mathtutor
functions & graphs, sequences and series

Inverse functions

An inverse function is a second function which undoes the work of the first one. In this unit we describe two methods for finding inverse functions, and we also explain that the domain of a function may need to be restricted before an inverse function can exist.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- understand the difference between inverse functions and reciprocal functions,
- find an inverse function by reversing the operations applied to x in the original function,
- find an inverse function by algebraic manipulation,
- understand how to restrict the domain of a function so that it can have an inverse function,
- sketch the graph of an inverse function using the graph of the original function.

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1. Introduction

Suppose we have a function f that takes x to y, so that

$$f(x) = y.$$

An inverse function, which we call f^{-1}, is another function that takes y back to x. So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f, this needs to work for every x that f acts upon.

Key Point

The inverse of the function f is the function that sends each $f(x)$ back to x. We denote the inverse of f by f^{-1}.

2. Working out f^{-1} by reversing the operations of f

One way to work out an inverse function is to reverse the operations that f carries out on a number. Here is a simple example. We shall set $f(x) = 4x$, so that f takes a number x and multiplies it by 4:

$$f(x) = 4x \qquad \text{(multiply by 4)}.$$

We want to define a function that will take 4 times x, and send it back to x. This is the same as saying that $f^{-1}(x)$ divides x by 4. So

$$f^{-1}(x) = \tfrac{1}{4}x \qquad \text{(divide by 4)}.$$

There is an important point about notation here. You should notice that $f^{-1}(x)$ does not mean $1/f(x)$. For this example, $1/f(x)$ would be $1/4x$ with the x in the denominator, and that is not the same as $\frac{1}{4}x$.

Here is a slightly more complicated example. Suppose we have

$$f(x) = 3x + 2\,.$$

We can break up this function into a series of operations. First the function multiplies by 3, and then it adds on 2.

$$\begin{array}{cc} x & \\ \downarrow & \times 3 \\ 3x & \\ \downarrow & +2 \\ 3x+2 & \end{array}$$

To get back to x from $f(x)$, we would need to reverse these operations. So we would need to take away 2, and then divide by 3. When we undo the operations, we have to reverse the order as well.

$$\begin{array}{ccccc} x & & & (x-2)/3 & \\ \downarrow & \times 3 & & \uparrow & \div 3 \\ 3x & & & x-2 & \\ \downarrow & +2 & & \uparrow & -2 \\ 3x+2 & & & x & \end{array}$$

Now we have reversed all the operations carried out by f, and so we are left with

$$f^{-1}(x) = \frac{x-2}{3}\,.$$

Here is one more example of how we can reverse the operations of a function to find its inverse. Suppose we have

$$f(x) = 7 - x^3.$$

It is easier to see the sequence of operations to be carried out on x if we rewrite the function as

$$f(x) = -x^3 + 7.$$

So the first operation performed by f takes x to x^3; then the result is multiplied by -1; and finally 7 is added on.

$$\begin{array}{cc} x & \\ \downarrow & \text{(cube)} \\ x^3 & \\ \downarrow & \times(-1) \\ -x^3 & \\ \downarrow & +7 \\ -x^3+7 & \end{array}$$

So to get from $f(x)$ to x, we need to start by taking away 7. Then we need to undo the operation 'multiply by -1', so we divide by -1. And finally we undo the first operation by taking the cube root.

$$\begin{array}{ccccc} x & & & \sqrt[3]{7-x} & \\ \downarrow & \text{(cube)} & & \uparrow & \text{(cube root)} \\ x^3 & & & 7-x & \\ \downarrow & \times(-1) & & \uparrow & \div(-1) \\ -x^3 & & & x-7 & \\ \downarrow & +7 & & \uparrow & -7 \\ -x^3+7 & & & x & \end{array}$$

Now we have reversed every operation carried out by f. So

$$f^{-1}(x) = \sqrt[3]{7-x}\,.$$

Key Point

We can work out f^{-1} by reversing the operations of f. If there is more than one operation then we must reverse the order as well as reversing the individual operations.

Exercises

1. Work out the inverses of the following functions:

(a) $f(x) = 6x$, (b) $f(x) = 3 + 4x^3$, (c) $f(x) = 1 - 3x$.

3. Using algebraic manipulation to work out inverse functions

Another way to work out inverse functions is by using algebraic manipulation. We can demonstrate this using our second example, $f(x) = 3x + 2$.

Now the inverse function takes us from $f(x)$ back to x. If we set

$$y = f(x) = 3x + 2,$$

then f^{-1} is the function that takes y to x. So to work out f^{-1} we need to know how to get to x from y. If we rearrange the expression for y we obtain

$$\begin{aligned} y &= 3x + 2\,, \\ y - 2 &= 3x \\ \text{so that} \quad x &= \frac{y-2}{3}\,. \end{aligned}$$

So we want $f^{-1}(y) = (y - 2)/3$, and this is exactly the same as saying that the function f^{-1} is given by $f^{-1}(x) = (x - 2)/3$.

