Homework 2

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Chapter 14: The Riemann-Stieltjes Integral

1. If $f, g \in \mathcal{R}_\alpha[a,b]$ with $f \le g$, show that $\int_a^b f\,d\alpha \le \int_a^b g\,d\alpha$.

Proof. We first show that $L(f,P) \le L(g,P)$ for any fixed partition P. We have

$$\begin{aligned} L(f,P) &= \sum_{i=1}^n \inf\{f(x) : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i \\ &\le \sum_{i=1}^n \inf\{g(x) : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i = L(g,P) \end{aligned}$$

as desired. Now, fix partitions P and Q. We have

$$L(f,P) \le L(f, P\cup Q) \le U(g, P\cup Q) \le U(g,Q)$$

Since P and Q were arbitrary, and since $f, g \in \mathcal{R}_\alpha[a,b]$, we have

$$\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \sup_P L(f,P) \le \inf_Q U(g,Q) = \overline{\int_a^b} g\,d\alpha = \int_a^b g\,d\alpha$$

□

3. If $f \in \mathcal{R}_\alpha[a,b]$, show that $|f| \in \mathcal{R}_\alpha[a,b]$ and that $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$. (Hint: $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$. Why?)

Proof. Since $f \in \mathcal{R}_\alpha[a,b]$, given any $\varepsilon > 0$, we can find a partition P such that $U(f,P) - L(f,P) < \varepsilon$. Let P be such a partition of $[a,b]$. We have

$$\begin{aligned} U(|f|,P) &= \sum_{i=1}^n \sup\{|f(x)| : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i \\ L(|f|,P) &= \sum_{i=1}^n \inf\{|f(x)| : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i \end{aligned}$$

Now, on any interval $[x_{i-1}, x_i]$, we have

$$\sup|f(x)| - \inf|f(x)| \le \sup f(x) - \inf f(x)$$

which is clear by checking signs. Thus,

$$U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P) < \varepsilon$$

so $|f|$ is RS-integrable by Riemann's condition. □

6. Define increasing functions α, β, and γ on $[-1,1]$ by $\alpha = \chi_{(0,1]}, \beta = \chi_{[0,1]}$, and $\gamma = \frac{1}{2}(\alpha+\beta)$. Given $f \in B[-1,1]$, show that:

   (a) $f \in \mathcal{R}_\alpha[-1,1]$ if and only if $f(0+) = f(0)$.

   Proof. ($\implies$) : If $f \in \mathcal{R}_\alpha[-1,1]$, then given $\varepsilon > 0$, there exists a partition P WLOG with $x_k = 0$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

   Now, $\Delta\alpha_i = 1$ only when $i = k+1$, so we havee

$$U(f,P) = \sup_{[0,x_{k+1}]} f(x)$$
$$L(f,P) = \inf_{[0,x_{k+1}]} f(x)$$
$$U(f,P) - L(f,P) < \varepsilon \implies |f(x) - f(0)| < \varepsilon, \forall x \in [0, x_{k+1}]$$

   Thus, given ε, we have $|f(x) - f(0)| < \varepsilon$ whenever $0 < x < \frac{x_{k+1}}{2}$, so $f(0+) = f(0)$.
   ($\impliedby$) : If $f(0+) = f(0)$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(0) - \frac{\varepsilon}{2} < f(x) < f(0) + \frac{\varepsilon}{2}$ whenever $0 < x < \delta$. Let P be a partition of $[-1,1]$, with $0 = x_k \in P$ and $\delta/2 = x_{k+1}$. Then $\Delta\alpha_i = 1$ only when $i = k+1$, so

$$U(f,P) = \sup_{[0,\delta/2]} f(x) < f(0) + \frac{\varepsilon}{2}$$
$$L(f,P) = \inf_{[0,\delta/2]} f(x) > f(0) - \frac{\varepsilon}{2}$$
$$\implies U(f,P) - L(f,P) < \left(f(0) + \frac{\varepsilon}{2}\right) - \left(f(0) - \frac{\varepsilon}{2}\right) = \varepsilon$$

   so $f \in \mathcal{R}_\alpha[-1,1]$. □

   (b) $f \in \mathcal{R}_\beta[-1,1]$ if and only if $f(0-) = f(0)$.

