Two theorems on dissipative energy losses in capacitor systems



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# Two theorems on dissipative energy losses in capacitor systems

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**Abstract**

This article examines energy losses in charge motion in two capacitor systems. In the first charge is transferred from a charged capacitor to an uncharged one through a resistor. In the second a battery charges an originally uncharged capacitor through a resistance. Analysis leads to two surprising general theorems. In the first case the fraction of energy dissipated in the resistor depends solely on the ratio of the two capacitances. The values of the original charge and the resistance play no role. In the second case half of the energy supplied by the battery is dissipated and half is stored in the capacitor. The values of the battery emf and the resistance play no role.

## Introduction

We wish to examine two instances of energy losses in capacitor systems. In the first a capacitor $C_1$ carrying a charge $Q$ is connected to an originally uncharged capacitor $C_2$ through a resistor $R$. When equilibrium is reached, a fraction of the original energy is, as is well known, lost to ohmic heating. In the second a battery with emf $\mathcal{E}$ charges a capacitor through a resistor $R$. When the capacitor is charged, a fraction of the energy supplied by the battery is again lost to ohmic heating.

Calculation of the fractions of energy lost leads to two rather surprising general theorems. For the two-capacitor problem the fraction depends solely on the ratio of the two capacitances, $C_2/C_1$. The values of the resistance and the total charge have no effect. In charging the single capacitor the result is even more surprising. Half of the energy supplied by the battery is stored as electric potential energy in the capacitor, and half is lost to ohmic heating. This result is true no matter what the values of $\mathcal{E}$, $C$ and $R$ may be.

## The two-capacitor problem

Figure 1 shows the arrangement of the capacitors. The capacitor $C_1$ has an original charge $Q$ and an initial potential $V_0$. A second capacitor $C_2$, originally uncharged, is connected through a resistor $R$ to $C_1$. At $t = 0$ the switch S is closed. Charge flows from $C_1$ to $C_2$ until the steady state is reached and the final potentials are equal.

Let the capacitance $C_2$ be related to $C_1$ such that

$$C_2 = C_1/n \qquad (n \geqslant 0). \tag{1}$$

This allows for the limiting cases in which $C_2$ is zero ($n = \infty$) and $C_2$ is infinite ($n = 0$). Since the total charge $Q$ is conserved and the final potentials, $V_{1F}$ and $V_{2F}$, are equal, we readily find the final steady-state charges, $Q_{1F}$ and $Q_{2F}$, to be

$$Q_{1F} = \frac{Qn}{n+1} \qquad Q_{2F} = \frac{Q}{n+1}. \tag{2}$$

Note that $Q_{1F} + Q_{2F} = Q$ and that $Q_{2F}$ is the amount of charge that has flowed from the first capacitor to the second.

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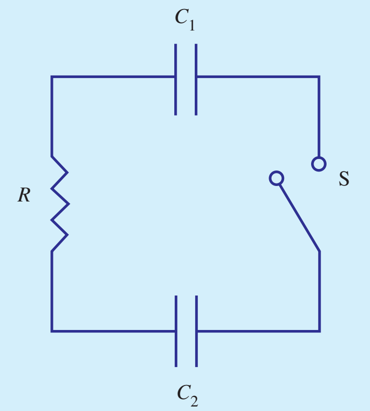


**Figure 1.** Two capacitors, $C_1$ and $C_2$, are connected through a switch S and a resistance $R$. When S is open, $C_1$ has a charge $Q$ and $C_2$ is uncharged. At $t = 0$ the switch is closed, and charge flows until the potentials across the two capacitors are the same.

At any instant let $q$ be the amount of charge that has left $C_1$ and arrived at $C_2$. Since the total charge is conserved, the instantaneous charges and voltages on the two capacitors are

$$Q_1 = Q - q \qquad Q_2 = q \tag{3}$$

$$V_1 = \frac{Q-q}{C_1} \qquad V_2 = \frac{q}{C_2}. \tag{4}$$

Using the Kirchhoff loop rule, we write

$$V_1 - IR - V_2 = 0 \tag{5}$$

where $I$ is the current, $I = \mathrm{d}q/\mathrm{d}t$. Substituting the values for voltage from equation (4), we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{RC_1}. \tag{6}$$

The equivalent capacitance of the circuit, $C_\mathrm{E}$, is given by

$$\frac{1}{C_\mathrm{E}} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{7}$$

Therefore our first-order differential equation for the charge is

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC_\mathrm{E}} = \frac{Q}{RC_1}. \tag{8}$$

