Exercise 1: (a) You want to live a happy life and tie it to people and things, not a goal

(b) The plane leaves and you are not on it, and you do not regret it.

(c) There is someone for which there isn't someone that makes them unhappy.

Exercise 2: Base Case:

$$2(1) - 1 = 1 = 1^2$$

Induction Hypothesis: Assume for some $n \in \mathbb{N}$ that $1 + \cdots + (2n-1) = n^2$.

Induction Step: We want to show that it holds for $n+1$.

Expanding the left hand side (line 1) and using the induction hypothesis (line 3),

$$\begin{aligned} 1 + \cdots + (2(n+1) - 1) &= 1 + \cdots + (2n-1) + (2(n+1) - 1) \\ &= 1 + \cdots + (2n-1) + (2n + 2 - 1) \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Hence, the statement holds for $n+1$.

Exercise 3: Base Case:

$$\begin{aligned} 1^3 + 2^3 + 3^3 &= 1 + 8 + 27 \\ &= 36 \end{aligned}$$

36 is divisble by 9

Induction Hypothesis: Assume for some $n \in \mathbb{N}$ that $(n)^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. So, $(n)^3 + (n+1)^3 + (n+2)^3 = 9k$ for some $k \in \mathbb{N}$.

Induction Step: We want to show that it holds for $n+1$.

Expanding the term and using the induction hypothesis, we get the following

$$\begin{aligned} (n+1)^3 + (n+2)^3 + (n+3)^3 &= (n+1)^3 + (n+2)^3 + n^3 + 3n^2 + 9n + 27 \\ &= n^3 + (n+1)^3 + (n+2)^3 + 9(n + 27) \\ &= 9k + 9(n+27) \\ &= 9(k + n + 27) \end{aligned}$$

Since $k, n, 27 \in \mathbb{N}$, the sum is also divisible by 9.

Exercise 4: Base Case:

$$1^5 - 1 = 0$$

0 is divisible by 30.

Induction Hypothesis: Assume for some $n \in \mathbb{N}$ that $n^5 - n$ is divisible by 5. That is $n^5 - n = 30k$ for some $k \in \mathbb{N}$.

Induction Step: We want to show that it holds for $n+1$.

Expanding the $(n+1)^5$ term and applying the induction hypothesis, we get the following.

$$\begin{aligned}(n+1)^5-(n+1) &= n^5+5n^4+10n^3+10n^2+5n+1-n-1\\ &= n^5-n+5n^4+10n^3+10n^2+5n\\ &= 30k+5(n^4+2n^3+2n^2+n)\\ &= 30k+5n(n+1)(n^2+n+1)\\ &= 30k+5n(n+1)(n^2+4n+4-3n-3)\\ &= 30k+5n(n+1)(n+2)^2-15n(n+1)\end{aligned}$$

We know that $n(n+1)$ must be even because either n or $n+1$ is even, so the third term is divisible by 30. We also know that $n(n+1)(n+2)$ must be divisible by 6 because at least one of the terms in the product must be divisible by 3 and one of the terms in the product must be divisible by 2. Therefore, it holds for $n+1$

Exercise 5: We want to show that

$$\frac{1^2}{1\cdot 3}+\cdots+\frac{n^2}{(2n-1)(2n+1)}=\frac{\sum_{i=1}^n i}{2n+1}$$

for all natural numbers $n\geq 1$.

Base Case:

$$\frac{1}{1\cdot 3}=\frac{1}{3}=\frac{2(1)-1}{2(1)+1}$$

Induction Hypothesis: Assume for some $n\in\mathbb{N}$ that

$$\frac{1^2}{1\cdot 3}+\cdots+\frac{n^2}{(2n-1)(2n+1)}=\frac{\sum_{i=1}^n i}{2n+1}$$

Induction Step: We want to show that it holds for $n+1$.

Note that $\sum_{i=1}^n i=\frac{n(n+1)}{2}$. Expanding the sum and using the induction hypothesis,

$$\begin{aligned}\frac{1^2}{1\cdot 3}+\cdots+\frac{n^2}{(2n-1)(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)} &= \frac{\sum_{i=1}^n i}{2n+1}+\frac{(n+1)^2}{(2n+1)(2n+3)}\\ &= \frac{n(n+1)}{2(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}\\ &= \frac{n(n+1)(2n+1)+2(n+1)^2}{2(2n+1)(2n+3)}\\ &= \frac{(n+1)(n+3)}{2(2n+3)}\\ &= \frac{\sum_{i=1}^{n+1} i}{2(n+1)+1}\end{aligned}$$

Therefore, it holds for $n+1$.

Exercise 6: Let $a, b \in \mathbb{Z}$ where $GCD(a,b) = 1$. Assume that $(\frac{a}{b})^2 = 6$. So, $a^2 = 6b^2$. So, a^2 is divisible by 6, meaning that a also must be divisible by 6, so $a^2 = 6k$ for some $k \in \mathbb{Z}$. Thus,

$$\begin{aligned} a^2 &= (6k)^2 \\ &= 36k^2 \\ 36k^2 &= 6b^2 \\ 6k^2 &= b^2 \end{aligned}$$

This implies that b^2 and b are divisible by 6. So, a and b are both divisible by 6, which contradicts our claim that $GCD(a,b) = 1$. Therefore there cannot exist a rational number whose square is 6.

Exercise 7: b

Exercise 8: b

Exercise 9: e

Exercise 10: g

Exercise 11: b