Homework 2

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(1) In a simple symmetric random walk, let T denote the time of the first return to the origin. Use the tail probability representation of the expectation to show that $E[T] = +\infty$.

Proof. From last time, we have

$$E[T] = \sum_{n=0}^{\infty} P[T > n]$$

Note that $P[T > 2k] = P[T > 2k+1]$ for all k, and $P[T > 0] = P[T > 1] = 1$, so

$$\begin{aligned} E[T] &= 2\sum_{k=0}^{\infty} P[T > 2k] = 2(1) + 2\sum_{k=1}^{\infty} P[S_1 \neq 0, S_2 \neq 0, \cdots, S_{2k} \neq 0] \\ &= 2 + 2\sum_{k=1}^{\infty} u_{2k} = 2 + 2\sum_{k=1}^{\infty} \binom{2k}{k} 2^{-2k} = 2 + 2\sum_{k=1}^{\infty} \frac{(2k)!}{k!k!} 2^{-2k} \end{aligned}$$

By Stirling's Formula, this is asymptotic to

$$2 + 2\sum_{k=1}^{\infty} \frac{\sqrt{2\pi(2k)}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{k}} \cdot 2^{-2k} = 2 + 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi k}} \to \infty$$

as desired. □

(2) Let X denote a random variable which has the arc sine distribution.

(a) Calculate $P\left[\frac{1}{4} < X < \frac{3}{4}\right]$.

Solution. The CDF for a X is given by

$$F(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$$

so the desired probability is

$$F(3/4) - F(1/4) = \frac{2}{\pi}\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{2}{\pi}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{3}$$

□

(b) Calculate $E[X]$.

Solution. The distribution for X is given by

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 < x < 1$$

so the expectation is

$$\begin{aligned}\int_0^1 x\cdot\frac{1}{\pi\sqrt{x(1-x)}}\,dx = \frac{1}{2\pi}\int_0^1 \frac{2x}{\sqrt{x-x^2}}\,dx &= \frac{1}{2\pi}\int_0^1\left(\frac{2x-1}{\sqrt{x-x^2}}+\frac{1}{\sqrt{x(1-x)}}\right)dx\\ &= \frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2}\int_0^1\frac{1}{\pi\sqrt{x(1-x)}}\,dx\\ &= \frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2}\end{aligned}$$

Using the substitution

$$u = x - x^2 \implies du = 1 - 2x\,dx \implies -du = 2x - 1\,dx$$

the expectation becomes

$$\frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2} = -\frac{1}{2\pi}\int_0^0 \frac{1}{\sqrt{u}}\,du + \frac{1}{2} = \boxed{\frac{1}{2}}$$

□

(c) Calculate $\mathrm{Var}(X)$.

Solution. We have the relation $\mathrm{Var}(X) = E[X^2] - (E[X])^2$. For $E[X^2]$, we have

$$\begin{aligned}E[X^2] &= \int_0^1 x^2\cdot\frac{1}{\pi\sqrt{x(1-x)}}\,dx = \frac{1}{\pi}\int_0^1\frac{x^2-x}{\sqrt{x-x^2}}\,dx + \int_0^1\frac{x}{\pi\sqrt{x(1-x)}}\,dx\\ &= -\frac{1}{\pi}\int_0^1\sqrt{x(1-x)}\,dx + \frac{1}{2}\end{aligned}$$

Completing the square, we have

$$\sqrt{x-x^2} = \sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}$$

so using the substitution

$$x - \frac{1}{2} = \frac{1}{2}\cos\theta \implies dx = -\frac{1}{2}\sin\theta\,d\theta$$

the integral becomes

$$\begin{aligned}-\frac{1}{\pi}\int_0^1\sqrt{x(1-x)}\,dx = -\frac{1}{\pi}\int_\pi^0 \frac{1}{2}\sin\theta\left(-\frac{1}{2}\sin\theta\,d\theta\right) &= \frac{1}{4\pi}\int_\pi^0 \sin^2\theta\,d\theta\\ &= \frac{1}{4\pi}\int_\pi^0\left(\frac{1}{2}-\frac{\cos 2\theta}{2}\right)d\theta = \frac{1}{4\pi}\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)\Bigg|_\pi^0\\ &= -\frac{1}{8}\end{aligned}$$

Thus, we have

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \left(\frac{1}{2}-\frac{1}{8}\right) - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{8}}$$

□

(3) Consider a simple symmetric random walk of length 12. Let L_{12} denote the amount of time that the random walk is positive.

(a) Use the formula given in class to calculate the values of the frequency function of L_{12} to three decimal places.

Solution. We have

$$P[L_{2n} = 2k] = u_{2k}u_{2n-2k} = \binom{2k}{k}2^{-2k}\binom{2n-2k}{n-k}2^{-2n+2k} = \binom{2k}{k}\binom{2n-2k}{n-k}2^{-2n}$$
$$P[L_{12} = 2k] = \binom{2k}{k}\binom{12-2k}{6-k}2^{-12}$$

Using $k = 0, 1, \cdots, 6$, we have

$$P[L_{12} = 0] = \binom{0}{0}\binom{12}{6}2^{-12} \approx 0.226$$
$$P[L_{12} = 2] = \binom{2}{1}\binom{10}{5}2^{-12} \approx 0.123$$
$$P[L_{12} = 4] = \binom{4}{2}\binom{8}{4}2^{-12} \approx 0.103$$
$$P[L_{12} = 6] = \binom{6}{3}\binom{6}{3}2^{-12} \approx 0.098$$
$$P[L_{12} = 8] = \binom{8}{4}\binom{4}{2}2^{-12} \approx 0.103$$
$$P[L_{12} = 10] = \binom{10}{5}\binom{2}{1}2^{-12} \approx 0.123$$
$$P[L_{12} = 12] = \binom{12}{6}\binom{0}{0}2^{-12} \approx 0.226$$

□

(b) To see how good the asymptotic approximation is, find the difference

$$\left| P\left[\frac{1}{4} < \frac{L_{12}}{12} < \frac{3}{4}\right] - P\left[\frac{1}{4} < X < \frac{3}{4}\right] \right|$$

where the latter value was calculated in problem 2a.

