Homework 5

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Chapter 4: Open Sets and Closed Sets

3. Some authors say that two metrics d and ρ on a set M are equivalent if they generate the same open sets. Prove this.

   Proof. Suppose $(x_n) \to x$ under d. Then let U be an open set under ρ containing x. Since d and ρ are equivalent, U is also open under d, so x_n is eventually in U, which means that $(x_n) \to x$ under ρ as well. Thus, if d and ρ generate the same open sets, they generate the same convergent sequences, so they are equivalent. □

18. Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are elements of $\overline{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is closed.

    Proof. From HW1, there exists $a, b \in E$ such that $a > \sup E - \varepsilon$ and $b < \inf E + \varepsilon$ for any $\varepsilon > 0$, thus

$$a \in B_\varepsilon(\sup E) \cap E \implies \sup E \in \overline{E}$$
$$b \in B_\varepsilon(\inf E) \cap E \implies \inf E \in \overline{E}$$

   If E is closed, then $E = \overline{E}$ and the conclusion follows. □

33. Let A be a subset of M. A point $x \in M$ is called a limit point of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \varnothing$ for every $\varepsilon > 0$. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

    Proof. Suppose there exists $\varepsilon > 0$ such that $(B_\varepsilon(x) \setminus \{x\}) \cap A = \{x_1, x_2, \cdots, x_n\}$ is a finite set. Take

$$r = \min_{1 \le i \le n} \{d(x, x_i)\}$$
$$\implies \varnothing = (B_r(x) \setminus \{x\}) \cap A$$

   which contradicts the fact that x is a limit point. Thus, any intersection must be infinite, as desired. □

41. Related to the notion of limit points and isolated points are boundary points. A point $x \in M$ is said to be a boundary point of A if each neighborhood of x hits both A and A^c. In symbols, x is a boundary point of A if and only if $B_\varepsilon(x) \cap A \neq \varnothing$ and $B_\varepsilon(x) \cap A^c \neq \varnothing$ for every $\varepsilon > 0$. Verify each of the following formulas, where $\partial(A)$ denotes the set of boundary points of A :

    (a) $\partial(A) = \partial(A^c)$

    Proof.

$$\begin{aligned} x \in \partial(A) &\iff B_\varepsilon(x) \cap A \neq \varnothing \text{ and } B_\varepsilon \cap A^c \neq \varnothing \\ &\iff B_\varepsilon(x) \cap A^c \neq \varnothing \text{ and } B_\varepsilon(x) \cap (A^c)^c \neq \varnothing \\ &\iff x \in \partial(A^c) \end{aligned}$$

   □

(b) $\overline{A} = \partial(A) \cup A^\circ$

Proof. $(\supset)$: Suppose $x \in \partial(A)$ but $x \notin A$. Then x is not in some set B containing A, so $x \in B^c$ which is open. Thus, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \varnothing$ since $A \subset B$. This contradicts $x \in \partial(A)$, so x must be in every closed set containing A, so $x \in \overline{A}$.

$(\subset)$: Suppose $x \in \overline{A}$ but $x \notin \partial(A) \cup A^\circ$. Then $x \notin A^\circ$ so $B_\varepsilon(x) \not\subset A \implies B_\varepsilon(x) \cap A^c \neq \varnothing$ for all $\varepsilon > 0$. Since $x \notin \partial(A)$, it must be that $B_\delta(x) \cap A = \varnothing$ for some δ. Then $(B_\delta(x))^c \supset A$ is a closed set containing A but not x, which contradicts $x \in \overline{A}$. Thus, $x \in \partial(A) \cup A^\circ$. ☐

(c) $M = A^\circ \cup \partial(A) \cup (A^c)^\circ$

Proof. From part (b), this is $M = \overline{A} \cup (A^c)^\circ$.

