UNIT IV

[Topics mentioned in Syllabus:

Electromagnetism: Displacement current, Three electric vectors (**E**, **P**, **D**,), Maxwell's equations in integral and differential forms. Electromagnetic wave propagation in free space.]

- **Note: This video link will also help you to understand the topic**

https://www.youtube.com/watch?v=XUR-dnDa7eI

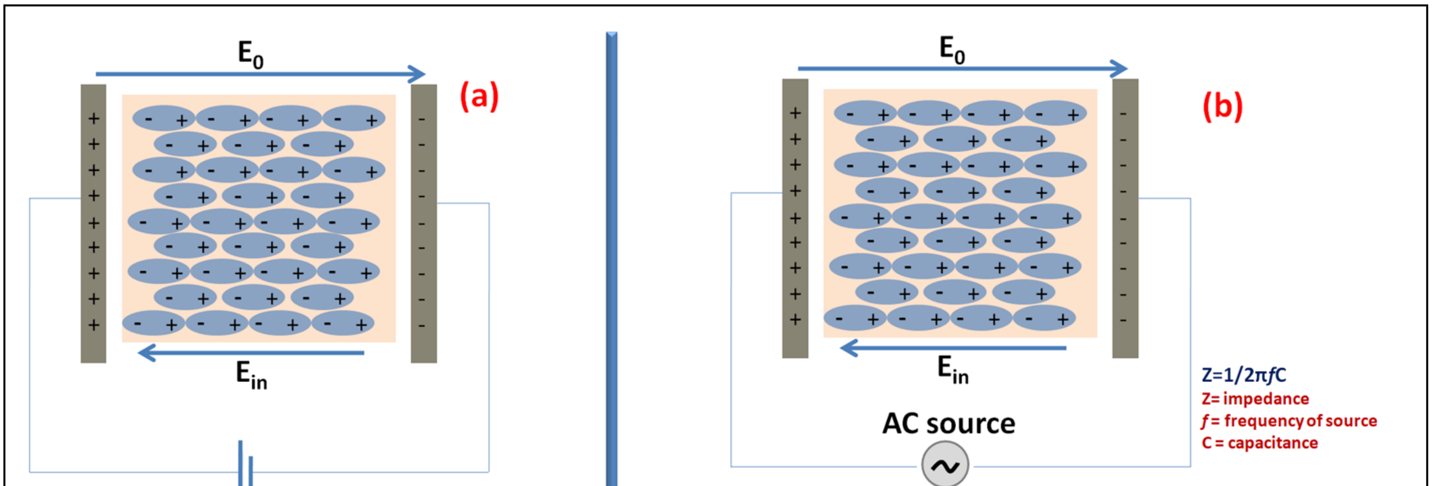
ELECTROMAGNETISM

EM waves are the waves on EM spectrum whether visible or invisible to human.

Application of Maxwell's equations

- Help to analyze the permittivity (ε) and permeability (μ) of any medium for propagation of electromagnetic waves.
- Maxwell Equations proves that in vacuum electromagnetic waves (EM waves) travels with the speed of light.
- To calculate the energy dissipated due to propagation of EM waves in any medium
- To calculate the depth of penetration of EM waves

- If we consider a parallel plate capacitor, it is well known that it works for AC source (as it used in ceiling fan to give initial gust ofcharge to motor to start the rotations). When a DC source is applied across it, it gets charged but further conduction stops because this device offers infinite impedance for DC source (refer Fig. (a) and (b) below).



Capacitor **works on applying AC source** because it offer a finite impedance **(Z=1/2πfC)** for AC **but infinite impedance for DC source (since f = 0 for DC source so Z = α)**

Concept of Displacement Current:

- As discussed above that a capacitor works on applying AC source; although the circuit is open due to gap between the plates still the circuit is completed means a current between the plates exist. The polarization of dielectric medium established the field across the plates. Maxwell assumed the **existence of a current called displacement current between the plates**. $\mathbf{J_D = dD/dt}$ (i.e. current density of displacement current ($\mathbf{J_D}$) is equal to the rate of change of displacement vector (**D**). This concept is used in deriving Maxwells fourth equation.

Things to understand: (not part of syllabus)

Following basics and identities are essential to start with derivation of Maxwell's equations.

Function of two variables :

eg: $z = x^2+y^2$. When z=r(a constant), we get : $r^2 = x^2+y^2$ which is equation of a circle.

Types of functions

a) **Scalar function** f(x,y,z)(eg: $f(x,y,z)=x^2+y^2+z^2$)

Partial derivative of a scalar function:

Consider the function $f(x, y) = x\cos(y) + y$, its derivative with respect to the variable x is

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x}(x\cos(y) + y) \\ &= \frac{\partial}{\partial x}(x) \times \cos(y) + \frac{\partial}{\partial x}(y) \\ &= 1 \times \cos(y) + 0 = \cos(y) . \end{aligned}$$

b) **Vector function**: $\vec{A} = A_x i + A_y j + A_z k$; A_x, A_y and A_z are the three vector components of the function.Each component can be function of a single variable or all three variables/coordinates.

example of a vector function: $(x^2-1)\, i + xy^2\, j + z^3\, k$

$\vec{r}$ is position vector for a point (x,y,z) distance from origin is |r|;

$$|\vec{r}| = \sqrt{(x^2+y^2+z^2)}$$

Gradient of a scalar function

Define a scalar function f: f(x,y,z)

Gradient of f is $\vec{\nabla}\mathrm{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)f$

is a vector quantity (*i,j,k* are unit vectors) ($\vec{\nabla}$ **:** is known as "del operator").

$$f(x,y) = 3x + 3$$

$$\begin{aligned}\boldsymbol{\nabla} f &= \frac{\partial}{\partial x}(3x+3)\boldsymbol{i} + \frac{\partial}{\partial y}(3x+3)\boldsymbol{j} \\ &= 3\boldsymbol{i} + 0\boldsymbol{j}\ .\end{aligned}$$

Physically it indicates "rate at which magnitude of "*f*" increases or decreases at a point (x,y,z) along the normal.