We can use the method of algebraic manipulation to work out inverses when we have slightly trickier functions than the ones we have seen so far. Let us take

$$f(x) = \frac{x}{x-1}, \qquad x > 1.$$

We have made the restriction $x > 1$ because at $x = 1$ the function does not have a value. This is because the denominator is zero when $x = 1$.

Now we set $y = x/(x-1)$. Multiplying both sides by $x-1$ we get

$$y(x-1) = x\,,$$

and then multiplying out the bracket gives

$$yx - y = x\,.$$

We want to rearrange this equation so that we can express x as a function of y, and to do this we take the terms involving x to the left-hand side, giving

$$yx - x = y\,.$$

Now we can then take out x as a factor on the left-hand side to get

$$x(y-1) = y\,,$$

and dividing throughout by $y-1$ we finally obtain

$$x = \frac{y}{y-1}\,.$$

So the inverse function is $f^{-1}(y) = y/(y-1)$, and this is exactly the same as saying that the function f^{-1} is given by $f^{-1}(x) = x/(x-1)$. So in this case f^{-1} happens to be the same as f.

In the last example, it would not have been possible to work out the inverse function by trying to reverse the operations of f. This example shows how useful it is to have algebraic manipulation to work out inverses.

Key Point

Algebraic manipulation is another method that can be used to work out inverse functions.

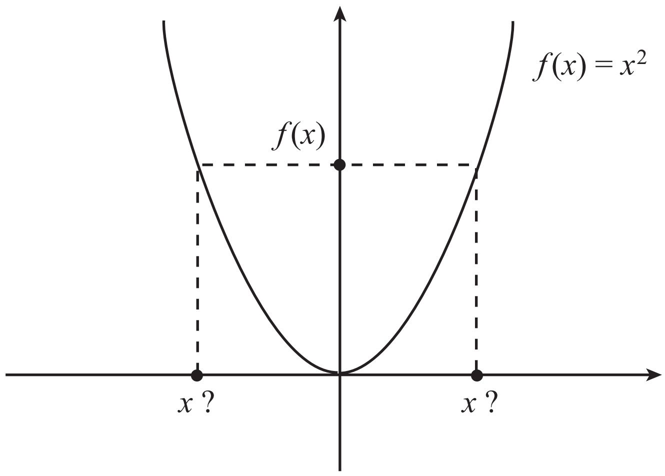
Exercises

2. Use algebraic manipulation to work out f^{-1} for each of the following functions:

(a) $f(x) = \dfrac{3}{4x-4}$ for $x > 1$, (b) $f(x) = \dfrac{x+1}{x+2}$ for $x > -2$.

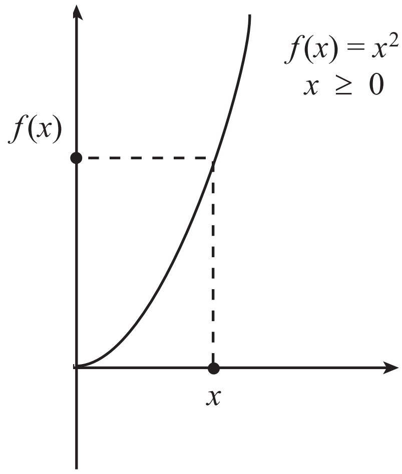
4. Restricting domains

Not all functions have inverses. For example, let us see what happens if we try to find an inverse for $f(x) = x^2$.



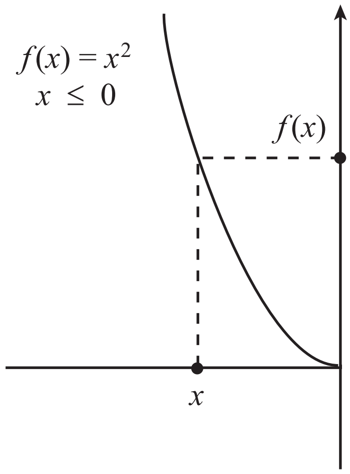
When we define an inverse function for f, we look for another function that takes the values $f(x)$ and gives us back x. But in the case of $f(x) = x^2$ there are two values of x that give the same $f(x)$. This is because both $f(x) = x^2$ and also $f(-x) = x^2$. We cannot define f^{-1} of something to be two different things.

To get around this problem, we restrict the domain of the function. So for example with $f(x) = x^2$, if we define the function only for $x \geq 0$ then the graph looks like this.

