Homework 6

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October 27, 2016

Section 2.6: Cosets and Lagrange's Theorem

4. If $K \subseteq H \subseteq G$ are finite groups, show that $|G : K| = |G : H| \cdot |H : K|$.

   Proof. For finite groups, we have $|G : K| = |G| / |K|$ and similarly for the other two, so we have

   $$\frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|}$$

   as desired. □

15. If H and K are subgroups of a group and $|H|$ is prime, show that either $H \subseteq K$ or $H \cap K = \{1\}$.

    Proof. Let $|H| = p$ where p is a prime. Thus, H must be a cyclic group, and is the only one of order p. We have $H \cap K$ is a subgroup of H, so $|H \cap K|$ divides $|H|$, so either $|H \cap K| = p$ or $|H \cap K| = 1$. In the first case, $H \cap K = H$, so $H \subseteq K$, and in the second case, $H \cap K = \{1\}$, as desired. □

27. Is $D_5 \times C_3 \cong D_3 \times C_5$? Prove your answer.

    Solution. The element $(\theta, g) \in D_3 \times C_5$ has order $\mathrm{lcm}(2, 5) = 10$, but there is no element of order 10 in $D_5 \times C_3$. The maximum order of any element in D_5 is 5, and elements in C_3 have order 3, except the identity. Thus, the orders of elements in $D_5 \times C_3$ are 5 and 15, but none have order 10. □

Section 2.8: Normal Subgroups

4. If $D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$, $K = \{1, b\}$ and $H = \{1, a^2, b, ba^2\}$ show that $K \trianglelefteq H \trianglelefteq D_4$, but $K \not\trianglelefteq D_4$.

   Proof. Since $|H : K| = 2$, by section 2.8 theorem 4, K is normal in H. Similarly, $|D_4 : H| = 2$, so H is normal in D_4. However, we have $aK = \{a, ab\} \neq \{a, ba\} = Ka$ since $ab \neq ba$. □

11. Let p and q be distinct primes. If G is a group of order pq that has a unique subgroup of order p and a unique subgroup of order q, show that G is cyclic.

Proof. Let H and K be the subgroups with order p and q. Then H and K are both cyclic and normal because they are the only ones with these orders. The intersection $H \cap K = \{1\}$ because it is a subgroup of both H and K, thus its order must divide both primes p and q, so the only possible order is 1. Thus, by Corollary 2 of Theorem 6, $G \cong H \times K$, so $G = C_p \times C_q$. This is a cyclic group of order pq, as desired. $\square$

16. Show that $\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G$ for any group G.

    Proof. Let $\varphi \in \operatorname{Aut} G$ be an isomorphism and $\sigma_a \in \operatorname{Inn} G$ be an inner automorphism. Then consider for some $g \in G$,

$$\begin{aligned}
(\varphi\sigma_a\varphi^{-1})(g) &= \varphi(\sigma_a(\varphi^{-1}(g))) \\
&= \varphi(a\varphi^{-1}(g)a^{-1}) \\
&= \varphi(a)\varphi(\varphi^{-1}(g))\varphi(a^{-1}) \\
&= \varphi(a)g\varphi(a^{-1}) \\
&= \varphi(a)g(\varphi(a))^{-1} \\
&= \sigma_{\varphi(a)}(g)
\end{aligned}$$

    so $\varphi\sigma_a\varphi^{-1} \in \operatorname{Inn} G$, so by part 2 of the Normality test, $\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G$ as desired. $\square$

25. If X is a nonempty subset of a group G, define the **normalizer** $N(X)$ of X by

$$N(X) = \{ a \in G \mid aXa^{-1} = X \}.$$

    (a) Show that $N(X)$ is a subgroup of G.

    Proof. Clearly $1_G X 1_G^{-1} = X$, so $1_G \in N(X)$. Then if $a, b \in N(X)$, we have

$$\begin{aligned}
aXa^{-1} &= X \\
bXb^{-1} &= X \\
\implies a(bXb^{-1})a^{-1} &= X \\
\implies (ab)X(ba)^{-1} &= X
\end{aligned}$$

    so $ab \in N(X)$. Then if $a \in N(X)$, we have

$$\begin{aligned}
aXa^{-1} &= X \\
aX &= Xa \\
X &= a^{-1}Xa
\end{aligned}$$

    so $a^{-1} \in N(X)$ as well. Thus, $N(X)$ is a subgroup of G, as desired. $\square$

    (b) If H is a subgroup of G, show that $H \trianglelefteq N(H)$.

