Notation for a generic continuous-time deterministic optimization problem:

State variables	$s_i(t)$ for $i \in \{1, \dots, n_s\}$ and $s(t) = \begin{bmatrix} s_1(t) \\ \vdots \\ s_{n_s}(t) \end{bmatrix}$
Action/Control variables	$c_i(t)$ for $i \in \{1, \dots, n_c\}$ and $c(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_{n_c}(t) \end{bmatrix}$ choice set $c_i \in C_i\big(s(t)\big)$
Transition function	$\mu_i\big(s(t), c(t)\big)$ for $i \in \{1, \dots, n_s\}$ and $\mu\big(s(t), c(t)\big) = \begin{bmatrix} \mu_1\big(s(t), c(t)\big) \\ \vdots \\ \mu_{n_s}\big(s(t), c(t)\big) \end{bmatrix}$ $ds(t) = \mu\big(s(t), c(t)\big)dt$
Return function	$r\big(s(t), c(t)\big)$
Value function	$V\big(s(t)\big)$
Policy function	$c_i\big(s(t)\big) = \pi_i\big(s(t)\big)$ for $i \in \{1, \dots, n_c\}$

"Sequential problem" (**SP**):

$V\big(s(0)\big) = \max_{c}\{\int_0^\infty e^{-\rho t} r\big(s(t), c(t)\big)dt\}$

$ds_i(t) = \mu_i\big(s(t), c(t)\big)dt$ for $i \in \{1, \dots, n_s\}$

(**HJB**):

$\rho V(s) = \max_{c}\{r\big(s(t), c(t)\big) + \nabla V \cdot \mu\big(s(t), c(t)\big)\}$

FOC: $r_{c_i} = \nabla V \cdot \big(-\mu_{c_i}\big)$ for $i \in \{1, \dots, n_c\}$

$$\begin{bmatrix} \rho V(s) = r(s, c) + \nabla V \cdot \mu(s, c) & HJB \\ r_{c_i} = \nabla V \cdot \big(-\mu_{c_i}\big) & FOC \\ V_{s_i} > 0, V_{s_i, s_i} < 0 & Conditions\ for\ max \end{bmatrix}$$

Neoclassical growth model (NGM): (slides 16+, Moll Julia code)

$r\big(s(t), c(t)\big) = \frac{\big(c(t)\big)^{1-\sigma}}{1-\sigma}$

$\mu_1\big(s(t), c(t)\big) = As(t)^\alpha - \delta s(t) - c(t)$

Consumption saving model: (Moll notes, Julia code)

$r\big(s(t), c(t)\big) = \frac{\big(c(t)\big)^{1-\sigma}}{1-\sigma}$

$\mu_1\big(s(t), c(t)\big) = w + r \times s(t) - c(t)$

Can make more "generic":
-Finite horizon "T"
-$r(t, s, c)$
-$\mu(t, s, c)$
-Non time/state separable return (Epstein-Zin etc)

In examples w/ 1 state & 1 choice we can solve the transition function for the choice variable:
$\dot{s} = \mu(s, c) = w + rs - c \quad \Rightarrow c = \hat{\mu}(s, \dot{s}) = w + rs - \dot{s}$
$\hat{\mu}(\underline{s}, \dot{s} = 0) = w + r\underline{s}$ {cons @ <u>lowest wealth if not borrowing</u>}
$V_s(\underline{s}) \geq u'\left(\hat{\mu}(\underline{s}, 0)\right)$ {borrowing @ lowest wealth should make you weakly better off, MU is high}
$\hat{\mu}(\overline{s}, \dot{s} = 0) = w + r\overline{s}$ {cons @ <u>highest wealth if not saving</u>}
$V_s(\overline{s}) \leq u'\left(\hat{\mu}(\overline{s}, 0)\right)$ {saving @ highest wealth should make you weakly better off, MU is low}

Moll code:
$V_s(\overline{s}) = u'\left(\hat{\mu}(\overline{s}, 0)\right)$ {should be LEQ?}
$V_s(\underline{s}) = u'\left(\hat{\mu}(\underline{s}, 0)\right)$ {should be GEQ?}

More generically:
$\dot{s} = \mu(s, c) \Rightarrow c = \hat{\mu}(s, \dot{s})$
-Consumption @ $\dot{s} = 0$: $c = \hat{\mu}(s, 0)$
$V_s(\underline{s}) \geq u'\left(\hat{\mu}(\underline{s}, 0)\right)$