

# Limits of sequences

In this unit, we recall what is meant by a simple sequence, and introduce infinite sequences. We explain what it means for two sequences to be the same, and what is meant by  $n$ -th term of a sequence. We also investigate the behaviour of infinite sequences, and see that they might tend to plus or minus infinity, or to a real limit, or behave in some other way,

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use the notation for a sequence in terms of a formula for its  $n$ -th term;
- decide whether a sequence tends to infinity;
- decide whether a sequence tends to minus infinity;
- decide whether a sequence tends to a real limit;
- decide whether a sequence diverges;
- use the notation for the limit of a sequence.

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# 1. Introduction

A simple sequence is a finite list of numbers. For example,

$$(1, 3, 5, \dots, 19)$$

is a simple sequence. So is

$$(4, 9, 16, \dots, 81).$$

We call the numbers in the sequence the *terms* of the sequence. So in our second example we say that the first term is 4, the second term is 9, and so on.

Not all sequences have to be finite. We can also define infinite sequences. These are lists of numbers, like simple sequences, but the difference is that the terms go on for ever. We write infinite sequences like this:

$$(2, 5, 8, \dots).$$

In a finite sequence, the three dots in the middle followed by the final number just mean that we have omitted some of the terms. But if you see three dots without anything after them, it means that the sequence goes on for ever.

We say two sequences are the same if all the terms are the same. This means that the sequences have to contain the same numbers, in the same places, throughout the sequence. So

$$(1, 2, 3, 4, \dots) \neq (2, 1, 4, 3, \dots)$$

even though they contain the same numbers. The positions of the numbers are different, so the sequences are not the same.

Our first two examples of sequences have obvious rules for obtaining each term. In the first example, we obtain the  $n$ -th term by taking  $n$ , multiplying by 2, and taking away 1. We say that the  $n$ -th term is  $2n - 1$ . Similarly, the  $n$ -th term of the second sequence is  $(n + 1)^2$ .

Our third example, the infinite sequence, also has rules for the  $n$ -th term. In this case we take  $n$ , multiply by 3, and take away 1. So the  $n$ -th term is  $3n - 1$ .

But not all sequences have clear rules for the  $n$ -th term. For example, the sequence that starts

$$(\sqrt{3}, -5, 99.7, \dots)$$

is still a sequence, even though it looks random.



## Key Point

A simple sequence is a finite list of numbers, and an infinite sequence is an infinite list of numbers. The numbers in the sequence are called the terms of the sequence.

Two sequences are the same only if they contain the same numbers in the same positions.

## 2. Some notation for sequences

You might have noticed that we have written out our sequences in brackets. For example, we have written

$$(1, 3, 5, 7, \dots, 19)$$

for our first sequence. If a sequence is given by a rule then another, more concise, way to denote the sequence is to write down the rule for the  $n$ -th term, in brackets. So we might write this sequence as

$$(2n - 1).$$

But we also need to show how many terms are included in the sequence, and we do this by writing the index of the first term just underneath the bracket, and the index of the last term just above, like this:

$$(2n - 1)_{n=1}^{10}.$$

Similarly, we could write the finite sequence  $(4, 9, 16, 25, \dots, 81)$  as

$$((n + 1)^2)_{n=1}^8.$$

Notice that we use two sets of brackets for this sequence. The inner set is used to denote the formula  $(n + 1)^2$  for the  $n$ -th term. The outer set indicates that we have a sequence, including all the terms with this formula from  $n = 1$  to  $n = 8$ .

We can use the same notation for infinite sequences. Our example of an infinite sequence can be written as

$$(3n - 1)_{n=1}^{\infty}.$$

Here, the superscript ' $\infty$ ' is used to indicate that the sequence goes on for ever.



### Key Point

We denote a sequence by writing the  $n$ -th term in brackets and indicating how many terms are included in the sequence.