   Proof. ($\implies$) : If $f \in \mathcal{R}_\beta[-1,1]$, then given $\varepsilon > 0$, there exists a partition P WLOG with $x_k = 0$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

   Now, $\Delta\beta_i = 1$ only when $i = k$, so we havee

$$U(f,P) = \sup_{[x_{k-1},0]} f(x)$$
$$L(f,P) = \inf_{[x_{k-1},0]} f(x)$$
$$U(f,P) - L(f,P) < \varepsilon \implies |f(x) - f(0)| < \varepsilon, \forall x \in [x_{k-1}, 0]$$

   Thus, given ε, we have $|f(x) - f(0)| < \varepsilon$ whenever $\frac{x_{k-1}}{2} < x < 0$, so $f(0-) = f(0)$.
   ($\impliedby$) : If $f(0-) = f(0)$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(0) - \frac{\varepsilon}{2} < f(x) < f(0) + \frac{\varepsilon}{2}$ whenever $-\delta < x < 0$. Let P be a partition of $[-1,1]$, with $0 = x_k \in P$ and $-\delta/2 = x_{k-1}$. Then $\Delta\beta_i = 1$ only when $i = k$, so

$$U(f,P) = \sup_{[-\delta/2,0]} f(x) < f(0) + \frac{\varepsilon}{2}$$
$$L(f,P) = \inf_{[-\delta/2,0]} f(x) > f(0) - \frac{\varepsilon}{2}$$
$$\implies U(f,P) - L(f,P) < \left(f(0) + \frac{\varepsilon}{2}\right) - \left(f(0) - \frac{\varepsilon}{2}\right) = \varepsilon$$

   so $f \in \mathcal{R}_\beta[-1,1]$. □

(c) $f \in \mathcal{R}_\gamma[-1,1]$ if and only if f is continuous at 0.

Proof. ($\implies$) : If $f \in \mathcal{R}_\gamma[-1,1]$, then given $\varepsilon > 0$, there exists a partition P WLOG with $x_k = 0$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

Now, $\Delta\gamma_i = \frac{1}{2}$ when $i = k, k+1$, so we have

$$\begin{aligned} U(f,P) &= \frac{1}{2}\left(\sup_{[x_{k-1},0]} f(x) + \sup_{[0,x_{k+1}]} f(x)\right) \leq \sup_{[x_{k-1},x_{k+1}]} f(x) \\ L(f,P) &= \frac{1}{2}\left(\inf_{[x_{k-1},0]} f(x) + \inf_{[0,x_{k+1}]} f(x)\right) \geq \inf_{[x_{k-1},x_{k+1}]} f(x) \\ U(f,P) - L(f,P) < \varepsilon &\implies |f(x) - f(0)| < \varepsilon, \forall x \in [x_{k-1}, x_{k+1}] \end{aligned}$$

If we let $\delta = \frac{1}{2}\min\{|x_{k-1}|, |x_{k+1}|\}$, we get the necessary condition for continuity of f at 0.
($\impliedby$) : If f is continuous at 0, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(0)| < \frac{\varepsilon}{2}$ whenever $|x| < \delta$. Let P be a partition of $[-1,1]$, with $0 = x_k \in P$ and $-\delta/2 = x_{k-1}$ and $\delta/2 = x_{k+1}$. Then $\Delta\gamma_i = \frac{1}{2}$ when $i = k, k+1$, so

$$\begin{aligned} U(f,P) &= \frac{1}{2}\left(\sup_{[-\delta/2,0]} f(x) + \sup_{[0,\delta/2]} f(x)\right) \leq \sup_{[-\delta/2,\delta/2]} f(x) < f(0) + \frac{\varepsilon}{2} \\ L(f,P) &= \frac{1}{2}\left(\inf_{[-\delta/2,0]} f(x) + \inf_{[0,\delta/2]} f(x)\right) \geq \inf_{[-\delta/2,\delta/2]} f(x) > f(0) - \frac{\varepsilon}{2} \\ \implies U(f,P) - L(f,P) &< \left(f(0) + \frac{\varepsilon}{2}\right) - \left(f(0) - \frac{\varepsilon}{2}\right) = \varepsilon \end{aligned}$$

so $f \in \mathcal{R}_\gamma[-1,1]$. $\square$

(d) If $f \in \mathcal{R}_\gamma[-1,1]$, then $\int_{-1}^1 f\,d\alpha = \int_{-1}^1 f\,d\beta = \int_{-1}^1 f\,d\gamma = f(0)$.

