Problem 1: Prove that $\sup S = \sup A + \sup B$.

First, we want to show that $\sup A + \sup B$ is an upper bound of S. Let $s \in S$. Then, we know $s = a + b$ for some $a \in A$ and $b \in B$. Since $a \leq \sup A$ and $b \leq \sup B$, we have $s = a + b \leq \sup A + \sup B$. Therefore $\sup A + \sup B$ is an upper bound of S.

Now, we want to show that $\forall L \in S$ such that $L < \sup A + \sup B$, there exists $c \in S$ such that $L < c$.

Let $L \in S$ such that $L < \sup A + \sup B$. So, $L = a + b$ for some $a \in A$ and $b \in B$.

Define $\epsilon = \sup A + \sup B - L$. Since, $L < \sup A + \sup B$, we have $\epsilon > 0$.

By the definition of supremum, there exists $a' \in A$ such that $\sup A - \frac{\epsilon}{2} < a'$. Similarly, there exists $b' \in B$ such that $\sup B - \frac{\epsilon}{2} < b'$.

Then, we have $c = a' + b' > \sup A + \sup B - \epsilon = L$. Since $c \in S$, we have found a c such that $L < c$. Hence $\sup A + \sup B$ must be the supremum of S.

Problem 2: Let $a \in \mathbb{R}$. By definition, a is an upper bound of A. We want to show that for $L < a, L \in \mathbb{Q}$ there exists $c \in A$ such that $L < c$.

Let $L \in \mathbb{Q}$ and $L < a$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can always find a rational number between two real numbers. Thus, there exists $r \in \mathbb{Q}$ such that $L < r < a$. So, L cannot be an upper bound of A. Therefore, a must be the supremum of A.

Problem 3: Prove $\sup P = \sup A \cdot \sup B$.

First, we want to show that $\sup A \cdot \sup B$ is an upper bound of P.

Let $c \in P$. By definiton, there exists some $a \in A$ and $b \in B$ such that $c = a \cdot b$. Since $a \leq \sup A$ and $b \leq \sup B$, we have $a \cdot b \leq \sup A \cdot \sup B$. Therefore, $\sup A \cdot \sup B$ is an upper bound of P.

Now, let $L \in P$ such that $L < \sup A \cdot \sup B$. We want show there exists a $p \in P$ such that $L < p$.

Choose a $a \in A$ and $b \in B$ such that

$$\begin{aligned} a &> \sup A - \epsilon \\ b &> \sup B - \epsilon \end{aligned}$$

for some $\epsilon > 0$.

Let $p = a \cdot b$. Then,

$$\begin{aligned} p &= (\sup A - \epsilon) \cdot (\sup B - \epsilon) \\ &= \sup A \cdot \sup B - \epsilon \sup A - \epsilon \sup B + \epsilon^2 \\ &= \sup A \cdot \sup B - \epsilon(\sup A + \sup B - \epsilon) \\ &> \sup A \cdot \sup B - \epsilon(\sup A + \sup B) \\ &> L \end{aligned}$$

Rearranging this inequality, we get

$$\sup A \cdot \sup B - L > \epsilon(\sup A + \sup B)$$
$$\frac{\sup A \cdot \sup B - L}{\sup A + \sup B} > \epsilon$$

Since A and B are sets of positive numbers, we know that $\sup A + \sup B > 0$. By assumption $\sup A \cdot \sup B - L > 0$, so $\epsilon > 0$.

Therefore, we have found a $p \in P$ such that $L < p$. Hence, $\sup A \cdot \sup B$ must be the supremum of P.

Problem 4:

- Let $\alpha, \beta \in F_+$.

  We want to show that $\alpha \cdot \beta \neq \emptyset$. Since $\alpha \neq \emptyset$, there exists $a \in \alpha$. Similarly, there exists $b \in \beta$. Then, $a \cdot b \in \alpha \cdot \beta$. Therefore, $\alpha \cdot \beta \neq \emptyset$.

