
3. Use the binomial theorem to expand $(1 - 3x)^4$.
4. Use the binomial theorem to find the first three terms in ascending powers of x of $(1 - \frac{x}{2})^8$.
5. Find the coefficient of x^5 in the expansion of $(1 + 4x)^9$.
6. In the expansion of $(1 - x)^8$ find the coefficient of x^7.
7. Find the first four terms in the expansion of $\left(2 + \frac{x}{3}\right)^{12}$.

Answers

Exercise 1

1. Seventh row: 1,6,15,20,15,6,1. Eighth row: 1,7,21,35,35,21,7,1. Ninth row: 1,8,28,56,70,56,28,8,1.

Exercise 2

1. $1 + 6x + 9x^2$ 2. $8 + 12x + 6x^2 + x^3$ 3. $1 - 3x + 3x^2 - x^3$

4. $1 - 25x + 250x^2 - 1250x^3 + 3125x^4 - 3125x^5$

5. $x^3 + 18x^2 + 108x + 216$

6. $a^7 - 7\,a^6\,b + 21\,a^5\,b^2 - 35\,a^4\,b^3 + 35\,a^3\,b^4 - 21\,a^2\,b^5 + 7\,a\,b^6 - b^7$.

7. $1 + \dfrac{12}{a} + \dfrac{54}{a^2} + \dfrac{108}{a^3} + \dfrac{81}{a^4}$

8. $x^6 - 6\,x^4 + 15\,x^2 - 20 + \dfrac{15}{x^2} - \dfrac{6}{x^4} + \dfrac{1}{x^6}$

Exercise 3

1. (a) $1 + 4\,x + 6\,x^2 + 4\,x^3 + x^4$ (b) $1 + 5\,x + 10\,x^2 + 10\,x^3 + 5\,x^4 + x^5$.
2. $1 + 6\,x + 12\,x^2 + 8\,x^3$.
3. $1 - 12\,x + 54\,x^2 - 108\,x^3 + 81\,x^4$.
4. $1 - 4\,x + 7\,x^2 \ldots$.
5. 129024.
6. -8.
7. $4096 + 8192\,x + \dfrac{22528}{3}\,x^2 + \dfrac{112640}{27}\,x^3 \ldots$