

# Homework 3

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1. Consider the linear program (LP)  $\min c^T x$  such that  $Ax = b, x \geq 0$  where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad c = [3 \quad -4 \quad 5]$$

- a) Find all basic feasible solutions. (There are three possibilities, two of which are basic feasible solutions and one isn't. Be clear why the third possibility fails to be a BFS.)

*Solution.* Suppose  $x_3 = 0$ , then

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \implies B^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \\ x_B &= B^{-1}b = -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix} \\ x &= \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \end{bmatrix} \end{aligned}$$

is a BFS.

Now suppose  $x_2 = 0$ , then

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \implies B^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \\ x_B &= B^{-1}b = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

is a BFS.

Now suppose  $x_1 = 0$ , then

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ x_B &= B^{-1}b = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix} \end{aligned}$$

and this is not a BFS because  $x_B \not\geq 0$ .

□

- b) Evaluate  $r_N$  for each of the two basic feasible solutions.

*Solution.* We have

$$r_N^T = c_N^T - c_B^T B^{-1} N$$

as the expression for  $r_n^T$ .

For  $x_B = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$ , we have  $c_N^T = [5]$  and  $c_B^T = [3 \quad -4]$  and  $N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , so

$$\begin{aligned} r_N^T &= [5] - [3 \quad -4] \left( -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [5] - [3 \quad -4] \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \\ &= [5] - [2/3] = [13/3] \end{aligned}$$

For  $x_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , we have  $c_N^T = [-4]$  and  $c_B^T = [3 \quad 5]$  and  $N = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , so

$$\begin{aligned} r_N^T &= [-4] - [3 \quad 5] \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= [-4] - [3 \quad 5] \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ &= [-4] - [9] = [-13] \end{aligned}$$

□

- c) Identify the optimal solution to the (LP) and explain why it is the optimal solution.

*Solution.* Since we have

$$c^T x = c_B^T x_B + r_N^T x_N,$$

the optimal solution corresponds with the  $r_N^T$  value that is non-negative, since any increases to  $x_N$  will increase the objective function value. This occurs when  $r_N^T = [13/3]$ , and the solution that corresponds with that is

$$x = \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \end{bmatrix}$$

The corresponding objective value function at this solution is  $c^T x = \boxed{11/3}$ .

□

2. Consider the linear program (LP)  $\min c^T x$  such that  $Ax = b, x \geq 0$  where

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

- a) Compute the BFS that comes from using the second, third, and fourth columns of  $A$  as the basis, and compute the associated vector  $r_N$ .

*Solution.* If we use the second, third, and fourth columns as a basis, we have

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \\ x_B &= B^{-1}b = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

where the inverse  $B^{-1}$  was calculated with MATLAB.

Now, we have

$$c_N^T = [1 \quad 1 \quad 1 \quad 1], \quad c_B^T = [1 \quad 1 \quad 1], \quad N = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so the vector  $r_N^T$  is given by

$$\begin{aligned} r_N^T &= [1 \quad 1 \quad 1 \quad 1] - [1 \quad 1 \quad 1] \left( \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \right) \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= [1 \quad 1 \quad 1 \quad 1] - [1 \quad 1 \quad 1] \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 7 & 1 & 1 & 1 \end{bmatrix} \\ &= [1 \quad 1 \quad 1 \quad 1] - [5 \quad 1 \quad 1 \quad 1] \\ &= [-4 \quad 0 \quad 0 \quad 0] \end{aligned}$$

□

- b) By examining the vector  $r_N$  from part a, explain how changing the different nonbasic variables will affect the objective function.

*Solution.* A unit increase of  $x_1$  will decrease the objective function value by 4 units (and increase by 4 if  $x_1$  is decreased by 1 unit). Changing the other non-basic variables  $x_5, x_6, x_7$  will not affect the objective function value.

□

- c) Now compute the BFS that comes from using the first three columns of  $A$  as the basis, and compute the associated vector  $r_N$ .

*Solution.* Now, we have

$$\begin{aligned} B &= \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix} \implies B^{-1} = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \\ x_B &= B^{-1}b = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix} \end{aligned}$$

where the inverse  $B^{-1}$  was calculated with MATLAB.

Now, we have

$$c_N^T = [1 \quad 1 \quad 1 \quad 1], \quad c_B^T = [1 \quad 1 \quad 1], \quad N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so the vector  $r_N^T$  is given by

$$\begin{aligned}
 r_N^T &= [1 \quad 1 \quad 1 \quad 1] - [1 \quad 1 \quad 1] \left( \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
 &= [1 \quad 1 \quad 1 \quad 1] - [1 \quad 1 \quad 1] \left( \frac{1}{7} \right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
 &= [1 \quad 1 \quad 1 \quad 1] - \frac{3}{7} [1 \quad 1 \quad 1 \quad 1] \\
 &= [4/7 \quad 4/7 \quad 4/7 \quad 4/7]
 \end{aligned}$$

□

d) Find the optimal solution for (LP).

*Solution.* Since  $r_N^T$  corresponding to the basis being the first 3 columns is entirely non-negative, the optimal solution occurs when

$$x_B = \begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix} \implies x = \begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the objective function value is  $3/7$ .

□

3. Suppose you are handed a linear program which is given to you in the form  $\min c^T x$  such that  $Ax \leq b, x \geq 0$ , and suppose that the vector  $b$  happens to be non-negative. This linear program is currently not in standard form; explain how to then immediately identify a BFS once it is converted to standard form. Then explain what  $r_N$  will be for this BFS.

*Solution.* We add a dummy variable  $z$ , so that

$$[A \mid I] \begin{bmatrix} x \\ z \end{bmatrix} = b$$

where  $I$  is the identity matrix. Then if we simply let  $I$  be the basis, we have  $Iz = b$  so then  $\begin{bmatrix} \vec{0} \\ z \end{bmatrix}$  will be a BFS.

Then the objective function will be  $\begin{bmatrix} c^T \\ \vec{0} \end{bmatrix}^T \begin{bmatrix} x \\ z \end{bmatrix}$ . Let the combined vector  $\begin{bmatrix} c^T \\ \vec{0} \end{bmatrix} = d^T$ . Here,

$$B = I \implies B^{-1} = I,$$

and we have

$$d_N^T = c^T, \quad d_B^T = \vec{0}, \quad N = A$$

so  $r_N^T$  is given by

$$\begin{aligned}
 r_N^T &= d_N^T - d_B^T B^{-1} N \\
 &= c^T - \vec{0} I A \\
 &= c^T.
 \end{aligned}$$

□