Problem 1: Using induction on i we can show that Z_i are mutually independent and each have exponential distribution with rate parameter of λ.

Base Case: Let's show that Z_1 and Z_2 are independent and are distributed exponentially.

Note that $Z_1 = X_1 - X_0 = X_1$. So,

$$\begin{aligned} F_{Z_1}(t) &= P(Z_1 \leq t) \\ &= P(X_1 \leq t) = P(N_t \geq 1) \\ &= P(N_t - N_0 \geq 1) \\ &= 1 - P(N_t - N_0 < 1) \\ &= 1 - \text{Pois}_{\lambda t}(0) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

This is the CDF of the exponential distribution.

We know that $Z_2 = X_2 - X_1$. Using similar logic (with some handwaviness),

$$\begin{aligned} F_{Z_2}(t) &= P(Z_2 \leq t) \\ &= P(Z_2 \leq t) \\ &= 1 - P(Z_2 > t) \quad \text{(here comes the handwaviness)} \\ &= 1 - P(X_2 > X_1 + t \mid X_1 = s) \\ &= 1 - P(N_{t+s} - N_s = 0) \\ &= 1 - P(N_t = 0) \quad \text{(by poisson)} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Thus we get that Z_2 is similarly distributed exponentially.

Let's show that Z_1 and Z_2 are independent, so that $P(Z_1 \leq t \cap Z_2 \leq t) = P(Z_1 \leq t)P(Z_2 \leq t)$.

We know that by the independent increments of N,

$$P(Z_1 \leq t_1 \cap Z_2 \leq t_2) = P(N_{t_1}) \cdot P(N_{t_2}) = P(Z_1 \leq t_1)P(Z_2 \leq t_2)$$

Induction: Assume that all $Z_1, ..., Z_k$ are mutually independent and distributed exponential with λ.

We want to show that Z_{k+1} is independent from $Z_1, ..., Z_k$ and exponentially distributed.

Using the same idea as before,

$$P(Z_{k+1} \leq t) = 1 - P(Z_{k+1} > t) = 1 - P(N_{X_{k+t}} - N_{X_k} = 0) = 1 - e^{-\lambda t}$$

Using the independent increments of the poisson process, Z_{k+1} must be independent of the previous time slices.

Problem 2: We want to show that $P(Z > z + w|Z > z) = P(Z > w)$.

Using the definition of conditional probability

$$\begin{aligned} P(Z > z + w|Z > z) &= \frac{P(Z > z + w \cap Z > z)}{P(Z > z)} \\ &= \frac{P(Z > z + w)}{P(Z > z)} \\ &= \frac{1 - (1 - e^{-\lambda(z+w)})}{1 - (1 - e^{-\lambda(z)})} \\ &= \frac{e^{-\lambda(z+w)}}{e^{-\lambda(z)}} \\ &= e^{-\lambda w} \\ &= P(Z > z) \end{aligned}$$

Problem 3: Let X be a continuous random variable that satisfies the memoryless property.

Then we get the following expression: for all $z, w \in \mathbb{R}_{\geq 0}$

$$\begin{aligned} P(X > w) &= \frac{P(X > z + w)}{P(X > z)} \\ P(X > z + w) &= P(X > z)P(x > w) \end{aligned}$$

Notice that $0 \leq P \leq 1$. Define $G(y) = 1 - f_X(y) = P(X > y)$. G must also be continuous and bounded by 0 and 1.

So, $G(z + w) = G(z)G(w)$. Taking the log of this, we get $\ln(G(z + w)) = \ln(G(z)) + \ln(G(w))$. Notice that $\ln \circ G$ is a continuous Cauchy function. By a theorem discussed in discussion, $\ln G(y)$ must be linear. Moreover, the linear coefficient must be negative because G is bounded by 0 and 1.

Hence we get that $\ln \circ G(y) = -\lambda y$ for some $\lambda > 0$.

Thus we get that $G(y) = e^{-\lambda y}$. So $f_X(y) = 1 - e^{-\lambda y}$. This is precisely the cdf of a exponential distribution. Hence, X must be exponentially distributed.

Problem 4: We want to show that H_t, T_t are Poisson Processes with rate $p\lambda$ and $(1 - p)\lambda$ respectively.

Obviously, $H_0 = 0$ and $T_0 = 0$, since $N_0 = 0$ (the underlying poisson process) and we haven't had an event where we could flip a coin.

$H_t - H_s$ must be non-decreasing for some $t > s$ because we cannot remove a decrement of heads/tails. It is always the sum of the bernoulli variable whose output is either 0 or 1.

Now, we want to show that H_t has independent increments. Let $0 \leq s_0 < s_1 < t$.

H_t is constructed by flipping a coin at every arrival of N_t. Let X be a random variable that is 1 when a heads is flipped and 0 when a tails is flipped, at the rates $p, (1-p)$. If we split the intervals $[s_0, s_1), [s_1, t]$ into smaller a set of intervals, let's say like a partition $\{s_0, s_0+1, s_0+2, ..., s_1\}$ and $\{s_1, s_1+1, s_1+2, ..., t\}$

We can define X_i for some time i to be the same as X but conditioned on the whether or not an arrival happened during the time period. In other words if $N_i - N_{i-1} = 1$ and $X = 1$ then $X_i = 1$. Since, X is independent of previous flips and the poisson process, each X_i are mutually independent.

Notice that H_t are constructed by the sum of these mutually independent events. Hence, by the same reasoning as homework 2, $H_t - H_{s_1}$ and $H_{s_1} - H_{s_0}$ are mutually independent, and there are independent increments.

Using the ideas of the previous proof, since the coin flip X and the poisson process are independent and we are summing the X_i it should follow that the rate of H_t would be $p\lambda$. Similarly can be said about T_i.