Homework 2

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Chapter 2: Countable and Uncountable Sets

3. Given finitely many countable sets $A_1, \cdots, A_n$, show that $A_1 \cup \cdots \cup A_n$ and $A_1 \times \cdots \times A_n$ are countable sets.

   Proof. Since A_i are countable, we have

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, \cdots\} \\ A_2 &= \{a_{21}, a_{22}, a_{23}, \cdots\} \\ &\vdots \\ A_n &= \{a_{n1}, a_{n2}, a_{n3} \cdots\} \end{aligned}$$

   The set of all the a_{ij} (including possible duplicates) is countable by Cantor's diagonal argument, so the union $A_1 \cup \cdots \cup A_n$ is countable.

   Since each of the A_i is countable, there exists bijections $f_i : \mathbb{N} \to A_i$. We can construct a map

$$\begin{aligned} g : \mathbb{N}^n &\to A_1 \times \cdots \times A_n \\ (b_1, b_2, \cdots, b_n) &\mapsto (f_1(b_1), f_2(b_2), \cdots, f_n(b_n)) \end{aligned}$$

   which is clearly bijective since each of the f_i is bijective. Since $\mathbb{N}^n$ is countable by Cantor's diagonal argument, the product $A_1 \times \cdots \times A_n$ is countable. □

7. Let A be countable. If $f : A \to B$ is onto, show that B is countable; if $g : C \to A$ is 1-1, show that C is countable.

   Proof. Since f is onto, $S_b := \{a \in A : f(a) = b\}$ is nonempty for all $b \in B$. Since $S_b \subset A$, it must be countable, so its elements can be enumerated with $\mathbb{N}$. Now, for any b, choose the element $a_b \in S_b$ with the smallest index, and $b \mapsto a_b$ is injective, since the sets are disjoint. This is easy to see because $a \in S_b \cap S_c \implies f(a) = b = c \implies S_b = S_c$. Since A is countable it can be injected into $\mathbb{N}$, and thus $B \hookrightarrow A \hookrightarrow \mathbb{N}$ implies B is countable.

   Since g is injective, it is a bijection from $C \to g(C) = \{g(c) : c \in C\}$. Since $g(C) \subset A$, it is countable, so there exists a bijection $h : g(C) \to A$. Thus, the composition $h \circ g : C \to g(C) \to A$ is a bijection, so $C \sim A$ and thus C is countable. □

8. Show that $(0, 1)$ is equivalent to $[0, 1]$ and to $\mathbb{R}$.

   Proof. Consider the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \tan\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)$. This is bijective since it has a well defined inverse $f^{-1} = \frac{2}{\pi}\left(\arctan x + 1\right)$. Thus, $(0, 1) \sim \mathbb{R}$.

To show that $[0,1] \sim (0,1)$, consider a countably infinite subset $(0,1) \supset S = \{x_1, x_2, x_3, \cdots\}$. Let $T = \{0,1\}$. Then since $S \cup T$ is a countable union of countable sets, it is countable, and therefore $(S \cup T) \sim S$, so there exists a bijection $h : (S \cup T) \to S$. Now, define $g : [0,1] \to (0,1)$ as

$$g(x) = \begin{cases} x & \text{if } x \in [1,0] \setminus (S \cup T) \\ h(x) & \text{if } x \in (S \cup T) \end{cases}$$

Here, g is defined on $[0,1]$, its image is $[1,0] \setminus T = (0,1)$, and is bijective because of how we have defined it. Thus, $[0,1] \sim (0,1)$. ☐

16. The algebraic numbers are those real or complex numbers that are the roots of polynomials having integer coefficients. Prove that the set of algebraic numbers is countable.

Proof. Consider the set of polynomials with integer coefficients, $\mathbb{Z}[x]$. Within $\mathbb{Z}[x]$, consider the set S_n of polynomials of degree n. Now, we construct a bijective map $g : \mathbb{Z}^{n+1} \to S_n$:

$$(a_0, a_1, \cdots, a_n) \mapsto \begin{cases} a_0 + a_1 x + \cdots + (a_n + 1)x^n, & a_n \geq 0 \\ a_0 + a_1 x + \cdots + a_n x^n, & a_n < 0 \end{cases}$$

This is obviously injective, and it is surjective because if

$$g = b_0 + b_1 x + \cdots + b_n x^n$$

then if $b_n > 0$, we can recover $(b_0, b_1, \cdots, b_n - 1)$ and if $b_n < 0$, we can recover $(b_0, b_1, \cdots, b_n)$. The case where $b_n = 0$ is impossible since the polynomial has degree n. Thus, S_n is countable since $\mathbb{Z}^{n+1}$ is a countable union of countable sets $\mathbb{Z}$. Since S_n is countable, we can enumerate the polynomials $f_i \in S_n$ with $\mathbb{N}$ and their corresponding set of n (possibly repeated) roots R_i:

$$\begin{aligned} R_1 &= \{r_{11}, r_{12}, r_{13}, \cdots, r_{1n}\} \\ R_2 &= \{r_{21}, r_{22}, r_{23}, \cdots, r_{2n}\} \\ R_3 &= \{r_{31}, r_{32}, r_{33}, \cdots, r_{3n}\} \\ &\vdots \end{aligned}$$

The set of all roots is countable by Cantor's diagonal argument, and denote this set by T_n for S_n. Now, the set of all algebraic numbers is

$$\bigcup_{i=1}^{\infty} T_i$$

which is a countable union of countable sets T_i, and therefore countable, as desired. ☐

17. If A is uncountable and B is countable, show that A and $A \setminus B$ are equivalent. In particular, conclude that $A \setminus B$ is uncountable.

Proof. Let $S \subset A \setminus B$ be a countably infinite set. Then $B \cup S$ is a countable union of countable sets, and therefore countable, so there exists a bijection $g : (B \cup S) \to S$. Now, define the mapping

$$\begin{aligned} f : A &\to A \setminus B \\ a &\mapsto \begin{cases} a & \text{if } a \in A \setminus (B \cup S) \\ g(a) & \text{if } a \in (B \cup S) \end{cases} \end{aligned}$$

This function is bijective as defined, so $A \sim A \setminus B$, as desired. Equivalent sets have the same cardinality, so it follows that $A \setminus B$ is also uncountable. ☐

18. Show that the set of all real numbers in the interval $(0, 1)$ whose base 10 decimal expansion contains no 3s or 7s is uncountable.

Proof. Suppose the set is countable. Then elements can be indexed with $\mathbb{N}$, so suppose the list

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \end{aligned}$$

is exhaustive, where $a_{ij} \in \{0, 1, 2, 4, 5, 6, 8, 9\}$. Now, construct a new number

$$y = 0.b_1b_2b_3\cdots, \quad b_i = \begin{cases} 4, & a_{ii} = 5 \\ 5, & a_{ii} \neq 5 \end{cases}$$

This element is not equal to any of the x_i, but it still fits the criteria of the set. Contradiction, so such a set must be uncountable. $\square$