The electric field is the gradient of the scalar potential Φ (potential difference).

If Φ is scalar function, $\vec{E} = \vec{\nabla}\Phi(x,y,z)$

Divergence of a vector function :

1. It is denoted by notation Del dot($\vec{\nabla}$.). since $\vec{\nabla}$(Del Operator) $= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

When a Del operator is operated over a **vector function $\vec{A}$** which has three components

(A_x, A_y, A_z);

therfore $\vec{\nabla}.\vec{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(iA_x + jA_y + kA_z\right)$

$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$ which is a scalar quantity ($i.\ i$ =1 ; $j.\ j$ =1; $k.\ k$ = 1)

Physically it measures how much a **vector function is converging or diverging at a point.**

Example:

$$F = xyi + yzj + zxk,$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$
$$= y + z + x = x + y + z.$$

Complicated example below!!

$F = \frac{r}{|r|^3}$ *for all* $r \neq 0$. *Find* $\nabla \cdot F$.

$$F = F_1 i + F_2 j + F_3 k = \frac{x}{(x^2+y^2+z^2)^{3/2}} i + \frac{y}{(x^2+y^2+z^2)^{3/2}} j + \frac{z}{(x^2+y^2+z^2)^{3/2}} k.$$

Using the quotient rule for the derivative, we have

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{1}{|r|^3} - \frac{3x^2}{|r|^5},$$

and analogous results for $\partial F_2/\partial y$ and $\partial F_3/\partial z$. Adding the three derivatives results in

$$\nabla \cdot F = \frac{3}{|r|^3} - \frac{3(x^2+y^2+z^2)}{|r|^5} = \frac{3}{|r|^3} - \frac{3}{|r|^3} = 0,$$

3 Curl of a vector function:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(i\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) - j\left(\frac{\partial Az}{\partial x} - \frac{\partial Ax}{\partial z}\right) + k\left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right)\right)$$

It is also a vector function. Physically it measures the "rotational" characteristic of a function at a point.(tendency to swirl)

$$u = u_1(x,y,z)i + u_2(x,y,z)j + u_3(x,y,z)k.$$

$$\nabla \times u = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) i + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) j + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) k.$$

Example:

$$F = xyi + yzj + zxk,$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix} = -yi - zj - xk.$$

Show :

$$\nabla \times (\nabla f) = 0.$$

Curl of gradient of function is zero!!

$$\nabla \times (\nabla f) = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{pmatrix}$$
$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) i + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) j + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) k = 0,$$

$$\nabla \cdot (\nabla \times u) = 0.$$

Divergence of a curl is zero

4 **Laplacian of a function:**

Divergence of gradient of function is known as Laplacian. It is a second order derivative.

It is denoted as $(\vec{\nabla}.)\vec{\nabla}\boldsymbol{f} = \nabla^2\boldsymbol{f} = \left(\frac{\partial^2}{\partial \boldsymbol{x}^2} + \frac{\partial^2}{\partial \boldsymbol{y}^2} + \frac{\partial^2}{\partial \boldsymbol{z}^2}\right)\boldsymbol{f}$

It is defined for scalar as well as vector functions.

For a vector function $\vec{A}$ $\left(\frac{\partial^2}{\partial \boldsymbol{x}^2} + \frac{\partial^2}{\partial \boldsymbol{y}^2} + \frac{\partial^2}{\partial \boldsymbol{z}^2}\right)$ acts on A_x, A_y and A_z individually

Find the Laplacian of $f(x, y, z) = x^2 + y^2 + z^2$.

We have $\nabla^2 f = 2 + 2 + 2 = 6.$

The vector signs → are dropped henceforth

Theorems in vector calculus: (Proof not required)

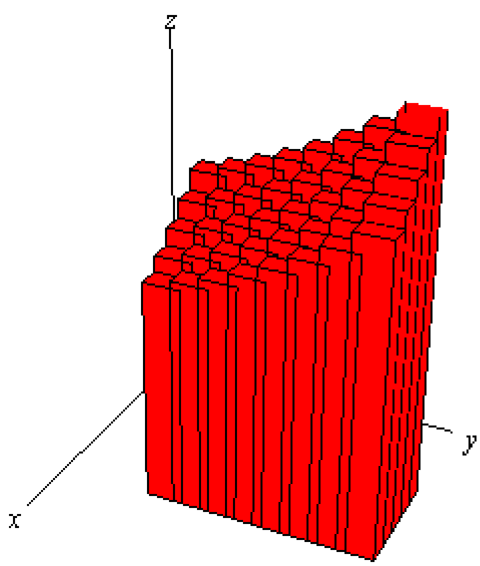
$\nabla.(\nabla \times \mathbf{A}) = 0$ [div curl $\mathbf{A} = 0$, Divergence of a curl is always zero]

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A})$ - $\nabla^2\mathbf{A}$ [Curl of Curl A = grad div A - ∇^2A]

$\nabla(\nabla.\mathbf{A}) = 0$ (gradient of a divergence is always zero

Multi-dimensional integrals (scalar)

$$\int_A f dA = \iint_A f(x, y)\, dx\, dy, \qquad \int_V f dV = \iiint_V f(x, y, z)\, dx\, dy\, dz,$$



Double integral is a volume enclosed between two surfaces (top and bottom)

Types of integrals on vector functions:

1) **Line integral** :

$\int$ E.dl : The line integral of a vector function is defined over a particular path for function **A**(x,y,z), between two end points P(x_1,y_1,z_1) Q(x_2,y_2,z_2).

