Problem 1: (7.1.1) Let G be such graph. We know that each face is bounded by exactly 3 edges. Consider a subset of the dual graph of G, where an edge exists between two vertices in the dual graph G' if the edge between their corresponding faces in G have different labels.

With this construction, a vertex in G' can only have degree 0 (the face is bounded by 3 vertices of different labels), 2 (there are two vertices with one label and one vertex with another), or 3 (all three vertices have different labels).

By, hand shake lemma, there can only be an even number of vertices in a graph with odd degree. So, there must be an even number of vertices of degree 3. Thus, there are an even number of faces with three labels.

Problem 2: (7.2.1)

(a) We can partition $\mathcal{N}$ into two sets $\mathcal{N}_1$ and $\mathcal{N}_2$. We define

$$\mathcal{N}_1 = \{N \in \mathcal{N} \text{ s.t. } M \subseteq N \text{ for some } M \in \mathcal{N}\}$$
$$\mathcal{N}_2 = \mathcal{N} \setminus \mathcal{N}_1$$

Note that $|\mathcal{N}| = |\mathcal{N}_1| + |\mathcal{N}_2|$.

We claim that $\mathcal{N}_1$ and $\mathcal{N}_2$ are both independent systems of X.

$\mathcal{N}_2$ is an independent system by construction.

Let $A, B \in \mathcal{N}_1$ where $A \neq B$. We want to show that $A \not\subseteq B$.

Assume that $A \subseteq B$. We know there exists some $M \in \mathcal{N}$ such that $M \subseteq A$. Note that $M \neq B$ since $A \neq B$. So, M, A, B are different subsets but $M \subseteq A \subseteq B$. This contradicts the fact that $\mathcal{N}$ is semi-independent.

Hence, $A \not\subseteq B$.

Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are both independent systems, we can use Sperner's lemma to bound both of them. So, $|\mathcal{N}_1| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $|\mathcal{N}_2| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Thus, $\mathcal{N} \leq 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

(b) If n is odd, $\binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$?

Problem 3: (7.2.6)

(a) Let $\mathcal{M} = \{I \subseteq \{1, \ldots, n\} \mid -1 < \sum_{i \in I} a_i - \sum_{j \notin I} a_j < 1\}$. We want to show that $\mathcal{M}$ is an independent system. If we do, it follows the conclusion follows from Sperner's Lemma.

We can assume that all a_i are non negative (since we can just flip the corresponding ϵ_i to correct the sign).

Let $A, B \in \mathcal{M}$ where $A \neq B$ and $A \subseteq B$.

Let $C = B \setminus A$. So,

$$S_A = \sum_{i \in A} a_i - \sum_{j \notin A} a_j$$
$$S_B = \sum_{i \in A} a_i + \sum_{k \in C} a_k - \sum_{l \notin B} a_l$$

Notice that

$$\sum_{j \notin A} a_j = \sum_{k \in C} a_k + \sum_{l \notin B} a_l$$

So,

$$S_A - S_B = -2 \sum_{k \in C} a_k$$

We know that $C \neq \emptyset$ and each $|a_k| > 1$. So, $\sum_{k \in C} a_k \geq 2|C| \geq 2$. Hence, both $-1 < S_A < 1$ and $-1 < S_B < 1$ cannot both happen. Thus, $A \not\subset B$ and $\mathcal{M}$ is an independent system.

(b) Let $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$ (a sequence of only 1s). Choosing two out of 4 to be positive makes $\binom{4}{2} = \binom{4}{\lfloor \frac{4}{2} \rfloor}$

Problem 4: (7.2.7) By definition, n must only have prime factors. So, $n = p_1 p_2 p_3 ... p_k$ We know that $k \leq \log_2(n)$. So, our system $\mathcal{M}$ would be subsets of $\{p_1, p_2, \ldots, p_k\}$. Using Sperner's lemma, $|\mathcal{M}| \leq \binom{k}{\lfloor \frac{k}{2} \rfloor} \leq \binom{\lceil \log_2(n) \rceil}{\lfloor \frac{\lceil \log_2(n) \rceil}{2} \rfloor}$

Problem 5: (7.3.5bc)

(b) **Case 1:** $x, y \leq 1$. Then $f(\lambda x + (1-\lambda)y) = 0 \leq \lambda f(x) + (1-\lambda) f(y) = 0$

Case 2: $x \leq 1, y > 1$. if $\lambda x + (1-\lambda)y \leq 1$ then $f(\lambda x + (1-\lambda)y) = 0 \leq (1-\lambda)\frac{y(y-1)}{2}$

Otherwise, we need to check that $\frac{(\lambda x + (1-\lambda)y)(\lambda x + (1-\lambda)y - 1)}{2} \leq (1-\lambda)\frac{y(y-1)}{2}$.

With some algebraic manipulation, we can conclude this.

Case 3: $x > 1$, $y > 1$. The second derivative of $\frac{x(x-1)}{2} = 1 > 0$ for all $x > 1$.

Thus, $f(x)$ is convex.

(c) Note that $m = |E(G)| = \frac{1}{2}\sum_{v \in V} deg(v)$. So $2m = \sum_{v \in V} deg(v)$.

From part a and b,

$$f(\frac{1}{n}\sum_{v \in V} deg(v)) \leq \frac{1}{n}\sum_{v \in V} f(deg(v))$$

So,

$$nf(\frac{2m}{n}) \leq \sum_{v \in V} f(deg(v))$$

From before, we know that for $deg(v) > 1$, $f(deg(v)) = \binom{deg(v)}{2}$. Since G cannot contain $K_{2,2}$,

$$\sum_{v \in V} \binom{deg(v)}{2} \leq \binom{n}{2}$$

Hence, $nf(\frac{2m}{n}) \leq \binom{n}{2}$.

Expanding this out, we get

$$n \cdot \frac{\frac{2m}{n}(\frac{2m}{n} - 1)}{2} \leq \frac{n(n-1)}{2}$$

Simplifying, we eventually get $4m^2 - 2mn \leq n^3 - n^2$.

Solving for m, we get that $m \leq \frac{1}{2}(n^{\frac{3}{2}} + n)$

Problem 6: (7.3.6) Let $f(x) = \frac{x(x-1)(x-2)}{6}$. On $x > 2$ if we take the second derivative, we can see that is clearly postive (1) for all $x > 2$. Hence, f is convex.

Using a similar argument,

$$nf(\frac{2m}{n}) \leq \sum_{v \in V} \binom{deg(v)}{3} \leq \binom{n}{3}$$

Expanding this out we get

$$2m^3 - 6mn^2 + 6mn^2 \leq n^3(n-1)(n-2)$$

Taking the cubed root, we get $m \leq O(n^{\frac{5}{3}})$ as desired.