Homework 10

Aleck Zhao

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Section 3.4: Homomorphisms

3. Show that a general ring homomorphism $\theta : \mathbb{Z} \to \mathbb{Z}$ is either a ring isomorphism or $\theta(k) = 0$ for all $k \in \mathbb{Z}$.

   Proof. A homomorphism is determined entirely by the value of $\theta(1)$. Suppose $\theta(1) = a \in \mathbb{Z}$, so $\theta(k) = ka$ for all $k \in \mathbb{Z}$. If $a \neq =$, then the image $\theta``(\mathbb{Z}) \cong a\mathbb{Z}$, but we know that $\mathbb{Z} \cong a\mathbb{Z}$ by the isomorphism $\varphi(n) = an$ for $n \in \mathbb{Z}$. Thus, the image is isomorphic to $\mathbb{Z}$ if $a \neq 0$. Otherwise, $a = 0$ and $\theta(k) = 0$ for all $k \in \mathbb{Z}$, as desired. □

4. Determine all onto general ring homomorphisms $\mathbb{Z}_{12} \to \mathbb{Z}_6$.

   Solution. The homomorphism is determined entirely by the value that 1 is mapped to. The homomorphism given by $1 \mapsto 1$ is obviously onto and a homomorphism. We can't have $1 \mapsto 2$ because then there is no element that maps to 1. Similarly we can't have $1 \mapsto 3$ or $1 \mapsto 4$ otherwise nothing maps to 1. However, we can have $1 \mapsto 5$, and this determines another homomorphism. Finally, $1 \mapsto 0$ is obviously not onto. Thus, $1 \mapsto 1$ and $1 \mapsto 5$ are the only onto general ring homomorphisms. □

20. If $n > 0$ in $\mathbb{Z}$, describe all the ideals of $\mathbb{Z}$ that contain $n\mathbb{Z}$.

    Solution. The only ideals of $\mathbb{Z}$ are of the form $k\mathbb{Z}$. If an ideal $k\mathbb{Z}$ contains $n\mathbb{Z}$, then we must have $k \mid n$, and these are all ideals that contain $n\mathbb{Z}$. □

Section 4.1: Polynomials

2. (c) Compute $(1 + x)^5$ in $\mathbb{Z}_5[x]$.

   Solution. This expands to

   $$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \equiv 1 + x^5$$

   in $\mathbb{Z}_5[x]$. □

4. (a) Find all roots of $(x - 4)(x - 5)$ in $\mathbb{Z}_6$; in $\mathbb{Z}_7$.

   Solution. In $\mathbb{Z}_6$, the possibilities are

   $$\begin{aligned} x - \bar{4} = \bar{0} &\implies x = \bar{4} \\ x - \bar{5} = \bar{0} &\implies x = \bar{5} \\ (x - \bar{4}) = \bar{2}, (x - \bar{5}) = \bar{3} &\implies \text{no solution} \\ (x - \bar{4}) = \bar{3}, (x - \bar{5}) = \bar{2} &\implies x = \bar{1} \end{aligned}$$

   Thus, in $\mathbb{Z}_6$, the solutions are $x = \bar{1}, \bar{4}, \bar{5}$.
   Since $\mathbb{Z}_7$ is an integral domain, the only possibilities are $x = \bar{4}, \bar{5}$. □

13. Divide $x^3 - 4x + 5$ by $2x + 1$ in $\mathbb{Q}[x]$. Why is it impossible in $\mathbb{Z}[x]$?

Solution. We have
$$x^3 - 4x + 5 = \left(\frac{1}{2}x^2 - \frac{1}{4}x - \frac{15}{8}\right) \cdot (2x+1) + \frac{55}{8}$$
The division is impossible in $\mathbb{Z}[x]$ because $2x+1$ is not monic, and quotients don't make sense in $\mathbb{Z}$. □

24. If R is a commutative ring, a polynomial f in $R[x]$ is said to **annihilate** R if $f(a) = 0$ for every $a \in R$.

(a) Show that $x^p - x$ annihilates $\mathbb{Z}_p$.

Proof. By Fermat's Little Theorem, we have $x^p \equiv x \pmod p$, so $x^p - x \equiv 0 \pmod p$, thus $x^p - x$ annihilates $\mathbb{Z}_p$, as desired. □

Section 4.2: Factorization of Polynomials over a Field

5. (a) Determine whether the polynomial $x^2 - 3$ is irreducible over each of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7$.

Solution. This degree 2 polynomial has no solution in $\mathbb{Q}$, so it is irreducible over $\mathbb{Q}$. In $\mathbb{R}$, it factors as $(x - \sqrt{3})(x + \sqrt{3})$, so it is reducible in $\mathbb{R}$, and also in $\mathbb{C}$.
In $\mathbb{Z}_2$, if $x = \bar{1}$, then $x^2 - 3 = 0$, so the polynomial has a root and is not irreducible. In $\mathbb{Z}_3$, we have $x = \bar{0}$ is a solution, so it is not irreducible. In $\mathbb{Z}_5$, the values of x^2 are $0, 1, 4$, so we can't have $x^2 - 3 = 0$, thus it is irreducible. In $\mathbb{Z}_7$, the values of x^2 are 0, 1, 2, 4, so we can't have $x^2 - 3 = 0$, thus it is irreducible. □

9. Show that an odd degree polynomial has a real root.

Proof. By Theorem 4, every polynomial in $\mathbb{R}[x]$ factors as
$$f = a(x - r_1)(x - r_2)\cdots(x - r_m)q_1q_2\cdots q_k$$
where $r_i \in \mathbb{R}$ and q_i are irreducible quadratics. The degree of the RHS is $m + 2k$, which is also the degree of f, which is odd. Thus, m must be odd, so it is at least 1, and there is at least 1 real root, as desired. □

10. Find all monic irreducible cubics in $\mathbb{Z}_2[x]$.

Solution. Let $f(x) = x^3 + ax^2 + bx + c$ be an irreducible cubic in $\mathbb{Z}_2[x]$. Since f has degree 3, it is irreducible if and only if it has no roots in $\mathbb{Z}_2$. That is, $f(\bar{0}) \neq \bar{0}$ and $f(\bar{1}) \neq \bar{0}$, so they must be equal to $\bar{1}$. This means
$$\begin{aligned} 1 + a + b + c &= 1 \\ c &= 1 \\ \implies a + b &= 1 \end{aligned}$$
Thus, either $a = 1, b = 0$ or $a = 0, b = 1$, so the irreducible monic cubics in $\mathbb{Z}_2[x]$ are
$$x^3 + x^2 + 1, \qquad x^3 + x + 1$$
□