University of the Pacific ENGR 250 Notes

Mari Anderson

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Contents

1 Week 1

1.1 Set Theory

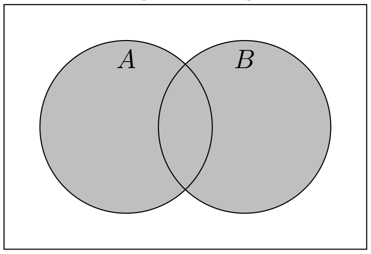
A **set** is a collection of things. For example:

- Collection of all natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$
- Even natural numbers less than or equal to 6: $E = \{2, 4, 6\}$.

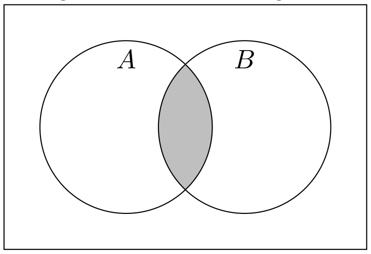
Elements of a set are denoted using lowercase letters. For example if $x = 4$ belngs to E we would denote that with $x \in E$. To denote an element is **not** in a set you dash the epsilon: $5 \notin E$.

1.1.1 Set Operations

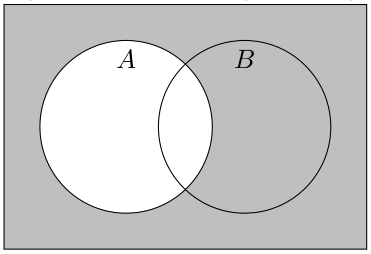
Set Union:
The union of two sets A and B is the set of all elements which is either in A or B. Behaves similar to logical OR from digital design. Formal Definiton: $A \cup B = \{x|(x \in A) \vee (x \in B)\}$.



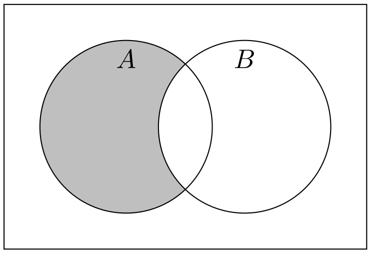
Set Intersection:
$A \cap B$ is the intersection of two sets, and contains every element that is both in A and B. Behaves similarly to logical AND from digital design. Formal Defintion: $A \cap B = \{x|(x \in A) \wedge (x \in B)\}$



Set Compliment: A^c is the compliment of A and contains every element not in A. Behaves similar to logical NOT from digital design. Formal Definition: $A^c = \{x|x \notin A\}$.



Set Difference: $A - B = A \cap B^c$. Contains every element of A that is not in B.



1.1.2 Other Definitions

A collection of sets $A_1, ..., A_n$ is **mutually exlusive** if and only if:

$$A_i \cap A_j = \emptyset \qquad i \neq j$$

A collection of sets $A_1, \ldots, A_n$ is **collectively exhaustive** if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$

Two sets are equal to each other if and only if

$$(A \subseteq B) \wedge (B \subseteq A)$$

De Morgan's Law: De Morgan's law relates all three basic set operations

$$(A \cup B)^c = A^c \cap B^c$$

Proof: Let $x \in (A \cup B)^c$. Then as $x \notin A \cup B$ therefore $x \notin A$ and $x \notin B$ Therefore $x \in A^c \cap B^c$ Therefore $(A \cup B)^c \subseteq A^c \cap B^c$. Now assume $x \in A^c \cap B^c$. Then $x \notin A$ and $x \notin B$, and therefore $x \notin (A \cup B)$. Thus $x \in (A \cup B)^c$. Therefore $(A \cup B)^c = A^c \cap B^c$. $\square$

$$(A \cap B)^C = A^c \cup B^c$$

Proof: Let $x \in (A \cap B)^c$. Then x is either in A not in B, in B not in A, or not in either A or B. Therefore $x \in A^c \cup B^c$ and thus $(A \cap B)^c \subseteq A^c \cup B^c$. Now let $x \in A^c \cup B^c$. Then by definition x is either in A^c or B^c. Thus x is not in both A and B. Therefore $x \in (A \cap B)^c$ and thus $A^c \cup B^c \subseteq (A \cap B)^c$. As $A^c \cup B^c \subseteq (A \cap B)^c$ and $(A \cap B)^c \subseteq A^c \cup B^c$, $(A \cap B)^c = A^c \cup B^c$. $\square$

1.2 Applying Set Theory to Probability

An **experiment** consists of a procedure and observations.