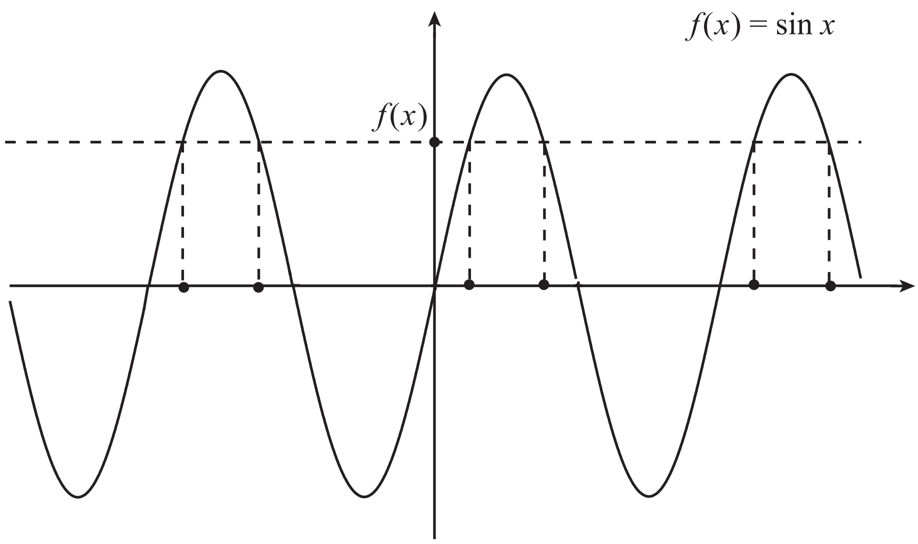


So now we have exactly one value of x giving each value of $f(x)$. This restricted version of $f(x)$ can have an inverse. The inverse is $f^{-1}(x) = +\sqrt{x}$.

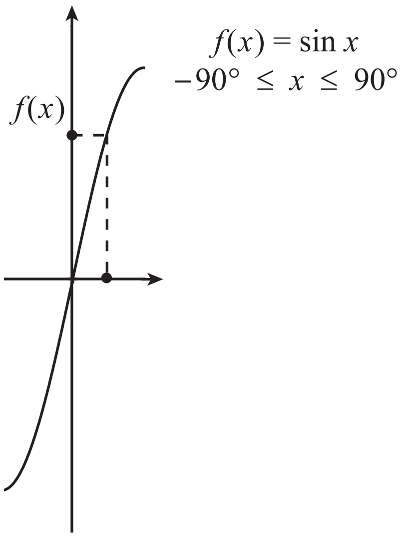
We could instead have restricted $f(x)$ to $x \leq 0$. This still gives only one value of x for each value of x^2. This time the inverse is $-\sqrt{x}$.



Here is another example of a function that we need to restrict in order to define an inverse. Let us look at $f(x) = \sin x$. The graph of the function looks like this.



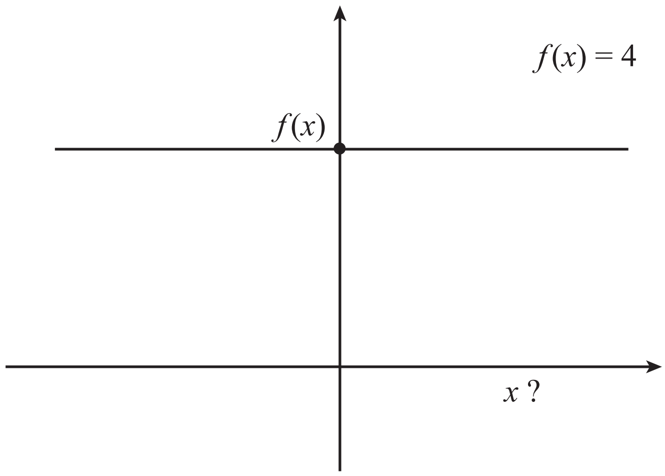
This time if we try to define the inverse function, we see that there are many possible values of x for each $f(x)$. We need to restrict the domain of our function so that we are looking at a section with only a single value of x for each value of $f(x)$. We do this by setting the domain of $\sin x$ to be $-90° \leq x \leq 90°$.



It is particularly important here to remember that $\sin^{-1} x$ is the inverse function to $\sin x$, and that it does not mean $(\sin x)^{-1}$, even though the similar expression $\sin^2 x$ does mean $(\sin x)^2$. For this reason, the inverse function $\sin^{-1} x$ is sometimes called $\arcsin x$.

It is also possible to define the inverse functions $\cos^{-1} x$ and $\tan^{-1} x$ by restricting the domains of the functions $\cos x$ and $\tan x$. These inverse functions are also called $\arccos x$ and $\arctan x$, and you can find out more about them in the unit on Trigonometric Functions.

Some functions cannot have inverses, even if we restrict their domains. For example, a constant function cannot have an inverse.



