Homework 4

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October 9, 2017

Chapter 3: Metrics and Norms

6. If d is any metric on M, show that $\rho(x,y) = \sqrt{d(x,y)}, \sigma(x,y) = \frac{d(x,y)}{1+d(x,y)}$, and $\tau(x,y) = \min\{d(x,y),1\}$ are also metrics on M.

Proof. ρ : Clearly ρ is non-negative since d is non-negative by being a metric, and

$$\rho(x,y) = 0 = \sqrt{d(x,y)} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because d is symmetric, and finally

$$\begin{aligned}
\rho(x,y) + \rho(y,z) &= \sqrt{d(x,y)} + \sqrt{d(y,z)} \\
\implies [\rho(x,y) + \rho(y,z)]^2 &= d(x,y) + d(y,z) + 2\sqrt{d(x,y)d(y,z)} \\
&\geq d(x,z) + 2\sqrt{d(x,y)d(y,z)} \geq d(x,z) \\
\implies \rho(x,y) + \rho(y,z) &\geq \sqrt{d(x,z)} = \rho(x,z)
\end{aligned}$$

σ : Clearly σ is non-negative since d is non-negative, and

$$\sigma(x,y) = 0 = \frac{d(x,y)}{1+d(x,y)} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because d is symmetric. Now, define $F(t) = \frac{t}{1+t}$. Then $F'(t) = \frac{1}{(1+t)^2} > 0$ so F is increasing, and we have

$$\begin{aligned}
F(t) + F(s) &= \frac{t}{1+t} + \frac{s}{1+s} = \frac{t+ts+s+st}{(1+t)(1+s)} = \frac{s+t+2st}{1+s+t+st} \\
&= \frac{s+t+st}{1+s+t+st} + \frac{st}{1+s+t+st} = F(s+t+st) + \frac{st}{1+s+t+st} \\
&\geq F(s+t)
\end{aligned}$$

since F is increasing since $F'(t) = (1+t)^{-2} > 0$. Thus,

$$\begin{aligned}
\sigma(x,y) + \sigma(y,z) &= F(d(x,y)) + F(d(y,z)) \geq F\left(d(x,y) + d(y,z)\right) \\
&\geq F(d(x,z)) = \sigma(x,z)
\end{aligned}$$

τ : Clearly τ is non-negative since d and 1 are non-negative, and

$$\tau(x,y) = 0 = \min\{d(x,y),1\} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because d is symmetric. Suppose that

$$\begin{aligned}
\tau(x,y) + \tau(y,z) &< \tau(x,z) \\
\min\{d(x,y),1\} + \min\{d(y,z),1\} = m_1 + m_2 &< \min\{d(x,z),1\} \\
\implies m_1 + m_2 < 1, &\quad m_1 + m_2 < d(x,z)
\end{aligned}$$

If $m_1 + m_2 < 1$, then we must have $m_1 = d(x,y)$ and $m_2 = d(y,z)$, but since d is a metric, $m_1 + m_2 = d(x,y) + d(y,z) \geq d(x,z)$, so it is impossible for both conditions to be true. Contradiction, so $\tau(x,y) + \tau(y,z) \geq \tau(x,z)$, and τ is a metric. $\square$

15. We define the diameter of a nonempty subset A of M by $\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Show that A is bounded if and only if $\operatorname{diam}(A)$ is finite.

Proof. ($\Longrightarrow$) : If A is bounded, then $\exists x_0 \in M$ and $C < \infty$ such that $d(a, x_0) \le C$ for all $a \in A$. Then

$$\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\} \le \sup\{d(a,x_0) + d(x_0,b) : a, b \in A\} \le 2C < \infty$$

($\Longleftarrow$) : If $\operatorname{diam}(A)$ is finite, say $s = \operatorname{diam}(A)$. Then take any $x_0 \in A \subset M$, and take $C = s$. Since s is the supremum, it follows that

$$C = s = \sup\{d(a,b) : a, b \in A\} \ge d(a, x_0)$$

for any $a \in A$, so A is bounded, as desired. □

22. Show that $\|x\|_\infty \le \|x\|_2$ for any $x \in \ell_2$, and that $\|x\|_2 \le \|x\|_1$ for any $x \in \ell_1$.

Proof. We have

$$|x_k| \le \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$$

for all k (obvious once we square both sides), so it follows that

$$\|x\|_\infty = \sup|x_k| \le \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} = \|x\|_2$$

