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The most energy efficient way to charge the capacitor in a RC circuit

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Abstract

The voltage waveform that minimize the energy loss in the resistance when charging the capacitor in a resistor–capacitor circuit is investigated using the calculus of variation. A linear voltage ramp gives the best efficiency, which means a constant current source should be used for charging. Comparison between constant current source and battery-powered system is made to illustrate the energy advantage of the former.

Capacitors play indispensable roles in applications of electrical energy storage, which is recognized as the underpinning technology for renewable energy applications. Charging capacitors gives rise to the electric current which, in turn, results in energy waste in resistance of the system. For electrical energy storage, the energy efficiency of capacitor charging is obviously a crucial performance factor.

Given a resistor–capacitor (RC) circuit shown in figure 1(a) where the power source is capable of generating any output voltage waveforms that are physically possible, what kind of waveform can minimize the energy loss in the resistance? The answer to this question is the method of adiabatic charging, namely, increasing the voltage applied across the capacitor at an infinitely slow rate so that the electric current in the circuit is kept at minimum [1].

The obvious problem of this method is that it will take forever to charge any capacitor to a finite voltage or to store a finite amount energy in it. Therefore, the practically more relevant and theoretically more interesting question is not about the waveform to minimize the energy loss; but rather, what kind of waveform can minimize the energy loss under the constraint of charging a fixed amount energy in the capacitor within a fixed amount of time?

Using the voltage across the capacitor V_c as the state variable, we can describe different charging processes using V_c as a function of time as illustrated in figure 1(b) for the three voltage waveforms: the linear ramp, the exponential growth and the inverted exponential decay which is the familiar voltage waveform when the capacitor is charged by a constant voltage source, i.e. a battery.

The energy ends up in the capacitor E_c, is a conservative quantity that only depends on the final voltage: $E_c = \frac{1}{2}CV_c^2$. The energy loss in the resistor E_R, on the other hand, depends on the manner in which the final voltage is reached, as should be expected for a dissipative process:

$$E_R(t) = \int_0^t I^2 R\mathrm{d}t = \int_0^t RC^2 \left(\frac{\mathrm{d}V_c(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t \qquad (1)$$

where, $I(t) = \frac{\mathrm{d}Q_c(t)}{\mathrm{d}t} = C\frac{\mathrm{d}V_c(t)}{\mathrm{d}t}$ is used to get the final integral in equation (1).

By referring to figure 1(b), the physics problem can be translated into a mathematic one, namely, how to find a curve of $V_c(t)$, which

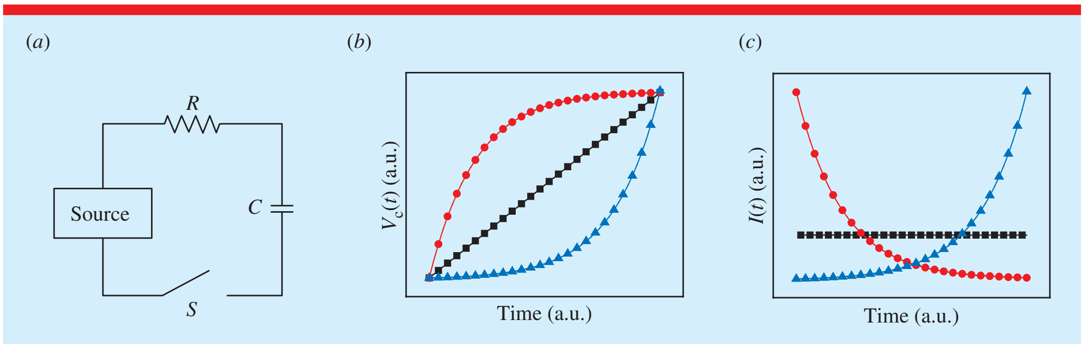

Figure 1. (a) A RC circuit charged by a power source which could be a battery, a constant current source or another charged capacitor. Here, R stands for the equivalent resistance of the system. (b) Diagram of voltage across the capacitor as a function of time, illustrating three different charging processes: the squared line is a linear ramp; the circled curve is the typical inverted exponential decay observed in battery-powered circuit; and the triangle line of an exponentially-growing voltage. (c) The corresponding electric current as a function of time for the three charging processes shown in (b). The linear ramp of voltage can be implemented by using a constant current source.

