Homework 6

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1. *Solution.* If X and Y are the results of the two dice, then $Z = XY$. The rolls are independent of each other, and we have

$$E[X] = E[Y] = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$
$$E[X^2] = E[Y^2] = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$

Now, since X and Y are independent, it also holds that X^2 and Y^2 are independent, so

$$\begin{aligned} E[Z] &= E[XY] = E[X]E[Y] = \frac{7}{2}\cdot\frac{7}{2} = \frac{49}{4} \\ \mathrm{Var}(Z) &= E[Z^2] - (E[Z])^2 = E[X^2Y^2] - (E[XY])^2 = E[X^2]E[Y^2] - (E[XY])^2 \\ &= \frac{91}{6}\cdot\frac{91}{6} - \left(\frac{49}{4}\right)^2 = \frac{11515}{144} \end{aligned}$$

□

2. (a) *Solution.* We wish to find α that minimizes

$$\begin{aligned} L &= \frac{1}{2}\mathrm{Var}[\alpha r_A + (1-\alpha)r_B] = \frac{1}{2}\left(\mathrm{Var}(\alpha r_A) + \mathrm{Var}[(1-\alpha)r_B] + 2\mathrm{Cov}(\alpha r_A, (1-\alpha)r_B)\right) \\ &= \frac{1}{2}\alpha^2\sigma_A^2 + \frac{1}{2}(1-\alpha)^2\sigma_B^2 + \alpha(1-\alpha)\rho\sigma_A\sigma_B \end{aligned}$$

Taking the derivative with respect to α, we have

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \alpha\sigma_A^2 - (1-\alpha)\sigma_B^2 + (1-2\alpha)\rho\sigma_A\sigma_B = 0 \\ \implies \alpha &= \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2+\sigma_B^2 - 2\rho\sigma_A\sigma_B} = 0.8261 \\ \implies 1-\alpha &= 0.1739 \end{aligned}$$

□

(b) *Solution.* Evaluating at α, we have

$$\sigma = \sqrt{2L(\alpha)} = 1.94\%$$

□

(c) *Solution.* The expected return is

$$E[r] = E[\alpha r_A + (1-\alpha)r_B] = \alpha\bar{r}_a + (1-\alpha)\bar{r}_B = 11.39\%$$

□

3. *Solution.* If α is the weight of asset 1 and $1-\alpha$ is the weight of asset 2, then from above, we have

$$\alpha = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

minimizes the variance of the portfolio. The expected return is

$$\begin{aligned} E[r] &= E[\alpha r_1 + (1-\alpha) r_2] = \alpha \bar{r}_1 + (1-\alpha)\bar{r}_2 \\ &= \bar{r}_1 + \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}(\bar{r}_1 - \bar{r}_2) \end{aligned}$$

□

4. (a) *Solution.* Each asset has the same return $\bar{r}_0$, so if the portfolio was entirely a single asset, all of these points would lie on the horizontal line $\bar{r} = \bar{r}_0$. Since they are uncorrelated, this line is exactly the minimum variance set, and the efficient frontier is the entire line. □

   (b) *Solution.* We have the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{n} w_i^2\sigma_i^2 - \lambda\left(\sum_{i=1}^{n} w_i\bar{r}_0 - \bar{r}\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

   Taking the partial derivatives with respect to each of the w_i, we have

$$\begin{aligned} \frac{\partial L}{\partial w_i} &= w_i\sigma_i^2 - \lambda\bar{r}_0 - \mu = 0 \\ \implies w_i &= \frac{\lambda\bar{r}_0 + \mu}{\sigma_i^2} \end{aligned}$$

   The constraints are thus

$$\begin{aligned} \sum_{i=1}^{n} w_i &= \sum_{i=1}^{n} \frac{\lambda\bar{r}_0 + \mu}{\sigma_i^2} = (\lambda\bar{r}_0 + \mu)\bar{\sigma}^2 = 1 \\ \sum_{i=1}^{n} w_i\bar{r}_i &= \sum_{i=1}^{n} w_i\bar{r}_0 = r_0 = \bar{r} \end{aligned}$$

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