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = \int_P^Q (A_x\, dx + A_y\, dy + A_z\, dz).$$

For instance, if **A** is a force then the line integral is the work done in going from P to Q.

Example of a line integral

The line integral of electric field is the scalar potential (emf)

(**Voltage**)V = $\int$ E.dl

2) **Surface integral :**

$\iint$ E.ds : It is an integral a vector function defined over a surface

Line and surface integrals are dot products of two vector quantities.

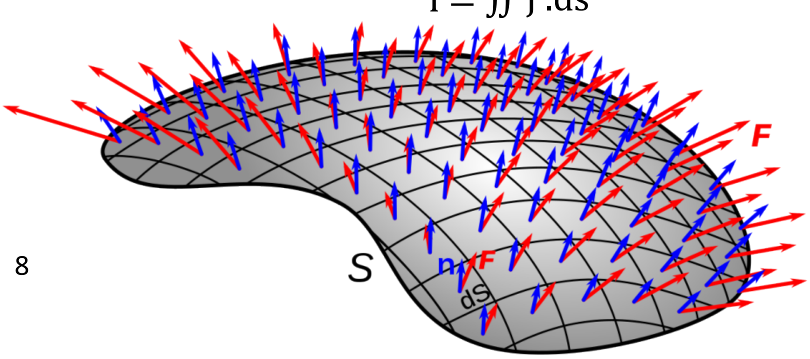
Let $u = u(r)$ be a vector field, and let S be a surface with normal unit vector $\hat{n}$. The surface integral of u over S is defined to be

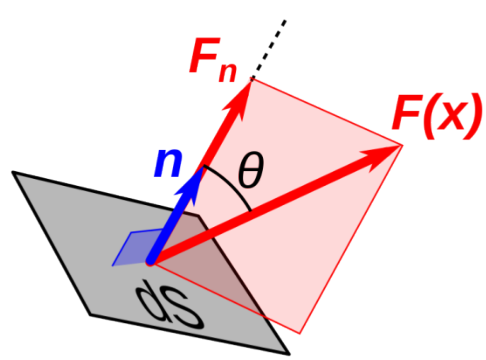
$$\int_S u \cdot dS = \int_S u \cdot \hat{n}\, dS,$$

Example of a surface integral

Let I be total current in a wire which is a scalar quantity. Let $\vec{J}$ be current density(current flowing through a cross-sectional area) which is a vector quantity. Then for an infinitesimal surface $\overrightarrow{ds}$ on the cross-section of the wire (refer to figure below)

I = $\iint$ J .ds



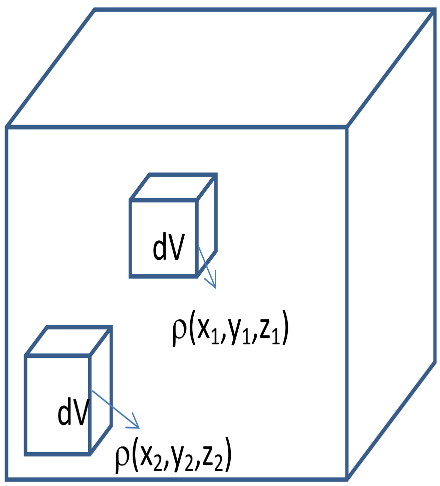


3) Volume integral :

$$\iiint f(x,y,z)\, dV$$

It is integral of a scalar function over a volume.V and dV=dxdydz

[For example: Let q be total charge enc Charge density = charge/volume



ρ=Q/V or Q=ρV if charge is constant, if not,

$$\text{Total charge } Q = \iiint \rho(x,y,z)\, dV$$

Conversion of Line integral to surface integral(important to remember)

a) Stokes' theorem enables to convert line integral into a surface integral as

$$\int \mathbf{E.dl} = \iint (\nabla \times \mathbf{E}).\mathbf{ds}$$

For this we require to take curl of that vector (say E) and then write it in surface integral form.

b) Divergence theorem enables to convert surface integral into a volume integral as

$$\iint \mathbf{E.ds} = \iiint \nabla.\mathbf{E}\, \mathbf{dv}$$

For this we require to take divergence of that vector (say E) and then write it in volume integral form.

