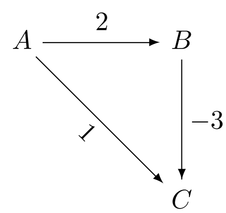
Homework 8

Aleck Zhao

December 5, 2016

1. Give an example of a shortest path instance with no negative length cycles such that Dijkstra's Algorithm fails to give the correct shortest paths.

   Solution. Consider the graph



   If we want to find the shortest $A - C$ path, we execute Dijkstra's Algorithm as follows:

nodes	distance
*A	0
B	∞
C	∞

$\Longrightarrow$

nodes	distance
A	0
B	2
*C	1

$\Longrightarrow$

nodes	distance
A	0
C	1
*B	2

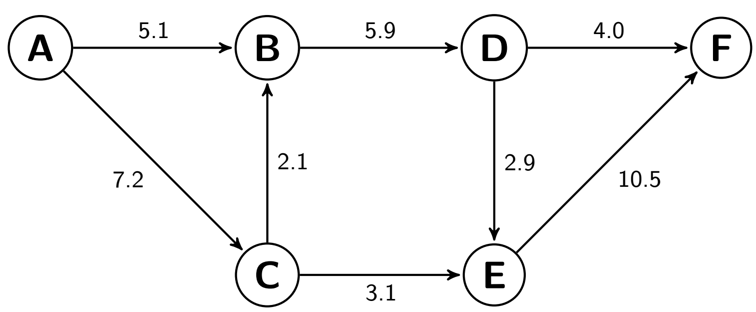
$\Longrightarrow$

nodes	distance
A	0
C	1
B	2

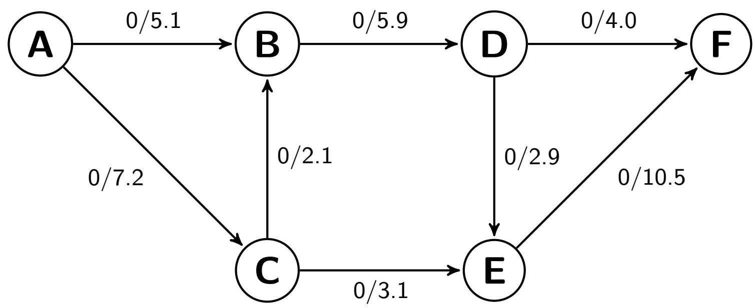
   At the conclusion, the shortest discovered $A - C$ path is $A \to C$ with length 1, but the real shortest length path is $A \to B \to C$ with length -1, so Dijkstra's Algorithm fails. $\square$

2. Solve the Max Flow problem from HW1 using the Ford-Fulkerson Algorithm. Be sure to provide the flow and residual network at each algorithm iteration. Then, at the conclusion, provide the Max Flow and the Min Cut.

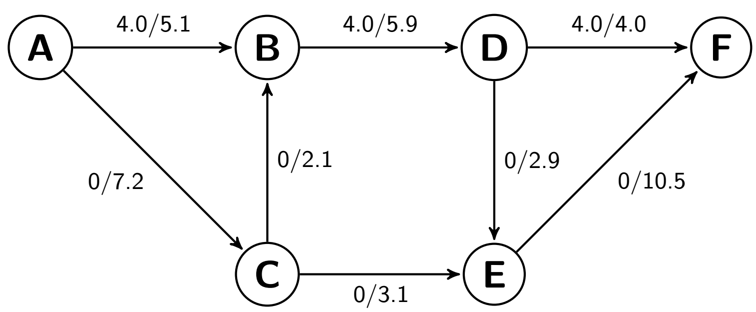
   Solution. The network and its capacities are given as



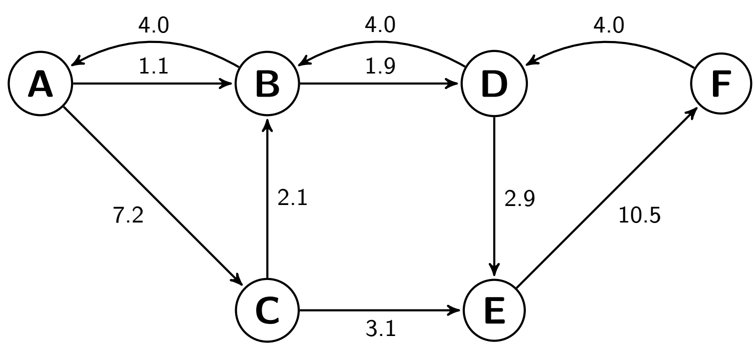
We start with the feasible flow x given by



