Problem 1: Let $\mathcal{B} = \mathcal{P}(\{P_1, ..., P_n\})$ be the power set of the partitions of Ω. We can construct $\mathcal{F}_P$ to be the union of the partitions in every set in $\mathcal{B}$. Or

$$\mathcal{F}_P = \{\bigcup_{i=1}^{j} a_i | a_i \in A, A \in \mathcal{B}, \text{ where } |A| = j\}$$

This is the smallest $\sigma-$algebra containing each of the subsets P_i.

(a) $\mathcal{F}_P$ is non-empty because $\mathcal{B}$ contains $\emptyset$, so $\mathcal{F}_P$ does as well.

(b) $\mathcal{F}_P$ is compliment closed. Let $B \in \mathcal{F}_P$. Then, $B = \bigcup_{i \in \mathcal{J}} P_i$ for some $\mathcal{J} \subseteq \{1, ..., n\}$ by how $\mathcal{F}_P$ is constructed.
We know that $B \cup B^C = \Omega$. So

$$(\bigcup_{i \in \mathcal{J}} P_i) \cup B^C = \Omega$$

Since the the P_i's are partitions, then

$$B^C = \bigcup_{j \notin \mathcal{J}} P_j$$

We know that $\bigcup_{j \notin \mathcal{J}} P_j$ is in $\mathcal{F}_P$ by construction. So, $B^C \in \mathcal{F}_P$ and $\mathcal{F}_P$ is compliment closed.

(c) By construction, $\mathcal{F}_P$ is countable $\cup$ closed

(d) Since $P_i \cap P_j = \emptyset$ for $i \neq j$, $\mathcal{F}_P$ is countable $\cap$ closed

Since there are 2^n elements in the power set $\mathcal{B}$ and there is a one to one correspondence between elements in $\mathcal{B}$ and $\mathcal{F}_P$, there are 2^n subsets of Ω in $\mathcal{F}_P$.

Also, this must be the smallest σ-algebra because the smallest sigma algebra only contains the union of all the partitions of Ω.

Problem 2: ($\Rightarrow$) Assume that f is measurable. We know for all $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}_P$.

Let $e \in E$. Since $\mathcal{E}$ is a power set, $\{e\} \in \mathcal{E}$. So $f^{-1}(\{e\}) \in \mathcal{F}_P$ and $f^{-1}(\{e\}) = \bigcup_{i \in \mathcal{I}} P_i$ for some $\mathcal{I} \subseteq \{1, ..., n\}$. Let $P_j \subseteq \bigcup_{i \in \mathcal{I}} P_i$ for the same index set $\mathcal{I}$. Then, for all $p \in P_j$, $f(p) = e$. Therefore the restriction of f to each P_j is a constant function.

($\Leftarrow$) Assume that for each $i = 1, ..., n$, $f|_{P_i}$ is a constant function. We want to show that f is measurable.

Let $A \in \mathcal{E}$. Let $A' \subseteq A$ where $A' = A \cap Im(f)$. Then, $f^{-1}(A') = \bigcup\{P_i | f(P_i) = e, \forall e \in A'\}$ or in other words union of the set of partitions whose image maps to an element of A. Note that $f^{-1}(A') = f^{-1}(A)$ because the elements not in $Im(f)$ map to the empty set, and the union of the empty set to another set S is S.

Since $\mathcal{F}_P$ contains all the unions of the partitions, $f^{-1}(A) \in \mathcal{F}_P$.

Problem 3: ($\Rightarrow$) Assume X_1 is measurable with respect to $\mathcal{F}_{X_0}$. So, we know that for all $A \in \mathcal{E}_1$, $X_1^{-1}(A) \in \mathcal{F}_{X_0}$. Since $\mathcal{F}_{X_0} := \{X_0^{-1}(B) | B \in \mathcal{E}_0\ \}$ that means for all A there exists a $B \in \mathcal{E}_0$ such that $X_1^{-1}(A) = X_0^{-1}(B)$.

Therefore, we can define h by the following. Let $x_0 \in X_0$. Then, $h(x_0) = X_1(\omega)$ for some $\omega \in \Omega$ where $X_0(\omega) = x_0$. We can assume that this ω exists by the problem statement.

($\Leftarrow$) Let $A \in \mathcal{E}_1$. We know that $X_1^{-1}(A) = (h \circ X_0)^{-1}(A)$.