$(\supset)$: If $x \in \overline{A}$, then since M is closed in M, we have $\overline{A} \subset M$, so $x \in M$. Since $(A^c)^\circ \subset A^c \subset M$, it follows that if $x \in (A^c)^\circ$, it must be that $x \in M$.

$(\subset)$: Suppose $x \in M$ but $x \notin \overline{A}$ and $x \notin (A^c)^\circ$. Since $x \notin (A^c)^\circ$, we have $B_\varepsilon(x) \not\subset A^c \implies B_\varepsilon(x) \cap A \neq \varnothing$ for all $\varepsilon > 0$. Since $x \notin \overline{A}$, there exists some $\delta > 0$ such that $B_\delta(x) \cap A = \varnothing$. Contradiction, so $x \in \overline{A} \cup (A^c)^\circ$. ☐

48. A metric space is called separable if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}$, in $\mathbb{R}^2$, and in $\mathbb{R}^n$.

Solution. $\mathbb{Q} \subset \mathbb{R}$ is a countable dense subset. Then under the Euclidean metric, $\mathbb{Q}^2 \subset \mathbb{R}^2$ is also dense, and countable. Consider any $x = (x_1, x_2) \in \mathbb{R}^2$. Then consider sequences $(p_n) \to x_1$ and $(q_n) \to x_2$ where $p_i, q_i \in \mathbb{Q}$, so $d(x, (p_n, q_n)) \to 0$ and thus $(p_n, q_n) \to x$. Extending this argument, $\mathbb{Q}^n \subset \mathbb{R}^n$ is a countable dense subset. ☐

Chapter 5: Continuity

17. Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

Proof. Suppose $x \in M \setminus D$. Then since D is dense in M, there exists a sequence $(x_n) \to x$ in D. Since f is continuous, we have $f(x_n) \to f(x)$ and since f and g agree on D, we have $g(x_n) \to f(x)$, and thus $f(x) = g(x)$. If $x \in D$, then the conclusion is obviously true, so $f(x) = g(x)$ for all $x \in M$, as desired.

If f is surjective, for any $y \in N$ there exists $x \in M$ such that $f(x) = y$. Then since D is dense, there exists a sequence $(x_n) \to x$ in D, and since f is continuous, we have $f(x_n) \to f(x) = y$, so the sequence $(f(x_n)) \to y$ where $f(x_i) \in f(D)$, and thus $f(D)$ is dense in N, as desired. ☐

42. Suppose that $f : \mathbb{Q} \to \mathbb{R}$ is Lipschitz. Show that f extends to a continuous function $h : \mathbb{R} \to \mathbb{R}$. Is h unique? Explain. (Hint: Given $x \in \mathbb{R}$, choose a sequence of rationals (r_n) converging to x and argue that $h(x) = \lim_{n\to\infty} f(r_n)$ exists and is actually independent of the sequence (r_n).)

Proof. If f is Lipschitz, then let $K \in \mathbb{R}$ such that $|f(x) - f(y)| \leq K|x - y|$. If $x \in \mathbb{Q}$, set $h(x) = f(x)$. Otherwise, if $x \in \mathbb{R} \setminus \mathbb{Q}$, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rationals $(r_n) \to x$. Thus, since convergent sequences are Cauchy, for any ε, there exists $N \in \mathbb{N}$ such that $|r_n - r_m| < \frac{\varepsilon}{K}$ for all $n, m \geq N$. Then since f is Lipschitz, we have

$$|f(r_n) - f(r_m)| \leq K|r_n - r_m| < \varepsilon$$

for all $n, m \geq N$, so $f(r_n)$ is Cauchy. Since Cauchy sequences in $\mathbb{R}$ converge, set $h(x) = \lim_{n\to\infty} f(r_n)$. Since f is continuous, it doesn't matter which sequence (r_n) we choose, and h is unique. ☐

46. Show that every metric space is homeomorphic to one of finite diameter. (Hint: Every metric is equivalent to a bounded metric.)

