Problem 1: Let $a_1 = 1$ and $a_{n+1} = [1 - \frac{1}{(n+1)^2}]a_n$ for all $n \geq 1$.

(a) We want to show that $\{a_n\}$ is monotonically decreasing and bounded by 0.

Claim: Showing that $0 \leq 1 - \frac{1}{(n+1)^2} \leq 1$ or $0 \leq \frac{1}{(n+1)^2} \leq 1$ is sufficient.

Note, that $(n+1) \neq 0$ since $n \geq 1$ so $n \neq -1$. Also notice that $(n+1)^2 > 0$ for all $n \in \mathbb{N}$, so $\frac{1}{(n+1)^2} > 0$ for all $n \in \mathbb{N}$.

Now let's show that $\frac{1}{(n+1)^2} \leq 1$ for all $n \in \mathbb{N}$.

Let's procesed by induction.

Base Case: $n = 1 \Rightarrow \frac{1}{2^2} = \frac{1}{4} \leq 1$

Induction Hypothesis: Let $n \in \mathbb{N}$. Assume that $\frac{1}{(n+1)^2} \leq 1$.

We know that $0 < n+1 < n+2$, so $(n+1)^2 < (n+2)^2$ and $\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}$. Hence, $\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2} \leq 1$. Therefore, $0 \leq \frac{1}{(n+1)^2} \leq 1$ for all $n \in \mathbb{N}$.

Because $\{a_n\}$ is monotonically decreasing and bounded by 0, it must converge.

(b) Notice that

$$a_{n+1} = [\frac{(n+1)^2 - 1}{(n+1)^2}]a_n = [\frac{n^2(n+2)^2}{(n+1)^2}]a_n$$

By a telescoping argument, we can show that $a_{n+1} = \frac{1}{2}(\frac{n+1}{n})\ldots$ Since we know that $\lim_{n\to\infty} \frac{n+1}{n} = 1$, $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Problem 2: Let A be a non-empty bounded subset of $\mathbb{R}$. Suppose $\sup A \notin A$.

First, let's show that there exists an increasing sequence of points. Choose an arbitrary $a \in A$. Let $a_1 = a$. There must exist an a_2 such that $a_1 < a_2 < \sup A$, otherwise $a_1 = \sup A$ and $\sup A \in A$ which contradicts the assumption... If we iteratively do this, we can find a subsequence, which by construction is monotonically increasing.

Since $\{a_n\}$ is monotonically increasing and bounded above the sequence must converge. Let $\{a_n\}$ converge to b. We want to show that $b = \sup A$.

Assume otherwise.

Case 1: $b > \sup A$. Then we can choose a ϵ such that $0 < b - \sup A = \epsilon\ldots$ This means there exists an N such that $n > N$, $|a_n - b| < \epsilon$. So,

$$\begin{aligned} -\epsilon < a_n - b < \epsilon \\ \Rightarrow b - \epsilon < a_n \\ \Rightarrow \sup A < a_n \end{aligned}$$

This contradicts our construction of a_n, so b cannot be greater than $\sup A$.

Case 2: $b < \sup A$. Let $\epsilon = \frac{b - \sup A}{2} > 0$.

Then $b+\epsilon < \sup A$. so $b+\epsilon$ is not an upper bound of A. Thus, there exists a $a \in A \cap \{a_n\}$ such that $b+\epsilon < a$. So, b would not be the limit of $\{a_n\}$, once again contradicting our claim.

Therefore, $b = \sup A$.

Problem 3: Let C be a the set of Cauchy sequences of $\mathbb{Q}$.

(a) **Reflexive:** Let $\{a_n\} \in C$. $\lim_{n\to\infty}(a_n - a_n) = \lim_{n\to\infty} 0 = 0$.
Symmetric: Let $\{a_n\}, \{b_n\} \in C$. Assume that $\{a_n\} \sim \{b_n\}$. Then

$$\lim_{n\to\infty}(a_n - b_n) = 0 \Rightarrow \lim_{n\to\infty}(b_n - a_n) = 0 \Rightarrow \{b_n\} \sim \{a_n\}$$

Transitive: Let $\{a_n\}, \{b_n\}, \{c_n\} \in C$ and assume that $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$.
So,

$$\lim_{n\to\infty} a_n - b_n = 0$$
$$\lim_{n\to\infty} b_n - c_n = 0$$

Adding these together,

$$\begin{aligned} &\lim_{n\to\infty}(a_n - b_n) + \lim_{n\to\infty}(b_n - c_n) = 0 \\ \Rightarrow &\lim_{n\to\infty} a_n - c_n = 0 \\ \Rightarrow &\{a_n\} \sim \{c_n\} \end{aligned}$$