Experiment	Procedure	Observation
Coin Flip	Flip the coin	heads or tails
Dice Rolls	Roll the die	the number face up on the die
Networking	Send packets	Record the packets that successfully get transmitted

The **sample space** of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.

Roll a die	$S = D = \{1, 2, 3, 4, 5, 6\}$
Flip a coin	$S = C = \{H, T\}$
Flip a coin twice	$S = F = \{HH, HT, TH, TT\}$

An **event** is a set of desired outcomes of an experiment. Example: Roll a die, you win if you roll an even number. $E = \{2, 4, 6\}$.

1.3 Axioms

Probability P maps the events from a sample space to real numbers such that

1. $P(A) \geq 0$ where A is an event in the sample space S
2. $P(S) = 1$ where S is the universal set
3. For a countable collections of mutually exclusive sets $A_1, A_2, A_3, ...A_n \in S$, $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

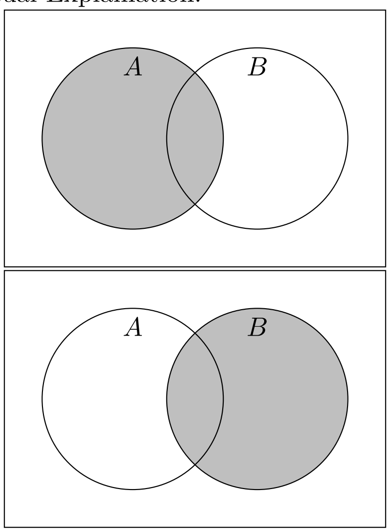
1.4 Theorems

Theorem 1.4
$P(A^c) = 1 - P(A)$.

For any two sets A and B not necessarily mutually exclusive:

$$P(A \cup B) = P(A) + B(B) - P(A \cap B)$$

Visual Explaination:



Here we see the intersection of A and B could be counted twice if we add $P(A)$ and $P(B)$ so we have to subtract the intersection so it is only counted once.

1.4.1 Theorem 1.5

The probability of event $B = \{s_1, s_2, \ldots s_m\}$ is the sum of probabilities contained in the event:

$$P(B) = \sum_{i=1}^{m} P(\{s_i\})$$

Follows from axiom 3 as each s_i is mutually exclusive.

1.4.2 Theorem 1.6

For an experiment with sample space $S = \{s_1, \ldots, s_n\}$ in which each outcome s_i is equally likely,

$$P(s_i) = 1/n \qquad 1 \leq i \leq n$$

Where n is the number of outcomes in the sample space (same as n is equal to the cardinality of S)

1.4.3 Theorem 1.7

If outcomes in an experiment are equally likely, then probability of event A is given by:

$$P(A) = \frac{|A|}{|S|}$$

Where || denotes the cardinality (number of elements) of the set.

1.5 Conditional Probability

The **conditional probability** of the event A given the occurance occurance of the event B is

$$P(A|B) = \frac{P(A \cap B)}{B}$$

1.6 Law of Total Probability

You use the law of total probability when your desired outcome may come from muliple sources with certain probability and the sources themselves have probability of being chosen.

1.6.1 Theorem 1.10:

For an event space $\{B_1, B_2, \ldots, B_m\}$ with $P(B_i) > 0$ for all i

$$P(A) = \sum_{i=1}^{m} P(A|B_i)P(B_i)$$

This can be shown using the conditional probability formula as

$$P(A) = \sum_{i=1}^{m} P(A|B_i)P(B_i) = \sum_{i=1}^{m} \frac{A \cap B_i}{B_i} P(B_i) = \sum_{i=1}^{m} P(A \cap B_i)$$