---------------------------------END OF MATHEMATICAL PRELIMINARIES-----------------------

--------------------------------------Basics from 12th class --

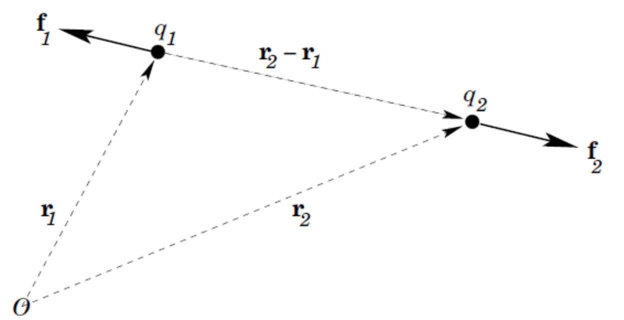
a) Coulomb`s law of force between electric charges

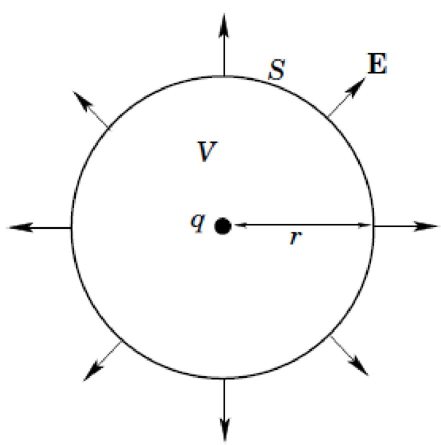
$$\mathbf{f}_2 = \frac{q_1\, q_2}{4\pi\,\epsilon_0} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \qquad \epsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C}^2\,\mathrm{N}^{-1}\mathrm{m}^{-2}.$$

and the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \sum_i$$

$$E_r(r) = \frac{q}{4\pi\,\epsilon_0\, r^2},$$





Electric field due to charge enclosed in sphere.

Electric flux: Total No of lines of electric field, normal to a given surface.

b) Gauss law in electrostatics:

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S E_r\, dS_r = E_r(r)\, 4\pi\, r^2 = \frac{q}{4\pi\epsilon_0\, r^2}\, 4\pi\, r^2 = \frac{q}{\epsilon_0},$$

Total electric flux due to charge enclosed by surface is equal to the charge within the surface divided by permittivity of the medium.

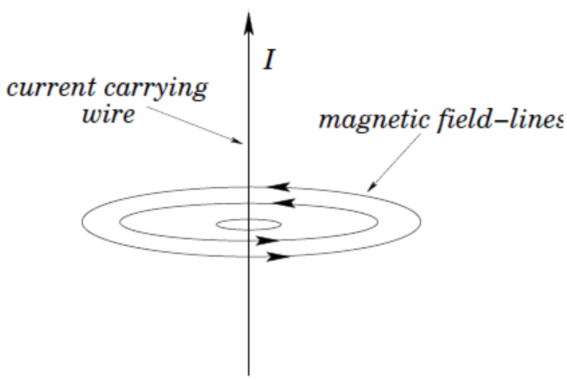
c) Scalar potential (Voltage):

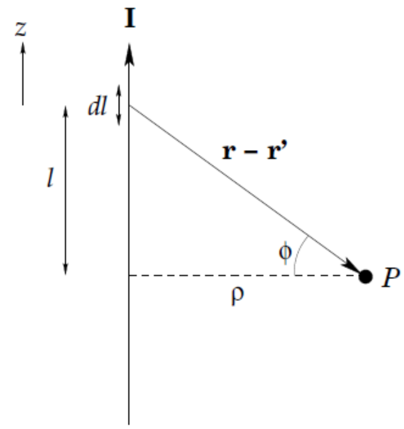
$$\mathbf{E} = -\nabla \phi,$$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}'.$$

d) Ampere`s theorem and magnetic field:





$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}$$

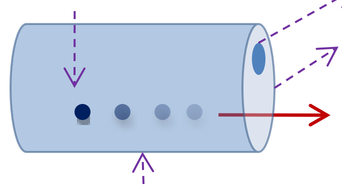
Figure 33:

Actual course content begins in next page

Current density: A vector quantity indicating directional flow of charges perpendicular to cross sectional area.

Q1. Derive the equation of continuity $\nabla . J = -\frac{d\rho}{dt}$ and interpret the expression.

As shown, a current is constituted only when a charge moves or leaves the volume V enclosed by the surface S.



From (see example of surface integral) definition of current,

$$I = \iint J.ds \quad \ldots\ldots\ldots\ldots[1]$$

And as we know that current is rate of change of charge $I = -\frac{dq}{dt}$ $\quad\ldots\ldots\ldots\ldots[2]$

Negative sign shows that charge (q) leaves the volume V enclosed by the Gaussian surface S.

$I = \iint J.ds = -\frac{dq}{dt}$ since $q = \iiint \rho(x,y,z)dV$ (total charge is integral of charge density)

Therefore $\iint J.ds = -\frac{d(\iiint \rho.dV)}{dt}$

The time differentiation can be performed inside the integral

$$\iint J.ds = \iiint -\frac{d\rho}{dt} dV$$

Using **Gauss divergence** theorem to convert surface integral into volume integral $\iint J.ds = \iiint \nabla.Jdv$

$$\iiint \nabla.J\, dV = \iiint -\frac{d\rho}{dt} dV$$

From above we find that

$$\nabla.\mathbf{J} = -\frac{\mathbf{d\rho}}{\mathbf{dt}}$$

Interpretation of the expression is that divergence of current density (J) is equal to rate of change of volume charge density (ρ). Negative sign shows that charge (q) leaves the unit volume.

$$\nabla.\mathbf{J} + \frac{\mathbf{d\rho}}{\mathbf{dt}} = \mathbf{0}$$

As special case, If $\nabla.\mathbf{J} = \mathbf{0}$

It shows that rate of flow charge from unit volume is zero means there is no flow of charge therefore we can say charge is stagnated or not moving.

