Problem 1: Note that aX has mean $a\mu_X$ and variance $a^2\sigma_X^2$. Similarly, bY has mean $b\mu_Y$ and variance $b^2\sigma_Y^2$. Since, X and Y are independent, aX and bY are independent. So, we can use the theorem that the density function of $Z = aX + bY$ is $f_X * f_Y$.

So,

$$f_X * f_Y = \frac{1}{2\pi a\sigma_X b\sigma_Y}\int_{-\infty}^{\infty}\exp(J)dx$$

where

$$J = -\frac{1}{2}((\frac{(x-a\mu_X)^2}{2a^2\sigma_X^2}) + \frac{(t-x-b\mu_Y)^2}{2b^2\sigma_Y^2})$$

Doing some rearranging and expanding of terms,

$$J = -\frac{1}{2}(x^2(\frac{1}{a^2\sigma_X^2} + \frac{1}{b^2\sigma_Y^2}) - 2x(\frac{a\mu_X}{a^2\sigma_X^2} + \frac{(t-b\mu_Y)}{b^2\sigma_Y^2}) + C)$$

where

$$C = \frac{a\mu_X}{a^2\sigma_X^2} + \frac{(t-b\mu_Y)^2}{b^2\sigma_Y^2}$$

Notice that C doesn't depend on x so we can take it out of the integral.

Let $A = \frac{1}{a^2\sigma_X^2} + \frac{1}{b^2\sigma_Y^2}$ and $B = \frac{a\mu_X}{a^2\sigma_X^2} + \frac{(t-b\mu_Y)}{b^2\sigma_Y^2}$. Using these substitutions and completing the square,

$$J = -\frac{1}{2}C + \frac{B^2}{2A} - \frac{1}{2}A(x-\frac{B}{A})^2$$

Next, using u substitution where $u = (\sqrt{\frac{A}{2}}(x-\frac{B}{A}))$ we get that $du = \sqrt{\frac{A}{2}}dx$. So,

$$\begin{aligned}\int_{\infty}^{-\infty}\exp(J)dx &= \int_{\infty}^{-\infty}\exp(-u^2 - \frac{C}{2} + \frac{B^2}{2A})du \\ &= \exp(-\frac{C}{2} + \frac{B^2}{2A})\int_{\infty}^{-\infty}\exp(-u^2)\sqrt{\frac{2}{A}}du \\ &= \sqrt{\frac{2\pi}{A}}\exp(-\frac{C}{2} + \frac{B^2}{2A})\end{aligned}$$

Simplifying some things, we get that

$$f_X * f_Y = \frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_X^2 + b^2\sigma_Y^2}}\exp(-\frac{C}{2} + \frac{B^2}{2A})$$

Notice that the $a^2\sigma_X^2 + b^2\sigma_Y^2$ in the denominator is exactly the variance we desire.

Expanding $-\frac{C}{2} + \frac{B^2}{2A}$, we get $\frac{(t-(a\mu_X + b\mu_Y))^2}{2(a^2\sigma_X^2 + b^2\sigma_Y^2)}$. Thus, $\mu_Z = a\mu_X + b\mu_Y$ as desired

Problem 2: **Case 1:** Let $s < t$. Note that $W_t = (W_t - W_s) + W_s$.

So, $\sigma(W_s, W_t) = \sigma(W_s, (W_t - W_s) + W_s)$. By linearity of Covariance,

$$\sigma(W_s, (W_t - W_s) + W_s) = \sigma(W_s, (W_t - W_s)) + \sigma(W_s, W_s)$$

Note that $W_s = W_s - W_0$. So, by independent increments

$$\sigma(W_s, W_t - W_s) = \sigma(W_s - W_0, W_t - W_s) = 0$$

Also note that $\sigma(W_s, W_s) = \sigma^2(W_s) = s = \min(s,t)$.

$t < s$ can be solved in a similar fashion.

Case 2: Let $s = t$. Then $\sigma(W_s, W_t) = \sigma^2(W_s) = s = t = \min(s,t)$.

Problem 3: $Z_1 = \frac{W_s}{\sqrt{s}}$ and $Z_2 = \frac{W_t - W_s}{\sqrt{t-s}}$ are independent and distributed $N(0,1)$. This is because the means of W_s and $W_t - W_s$ are 0 and the variance are s and $t - s$ respectively.

If there exists a A such that $X = \begin{bmatrix} W_s \\ W_t \end{bmatrix} = A \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \vec{\mu}$, then X is bivariate normal.

Substituting the $\frac{W_s}{\sqrt{s}}$ and $\frac{W_t - W_s}{\sqrt{t-s}}$, we get that

$$A = \begin{bmatrix} \sqrt{s} & 0 \\ \sqrt{s} & \sqrt{t-s} \end{bmatrix}$$

Hence, the joint distribution of W_s and W_t is bivariate normal

Problem 4: Note that for X, Y independent, $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$. We know that by independent increments

$$\mathbb{E}(W_t \mid W_s) = \mathbb{E}(W_t \mid W_{s_0}, W_{s_1}, \ldots, W_{s_n} \text{ for } s_i < s)$$

Moreover,

$$\mathbb{E}(W_t \mid W_s) = \mathbb{E}((W_t - W_s) + W_s \mid W_s)$$

By Linearity of expectation,

$$= \mathbb{E}(W_t - W_s \mid W_s) + \mathbb{E}(W_s \mid W_s)$$

Since $W_t - W_s$ and W_s are independent,

$$\mathbb{E}(W_t - W_s \mid W_s) = \mathbb{E}(W_t - W_s) = 0$$

$\mathbb{E}(W_s \mid W_s) = W_s$. Thus, $\mathbb{E}(W_t \mid W_s) = W_s$.

I think by this stack exchange post, this is sufficient: Stack Exchange