  Next, we want to show that $\alpha \cdot \beta \neq \mathbb{Q}$. We know that there exists a $p > a$ for all $a \in \alpha$ and $q > b$ for all $b \in \beta$ such that $p, q \in \mathbb{Q}$ since they are both not equal to $\mathbb{Q}$. Hence, $p \cdot \sup q > a \cdot b$ for all $a \in \alpha$ and $b \in \beta$. Therefore, $p \cdot q \notin \alpha \cdot \beta$ and $\alpha \cdot \beta \neq \mathbb{Q}$.

  Let $p \in \alpha \cdot \beta$ and let $q < p$. We want to show that $q \in \alpha \cdot \beta$.

  We know that $p = a \cdot b$ for some $0 < a \in \alpha$ and $0 < b \in \beta$. Since $q < p$, we have $q < a \cdot b$. So, by transitivity, $q \in \alpha \cdot \beta$.

  Let $p \in \alpha \cdot \beta$. We want to show that there exists $q \in \alpha \cdot \beta$ such that $q > p$. We know that $p = a \cdot b$ for some $0 < a < \sup A \in \alpha$ and $0 < b < \sup B \in \beta$. Using the property that α and β are positive dedekind cuts, there must exist a $a < a' \in \alpha$ and $b < b' \in \beta$. Hence, $a' \cdot b' > a \cdot b = p$. Since $a' \cdot b' < \sup A \cdot \sup B$, $a' \cdot b' \in \alpha \cdot \beta$ and $a' \cdot b' > p$.

  Therefore, $a \cdot b \in F_+$.

- Let $\alpha, \beta \in F_+$. We want to show that $\alpha \cdot \beta \subseteq \beta \cdot \alpha$.

  Let $p \in \alpha \cdot \beta$. Then, $p = a \cdot b$ for some $0 < a \in \alpha$ and $0 < b \in \beta$. It follows from the commutatvity of $\mathbb{Q}$ that $p < ba$. Hence, $p \in \beta \cdot \alpha$. Therefore, $\alpha \cdot \beta \subseteq \beta \cdot \alpha$.

  The other direction is similar. Therefore, $\alpha \cdot \beta = \beta \cdot \alpha$.

- Let $\alpha, \beta, \gamma \in F_+$. We want to show that $(\alpha \cdot \beta) \cdot \gamma \subseteq \alpha \cdot (\beta \cdot \gamma)$. Let $p \in (\alpha \cdot \beta) \cdot \gamma$. Then, $p = n \cdot c$ for some $0 < n \in \alpha \cdot \beta$ and $0 < c \in \gamma$. We can further expand this to say that $n = ab$ for some $0 < a \in \alpha$ and $0 < b \in \beta$. Hence, $p < (a \cdot b) \cdot c$. Using the associativity of $\mathbb{Q}$, we have $p < a \cdot (b \cdot c)$. Note that $a > 0$, and $0 < (b \cdot c) \in \beta \cdot \gamma$. Therefore, $p \in \alpha \cdot (\beta \cdot \gamma)$. Hence, $(\alpha \cdot \beta) \cdot \gamma \subseteq \alpha \cdot (\beta \cdot \gamma)$.

  The other direction can be proven in the same fashion. Therefore, $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

- We want to show that there exists a $1 \in F_+$ such that $1 \cdot \alpha = \alpha$ for all $\alpha \in F_+$.

  Define $1 := \{q \in \mathbb{Q} \mid q < 1\}$.

($\subseteq$) Let $p \in \alpha \cdot 1$. We want to show that $p \in \alpha$. We know that $p = a \cdot b$ for some $0 < a \in \alpha$ and $0 < b \in 1$. Since $b < 1$, $b < \frac{a}{a}$ so $p = ab < a$. So, $p \in \alpha$ since α is a dedekind cut. Hence, $\alpha \cdot 1 \subseteq \alpha$.

($\supseteq$) Let $p \in \alpha$. We want to show that $p \in \alpha \cdot 1$.

Let $p \in \alpha$. We know that there exists $a \in \alpha$ such that $p < a$. So, $a - p > 0$. We also know that since $a > p$ that $\frac{p}{a} < 1$. So, $\frac{a}{a} > \frac{p}{a} \Rightarrow 1 > \frac{a-p}{a} > 0$ and $1 > 1 - \frac{a-p}{a} > 0$.

