Homework 8

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Chapter 17: Measurable Functions

14. If f is measurable and B is a Borel set, show that $f^{-1}(B)$ is measurable. (Hint: $\{A : f^{-1}(A) \in \mathcal{M}\}$ is a σ-algebra containing the open sets.)

17. If $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable, show that $f \circ g$ is Borel measurable. If f is Borel measurable and g is Lebesgue measurable, show that $f \circ g$ is Lebesgue measurable.

21. Let f be a non-negative, bounded, measurable function on $[a, b]$ with $0 \le f \le M$. Let

$$E_{n,k} = \left\{ \frac{kM}{2^n} \le f \le \frac{(k+1)M}{2^n} \right\}$$

for each $n = 1, 2, \cdots$, and $k = 0, 1, \cdots, 2^n$, and set

$$\varphi_n = \sum_{k=0}^{2^n} \frac{kM}{2^n} \chi_{E_{n,k}}$$

Prove that $0 \le \varphi_n \le \varphi_{n+1} \le f$ and that $0 \le f - \varphi_n \le 2^{-n}M$ for each n. Thus, (φ_n) is a sequence of simple functions that converges uniformly to f on $[a, b]$. (Hint: Notice that $E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}$.)

31. Let (f_n) be a sequence of measurable functions, all defined on some measurable set D. Show that the set $C = \{x \in D : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable. (Hint: C is the set where $(f_n(x))$ is Cauchy.)

35. Give an example showing that the requirement that $m(D) < \infty$ cannot be dropped from Egorov's theorem.

36. If (f_n) converges almost uniformly to f, prove that (f_n) converges almost everywhere to f. (Hint: For each k, choose a set E_k such that $m(E_k) < 1/k$ and $f \implies f$ off E_k. Then $m\left(\bigcap_{k=1}^{\infty} E_k\right) = 0$.)

Chapter 18: The Lebesgue Integral

1. If ψ s a non-negative simple function, check that

$$\int \psi = \sup \left\{ \int \varphi : 0 \le \varphi \le \psi : \varphi \text{ simple and integrable} \right\}$$

3. Prove that $\int_1^{\infty} (1/x)\, dz = \infty$ (as a Lebesgue integral).