Homework 3

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1. Prove that the Fejer kernel is given by

$$F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

Hint: Remember that $NF_N(x) = D_0(x) + \cdots + D_{N-1}(x)$ where $D_n(x)$ is the Dirichlet kernel. Therefore, if $\omega = e^{ix}$ we have

$$NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1 - \omega}$$

Proof. From section 1.1 we have the closed form of the Dirichlet kernel

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin(x/2)}$$

Also note the following result:

$$\begin{aligned}
\cos nx &= \cos\left[\left(n + \frac{1}{2}\right)x - \frac{x}{2}\right] = \cos\left[\left(n + \frac{1}{2}\right)x\right]\cos\frac{x}{2} + \sin\left[\left(n + \frac{1}{2}\right)x\right]\sin\frac{x}{2} \\
\cos(n+1)x &= \cos\left[\left(n + \frac{1}{2}\right)x + \frac{x}{2}\right] = \cos\left[\left(n + \frac{1}{2}\right)x\right]\cos\frac{x}{2} - \sin\left[\left(n + \frac{1}{2}\right)x\right]\sin\frac{x}{2} \\
\implies \cos nx - \cos(n+1)x &= 2\sin\left[\left(n + \frac{1}{2}\right)x\right]\sin\frac{x}{2}
\end{aligned}$$

Thus, our sum becomes

$$\begin{aligned}
F_N(x) &= \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\sum_{n=0}^{N-1} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin(x/2)} = \frac{1}{2N\sin^2(x/2)}\sum_{n=0}^{N-1} 2\sin\left[\left(n + \frac{1}{2}\right)x\right]\sin\frac{x}{2} \\
&= \frac{1}{2N\sin^2(x/2)}\sum_{n=0}^{N-1}\left[\cos nx - \cos(n+1)x\right] = \frac{1 - \cos Nx}{2N\sin^2(x/2)} \\
&= \frac{1 - \cos\left(2 \cdot \frac{Nx}{2}\right)}{2N\sin^2(x/2)} = \frac{1 - \left(1 - 2\sin^2\frac{Nx}{2}\right)}{2N\sin^2(x/2)} = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}
\end{aligned}$$

as desired. □

2. Solve Laplace's equation $\Delta u = 0$ on the semi infinite strip

$$S = \{(x, y) : 0 < x < 1, 0 < y\}$$

subject to the following boundary conditions

$$\begin{cases} u(0, y) = 0 & 0 \le y \\ u(1, y) = 0 & 0 \le y \\ u(x, 0) = f(x) & 0 \le x \le 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x, y) = e^{-n\pi y} \sin(n\pi x)$$

Express u as an integral involving f, analogous to the Poisson kernel formula (6).

Solution. Note that $u_n(x, 0) = \sin(n\pi x)$ and it also satisfies the boundary conditions $u_n(x, 0) = u_n(x, 1) = 0$. Thus, since f can be written as a sum of sines, we have

$$u(x, y) = \sum_{n=1}^{\infty} a_n u_n(x, y)$$

is the general solution because it satisfies the 0 boundary conditions since each of the individual terms is 0, and satisfies the f boundary condition as well.

Now, we can also write a_n as the Fourier coefficient

$$a_n = \int_0^1 f(z) \sin(n\pi z)\, dz$$

and thus the solution can be written as

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} \int_0^1 f(z) \sin(n\pi z)\, dz \cdot e^{-n\pi y} \sin(n\pi x) \\ &= \int_0^1 f(z) \left(\sum_{n=1}^{\infty} e^{-n\pi y} \sin(n\pi x) \sin(n\pi z) \right) dz \end{aligned}$$

□