Proof. Let (M, d) be a metric space, and set $\rho(x, y) = \min\{1, d(x, y)\}$ for $x, y \in M$. Then ρ is a metric on M from HW4, so (M, ρ) is a bounded metric space since $\rho(x, y) \leq 1$ for any $x, y \in M$, and from exercise 3.42, d is equivalent to ρ. Since d and ρ are equivalent, we have $i : (M, d) \to (M, \rho)$ the identity map and its inverse i^{-1} are both continuous, so (M, d) is homeomorphic to (M, ρ). $\square$

48. Prove that $\mathbb{R}$ is homeomorphic to $(0, 1)$ and that $(0, 1)$ is homeomorphic to $(0, \infty)$. Is $\mathbb{R}$ isometric to $(0, 1)$? to $(0, \infty)$? Explain.

Proof. Let $f : (0, 1) \to \mathbb{R}$ be given by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$. Clearly f is continuous on $(0, 1)$ under the usual metric. Then $f^{-1}(y) = \frac{1}{\pi}\arctan(y) + \frac{1}{2}$ is its continuous inverse on $\mathbb{R}$, so $\mathbb{R}$ is homeomorphic to $(0, 1)$, under the usual metric for both.

Let $g : (0, 1) \to (0, \infty)$ be given by $g(x) = \frac{x}{1-x}$. Clearly g is continuous on $(0, 1)$ under the usual metric. Then $g^{-1}(y) = \frac{y}{1+y}$ is also continuous on $(0, \infty)$, so $(0, 1)$ is homeomorphic to $(0, \infty)$, under the usual metric for both.

$\mathbb{R}$ is not isometric to $(0, 1)$ or $(0, \infty)$, or any proper subset of itself. Suppose $X \subsetneq \mathbb{R}$ and $f : \mathbb{R} \to X$ was isometric. Then $|f(x) - f(0)| = |x|$ so either $f(x) = f(0) + x$ or $f(x) = f(0) - x$. Since X is a proper subset, there exists $y \in \mathbb{R} \setminus X$. Then if $f(y - f(0)) = f(0) + (y - f(0)) = y$, we reach a contradiction since $y \notin X$, so we must have $f(y - f(0)) = f(0) - (y - f(0)) = 2f(0) - y$. Similarly, we must have $f(f(0) - y) = 2f(0) - y = f(y - f(0))$. Since f is an isometry, it must be injective, so $f(0) - y = y - f(0)$ so $y = f(0)$, which is a contradiction since $y \notin X$. Thus, an isometry does not exist. $\square$

56. Let $f : (M, d) \to (N, \rho)$.

(i) We say that f is an open map if $f(U)$ is open in N whenever U is open in M; that is, f maps open sets to open sets. Give examples of a continuous map that is not open and an open map that is not continuous.

Solution. Let $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = n$. Then f is continuous, and $\{n\}$ is open in $\mathbb{N}$, but $f(\{n\}) = \{n\}$, which is not open in $\mathbb{R}$.
Let $g : \mathbb{R} \to \mathbb{Z}$ where $g(x) = \lfloor x \rfloor$ and $\mathbb{Z}$ is endowed with the discrete metric. Then any subset of $\mathbb{Z}$ is open, but g is not continuous. $\square$

(ii) Similarly, f is called closed if it maps closed sets to closed sets. Give examples of a continuous map that is not closed and a closed map that is not continuous.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x$. Then f is continuous on $\mathbb{R}$, and $\mathbb{R}$ is closed in $\mathbb{R}$, but $f(\mathbb{R}) = (0, \infty)$ is not closed in $\mathbb{R}$.
Let $g : \mathbb{R} \to \mathbb{Z}$ where $g(x) = \lfloor x \rfloor$ and $\mathbb{Z}$ is endowed with the discrete metric. Then any subset of $\mathbb{Z}$ is closed, but g is not continuous. $\square$