Homework 7

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October 29, 2016

Section 2.10: The Isomorphism Theorem

22. Show that $\mathbb{R}^*/\{1,-1\} \cong \mathbb{R}^+$.

29. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$.

    (a) Show that G is a subgroup of $M_3(\mathbb{R})^*$ and that $Z(G) \cong \mathbb{R}$.

    (b) Show that $G/Z(G) \cong \mathbb{R} \times \mathbb{R}$.

Section 8.2: Cauchy's Theorem

7. If H and K are conjugate subgroups in G, show that $N(H)$ and $N(K)$ are conjugate.

14. Let $D_3 = \left\{1, a, a^2, b, ba, ba^2\right\}$ where $o(a) = 3, o(b) = 2, aba = b$. If $H = \{1, b\}$, show that $N(H) = H$.

23. Let G^ω be the group of sequences $[g_i) = (g_0, g_1, \cdots)$ from a group G with component-wise multiplication $[g_i) \cdot [h_i) = [g_i h_i)$. Show that if $G \neq \{1\}$ is a finite p-group, then G^ω is an infinite p-group.

26. Let G be a non-abelian group of order p^3 where p is a prime. Show that

    (a) $Z(G) = G'$ and this is the unique normal subgroup of G of order p.

    (b) G has exactly $p^2 + p - 1$ distinct conjugacy classes.

Section 8.3: Group Actions

3. If p and q are primes, show that no group of order pq is simple.

13. Let $G = (\mathbb{R}, +)$ and define $a \cdot z = e^{ia} z$ for all $z \in \mathbb{C}$ and $a \in G$. Show that $\mathbb{C}$ is a G-set, describe the action geometrically, and find all orbits and stabilizers.

21. If H is a subgroup of G, find a G-set X and an element $x \in X$ such that $H = S(x)$.

23. Let X be a G-set and let x and y denote elements of X.

    (a) Show that $S(X)$ is a subgroup of G.

    (b) If $x \in X$ and $b \in G$, show that $S(b \cdot x) = bS(x)b^{-1}$.

    (c) If $S(x)$ and $S(y)$ are conjugate subgroups, show that $|G \cdot x| = |G \cdot y|$.