Solution. We have

$$P\left[\frac{1}{4} < \frac{L_{12}}{12} < \frac{3}{4}\right] = P\left[3 < L_{12} < 9\right] = P[L_{12} = 4] + P[L_{12} = 6] + P[L_{12} = 8]$$
$$\approx 0.103 + 0.098 + 0.103 = 0.304$$

From part 2a, we have $P[\frac{1}{4} < X < \frac{3}{4}] = \frac{1}{3}$, so the difference is

$$\left|0.304 - \frac{1}{3}\right| \approx \boxed{0.0293}$$

□

(4) Find the conditional probability that a simple symmetric random walk of length $2n$ is always nonnegative, given that it ends at 0.

(a) Write an expression in terms of $S_1, S_2, S_3, \cdots, S_{2n}$ for the desired conditional probability, as a ratio of two unconditional probabilities, using the definition of conditional probability.

Solution. This probability is

$$P[S_1 \geq 0, S_2 \geq 0, \cdots, S_{2n-1} \geq 0 \mid S_{2n} = 0] = \frac{P[S_1 \geq 0, S_2 \geq 0, \cdots, S_{2n-1} \geq 0, S_{2n} = 0]}{P[S_{2n} = 0]}$$

□

(b) Write an exact formula for the denominator of the fraction in (a).

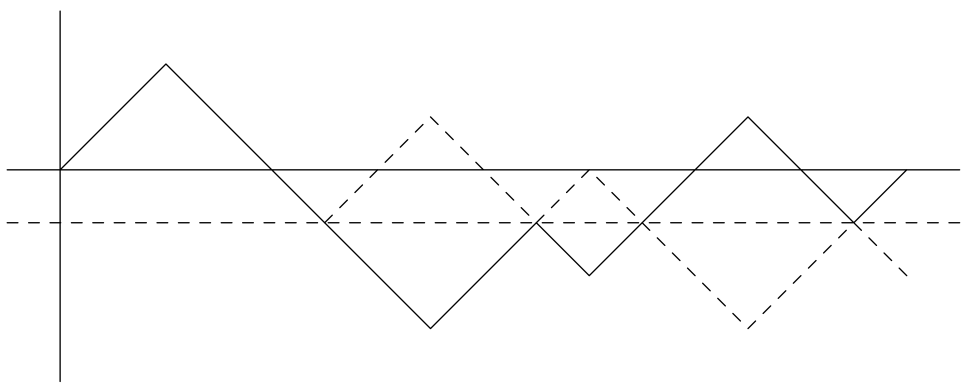
Solution. We have

$$P[S_{2n} = 0] = u_{2n} = \binom{2n}{n} 2^{-2n}$$

□

(c) To derive an expression for the numerator consider the (relative) complementary event that the random walk goes below the x-axis at some time but ends at 0.

Solution. Consider a path that goes below the x-axis but ends at 0:



If we reflect the path about $y = -1$ at the first time the path becomes negative, we get a path that ends at -2, which is guaranteed because the original path ended at 0. This is a 1-1 correspondence because anytime a path ends at -2, it must have passed through -1 at some point, so reflect the path after the first time that happened to get a path ending at 0. □

(d) Calculate an expression for the probability that a simple symmetric random walk of length $2n$ ends at height -2.

Solution. If the path of length $2n$ ends at -2, it must have had $n-1$ positive and $n+1$ negative results. Thus, the probability is

$$P[S_{2n} = -2] = \binom{2n}{n+1} 2^{-2n}$$

□

(e) Use parts (b), (c), and (d) to calculate the desired numerator.

Solution. Of all paths that end at 0, either the path is nonnegative, or it crosses the x-axis and goes negative at some point. The probability a path ends at 0 is $\binom{2n}{n}2^{-2n}$. We just showed that there is a 1-1 correspondence between paths that cross the x-axis and paths that end at -2, so the numerator is

$$\binom{2n}{n}2^{-2n} - \binom{2n}{n+1}2^{-2n}$$

□

(f) Calculate the answer to the original question.

Solution. The answer to the original question is

$$\begin{aligned} P[S_1 \geq 0, S_2 \geq 0, \cdots, S_{2n-1} \geq 0 \mid S_{2n} = 0] &= \frac{P[S_1 \geq 0, S_2 \geq 0, \cdots, S_{2n-1} \geq 0, S_{2n} = 0}{P[S_{2n} = 0]} \\ &= \frac{\frac{(2n)!}{n!n!}2^{-2n} - \frac{(2n)!}{(n-1)!(n+1)!}2^{-2n}}{\frac{(2n)!}{n!n!}2^{-2n}} \\ &= 1 - \frac{n}{n+1} = \boxed{\frac{1}{n+1}} \end{aligned}$$

□