Exercise 1: Let $n \geq 1$. We want to show that if $A_1, \ldots, A_n$ are countable then $A_1 \times \cdots \times A_n$ is countable. We will prove this by induction on n.

Base Case: $n = 1$. By the given, A_1 is countable.

$n = 2$. We want to show that $A_1 \times A_2$ is countable.

Since A_1 and A_2 are countable there exists bijective functions $f_1 : A_1 \to \mathbb{N}$ and $f_2 : A_2 \to \mathbb{N}$. We know that $g(n, m) = \frac{(n+m-1)(n+m-2)}{2} + n$ is a bijective function from homework 7.

We also know that the composition of bijective functions are bijective.

Hence, we can a bijective $h_1 : A_1 \times A_2 \to \mathbb{N}$ where $h_2(a_1, a_2) = g(f_1(a_1), f_2(a_2))$.

Induction Step: Assume that $A_1 \times \cdots \times A_n$ is countable. We want to show that $A_1 \times \cdots \times A_n \times A_{n+1}$ is countable.

Since $A_1 \times \cdots \times A_n$ is countable, there exists a bijective function $h_n : A_1 \times \cdots \times A_n \to \mathbb{N}$.

Since A_{n+1} is countable, there exists a bijective function $f_{n+1} : A_{n+1} \to \mathbb{N}$.

So, we can define $h_{n+1} = g(h_n, f_{n+1})$ where g is the bijective function from the base case. Since the composition of bijective functions are bijective, h_{n+1} is bijective.

Thus, there exists a bijective function $h_{n+1} : A_1 \times \cdots \times A_n \times A_{n+1} \to \mathbb{N}$. So, $A_1 \times \cdots \times A_n \times A_{n+1}$ is countable.

Exercise 2: Assume $A \sim B$. We want to show $\mathcal{P}(A) \sim \mathcal{P}(B)$.

We know there exists $f : A \to B$ that is bijective. Define $g : \mathcal{P}(A) \to \mathcal{P}(B)$ where $g(S) = \{f(s) \mid s \in S\}$ for all $S \in \mathcal{P}(A)$.

Take $X, Y \in \mathcal{P}(A)$. We want to show that $g(X) = g(Y) \implies X = Y$.

Assume otherwise, $g(X) = g(Y)$ and $X \neq Y$. Then there exists $x \in X$ such that $x \notin Y$ or $y \in Y$ such that $y \notin X$. Without loss of generality, assume that there exists $x \in X$ such that $x \notin Y$. Then $f(x) \in g(X)$ by definition of g. We also know that $f(x) \in g(Y)$ since $g(X) = g(Y)$. So, $f(x) = x'$ for some $x' \in Y$. However, f is bijective so $f^{-1}(x') = x$. Thus, $x \in Y$, which is a contradiction.

Thus, $X = Y$. So, g is injective.

Now we want to show that g is surjective. Take $Y \in \mathcal{P}(B)$. We know that for all $y \in Y$, there exists a $x \in X$ such that $f(x) = y$ since f is bijective.

So, $X = \{x \in A \mid f(x) \in Y\}$. Then $g(X) = \{f(x) \mid x \in X\} = \{y \in B \mid y \in Y\} = Y$.

Thus, g is surjective and bijective.

Exercise 3: Let $S \subseteq \mathbb{N}$. We define $f : \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}}$ by $f(S) = f_S(x) = \{1 \text{ if } x \in S, 0 \text{ if } x \notin S\}$. We want to show that f is bijective.

Take $S, T \in \mathcal{P}(\mathbb{N})$. We want to show that $f(S) = f(T) \implies S = T$.

Assume otherwise, $f(S) = f(T)$ and $S \neq T$. Then there exists $x \in S$ such that $x \notin T$ or $y \in T$ such that $y \notin S$. Without loss of generality, assume that there exists $x \in S$ such that $x \notin T$. Then $f_S(x) = 1$ and $f_T(x) = 0$. This contradicts the fact that $f(S) = f(T)$. Hence, $S = T$, and f is injective.

Now we want to show that f is surjective. Take $g \in 2^{\mathbb{N}}$. We want to show that there exists $S \in \mathcal{P}(\mathbb{N})$ such that $f(S) = g$. Let $S = \{x \in \mathbb{N} \mid g(x) = 1\}$. Then, $f(S) = g$ by definition of f.

Hence, f is bijective.

Exercise 4: Notice that $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$. So the identity function from $2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is injective. Now we want to show that there exists a function $f : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ that is injective.

Let $\{a_n\} \in \mathbb{N}^{\mathbb{N}}$. We can define f as the encoding of $\{a_n\}$ by the following: For each $a_i \in \{a_n\}$, we map it to a string of 0's of length a_i, followed by a 1. We then concatenate all of these strings together.

This is an injective function.

Hence, since we can define two injective functions from $2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$, we have that $2^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}}$.

Exercise 5: We want to show that the set of all real roots of all polynomials in $\mathcal{P}$ is countable. Let $n \geq 1$. We know that a polynomial of degree n has at most n real roots.

Thus, for all $p \in \mathcal{P}$, there is a finite number of real roots of p.

Note that we can map $\mathcal{P}$ to $\mathbb{N}^n$ using the coefficients of the polynomial. This means that $\mathcal{P}$ is countable. Hence $\mathcal{A}$ can be defined as the countable union of finite sets. By some theorem proven in class, this is a countable set. Hence, $\mathcal{A}$ is countable.

Exercise 6: Fix a $n \geq 1$. We want to show that the set of all subsets of $\mathbb{N}$ with n distinct elements is countable. Let's call this set $\mathcal{M}$.

We can use exercise 1 to show that S is countable, since we know that $\mathbb{N} \times \cdots \times \mathbb{N}$ is countable.

We can define an injective function $f : \mathcal{M} \to \mathbb{N} \times \cdots \times \mathbb{N}$. Let $S \in \mathcal{M}$. Define f by sorting the elements of S in ascending order. Then, make a tuple of the elements of S. This is obviously an element of $\mathbb{N} \times \cdots \times \mathbb{N}$.

Let $S, T \in \mathcal{M}$ such that $f(S) = f(T)$. Then, S and T are the same $n-$tuple, so they contain all the same elements. Hence $S = T$.

We want to define a surjective function $g : \mathbb{N} \times \cdots \times \mathbb{N} \to \mathcal{M}$.

Exercise 7: We want to show that $\mathbb{R} \sim (0, 1)$. Note that strictly increasing function are injective. $\tanh(x)$ is strictly increasing on $\mathbb{R}$. If we define $f(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$ for all $x \in \mathbb{R}$, then f is injective and is restricted to $(0, 1)$.

Since $(0, 1) \subseteq \mathbb{R}$, the identity map suffices as an injective function.

Exercise 8: Denote the set of irrational numbers as $\mathcal{I} = \mathbb{R} \setminus \mathbb{Q}$. We want to show that $\mathbb{R} \sim \mathcal{I}$.

The identity map $f : \mathcal{I} \to \mathbb{R}$ is injective because $\mathcal{I} \subseteq \mathbb{R}$.

We can define an injective function $g : \mathbb{R} \to \mathcal{I}$ by the following: $g(x) = x + \pi$ for all $x \in \mathbb{R}$.

Let $x, y \in \mathbb{R}$ such that $g(x) = g(y)$. Then $x + \pi = y + \pi$, so $x = y$. Hence g is injective. Thus, $\mathbb{R} \sim \mathcal{I}$.