Problem 1: (7.1.5) Note that $x^2 + y^2 = r^2$. For x,

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\sin\theta\dot{\theta} \\ &= r(1-r^2)\cos\theta - r\sin\theta \\ &= (1-r^2)x - y \\ &= x - y - x(x^2+y^2) \end{aligned}$$

Similarly, expanding $\dot{y}$,

$$\begin{aligned} \dot{y} &= \dot{r}\sin\theta + r\cos\theta\dot{\theta} \\ &= r(1-r^2)\sin\theta + r\cos\theta \\ &= y(1-r^2) + x \\ &= x + y - y(x^2+y^2) \end{aligned}$$

Problem 2: (7.2.10) Take $\dot{V}$.

$$\begin{aligned} \dot{V} = 2ax\dot{x} &= 2by\dot{y} \\ &= 2ax(y-x^3) + 2by(-x-y^3) \\ &= 2axy - 2ax^4 - 2bxy - 2b^4 \\ &= (2a-2b)xy - 2ax^4 - 2by^4 \end{aligned}$$

Notice that if $2a - 2b = 0$ and $a, b > 0$, then the rest of the terms would be strictly negative for all $x, y \in \mathbb{R}$ except the fixed point. Let $a = b = 1$. Then $V = x^2 + y^2 > 0$ for all $(x, y) \neq (0, 0)$, and $\dot{V} = -2x^4 - 2y^4 < 0$. Therefore, since we can construct a Liapunov function, there cannot exist a closed orbit.

Problem 3: (7.2.12) let $V = x^m + ay^n$. Then,

$$\begin{aligned} \dot{V} &= mx^{m-1}\dot{x} + any^{n-1}\dot{y} \\ &= -mx^m + (2my^3 - 2my^4)x^{m-1} - anxy^{n-1} + (an + axn)y^n \end{aligned}$$

Let $n = 4, m = 2, a = 1$. Then,

$$= -2x^2 - 4y^2 < 0$$

for all $(x, y) \neq (0, 0)$. Also note that $V = x^2 + y^4 > 0$ for all $(x, y) \neq (0, 0)$. Hence, since we can construct a Liapunov function, there cannot exist a closed orbit.

Problem 4: (7.2.18) By Dulac's Criterion, we want to show that $\nabla \cdot (g\dot{x})$ is one sign on the first quadrant.

$$\begin{aligned}\nabla \cdot (g\dot{x}) &= \frac{d}{dx}(g\dot{x}) + \frac{d}{dy}(g\dot{y}) \\ &= y^{\alpha-1}(-rx + \frac{r}{2} + \alpha - \frac{\alpha}{x})\end{aligned}$$

For $x, y \geq 0$, $r > 0$, we have $y^{\alpha-1} > 0$. So we want $-rx + \frac{r}{2} + \alpha - \frac{\alpha}{x} < 0$.

According to ChatGPT (Reasoning Model), if we set $\alpha = \frac{r}{2}(1 - \frac{1}{\sqrt{2}})^2$ then $-rx + \frac{r}{2} + \alpha - \frac{\alpha}{x} < 0$ for all x. Thus, by Dulac's Criterion, we cannot have a closed orbit in the first quadrant...

Problem 5: (8.2.12) (a) Note that $f(x,y) = xy^2, g(x,y) = -x^2, \omega = 1$. Note the following derivations of f and g.

$$\begin{aligned} f_{xxx} &= 0 \\ f_{xyy} &= 2 \\ f_{xy} &= 2y \\ f_{xx} &= 0 \\ f_{yy} &= 2x \\ g_{xxy} &= 0 \\ g_{yyy} &= 0 \\ g_{xy} &= 0 \\ g_{xx} &= -2 \\ g_{yy} &= 0 \end{aligned}$$

So, $16a = 2 + 4xy$. Evaluated at $(0,0)$, $a = \frac{2}{16}$.

(b) This is greater than 0, so this is a subcritical Hopf bifurcation.