Problem 1: Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} \inf |a_n| = 0$.

If we show that $\{|a_{K_n}|\}$ converges, then $\{a_{K_n}\}$ must also converge.

Since $\lim_{n\to\infty} \inf |a_n| = 0$, we know that for all $\epsilon > 0$, there exists a $N_\epsilon \in \mathbb{N}$ such that

$$|\inf\{a_n : n \geq N\}| < \epsilon$$

for all $N \geq N_\epsilon$. Let's construct our subsequence by choosing the kth element in the subsequence by finding a a_n such that $|a_n| < \frac{1}{2^k}$. By setting our epsilon to $\frac{1}{2^n}$, we can such a_n by assumption.

Since the absolute sum of this subsequence is less than the geometric sum with ratio $\frac{1}{2}$ it must also converge. Since it converges absolutely, it also converges (not absolutely). Hence, we have found such subseqeunce.

Problem 2: (a) By the ratio test, $\sum_{n\geq 1} \frac{n^4}{2^n}$ converges. Notice that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{(n)^4}$$

Taking the limit of this, we get that $\lim_{n\to\infty} \frac{(n+1)^4}{2n^4} = \frac{1}{2}$. Since this sequence converges, $\lim_{n\to\infty} \sup \frac{(n+1)^4}{2n^4} = \frac{1}{2} < 1$. Hence, the sequence converges.

(b) By ratio test, $\sum \frac{2^n}{n!}$ converges. Notice that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1}$$

This sequence converges, and its limit is 0 which is less than 1. Hence, the series must converge.

(c) and (d) both terms do not converge to 0 as the n approaches ∞, hence the series both do not converge.

Problem 3: (a) Expanding the first few terms of the sum, we find that

$$\sum_{n\geq 2} \frac{1}{[n + (-1)^n]} = \sum_{n\geq 2} \frac{1}{n^2}$$

Since $\frac{1}{n^2}$ is a convergent series, then $\sum_{n\geq 2} \frac{1}{[n+(-1)^n]}$ must also be convergent.

(b) Using a telescoping argument, $s_n = \sqrt{n+1} - 1$. Since $\lim_{n\to infty} \sqrt{n} = \infty$ the series must also diverge to infinity

(c) The series converges by the ratio test.

$$\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)(n)^n}{(n+1)^{n+1}} = (\frac{n}{n+1})^n$$

This is less than 1, for all n because $\frac{n}{n+1} < 1$. Hence, $\lim_{n\to\infty} \frac{n}{n+1} < 1$ so the series converges

Problem 4: (a) Note that $n^{\ln n} = e^{\ln n^{\ln n}} = e^{\ln n \cdot \ln n}$. Similarly, $(\ln n)^n = e^{n \ln(\ln n)}$.
So,

$$\frac{n^{\ln n}}{(\ln n)^n} = \frac{e^{\ln n \cdot \ln n}}{e^{n \ln(\ln n)}} = e^{\ln n \cdot \ln n - n \ln(\ln n)}$$

It can be shown that $\frac{1}{e^{n \ln(\ln n) - \ln n \cdot \ln n}} < \frac{1}{e^n}$ for all $n \geq N$ for some $N \in \mathbb{N}$. Thus, since $\sum \frac{1}{e^n}$ converges, then $\sum \frac{1}{e^{n \ln(\ln n) - \ln n \cdot \ln n}}$ converges by comparison test

(b) Let's use the same idea. $(\ln n)^{\ln n} = e^{\ln((\ln n)^{\ln n})} = e^{\ln n \cdot \ln(\ln n)}$. For some $N \in N$ (around 1600), $e^{\ln n \cdot \ln(\ln n)} > n^2$ for all $n > N$. So, $\frac{1}{e^{\ln n \cdot \ln(\ln n)}} < \frac{1}{n^2}$ for all such n... Since $\sum \frac{1}{n^2}$ converges, by the comparison test, $\sum \frac{1}{e^{\ln n \cdot \ln(\ln n)}}$ also converges.

(c) Let's use the ratio test.

$$\left| (-1) \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{(n)!} \right| = \frac{n+1}{2}$$

Evidentally, $\lim_{n\to\infty} \inf \frac{n+1}{2} = \infty > 1$. Therefore, the series diverges.

Problem 5: (a) Take the series $a_n = \frac{1}{n^{0.6}}$. $0.6 < 1$ so the series must diverge. Note that $a_n^2 = \frac{1}{n^{1.2}}$. Since $1.2 > 1$ so the series must converge.

(b) Assume that $\sum |a_n|$ converges. We want to show that $\sum a_n^2$ converges. Since $\sum |a_n|$ converges we know that $\lim_{n\to\infty} |a_n| = 0$. Hence, there exists a $N \in \mathbb{N}$ such that $|a_n| < 1$ for all $n \geq N$. So, $|a_n| > |a_n|^2 = |a_n^2|$ for all $n \geq N$.

By comparison test, since $|a_n^2| < |a_n|$ and $\sum |a_n|$ converges, then $\sum |a_n^2|$ must converge. Since, the series converges absolutely, then $\sum a_n^2$ must also converge.

(c) Using facts from normal calculus (alternating series test), $\sum (-1)^n \frac{1}{n^{0.5}}$ converges. However, $((-1)^n \frac{1}{n^{0.5}})^2 = \frac{1}{n}$ which we know doesn't converge.

