Problem 1: (10.1.2a) We want to show that $m(4) \geq 15$.

First, let's see what cases we should consider. From theorem 10.1.5, we can use the gluing argument and replace $x, y \in X$ where $\{x, y\} \not\subseteq M$ for all $M \in \mathcal{M}$ with some z. If we color z with the same color as x and y, then we get the same coloring as the unaltered graph.

Notice that in systems of size 4 subsets, there are $\binom{4}{2} = 6$ pairs. In 14 sets of size 4, there are $14 \cdot 6 = 84$ pairs.

So we want the number of total pairs in X to be at least 84. Let $n = |X|$. This means $\binom{n}{2} \geq 84$. The smallest such n is 14.

So if $|X| \geq 14$, then we can inductively "remove" a pair of elements from X and add a new element to get a new system of size 4 subsets with one less element, until $|X| \leq 14$.

Now, let's show that for $|X| \leq 14$, the probability that at least one of the 14 sets of size 4 subsets is monochromatic is non-zero.

Add vertices until the graph has 14 vertices. Color 7 vertices red and 7 vertices white.

There are $\binom{14}{7} = 3432$ such colorings. If we let a single 4-tuple be monochromatic, (let's say white) then there must be 3 other white vertices in the 10 remaining. There are $\binom{10}{3} = 120$ ways to choose these vertices. Since, we can say the same about red, the probability that one 4 tuple is $2 \cdot \frac{\binom{10}{3}}{3432}$.

So, the probability at least one of 14 sets of size 4 subsets is monochromatic is less than the union bound $14 \cdot 2 \cdot \frac{\binom{10}{3}}{3432} = \frac{28 \cdot 120}{3432} = \frac{3360}{3432} < 1$.

The complement of this is non-zero so the probability that no 4-tuple is monochromatic is non-zero.

Problem 2: (10.1.4) We want to show that if n is high enough then there is some order of the cars that cannot be achieved on track B. In other word, we want to show that the total number of permutations is less than $n!$.

For a set of n trains, we can have $4n$ total moves, where a move is defined as a train moving from one track to another.

At each move, we have 5 places that may have trains, and we want to choose 2 of them, so that we can take a train from one location and move it to another location. There are $\binom{5}{2} = 10$ ways to choose these locations.

Hence, there are a total of 10^{4n} possible moves if we count this way. Since factorial grows faster than exponential, there must be some order of the cars that cannot be achieved on track B.

Problem 3: (10.2.3) We want to show that a random graph almost surely contains a triangle. Notice that the probability of two sets of three distinct vertices containing a triangle are independent events. There are roughly $\frac{n}{3}$ sets of three distinct vertices. The probability that a set of three vertices doesn't contain a triangle is $1-\frac{1}{8}=\frac{7}{8}$. So the probability that no set of three vertices contains a triangle is $\left(\frac{7}{8}\right)^{\frac{n}{3}}$.

So, the probability that at least one set of three vertices contains a triangle is $1-\left(\frac{7}{8}\right)^{\frac{n}{3}}$. As we take the limit, this probability approaches 1.

Problem 4: (10.2.9) $\frac{1}{3}$

Problem 5: (10.3.1)

- Let $\mathbb{P}(f=1)=\frac{1}{2}$ and $\mathbb{P}(f=0)=\frac{1}{2}$. Let $\mathbb{P}(g=1)=\frac{3}{4}$ and $\mathbb{P}(g=0)=\frac{1}{4}$. Then $E[f]=\frac{1}{2}$ and $E[g]=\frac{3}{4}$. So, $E[f]E[g]=\frac{3}{8}$.
  Let's look at the random variable fg. Assume that $\mathbb{P}(f=1,g=1)=\frac{1}{2}$, $\mathbb{P}(f=1,g=0)=0$, $\mathbb{P}(f=0,g=1)=\frac{1}{4}$, and $\mathbb{P}(f=0,g=0)=\frac{1}{4}$. Then $E[fg]=\frac{1}{2}$.
- Let $\mathbb{P}(f=2)=\frac{1}{2}$ and $\mathbb{P}(f=1)=\frac{1}{2}$. Then $E[f]=\frac{3}{2}$ so $E[f]^2=\frac{9}{4}$. On the other hand $E[f^2]=2^2(\frac{1}{2})+1^2(\frac{1}{2})=\frac{5}{2}$.
- Let $\mathbb{P}(f=2)=\frac{1}{2}$ and $\mathbb{P}(f=1)=\frac{1}{2}$. Then $E[f]=\frac{3}{2}$ so $\frac{1}{E[f]}=\frac{2}{3}$. On the other hand $\frac{1}{E[f]}=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot 1=\frac{3}{4}$.

Problem 6: (10.3.3) We want compute the expected number of fixed points of a permutation π in the space $\mathcal{S}_n$. We note that $f(\pi)=\sum_{i=1}^n X_i$ where each $X_i=\{1 \text{ if } \pi(i)=i \text{ and } 0 \text{ otherwise}\}$. So, $E[f(\pi)]=E[\sum_{i=1}^n X_i]=\sum_{i=1}^n E[X_i]=\sum_{i=1}^n \mathbb{P}(\pi(i)=i)=\sum_{i=1}^n \frac{1}{n}=1$.