1.6.2 Example: Problem 2.2

Problem: A company has three machines B_1, B_2, B_3 making 1k resistors. It is observed that 80% of the resitors by B_1, 90% of resistors by B_2 and 60% of resistors are acceptable. Each hour B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. What is the probability that the company ships an acceptable resistor.
Solution: Let $P(B_i)$ be the probability of a resistor coming from a specific machine, and let $P(A_i)$ be the probability that machine produced an acceptable resistor. Then from the problem statement we get that $10000 = 3000 + 4000 + 3000$ resistors are produced every hour, which gives us our $P(B_i)$'s as

$$P(B_1) = \frac{3000}{10000} = 0.3 \qquad P(B_2) = \frac{4000}{10000} = 0.4 \qquad P(B_3) = \frac{3000}{10000} = 0.3$$

Then also from the problem statement we get our A_i's to be that

$$P(A_1) = 0.9 \qquad P(A_2) = 0.8 \qquad P(A_3) = 0.6$$

From this we can compute our probability of an acceptable resistor $P(R)$ as

$$P(R) = P(B_1)P(A_1)+P(B_2)P(A_2)+P(B_3)P(A_3) = (0.3)(0.9)+(0.4)(0.8)+(0.3)(0.6) = 0.27+0.32+0.18 = 0.77$$

1.7 Bayes' theorem

1.7.1 Theorem 1.11

Bayes's theorem is that the probability of an event B given event A occured can be given by

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Which we can easily confirm as $P(A|B)P(B) = P(A \cap B) = P(B \cap A)$ and $P(B|A) = \frac{P(B \cap A)}{P(A)}$ by defintion.

1.7.2 Example: Problem 2.3

Problem: In the previous problem (2.2) what is the probability that the acceptable resistor came from B_3.
Solution: From Problem 2.2 as $P(R|P(B_3)) = 0.60$ as 0.60 of the resistors produced by B_3 are acceeptable. From this we can apply bayes' theorem to get

$$P(B_3|R) = \frac{P(R|B_3)(P(B_3))}{P(R)} = \frac{0.18}{0.77} \approx 0.234$$

1.8 Independence

Definition: Two events are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Defintion: Three events A_1, A_2 and A_3 are independent if and only if

1. A_1 and A_2 are independent
2. A_2 and A_3 are independent
3. A_1 and A_3 are independent
4. $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

Definition: Multiple events $A_1, A_2, \ldots, A_n$ are independent if and only if both

$$P(A_i \cap P_j) = P(A_i)P(A_j) \text{ and } P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$$

For all pairs i, j $1 \leq i \leq n$, $1 \leq j \leq n$ where $i \neq j$.

1.9 Independent Trials

Properties:

1. They are identical subexperiments in a sequential experiment
2. The probability models of all subexperiments are identical and independent of the outcomes in other subexperiments (iid)

If you have an event that is $S = \{0, 1\}$ and you want to find the outcome of k sucesses in n trials with $P(1) = p$ in a specific ordering

$$P(n_1 = k) = p^k(1-p)^{n-k}$$

If you don't care about the order of the successes you will add a binomial term, as you will be choosing k successes out of n trials

$$P(n_1 = k) = \binom{n}{k} p^k(1-p)^{n-k}$$

The probability of at least k successes in n trials with each event having probability of sucesses p is

$$P(n_1 \geq k) = \sum_{i=0}^{n-k} \binom{n}{k+i} (p)^{k+i}(1-p)^{n-k-i}$$

Extending this to the multinomial case we get that if a sub experiment has sample spaces $S = \{s_0, \ldots s_{m-1}\}$, the probability of s_0 appearing n_0 times, s_1 appearing n_1 times and s_m appearing n_m times is:

$$P(E_{n_0, n_1, \ldots, n_m}) = \binom{n}{n_0, n_1, \ldots, n_m} P(s_0)^{n_0} P(s_1)^{n_1} \ldots P(s_m)^{n_m}$$

Where $n = \sum_{i=0}^{m} n_i$.