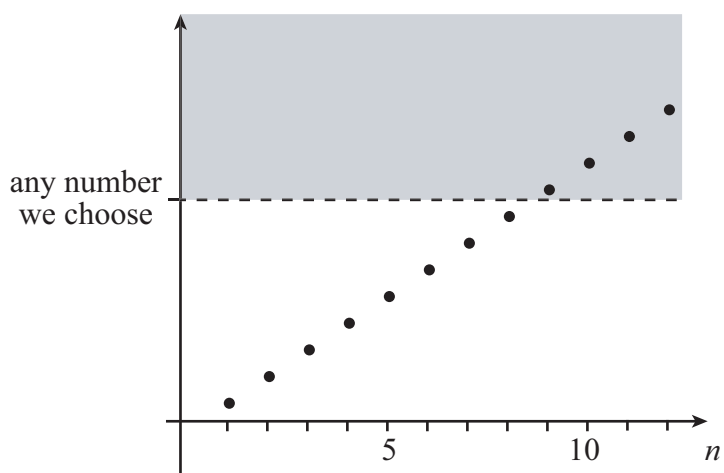
## 3. The behavior of infinite sequences

We shall now concentrate on infinite sequences. It is often very important to examine what happens to a sequence as  $n$  gets very large. There are three types of behaviour that we shall wish to describe explicitly. These are

- sequences that 'tend to infinity';
- sequences that 'tend to minus infinity';
- sequences that 'converge to a real limit'.

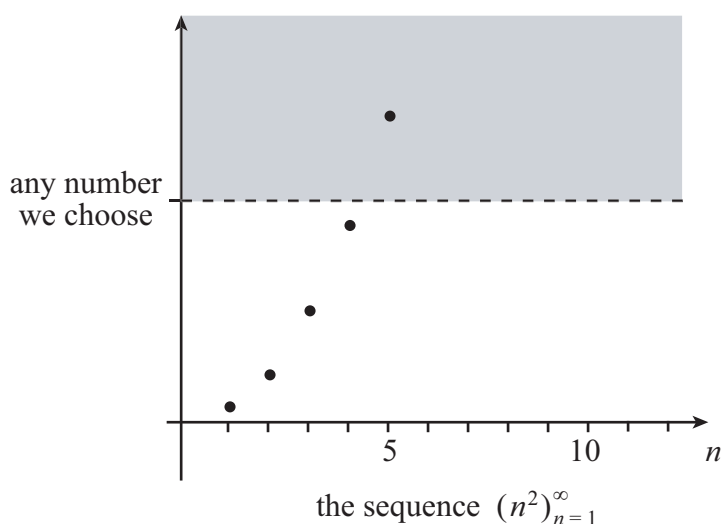
First we shall look at sequences that tend to infinity. We say a sequence *tends to infinity* if, however large a number we choose, the sequence becomes greater than that number, and stays greater.

So if we plot a graph of a sequence tending to infinity, then the points of the sequence will eventually stay above any horizontal line on the graph.

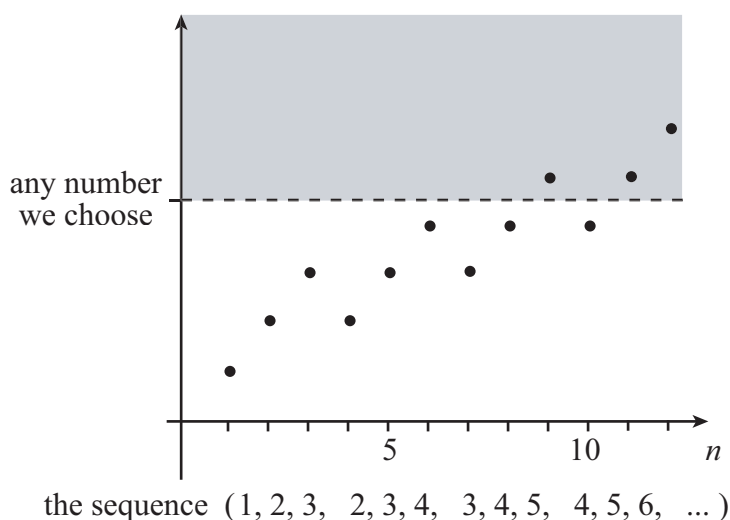


Here are some examples of sequences that tend to infinity.

At some point this sequence will be greater than any number we choose, and stay greater, so the sequence tends to infinity.

