Homework 2

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1. Consider the statement you are asked to prove in Scheinerman Problem 5.21.

   (a) Rewrite the statement in if-then format.

   Answer. Prove that if a, b are distinct, nonconsecutive perfect squares, then their difference is composite.

   (b) Explain how the term "without loss of generality" could be used in the proof of this statement.

   Answer. There is nothing special about a or b, so WLOG we can assume $a > b$.

   (c) Use direct proof to prove the statement.

   Proof. WLOG, $a > b$. Let $a = n^2$ and $b = m^2$, so

   $$a - b = n^2 - m^2 = (n - m)(n + m)$$

   By assumption, a and b are not consecutive squares, so $n - m \neq 1$. Clearly $n + m \neq 1$, so $a - b$ is the product of two factors, neither of which is 1, so it is composite, as desired. □

2. Let a, b be nonequal positive integers.

   (a) Explain why $(a - b)^2 > 0$.

   Answer. Since $a \neq b$, we have $a - b \neq 0$. If $a > b$, then $a - b > 0 \implies (a - b)^2 > 0$. On the other hand, if $a < b$, then $a - b < 0 \implies (a - b)^2 > 0$.

   (b) Use direct proof to show that $\frac{a}{b} + \frac{b}{a} > 2$.

   Proof. We wish to show that

   $$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} > 2$$

   Since a, b are positive, we can multiply by ab on both sides of the inequality.

   $$\begin{aligned} \frac{a^2 + b^2}{ab} &> 2 \\ \implies a^2 + b^2 &> 2ab \\ \implies a^2 - 2ab + b^2 = (a - b)^2 &> 0 \end{aligned}$$

   This is true by part (a), so the statement is proven. □

3. Prove/Disprove the following statement: Let $x, y \in \mathbb{Z}$. Let n be a positive integers. If $x \mid y^n$, then $x \mid y$.

Proof. This is false. Let $x = 4, y = 6, n = 2$. Then $4 \mid 6^2$, but $4 \nmid 6$. □

4. Suppose $a, b, c \in \mathbb{Z}$. Prove the following

   (a) If $a \mid b$ and $b \neq 0$, then $a = \pm b$ or $|a| < |b|$.

   Proof. If $a \mid b$, then $b = na$ for some $n \in \mathbb{Z}$. If $n = \pm 1$, then $a = \pm b$. Otherwise, $|n| > 1$, so

   $$b = na \implies |b| = |na| = |n|\,|a| > |a|$$

   as desired. $\square$

   (b) If $a \mid b$ and $b \mid a$ then $|a| = |b|$.

   Proof. If $a \mid b$, then $b = na$ for some $n \in \mathbb{Z}$. Dividing by n, we have $a = \frac{1}{n}b$. Since $b \mid a$, we must also have $\frac{1}{n} \in \mathbb{Z}$, so it must be that either $n = 1$ or $n = -1$. If $n = 1$, then $a = b \implies |a| = |b|$, and if $n = -1$, then $a = -b \implies |a| = |b|$, as desired. $\square$

   (c) If $a \mid b$ and $a \mid c$ then for any $x, y \in \mathbb{Z}, a \mid (bx + cy)$.

   Proof. If $a \mid b$ and $a \mid c$, then $b = na$ and $c = ma$ for some $n, m \in \mathbb{Z}$. Then

   $$bx + cy = (na)x + (ma)y = a(nx + my)$$

   so $a \mid (bx + cy)$, as desired. $\square$

8.12 A U.S. Social Security number is a nine-digit number. The first digit(s) may be 0.

a. How many Social Security numbers are available?

   Answer. There are 10 possibilities for each digit, so there are 10^9 possible numbers.

b. How many of these are even?

   Answer. For the last digit, there are only 5 even possibilities, and there are 10 possibilities for each of the other 8, so there are $5 \cdot 10^8$ even numbers.

c. How many have all of their digits even?

   Answer. There are 5 even digits, so there are 5^9 numbers with all even digits.

d. How many read the same backward and forward?

