mathtutor
integration

Integration by substitution

There are occasions when it is possible to perform an apparently difficult piece of integration by first making a substitution. This has the effect of changing the variable and the integrand. When dealing with definite integrals, the limits of integration can also change. In this unit we will meet several examples of integrals where it is appropriate to make a substitution.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- carry out integration by making a substitution
- identify appropriate substitutions to make in order to evaluate an integral

Contents

1. Introduction

There are occasions when it is possible to perform an apparently difficult piece of integration by first making a **substitution**. This has the effect of changing the variable and the integrand. When dealing with definite integrals, the limits of integration can also change. In this unit we will meet several examples of this type. The ability to carry out integration by substitution is a skill that develops with practice and experience. For this reason you should carry out all of the practice exercises. Be aware that sometimes an apparently sensible substitution does not lead to an integral you will be able to evaluate. You must then be prepared to try out alternative substitutions.

Before looking at the first example, it is worth recording a preliminary result which will be very useful in all the examples.

Suppose we have a variable $u = u(x)$ which can be differentiated, to give $\frac{\mathrm{d}u}{\mathrm{d}x}$. Then a quantity called the differential is given by

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x\,.$$

For example, if we have $u = 1 - 2x$, we can differentiate u to give

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2$$

so that $\mathrm{d}u = -2\,\mathrm{d}x$.

2. Integration by substituting $u = ax + b$

We introduce the technique through some simple examples for which a linear substitution is appropriate.

Example

Suppose we want to find the integral

$$\int (x+4)^5\,\mathrm{d}x\,. \qquad (1)$$

You will be familiar already with finding a similar integral $\int u^5\,\mathrm{d}u$ and know that this integral is equal to $\frac{u^6}{6} + c$, where c is a constant of integration. This is because you know that the rule for integrating powers of a variable tells you to increase the power by 1 and then divide by the new power.

In the integral given by Equation (1) there is still a power 5, but the integrand is more complicated due to the presence of the term $x + 4$. To tackle this problem we make a **substitution**. We let $u = x + 4$. The point of doing this is to change the integrand into the much simpler u^5. However, we must take care to substitute appropriately for the term $\mathrm{d}x$, too.

In terms of differentials we have

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x\,.$$

Now, in this example, because $u = x + 4$ it follows immediately that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ and so

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = 1\,\mathrm{d}x = \mathrm{d}x\,.$$

So, substituting both for $x + 4$ and for $\mathrm{d}x$ in Equation (1) we have

$$\int (x+4)^5\,\mathrm{d}x = \int u^5\mathrm{d}u\,.$$

The resulting integral can be evaluated immediately to give $\dfrac{u^6}{6} + c$. We can revert to an expression involving the original variable x by recalling that $u = x + 4$, giving

$$\int (x+4)^5\,\mathrm{d}x = \frac{(x+4)^6}{6} + c\,.$$

We have completed the integration by substitution.

Example

Suppose now we wish to find the integral

$$\int \cos(3x+4)\,\mathrm{d}x\,. \qquad (2)$$

Observe that if we make a substitution $u = 3x + 4$, the integrand will then contain the much simpler form $\cos u$ which we will be able to integrate.

As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

and so

$$\text{with } u = 3x + 4 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

it follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = 3\,\mathrm{d}x\,.$$

So, substituting u for $3x + 4$, and with $\mathrm{d}x = \frac{1}{3}\mathrm{d}u$ in Equation (2) we have

$$\begin{aligned}\int \cos(3x+4)\,\mathrm{d}x &= \int \frac{1}{3}\cos u\,\mathrm{d}u \\ &= \frac{1}{3}\int \cos u\,\mathrm{d}u \\ &= \frac{1}{3}\sin u + c\,.\end{aligned}$$

We can revert to an expression involving the original variable x by recalling that $u = 3x + 4$, giving

$$\int \cos(3x+4)\,\mathrm{d}x = \frac{1}{3}\sin(3x+4) + c\,.$$

We have completed the integration by substitution.