Proof. If $f \in \mathcal{R}_\gamma[-1,1]$, then f is continuous at 0 by part (c), so it is right and left continuous at 0, so all three integrals exist by parts (a) and (b).
Let P be a partition WLOG with $0 = x_k$. Then

$$\begin{aligned} L_\alpha(f,P) = \inf_{[0,x_{k+1}]} f(x) &\implies \int_{-1}^1 f\,d\alpha = \sup_{x_{k+1}}\left(\inf_{[0,x_{k+1}]} f(x)\right) \geq \inf_{[0,0]} f(x) = f(0) \\ U_\alpha(f,P) = \sup_{[0,x_{k+1}]} f(x) &\implies \int_{-1}^1 f\,d\alpha = \inf_{x_{k+1}}\left(\sup_{[0,x_{k+1}]} f(x)\right) \leq \sup_{[0,0]} f(x) = f(0) \\ \implies \int_{-1}^1 f\,d\alpha = f(0) \end{aligned}$$

Where we can take any sequence $x_{k+1} \to 0$. Similarly, $\int_{-1}^1 f\,d\beta = f(0)$. For $\int_{-1}^1 f\,d\gamma$, we have

$$\begin{aligned} L_\gamma(f,P) &= \frac{1}{2}\left(\inf_{[x_{k-1},0]} f(x) + \inf_{[0,x_{k+1}]} f(x)\right) \geq \inf_{[x_{k-1},x_{k+1}]} f(x) \\ \implies \int_{-1}^1 f\,d\gamma &= \sup_{x_{k-1},x_{k+1}}\left(\inf_{[x_{k-1},x_{k+1}]} f(x)\right) \geq \inf_{[0,0]} f(x) = f(0) \end{aligned}$$

and similarly with $U_\gamma(f,P)$, so we get $\int_{-1}^1 f\,d\gamma = f(0)$. $\square$

7. Let $P = \{x_0, \cdots, x_n\}$ be a (fixed) partition of $[a, b]$, and let α be an increasing step function on $[a, b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the x_i, where $\alpha_0 = \alpha(a+) - \alpha(a)$ and $\alpha_n = \alpha(b) - \alpha(b-)$. If $f \in B[a, b]$ is continuous at each of the x_i, show that $f \in \mathcal{R}_\alpha$ and $\int_a^b f\,d\alpha = \sum_{i=1}^n f(x_i)\alpha_i$.

   Proof. We have

$$L(f, P) = \sum_{i=1}^n \inf\{f(x) : x_{i-1} \le x \le x_i\}\,\alpha_i = \sum_{i=1}^n f(x_i)\alpha_i$$
$$U(f, P) = \sum_{i=1}^n \sup\{f(x) : x_{i-1} \le x \le x_i\}\,\alpha_i = \sum_{i=1}^n f(x_i)\alpha_i$$

   since $f(x)$ is constant on each interval $[x_{i-1}, x_i]$. Thus, for any $\varepsilon > 0$, we have $U(f, P) - L(f, P) = 0 < \varepsilon$ so $f \in \mathcal{R}_\alpha[a, b]$ by Riemann's condition.

   If $\sup_Q L(f, Q) > L(f, P) = U(f, P)$, then we would have a contradiction since $L(f, P) \le U(f, Q)$ for any partitions P and Q. Thus, $\sup_Q L(f, Q) = L(f, P) = \int_a^b f\,d\alpha = \sum_{i=1}^n f(x_i)\alpha_i$. □