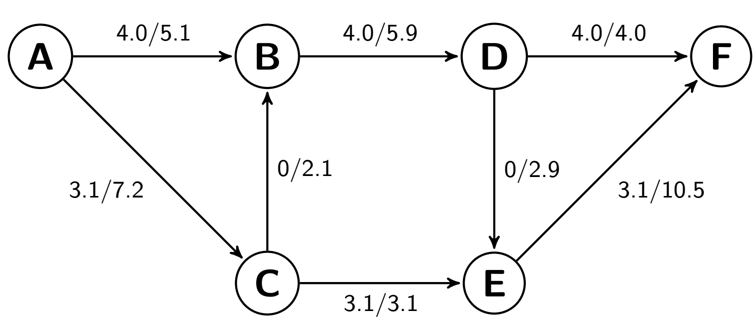
and the associated residual network G^x is just the original network. Use the path $P = A \to B \to D \to F$, where $\mu^x(P) = 4.0$, and augment along P by 4.0 to get the new flow x



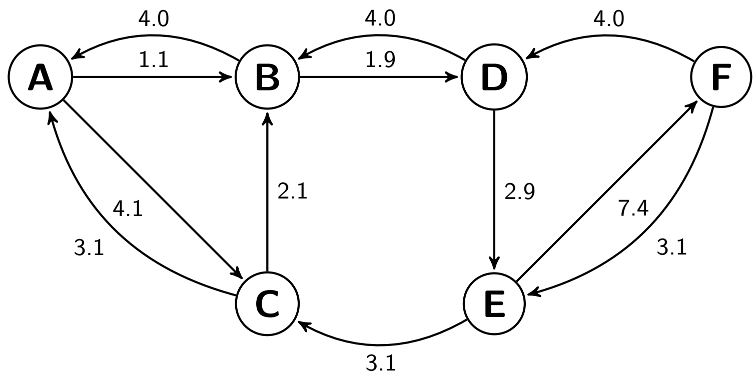
and the residual network



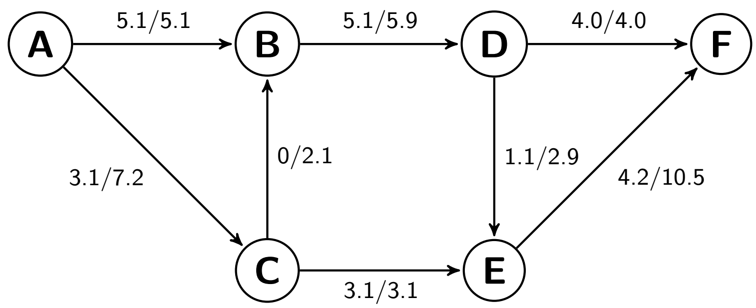
Next, choose the path $A \to C \to C \to F$, where $\mu^x(P) = 3.1$, so augmenting by 3.1 gives the new flow x