It is very easy to generalise the result of the previous example. If we want to find $\int \cos(ax+b)\mathrm{d}x$ where a and b are constants, the substitution $u = ax + b$ gives $\mathrm{d}u = a\,\mathrm{d}x$, so that $\mathrm{d}x = \frac{1}{a}\mathrm{d}u$. The integral is then

$$\frac{1}{a}\int \cos u\,\mathrm{d}u,$$

which equals $\frac{1}{a}\sin u + c$, and in terms of x this is $\frac{1}{a}\sin(ax+b) + c$. For instance, we can use this result to see straight away that

$$\int \cos(7x+3)\mathrm{d}x = \frac{1}{7}\sin(7x+3) + c.$$

A similar argument, which you should try, shows that $\int \sin(ax+b)\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c.$

Key Point

$$\int \sin(ax+b)\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c \qquad \int \cos(ax+b)\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$$

Example

Suppose we wish to find $\int \frac{1}{1-2x}\mathrm{d}x.$

We make the substitution $u = 1 - 2x$ in order to simplify the integrand to $\frac{1}{u}$. Recall that the integral of $\frac{1}{u}$ with respect to u is the natural logarithm of u, $\ln|u|$. As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

and so

$$\text{with} \quad u = 1 - 2x \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -2$$

it follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = -2\,\mathrm{d}x\,.$$

The integral becomes

$$\begin{aligned}\int \frac{1}{u}\left(-\frac{1}{2}\mathrm{d}u\right) &= -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \\ &= -\frac{1}{2}\ln|u| + c \\ &= -\frac{1}{2}\ln|1-2x| + c\,.\end{aligned}$$

The result of the previous example can be generalised: if we want to find $\int \frac{1}{ax+b}\mathrm{d}x$, the substitution $u = ax+b$ gives $\mathrm{d}u = a\,\mathrm{d}x$, so that $\mathrm{d}x = \frac{1}{a}\mathrm{d}u$. So the integral is

$$\frac{1}{a}\int \frac{1}{u}\,\mathrm{d}u,$$

which equals $\frac{1}{a}\ln|u| + c$, and in terms of x this is $\frac{1}{a}\ln|ax+b| + c$. For instance, we can use this result to see straight away that

$$\int \frac{1}{x+1}\,\mathrm{d}x = \ln|x+1| + c$$

and

$$\int \frac{1}{3x-2}\,\mathrm{d}x = \frac{1}{3}\ln|3x-2| + c.$$

Key Point

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

A little more care must be taken with the limits of integration when dealing with definite integrals. Consider the following example.

Example
Suppose we wish to find

$$\int_1^3 (9+x)^2\,\mathrm{d}x\,.$$

We make the substitution $u = 9+x$. As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

and so

$$\text{with} \quad u = 9 + x \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

it follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = \mathrm{d}x\,.$$

The integral becomes

$$\int_{x=1}^{x=3} u^2\,\mathrm{d}u$$

where we have explicitly written the variable in the limits of integration to emphasise that those limits were on the variable x and not u. We can write these as limits on u using the substitution $u = 9 + x$. Clearly, when $x = 1$, $u = 10$, and when $x = 3$, $u = 12$. So we require

$$\begin{aligned}
\int_{x=1}^{x=3} u^2\,\mathrm{d}u &= \int_{u=10}^{u=12} u^2\,\mathrm{d}u \\
&= \left[\frac{1}{3}u^3\right]_{10}^{12} \\
&= \frac{1}{3}\left(12^3 - 10^3\right) \\
&= \frac{1}{3}(1728 - 1000) \\
&= \frac{728}{3}\,.
\end{aligned}$$

Note that in this example there is no need to convert the answer given in terms of u back into one in terms of x because we had already converted the limits on x into limits on u.

Exercises 1.

1. In each case use a substitution to find the integral:

(a) $\int (x-2)^3\mathrm{d}x$ (b) $\int_0^1 (x+5)^4\mathrm{d}x$ (c) $\int (2x-1)^7\mathrm{d}x$ (d) $\int_{-1}^1 (1-x)^3\mathrm{d}x$.

2. In each case use a substitution to find the integral:

(a) $\int \sin(7x-3)\mathrm{d}x$ (b) $\int \mathrm{e}^{3x-2}\mathrm{d}x$ (c) $\int_0^{\pi/2} \cos(1-x)\mathrm{d}x$ (d) $\int \frac{1}{7x+5}\mathrm{d}x$.

3. Finding $\displaystyle\int f(g(x))g'(x)\,\mathrm{d}x$ by substituting $u = g(x)$

We are now going to extend the method of integrating by substitution by looking at integrals which can be written in the form

$$\int f(g(x))g'(x)\,\mathrm{d}x\,.$$

Now this expression looks very complicated, so we shall try to understand it by taking a particular example.

Example

Suppose we write

$$g(x) = 1 + x^2 \qquad \text{and} \qquad f(u) = \sqrt{u}\,.$$

Then we note that the composition[1] of the functions f and g is $f(g(x)) = \sqrt{1+x^2}$.