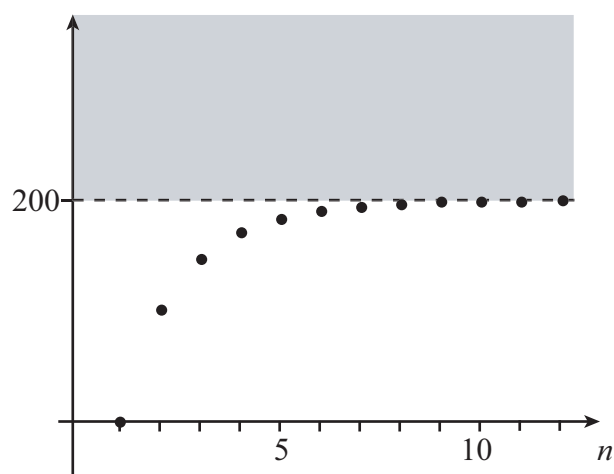


Even though this sequence sometimes decreases, it will still eventually become greater, and stay greater, than any number we choose.



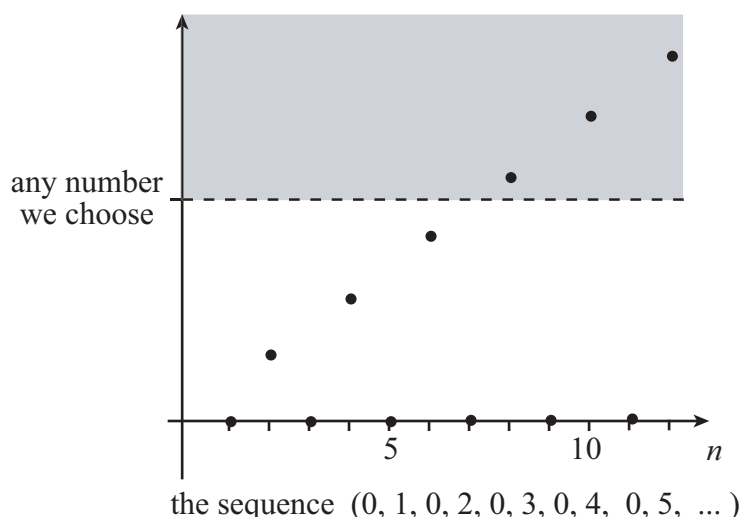
Here are some sequences that do not tend to infinity.

For this sequence we obtain the second term by adding 100 to the first term. Then, to get the next term, we add 50, and then we add 25, and so on. Each time we add half as much as we did before. Even though this sequence starts by increasing very quickly, it will never be larger than 200. So we can choose a number that will never exceed.



the sequence  $(0, 100, 150, 175, 187.5, \dots)$

This sequence does get larger than any number we choose. But it does not stay larger, because it always returns to zero. So we cannot say that this sequence tends to infinity.



the sequence  $(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots)$

There is some notation we can use if a sequence tends to infinity. If the sequence  $(x_n)$  tends to infinity, we write either

