Theorem 1 (Mean Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there exists some $c \in (a,b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define the auxiliary function:

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

Notice that g is continuous on $[a,b]$ and differentiable on (a,b), because f is and the other terms are polynomials. Moreover,

$$g(a) = f(a)$$
$$g(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a}\right)(b - a) = f(b) - (f(b) - f(a)) = f(a)$$

Thus, $g(a) = g(b)$. By Rolle's Theorem, there exists some $c \in (a,b)$ such that $g'(c) = 0$.

Calculating $g'(x)$:

$$g'(x) = f'(x) - \left(\frac{f(b) - f(a)}{b - a}\right)$$

Setting $g'(c) = 0$ gives:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as desired. □