9. If f is monotone and α is continuous (and still increasing), show that $f \in \mathcal{R}_\alpha[a, b]$.

   Proof. Let P be a partition of $[a, b]$. Then WLOG f is monotone increasing, so we have

$$L(f, P) = \sum_{i=1}^n \inf\{f(x) : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i = \sum_{i=1}^n f(x_{i-1})\,(\alpha(x_i) - \alpha(x_{i-1}))$$
$$U(f, P) = \sum_{i=1}^n \sup\{f(x) : x_{i-1} \le x \le x_i\}\,\Delta\alpha_i = \sum_{i=1}^n f(x_i)\,(\alpha(x_i) - \alpha(x_{i-1}))$$
$$\implies U(f, P) - L(f, P) = f(x_n)\,(\alpha(x_n) - \alpha(x_{n-1})) = f(b)\,(\alpha(b) - \alpha(x_{n-1}))$$

   Since α is continuous, given $\varepsilon > 0$, we can find δ such that

$$|b - x_{n-1}| < \delta \implies |\alpha(b) - \alpha(x_{n-1})| < \frac{\varepsilon}{f(b)}$$

   Thus, as long as the partition P has $|b - x_{n-1}| < \delta$, we will have

$$U(f, P) - L(f, P) = f(b)\,(\alpha(b) - \alpha(x_{n-1})) < f(b) \cdot \frac{\varepsilon}{f(b)} = \varepsilon$$

   so $f \in \mathcal{R}_\alpha[a, b]$ by Riemann's condition. □

10. If $f \in \mathcal{R}_\alpha[a, b]$, show that $f \in \mathcal{R}_\alpha[c, d]$ for every subinterval $[c, d]$ of $[a, b]$. Moreover, $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ for every $a < c < b$. In fact, if any two of these integrals exist, then so does the third and the equation above still holds.

    Proof. Fix $\varepsilon > 0$. Since $f \in \mathcal{R}_\alpha[a, b]$, there exists a partition P of $[a, b]$ with $U(f, P) - L(f, P) < \varepsilon$. Now, let $P' = P \cup \{c, d\}$ and $Q = P' \cap [c, d]$, so P' is a refinement of P and Q is a partition of $[c, d]$. Then we have

$$U(f, P') - L(f, P') \le U(f, P) - L(f, P) < \varepsilon$$

    since $P' \supset P$. Then since Q is a partition of $[c, d]$ contained in P', we have

$$U(f, Q) - L(f, Q) \le U(f, P') + L(f, P') < \varepsilon \implies f \in \mathcal{R}_\alpha[c, d]$$

Let P, Q be partitions of $[a, c]$ and $[c, b]$, respectively. Then $P \cup Q$ is a partition of $[a, b]$. We have

$$L(f, P) + L(f, Q) = L(f, P \cup Q) \leq \int_a^b f\, d\alpha$$

Taking supremums over P and Q, we find that $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha \leq \int_a^b f\, d\alpha$.

If R is a partition of $[a, b]$, then let $R' = R \cup \{c\}$ be a refinement. Then if $P = R' \cap [a, c]$ and $Q = R' \cap [c, b]$, we have

$$L(f, R) \leq L(f, R') = L(f, P) + L(f, Q)$$

then taking supremums, we have $\int_a^b f\, d\alpha \leq \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$, so combining with the inequality from above, we have equality.

Suppose $\int_a^c f\, d\alpha$ and $\int_c^b f\, d\alpha$ exist, so $f \in \mathcal{R}_\alpha[a, c]$ and $f \in \mathcal{R}_\alpha[c, b]$. Fix $\varepsilon > 0$. Then there exist partitions P, Q of $[a, c]$ and $[c, b]$, respectively, such that

$$\begin{aligned} U(f, P) - L(f, P) &< \frac{\varepsilon}{2} \\ U(f, Q) - L(f, Q) &< \frac{\varepsilon}{2} \\ \implies [U(f, P) + U(f, Q)] - [L(f, P) + L(f, Q)] &= U(f, P \cup Q) - L(f, P \cup Q) \\ &< \varepsilon \end{aligned}$$

Thus, since $P \cup Q$ is a partition of $[a, b]$, it follows that $f \in \mathcal{R}_\alpha[a, b]$, so $\int_a^b f\, d\alpha$ exists.

