Problem 1: ($\Rightarrow$) Assume that $\{a_n\}_{n\in\mathbb{N}}$ converges to a. We want to show that $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

Let $\epsilon > 0$. Since $\{a_n\}_{n\in\mathbb{N}}$ converges to a, there exists an $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $|a_n - a| < \epsilon$. We know that $||a_n| - |a|| \leq |a_n - a|$. Thus, $||a_n| - |a|| < \epsilon$ for all $n \geq n_\epsilon$. Hence, $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

($\not\Leftarrow$) Let $a_n = -1$ for all $n \in \mathbb{N}$. Evidentally, $\{|a_n|\}$ converges to 1 but $\{a_n\}$ converges to -1. Hence, the converse is not true.

Problem 2: Let $\epsilon > 0$. Note that $a_n = 1 + \sum_{i=1}^{n-1} \frac{1}{3^i}$.

Using some facts about sums and geometric series, we know that

$$\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$
$$\sum_{i=1}^{n} ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Subtracting the two and plugging in $a = 1$ and $r = \frac{1}{3}$, we get

$$\sum_{i=n}^{\infty} \frac{1}{3^i} = \frac{1}{1-\frac{1}{3}} - \frac{1-\frac{1}{3^n}}{1-\frac{1}{3}} = \frac{1-(1-\frac{1}{3^n})}{\frac{2}{3}} = \frac{3}{2}\frac{1}{3^n}$$

Note the limit of the series is $\sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{3}{2}$, and the difference from that value from a_n is $\frac{3}{2}\frac{1}{3^n}$. Thus, we want to choose a n_ϵ such that $\frac{3}{2}\frac{1}{3^{n_\epsilon}} < \epsilon$.

Solving for n_ϵ, we get $n_\epsilon > \log_3(\frac{3}{2\epsilon})$. Hence, for all $n \geq n_\epsilon$, $|a_n - \frac{3}{2}| < \epsilon$. Thus, $\{a_n\}$ converges to $\frac{3}{2}$.

Problem 3: Since $\{a_n\}$ is bounded, we know there exists some M such that $a_n \leq M$ for all $n \in \mathbb{N}$.

Hence, $|a_n b_n| = |a_n||b_n| \leq |M||b_n|$.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} b_n = 0$, we can find a $n_\epsilon \in \mathbb{N}$ such that $|b_n - 0| = |b_n| < \frac{\epsilon}{|M|}$. Note that $\frac{\epsilon}{|M|} > 0$.

Hence, for all $n \geq n_\epsilon$, $|a_n b_n - 0| = |a_n b_n| \leq |M||b_n| < |M|\frac{\epsilon}{|M|} = \epsilon$. Thus, $\lim_{n\to\infty} a_n b_n = 0$.

Problem 4: Since we know $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty}(a_n - c_n) = 0$.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} b_n = 0$, we can find a $n_\epsilon \in \mathbb{N}$ such that $|a_n - c_n| < \epsilon$. Note that since we know $c_n \geq a_n$ for all $n \in \mathbb{N}$, $|a_n - c_n| = c_n - a_n$.

Using the fact that $a_n \leq b_n$, we know that $c_n - a_n \geq c_n - b_n$ for all $n \in \mathbb{N}$.

Hence, $|c_n - b_n| \leq |c_n - a_n|$ for all $n \in \mathbb{N}$. Thus, for all $n \geq n_\epsilon$, $|c_n - b_n| \geq |c_n - a_n| < \epsilon$. So, $\lim_{n\to\infty}(c_n - b_n) = 0$. So, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n$.

