Homework 9

Aleck Zhao

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Chapter 14: Wiener Processes and Ito's Lemma

3. A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of negative cash position by the end of 1 year?

   Solution. After 4 quarters, the company will have a position that follows a normal distribution with mean $4 \cdot 0.5 = 2$ and variance $4.0 \cdot 4 = 4^2$. If X is the distribution of their position, we have

   $$P(X < 0) = P\left(\frac{X-2}{4} < -\frac{0-2}{4}\right) = \Phi\left(-\frac{1}{2}\right)$$

   □

5. Consider a variable S that follows the process

   $$dS = \mu\, dt + \sigma\, dz$$

   For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

   Solution. After the first 3 years, S has normal distribution with mean $3 \cdot 2 = 6$ and variance $3 \cdot 3 = 9$. After the next 3 years, S has normal distribution with mean $6 + 3 \cdot 3 = 15$ and variance $9 + 3 \cdot 4 = 21$. □

9. It has been suggested that the short-term interest rate r follows the stochastic process

   $$dr = a(b - r)\, dt + rc\, dz$$

   where a, b, c are positive constants and dz is a Wiener process. Describe the nature of this process.

   Solution. When $r < b$, the drift rate is positive, while when $r > b$, the drift rate is negative, so the rate is attracted to b. The larger the value of a, the faster the rate approaches b. Then c is a volatility rate. □

11. Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

    $$dx = a(x_0 - x)\, dt + sx\, dz$$

    where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

Solution. The bond price is $G(x,t) = e^{-x(T-t)}$. Here $a(x,t) = a(x_0 - x)$ and $b(x,t) = sx$, so by Ito's lemma we have

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial x} a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2(x,t) \right) dt + \frac{\partial G}{\partial x} b(x,t)\, dz \\ &= \left(-(T-t)e^{-x(T-t)} \cdot a(x_0 - x) + xe^{-x(T-t)} + \frac{1}{2}(T-t)^2 e^{-x(T-t)} s^2 x^2 \right) dt - (T-t)e^{-x(T-t)} sx\, dz \end{aligned}$$

□

13. Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:

   (a) The expected stock price at the end of the next day.

   Solution. Using $t = 1/365$, we have

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} = 0.16 \cdot 50 \cdot \frac{1}{365} + 0.30 \cdot 50\varepsilon \sqrt{\frac{1}{365}} = 0.0219 + 0.7851\varepsilon$$

   so the expected stock price at the end of the next day is $50 + 0.0219 = 50.0219$. □

   (b) The standard deviation of the stock price at the end of the next day

   Solution. The standard deviation of the stock price at the end of the next day is 0.7851. □

   (c) The 95% confidence limits for the stock price at the end of the next day

   Solution. For the 95% confidence interval, we have $z_{5/2} = 1.96$, so the confidence interval is

$$(50.0219 - z_{5/2} \cdot 0.7851, 50.0219 + z_{5/2} \cdot 0.7851) = (48.4830, 51.5608)$$

□