Problem 1: At time step t, $V_x = \{-t, -t+2, \ldots, t-2, t\}$ and $\mathbb{P}(x = -t+2k) = \binom{t}{k} p^k q^{t-k}$ because to be at step $-t+2k$, you have to choose $+1$ k times out of t

Problem 2: We are given that Z_t are all mutually independent.

$$X_t - X_{s_1} = \sum_{i=s_1+1}^{t} Z_i$$

$$X_{s_1} - X_{s_0} = \sum_{i=s_0+1}^{s_1} Z_i$$

So, the random variable $X_t - X_{s_1}$ dependent on $\{Z_{s_{1+1}}, \ldots, Z_t\}$ while the random variable $X_{s_1} - X_{s_0}$ dependents on $\{Z_{s_0+1}, \ldots, Z_s\}$. Notice that these two sets are disjoint and are constructed from mutually independent events . By some theorem or by intuition, since the events are all mutually independent and the sets are disjoint, the sets of events are also independent (think about how the union of the two sets can be separated into the probabilities of each set occuring). Hence $X_t - X_{s_1}$ and $X_{s_1} - X_{s_0}$ are independent random variables

Problem 3: From question 2, we know that $X_t - X_{s_1}$ and $X_{s_1} - X_{s_0}$ are independent random variables. So, by the definition of independence,

$$\mathbb{P}(X_t - X_{s_1} = x_t - x_{s_1} | X_{s_1} - X_{s_0} = x_{s_1} - x_{s_0}) = \mathbb{P}(X_t - X_{s_1} = x_t - x_{s_1})$$

Intuitively, this equation is equivalent to

$$\mathbb{P}(X_t - X_{s_1} = x_t - x_{s_1} | X_{s_1} = x_{s_1}, X_{s_0} = x_{s_0}) = \mathbb{P}(X_t - X_{s_1} = x_t - x_{s_1})$$

because the given on the left side of the equation can be derived from knowing the values of the random variables X_{s_1}, X_{s_0}. Since, we are given that $X_{s_1} = x_{s_1}$, it is redundant on the left side of the probabilities. So, intuitively, this is equivalent to

$$\mathbb{P}(X_t = x_t | X_{s_1} = x_{s_1}, X_{s_0} = x_{s_0}) = \mathbb{P}(X_t = x_t | X_{s_1} = x_{s_0})$$

Problem 4:

- Note that for $x \notin \{x_{t-1}+1, x_{t_1}-1\}$, $\mathbb{P}(X_t = x | X_{t-1} = x_{t-1}) = 0$. Therefore, the sum on the right side is equal to

$$\mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_{t-1}+1)\mathbb{P}(X_t = x_{t-1}+1 | X_{t-1} = x_{t-1})$$
$$+ \mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_{t-1}-1)\mathbb{P}(X_t = x_{t-1}-1 | X_{t-1} = x_{t-1})$$

  Assume that $X_{t+1} = x_{t-1}+2$. Then, $\mathbb{P}(X_{t+1} = x_{t-1}+2 | X_t = x_{t-1}-1) = 0$. So, $\mathbb{P}(X_{t+1} = x_{t-1}+2 | X_{t-1} = x = t_1) = p \cdot p + 0 = p^2$.
  For $X_{t+1} = x_{t-1} - 2$, $\mathbb{P}(X_{t+1} = x_{t-1} - 2 | X_t = x_{t-1}+1) = 0$. So, $\mathbb{P}(X_{t+1} = x_{t-1} - 2 | X_{t-1} = x = t_1) = 0 + q \cdot q = q^2$.
  For $X_{t+1} = x_{t-1}$, we can get to $X_{t+1} = x_{t-1}$ from $X_{t-1} = x_{t-1}$ by subtracting one first and then adding one, and then reversing the order. Hence, $\mathbb{P}(X_{t+1} = x_{t-1} | X_{t-1} = x = t_1) = p(1-q) + (1-q)p = 2p(1-q)$.
  Therefore, the sum is equal to the piece wise function.

- The general form for n time steps is

$$\mathbb{P}(X_{t+n} = x_{t+n} | X_t = x_t) = \sum_{x_t \in \mathbb{Z}} \sum_{x_{t+1} \in \mathbb{Z}} \cdots \sum_{x_{t+n-1} \in \mathbb{Z}} \prod_{k=t+1}^{t+n} \mathbb{P}(X_k = x_k | X_{k-1} = x_{k-1})$$