Homework 2

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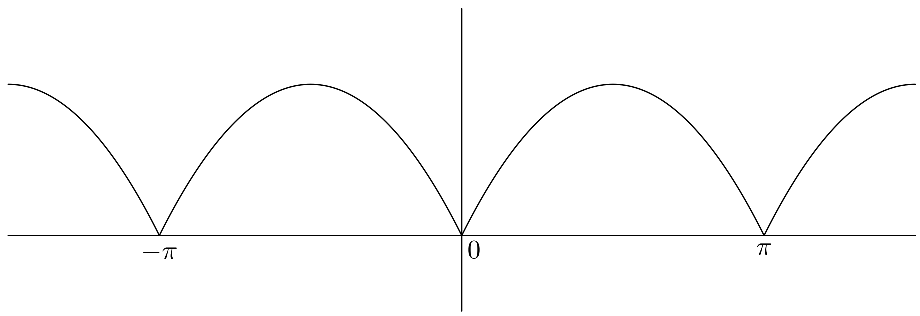
1. Consider the 2π-period odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. NOTE: Discussed with Mauro, I am using the π-periodic even function defined on just $[0, \pi]$. The results will not be identical but they should be okay.

   (a) Draw the graph of f



   (b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi} \sum_{k \text{ odd } \geq 1} \frac{\sin k\theta}{k^3}$$

Is this function continuous on the circle? Continuously differentiable on the circle? In L^2 of the circle?

Solution. We have

$$\hat{f}(n) = \frac{1}{\pi} \int_0^\pi \theta(\pi - \theta) e^{-2in\theta}\, d\theta = \int_0^\pi \theta e^{-2in\theta}\, d\theta - \frac{1}{\pi} \int_0^\pi \theta^2 e^{-2in\theta}\, d\theta$$

Integrating by parts, we have

$$\begin{aligned}
\int_0^\pi \theta e^{-2in\theta}\, d\theta &= -\frac{\theta}{2in} e^{-2in\theta} \Big|_0^\pi - \int_0^\pi -\frac{1}{2in} e^{-2in\theta}\, d\theta \\
&= -\frac{\pi}{2in} + \frac{1}{4n^2} e^{-2in\theta} \Big|_0^\pi = -\frac{\pi}{2in} \\
\int_0^\pi \theta^2 e^{-2in\theta}\, d\theta &= -\frac{\theta^2}{2in} e^{-2in\theta} \Big|_0^\pi - \int_0^\pi -\frac{1}{2in} e^{-2in\theta} \cdot 2\theta\, d\theta \\
&= -\frac{\pi^2}{2in} + \frac{1}{in} \int_0^\pi \theta e^{-2in\theta}\, d\theta = -\frac{\pi^2}{2in} + \frac{1}{in} \cdot \left(-\frac{\pi}{2in}\right) = -\frac{\pi^2}{2in} + \frac{\pi}{2n^2} \\
\implies \hat{f}(n) &= -\frac{\pi}{2in} - \frac{1}{\pi}\left(-\frac{\pi^2}{2in} + \frac{\pi}{2n^2}\right) = -\frac{1}{2n^2}
\end{aligned}$$

Given a fixed $n \neq 0$, we have

$$\hat{g}(n,\theta) = \hat{f}(n)e^{2in\theta} - \hat{f}(-n)e^{-2in\theta} = -\frac{1}{2n^2}\left(\cos 2n\theta + i\sin 2n\theta\right) + \frac{1}{2(-n)^2}\left(\cos(-2n\theta) + i\sin(-2n\theta)\right)$$
$$= -\frac{1}{n^2}\cos 2n\theta$$

due to the even and odd properties of cos and sin, respectively. We also have

$$\hat{f}(0) = \frac{1}{\pi}\int_0^\pi \theta(\pi-\theta)\,d\theta = \frac{1}{\pi}\left(\frac{\theta^2\pi}{2} - \frac{\theta^3}{3}\right)\Big|_0^\pi = \frac{1}{\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) = \frac{\pi^2}{6}$$

and thus the Fourier series for f is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2in\theta} = \frac{\pi^2}{6} + \sum_{k=1}^{\infty}\hat{g}(k,\theta) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{1}{n^2}\cos 2n\theta$$

Now, since f is continuous, we have

$$\sum_{k=-\infty}^{\infty}\left|\hat{f}(k)\right| = 2\sum_{k=1}^{\infty}\left|\frac{1}{n^2}\right| + \left|\hat{f}(0)\right| = 2\cdot\frac{\pi^2}{6} + \frac{\pi^2}{6} < \infty$$

and thus the Fourier series converges uniformly to f.
It is not continuously differentiable, it has a cusp at $k\pi$ for $k \in \mathbb{Z}$. It is also in L^2 (trivial to show, just integrating a quartic polynomial). □

2. Prove that if f is an even 2π-period function, i.e. $f(\theta) = f(-\theta)$ for all $\theta \in [-\pi,\pi]$, then the Fourier series can be written as a cosine series.

Proof. We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$
$$\hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{iny}\,dy$$

We can see that these two integrals are equal with the change of variable $y = -x, dy = -dx$, since

$$\hat{f}(-n) = \frac{1}{2\pi}\int_{\pi}^{-\pi} -f(-x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)e^{-inx}\,dx = \hat{f}(n)$$

due to the even-ness of f. Thus, for any $n \neq 0$, we have

$$\begin{aligned}\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta} &= \hat{f}(n)\left[(\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))\right]\\ &= \hat{f}(n)\left[(\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)\right]\\ &= 2\hat{f}(n)\cos n\theta\end{aligned}$$

Thus, in the infinite sum expansion of the Fourier series for f, we see that it will only contain cosine terms, as desired. □