Homework 7

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1. (a) *Solution.* We have

$$\begin{aligned}
\sigma_M^2 &= \text{Var}\left(\frac{1}{2}r_A + \frac{1}{2}r_B\right) = \frac{1}{4}\sigma_A^2 + \frac{1}{4}\sigma_B^2 + \frac{1}{4}\sigma_{AB} \\
\beta_A &= \frac{\text{Cov}\left(r_A, \frac{1}{2}r_A + \frac{1}{2}r_B\right)}{\sigma_M^2} = \frac{\frac{1}{2}\sigma_A^2 + \frac{1}{2}\sigma_{AB}}{\frac{1}{4}\sigma_A^2 + \frac{1}{4}\sigma_B^2 + \frac{1}{4}\sigma_{AB}} = \frac{2\sigma_A^2 + 2\sigma_{AB}}{\sigma_A^2 + \sigma_B^2 + \sigma_{AB}} \\
\beta_b &= \frac{2\sigma_B^2 + 2\sigma_{AB}}{\sigma_A^2 + \sigma_B^2 + \sigma_{AB}}
\end{aligned}$$

□

(b) *Solution.* According to the CAPM, we have

$$\begin{aligned}
\bar{r}_A &= r_f + \beta_A(\bar{r}_M - r_f) = 0.10 + \frac{2 \cdot 0.04 + 2 \cdot 0.01}{0.04 + 0.02 + 0.01}(0.18 - 0.10) = 21.4\% \\
\bar{r}_B &= r_f + \beta_B(\bar{r}_M - r_f) = 0.10 + \frac{2 \cdot 0.02 + 2 \cdot 0.01}{0.04 + 0.02 + 0.01}(0.18 - 0.10) = 16.9\%
\end{aligned}$$

□

2. (a) *Solution.* Since the market portfolio is efficient, its weights can be represented as a linear combination of the two portfolios on the minimum variance set

$$\alpha \begin{bmatrix} 0.60 \\ 0.20 \\ 0.20 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0.80 \\ -0.20 \\ 0.40 \end{bmatrix} = \begin{bmatrix} 0.8 - 0.2\alpha \\ -0.2 + 0.4\alpha \\ 0.4 - 0.2\alpha \end{bmatrix}$$

Market portfolio weights must be non-negative, so from the above, we get the bounds $0.5 \le \alpha \le 2$. The expected return of the market portfolio is bounded by

$$\begin{aligned}
\begin{bmatrix} 0.8 - 0.2\alpha \\ -0.2 + 0.4\alpha \\ 0.4 - 0.2\alpha \end{bmatrix}^T \begin{bmatrix} 0.10 \\ 0.20 \\ 0.10 \end{bmatrix} &= 0.08 + 0.04\alpha \\
\implies 0.10 = 0.08 + 0.04 \cdot 0.5 \le \bar{r}_M &\le 0.08 + 0.04 \cdot 2 = 0.16
\end{aligned}$$

□

(b) *Solution.* We have

$$r_w = 0.6 \cdot 0.1 + 0.2 \cdot 0.2 + 0.2 \cdot 0.1 = 12\%$$

Since w is the minimum variance point and the market portfolio is efficient, its lower bound is 12%, and the upper bound is still 16%. □

3. *Solution.* If r_i is the rate of return for asset i, then we have

$$\sigma_M^2 = \mathrm{Var}\left(\sum_{i=1}^n x_i r_i\right) = \sum_{i=1}^n x_i \sigma_i^2$$

since the assets are uncorrelated. We also have

$$\sigma_{jM} = \mathrm{Cov}\left(r_j, \sum_{i=1}^n x_i r_i\right) = \sum_{n=1}^n \mathrm{Cov}(r_j, x_i r_i) = x_j \sigma_j^2$$

so then

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2} = \frac{x_j \sigma_j^2}{\sum_{i=1}^n x_i \sigma_i^2}$$

□

4. (a) *Solution.* The market consists of 40% stock A and 60% stock B, so we have

$$\bar{r}_M = 40\% \cdot 15\% + 60\% \cdot 12\% = 13.2\%$$

□

(b) *Solution.* As above, the market portfolio is 0.4 stock and 0.6 stock B, so we have

$$\begin{aligned}\sigma_M &= \sqrt{\mathrm{Var}(0.4r_A + 0.6r_B)} = \sqrt{0.16\sigma_A^2 + 0.36\sigma_B^2 + 0.24\sigma_{AB}} \\ &= \sqrt{0.16 \cdot 0.15^2 + 0.36 \cdot 0.09^2 + 0.24 \cdot 0.15 \cdot 0.09 \cdot \frac{1}{3}} = 8.72\%\end{aligned}$$

□

(c) *Solution.* We have

$$\begin{aligned}\beta_A &= \frac{\mathrm{Cov}(r_A, r_M)}{\sigma_M^2} = \frac{\mathrm{Cov}(r_A, 0.4r_A + 0.6r_B)}{\sigma_M^2} = \frac{0.4\sigma_A^2 + 0.6\sigma_{AB}}{\sigma_M^2} \\ &= \frac{0.4 \cdot 0.15^2 + 0.6 \cdot 0.15 \cdot 0.09 \cdot \frac{1}{3}}{0.007596} = 4.74\end{aligned}$$

□

(d) *Solution.* According to CAPM, we have

$$\bar{r}_A - r_f = \beta_A(\bar{r}_M - r_f) \implies r_f = \frac{\beta_A \bar{r}_M - \bar{r}_A}{\beta_A - 1} = 11.2\%$$

□

5. (a) *Solution.* Consider the portfolio $(1-\alpha)w_0 + \alpha w_1$. The variance of its return is

$$\sigma^2 = \mathrm{Var}((1-\alpha)r_0 + \alpha r_1) = (1-\alpha)^2\sigma_0^2 + \alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{01}$$

Taking its derivative with respect to α and evaluating at $\alpha = 0$, we have

$$\begin{aligned}&\frac{\partial \sigma^2}{\partial \alpha} = -2(1-\alpha)\sigma_0^2 + 2\alpha\sigma_1^2 + (2-4\alpha)\sigma_{01} = -2\sigma_0^2 + 2\sigma_{01} = 0 \\ &\implies \sigma_{01} = \sigma_0^2 \implies A = 1\end{aligned}$$

□

(b) *Solution.* Using $w_z = (1-\alpha)w_0 + \alpha w_1$, we have

$$\begin{aligned} \sigma_{1,z} &= \operatorname{Cov}\left(r_1, (1-\alpha)r_0 + \alpha r_1\right) = 0 \\ \implies (1-\alpha)\sigma_{01} + \alpha\sigma_1^2 &= 1 - \alpha\sigma_0^2 + \alpha\sigma_1^2 = 0 \\ \implies \alpha &= \frac{1}{\sigma_0^2 - \sigma_1^2} \end{aligned}$$

□

(c) *Solution.* w_0 is the minimum variance point, so it is the left-most point on the feasible region. w_1 is any point on the efficient frontier, and w_z is a point on the bottom half of the minimum variance set. □

(d) *Solution.* We have

$$\begin{aligned} \bar{r}_i &= \bar{r}_z + \beta_{iM}(\bar{r}_M - \bar{r}_z) = \bar{r}_z + \frac{\rho_{iM}\sigma_i\sigma_M}{\sigma_M^2}(\bar{r}_M - \bar{r}_z) \\ &= 0.09 + \frac{0.5 \cdot 0.15 \cdot 0.05}{0.15^2}(0.15 - 0.09) = 10\% \end{aligned}$$

□