Homework 7

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Section 2.10: The Isomorphism Theorem

22. Show that $\mathbb{R}^*/\{1,-1\} \cong \mathbb{R}^+$.

Solution. Define the mapping $\varphi : \mathbb{R}^* \to \mathbb{R}^+$ given by $\varphi(x) = x^2$ for $x \in \mathbb{R}^*$. This is indeed a homomorphism:

$$\varphi(xy) = (xy)^2 = x^2y^2 = \varphi(x)\varphi(y)$$

and the kernel is the set $\{1,-1\}$ since $\varphi(1) = \varphi(-1) = 1$. Here, the image of $\mathbb{R}^*$ under φ is exactly $\mathbb{R}^+$, since the square of non-zero elements of $\mathbb{R}$ are positive. Thus, by the Isomorphism theorem,

$$\varphi``(\mathbb{R}^*) = \mathbb{R}^+ \cong \mathbb{R}^*/\ker\varphi = \mathbb{R}^*/\{1,-1\}$$

as desired.

□

29. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$.

(a) Show that G is a subgroup of $M_3(\mathbb{R})^*$ and that $Z(G) \cong \mathbb{R}$.

Proof. Clearly $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G$ which is the identity in $\mathrm{GL}_3(\mathbb{R})$. Then let

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$
$$M = \begin{bmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$$

be in G, so their product

$$AM = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m+a & n+ap+b \\ 0 & 1 & p+c \\ 0 & 0 & 1 \end{bmatrix}$$

is also in G. Finally, the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

which is also in G. Thus, G is a subgroup of $\mathrm{GL}_3(\mathbb{R}_)$, as desired.
Let $M \in Z(G)$. Then we have

$$AM = \begin{bmatrix} 1 & m+a & n+ap+b \\ 0 & 1 & p+c \\ 0 & 0 & 1 \end{bmatrix}$$
$$MA = \begin{bmatrix} 1 & a+m & b+mc+n \\ 0 & 1 & c+p \\ 0 & 0 & 1 \end{bmatrix}$$

so since $M \in Z(G)$, we must have $AM = MA$, which is equivalent to having $n+ap+b = b+mc+n$ or $ap = mc$. Since a and c can be anything, it must be the case that $m = p = 0$. Thus, the general form of $M \in Z(G)$ is

$$M = \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad n \in \mathbb{R}$$

and we can construct a mapping $\varphi : Z(G) \to \mathbb{R}$ where

$$\varphi\left(\begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = n$$

which is obviously bijective. It is also a homomorphism because

$$\varphi\left(\begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & m \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & 0 & m+n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = m+n$$
$$\varphi\left(\begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + \varphi\left(\begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = m+n$$

Thus $Z(G) \cong \mathbb{R}$, as desired.

$\square$

(b) Show that $G/Z(G) \cong \mathbb{R} \times \mathbb{R}$.

Proof. Construct a mapping $\varphi : G \to \mathbb{R} \times \mathbb{R}$ where

$$\varphi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, c)$$

If we take A and M as above, then

$$AM = \begin{bmatrix} 1 & m+a & n+ap+b \\ 0 & 1 & p+c \\ 0 & 0 & 1 \end{bmatrix}$$

so $\varphi(AM) = (m+a, p+c)$ while

$$\varphi(A) + \varphi(M) = (a, c) + (m, p) = (a+m, c+p)$$

so φ is a homomorphism where $\varphi``(G) = \mathbb{R} \times \mathbb{R}$ since $a, c \in \mathbb{R}$. Here, $\ker \varphi$ is exactly the set of matrices where $b \in \mathbb{R}$ and $(a, c) = (0, 0)$, so $a = c = 0$, which is exactly $Z(G)$. Thus, by the Isomorphism theorem,

$$\varphi``(G) = \mathbb{R} \times \mathbb{R} \cong G/\ker \varphi = G/Z(G)$$

as desired.

$\square$

Section 8.2: Cauchy's Theorem

7. If H and K are conjugate subgroups in G, show that $N(H)$ and $N(K)$ are conjugate.

Proof. Let $H = g_0 K g_0^{-1}$ for some $g_0 \in G$. Then we have

$$\begin{aligned} N(H) &= \{\, g \in G \mid gHg^{-1} = H \,\} \\ &= \{\, g \in G \mid g(g_0 K g_0^{-1})g^{-1} = g_0 K g_0^{-1} \,\} \\ &= \{\, g \in G \mid (g_0^{-1} g g_0)K(g_0^{-1} g^{-1} g_0) = K \,\} \\ &= \{\, g \in G \mid (g_0^{-1} g g_0)K(g_0^{-1} g g_0)^{-1} = K \,\} \end{aligned}$$

so the set given by $g_0^{-1}N(H)g_0$ is exactly

$$g_0^{-1}N(H)g_0 = \{\, g_0^{-1} g g_0 \in G \mid (g_0^{-1} g g_0)K(g_0^{-1} g g_0) = K \,\} = \{\, g_1 \in G \mid g_1 K g_1^{-1} = K \,\} = N(K)$$

so $N(H) = g_0 N(K) g_0^{-1}$, thus $N(H)$ and $N(K)$ are conjugate, as desired.

□

14. Let $D_3 = \left\{1, a, a^2, b, ba, ba^2\right\}$ where $o(a) = 3, o(b) = 2, aba = b$. If $H = \{1, b\}$, show that $N(H) = H$.

Proof. Since $N(H)$ is a subgroup of D_3 so its order must divide 6. Since H is a subgroup of D_3, it is also a subgroup of $N(H)$, so $|H| = 2 \mid |N(H)|$. Thus, $|N(H)|$ is even and divides 6, so $|N(H)| = 2$ or 6. $N(H)$ can't possibly be all of D_3 since $ab \neq ba$, so we must have $|N(H)| = 2$, so in fact $N(H) = H$ since $N(H)$ contains H.

□

23. Let G^ω be the group of sequences $[g_i) = (g_0, g_1, \cdots)$ from a group G with component-wise multiplication $[g_i) \cdot [h_i) = [g_i h_i)$. Show that if $G \neq \{1\}$ is a finite p-group, then G^ω is an infinite p-group.

26. Let G be a non-abelian group of order p^3 where p is a prime. Show that
    (a) $Z(G) = G'$ and this is the unique normal subgroup of G of order p.
    (b) G has exactly $p^2 + p - 1$ distinct conjugacy classes.

Section 8.3: Group Actions

3. If p and q are primes, show that no group of order pq is simple.

13. Let $G = (\mathbb{R}, +)$ and define $a \cdot z = e^{ia}z$ for all $z \in \mathbb{C}$ and $a \in G$. Show that $\mathbb{C}$ is a G-set, describe the action geometrically, and find all orbits and stabilizers.

21. If H is a subgroup of G, find a G-set X and an element $x \in X$ such that $H = S(x)$.

23. Let X be a G-set and let x and y denote elements of X.
    (a) Show that $S(X)$ is a subgroup of G.
    (b) If $x \in X$ and $b \in G$, show that $S(b \cdot x) = bS(x)b^{-1}$.
    (c) If $S(x)$ and $S(y)$ are conjugate subgroups, show that $|G \cdot x| = |G \cdot y|$.