The equation is readily solved, yielding

$$q(t) = K\exp(-t/RC_\mathrm{E}) + QC_\mathrm{E}/C_1. \tag{9}$$

The constant $K$ is found from the initial conditions for $q$ ($q = 0$ at $t = 0$) and equals $-QC_\mathrm{E}/C_1$. We may now write equation (9) as

$$q(t) = Q(C_\mathrm{E}/C_1)[1 - \exp(-t/RC_\mathrm{E})]. \tag{10}$$

Differentiating equation (10) gives the current $I$ as

$$I(t) = (Q/RC_1)\exp(-t/RC_\mathrm{E}). \tag{11}$$

The original energy in the system, $U_0$, is

$$U_0 = Q^2/2C_1. \tag{12}$$

The power dissipated in the resistor is $I^2R$. The energy dissipated in the resistor is the integral of the power taken from $t = 0$ to $t = \infty$. This gives the value of the dissipated energy $U_\mathrm{D}$ as

$$U_\mathrm{D} = Q^2C_\mathrm{E}/2C_1^2. \tag{13}$$

From our definition of $n$ we can write $C_\mathrm{E}$ as $C_1/(n+1)$. Therefore our final expression for the dissipated energy is

$$U_\mathrm{D} = \frac{Q_2}{2C_1(n+1)} = \frac{U_0}{n+1}. \tag{14}$$

The fraction of energy lost is

$$\frac{U_\mathrm{D}}{U_0} = \frac{1}{n+1}. \tag{15}$$

Comparing this with equation (2) we note that the fraction of charge that left the first capacitor and arrived at the second is

$$\frac{Q_{2\mathrm{F}}}{Q} = \frac{1}{n+1}. \tag{16}$$

This fraction depends solely on the ratio of the two capacitances. It is independent of $Q$ and $R$. Such a result was hardly anticipated.

## Charging a capacitor

Figure 2 shows an uncharged capacitor $C$ connected to a battery of emf $\mathcal{E}$ through a resistor $R$. At $t = 0$ the switch is closed. Charge begins to flow, and, as we know from capacitor theory, the final charge $Q$ on the capacitor when steady state is reached is

$$Q = C\mathcal{E}. \tag{17}$$

The charge $q$ on the capacitor varies with time and is

$$q(t) = Q[1 - \exp(-t/RC)]. \tag{18}$$

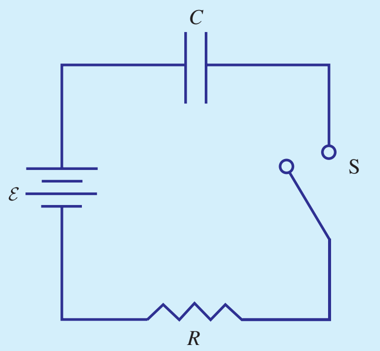


**Figure 2.** A capacitor $C$ is connected to a battery with emf $\mathcal{E}$ through a switch S and a resistance $R$. The capacitor is originally uncharged. At $t = 0$ the switch is closed, and charge flows until the potential across the capacitor equals $\mathcal{E}$.

The initial current $I_0$ is given as

$$I_0 = \mathcal{E}/R. \tag{19}$$

The instantaneous current is a function of time,

$$I(t) = (\mathcal{E}/R)\exp(-t/RC). \tag{20}$$

The power delivered by the battery is

$$P_{\mathrm{B}} = \mathcal{E}I. \tag{21}$$

Integrating equation (21) from $t = 0$ to $t = \infty$ gives the total energy $U_{\mathrm{B}}$ supplied by the battery,

$$\begin{aligned} U_{\mathrm{B}} &= \int_0^\infty \mathcal{E}I\,\mathrm{d}t \\ &= \int_0^\infty (\mathcal{E}^2/R)\exp(-t/RC)\,\mathrm{d}t \\ &= C\mathcal{E}^2. \end{aligned} \tag{22}$$

The ohmic power $P_{\mathrm{D}}$ dissipated in the resistor is

$$P_{\mathrm{D}} = I^2R. \tag{23}$$

Its integral taken from from $t = 0$ to $t = \infty$ gives the dissipated energy $U_{\mathrm{D}}$:

$$\begin{aligned} U_{\mathrm{D}} &= \int_0^\infty I^2R\,\mathrm{d}t \\ &= \int_0^\infty (\mathcal{E}^2/R^2)\exp(-2t/RC)\,\mathrm{d}t \\ &= C\mathcal{E}^2/2. \end{aligned} \tag{24}$$

Finally we have the final electrical energy $U_{\mathrm{C}}$ stored in the capacitor,

$$U_{\mathrm{C}} = C\mathcal{E}^2/2. \tag{25}$$

## Discussion

These results are somewhat surprising. Intuition (mine, at least) would predict a dependence on the value of the resistance for the fraction of energy dissipated. Yet in the two-capacitor problem it depends solely on the ratio of the two capacitances. In the charging of a single capacitor, half the energy supplied by the battery is stored in the capacitor and half is dissipated in the resistor—this is so no matter what our values of $C$, $\mathcal{E}$ and $R$ may be.

This is the consequence of the (theoretically) infinite time required to complete these processes. Therefore the integrals of the power are taken from zero to infinity. As a consequence, the value of the resistance disappears in the final answers. If the processes were stopped before they came to completion, then, of course, the results would indeed depend on the value of the resistance.

The fact that charging a single capacitor leads always to a 50% energy loss is because the power delivered by the battery is a function of the first power of the current (see (21)) whereas the power dissipated by the resistance is a function of the current squared (see (23)). Since the current is present to the first power in (21) and to the second power in (23), and has moreover an exponential dependence, the integral of (23) must, of necessity, have a value that is half of the integral of (21).

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