Let $b > 1 - \frac{a-p}{a}$. Then, $b \in 1$ and $b > 0$.

So, $ab > a(1 - \frac{a-p}{a}) = a - (a-p) = p$ and $ab \in \alpha \cdot \beta$. Hence, $p \in \alpha \cdot 1$. Therefore, $\alpha \subseteq \alpha \cdot 1$ and $\alpha = \alpha \cdot 1$.

- Let $\alpha \in F_+ \setminus \{0\}$. Define the multiplicative inverse by the following:

$$\alpha^{-1} := \{p \in \mathbb{Q} \mid ap < 1 \text{ s.t. } a \notin \alpha, p > 0\}$$

We will know show that α^{-1} is a dedekind cut.

  - Let $M = \sup \alpha$ and let $0 < M < a \in \mathbb{Q}$. We also know that $a + 1 \in \mathbb{Q}$ and that $a, a+1 \notin \alpha$. Note that $a + 1 > a$ so $(a+1)^{-1} < a^{-1}$. Hence, $(a+1)^{-1} a < 1$. So, $(a+1)^{-1} \in \alpha^{-1}$ and $a^{-1} \neq \emptyset$.
  - Let $M = \sup \alpha$. Suppose $M^{-1} \in \alpha^{-1}$. Then there would exist a $p \notin \alpha$ such that $pM^{-1} < 1$. So, $p < M$. However, M is the supremum of α so $p \in \alpha$. This is a contradiction. Hence, $M^{-1} \notin \alpha^{-1}$ and $\alpha^{-1} \neq \mathbb{Q}$.
  - Let $p \in \alpha^{-1}$. We want to show that there exists a $q \in \alpha^{-1}$ such that $q > p$.

    We know that there exists some $a \notin \alpha$ such that $ap < 1$ and $a > 0$. Let $0 < \epsilon < 1 - ap$. Define $q = p + \frac{\epsilon}{a}$. Evidentally, $q > p$. Now, we want to show that $aq < 1$.

$$aq = a(p + \frac{\epsilon}{a}) = ap + \epsilon < ap + 1 - ap < 1$$

  - Let $p \in \alpha^{-1}$. Let $q < p$. We want to show that $q \in \alpha^{-1}$.

    We know that $p \cdot a < 1$ for some $a \notin \alpha$. Since $q < p$, we have $q \cdot a < p \cdot a < 1$. Hence, $q \in \alpha^{-1}$.

Now we will show that $\alpha \cdot \alpha^{-1} = 1$.

($\subseteq$) Let $p \in \alpha \cdot \alpha^{-1}$. We want to show that $p \in 1$. So, $p = a \cdot b$ for some $0 < a \in \alpha$ and $0 < b \in \alpha^{-1}$.

Since $b \in \alpha^{-1}$, we know there exists some $c \notin \alpha$ such that $bc < 1$. Since $c \notin \alpha$, we have $c > \sup \alpha$. So, $c > a$ for all $a \in \alpha$. Hence $p = ab < bc < 1$. Therefore, $p \in 1$.

($\supseteq$) Let $p \in 1$. So, $0 < p < 1$. We want to show that $p \in \alpha \cdot \alpha^{-1}$.

Let $a \in \alpha$ such that $a > \sup \alpha \cdot p$. This must exist since $p < 1$ so $\sup \alpha \cdot p < \sup \alpha$.

Let $b = \frac{p}{a}$. We know show that $b \in \alpha^{-1}$. So, we know that $a > \sup \alpha \cdot p \Rightarrow \frac{a}{p} > \sup \alpha$. By density of $\mathbb{Q}$ in the $\mathbb{R}$, we can find a c such that $\frac{a}{p} > c > \sup \alpha$.

Obviously, $c \notin \alpha$, and

$$bc = \frac{p}{a} \cdot c < \frac{p}{a}\frac{a}{p} = 1$$

So, $b \in \alpha^{-1}$ and $p = a \cdot b \in \alpha \cdot \alpha^{-1}$. Therefore, $1 = \alpha \cdot \alpha^{-1}$.