connects the fixed starting and ending points while minimizing the integral in equation (1)? Such a problem can be solved by the Euler–Lagrange equation[1] [2]:

$$\frac{\partial f}{\partial V_c} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial V_c'} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}V_c(t)}{\mathrm{d}t}\right] = 0 \qquad (2)$$

where, $f\,(t,\ V_c, V_c') = RC^2\left(\frac{\mathrm{d}V_c(t)}{\mathrm{d}t}\right)^2$ is the integrand in equation (1), and $V_c' = \frac{\mathrm{d}V_c(t)}{\mathrm{d}t}$.

Equation (2) is subjected to the boundary condition imposed by the constraint of charging the capacitor to an energy of ΔE_c in a time of T:

$$V_c(0) = 0 \text{ and } V_c\,(T) = \sqrt{\frac{2\Delta E_c}{C}}. \qquad (3)$$

Because we seek a functional form of $V_c(t)$, instead of the values of system parameters, the Lagrange multiplier method is not applicable here.

Equation (2) yields a linear function of time for $V_c(t)$ with two constants of integral, which can be determined by applying equation (3) to get:

$$V_c(t) = \sqrt{\frac{2\Delta E_c}{CT^2}}\,t. \qquad (4)$$

With $V_c(t)$ determined, the current in the circuit is readily obtained:

$$I(t) = \frac{\mathrm{d}Q_c(t)}{\mathrm{d}t} = C\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} = \frac{\sqrt{2C\Delta E_c}}{T}. \qquad (5)$$

[1] For example, see [2].

Namely, the most efficient way to charge is to use a constant current source which gives rise to a linear voltage ramp across the capacitor. Once the amount of energy and the time of charging required for an application are specified, the output of the constant current source can be set at the level given by equation (5) to minimize the energy loss.

The merit of a constant current source lies in the elimination of any current surge during the charging process. To see this point, we compare, in figure 2(c), the currents generated by three different $V_c(t)$. Both the exponential growth and the inverted exponential decay involve large current surge: the former at the end of the charging process and the latter at the beginning of the charging. There is nothing unique about the exponential waveforms: any superlinear or sublinear functionals will suffer similar current surge. By contrast, the linear waveform generates a constant current. Although the area covered underneath each current curve is the same (i.e. the capacitor ends up with the same amount charge in each case), the square on the current in equation (1) penalizes any large surge. This result should not appear surprising if one realize that the essence of Euler–Lagrange and many other variational methods is simply to find the stationary state.

To further illustrate the difference between the constant current and constant voltage source, let us take a more detailed look at the energy composition change during charging. E_R, E_C and the