Problem 6: We can separate the fraction into two since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. By a telescoping argument, we can see that

$$\sum_{n\geq 1} (\frac{1}{n} - \frac{1}{n+1}) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1$$

Problem 7: (a) Using the hint, we can use a telescoping argument to show that

$$\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \frac{1}{2} - \lim_{n\to\infty} \frac{n+1}{2^{n+1}} = \frac{1}{2}$$

(b) We know that

$$\sum \frac{n-1}{2^{n+1}} = \frac{1}{2} \sum \frac{n-1}{2} = \frac{1}{2} (\sum \frac{n}{2^n} - \sum \frac{1}{2^n})$$

Using the geometric sum formula, $\sum_{n\geq 1} \frac{1}{2^n} = 1$. So,

$$\frac{1}{2}(\sum_{n\geq 1} \frac{n}{2^n} - 1) = \frac{1}{2}$$

Hence, $\sum_{n\geq 1} \frac{n}{2^n} = 2$.

Problem 8: (a) Assume that $\sum a_n$ diverges. We want to show that $\sum \frac{a_n}{a_n+1}$ diverges. Let's split this into two cases.

Case 1: Assume that $a_n \to 0$. Then at some N, $a_n < 1$ for all $n \geq N$.
Hence $a_n + 1 < 2$ and $\frac{a_n}{a_n+1} > \frac{a_n}{2}$. Note that because $\sum a_n = \infty$ then $\sum \frac{1}{2} a_n = \frac{1}{2} \sum a_n = \infty$.
Hence, by the comparison test, since $\frac{a_n}{a_n+1} > \frac{a_n}{2}$ for all $n \geq N$, then $\sum \frac{a_n}{a_n+1}$ must diverge.

Case 2: Assume that $a_n \not\to 0$. Then, there exists a $\epsilon > 0$ there are an infinitely many $a_n > \epsilon$. So, we can found a lower boudn for $\frac{a_n}{a_n+1} > \frac{\epsilon}{\epsilon+1}$.
We know that $\frac{a_n}{a_n+1} > \frac{\epsilon}{\epsilon+1} > 0$. So $\lim_{n\to\infty} \frac{a_n}{a_n+1} \neq 0$.
Hence, $\sum \frac{a_n}{a_n+1}$ diverges.

(b) Recall that $s_n = a_1 + ... + a_n$. So,

$$\sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+n}}{s_{N+n}}$$

and

$$1 - \frac{S_N}{S_{N+n}} = \frac{S_{N+n} - S_n}{S_{N+n}} = \frac{a_{N+n} + ... + a_{N+1}}{S_{N+n}}$$

Note that $S_{N+n} \geq S_{N+k}$ for all such k in the sum because $a_{n+k} \geq 0$.
Hence each, $\frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+k}}{S_{N+n}}$. Thus,

$$\sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+n}}{s_{N+n}} \geq \frac{a_{N+n} + ... + a_{N+1}}{S_{N+n}} = 1 - \frac{S_N}{S_{N+n}}$$

Notice that S_{N+n} approaches infinity since $\sum a_n$ diverges by assumption. Since S_N is a constant, $\lim_{n\to\infty} \frac{S_N}{S_{N+n}} = 0$.
Hence, we can find some n_ϵ such that $\frac{S_N}{S_{N+n}} < \frac{1}{2}$. So, $1 - \frac{S_N}{S_{N+n}} \geq \frac{1}{2}$.
Thus, if we partition the sum $\sum \frac{a_n}{s_n}$ into sets of n_ϵ values, for any $M > 0$, the sum of $2M$ such groupings must add up to greater than or equal to M. Therefore, the $\sum \frac{a_n}{s_n}$ diverges to infinity.

(c) Using some basic math, and the fact that $s_{n-1}s_n \leq s_n s_n$,

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{(s_{n-1})s_n} \geq \frac{a_n}{(s_n)^2}$$

Using this and a telescoping argument, we find that

$$\begin{aligned}\sum_{n\geq 1}\frac{a_n}{s_n^2} &= \frac{a_1}{s_1^2}+\sum_{n\geq 2}\frac{a_n}{s_n^2}\\ &\leq \frac{1}{a_n^2}+\sum_{n\geq 2}(\frac{1}{s_{n-1}}-\frac{1}{s_n})\\ &= \frac{1}{a_1^2}+\frac{1}{s_1^2}-\lim_{n\to\infty}\frac{1}{s_n^2}=1+\frac{2}{a_1^2}\end{aligned}$$

Assuming that $a_1 \neq 0$, this converges.

Problem 9: Let $s_n = \sum_{k=1}^n a_k$. We know that $\lim_{N\to\infty}\sum_{n\geq N} a_n = 0$ since $\lim_{n\to\infty} s_n$ converges.

Notice that

$$\sum_{n\geq N} a_n \geq \sum_{n=N}^{N+M} a_n \geq \sum_{n=N}^{N+M} a_{N+M} = (N+M)a_{N+M}$$

for all $N, M \in \mathbb{N}$ because the sequence is decreasing and non-negative.

Using the fact that $\lim_{N\to\infty}\sum_{n\geq N} a_n = 0$, for every $\epsilon > 0$, we can bound na_n this way. Thus, $\lim_{n\to\infty} na_n = 0$.