Advanced Algebra I **Print Name: Aleck Zhao**
Fall 2016
2nd Midterm Exam (take home)
12/9-12/2016
Time Limit: 72 Hours **Johns Hopkins University**

This exam contains 7 pages (including this cover page) and 6 questions. Total of points is 100.

You must answer the first 4 questions, and then answer <u>one</u> of question 5 or 6. Do not answer both. No extra points will be rewarded. Place an "X" through the question-page you are not going to answer.

The use of books, notes and any sort of human or web-based external help are NOT allowed.
Be sure to show all work for all problems. No credit will be given for answers without work shown.

Academic Honesty Certification

I certify that I have taken this exam without the aid of unauthorized people or objects.

Signature: Aleck Zhao

Grade Table (for instructor use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	100	

1. (20 points) Let G be a group of order 6.

   (a) (5 points) How many 3-Sylow subgroups are there in G?

   (b) (5 points) Show that G contains at least one subgroup of order 2.

   Assume, for the remaining part of the exercise, that G is not cyclic.

   (c) (5 points) Let H be a subgroup of G of order 2. Consider the set $\Omega = \{aH | a \in G\}$ of left cosets of H in G. G acts on Ω as follows:

   $$G \times \Omega \to \Omega, \qquad (g, aH) \mapsto gaH, \qquad \forall g \in G,\ \forall a \in G.$$

   Determine the cardinality $|\Omega|$ of Ω.

   (d) (5 points) Let

   $$\varphi : G \to S_{|\Omega|}, \qquad \varphi(g)(aH) = gaH$$

   be the group homomorphism of G into the group of permutations of Ω. Determine $\mathrm{Ker}(\varphi)$.

2. (20 points) Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 10 & 9 & 8 & 11 & 7 & 3 & 2 & 6 & 12 & 5 & 4 & 1 \end{pmatrix}$$

be a permutation of the set $X_{12} = \{1, 2, 3, \ldots, 12\}$. Compute σ^{2000}.

3. (20 points) Let $A = C([0,1], \mathbb{R})$ be the ring of continuous (for the Euclidean topology) functions $f : [0,1] \to \mathbb{R}$ and let $I \subset A$ be the subset of functions $f \in A$ such that $f(1/2) = 0$.
   (a) (5 points) Show that I an ideal of A.
   (b) (5 points) Is I a prime ideal? Prove it or disprove it.
   (c) (10 points) Is I a maximal ideal? Prove it or disprove it.

4. (20 points) Consider the polynomial $f(X) = X^2 + 2X + 3$ in $\mathbb{Z}_5[X]$.
   (a) (5 points) Is $f(X)$ irreducible in $\mathbb{Z}_5[X]$? If yes, prove it, if not determine a proper factorization of $f(X)$ in $\mathbb{Z}_5[X]$.
   (b) (10 points) Let $I = (f(X))$ be the principal ideal in $\mathbb{Z}_5[X]$ generated by $f(X)$. Consider the factor ring $F = \mathbb{Z}_5[X]/I$.
   Prove that the coset $\overline{X} := X + I$ is invertible in F (i.e. find its multiplicative inverse) and determine the order of $\overline{X}$ in the multiplicative group $F^\times$ of units of F.
   (c) (5 points) Find, if exists, a coset of order 3 in $F^\times$.

5. (20 points) **Answer this question OR 6.**

   A *local ring* A is a commutative, unital ring with a unique maximal ideal. Which of the following rings is local? For each ring, show or provide a counterexample to the statement: "the ring is local".

   (a) (10 points) $A = \mathbb{Z}/p^r\mathbb{Z}$ (p = prime number, $r \in \mathbb{N}$).

   (b) (10 points) $A_1 = \mathbb{Z}_p[X]$ ring of polynomials in X with coefficients in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

6. (20 points) **Answer this question OR 5.**

   (a) (10 points) Let p be a prime number. Consider the polynomial $f(X) = X^p - pX - 1$. Prove or disprove the following statement:

   $$f(X) \text{ is irreducible in } \mathbb{Q}[X].$$

   (b) (10 points) Consider the polynomial $g(X) = X^4 + 5X^2 + 3X + 2$. Prove or disprove the following statement:

   $$g(X) \text{ is irreducible in } \mathbb{Q}[X]$$

   .