Problem 1: We want to show that $\mathbb{P}_1$, the measure induced by $Y = h \circ X$ on Ω is the same as $\tilde{\mathbb{P}}_1$, the measure induced on h on E_0. So,

$$\mathbb{P}_1(A) = \mathbb{P}_Y(A) \qquad \forall A \in \mathcal{E}_1$$
$$\tilde{\mathbb{P}}_1(A) = \mathbb{P}_X(h^{-1}(A)) \qquad \forall A \in \mathcal{E}_0$$

We know that $Y = h \circ X$. Let $A \in \mathcal{E}_1$. Stating some definitions of sets and functions, we get $Y^{-1}(A) = (h \circ X)^{-1}(A) = X^{-1}(h^{-1}(A))$.

Hence,

$$\tilde{\mathbb{P}}_1(A) = \mathbb{P}_X(h^{-1}(A)) = \mathbb{P}_X(X^{-1}(h^{-1}(A))) = \mathbb{P}_X(Y^{-1}(A)) = \mathbb{P}_Y(A) = \mathbb{P}_1(A)$$

Problem 2: Let $E_0 = V_{X_1} \times V_{X_2}$. Let $X : \Omega \to E_0$, so $X = (X_1, X_2)$ and let $h : V_{X_1} \times V_{X_2} \to V_Y$. Define $Y = h \circ X$. So, we know that $\mathbb{E}(h(X_1, X_2)) = \mathbb{E}(h(X)) = \mathbb{E}(Y)$.

By question 1, we know that $\mathbb{P}_Y(y) = \mathbb{P}_X(h^{-1}(y))$ for all $y \in V_Y$, since V_Y is discrete by assumption. So,

$$\sum_{y \in V_Y} y \mathbb{P}_Y(y) = \sum_{y \in V_Y} y \mathbb{P}_X(h^{-1}(y))$$

Realize that $\mathbb{P}_X(h^{-1}(y)) = \sum_{(x_1,x_2) \in E_0 \text{ s.t. } h(x_1,x_2=y)} \mathbb{P}_{\mathbb{X}}(x_1, x_2)$.

So,

$$\begin{aligned}
&= \sum_{y \in V_Y} y \cdot (\sum_{(x_1,x_2) \in E_0 \text{ s.t. } h(x_1,x_2=y)} \mathbb{P}_{\mathbb{X}}(x_1, x_2)) \\
&= \sum_{(x_1,x_2) \in E_0} h(x_1, x_2) \sum_{y \in V_Y \text{ s.t. } y=h(x_1,y_1)} \mathbb{P}_{\mathbb{X}}(x_1, x_2) \\
&= \sum_{(x_1,x_2) \in E_0} h(x_1, x_2) \mathbb{P}_{\mathbb{X}}(x_1, x_2) \\
&= \sum_{(x_1,x_2) \in E_0} h(x_1, x_2) f_{X_1,X_2}(x_1, x_2)
\end{aligned}$$

y/x	0	1	2
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$
1	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$

Problem 3:

$$g(x = 1 \mid y = 1) = \frac{2}{9}$$
$$g(x = 2 \mid y = 1) = \frac{3}{9}$$
$$g(x = 3 \mid y = 1) = \frac{4}{9}$$
$$g(x = 1 \mid y = 2) = \frac{3}{12}$$
$$g(x = 2 \mid y = 2) = \frac{4}{12}$$
$$g(x = 3 \mid y = 2) = \frac{5}{12}$$
$$h(y = 1 \mid x = 1) = \frac{2}{5}$$
$$h(y = 2 \mid x = 1) = \frac{3}{5}$$
$$h(y = 1 \mid x = 2) = \frac{3}{7}$$
$$h(y = 2 \mid x = 2) = \frac{4}{7}$$
$$h(y = 1 \mid x = 3) = \frac{4}{9}$$
$$h(y = 2 \mid x = 3) = \frac{5}{9}$$

Problem 4: Recall $\mathbb{P}(X = x) = \sum_{y \in V_Y} \mathbb{P}(X = x, Y = y)$. So, $\mathbb{P}(X = 1) = \frac{5}{21}$, $\mathbb{P}(X = 2) = \frac{7}{21}$, $\mathbb{P}(X = 3) = \frac{9}{21}$. Similarly, $\mathbb{P}(Y = 1) = \frac{9}{21}$ and $\mathbb{P}(Y = 2) = \frac{12}{21}$.

Hence,

$$\mathbb{E}(X) = \frac{5}{21} + 2 \cdot \frac{7}{21} + 3 \cdot \frac{9}{21} = \frac{5}{21} + \frac{14}{21} + \frac{27}{21} = \frac{46}{21} = \frac{46}{21}$$
$$\mathbb{E}(Y) = \frac{9}{21} + 2 \cdot \frac{12}{21} = \frac{33}{21} = \frac{11}{7}$$

So,

$$\begin{aligned}
\sigma^2(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{5}{21} + 4 \cdot \frac{7}{21} + 9 \cdot \frac{9}{21} - (\frac{46}{21})^2 \\
&= \frac{5}{21} + \frac{28}{21} + \frac{81}{21} - \frac{2116}{441} \\
&= \frac{114}{21} - \frac{2116}{441} = \frac{278}{441} \\
\sigma^2(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\
&= \frac{9}{21} + 4 \cdot \frac{12}{21} - (\frac{11}{7})^2 \\
&= \frac{9}{21} + \frac{48}{21} - \frac{121}{49} \\
&= \frac{19}{7} - \frac{121}{49} = \frac{12}{49} \\
\sigma(X,Y) &= \mathbb{E}((X-\mu_X)(Y-\mu_Y)) \\
&= \sum_{(x,y)\in V_X \times V_Y} (X-\mu_X)(Y-\mu_Y) f_{X,Y}(x,y) \\
&= (1-\frac{46}{21})(1-\frac{11}{7})\frac{2}{21} + (1-\frac{46}{21})(2-\frac{11}{7})\frac{3}{21} \\
&+ (2-\frac{46}{21})(1-\frac{11}{7})\frac{3}{21} + (2-\frac{46}{21})(2-\frac{11}{7})\frac{4}{21} \\
&+ (3-\frac{46}{21})(1-\frac{11}{7})\frac{4}{21} + (3-\frac{46}{21})(2-\frac{11}{7})\frac{5}{21} \\
&= -\frac{2}{147}
\end{aligned}$$