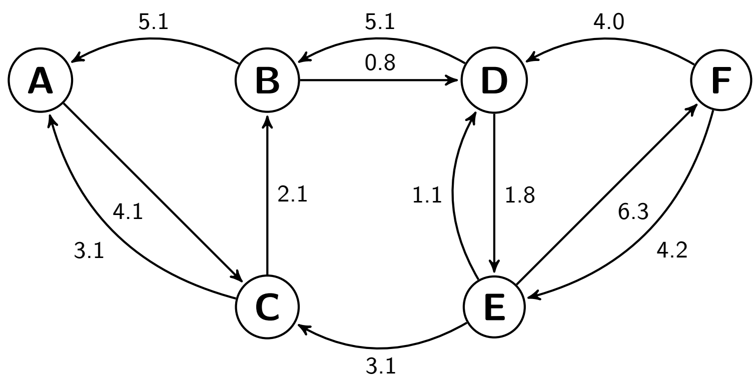
and the residual network



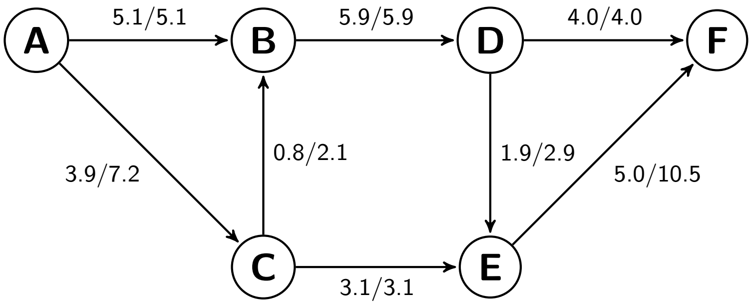
Next, choose the path $A \to B \to D \to E \to F$ where $\mu^x(P) = 1.1$, so augmenting by 1.1 gives the new flow x



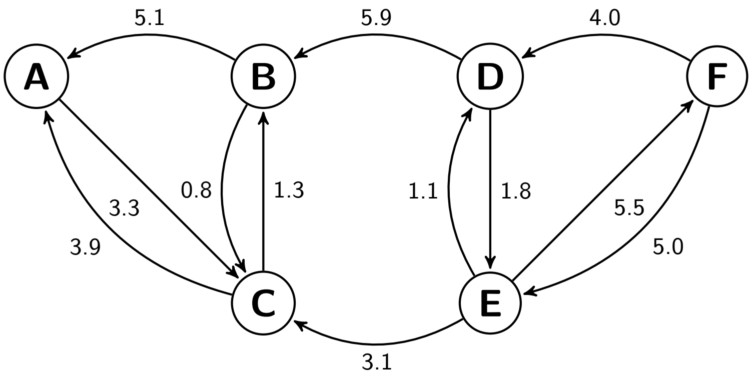
and the residual network



Next, choose the path $A \to C \to B \to D \to E \to F$, with $\mu^x(P) = 0.8$, so augmenting by 0.8 gives the new flow x



with residual network



At this point, there no longer exists an $A - F$ path, so the algorithm concludes. The effective flow is 9.0. The Min Cut is also 9.0, where we cut the arcs from $B \to D$ and $C \to E$. $\square$

3. Suppose families $F_1, F_2, \cdots, F_n$ each respectively consist of $f_1, f_2, \cdots, f_n$ people, and cars $C_1, C_2, \cdots, C_m$ can respectively seat $c_1, c_2, \cdots, c_m$ people. Everyone needs to travel from Baltimore to Pittsburgh in these cars to see the Ravens-Steelers game, but no two members of any family can ride in the same car, as they will fight the whole time. Formulate a max flow instance such that the max flow will provide everyone with seating assignments for this very important trip.

   Solution. I will attempt to describe this because drawing seems disgusting.

   There is a starting node from Baltimore and the ending node at Pittsburgh. There is a node for each family $F_1, \cdots, F_n$, and there exists an arc from Baltimore to each F_i with capacity f_i. Then there is a node for each car $C_1, \cdots, C_m$, and for every pair F_i, C_j, there exists an arc from $F_i \to C_j$ with capacity 1. Then from each node C_i to Pittsburgh, there exists an arc with capacity c_i. This max flow problem will have a solution that transports everyone. $\square$

4. Suppose that $G = (V, E), s, t \in V$, and $\ell : E \to \mathbb{R}$ is a shortest path problem instance such that there is a negative cycle. Prove that there do not exist distance labels $d : V \to \mathbb{R}$ such that d satisfies the triangle inequality.

Proof. Suppose a distance label d exists that satisfies the triangle inequality. Suppose a cycle $C = (a_1, a_2, \cdots, a_n, a_1)$ has a negative length. If d satisfies the triangle inequality, then we must have

$$\begin{aligned} d(a_2) - d(a_1) &\leq \ell(a_1, a_2) \\ d(a_3) - d(a_2) &\leq \ell(a_2, a_3) \\ &\vdots \\ d(a_n) - d(a_{n-1}) &\leq \ell(a_{n-1}, a_n) \\ d(a_1) - d(a_n) &\leq \ell(a_n, a_1) \end{aligned}$$

If we sum all of these inequalities, the LHS sums to 0 because everything cancels, whereas the RHS becomes $\ell(C)$, which we know is negative. This is a contradiction: 0 can't be less than or equal to a negative number. Thus, no such d exists that satisfies the triangle inequality,a $\square$