1.10 Problem 3.1

To communicate one bit information reliably, cellular phones trasmit the same binary symbol 5 times. The reciever detects the correct information if three or more binary symols are received correctly. What is the information error probability $P(E)$ if the binary symbol error is probability $q = 0.1$

$$P(E) = 1 - \binom{5}{3}(0.9)^3(0.1)^2 + \binom{5}{4}(0.9)^4(0.1) + (0.9)^5 = \binom{5}{1}(0.9)(0.1)^4 + \binom{5}{2}(0.9)^2(0.1)^3$$

1.11 Problem 3.2

There are 8 VMs in a lab. Each computer can be in idle (I) with probability 0.2, suspended (S) with probability 0.2 and probability of active (A) with probability 0.6 What is the probability there are an equal number of idle and suspended VMs and at least 4 active VMs?

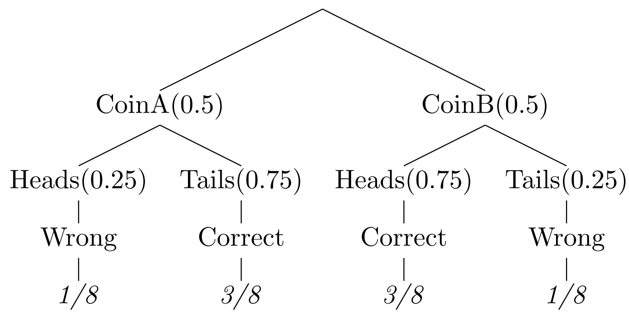
$$P(I = S \wedge A \geq 4) = P(I = 2, S = 2, A = 4) + P(I = 1, S = 1, A = 6) + P(I = 0, S = 0, A = 8)$$

$$= \binom{8}{2,2,4}(0.2)^2(0.2)^2(0.6)^4 + \binom{8}{1,1,6}(0.2)(0.2)(0.6)^6 + \binom{8}{0,0,8}(0.6)^8$$

2 Week 2

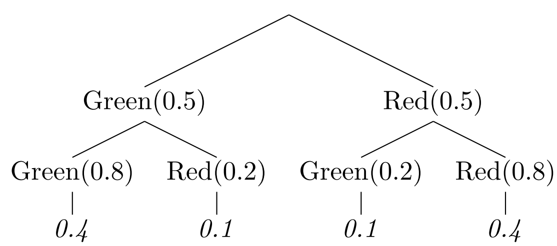
2.1 Problem 2.4

You have two biased coins. Coin A comes up with heads with probability 0.25. Coin B comes up heads with probability 0.75. However you are not sure which is which so you choose a coin randomly and you flip it. If the flip is heads you guess Coin B. If tails you guess Coin A. What is the probability $P(C)$ that your guess is correct.



2.2 Problem 2.5:

Traffic enginers have coordinated the timing of two traffic lights to encourage the run of green lights. With probability of 0.8 a driver will find the 2nd light to have the same color as the first. Assuming that the first light is equally likely to be red or green:



Part A: What is the probability $P(G2)$ that the second light is green?

$$P(G2) = P(G2 \cap G1) + P(G2 \cap R1) = 0.4 + 0.1 = 0.5$$

Part B: What is the probability $P(W)$ that you wait for at least one of the first two lights:

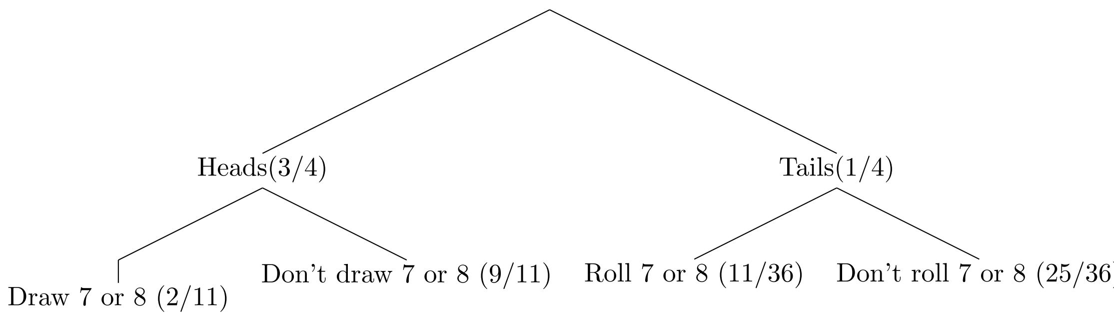
$$P(W) = P(R2 \cap G1) + P(R1) = 0.1 + 0.5 = 0.6$$

