Homework 6

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Section 2.6: Cosets and Lagrange's Theorem

4. If $K \subseteq H \subseteq G$ are finite groups, show that $|G : K| = |G : H| \cdot |H : K|$.

   Proof. For finite groups, we have $|G : K| = |G| / |K|$ and similarly for the other two, so we have

   $$\frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|}$$

   as desired. $\square$

15. If H and K are subgroups of a group and $|H|$ is prime, show that either $H \subseteq K$ or $H \cap K = \{1\}$.

27. Is $D_5 \times C_3 \cong D_3 \times C_5$? Prove your answer.

Section 2.8: Normal Subgroups

4. If $D_4 = \left\{1, a, a^2, a^3, b, ba, ba^2, ba^3\right\}, K = \{1, b\}$ and $H = \left\{1, a^2, b, ba^2\right\}$ show that $K \trianglelefteq H \trianglelefteq D_4$, but $K \not\trianglelefteq D_4$.

   Proof. Since $|H : K| = 2$, by section 2.8 theorem 4, K is normal in H. Similarly, $|D_4 : H| = 2$, so H is normal in D_4. However, we have $aK = \{a, ab\} \neq \{a, ba\} = Ka$ since $ab \neq ba$. $\square$

11. Let p and q be distinct primes. If G is a group of order pq that has a unique subgroup of order p and a unique subgroup of order q, show that G is cyclic.

16. Show that $\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G$ for any group G.

25. If X is a nonempty subset of a group G, define the **normalizer** $N(X)$ of X by

    $$N(X) = \{\, a \in G \mid aXa^{-1} = X \,\}.$$

    (a) Show that $N(X)$ is a subgroup of G.

    Proof. Clearly $1_G X 1_G^{-1} = X$, so $1_G \in N(X)$. Then if $a, b \in N(X)$, we have

    $$\begin{aligned} aXa^{-1} &= X \\ bXb^{-1} &= X \\ \implies a(bXb^{-1})a^{-1} &= X \\ \implies (ab)X(ba)^{-1} &= X \end{aligned}$$

so $ab \in N(X)$. Then if $a \in N(X)$, we have

$$\begin{aligned} aXa^{-1} &= X \\ aX &= Xa \\ X &= a^{-1}Xa \end{aligned}$$

so $a^{-1} \in N(X)$ as well. Thus, $N(X)$ is a subgroup of G, as desired.

□

(b) If H is a subgroup of G, show that $H \trianglelefteq N(H)$.

(c) If H is a subgroup of G, show that $N(H)$ is the largest subgroup of G in which H is normal. That is, if $H \trianglelefteq K$, and K if a subgroup of G, then $K \subseteq N\left(H\right)$.

Section 2.10: The Isomorphism Theorem

7. If $\alpha : G \to G_1$ is a group homomorphism and both $\alpha(G)$ and $\ker \alpha$ are finitely generated, show that G is finitely generated.

9. If $K = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$, is there a group homomorphism $\alpha : S_4 \to A_4$ with $\ker \alpha = K$?

21. Show that $\mathbb{C}^*/\mathbb{C}^0 \cong \mathbb{R}^+$ where $\mathbb{C}^0 = \{\, z \mid |z| = 1 \,\}$ is the circle group.

Proof. Define the homomorphism $\varphi : \mathbb{C}^* \to \mathbb{R}^+$ where $\varphi(z) = \left|z\right|$. This is indeed a homomorphism because $\varphi(z_1z_2) = |z_1z_2| = |z_1|\,|z_2| = \varphi(z_1)\varphi(z_2)$.

Then the kernel of φ is the set $\{\, z \mid \varphi(z) = 1 \,\}$ which is exactly $\mathbb{C}^0$. Finally, $\varphi``(\mathbb{C}^*) = \mathbb{R}^+$ since invertible elements in $\mathbb{C}$ are all except 0, whose magnitudes are all positive.

Thus, by the first Isomorphism Theorem, since $\mathbb{C}^0$ is the kernel of a homomorphism, $\mathbb{C}^*/\mathbb{C}^0 \cong \mathbb{R}^+$, as desired.

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