- Let $\alpha, \beta, \gamma \in F_+$. We want to show that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

  ($\subseteq$) Let $p \in \alpha \cdot (\beta + \gamma)$. We want to show that $p \in \alpha \cdot \beta + \alpha \cdot \gamma$. So, $p = a \cdot m$ for some $0 < a \in \alpha$, $0 < m \in \beta + \gamma$. Further decompositing this, we have $m = b + c$ for some $0 < b \in \beta$ and $0 < c \in \gamma$. So, $p = a \cdot (b + c) = a \cdot b + a \cdot c$. Since $a \cdot b \in \alpha \cdot \beta$ and $a \cdot c \in \alpha \cdot \gamma$, we have $p \in \alpha \cdot \beta + \alpha \cdot \gamma$. Hence, $\alpha \cdot (\beta + \gamma) \subseteq \alpha \cdot \beta + \alpha \cdot \gamma$.

  ($\supseteq$) Let $q \in \alpha \cdot \beta + \alpha \cdot \gamma$. Then $q = p + r$ for some $p \in \alpha \cdot \beta$ and $r \in \alpha \cdot \gamma$. By definition of the product, there exists $0 < a_2, a_1 \in \alpha$, $0 < b \in \beta$, and $0 < c \in \gamma$ such that $p = a_1 \cdot b$ and $r = a_2 \cdot c$. If we take the max of a_1 and a_2, we have $p < max\{x_1, x_2\} \cdot b$ and $r < max\{x_1, x_2\} \cdot c$. Note that these products are still in $\alpha \cdot \beta$ and $\alpha \cdot \gamma$ respectively.

  So, $q = p + r < max\{x_1, x_2\} \cdot b + max\{x_1, x_2\} \cdot c = max\{x_1, x_2\} \cdot (b + c)$. Since $b + c \in \beta + \gamma$, we have $q \in \alpha \cdot (\beta + \gamma)$. Therefore, $\alpha \cdot \beta + \alpha \cdot \gamma \subseteq \alpha \cdot (\beta + \gamma)$.

  Hence, $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Problem 5: (a) Show that A_x is a non-empty set that is bounded above.

Let $x \in \mathbb{R}$. We know that there exists a $r \in \mathbb{Q}$ such that $r < x$. So, $\phi(r) \in A_x$ and $A_x \neq \emptyset$.

We want to show that A_x is bounded above. Choose a $x - 1 \leq r < x$ such that $r \in \mathbb{Q}$. Then, $r + 1 \geq x$. So, $\phi(r + 1) \geq x > \phi(p)$ for all $p \in A_x$. Hence, $r + 1$ is an upper bound of A_x. So, A_x is a non-empty set that is bounded above.

(b) Let $x, y \in \mathbb{R}$.

- We want to show that $\phi(x) + \phi(y)$ is an upper bound for A_{x+y}.
  Let $p \in A_{x+y}$. Then, $p = \phi(r)$ for some $r < x + y$ and $r \in \mathbb{Q}$. We can find $a < x$ and $b < y$ such that $r = a + b$ by the density of $\mathbb{Q}$.
  Note that $\phi(a) \in A_x$ and $\phi(b) \in A_y$. So, $\phi(a) < \sup A_x = \phi(x)$ and $\phi(b) < \sup A_y = \phi(y)$. So, $p = \phi(r) = \phi(a + b) = \phi(a) + \phi(b) < \phi(x) + \phi(y)$. Hence, $\phi(x) + \phi(y)$ is an upper bound for A_{x+y}.
  Let $L \in A_{x+y}$ such that $L < \phi(x) + \phi(y)$. We want to show that there exists $c \in A_{x+y}$ such that $L < c$.
  Let $\epsilon = \phi(x) + \phi(y) - L$. Since $L < \phi(x) + \phi(y)$, we have $\epsilon > 0$. We know we can find a $\phi(x) > \phi(p) > \phi(x) - \frac{\epsilon}{2}$ and $\phi(y) > \phi(q) > \phi(y) - \frac{\epsilon}{2}$ for some $p, q \in \mathbb{Q}$.
  So, $\phi(p) + \phi(q) > \phi(x) + \phi(y) - \epsilon = L$.
  Now, we must show that $c = \phi(p) + \phi(q) \in A_{x+y}$. Note that $c = \phi(p + q)$. Since $\phi(p) < \phi(x)$, then $p < x$ and similarly, $q < y$. So, $p + q < x + y$. Hence, $c \in A_{x+y}$.
  Thus $\phi(x) + \phi(y) = \sup A_{x+y} = \phi(x + y)$.