If $\int_a^c f\, d\alpha$ and $\int_a^b f\, d\alpha$ exist, then for a fixed $\varepsilon > 0$, there exists a partition P of $[a, b]$ and Q partition of $[a, c]$ such that

$$\begin{aligned} U(f, P) - L(f, P) &< \varepsilon \\ U(f, Q) - L(f, Q) &< \varepsilon \end{aligned}$$

Then let $Q' = (P \cap [a, c]) \cup Q$ be a partition of $[a, c]$ refining Q. Then we have

$$U(f, Q') - L(f, Q') \leq U(f, Q) - L(f, Q) < \varepsilon$$

Now, take $R = P \setminus Q' \cup \{c\}$ be a partition of $[c, b]$. We have

$$\begin{aligned} [U(f, R) + U(f, Q')] - [L(f, R) + L(f, Q')] &= U(f, P) - L(f, P) \\ \implies U(f, R) - L(f, R) &= [U(f, P) - L(f, P)] - [U(f, Q') - L(f, Q')] \\ &< \varepsilon \end{aligned}$$

so $f \in \mathcal{R}_\alpha[c, b]$, so the integral exists. A similar argument shows that $f \in \mathcal{R}_\alpha[a, c]$ when the other two integrals exist. □

23. Suppose that φ is a strictly increasing continuous function from $[c, d]$ onto $[a, b]$. Given $f \in \mathcal{R}_\alpha[a, b]$, show that $g = f \circ \varphi \in \mathcal{R}_\beta[c, d]$, where $\beta = \alpha \circ \varphi$. Moreover, $\int_c^d g\, d\beta = \int_a^b f\, d\alpha$.

Proof. Fix $\varepsilon > 0$. Since $f \in \mathcal{R}_\alpha[a, b]$, there exists a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$ such that $U_\alpha(f, P) - L_\alpha(f, P) < \varepsilon$. Then since φ is strictly increasing and continuous and onto $[a, b]$, it has a well defined inverse φ^{-1}, and $Q = \left\{c = \varphi^{-1}(x_0) < \cdots < \varphi^{-1}(x_n) = d\right\}$ is a partition of $[c, d]$.

Now, we have

$$\begin{aligned} U_\beta(f\circ\varphi, Q) &= \sum_{i=1}^{n} \sup\left\{f(\varphi(y)) : \varphi^{-1}(x_{i-1})) \le y \le \varphi^{-1}(x_i)\right\}\left[\alpha\circ\varphi\circ\varphi^{-1}(x_i) - \alpha\circ\varphi\circ\varphi^{-1}(x_{i-1})\right] \\ &= \sum_{i=1}^{n} \sup\left\{f(x) : x_{i-1} \le x \le x_i\right\}(\alpha(x_i) - \alpha(x_{i-1})) = U_\alpha(f, P) \end{aligned}$$

and similarly, $L_\beta(f\circ\varphi, Q) = L_\alpha(f, P)$, so

$$U_\beta(f\circ\varphi, Q) - L_\beta(f\circ\varphi, Q) = U_\alpha(f, P) - L_\alpha(f, P) < \varepsilon$$

so $g = f\circ\varphi \in \mathcal{R}_\beta[c, d]$.

Suppose the integrals were not equal, and WLOG $\int_c^d g\,d\beta > \int_a^b f\,d\alpha$. That is,

$$\begin{aligned} &\sup_P L_\alpha(f, P) < \sup_Q L_\beta(f\circ\varphi, Q) \\ \implies &\sup_P L_\alpha(f, P) < L_\beta(f\circ\varphi, Q) \end{aligned}$$

for some partition Q of $[c, d]$. But then applying φ to every element of Q, we will obtain a partition Q' of $[a, b]$, with $L_\alpha(f, Q') = L_\beta(f\circ\varphi, Q)$. This is a contradiction, because then $L_\alpha(f, Q') > \sup_P L_\alpha(f, P)$, so we cannot have $\int_c^d g\,d\beta > \int_a^b f\,d\alpha$. By a similar argument, we cannot have the reverse inequality, so the two integrals must be equal. □

27. Give an example of a sequence of Riemann integrable functions on $[0, 1]$ that converges pointwise to a non-integrable function.

Solution. Let $f_n = x^n$ on $[0, 1]$. Each of these is RS integrable. Then $f_n \to f$ where

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

which is not RS-integrable because the greatest value on the final interval including 1 is 1, while the smallest value is 0, and the greatest and smallest values everywhere else are all 0. □