Polarization of a dielectric medium: (underlined portion for understanding , not needed for exam).

When a dielectric medium(any material which is non-metal) is placed in an external electric field, its molecules gain electric dipole moment and the dielectric material is said to be polarized. The electric dipole moment induced per unit volume of the dielectric material is called the electric polarization of the dielectric. The polarization density is represented by a vector $\overline{\boldsymbol{P}}$.

Concept of displacement current and electric displacement field in a dielectric medium:

In dielectric materials, the total charge of an object can be separated into "free" and "bound" charges. Bound charges set up electric dipoles in response to an applied electric field E, and polarize other nearby dipoles tending to line them up, the net accumulation of charge from the orientation of the dipoles is the bound charge. They are called bound because they cannot be removed: in the dielectric material the charges are the electrons bound to the nuclei. Free charges are the excess charges which can move into electrostatic equilibrium, i.e. when the charges are not moving and the resultant electric field is independent of time, or constitute electric currents.

The part below is essential for derivation of Maxwell`s fourth equation:

Total charge densities in a medium:

In terms of volume charge densities, the total charge density is:

$$\rho=\rho_f+\rho_b$$

Or in terms of surface charge densities,

$$\sigma=\sigma_f+\sigma_b$$

Electric displacement vector **D**: The electric displacement field "D" is defined as

$$\overrightarrow{\boldsymbol{D}} = \varepsilon_0\vec{\boldsymbol{E}} + \vec{\boldsymbol{P}}$$

Based on this, we define the displacement current as

$$\vec{J}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t}$$

This will be used in derivation of correct form of Maxwell`s fourth equation.

E, P and D vectors: following above discussion. Refer to below figure, when DC voltge is applied across the capacitor the dielectric medium (insulating medium) between the plates gets polarized due to applied electric field (**E_0**). As shown in Fig (a), there exists an internal electric field (**E_{in}**) inside the dielectric medium. Hence a net electric field (**E**) is established across the plates.

(Note: The concept of electric displacement vector is used in Maxwell equation 1(Gauss theorem) and Maxwell equation 4)

- $\vec{E} = \overrightarrow{E_0} - \overrightarrow{E_{in}}$
- $\vec{E} = \frac{\rho}{\epsilon_0} - \frac{\rho in}{\epsilon_0}$
-
- Here **Displacement vector (D)** is defined as

$\vec{D} = \frac{q}{A\epsilon_0} = \frac{\rho}{\epsilon_0}$

- It is equal to the **free charge per unit area or equal to the surface density of free charges**.
- $\epsilon_0\vec{E} = \rho - \rho in$
- $\epsilon_0\vec{E} = \vec{D} - \vec{P}$
- $\vec{D} = \epsilon_0\vec{E} + \vec{P}$
- **This is the relation between E, P and D vectors**
- The representation of the displacement vector is shown in figure below

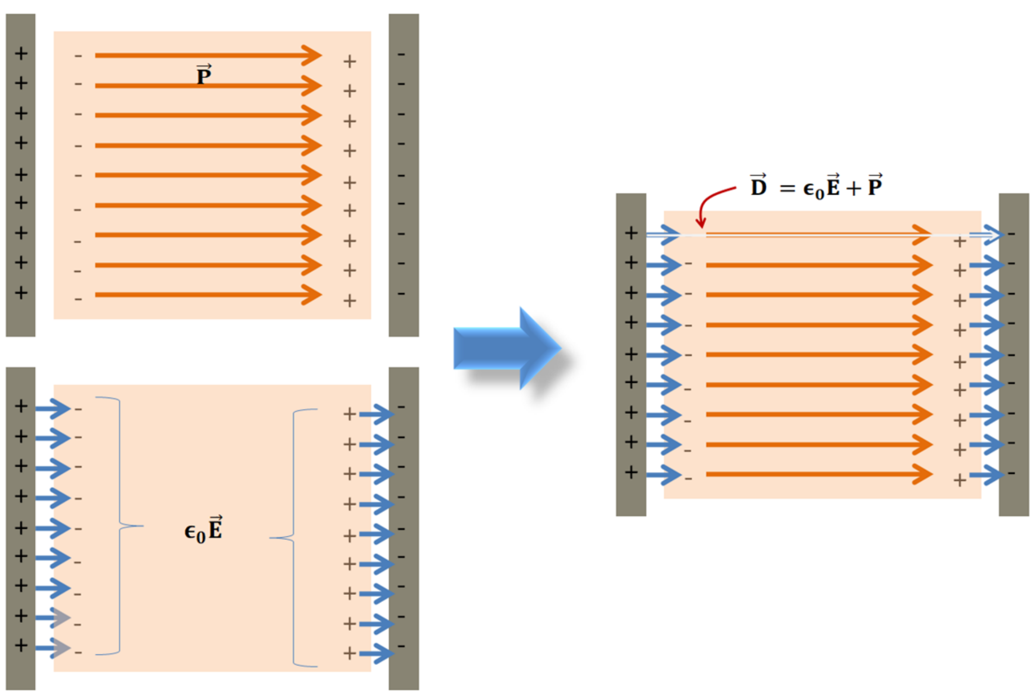


Figure: Effect of electric field on dielectric medium between plates of a capacitor

Basics of electrostatics: 1) Coulomb`s law (please refer to 12^{th} class textbook for detailed explanation),

Gauss law

The electric flux (φ_E) diverging from a closed surface S (enclosed by volume V) is equal to the $1/\epsilon_0$ times of charge enclosed by that surface. Intergral form of law is $\varphi_E = \iint E.ds = \frac{1}{\epsilon_0} q$.