- We want to show that $\phi(x) \cdot \phi(y)$ is an upper bound for A_{xy}.
  Let $p \in A_{xy}$. Then $a = \phi(r)$ for some $r < xy$ and $r \in \mathbb{Q}$. Using the density of $\mathbb{Q}$, we can find $a < x$ and $b < y$ such that $r < ab$ and $a, b \in \mathbb{Q}$.
  Note that $\phi(a) < \phi(x)$ and $\phi(b) < \phi(y)$. This is because we can find a $a' \in \mathbb{Q}$ such that $a < a' < x$. Since $\phi(x) = \sup A_x$ and $\phi(a') \leq \phi(x)$, we have $\phi(a) < \phi(a') \geq \phi(x)$. Hence, $\phi(r) < \phi(ab) = \phi(a) \cdot \phi(b) < \phi(x) \cdot \phi(y)$. So, $\phi(x) \cdot \phi(y)$ is an upper bound for A_{xy}.
  Let $L \in A_{xy}$ such that $L < \phi(x) \cdot \phi(y)$. We want to show that there exists $c \in A_{xy}$ such that $L < c$.
  Let $\epsilon = \phi(x) \cdot \phi(y) - L$. Since $L < \phi(x) \cdot \phi(y)$, we have $\epsilon > 0$. We know we can find a $\phi(x) > \phi(p) > \phi(x) - \frac{\epsilon}{2\phi(y)}$ and $\phi(y) > \phi(q) > \phi(y) - \frac{\epsilon}{2\phi(x)}$ for some $p, q \in \mathbb{Q}$. Note that

  $$\phi(p) \cdot \phi(q) = \phi(x)\phi(y) - \epsilon + \frac{\epsilon^2}{2\phi(x)\phi(y)} > \phi(x) \cdot \phi(y) - \epsilon = L$$

  Also, $p < x$ and $q < y$ by similar reasoning as before. So, $p \cdot q < xy$. Hence, $c = \phi(p) \cdot \phi(q) \in A_{xy}$. Thus, L cannot be an upper bound. So, $\phi(x) \cdot \phi(y) = \sup A_{xy}$.
- Assume $x < y$. We want to show that $\phi(x) < \phi(y)$. By the density of $\mathbb{R}$, we can find a $p \in \mathbb{Q}$ such that $x < p < y$. Let $q \in \mathbb{Q}$ such that $q < x$. Then, perservation of order in $\mathbb{Q}$, $\phi(q) < \phi(p)$. Hence, $\phi(p)$ is an upper bound for A_x. So, $\phi(x) = \sup A_x \leq \phi(p) < \phi(y)$. By the same argument as before, we know that $\phi(p) < \phi(y)$ because we can find a p' such that $p < p' < y$. Hence, $\phi(x) \leq \phi(p) < \phi(y)$, so $\phi(x) < \phi(y)$.

(c) We want to show that $\phi : \mathbb{R} \to F$ is one to one and onto.

The fact that ϕ is one to one follows from the fact that $\phi(x) < \phi(y)$ if $x < y$. This is because if $x < y$, then $\phi(x) < \phi(y)$ and if $x > y$, then $\phi(x) > \phi(y)$. By trichotomy of the order relation, then if $x = y$, then $\phi(x) = \phi(y)$.

To show that ϕ is onto, we want to show that for all $y \in F$, there exists $x \in \mathbb{R}$ such that $\phi(x) = y$.

Let $y \in F$. Define $A := \{r \in \mathbb{R} \mid \phi(x) < y\}$. Since this is a subset of $\mathbb{R}$, using the least upper bound property, we can find a $x \in \mathbb{R}$ such that $\sup A = x$. Let $\phi(A) = \{\phi(a) \mid a \in A\}$. Realize that $y = \sup(\phi(A))$. With some careful noticing, we can see that $\phi(x) = \sup(\phi(A))$ using preservation of the order relation. Hence, ϕ is onto.