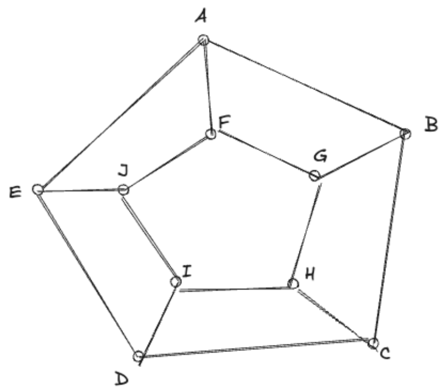
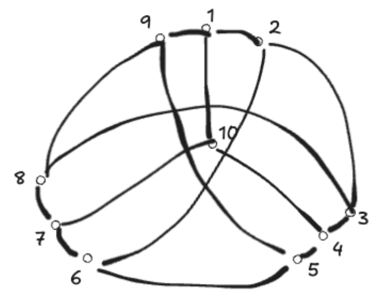
Problem 1: (4.1.1)

(a) Label the two graphs with the following:





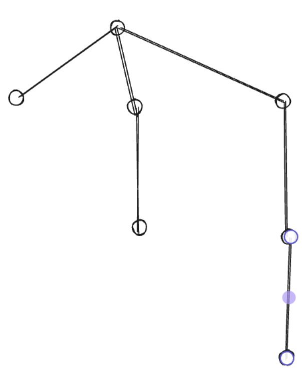
There exists a bijection between the graphs

$$\begin{pmatrix} A & B & C & D & E & F & G & H & I & J \\ 1 & 2 & 3 & 4 & 10 & 9 & 6 & 8 & 5 & 7 \end{pmatrix}$$

(b) At a glance, each vertex is connected to three other vertices, since $v = \{a, b\}$ is connected to the two element subsets of $\{1, 2, 3, 4, 5\} \setminus \{a, b\}$ and there are 10 vertices, both properties match the above two graphs. More explicitly, we can construct the following bijection

$$\begin{pmatrix} A & B & C & D & E & F & G & H & I & J \\ \{1,2\} & \{4,5\} & \{2,3\} & \{1,5\} & \{3,4\} & \{3,5\} & \{1,3\} & \{1,5\} & \{2,4\} & \{2,5\} \end{pmatrix}$$

Problem 2: (4.1.3)



(a)

(b) **Lemma 1:** Graphs with maximum degree 1 have a nontrivial automorphism. The graph would consist of connected components of two connected vertices, $v_1, v_2 \in V$. Mapping v_1 to v_2 and vice versa would be an isomorphism

Lemma 2: Graphs with maximum degree 2 have a non-trivial automorphism. These types of graphs can be split into the case where the graph has a cycle where vertices in the cycle are degree 2 and the case where the connected components are paths, with two vertices of degree 1.

(Cycle Case) If we start from any given vertex, v_0, it must be connected to two other vertices, v_1, v_2. We can map v_1 to v_2 and v_2 to v_1. These vertices must also be connected to another vertex v_3 and v_4 respectively (where v_3 isn't necessarily different from v_4). We can map v_3 to v_4 and v_4 to v_3. If we iteratively do this, we always get an isomorphism.

(Path Case) Let P be the path. Start from the two vertices, v_1, v_n with degree 1. We can map v_1 to v_n and vice versa. Take those two elements out of the path, and then we get two more vertices of degree 1. Iteratively, do this and we get another non-trivial automorphism.

Using Lemmas 1 and 2, we know that any graph with 3 or less vertices have nontrivial automorphisms.

Lemma 3: Disconnected graphs have non-trivial automorphisms. Intiutively, we can map two separate connected components to each other, and we still get an isomorphism.

An exhaustive search of all connected graphs with 4 and 5 vertices, whose max degree is at least 3 reveals that there doesn't exist a asymmetric graph.

Problem 3: (4.2.1) Let $G = (V, E)$ be a graph with k connected components ($k > 1$). Let $G' = (V, E' = \binom{V}{2}\ E)$ denote the complement of G.

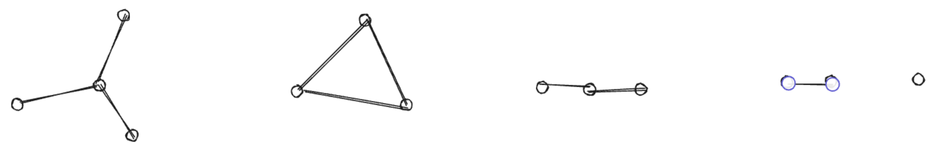
Let C_1 and C_2 be distinct connected components of G. Then, by definition that for all $v_1 \in C_1$ and $v_2 \in C_2$ there does not exist $\{v_1, v_2\} \in E$. Hence, for all $v_1 \in C_1$ and $v_2 \in C_2$, there exists $\{v_1, v_2\} \in E'$. This shows that every connected component in G is connected to each other in G'.

Now, we want to show that there still remains a path within every connected component in G'. Let $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$ be connected components of G. Let $w_1, w_2 \in V_1$ be distinct vertices in w_1 and w_2. We know that there is an edge between every vertex in C_1 and every vertex in C_2 in G'. Let $x \in V_2$. So, $\{w_1, x\} \in E'$ and $\{w_2, x\} \in E'$. Thus, we can form the path $(w_1\ x\ w_2)$. Therefore, every connected component in G remains connected in G'.

Therefore, since each connected component remains connected and there exists a path between connected components, then any two vertices in G' can be connected by a path. Therefore, G' is a connected graph.

Problem 4: (4.2.5) Let G be a graph containing no path of length 3 (this implies that G doesn't have a path of longer than 3).

Then, each connected component of G must look like the following.



Problem 5: (4.3.1) The first graph has 5 cycles of length 4. The second graph has 2 cycles of length 4. The third graph has 0 cycles of length 4. Hence, the graphs cannot be isomorphic.

Problem 6: (4.3.9) Let $G = (V, E)$ be a graph where all vertices have at least degree d. We want to show that G contains a path of length d.

Lemma: The complete graph K_{n+1} is a subgraph of any graph with degree at least n.

Idea: Each vertex in K_{n+1} has degree n. We can continuously remove edges and vertices from the graph until we obtain K_{n+1}.

Hence, if we show that K_{n+1} has a path of length n, then every graph that has degree at least n will have a path of length n.

Proof By Induction:

Base Case: Let $p = 1$. Evidentally, $K_{1+1} = K_2$ has a path of length 1, since it is a graph with two vertices, connected by a edge.

Induction Hypothesis: Let $p = n$ for some $n \in \mathbb{N}$. Assume that K_{n+1} has a path of length n. Let this path be $P = (v_0\, e_1\, \ldots e_n v_n)$

Induction Step: We want to show that K_{n+2} has a path of length $n+1$. We can construct K_{n+2} from K_{n+1} by adding a vertex v_{n+1}, and connecting the vertex to every $v \in K_{n+1}$. Since v_{n+1} is connected to v_n and v_{n+1} is not in K_{n+1}, we can add v_{n+1} to the path P, and it remains a path, $(v_0\, e_1\, \ldots e_n v_n\, e_{n+1} v_{n+1})$. Note that this path is now of length $n+1$.