(b) Let $\{a_n\}, \{a'_n\}, \{b_n\}, \{b'_n\} \in C$ and $\{a_n\} \sim \{a'_n\}$ and $\{b_n \sim b'_n\}$.
We want to show that $\{a_n + b_n\} \sim \{a'_n + b'_n\}$.
We know that

$$\lim_{n\to\infty} a_n - a'_n = 0$$
$$\lim_{n\to\infty} b_n - b'_n = 0$$

So,

$$\begin{aligned} \lim_{n\to\infty}(a_n - a'_n) + \lim_{n\to\infty}(b_n - b'_n) = 0 \\ \Rightarrow \lim_{n\to\infty}((a_n + b_n) - (a'_n + b'_n)) = 0 \\ \Rightarrow \{a_n + b_n\} \sim \{a'_n + b'_n\} \end{aligned}$$

We want to show that $\{a_n b_n\} \sim \{a'_n b'_n\}$.

Observe that

$$\begin{aligned} a_n b_n - a'_n b'_n &= a_n b_n - a_n b'_n + a_n b'_n - a'_n b'_n \\ &= a_n(b_n - b'_n) + b'_n(a_n - a'_n) \end{aligned}$$

Now we just need to bound a_n and b'_n. We know that $\{a_n\}$ and $\{b'_n\}$ are Cauchy, so we can bound above $\{a_n\}$ by some $|M_1|$ and $\{b'_n\}$ by $|M_2|$.

Let $\epsilon > 0$. Since we know $\lim_{n\to\infty} b_n - b'_n = 0$, there exists some $N_1 \in \mathbb{N}$, such that $|b_n - b_{n'}| < \frac{\epsilon}{|M_1|}$. Similarly, there exists some $N_2 \in \mathbb{N}$ such that $|a_n - a_{n'}| < \frac{\epsilon}{|M_2|}$

Hence, $|a_n(b_n - b'_n)| = |a_n|\,|(b_n - b'_n)| \leq |M_1| < \epsilon$. So we can choose the N_1 and $\lim_{n\to\infty} a_n(b_n - b'_n) = 0$. Similarly, we can choose N_2, and $\lim_{n\to\infty} b'_n(a_n - a'_n) = 0$.

Hence, $\lim_{n\to\infty} a_n b_n - a'_n b'_n = 0$, so $\{a_n b_n\} \sim \{a'_n b'_n\}$. So, $+, \cdot$ are both well defined operations on C.

Most of the field axioms on F follow from the axioms of $\mathbb{Q}$. Here are some of the interesting axioms.

- Additive Identity. The additive identity element of F can be the zero sequence since $a_n + 0 = a_n$ for all $n \in \mathbb{N}$ by the properties of $\mathbb{Q}$. This is also a Cauchy sequence, since for $N = 1$, the difference between any elements is 0.
- Additive Inverse. The additive inverse element of any $[a_n]$ is $[-a_n]$. Evidentally, $a_n + (-a_n) = 0$ for all a_n so the sum results in $[0]$. This is also a Cauchy sequence, since $|-a_n - (-a_m)| = |a_n - a_m|$, so we can use the same $N \in \mathbb{N}$ that works for $\{a_n\}$.
- Multiplicative Identity. The multiplicative identity is $[1]$ which is the sequence of 1. This is evidentally cauchy, since the difference between any two elements is, and by the properties of $\mathbb{Q}$, $a_n \cdot 1 = a_n$ for all $n \in \mathbb{N}$
- Multiplicative Inverse. We want to show that the sequence $\{\frac{1}{a_n}\}$ is Cauchy, given that $\{a_n\}$ is cauchy. This was a question on the midterm, so use the proof from there.

(c) Let's first show that $<$ is a well defined order.

- Let $a_n, b_n, a'_n, b'_n \in C$ where $[a_n] = [a'_n]$, $[b_n] = [b'_n]$. We want to show that $[a_b] < [b_n] \Leftrightarrow [a'_n] < [b'_n]$.
  So, $[a_n] = [a'_n]$ means that $\lim_{n\to\infty} a_n - a'_n = 0$. So, for all $\epsilon > 0$, there exists some $N_1 \in \mathbb{N}$ such that $|a_n - a'_n| < \epsilon$ for $n > N_1$. Similarly, there exists some N_2 such that $|b_n - b'_n| < \epsilon$.
  Let $N_3 \in \mathbb{N}$ be the value such that $a_n < b_n$ for all $n \geq N_3$. Define $N := max\{N_1, N_2, N_3\}$.
  Let $\epsilon = \frac{b_n - a_n}{2} > 0$.

