mathtutor
functions & graphs, sequences and series

The sum of an infinite series

In this unit we see how finite and infinite series are obtained from finite and infinite sequences. We explain how the partial sums of an infinite series form a new sequence, and that the limit of this new sequence (if it exists) defines the sum of the series. We also consider two specific examples of infinite series that sum to e and ¶ respectively.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise the difference between a sequence and a series;
- write down the sequence of partial sums of an infinite series;
- determine (in simple cases) whether an infinite series has a sum.

Contents

1. Introduction

A finite series is given by the terms of a finite sequence, added together. For example, we could take the finite sequence

$$(2k+1)_{k=1}^{10} = (3,\ 5,\ 7,\ \ldots,\ 21)\,.$$

Then the corresponding example of a finite series would be given by all of these terms added together,

$$3+5+7+\ldots+21\,.$$

We can write this sum more concisely using sigma notation. We write the capital Greek letter sigma, and then the rule for the k-th term. Below the sigma we write '$k=1$'. Above the sigma we write the value of k for the last term in the sum, which in this case is 10. So in this case we would have

$$\sum_{k=1}^{10} 2k+1 = 3+5+7+\ldots+21\,,$$

and in this case the sum of the series is equal to 120.

In the same way, an infinite series is the sum of the terms of an infinite sequence. An example of an infinite sequence is

$$\left(\frac{1}{2^k}\right)_{k=1}^{\infty} = (\tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \ldots)\,,$$

and then the series obtained from this sequence would be

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots$$

with a sum going on forever. Once again we can use sigma notation to express this series. We write down the sigma sign and the rule for the k-th term. But now we put the symbol for infinity above the sigma, to show that we are adding up an infinite number of terms. In this case we would have

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots\,.$$

Key Point

A finite series is given by all the terms of a finite sequence, added together.

An infinite series is given by all the terms of an infinite sequence, added together.

2. The sum of an infinite series

What could we mean by the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots \quad = \quad ?$$

Let us try adding up the first few terms and see what happens. If we add up the first two terms we get

$$\tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}\,.$$

The sum of the first three terms is

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8}\,.$$

The sum of the first four terms is

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16}\,.$$

And the sum of the first five terms is

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} = \tfrac{31}{32}\,.$$

These sums of the first terms of the series are called *partial sums*. The first partial sum is just the first term on its own, so in this case it would be $\frac{1}{2}$. The second partial sum is the sum of the first two terms, giving $\frac{3}{4}$. The third partial sum is the sum of the first three terms, giving $\frac{7}{8}$, and so on.

If we write down the partial sums from this example,

$$\left(\tfrac{1}{2},\ \tfrac{3}{4},\ \tfrac{7}{8},\ \tfrac{15}{16},\ \tfrac{31}{32},\ \ldots\right),$$

we can see they form the beginning of another infinite sequence. The n-th term of this sequence is the n-th partial sum. We can see that the partial sums here form a sequence that has limit 1. So it would make sense to say that this series has sum 1. We write

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1\,.$$

In general, we say that an infinite series *has a sum* if the partial sums form a sequence that has a real limit. If the series is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$$

then it has a sum if the sequence of partial sums

$$(a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots)$$

has a limit. If the sequence of partial sums does not have a real limit, we say the series does not have a sum.

Here is another infinite series that has a sum. It is the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{20} + \dots .$$

To find the sum of this series, we need to work out the partial sums. For this particular series, the best way to do this is to split each individual term into two parts:

$$\begin{aligned} \frac{1}{k(k+1)} &= \frac{k+1-k}{k(k+1)} \\ &= \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} \\ &= \frac{1}{k} - \frac{1}{k+1} . \end{aligned}$$

If we do this to each term, the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = (\tfrac{1}{1} - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \dots ,$$

and so the n-th partial sum is

$$(1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) .$$

As you can see, most of the terms in this expression cancel in pairs. We just get the outermost two terms,

$$1 - \frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = \frac{n}{n+1} .$$

So the sequence of partial sums is

$$\left(\frac{n}{n+1}\right)_{n=1}^{\infty} = \left(\tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ \dots\right) ,$$

and this sequence has limit 1. We say the infinite series sums to 1, and we write

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 .$$

Here is an example of an infinite series that does not have a sum. The series

$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$$

has the sequence of partial sums

$$(1,\ 2,\ 3,\ 4,\ \dots) .$$

This sequence of partial sums does not tend to a real limit. It tends to infinity. So the series does not have a sum.

You might have noticed that, whenever we have taken an infinite series with a sum, then the individual terms of the series have tended to zero. This is a general feature of infinite series. But this argument does not work in the opposite direction. It is possible to have a series with individual terms tending to zero, but with no sum. For example, the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots$$

is called the *harmonic series*, and it has terms that tend to zero. But the sequence of partial sums for this series tends to infinity. So this series does not have a sum.

Key Point

The n-th partial sum of a series is the sum of the first n terms.

The sequence of partial sums of a series sometimes tends to a real limit. If this happens, we say that this limit is the sum of the series. If not, we say that the series has no sum.

A series can have a sum only if the individual terms tend to zero. But there are some series with individual terms tending to zero that do not have sums.

3. Evaluating π and e with series

Some infinite series can help us to evaluate important mathematical constants. For example, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!}\,.$$

Written out term by term, this series is

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \quad = \quad 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \ldots\,.$$

If we use a calculator to work out the first few partial sums of this series, we get

$$(1,\ 2,\ 2{\cdot}5,\ 2{\cdot}66667,\ 2{\cdot}70833,\ 2{\cdot}71667,\ 2{\cdot}71806,\ \ldots)\,,$$

where we have written down some of the terms to five decimal places.

Now you might have noticed that this sequence of partial sums seems to be getting closer and closer to the number e, which is $2{\cdot}71828$ to five decimal places. In fact it can be shown that the partial sums do tend to e. So working out the partial sums of this series is a useful way of calculating e to a large number of decimal places.

Now let us look at the infinite series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{4}{2k-1}\right).$$

For this series, we need to recall the meaning of the power $(-1)^{k+1}$. If k is odd then $k+1$ is even, and so $(-1)^{k+1}=1$. On the other hand, if k is even then $k+1$ is odd, and so $(-1)^{k+1}=-1$. We can now write out the series term by term as

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{4}{2k-1}\right)=4-\tfrac{4}{3}+\tfrac{4}{5}-\tfrac{4}{7}+\tfrac{4}{9}+\ldots.$$

Again we can use a calculator to work out the first few partial sums of this series. We get

$$(4,\ 2{\cdot}6667,\ 3{\cdot}4667,\ 2{\cdot}8952,\ 3{\cdot}3397,\ 2{\cdot}9760,\ 3{\cdot}2837,\ \ldots)$$

where we have written down some of the terms to four decimal places.

This sequence of partial sums looks like it might be getting close to some number just greater than 3. In fact it can be shown that the partial sums tend to π, which is 3·1416 to four decimal places. If we kept on calculating the partial sums for this series, we would eventually obtain a value for π to several decimal places.

Key Point

Some infinite series can help us evaluate numbers like π and e as accurately as we choose.

Exercises

1.

The geometric series $1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\ldots$ sums to 3. How many terms n are required for the nth partial sum S_n to differ from 3 by less than (a) 1, (b) 0.2, (c) 0.05 ?

2.

The harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ does not have a sum. How many terms n are required for the nth partial sum S_n to be greater than

(a) 2, (b) 3, (c) 5 ?

Answers

1.

(a) 3 (b) 7 (c) 11

2.

(a) 4 (b) 11 (c) 83