$$x_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

or

$$\lim_{n \rightarrow \infty} (x_n) = \infty.$$

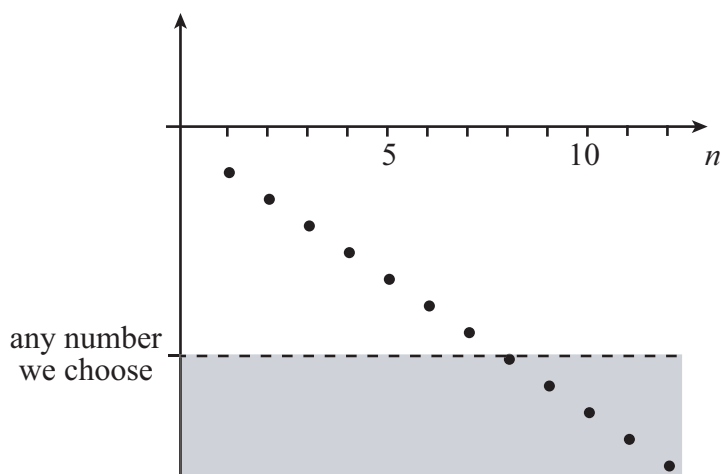


### Key Point

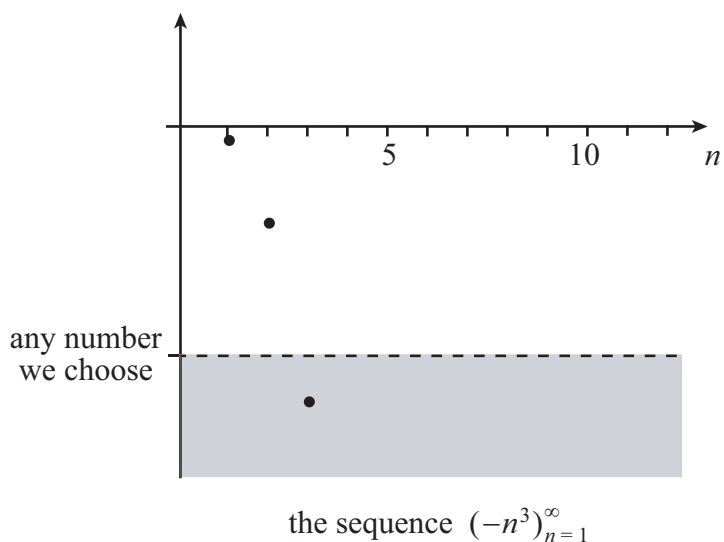
A sequence  $(x_n)$  tends to infinity if, however large a number we choose, the sequence will eventually become greater than that number, and stay greater.

We shall now look at sequences that tend to minus infinity. We say a sequence *tends to minus infinity* if, however large a negative a number we choose, the sequence eventually becomes less than that number and stays less than it.

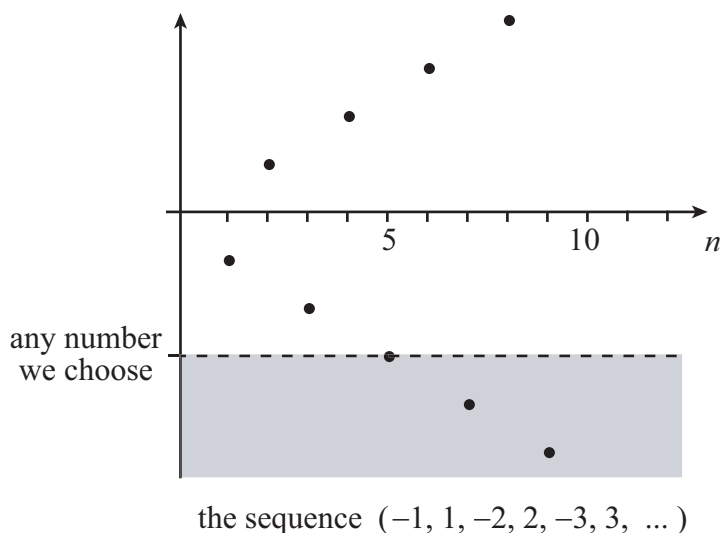
So if we plot a graph of a sequence tending to minus infinity, then the points of the sequence will eventually stay below any horizontal line on the graph.



For example, this sequence tends to minus infinity.



But this sequence does not tend to minus infinity. It does become less than any number we choose. But it does not stay less, because it always becomes positive again. So we cannot say that this sequence tends to minus infinity.



There is some notation we can use if a sequence tends to minus infinity. If the sequence  $(x_n)$  tends to minus infinity, we write either

$$x_n \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty$$

or

$$\lim_{n \rightarrow \infty} (x_n) = -\infty.$$

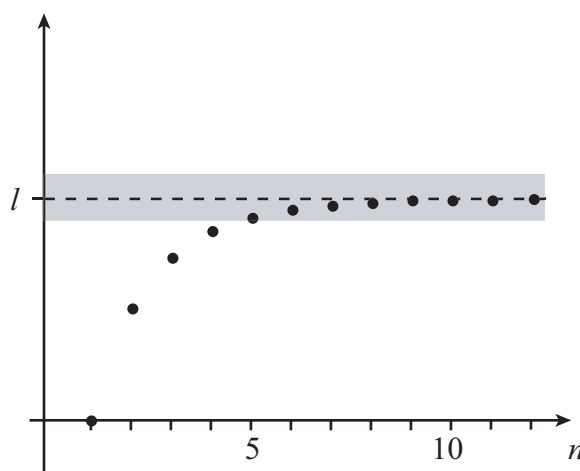


## Key Point

A sequence  $(x_n)$  tends to minus infinity if, however large a negative number we choose, the sequence will eventually become less than that number, and stay less.