So we get,

$$a'_n = a_n + (a'_n - a_n) < a_n + \epsilon$$
$$b'_n = b_n + (b'_n - b_n) > b_n + \epsilon$$

Ergo,

$$b'_n - a'_n > b_n + \epsilon + a_n + \epsilon = 0$$

- Trichotomy. Let $[a_n], [b_n] \in F$. Obviously by trichotomy of $\mathbb{Q}$, $[a_n] < [b_n]$ and $[b_n] < [a_n]$ cannot both be true. Now, let's assume that $[a_n] = [b_n]$ and $[a_n] < [b_n]$.
  Then, $\lim_{n\to\infty} a_n - b_n = 0$. We also know that for some $N_1 \in \mathbb{N}$, that $a_n < b_n$ for all $n \geq N_1$.
  Let $b_n - a_n = \epsilon > 0$ where this is true for all $n \geq N_1$, we can use some lim inf argument to show that this ϵ exists. Let $N_2 \in \mathbb{N}$ such that for all $|b_n - a_n| < \frac{\epsilon}{2}$. Take $N = max\{N_1, N_2\}$. Then, obviously the two statements cannot both be true. Hence, by contradiction only one can be true.
- Transitivity. Let $[a_n] < [b_n]$ and $[b_n] < [c_n]$. Then for some $N_1 \in \mathbb{N}$, $a_n < b_n$ for all $n > N_1$, and similarly can be done with a N_2. Taking $N = \max N_1, N_2$, we find that $a_n < b_n < c_n$ for all $n \geq N$. Hence, $[a_n] < [c_n]$.
- (01). If $[a_n] = [0]$, then by definition of P and the well defined ness of $<$, $[a_n] \notin P$. Assume $[a_n] \in P$ and $-[a_n] \in P$.
  Note that $-[a_n] = [-a_n]$. So at some N_1, $-a_n > 0$ for all $n \geq N_1$ and at some N_2, $a_n > 0$ for all $n \geq N_1$. This cannot happen by the trichotomy of the order relation on $\mathbb{Q}$. Hence, by contradiction, only one of $-[a_n], [a_n]$ can be in P.
- (02). They are both evident by choosing a $N_1, N_2 \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N_1$ and $b_n > 0$ for all $n \geq N_2$. Let $N = max\{N_1, N_2\}$. Then $a_n + b_n > 0$ and $a_n \cdot b_n > 0$ for all $n \geq N$. Therefore, both are elements of P still.

(d) With $01, 02$, we can use the equivalent definition of an ordered field, since $P \subseteq F$ and has those properties. Therefore, F is an ordered field.

Problem 4: Since $\{a_n\}$ and $\{b_n\}$ are bounded, we know that there exists a $N_1, N_2 \in \mathbb{N}$ such that for all $\epsilon > 0$ and $|\sup\{a_n : n \geq N_1\} - L_a| < \frac{\epsilon}{2}$ and $|\sup\{b_n : n \geq N_2\} - L_b| < \frac{\epsilon}{2}$ where $\lim_{n\to\infty} \sup a_n = L_a$ and $\lim_{n\to\infty} \sup b_n = L_b$.

Let $N = max\{N_1, N_2\}$. So, $\sup\{a_n : n \geq N\} < L_a + \frac{\epsilon}{2}$ and $\sup\{b_n : n \geq N\} < L_b + \frac{\epsilon}{2}$. This means

$$a_n + b_n \leq \sup\{a_n : n \geq N\} + \sup\{b_n : n \geq N\} \leq L_a + L_b + \epsilon$$

for all $n \geq N$, So, $\sup\{a_n : n \geq N\} + \sup\{b_n : n \geq N\}$ is an upper bound for $a_n + b_n$. Therefore, we can also conclude that

$$\sup\{a_n + b_n : n \geq N\} \leq \sup\{a_n : n \geq N\} + \sup\{b_n : n \geq N\} \leq L_a + L_b + \epsilon$$

We also know that $\lim_{n\to\infty} \sup c_n = \inf_{N\geq 1} \sup\{c_n : n \geq N\}$ so,

$$\lim_{n\to\infty} \sup(a_n + b_n) = \inf_{N\geq 1} \sup\{a_n + b_n : n \geq N\} \leq \sup\{a_n + b_n : n \geq N\}$$