The preimage of $h \circ X_0$ is defined by the following

$$(h \circ X_0)^{-1}(A) = \{\omega \in \Omega | h(X_0(\omega)) \in A\}$$

Another way to write this is

$$= \{\omega \in \Omega | X_0(\omega) \in h^{-1}(A)\}$$

where

$$h^{-1}(A) = \{x_0 \in E_0 | h(x_0) \in A\}$$

This shows that $h^{-1}(A) \subseteq E_0$. Since, $(E_0, \mathcal{E}_0)$ is a discrete space, every subset of E_0 must be contained in the space. Hence, $h^{-1}(A) \in \mathcal{E}_0$ and $X_0^{-1}(h^{-1}(A)) \in \mathcal{F}_{X_0}$

Note, $X_0^{-1} \circ h^{-1} = X_1^{-1}$. Hence, $X_1^{-1}(A) \in \mathcal{F}_{X_0}$. Therefore, X_1 is measurable with respect to $\mathcal{F}_{X_0}$

Problem 4: (a) $\mathbb{P}_X$ is non-negative.

Let $A \in \mathcal{E}$. Because X is a measureable function, we know that $X^{-1}(A) \in \mathcal{F}$. Since $\mathbb{P}$ is a probability measure over $(\Omega, \mathcal{F})$ that means $\mathbb{P}(X^{-1}(A)) \geq 0$. Thus, for all $A \in \mathcal{E}$, $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)) \geq 0$.

(b) $\mathbb{P}_X$ satisifes countable additivity.

First, we prove that the pre-image of disjoint sets are also disjoint.

Let $A_i, A_j \in \mathcal{E}$ and $A_i \cap A_j = \emptyset$. Then, by definition of preimage we know that

$$X^{-1}(A_i) = \{\omega \in \Omega | X(\omega) \in A_i\}$$
$$X^{-1}(A_j) = \{\omega \in \Omega | X(\omega) \in A_j\}$$

Let $\omega \in X^{-1}(A_i) \cap X^{-1}(A_j)$. Then, $X(\omega) \in A_i \cap A_j$. However, we assume that $A_i \cap A_j = \emptyset$. Thus, there cannot exist such ω, showing that $X^{-1}(A_i) \cap X^{-1}(A_j) \subseteq \emptyset$ and $X^{-1}(A_i) \cap X^{-1}(A_j) = \emptyset$

Another useful fact to know is that

$$X^{-1}(\bigcup_{i \in \mathcal{I}} A_i) = \bigcup_{i \in \mathcal{I}} X^{-1}(A_i)$$

This can be proven just by stating the definition of the two sides. Using these two facts, we get that

$$X^{-1}(\bigsqcup_{i\in\mathcal{I}} A_i) = \bigsqcup_{i\in\mathcal{I}} X^{-1}(A_i)$$

Now, we want to show that for $A_1, A_2, \dots \in \mathcal{E}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, that

$$\mathbb{P}_X(\bigsqcup_{i\in\mathcal{I}} A_i) = \sum_{i\in\mathcal{I}} \mathbb{P}_X(A_i)$$

Using the definition of $\mathbb{P}_X$,

$$\mathbb{P}_X(\bigsqcup_{i\in\mathcal{I}} A_i) = \mathbb{P}(X^{-1}(\bigsqcup_{i\in\mathcal{I}} A_i))$$

Using our lemma,

$$\mathbb{P}(X^{-1}(\bigsqcup_{i\in\mathcal{I}} A_i)) = \mathbb{P}(\bigsqcup_{i\in\mathcal{I}} X^{-1}(A_i))$$

Since, $X^{-1}(A_i) \in \mathcal{F}$ because X is measurable,

$$\mathbb{P}(\bigsqcup_{i\in\mathcal{I}} X^{-1}(A_i)) = \sum_{i\in\mathcal{I}} \mathbb{P}(X^{-1}(A_i))$$

Using the definition of $\mathbb{P}$, we find

$$\sum_{i\in\mathcal{I}} \mathbb{P}(X^{-1}(A_i)) = \sum_{i\in\mathcal{I}} \mathbb{P}_X(A_i)$$

Thus, $\mathbb{P}_X$ is countable additive.

(c) $\mathbb{P}_X(\emptyset) = 0$. We know that $X^{-1}(\emptyset_{\mathcal{E}}) = \emptyset_\Omega$. So, $\mathbb{P}(\emptyset_\Omega) = \mathbb{P}(X^{-1}(\emptyset_{\mathcal{E}})) = \mathbb{P}_X(\emptyset_{\mathcal{E}}) = 0$

(d) $\mathbb{P}_X(E) = 1$. We know that

$$X^{-1}(E) = \{\omega \in \Omega | X(\omega) \in E\}$$

By the definition of X, for all ω, $X(\omega) \in E$, so $X^{-1}(E) = \Omega$. Hence, $\mathbb{P}_X(E) = \mathbb{P}(X^{-1}(E)) = \mathbb{P}(\Omega) = 1$