    Proof. We must show that for all $n \in N(H)$, it holds that $nHn^{-1} = H$. However, by the way $N(H)$ is defined, $N(H)$ consists exactly of all elements $g \in G$ such that $gHg^{-1} = H$. Thus, for all $n \in N(H)$, it olds that $nHn^{-1} = H$, so $H \trianglelefteq N(H)$, as desired. $\square$

    (c) If H is a subgroup of G, show that $N(H)$ is the largest subgroup of G in which H is normal. That is, if $H \trianglelefteq K$, and K if a subgroup of G, then $K \subseteq N(H)$.

    Proof. By the normality test, if $k \in K$ since H is normal in K, we have $kHk^{-1} = H$. The normalizer of H is defined as all $g \in G$ such that $gHg^{-1} = H$. Thus, if $k \in K$, it must be that $k \in N(H)$, so $K \subseteq N(H)$, as desired. $\square$

Section 2.10: The Isomorphism Theorem

7. If $\alpha : G \to G_1$ is a group homomorphism and both $\alpha(G)$ and $\ker\alpha$ are finitely generated, show that G is finitely generated.

Proof. Let $\alpha(G) = \langle X\rangle = \langle x_1, x_2, \cdots, x_n\rangle$ and $\ker\alpha = \langle Y\rangle = \langle y_1, \cdots, y_m\rangle$ where $x_1, \cdots, x_n \in \alpha(G)$ and $y_1, \cdots, y_m \in G$. Then since x_i are in the image, let $x_i = \alpha(z_i)$ for some $z_i \in G$. Thus, for some $g \in G$, its image $\alpha(g) \in G$, so we can write it as

$$\begin{aligned}\alpha(g) &= x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \\ &= \alpha(z_1)^{k_1}\alpha(z_2)^{k_2}\cdots\alpha(z_n)^{k_n} \\ &= \alpha\left(z_1^{k_1} z_2^{k_2}\cdots z_n^{k_n}\right)\end{aligned}$$

since α is a homomorphism. Let $h = z_1^{k_1}\cdots z_n^{k_n}$, so

$$\begin{aligned}\alpha(gh^{-1}) &= \alpha(g)\alpha(h^{-1}) = \alpha(g)\alpha(h)^{-1} \\ &= \alpha(z_1^{k_1}\cdots z_n^{k_n})\alpha(z_1^{k_1}\cdots z_n^{k_n}) = 1 \\ \implies gh^{-1} &\in \ker\alpha \\ \implies gh^{-1} &= y_1^{j_1} y_2^{j_2}\cdots y_m^{j_m} \\ \implies g &= y_1^{j_1}\cdots y_m^{j_m} h \\ &= y_1^{j_1}\cdots y_m^{j_m} z_1^{k_1}\cdots z_n^{k_n}\end{aligned}$$

thus any $g \in G$ is in the set $\langle y_1, \cdots, y_m, z_1, \cdots, z_n\rangle$ which is finite. Thus, G is finitely generated, as desired.

□

9. If $K = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$, is there a group homomorphism $\alpha : S_4 \to A_4$ with $\ker\alpha = K$?

Solution. If such a group homomorphism exists, then $\alpha(S_4) \cong S_4/K$ by the isomorphism theorem. We have $|S_4/K| = 4!/4 = 6$, and the only groups of order 6 are C_6 and S_3. Clearly this quotient group is not cyclic, so it must be isomorphic to S_3. Thus, $\alpha(S_4) \cong S_3$ is a subgroup of A_4. However, $\sigma^2 = (312) \in S_3$ but $\sigma^2 \notin A_4$, so this is a contradiction, so no such homomorphism exists.

□

21. Show that $\mathbb{C}^*/\mathbb{C}^0 \cong \mathbb{R}^+$ where $\mathbb{C}^0 = \{\, z \mid |z| = 1 \,\}$ is the circle group.

Proof. Define the homomorphism $\varphi : \mathbb{C}^* \to \mathbb{R}^+$ where $\varphi(z) = |z|$. This is indeed a homomorphism because $\varphi(z_1 z_2) = |z_1 z_2| = |z_1|\,|z_2| = \varphi(z_1)\varphi(z_2)$.

Then the kernel of φ is the set $\{\, z \mid \varphi(z) = 1 \,\}$ which is exactly $\mathbb{C}^0$. Finally, $\varphi``(\mathbb{C}^*) = \mathbb{R}^+$ since invertible elements in $\mathbb{C}$ are all except 0, whose magnitudes are all positive.

Thus, by the first Isomorphism Theorem, since $\mathbb{C}^0$ is the kernel of a homomorphism, $\mathbb{C}^*/\mathbb{C}^0 \cong \mathbb{R}^+$, as desired.

□