Thus,

$$\lim_{n\to\infty} \sup(a_n + b_n) \leq L_a + L_b + \epsilon$$

If we assume that $\lim_{n\to\infty} \sup(a_n + b_n) > L_a + L_b$, then there must exist a $\delta > 0$ such that $\lim_{n\to\infty} \sup(a_n + b_n) = L_a + L_b + \delta$. If we choose $\epsilon = \frac{\delta}{123}$, then we get a contradiction, since

$$\lim_{n\to\infty} \sup(a_n + b_n) = L_a + L_b + \delta \leq L_a + L_b + \frac{\delta}{123}$$

Therefore, $\lim_{n\to\infty} \sup(a_n + b_n) \leq \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

Problem 5: Using the same idea as question 4 and using the fact that the sequence is non-negative, we can obtain the inequality

$$\lim_{n\to\infty} \sup(a_n b_n) \leq (L_a + \epsilon)(L_b + \epsilon)$$

for all $\epsilon > 0$. Let $L = \lim_{n\to\infty} \sup(a_n b_n)$.

Assume for contradiction that

$$L_a L_b < L \leq (L_a + \epsilon)(L_b + \epsilon)$$

We want to show that there exists an $\epsilon > 0$ such that $(L_a + \epsilon)(L_b + \epsilon) < L$.

Note that $(L_a + \epsilon)(L_b + \epsilon) = L_a L_b + \epsilon(L_a + L_b) + \epsilon^2$. So,

$$\epsilon^2 + \epsilon(L_a + L_b) - (L - L_a L_b) \leq 0$$

Using the quadratic formula, we get

$$\epsilon < \frac{-(L_a + L_b) + \sqrt{(L_a + L_b)^2 + 4(L - L_a L_b)}}{2}$$

Note that the expression on the right is greater than 0, since $L - L_a L_b > 0$ so $L_a + L_b < \sqrt{(L_a + L_b)^2 + 4(L - L_a L_b)}$ So by the density of the real numbers we can find a $\epsilon > 0$ that satisifies the inequality.

Substituting ϵ with its value, we get that $(L_a + \epsilon)(L_b + \epsilon) < L$. Hence, we have a contradiction. So $L \leq L_a L_b$ meaning

$$\lim_{n\to\infty} \sup(L_a L_b) \leq \lim_{n\to\infty} \sup L_a \lim_{n\to\infty} \sup L_b$$

Problem 6: ($\Rightarrow$) Assume that $\{|a_n|\}$ is bounded. Then for all $n \in \mathbb{N}$, $|a_n| \leq M$ for some $M > 0, M \in \mathbb{R}$.

So, for all $N \in n$, $\sup\{|a_n| : n \geq N\} \leq M$. Since $\lim_{n\to\infty} \sup |a_n| = \inf \sup\{|a_n| : n \geq N\}$, then

$$\inf \sup\{|a_n| : n \geq N\} \leq \sup\{|a_n| : n \geq N\} \leq M$$

($\Leftarrow$) It is sufficient to show that $\{|a_n|\}$ is bounded above. Assume that $\lim_{n\to\infty} \sup |a_n| = L < \infty$.

Let $\epsilon > 0$. Then, we know that $\sup\{|a_n| : n \geq N\} < L + \epsilon$ for some $N \in \mathbb{N}$. Recall that for $N' < N$, that $\sup\{|a_n| : n \geq N'\} \geq \sup\{|a_n| : n \geq N'\}$. So take the max of $\{\sup\{|a_n| : n \geq 1\}, ..., \sup\{|a_n| : n \geq N, L + \epsilon\}\}$. This will be an upper bound of $\{|a_n|\}$. Hence $\{|a_n|\}$ has an upper bound. This implies that $\{a_n\}$ is bounded.

Problem 7: Let's split the limit into cases.

Case 1: Assume $\lim_{n\to\infty} b_n = \infty$. This implies that A is not bounded above, since for every $a \in A$, we can find an element greater than it. Hence, we can abuse some notation and say that $\sup A = \infty$. From a theorem, we know that $\lim_{n\to\infty} a_n = \sup A = \infty$. We also know that $\lim_{n\to\infty} \sup a_n$ is a subsequence of $\{a_n\}$, so $\lim_{n\to\infty} \sup a_n \in A$.

Therefore, $\infty \in A$ so $\lim_{n\to\infty} b_n \in A$.

Case 2: $\lim_{n\to\infty} b_n = -\infty$ can be proven in a similar way.

Case 3: Assume that $\lim_{n\to\infty} b_n = L$ for some $L \in \mathbb{R}$.

Let $\epsilon > 0$. So we know for some $N \in \mathbb{N}$ that $|b_n - L| < \frac{\epsilon}{2}$ for all $n \geq \mathbb{N}$. This means there exists a subsequence K_n such that $\lim_{n\to\infty} a_{K_n} = b_n$.