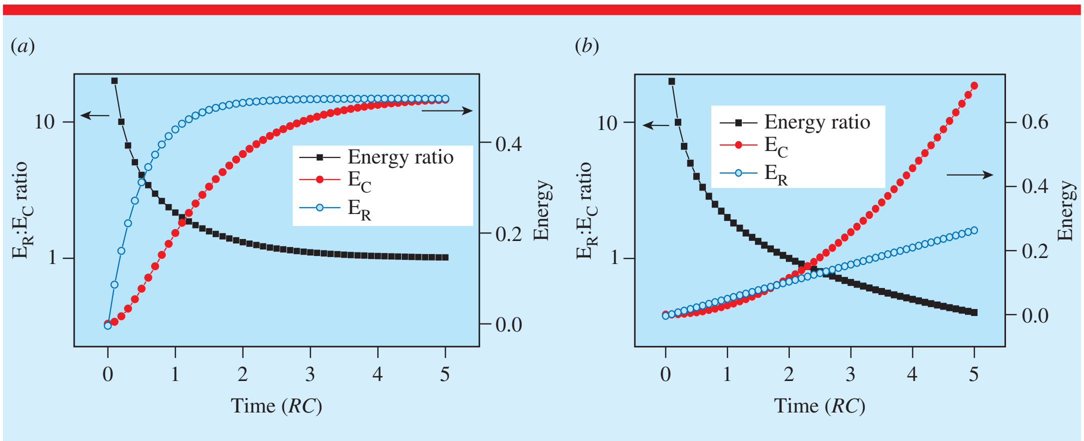

Figure 2. Energy stored in the capacitor (solid circle) and consumed in the resistor (empty circle) as functions of charging time for constant voltage source (a) and constant current source (b). $E_R : E_C$ ratios are also shown (solid square), which drop with increasing time and approach unity and zero for constant voltage and current source, respectively. The time of charging is normalized relative to the RC time constant, and the energies are normalized relative to the total energy delivered by the sources when the capacitor is either fully charged (a) or charged to the targeted energy level (b).

ratio of E_R to E_C are plotted against time of charging in figure 2(a) for a RC system with a constant voltage source. We see that the energy in the capacitor is always smaller than the energy consumed by the resistor; but the former approaches the latter asymptotically as the time increases. The $E_R : E_C$ ratio is very large at the initial stage; but drops to nearly unity beyond 5RC time.

In case of a constant current source, figure 2(b) shows that the energy consumption by the resistor increases linearly with time while the energy stored in the capacitor increases quadratically. The breakeven point, i.e. the time when E_R and E_C equalize, occurs at $t = 2$RC. Comparing this breakeven point against the infinite long time needed if a constant voltage source were used, the advantage of using a constant current source is obvious. In fact, the ratio of E_R to E_C will drop all the way to zero at infinite long time (figure 2(b)), although the highest voltage the capacitor can withstand and the highest output voltage of the source will place the ultimate limit on the energy efficiency in practice.

The conclusion that constant current sources offer the best energy efficiency has important implication in renewable energy application. Under constant illumination, photovoltaic devices behave like constant current sources [3], provided that the output voltage is far below the upper limits of their output voltages. In fact, experiments have shown that V_c increases linearly in the early stage of the charging. However, at later stage of charging, the V_c curve resembles the behavior of the circuit charged by battery, and starts to level off when approaches the output voltage limit of solar cells [4]. A Van de Graaff generator was used as a constant current source to charge capacitors, and V_c was observed to strictly follow the linear behavior [5].

The conclusion of this work is valid for narrowly-defined RC systems with only two degrees of freedom: resistance and capacitance. If additional channels of energy dissipation and conversion exist in the system, a constant current source may not be the best option. Numerous mechanisms, such as radiation loss and mechanical effects due to charge repulsion [6–8], have been proposed. Even in case of substantial radiation loss, a constant current flow can still minimize the acceleration of charges and the subsequent radiation loss. For typical RC systems used in undergraduate labs, Ohmic loss is likely to be the dominant mechanism.

In capacitor charging experiment using battery-powered RC circuits, the observed value of energy efficiency is consistently smaller than the theoretically predicted value, which is 50% for all battery-power RC systems[2] [9, 10]. Although

[2] For example, see [10].

it is convenient for teachers to point out other mechanisms of energy loss when explaining the discrepancy observed in the experiments, it is pedagogically necessary to remind students that the efficiency quoted in textbooks is the highest possible value achieved only when the capacitor is fully charged by the battery, i.e. at infinitely long time. In fact, as demonstrated in figure 2, for both constant current and constant voltage sources, the efficiencies always improve with increasing charging time relative to the RC time constant of the system.

In short, a constant current source offers the most energy efficient way to charge a RC system to certain energy level within a fixed time. Replacing the abrupt voltage switching by linear voltage ramp using a constant current avoids the current surge suffered by systems charged by constant voltage sources. The analysis presented in this work can be used for calculus-based introductory courses in electromagnetism. It can also be included as an example of application of Euler–Lagrange method when teaching the calculus of variations. The conclusion would be of importance for teaching topics in renewable energy applications.

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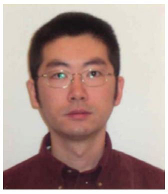

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