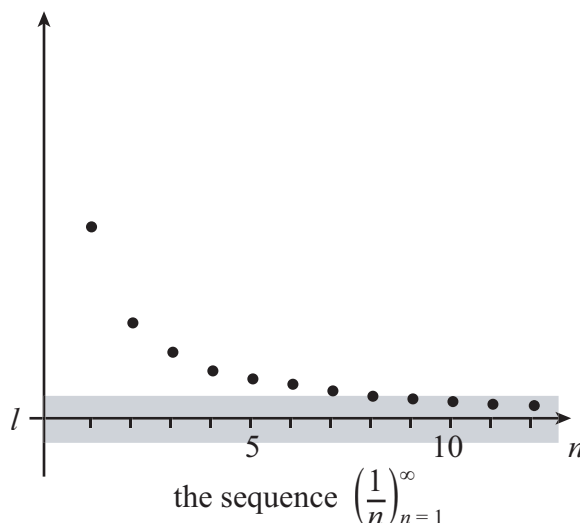
Finally we shall look at sequences with real limits. We say a sequence tends to a real limit if there is a real number,  $l$ , such that the sequence gets closer and closer to it. We say  $l$  is the *limit* of the sequence.

The sequence gets ‘closer and closer to  $l$ ’ if, whenever we draw an interval as narrow as we choose around  $l$ , the sequence eventually gets trapped inside the interval.

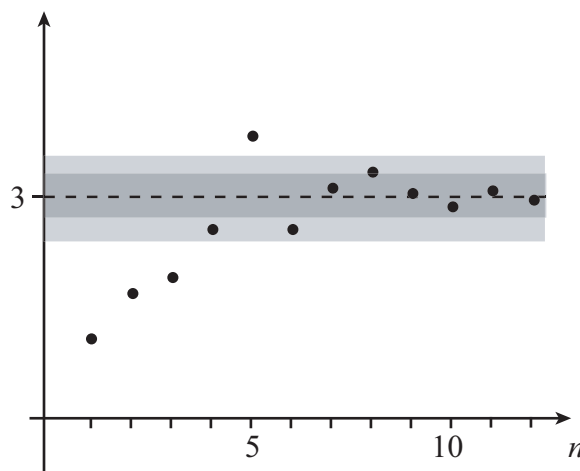


Notice that it does not matter whether or not the sequence eventually takes the value  $l$ . We just need it to get as close as we choose to  $l$ .

This sequence gets as close as we like to zero. So we say the sequence tends to zero as  $n$  tend to infinity.



This sequence tends to the limit 3. Even though the values of the sequence sometimes move away from 3, they eventually stay within any interval around 3 that we choose. The narrower the interval, the further along the sequence we might have to go before the values become trapped within the interval.



There is some notation we can use if a sequence tends to a real limit. If the sequence  $(x_n)$  tends to the limit  $l$ , we write either

$$x_n \rightarrow l \quad \text{as} \quad n \rightarrow \infty$$

or

$$\lim_{n \rightarrow \infty} (x_n) = l.$$



### Key Point

A sequence  $(x_n)$  tends to a real limit  $l$  if, however small an interval around  $l$  that we choose, the sequence will eventually take values within that interval, and remain there.

We say a sequence is *divergent* if it does not converge to a real limit. For example, if a sequence tends to infinity or to minus infinity then it is divergent. But not all divergent sequences tend to plus or minus infinity. For example, the sequence

$$(1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 1, 0, -1, -2, \dots)$$

is divergent, but it does not tend to either infinity or to minus infinity. A sequence like this, repeating itself over and over again, is called a periodic sequence.



### Key Point

A sequence that does not converge to a real limit is called a divergent sequence.



## Exercises

Decide whether each of the following sequences tends to infinity, tends to minus infinity, tends to a real limit, or does not tend to a limit at all. If a sequence tends to a real limit, work out what it is.

1.  $(2^n)_{n=1}^\infty$
2.  $(1000 - n)_{n=1}^\infty$
3.  $\left(\frac{n}{n+1}\right)_{n=1}^\infty$
4.  $(+\sqrt{n})_{n=1}^\infty$
5.  $\left(\sin \frac{\pi n}{4}\right)_{n=1}^\infty$
6.  $\left(5 - \frac{1}{n}\right)_{n=1}^\infty$

## Answers

1. infinity
2. minus infinity
3. 1
4. infinity
5. no limit
6. 5