Homework 6

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Chapter 6: Connectedness

5. If E and F are connected subsets of M with $E \cap F \neq \varnothing$, show that $E \cup F$ is connected.

   Proof. Suppose $E \cup F$ was disconnected. Then write $E \cup F = A \cup B$ where $A \cap B = \varnothing$ and $A, B \neq \varnothing$. Since $E \cap F \neq \varnothing$, take $x \in E \cap F$, and WLOG $x \in A$. Then since $B \neq \varnothing$, take $y \in B$. Then either $y \in E$ or $y \in F$, so WLOG $y \in E$, and thus $y \in B \cap E$. Now, $x \in E$ as well so $x \in A \cap E$. Thus we have $A \cap E \neq \varnothing$ and $B \cap E \neq \varnothing$, but $(A \cap E) \cup (B \cap E) = E$ is a disconnection for E. Contradiction, since E was assumed to be connected, and thus $E \cup F$ is connected. $\square$

12. If M is connected and has at least two points, show that M is uncountable. (Hint: Find a non-constant, continuous, real-valued function on M.)

    Proof. Let $x, y \in M$, then $d(x, y) > 0$. Consider $d(x, \cdot) : M \to \mathbb{R}$. If d takes on every value from 0 to $d(x, y)$, then M must be uncountable. If not, then suppose $d(x, a) \neq d_0$ for any $a \in M$. Then we have

    $$M = \{a : d(x, a) < d_0\} \cup \{a : d(x, a) > d_0\}$$

    is a disjoint union of open sets in M, which contradicts that M is connected. Thus, it follows that M is uncountable. $\square$

15. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and open, show that f is strictly monotone.

    Proof. Suppose f was not strictly monotonic. Then there exists $a < c < b$, where either

    $$f(a) \leq f(c) \geq f(b)$$

    or

    $$f(a) \geq f(c) \leq f(b)$$

    If the first condition is true, then f attains a maximum value over $[a, b]$ that is a least as large as $f(c)$, which means f attains a maximum value m over (a, b) as well since $f(a)$ and $f(b)$ are not unique maxima. Then the image of (a, b) contains its maximum, contradicting the fact that f is an open map.

    If the second condition is true, then f attains a minimum value of $[a, b]$ that is at most as large as $f(c)$, which means f attains a minimum value n over (a, b) as well since $f(a)$ and $f(b)$ are not unique minima. Then the image of (a, b) contains its minimum, contradicting the fact that f is an open map. Thus, f must be strictly monotonic. $\square$

26. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that although f is not continuous, the graph of f is a connected subset of $\mathbb{R}^2$. (Hint: Use exercise 9.)

    Proof. Let $A = \{(x, \sin(1/x)) : x \in (0, 1]\}$. Then $\overline{A} = A \cup \{(0, 0)\}$. Since A is connected, it follows that $\overline{A}$ is also connected, which is the graph of f in $\mathbb{R}^2$. $\square$

Chapter 7: Completeness

5. Prove that A is totally bounded if and only if $\overline{A}$ is totally bounded.

Proof. ($\implies$) : Since A is totally bounded, for any $\varepsilon > 0$, we have $A \subset \bigcup_{i=1}^{n} B_{\varepsilon/2}(x_i)$ for $x_i \in M$. Take $y \in \overline{A}$. Then there must exist some $x \in A$ such that $d(y, x) < \varepsilon/2$ since y is a limit point of A. Then we also have $d(x, x_i) < \varepsilon/1$ for some i, so

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so y is within ε of some x_i. Thus $\overline{A} \subset \bigcup_{i=1}^{n} B_{\varepsilon}(x_i)$, so $\overline{A}$ is totally bounded.