By Cauchy, we have

$$\sum_{i=1}^{\infty} |x_i \cdot x_i| \le \sum_{i=1}^{\infty} |x_i| \sum_{i=1}^{\infty} |x_i|$$
$$\Longrightarrow \|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} \le \sum_{i=1}^{\infty} |x_i| = \|x\|_1$$

□

23. The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. Note that c_0 is actually a linear subspace of ℓ_∞; thus c_0 is also a normed vector space under $\|\cdot\|_\infty$. Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

Proof. Suppose $x \in \ell_1$. Then $\sum_{i=1}^{\infty} |x_i| < \infty$. By Cauchy, we have

$$\sum_{i=1}^{\infty} |x_i \cdot x_i| \le \sum_{i=1}^{\infty} |x_i| \sum_{i=1}^{\infty} |x_i|$$
$$\Longrightarrow \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} \le \sum_{i=1}^{\infty} |x_i| < \infty$$

so $\ell_1 \subset \ell_2$. Then the reverse inclusion does not hold because for the sequence $x_n = \frac{1}{n}$, we have

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi}{6} \quad \text{but} \quad \sum_{i=1}^{\infty} \frac{1}{i} \to \infty$$

Suppose $y \in \ell_2$, but y_n does not converge to 0. Clearly y must be bounded, otherwise it would not be in ℓ_2. Then there exists some $\varepsilon > 0$ such that $|y_n| \geq \varepsilon$ infinitely often. Then $\sum_{i=1}^{\infty} |y_i|^2 \to \infty$, so $y \notin \ell_2$, contradiction. Thus $y_n \to 0 \implies y \in c_0$, as desired. The reverse inclusion does not hold for the sequence $y_n = \frac{1}{\sqrt{n}}$, since $y_n \to 0$ but $\sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{i}}\right)^2 \to \infty$.

Finally, c_0 is a linear subspace of ℓ_∞, and the reverse inclusion does not hold if we consider the sequence $(1, 0, 1, 0, 1, 0, \cdots)$. This is bounded, but does not converge to 0. □

25. The same techniques can be used to show that $\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p}$ defines a norm on $C([0,1])$ for any $1 < p < \infty$. State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for $p = 1$ and $q = \infty$?)

Lemma 3.7': Let $1 < p < \infty$ and let q be defined by $1/p + 1/q = 1$. Given $f, g \in C([0,1])$, we have $\int_0^1 |f(x)g(x)| \, dx \leq \|f\|_p \|g\|_q$.

Proof. From Young's inequality, we have

$$\begin{aligned}
\left|\frac{f(x)}{\|f\|_p}\right| \cdot \left|\frac{g(x)}{\|g\|_q}\right| &\leq \frac{1}{p}\left|\frac{f(x)}{\|f\|_p}\right|^p + \frac{1}{q}\left|\frac{g(x)}{\|g\|_q}\right|^q \\
\implies \int_0^1 \left|\frac{f(x)g(x)}{\|f\|_p \|g\|_q}\right| dx &\leq \int_0^1 \frac{1}{p}\left|\frac{f(x)}{\|f\|_p}\right|^p dx + \int_0^1 \frac{1}{q}\left|\frac{g(x)}{\|g\|_q}\right|^q dx \\
&= \frac{1}{p} + \frac{1}{q} = 1 \\
\implies \int_0^1 |f(x)g(x)| \, dx &\leq \|f\|_p \|g\|_q
\end{aligned}$$

as desired. If $p = 1$ and $q = \infty$, the statement still holds. We have

$$\begin{aligned}
\|g\|_q &= \sup_{0 \leq t \leq 1} |g(t)| \\
\implies |f(x)| |g(x)| &\leq |f(x)| \cdot \sup_{0 \leq t \leq 1} |g(t)| \\
\implies \int_0^1 |f(x)g(x)| \, dx &\leq \int_0^1 |f(x)| \cdot \sup_{0 \leq t \leq 1} |g(t)| \, dx \\
&= \|g\|_\infty \|f\|_1
\end{aligned}$$