Basics of magnetostatics: Biot Savart`s law, Amperes theorem, Faradays law of induction

(please refer to 12^{th} class textbook for detailed explanation)

Gauss law in magnetostatics

Magnetic flux (φ_B) diverging from a Gaussian surface of S always zero.

Intergral form of law is $\varphi_B = \iint B.ds = 0$.

Maxwell's equations in differential form:

Basically, these equations are four laws in differential form in Electrostatics and Magnetostatics. These laws govern the relation between the electric and magnetic fields in vacuum and in a medium.

Statements of the Maxwell`s equations(1-4) (Proofs are from page 17 to page 20)

1) **Gauss Law in Electrostatics**:

The electric flux (φ_E) diverging from a closed surface S (enclosed by volume V) is equal to the $1/\epsilon_0$ times of charge enclosed by that surface. Intergral form of law is $\varphi_E = \iint E.ds = \frac{1}{\epsilon_0} q$. Differential form of law is given below. This is known as Maxwell's I^{st} equation, which will be proved .

$$\nabla.E = \rho$$

2) **Gauss Law in Magnetostatics**: Magnetic flux (φ_B) diverging from a Gaussian surface of S always zero.

It is because magnetic flux has always two poles (North–south) thus any flux diverging from closed surface re-enters the surface therefore net flux diverging is zero.
Intergral form of law is $\varphi_B = \iint B.ds = 0$.

Differential form of law is given below. This is known as Maxwell's IInd equation.

$$\nabla.B = 0$$

3) **Faraday's induction law:** The rate of change of magnetic flux is equals to induced electromotive force (emf) in the circuit. Intergral form of law is **emf** ie$\int E.dl = -\frac{d\varphi_B}{dt}$.

In R.H.S Φ_B is the magnetic flux, whose rate of change is the emf induced.

Differential form of law is given below. This is known as Maxwell's IIIrd equation.

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

4) **Modified Ampere's law for time varying fields:**

Ampere`s law states that the total magnetic field over a current carrying wire is proportional to the strength of the current.

As the Ampere's law in integral form is given as $\int B.dl = \mu_0 I$. The modified **Ampere's law for time varying magnetic fields'** in differential form is given below. This is known as Maxwell's IVth equation.

$$\nabla \times H = J_c + \frac{\partial D}{\partial t}$$

Q.1 Derive all four Maxwell's equations.

Derivation of Maxwell's Ist equation:

The Gauss Law of Electrostatics in integral form is given as

$$\varphi_E = \iint E.ds = \frac{1}{\epsilon_0} q$$

$$\iint (\epsilon_0 E) .ds = q$$

$$\text{since} \iiint \rho dV = q \text{ therefore} \iint (\epsilon_0 E) .ds = \iiint \rho dV$$

(*Please refer to page 13 and 14 for definition of* $\vec{D}$)

$$\text{and } \epsilon_0 E = D$$

$$\iint D .ds = \iiint \rho dV$$

Using **Gauss divergence** theorem to convert surface integral into volume integral $\iint D.ds = \iiint \nabla.DdV$

$$\iiint \nabla.D\, dV = \iiint \rho dV$$

$$\text{or} \iiint (\nabla.D - \rho)\, dV = 0$$

$$\text{or } \nabla.D - \rho = 0$$

$$\text{or } \nabla.\mathbf{D} = \boldsymbol{\rho}$$

Derivation of Maxwell's II[nd] equation:

As the Gauss Law of Magnetostatics in integral form is given as

$$\varphi_B = \iint B.ds = 0.$$

Using **Gauss divergence** theorem to convert surface integral into volume integral $\iint B.ds = \iiint \nabla.Bdv$

$$\iint B.ds = \iiint \nabla.B\, dV = 0$$

$$\nabla.\mathbf{B} = \mathbf{0}$$

Derivation of Maxwell's III[rd] equation:

Faraday's induction law in intergral form is given as **emf** $= \int E.dl = -\frac{d\varphi_B}{dt}$

$$\int E.dl = -\frac{d\varphi_B}{dt} \quad \text{since} \quad \varphi_B = \iint B.ds$$

Magnetic flux is defined as the integral of magnetic field over a surface

$$\text{therefore} \int \mathrm{E.dl} = - \frac{d(\iint \mathrm{B.ds})}{dt}$$

Here operator–d/dt can be operated over B within the integral. Therefore

$$\int \mathrm{E.dl} = - \iint \frac{dB}{dt}.\mathrm{ds}$$

Using **Stokes' theorem** to convert line integral into surface integral $\int \mathrm{E.dl} = \iint (\nabla \times \mathrm{E}).\mathrm{ds}$, then above can be written as

$$\iint (\nabla \times \mathrm{E}).\mathrm{ds} \;\; = - \iint \frac{dB}{dt}.\mathrm{ds}$$

$$\iint \left[(\nabla \times \mathrm{E}) - \frac{dB}{dt}\right].\mathrm{ds} \;\; = 0$$

$$\left[(\nabla \times \mathrm{E}) - \frac{dB}{dt}\right] = 0$$

$$\nabla \times \mathbf{E} \; = - \frac{\mathbf{dB}}{\mathbf{dt}}$$

Derivation of Maxwell's IVth equation:

As the Ampere's law is given as $\int \mathrm{B.dl} = \mu_0 \mathrm{I}$

Or $\int \mathrm{B.dl} = \mu_0 \iint \mathrm{J.ds}$ *since* $I = \iint \mathrm{J.ds}$ *and* $B = \mu_0 H$ *or* $\frac{B}{\mu_0} = H$

$$\int \frac{B}{\mu_0}.\mathrm{dl} = \iint \mathrm{J.ds}$$

$$\text{or} \int \mathrm{H.dl} = \iint \mathrm{J.ds}$$

Using **Stokes' theorem** to convert line integral into surface integral $\int \mathrm{H.dl} = \iint (\nabla \times \mathrm{H}).\mathrm{ds}$

$$\iint (\nabla \times \mathrm{H}).\mathrm{ds} = \mu_0 \iint \mathrm{J.ds}$$

$or\ \nabla \times H = \mu_0 J$ …………[1]

Taking divergence of above expression as

$\nabla .(\nabla \times H) = \nabla .J$

As the divergence of a curl is always zero $[\nabla .(\nabla \times A) = 0]$ therefore, $\nabla .(\nabla \times H) = 0$

We get

$0 = \nabla .J$

Interpretation of $\nabla .\mathbf{J} = \mathbf{0}$ is that *charge does not flow* or *current is zero* but as the current is established therefore here the equation of continuity $\nabla .\mathbf{J} = -\frac{d\rho}{dt}$ ……….. [2] must be followed. To overcome this short coming,

Maxwell suggested adding the displacement current term $\mathbf{J_D}$ in RHS of equation [1] and then again take the divergence.

$\nabla \times H = J + J_D$ ……….. [3]

Take the divergence of above as $\nabla .(\nabla \times H) = \nabla .J + \nabla .J_D$. As above $\nabla .(\nabla \times H) = 0$ therefore

$\nabla .J_D = - \nabla .J$

Now, substituting the equation of continuity $\nabla .J = -\frac{d\rho}{dt}$ in above expression we get

$\nabla .J_D = \frac{d\rho}{dt}$

Now substituting the value of ρ (i.e. $\nabla .\mathbf{D} = \boldsymbol{\rho}$)

($\nabla .\mathbf{D} = \boldsymbol{\rho}$ was proved in Maxwell equation 1 – page 17)

$$\nabla .J_D = \frac{d(\nabla .\mathbf{D})}{dt}$$

Here operator d/dt can be operated over D in above expression.

$$\nabla .J_D = \frac{\nabla .}{\ } \frac{d(\mathbf{D})}{dt}$$

From above we get

$\mathbf{J_D} = \frac{\mathbf{d(D)}}{\mathbf{dt}}$

The value of density of **displacement current** (J_D) is equal to rate of change of displacement vector (**D**).

At last substituting the value of $\mathbf{J_D} = \frac{\mathbf{d(D)}}{\mathbf{dt}}$ into equation [3] we get the Maxwell's IVth equation as

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\mathbf{d(D)}}{\mathbf{dt}}$$

In a nutshell

Medium	Vacuum
$\nabla.\mathrm{D} = \rho$	$\nabla.\mathbf{D} = \mathbf{0}$
$\nabla.\mathrm{B} = 0$	$\nabla.\mathbf{B} = \mathbf{0}$
$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial \mathrm{t}}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$
$\nabla \times \mathrm{H} = \mathrm{J_c} + \frac{\partial \mathrm{D}}{\partial \mathrm{t}}$	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial \mathbf{t}}$

Q.3 show that electric field and magnetic fields travel with same speed (as that of c)

Or propagation of plane electromagnetic waves in free space:

(From Maxwell`s fourth equation we derive the wave equation of electric field. From third equation we can derive wave equation of magnetic field)

<u>Characteristics of free space:</u>

As there is no free charge in free space (vacuum) thus charge density (ρ) =0 and therefore conductivity (σ) =0 or current density (J) =σE=0. Consequently, Maxwells fourth equation reduces to:

$$\nabla \times \nabla \times \mathrm{H} = \nabla \times \left(\frac{\partial \mathrm{D}}{\partial \mathrm{t}}\right)$$

since $\nabla \times (\nabla \times \mathrm{A}) = \nabla(\nabla.\mathrm{A}) - \nabla^2\mathrm{A}$ therfore $\nabla \times (\nabla \times \mathrm{H}) = \nabla(\nabla.\mathrm{H}) - \nabla^2\mathrm{H}$

and $D = \epsilon_0 E$

$$\nabla \times (\nabla \times H) = \nabla(\nabla.H) - \nabla^2 H = \frac{\partial}{\partial t}\nabla \times (\epsilon_0 E)$$

$\nabla(\nabla.H) - \nabla^2 H = \epsilon_0 \frac{\partial}{\partial t}(\nabla \times E)$ since $\nabla(\nabla.H) = 0$ gradient of a divergance is always zero.