Further, we note that if $g(x) = 1 + x^2$ then $g'(x) = 2x$. So the integral

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x \qquad \text{is of the form} \qquad \int f(g(x))\,g'(x)\,\mathrm{d}x\,.$$

To perform an integration of this kind, we always try the substitution $u = g(x)$. Then

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = g'(x)\mathrm{d}x.$$

So in this case, we try the substitution $u = 1 + x^2$. Then $\mathrm{d}u = g'(x)\mathrm{d}x = 2x\,dx$.

Once the substitution is made, the resulting integral becomes $\displaystyle\int \sqrt{u}\,\mathrm{d}u$. In the general case, the integral will become $\displaystyle\int f(u)\,\mathrm{d}u$. This will always happen when the integrand is in the particular form $f(g(x))g'(x)$. Then, provided that the final integral can be found, the problem is solved. So now

$$\begin{aligned}
\int 2x\sqrt{1+x^2}\,\mathrm{d}x &= \int \sqrt{u}\,\mathrm{d}u \\
&= \int u^{1/2}\,\mathrm{d}u \\
&= \frac{u^{3/2}}{3/2} + c \\
&= \frac{2}{3}u^{3/2} + c\,.
\end{aligned}$$

We can revert to an expression involving the original variable x by recalling that $u = 1 + x^2$, giving

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \frac{2}{3}(1+x^2)^{3/2} + c\,.$$

We have completed the integration by substitution.

[1] When finding the composition of functions f and g it is the output from g which is used as input to f, resulting in $f(g(x))$.

For purposes of comparison the specific example and the general case are presented side-by-side:

$\displaystyle\int 2x\sqrt{1+x^2}\,\mathrm{d}x$	$\displaystyle\int f(g(x))g'(x)\mathrm{d}x$
let $u = 1+x^2$	let $u = g(x)$
$\mathrm{d}u = \left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = 2x\,\mathrm{d}x$	$\mathrm{d}u = \left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = g'(x)\,\mathrm{d}x$
$\displaystyle\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u$	$\displaystyle\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\,\mathrm{d}u$
$\dfrac{2}{3}u^{3/2}+c$	
$\dfrac{2}{3}(1+x^2)^{3/2}+c$	

It is worth pointing out that integration by substitution is something of an art — and your skill at doing it will improve with practice. Furthermore, a substitution which at first sight might seem sensible, can lead nowhere. For example, if you were try to find $\int\sqrt{1+x^2}\mathrm{d}x$ by letting $u = 1+x^2$ you would find yourself up a blind alley. Be prepared to persevere and try different approaches.

Example

Suppose we wish to evaluate

$$\int \frac{4x}{\sqrt{2x^2+1}}\,\mathrm{d}x\,.$$

By writing the integrand as $\dfrac{1}{\sqrt{2x^2+1}}\cdot 4x$ we note that it takes the form $\displaystyle\int f(g(x))g'(x)\mathrm{d}x$ where $f(u) = \dfrac{1}{\sqrt{u}}$, $g(x) = 2x^2+1$ and $g'(x) = 4x$.

The substitution $u = g(x) = 2x^2+1$ transforms the integral to

$$\int f(u)\,\mathrm{d}u = \int \frac{1}{\sqrt{u}}\,\mathrm{d}u\,.$$

This is evaluated to give

$$\begin{aligned}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u &= \int u^{-1/2}\,\mathrm{d}u\\ &= 2u^{1/2}+c\,.\end{aligned}$$

Finally, using $u = 2x^2+1$ to revert to the original variable gives

$$\int \frac{4x}{\sqrt{2x^2+1}}\,\mathrm{d}x = 2(2x^2+1)^{1/2}+c$$

or equivalently

$$2\sqrt{2x^2+1}+c\,.$$

Key Point

To evaluate

$$\int f(g(x))g'(x)\mathrm{d}x$$

substitute $u = g(x)$, and $\mathrm{d}u = g'(x)\mathrm{d}x$ to give

$$\int f(u)\,\mathrm{d}u$$

Integration is then carried out with respect to u, before reverting to the original variable x.

Example

Suppose we wish to evaluate

$$\int 2xe^{x^2}\mathrm{d}x\,.$$

We note that the integrand takes the form $\int f(g(x))g'(x)\mathrm{d}x$ where $f(u) = e^u$, $g(x) = x^2$ and $g'(x) = 2x$.

