Problem 1: **Lemma:** Removing a leaf from a tree (with at least 2 vertices) creates a new tree. I'm pretty sure this was proven in class, but the idea is that the tree still remains connected because it was a leaf, and there is still no cycles because there were non to begin with.

We will induce on the number of vertices in the rree.

Base Case: Let T be a tree with two vertices. The only such tree has two vertices, both of degree 1. So the sum would equal to $p_1 = 2$.

Induction Hypothesis: Assume that the equality holds for any tree with n vertices.

Induction Step: Let T be a tree with $n+1$ vertices. Let v be a leaf of T. Then, $T - v$ is a tree with n vertices. Hence, the equality

$$p_1' - p_3' - 2p_4' - \cdots - (n-3)p_{n-1}' = 2$$

holds. Let's add back the leaf v to $T - v$.

Case 1: The leaf was added to another leaf. Then, $p_n = 0$, and $p_1 = p_1'$. This is because the degree of the old leaf is now 2, and the degree of the new leaf is 1, so the number of vertices of degree 1 remains the same. The number of vertices of the other degrees remain the same, so $p_i = p_i'$. So,

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} - (n-2)p_n = p_1' - p_3' - 2p_4' - \cdots - (n-3)p_{n-1}' = 2$$

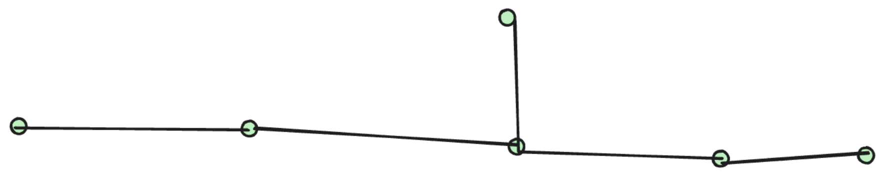
Case 2: The leaf was added back to the tree to a vertex u of degree $k > 1$. Then, $p_1 = p_1' + 1$ because we add a leaf, and the degree of the vertex that the leaf was added to is not of degree 1. Now, realize that the u was adding $-(k)$ to the sum in T'. But, now the vertex is adding $-(k+1)$ because an extra vertex was added. The difference in the previous and new sum of $p_k + p_{k+1}$ is -1. Combining these two results, we find that $-1 + 1 = 0$, so there is no change in the sum

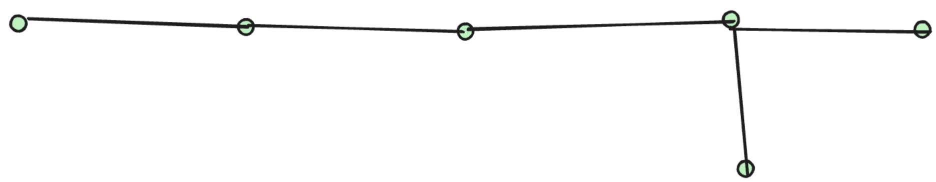
$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} - (n-2)p_n - p_1' - p_3' - 2p_4' - \cdots - (n-3)p_{n-1}' = 0$$

Hence,

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} - (n-2)p_n = 2$$

Problem 2: These two trees both have score of $(1, 1, 2, 2, 3)$ but are not isomorphic





Problem 3: By what we have proven in class, there is a unique encoding of every tree, that uses two bits to encode information about every vertex. Hence, if there are n vertices, there is a unique $2n$ string of bits to encode that tree. There are 2^{2n} possible strings of bits, so there are $2^{2n} = 4^n$ possible trees.