Problem 1: Let $|A| = n$ and $|B| = m$. We want to show there existss m^n many functions from A to B.

For every element in A, we can map it to any of the m elements of B. Since there are n elements in A, there are m^n many mappings/functions from A to B.

Problem 2: (a) We want to show that $\sim$ is an equivalence relation on $\mathbb{R}$.

Reflexive: For all $x \in \mathbb{R}$, $x \sim x$ since $x - x = 0$ is an integer.

Symmetric: Let $x, y \in \mathbb{R}$ and $x \sim y$. Then, $x - y = a$ for some $a \in \mathbb{Z}$. We also know that $-a \in \mathbb{Z}$. So, $y - x = -a$ is an integer. Thus, $y \sim x$.

Transitive: Let $x, y, z \in \mathbb{R}$ and $x \sim y$ and $y \sim z$. Then, $x - y = a$ and $y - z = b$ for some $a, b \in \mathbb{Z}$. Note that $a + b \in \mathbb{Z}$. So, $x - z = (x - y) + (y - z) = a + b$ is an integer. Thus, $x \sim z$.

(b) Let's show that Φ is injective. Choose a $x_1, x_2 \in [0, 1)$ such that $\Phi(x_1) = \Phi(x_2)$. Then $x_1 - x_2 = a$ for some $a \in \mathbb{Z}$. We also know that $|x_1 - x_2| < 1$ because $x_1, x_2 \in [0, 1)$.

Since a is an integer, $a = 0$. So, $x_1 = x_2$. Thus, Φ is injective.

Now let's show that Φ is surjective. Let $y \in \mathbb{R}$. We want to show that there exists an $x \in [0, 1)$ such that $\Phi(x) = [y]$. We can take the floor of y, to get $\lfloor y \rfloor \in \mathbb{Z}$. Subtracting $\lfloor y \rfloor$ from y gives us a number in $[0, 1)$. So, $\Phi(y - \lfloor y \rfloor) = y$. Thus, Φ is surjective.

Problem 3: We want to show that f is bijective where

$$f(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + n$$

Injective: Injective proof from math stack exchange "Showing a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is injective"

Let $n, m, u, v \in \mathbb{N}$ such that $f(n, m) = f(u, v)$. Consider two cases:

Case 1: $n + m = u + v$. Then

$$f(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + n = \frac{(u + v - 2)(u + v - 1)}{2} + u = f(u, v)$$

The two products are the same by assumption, so $n = u$. This implies that $m = v$ since $n + m = u + v$. Hence, f is injective.

Case 2: $n + m \neq u + v$. Without loss of generality, assume that $n + m < u + v$. We note that

$$\frac{(n + m - 2)(n + m - 1)}{2} = \sum_{k=1}^{n+m-2} k$$

Since this sum is just a function of $n + m$,

$$\frac{(n + m - 2)(n + m - 1)}{2} - \frac{(u + v - 2)(u + v - 1)}{2} = \sum_{j=1}^{n+m-2} j - \sum_{k=1}^{u+v-2} k$$

Expanding the k sum,

$$\sum_{j=1}^{n+m-2} j - \sum_{k=1}^{u+v-2} k = \sum_{j=1}^{n+m-2} j - (\sum_{j=1}^{n+m-2} j + \sum_{j=n+m-1}^{u+v-2} j) = -\sum_{j=n+m-1}^{u+v-2} j$$

Since $n+m-2 < u+v-2$, we can assume that the difference is at least $n+m-1 > 0$. So,

$$f(n,m) = \frac{(n+m-2)(n+m-1)}{2} + n \leq \frac{(u+v-2)(u+v-1)}{2} - (n+m-1) + n$$
$$< f(u,v)$$

Thus, by trichotomy $f(n,m) \neq f(u,v)$. This contradicts the assumtion

Surjective: Notice, with some algebraic manipulation, $f(n,m)+1 = f(1,n+1)$ if $m=1$ and $f(n,m)+1 = f(n+1,m-1)$ otherwise. Let $k = f(n,m)$. We can induce on k to show that for all $k \in \mathbb{N}$, $k = f(n,m)$ for some $n,m \in \mathbb{N}$.

Base Case: Let $k = 1$. Then, $f(1,1) = 1$.

Induction Hypothesis: Assume for some $k \in \mathbb{N}$ that $f(n,m) = k$ for some $n,m \in \mathbb{N}$.

Inductive Step: Using our observation, we notice that if $k+1 = f(n,m)+1$ so $k+1 = f(1,n+1)$ if $m = 1$ and $k+1 = f(n+1,m-1)$ otherwise. Thus, we have found a n,m such that $f(n,m) = k+1$, so f is surjective.

Problem 4: Let A be a non-empty finite set and let B be a proper subset of A. We want to show that there cannot exist a bijection between A and B so they cannot have the same cardinality.

Assume there exists a bijection $f : A \to B$. We know that there exists a $a_0 \in A \setminus B$. Construct a sequence $\{a_n\}_{n\geq 0}$ where $a_{n+1} = f(a_n)$. Note that each a_n except a_0 is in B.

Assume that there exist some n and k such that $a_{n+k} = a_n$. This means that $f^n(a_k) = f^{n+k}(a_0) = f^n(a_0)$. Note that the composition of bijective functions are still bijective. So, by injectivity $a_k = a_0$. However, $a_0 \notin B$, but $a_k \in B$. This is a contradiction, so there cannot be repeated elements in the sequence. However, this implies that B is infinite. This contradicts that B is a proper subset of A and is finite. Hence, there cannot exist a bijection between A and B. Therefore, A and B have different cardinalities.

Problem 5: Assume that A is a infinite set. We want to show that A contains a countable subset. Using an inductive argument we can select a countable subset of A.

We know that A is non-empty. So, we can select an element $a_1 \in A$. Since A is infinite, $A \setminus a_1 \neq \emptyset$.

Similarly, we can select $a_2 \in A \setminus \{a_1\}$. $A \setminus \{a_1, a_2\} \neq \emptyset$ since A is infinite.

We can continue this process, for n, and notice that $A \setminus \{a_1, \dots, a_n\} \neq \emptyset$. Hence, we can select a $a_{n+1} \in A \setminus \{a_1, \dots, a_n\}$.