Since $\nabla \times E = -\frac{\partial B}{\partial t}$

$$-\nabla^2 H = -\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial t}\right)$$

Since $B = \mu_0 H$

$$\nabla^2 H = -\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial \mu_0 H}{\partial t}\right) \rightarrow = \epsilon_0 \mu_0 \frac{\partial^2 H}{\partial t2}$$

$$\boldsymbol{\nabla^2 H = \epsilon_0 \mu_0 \frac{\partial^2 H}{\partial t2}}$$

Comparing it with standard equation of motion of wave

$$\boldsymbol{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t2}}$$

We find that $\frac{1}{v^2} = \epsilon_0 \mu_0$

or $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{(8.85 \times 10^{-12} \text{ farad }/\text{m}) \times (4\pi \times 10^{-7} \text{ Henry }/\text{m})}} = 2.99 \times 10^8 \text{ m/sec}$ = c (speed of light in vacuum)

Similarly from taking the curl of Maxwell's III^rd^ equation will give equation of motion of electric field (E) in vacuum

$$\nabla \times \nabla \times E = \frac{\nabla \times \partial B}{\partial t}$$

since $\nabla \times (\nabla \times A) = \nabla(\nabla.A) - \nabla^2 A$ therfore $\nabla \times (\nabla \times E) = \nabla(\nabla.E) - \nabla^2 E$

and $B = \mu_0 H$

$$\nabla \times (\nabla \times E) = \nabla(\nabla.E) - \nabla^2 E = \frac{\partial}{\partial t}\nabla \times (\mu_0 H)$$

$$\nabla(\nabla.E) - \nabla^2 E = \mu_0 \frac{\partial}{\partial t}(\nabla \times H)$$

Since $\nabla \times H = \frac{\partial D}{\partial t}$ *and* $\nabla(\nabla.E) = 0$ (gradient of a divergance is always zero).

$$- \nabla^2 E = - \mu_0 \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right)$$

Since $D = \epsilon_0 E$

$$\nabla^2 E = - \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right) \rightarrow = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial^2 t}$$

$$\boldsymbol{\nabla^2 E = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial^2 t}}$$

Comparing it with standard equation of motion

$$\boldsymbol{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial^2 t}}$$

We find that $\frac{1}{v^2} = \epsilon_0 \mu_0$ or $\frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \frac{\text{Henry}}{\text{m}} \ \text{x}\ 8.85 \times 10^{-12} \frac{\text{farad}}{\text{m}}}}$ v = c (speed of light)

Above, shows that electric field and magnetic field travels with the same speed in vacuum.

Maxwell also proved that

1. Electric field (E) and magnetic field (H) are bound with each other at 90^0 (or mutually perpendicular) and the direction flow of electromagnetic (EM) energy is again perpendicular to plane of E and B field.

2. The ratio of electric field (E) and magnetic field (H) is $E_0/H_0 = Z_0$ is defined as the *characteristic impedance or intrinsic impedance* or *wave impedance of the free space*.

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}\ \text{Henry/m}}{8.85 \times 10^{-12}\ \text{farad /m}}} = 376.72\ \Omega$$

$$\boldsymbol{\frac{E_0}{H_0} = 376.72\ \Omega}$$

It shows that vacuum offers 376.72 Ω resistance to the propagation of Electric field (E) and magnetic field (H).

$$\frac{E_0}{H_0} = \mu_0 c \qquad \text{or } \frac{\mathbf{E}}{\mathbf{B}} = \mathbf{c} \qquad\qquad \text{since } B = \mu_0 H$$

This relation shows that ratio of electric field and magnetic density (B) in vacuum is equal to c (speed of light)

Numerical problems relate d to **E/H =376** ohm, **E/B =c** and **S = E X H =EH**

1. If the magnitude of **H** (vector) in a plane wave is 1 amp/meter, find the magnitude of **E** (vector) for plane wave in free space. Then find the **B** (vector).
   [Hint: since **E/H =376** ohm, find E field. Units: **E =volt /meter** and **H= turns-amp/meter**]

2. The maximum electric field (E) in a plane electromagnetic wave is 10^2 Newton/coulomb. The wave is proceeding in X direction and the electric field is in the Y direction. Find the maximum magnetic field in the wave and its direction.
   [Hint: since **E/B =c =3x10^8**, find B field. **Units:** B = Weber/m^2. Units: **E =volt /meter** and **H= turns-amp/meter**]

3. If he earth receives 3 cal min^{-1} cm^2 solar energy, what are the amplitudes if electric (E) and magnetic (H) fields of radiation.
   [**Hint:** 1 calorie = 4.2 joule, 1m = 100cm and **E/H =376** ohm (known)
   Given rate of transmission/Flow of electromagnetic radiation **(S) = E X H =EH** =3 cal min^{-1} cm^2 = (3x4.2x10$^{4)}$/60 joule m^{-2} sec^{-1}.
   Find the field E using **E/H =376** and **S = EH** = (3x4.2x10$^{4)}$/60 joule m^{-2} sec^{-1}.
   Units: **E =volt /meter** and **H= turns-amp/meter**]

4. A lamp radiates energy of 1500 watt uniformly. Calculate the average values of electric (E) and magnetic (H) fields of radiation at a distance of 3 m from the lamp.
   [**Hint:** rate of transmission/Flow of electromagnetic radiation **(S) = E X H =EH** =power/total surface area of sphere of radius 3 meter **=P/4πr^2** = 1500/ (4. 3.14. 3^2) and **E/H =376** ohm (known).
   Using **E/H =376** ohm and **S** = 1500/ (4. 3.14. 3^2), find E and H fields]