□

Theorem 3.8': Let $1 < p < \infty$. If $f, g \in C([0,1])$. then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. Let q be the conjugate of p. By the triangle inequality and Holder's inequality, we have

$$\begin{aligned}
|f(x) + g(x)|^p &= |f(x) + g(x)| \cdot |f(x) + g(x)|^{p-1} \\
&\leq |f(x)| \cdot |f(x) + g(x)|^{p-1} + |g(x)| \cdot |f(x) + g(x)|^{p-1} \\
\implies \int_0^1 |f(x) + g(x)|^p \, dx &\leq \int_0^1 |f(x)| \cdot |f(x) + g(x)|^{p-1} \, dx + \int_0^1 |g(x)| \cdot |f(x) + g(x)|^{p-1} \, dx \\
&\leq \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \left\|(f+g)^{p-1}\right\|_q
\end{aligned}$$

Now, since $1/p + 1/q = 1 \implies q = \frac{p}{p-1}$, we have

$$\begin{aligned}\left\|(f+g))^{p-1}\right\|_q &= \left(\int_0^1 \left(f(x)+g(x)^{p-1}\right)^{\frac{p}{p-1}}\ dx\right)^{\frac{p-1}{p}} \\ &= \left(\left(\int_0^1 |f(x)+g(x)|^p\ dx\right)^{1/p}\right)^{(p-1)} \\ &= \|f+g\|_p^{p-1}\end{aligned}$$

so it follows that

$$\begin{aligned}\int_0^1 |f(x)+g(x)|^p\ dx = \|f+g\|_p^p &\leq \Big(\|f\|_p + \|g\|_p\Big) \cdot \|f+g\|_p^{p-1} \\ \implies \|f+g\|_p &\leq \|f\|_p + \|g\|_p\end{aligned}$$

as desired. □

31. Give an example where $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$. If $A \cap B \neq \varnothing$, show that $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

Proof. Let $A = \{0\}$ and $B = \{1\}$ under the discrete metric. Then $\operatorname{diam}(A) = 0$ and $\operatorname{diam}(B) = 0$, but $\operatorname{diam}(A \cup B) = \operatorname{diam}(\{0,1\}) = 1$.

Take two points $x, y \in A \cup B$. If x and y are both in A, then $d(x,y) \leq \operatorname{diam}(A)$, so

$$\operatorname{diam}(A \cup B) = \sup\{d(a,b) : a, b \in A \cup B\} \leq \operatorname{diam}(A)$$

and similarly if $x, y \in B$. WLOG $x \in A, y \in B$. Since $A \cap B \neq \varnothing$, take $z \in A \cap B \implies z \in A, z \in B$.

$$\begin{aligned}d(x,y) \leq d(x,z) + d(y,z) \leq \operatorname{diam}(A) + \operatorname{diam}(B) \\ \implies \operatorname{diam}(A \cup B) = \sup\{d(a,b) : a, b \in A \cup B\} \leq \operatorname{diam}(A) + \operatorname{diam}(B)\end{aligned}$$

as desired. □

37. A Cauchy sequence with a convergent subsequence converges.

Proof. Suppose (x_n) is a sequence with a convergent subsequence $(x_{k_j}) \to y$. Let $\varepsilon > 0$. Since (x_n) is Cauchy, choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m \geq N$. Next, since $(x_{k_j}) \to y$, choose M such that $d(x_{k_j}, y) < \varepsilon/2$ for all $k_j \geq M$. Take $K = \max\{K, M\}$, so that $d(x_n, x_{k_j}) < \varepsilon/2$ and $d(x_{k_j}, y) < \varepsilon/2$ for all $n, k_j \geq K$. By the triangle inequality, we have

$$d(x_n, y) \leq d(x_n, x_{k_j}) + d(x_{k_j}, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq K$, as desired. □