Problem 5:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{4n^2+n} - 2n &= \lim_{n\to\infty} (\sqrt{4n^2+n} - 2n) \cdot \frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} \\
&= \lim_{n\to\infty} \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} \\
&= \lim_{n\to\infty} \frac{n}{n(\sqrt{4+\frac{1}{n}}+2)} \\
&= \frac{1}{\sqrt{4+\lim_{n\to\infty}\frac{1}{n}}+2} \\
&= \frac{1}{4}
\end{aligned}$$

Problem 6: (a) Assume that for all but finitely many a_n we have $a_n \geq a$. Assume that $\lim_{n\to\infty} a_n = L < a$. Let $\epsilon = a - L > 0$. We know that there exists an $n_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_\epsilon$. So,

$$\begin{aligned}
-\epsilon < a_n - L &< \epsilon \\
L - a < a_n - L &< a - L \\
a_n &< a
\end{aligned}$$

Hence, for all $n \geq n_\epsilon$, $a_n < a$. So, there an infinite number of $a_n < a$. This is a contradiction. So, $\lim_{n\to\infty} a_n \geq a$.

(b) This can be shown similarly by contradiction.

(c) We know that finitely many a_n belong to the intervval $[a, b]$. So, we know that for finitely many $a_n \geq a$ and $a_n \geq b$. Using the previous two parts, we know that $\lim_{n\to\infty} a_n \geq a$ and $\lim_{n\to\infty} a_n \leq b$. Hence, $a \leq \lim_{n\to\infty} a_n \leq b$, so $\lim_{n\to\infty} a_n \in [a, b]$.

Problem 7: Let $\lim_{n\to\infty} a_n = L > a$. Let $\epsilon = L - a > 0$. We know there exists a n_ϵ such that $|a_n - L| < \epsilon$ for all $n \geq n_\epsilon$.

So, $|a_n - L| < L - a$. The case where $a_n > L$ is trivial, so we consider the case where $a_n < L$. $|a_n - L| = L - a_n$. So, $-a_n < -a$ which means $a_n > a$ for all $n \geq n_\epsilon$. Hence, we have found such n_ϵ.

Problem 8: Since $\{a_n\}$ is bounded, we know there exists some M such that $a_n \leq M$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Choose a n_ϵ such that $|a_n - a_m| < \frac{\epsilon}{M^2}$ for all $n, m \geq n_\epsilon$. So, $|a_n^2 - a_m^2| = |a_n - a_m||a_n + a_m| < \frac{\epsilon}{M^2}(2M) = \epsilon$ for all $n, m \geq n_\epsilon$. Hence, $\{a_n^2\}$ is Cauchy.

Problem 9: (a) Inductively, we can show that since $a_1 = 3$ and $a_{n+1} = \frac{1}{2}a_n + \frac{2}{a_n}$, that if a_n is a positive rational number, then a_{n+1} must be one as well. This is beacuse $\mathbb{Q}$ is closed under addition and multiplciation, and both terms in the reccurence must be positive.

Hence, since $a_n > 0$ for all $n \in \mathbb{N}$, we can conclude that 0 is a lower bound for the sequence.

(b) By the above, we know that each $a_n \in \mathbb{Q}$.

(c) We want to show that $\{a_n\}$ is monotonically decreasing.

So, we know that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} = \frac{a_n^2 + 2}{2a_n}$$

We want to show that $a_n \geq a_{n+1}$.

$$\begin{aligned} a_n &\geq a_{n+1} \\ a_n &\geq \frac{a_n^2 + 2}{2a_n} \\ a_n^2 &\geq 2 \end{aligned}$$

Using induction on n, we can show that $a_n^2 \geq 2$ for all $n \in \mathbb{N}$.

Hence, we have shown that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, so $\{a_n\}$ is monotonically decreasing.

(d) Since $\{a_n\}$ is monotonically decreasing and bounded below, we know that it must converge.

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a$, we can plug this into the reccurence to find that $a = \frac{a^2+2}{2a}$. Solving for a, we get $2a^2 - a^2 - 2 = 0$. So, $a^2 - 2 = 0$. Hence, $a = \pm\sqrt{2}$. However, we know that $a > 0$, so $a = \sqrt{2}$.

Problem 10: (a) By induction, we can show that for all $n \in \mathbb{N}$, $a_n < 2$.

Base Case: $a_1 = 2 < 2$.

Induction Hypothesis: Assume that $a_n < 2$ for some $n \in \mathbb{N}$.

Induction Step: We want to show that $a_{n+1} < 2$. We know that $2 + a_n < 4$ by the induction hypothesis. So, $\sqrt{2 + a_n} < 2$. Hence, $a_{n+1} = \sqrt{2 + a_n} < 2$.