   Answer. There are 10 choices for the first, second, third, fourth, and fifth digits. After that, the sixth, seventh, eighth, and ninth digits are already determined by the first four. Thus, there are 10^5 palindromic numbers.

e. How many have none of their digits equal to 8?

   Answer. Now there are only 9 choices for each digit, so there are 9^9 such numbers.

f. How many have at least one digit equal to 8?

   Answer. There are 10^9 total numbers, and 9^9 numbers without any digit equal to 8, so there are $10^9 - 9^9$ numbers with at least one digit equal to 8 since these two events partition the set of numbers.

g. How many have exactly one 8?

   Answer. There are 9 positions for the 8. For the remaining 8 digits, there are 9 possibilities for each since they can't be equal to 8, so there are $9 \cdot 9^8 = 9^9$ numbers with exactly one 8.

8.15 How many five-digit numbers are there that do not have two consecutive digits the same? Note: the first digit may not be 0.

Solution. There are 9 choices for the first digit, then 9 choices for the second since it can't be the same as the first (but can include 0 now), and likewise 9 choices for the third since it can't be the same as the second, and 9 for the fourth and fifth for the same reason. Thus, there are $9^5 = 54049$ such numbers. □

9.9 Please calculate the following:

a. $1 \times 1! = 1$
b. $1 \times 1! + 2 \times 2! = 5$
c. $1 \times 1! + 2 \times 2! + 3 \times 3! = 23$
d. $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 119$
e. $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + 5 \times 5! = 719$

Now make a conjecture. That is, guess the value of $\displaystyle\sum_{k=1}^{n} k \cdot k!$

Answer. This sum evaluates to $(n+1)! - 1$.

9.15 The double factorial $n!!$ is defined for odd positive integers n; it is the product of all the odd numbers from 1 to n inclusive.

a. Evaluate 9!!.

Answer. $9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 945$

b. For an odd integer n, are $n!!$ and $(n!)!$ equal?

Answer. No. For $n = 3$, we have $3!! = 3 \cdot 1 = 3$, but $(3!)! = 6! = 720$.

c. Write an expression for $n!!$ using product notation.

Answer. $\displaystyle\sum_{i=1}^{(n+1)/2} (2i - 1)$

d. Explain why this formula works:

$$(2k-1)!! = \frac{(2k)!}{k!2^k}$$

Solution. For $k = 1$, we have $(2k-1)!! = 1 = \frac{2!}{1! \cdot 2}$. Now suppose the formula holds for arbitrary $k = n$, that is

$$(2n-1)!! = \frac{(2n)!}{n!2^n}$$

Multiplying both sides by $2n + 1$, we have

$$\begin{aligned}(2n+1)(2n-1)!! = (2n+1)!! = (2(n+1)-1)!! \\ = \frac{(2n)!}{n!2^n} \cdot (2n+1) = \frac{(2n+1)!}{n!(2^n)} \cdot \frac{2n+2}{2(n+1)} = \frac{(2n+2)!}{(n+1)!2^{n+1}} \\ = \frac{(2(n+1))!}{(n+1)!2^{n+1}}\end{aligned}$$

Thus, the formula holds for $k = n + 1$, so by induction the formula holds for all $k \geq 1$. □

10.1 Write out the following sets by listing their elements between curly braces.

(a) $\{\, x \in \mathbb{N} \mid x \leq 10 \text{ and } 3 \mid x \,\} = \{0, 3, 6, 9\}$

(b) $\{\, x \in \mathbb{Z} \mid x \text{ is prime and } 2 \mid x \,\} = \{2\}$

(c) $\{\, x \in \mathbb{Z} \mid x^2 = 4 \,\} \{2, -2\}$

(d) $\{\, x \in \mathbb{Z} \mid x^2 = 5 \,\} = \{\}$

(e) $2^{\varnothing} = \{\varnothing\}$

(f) $\{\, x \in \mathbb{Z} \mid 10 \mid x \text{ and } x \mid 100 \,\} = \{-100, -50, -20, -10, 10, 20, 50, 100\}$

(g) $\{\, x \mid x \subseteq \{1, 2, 3, 4, 5\} \text{ and } |x| \leq 1 \,\} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$