($\impliedby$) : Since $A \subset \overline{A}$, it follows that A is totally bounded. □

9. Give an example of a closed bounded subset of ℓ_∞ that is not totally bounded.

Solution. Let $S = \left\{ e^{(n)} : n \geq 1 \right\}$ where $e^{(i)}$ is the sequence of 0s with a 1 in the ith position. Then clearly S is closed and bounded since $d(x, y) = 1$ for $x \neq y$. However, if $\varepsilon < 1$, then S cannot be covered by finitely many ε-balls since each ball could only cover a single element in S, so S is not totally bounded. □

10. Prove that a totally bounded metric space M is separable. (Hint: For each n, let D_n be a finite $(1/n)$-net for M. Show that $D = \bigcup_{n=1}^{\infty} D_n$ is a countable dense set.)

Proof. For $n \geq 1$, let D_n be a finite $(1/n)$-net for M, which must exist because M is totally bounded. Then since D_i is finite for all i, their union is countable.

Now, suppose $x \in M$ but $B_\varepsilon(x) \cap D = \varnothing$ for some $\varepsilon > 0$. Then that means $B_\varepsilon(x) \cap D_k = \varnothing$ for all k. However, if $k > 1/\varepsilon$ then since $B_\varepsilon(x) \cap D_k = \varnothing$, it follows that x is not within $1/k$ of any element in the net, which is a contradiction. Thus D is dense in M, so M is separable. □

18. Fill in the details of the proofs that ℓ_1 and ℓ_∞ are complete.

Proof. (ℓ_1): Let (f_n) be a sequence in ℓ_1, where $f_n = (f_n(k))_{k=1}^{\infty}$, and suppose (f_n) is Cauchy in ℓ_1.

$$|f_n(k) - f_m(k)| \leq \sum_{i=1}^{\infty} |f_n(i) - f_m(i)| = \|f_n - f_m\|_1$$

for any k, so $(f_n(k))_{n=1}^{\infty}$ is Cauchy for any k. Then set $f(k) := \lim_{n\to\infty} f_n(k)$ for each k.

Now, (f_n) is bounded in ℓ_1 since it is Cauchy, so suppose $\|f_n\|_2 \leq B$ for all n. Then

$$\sum_{k=1}^{N} |f(k)| = \lim_{n\to\infty} \sum_{k=1}^{N} |f_n(k)| \leq B$$

Since this holds for arbitrary N, it follows that $\|f\|_1 \leq B$.

Given $\varepsilon > 0$, choose n_0 such that $\|f_n - f_m\|_1 < \varepsilon$ whenever $m, n \geq n_0$. Then for any N and any $n \geq n_0$,

$$\sum_{i=k}^{N} |f(k) - f_n(k)| = \lim_{m\to\infty} \sum_{k=1}^{N} |f_m(k) - f_n(k)| < \varepsilon$$

and thus $\|f - f_n\|_1 < \varepsilon$ for all $n \geq n_0$, so $f_n \to f$.

(ℓ_∞) : Let (f_n) be a sequence in ℓ_∞, where $f_n = (f_n(k))_{k=1}^\infty$, and suppose (f_n) is Cauchy in ℓ_∞.

$$|f_n(k) - f_m(k)| \leq \sup_j |f_n(j) - f_m(j)| = \|f_n - f_m\|_\infty$$

for any k, so $(f_n(k))_{n=1}^\infty$ is Cauchy for any k. Then set $f(k) := \lim_{n\to\infty} f_n(k)$ for each k.
Now, (f_n) is bounded in ℓ_∞ since it is Cauchy, so suppose $\|f_n\|_\infty \leq B$ for all n. Then

$$\sup_{1\leq i\leq N} |f(i)| = \lim_{n\to\infty} \sup_{1\leq i\leq N} |f_n(i)| \leq B$$

Since this holds for arbitrary N, it follows that $\|f\|_1 \leq B$.
Given $\varepsilon > 0$, choose n_0 such that $\|f_n - f_m\|_\infty < \varepsilon$ whenever $m, n \geq n_0$. Then for any N and any $n \geq n_0$,

$$\sup_{1\leq i\leq N} |f(i) - f_n(i)| = \lim_{m\to\infty} \sup_{1\leq i\leq N} |f_m(i) - f_n(i)| < \varepsilon$$

and thus $\|f - f_n\|_\infty < \varepsilon$ for all $n \geq n_0$, so $f_n \to f$.

$\square$