Part C: What is $P(G1|R2)$.

$$P(G1|R2) = \frac{P(G1 \cap R2)}{P(R2)} = 0.1/0.5 = 0.2$$

2.3 Problem 2.6:

You have a shuffled deck of cards labeled 2 to 12. You also have two fair dice. You toss a baised coin (heads with probability 0.75). If the result isheads then you draw a card from the shuffled deck of cards. Otherwise, you roll two dice and add the numbers. What's the probability that you get a 7 or 8.



$$P(7/8) = P(H \cap (7/8)) + P(T \cap (7/8)) = \left(\frac{3}{4}\right)\left(\frac{2}{11}\right) + \left(\frac{1}{4}\right)\left(\frac{11}{36}\right) = \frac{6}{44} + \frac{11}{144} = 0.213$$

2.4 Counting Principles

2.4.1 Counting Principle 1

An experiment consists of two subexperiments. If one subexperiment hsa k outcomes and the other subexperiment has n outcomes, the the experiment has nk outcomes.

2.4.2 Counting Principle 2

A sampling without replacement technique where you pick k objects out of n distinguishable bjects such that the order of picking does matter.

$$P(n, k) = {}^{n}P_k = \frac{n!}{(n-k)!}$$

2.4.3 Counting Principle 3

When you pick k objects from n objections, each way contains k objects that can be permuted $k!$ ways. The number of ways to **choose** k objects out of n distinguishable objects is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

2.5 Problem 2.7

In a game of yummy you are dealt a seven card hand.

2.5.1 Part A

Q: what is the probability $P(R_7)$ that your hand only has red cards.

$$P(R_7) = \frac{\binom{26}{7}}{\binom{52}{7}} = \frac{26!}{(26-7)!7!} \cdot \frac{(52-7)!7!}{52!} = \frac{26!45!}{19!52!} = 0.00492$$

2.5.2 Part B

Q: What is the probability $P(F)$ that your hand has only face cards

$$P(F) = \frac{\binom{12}{7}}{\binom{52}{7}} = 5.91 \times 10^{-6}$$

2.6 Problem 2.8

2.6.1 Part A

Q: In a game of poker you are dealt a five of card hand. What is the probability of a full house: "Three of a kind and two of a kind"

$$P(FH) = \frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}} = 0.00144 = 0.14\%$$

This comes from there being 13 suites and you pick one of them, then choose 3 from the 4 cards in that suite, then for the pair you have a different suite and you pick 2 from the 4 in that new suite. Note $\binom{n}{1} = n$.

2.6.2 Part B

Q: In a game of poker you are dealt a five of card hand. What is the probability of a 4 of a kind

$$P(4oK) = \frac{13 * 48}{\frac{52}{5}} = 0.000240 = 0.02\%$$

This comes from there being 13 ways to make a 4 of a kind, and then 48 different cards remaining in your deck of 52

2.7 Counting Principle Number 4

The number of observation sequences for n subexperiments with sample space $S = \{0, 1\}$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is

$$\binom{n}{n_1}$$

2.7.1 Example Problem:

Consider a 32 digit binary value, and you test each bit. How many binary codes with exactly 8 1s exist.

$$P(n_1 = 8) = \binom{32}{8}$$

We get this as $n = 32$ from the fact its a 32 bit value and each bit is a 0 or 1 value.

2.8 Counting Principle 5 (Multinomial)

For n repeating subexperiments with Sample Space $S - \{s_0, \dots, s_{m-1}\}$ the number of length $n = n_0 + n_1 + \cdots + n_{m-1}$ obersvation sequences with s_i appearing s_i times is

$$\binom{n}{n_0, n_1, \dots, n_{m-1}} = \frac{n!}{n_0!n_1! \dots n_{m-1}!}$$

2.9 Problem 2.10

Q: Consider testing each of the 16 elements where each element can be in high voltage state (H), low holtage state (L) and high impedance state (Z). In how many ways can you observe $6H$, $8L$ and $2Z$ elements?

$$P(6H, 8L, 2Z) = \binom{16}{6, 8, 2} = \frac{16!}{6!8!2!}$$

3 Week 3

3.1 Discrete Random Variables