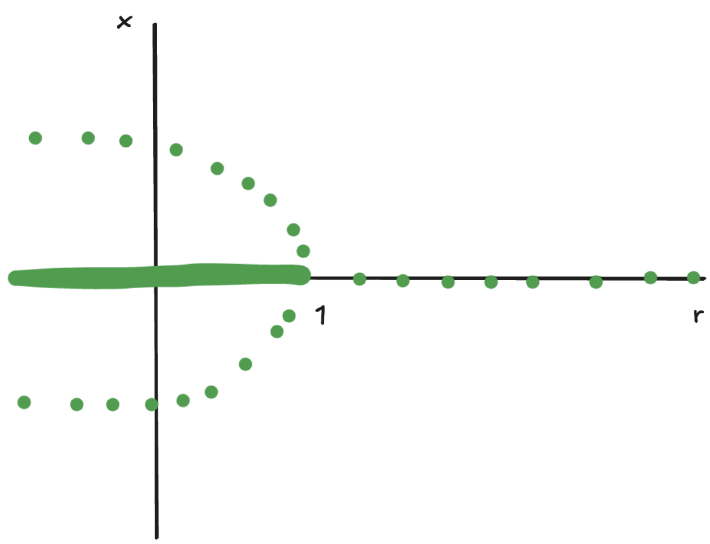
Problem 1: (3.4.8)

(a) $\frac{dx}{dt} = rx - \frac{x}{1+x^2} = 0$

$$x(r - \frac{1}{x^2+1}) = 0$$
$$x = 0, \quad r = \frac{1}{x^2+1}$$
$$\frac{df}{dt} = r + \frac{x^2-1}{(x^2+1)^2} = 0$$

Plugging in $x = 0$, we get $r = 1$. Thus, the only equilibrium point is $(0, 1)$. We get a similar result when plugging in $r = \frac{1}{x^2+1}$.

(b) Graphically, we can see that if $r < 1$ then there are two unstable equilibrium points and one stable equilibrium point at 0. If $r > 1$, only the $x = 0$ equilibrium point remains, and it is unstable. Hence, we have a subcritical pitchfork bifurcation at $r = 1$.



Problem 2: (4.4.1) $b\dot{\theta} + mgL\sin\theta = \tau - k\theta$

$$f(\theta) = \dot{\theta} = \frac{\tau - k\theta - mgL\sin\theta}{b}$$

If this was well defined on a circle, then $f(\theta) = f(\theta + 2\pi)$. However, because of the $k\theta$ term they are clearly different for all $k \neq 0$. Hence this vector field is only well defined on a circle if $k = 0$.

Problem 3: (a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x \\ y \end{bmatrix}$

Indeed, the characteristic polynomial is $\lambda^2 - 5\lambda + 6 = 0$ which has roots $\lambda = 2, 3$.

The eigenvector with eigenvalue 2 is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the eigenvector with eigenvalue 3 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Plotting the graph reveals a spiral. The vectors are pointing away from the origin, hence, the origin is a unstable spiral.