The substitution $u = g(x) = x^2$ transforms the integral to

$$\int f(u)\,\mathrm{d}u = \int e^u\,\mathrm{d}u\,.$$

This is evaluated to give

$$\int e^u\,\mathrm{d}u = e^u + c\,.$$

Finally, using $u = x^2$ to revert to the original variable gives

$$\int 2xe^{x^2}\mathrm{d}x = e^{x^2} + c.$$

Example

Suppose we wish to evaluate

$$\int \cos(3x^4)12x^3\mathrm{d}x\,.$$

We note that the integrand takes the form $\int f(g(x))g'(x)\mathrm{d}x$ where $f(u) = \cos u$, $g(x) = 3x^4$ and $g'(x) = 12x^3$.

The substitution $u = g(x) = 3x^4$ transforms the integral to

$$\int f(u)\,\mathrm{d}u = \int \cos u\,\mathrm{d}u\,.$$

This is evaluated to give

$$\int \cos u \, \mathrm{d}u = \sin u + c\,.$$

Finally, using $u = 3x^4$ to revert to the original variable gives

$$\int \cos(3x^4) 12x^3 \mathrm{d}x = \sin(3x^4) + c\,.$$

Exercises 2

1. In each case the integrand can be written as $f(g(x))\, g'(x)$. Identify the functions f and g and use the general result on page 7 to complete the integration.

(a) $\displaystyle\int 2x\mathrm{e}^{x^2-5}\mathrm{d}x$ (b) $\displaystyle-\int -2x\sin(1-x^2)\mathrm{d}x$ (c) $\displaystyle\int \frac{\cos x}{1+\sin x}\mathrm{d}x.$

2. In each case use the given substitution to find the integral:

(a) $\displaystyle\int -2x\mathrm{e}^{-x^2}\mathrm{d}x, \qquad u = -x^2.$

(b) $\displaystyle\int x\sin(2x^2)\mathrm{d}x, \qquad u = 2x^2.$

(c) $\displaystyle\int_0^5 x^3\sqrt{x^4+1}\mathrm{d}x, \qquad u = x^4+1.$

3. In each case use a suitable substitution to find the integral.

(a) $\displaystyle\int 5x\sqrt{1-x^2}\mathrm{d}x$ (b) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt{x})^2}$ (c) $\displaystyle\int x^4(1+x^5)^3\mathrm{d}x$

(d) $\displaystyle\int \frac{x^3}{\sqrt{x^4+16}}\mathrm{d}x$ (e) $\displaystyle\int \frac{\cos x}{(5+\sin x)^2}\mathrm{d}x$ (f) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+12}}\mathrm{d}x$

(g) $\displaystyle\int 5x^2\sqrt{1-x^3}\mathrm{d}x$ (h) $\displaystyle\int \mathrm{e}^{\cos x}\sin x\,\mathrm{d}x$ (i) $\displaystyle\int \mathrm{e}^{\sin x}\cos x\,\mathrm{d}x.$

Answers to Exercises

Exercises 1

1. (a) $\dfrac{(x-2)^4}{4}+c$ (b) $\dfrac{4651}{5}=930\dfrac{1}{5}$

   (c) $\dfrac{(2x-1)^8}{16}+c$ (d) 4.

2. (a) $-\dfrac{\cos(7x-3)}{7}+c$ (b) $\dfrac{e^{3x-2}}{3}+c$ (c) 1.382 (3dp) (d) $\dfrac{1}{7}\ln|7x+5|+c$

Exercises 2

1. (a) $f(u)=e^u$, $g(x)=x^2-5$, $e^{x^2-5}+c$,

   (b) $f(u)=\sin u$, $g(x)=1-x^2$, $-\cos(1-x^2)+c$

   (c) $f(u)=\frac{1}{u}$, $g(x)=1+\sin x$, $\ln|1+\sin x|+c$.

2. (a) e^{-x^2}+c (b) $-\dfrac{\cos(2x^2)}{4}+c$ (c) 2610 (4sf).

3. (a) $-\dfrac{5}{3}(1-x^2)^{3/2}+c$ (b) $-\dfrac{2}{1+\sqrt{x}}+c$ (c) $\dfrac{1}{20}(1+x^5)^4+c$

   (d) $\dfrac{1}{2}(x^4+16)^{1/2}+c$ (e) $-\dfrac{1}{5+\sin x}+c$ (f) 0.0707

   (g) $-\dfrac{10}{9}(1-x^3)^{3/2}+c$ (h) $-e^{\cos x}+c$ (i) $e^{\sin x}+c$.