In this subsequence, we can choose a $m > N'$ such that $|a_m - b_n| < \frac{\epsilon}{2}$ for some $N' \in \mathbb{N}$. Combining these statements, we get

$$|a_m - L| = |a_m - b_n + b_n - L| \leq |a_m - b_n| + |b_n - L| < \epsilon$$

Thus, we can construct a subsequence by selecting a a_m for each b_n, $n \geq N$ and by construction its limit will be L. Thus, $\lim_{n\to\infty} b_n = L \in A$.

Problem 8: (a) Since we know that $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$, we just need to prove that $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf s_n$ and $\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup s_n$.

Case 1: Assume that $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$.

Let $\epsilon > 0$. We get that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|\inf\{a_n\} - L| < \epsilon$. From this, we find

$$L - \epsilon < \inf\{a_n\} \leq a_n < L + \epsilon$$

Let's use this fact with s_n, where $n \geq N$.

$$s_n = \frac{a_1 + \cdots + a_N + (a_{N+1} + \cdots + a_n)}{n} > \frac{a_1 + \cdots + a_N}{n} + \frac{(n - (N + 1))(L - \epsilon)}{n}$$

As we take n to infinity, we see that the sum approaches $L-\epsilon$. So we see that $s_n > L-\epsilon$. Since, $L-e$ is a lower bound for all s_n, $\lim_{n\to\infty}\inf s_n \geq L-\epsilon$, and since ϵ vanishes, we get the desired inequality, $\lim_{n\to\infty}\inf s_n \geq \lim_{n\to\infty}\inf a_n$.

Using the same idea, we can show that $\lim_{n\to\infty}\sup s_n \leq \lim_{n\to\infty}\sup a_n$.

Thus, combining the inequalities together $\lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\inf s_n \leq \lim_{n\to\infty}\sup s_n \leq \lim_{n\to\infty}\sup a_n$

Case 2: Assume that $\lim_{n\to\infty} a_n = -\infty$.

Then for every $M < 0$ there exists an $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$. We can use the same idea as the first case and say that

$$s_n = \frac{a_1 + \cdots + a_N + (a_{N+1} + \cdots + a_N)}{n} < \frac{a_1 + \cdots + a_N}{n} + \frac{(n-(N+1))M}{n}$$

Taking the limit as n approaches ∞, we get that $s_n < M$. Hence $\lim_{n\to\infty} s_n = \infty$ as well.

The same logic can be applied for the lim sup and the positive infinity cases.

(b) We know that if a limit exists then its lim sup equals the lim inf. Since, s_n is bounded by the lim sup and lim inf of $\{a_n\}$ and they are equal, then we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\inf a_n = \lim_{n\to\infty}\inf s_n = \lim_{n\to\infty}\sup s_n = \lim_{n\to\infty}\sup a_n = \lim_{n\to\infty} s_n$$

Problem 9: (a) Let $\epsilon > 0$. Then there exists some $M \in \mathbb{N}$ such that for all $N \geq M$, $|\sup\{a_n : n \geq N\} - L| < \epsilon$. This means

$$\sup\{a_n : n \geq N\} \leq \sup\{a_n : n \geq M\} < L + \epsilon$$

Hence, we know that any $a_k \in \{a_n\}$ such that $a_k \geq L+\epsilon$ must be indexed with $k < M$. So, it must be in the set $\{a_k : k < M\}$. This set has finitely many elements, so there are only finitely many n for which $a_n > L+\epsilon$.

(b) Let $\epsilon > 0$. Then, there exists an $M \in \mathbb{N}$ such taht for all $N \geq M$,

$$|\sup\{a_n : n \geq N\} - L| < \epsilon$$

So

$$L - \epsilon < \sup\{a_n : n \geq N\}$$

Assume there are finitely many elements such that $a_k > L-\epsilon$. Then we could find a $N' \in \mathbb{N}$ such that for all $n' \geq N'$, $a_{n'} < L-\epsilon$. The sup of this tail would be less than $L-\epsilon$, meaning $\lim_{n\to\infty}\sup a_n < L-\epsilon$. This leads to a contradiction. Hence, there are an infinitely many n for which $a_n > L-\epsilon$.

Problem 10: Assume for contradiction there are two real numbers that satisfy both conditions, L_1, L_2. Assume that $L_1 < L_2$.

Choose $\epsilon = \frac{L_1+L_2}{2}$.

Note that $L_1 + \epsilon = L_2 - \epsilon$. By condition ii, we know that there must be infinitely many $a_n > L_2 - \epsilon$, and by condition i, there must be finitely many $a_n > L_1 + \epsilon$, leading to a contradiction. Hence, there can only exist one such L.