Thus, there exists an upper bound for a_n and it is 2.

(b) We want to show that $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_{n+1} = \sqrt{2 + a_n} &\geq a_n \\ 2 + a_n &\geq a_n^2 \\ a_n^2 - a_n - 2 &\leq 0 \\ (a_n - 2)(a_n + 1) &\leq 0 \\ -1 \leq a_n &\leq 2 \end{aligned}$$

Evidentally, a_n is greater than 0, since we can prove with induction that $a_n+2 > 0$ for all $n \in \mathbb{N}$, and by part 1, we know that 2 is an upper bound for a_n. Hence, a_n is monotonically increasing because the inequality holds for all n.

(c) Since $\{a_n\}$ is monotonically increasing and bounded above, we know that it must converge. Let $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. Then, $a = \sqrt{2+a}$. Solving for a, we get $a^2 - a - 2 = 0$. Hence, $a = 2, -1$. Since $a > 0$, we know that $a = 2$.

Problem 11: (a) We want to show that $\{a_n\}$ is monotonically increasing.

Lemma: $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Base Case: $0 < a_1 < b_1$ is given.

Induction Hypothesis: Assume that $a_n \leq b_n$ for some $n \in \mathbb{N}$.

Induction Step: We want to show that $a_{n+1} \leq b_{n+1}$.

We know that $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \frac{a_n+b_n}{2}$. We can use induction to show that $a_n > 0$ and $b_n > 0$. By the AM-GM inequality, we know that $\sqrt{a_n b_n} \leq \frac{a_n+b_n}{2}$. Hence, $a_{n+1} \leq b_{n+1}$.

Using the lemma, we know that $a_n \leq b_n$. So,

$$\begin{aligned} a_n &\leq b_n \\ a_n a_n &\leq a_n b_n \\ \sqrt{a_n a_n} &\leq \sqrt{a_n b_n} \\ a_n &\leq a_{n+1} \end{aligned}$$

Hence, $\{a_n\}$ is monotonically increasing.

We want to show that $\{b_n\}$ is monotonically decreasing.

We know that $a_n \leq b_n$ by the lemma. So,

$$\begin{aligned} a_n &\leq b_n \\ a_n + b_n &\leq 2b_n \\ \frac{a_n + b_n}{2} &\leq b_n \\ b_{n+1} &\leq b_n \end{aligned}$$

Therefore, $\{b_n\}$ is monotonically decreasing.

(b) We want to show that $a_n \leq b_1$, or in other words b_1 is an upper bound for $\{a_n\}$.

Base Case: $a_1 < b_1$ by the assumption.

Induction Hypothesis: Assume that $a_n \leq b_1$ for some $n \in \mathbb{N}$.

Induction Step: Note the following ineqaulity

$$a_n b_n \leq a_1 b_n \leq a_1 b_1 \leq b_1^2$$

Hence, $a_{n+1} = \sqrt{a_n b_n} \leq b_1$

We want to show that $b_n \geq a_1$, or in other words a_1 is a lower bound for $\{b_n\}$.

Base Case: $b_1 > a_1$ by the assumption.

Induction Hypothesis: Assume that $b_n \geq a_1$ for some $n \in \mathbb{N}$.

Induction Step: Note the following inequality We know that $a_1 \leq a_n$ since $\{a_n\}$ is monotone increasing. So, $a_1 + b_n \leq a_n + b_n$. So, $\frac{a_1+b_n}{2} \leq b_{n+1}$. By induction hypothesis, $a_1 \leq b_n$. Hence, $\frac{a_1+a_1}{2} = a_1 \leq b_n + 1$.

Thus a_1 is a lower bound for $\{b_n\}$.

(c) By the previous parts, we know that $\{a_n\}$ is bounded above and monotone increasing, so it must converge. Similarly, $\{b_n\}$ is bounded below and monotone decreasing, so it must converge.

Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then, $a = \sqrt{ab}$ and $b = \frac{a+b}{2}$. Solving for a and b, we get $a = b$.