However small we make the domain, there are always lots of values of x giving the same value of $f(x)$. The only way we can get a single value of x is by restricting the domain to a single point. So we say that this function has no inverse.

Exercises

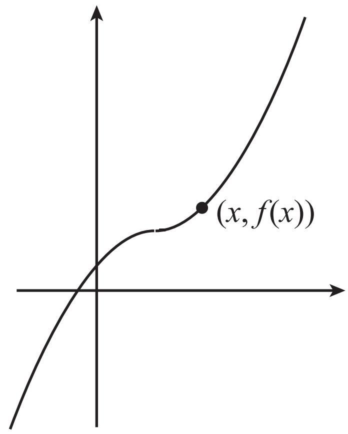
3. Which of these functions need to have their domains restricted in order to define an inverse? How would you restrict their domains?

(a) $f(x) = x^2 + 2x + 1$, (b) $f(x) = x^3$, (c) $f(x) = 5 - x^2$, (d) $f(x) = \sin 2x$.

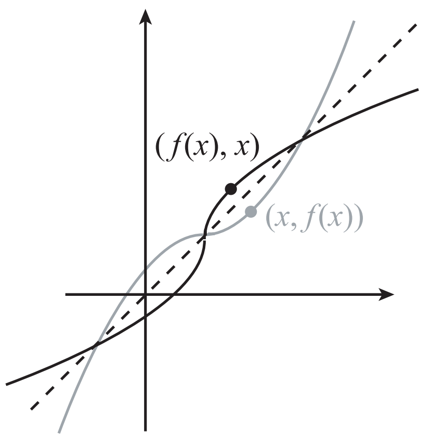
5. The graph of f^{-1}

There is an easy way to work out the graph of an inverse function f^{-1}, using the graph of the original function f.

Suppose that we have the graph of some function f. Then a point on the graph of f will have co-ordinates $(x, f(x))$.



Remember that an inverse function sends $f(x)$ back to x. So the graph of f^{-1} must contain the points $(f(x), x)$. So if we interchange the x and y axes, we will get the graph of the inverse function.



By going from the graph of f to the graph of f^{-1}, we are reflecting the graph in the diagonal line. But this diagonal line is the line $y = x$. So the graph of f^{-1} is just the graph of f reflected in the line $y = x$.

Exercises

4. For each of the following, sketch the graph of $f(x)$ and use it to sketch the graph of $f^{-1}(x)$:

(a) $f(x) = 3x$, (b) $f(x) = 2x + 5$, (c) $f(x) = x^2$ for $x \geq 0$,
(d) $f(x) = 1/x$ for $x > 0$, (e) $f(x) = x^3$.

Answers

1.

(a) $f^{-1}(x) = \dfrac{x}{6}$ (b) $f^{-1}(x) = \sqrt[3]{\dfrac{x-3}{4}}$ (c) $f^{-1}(x) = \dfrac{x-1}{-3} = \dfrac{1-x}{3}$

2.

(a) $f^{-1}(x) = \dfrac{3+4x}{4x}$ (b) $f^{-1}(x) = \dfrac{1-2x}{x-1}$

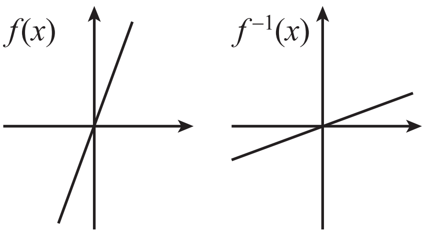
3.

(a) $x \le -1$ or $x \ge -1$ (b) no restriction needed (c) $x \le 0$ or $x \ge 0$

(d) $-45° \le x \le 45°$

4.

(a)



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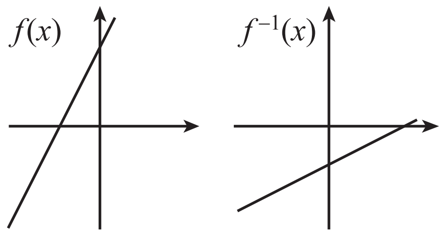

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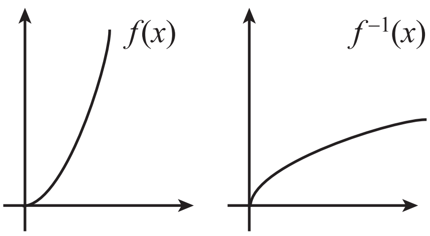

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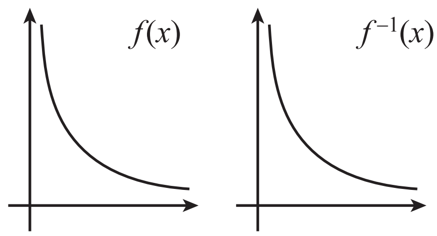

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