Homework 6

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1. (a) Let $F \to \overline{F}$ be an algebraic closure of F and let $F \to E$ be a finite field extension. Show that there exists an F-embedding of E into $\overline{F}$.
   (b) It can be shown that (a) continues to hold when E is only assumed to be algebraic over F. Assuming this fact, show that any two algebraic closures of F are isomorphic as F-algebras.
2. Let F be a field and let $F \to \overline{F}$ an algebraic closure. As a continuation of 6.3 Ex. 21, show that a finite field extension $F \to E$ is normal $\iff$ all F-embeddings of E into $\overline{F}$ have the same image.

Section 6.2: Algebraic Extensions

7. Find the minimal polynomial of $u = \sqrt{3} - i$

   (a) over $\mathbb{R}$.

   Solution. We have $\overline{u} = \sqrt{3} + i$, where

$$\begin{aligned} u + \overline{u} &= 2\sqrt{3} \\ u\overline{u} &= 4 \end{aligned}$$

   so the minimal polynomial over $\mathbb{R}$ is given by

$$m = x^2 - 2\sqrt{3}x + 4$$

□

   (b) over $\mathbb{Q}$.

19. Let $\mathbb{C} \supseteq E \supseteq \mathbb{Q}$, where E is a field, and assume that $[E : \mathbb{Q}] = 2$. Show that $E = \mathbb{Q}(\sqrt{m})$, where m is a square-free integer.

21. Let $E \supseteq F$ be fields, and let $u, v \in E$ be algebraic over F of degrees m, n.

   (a) Show that $[F(u, v) : F] \leq mn$.
   (b) If m and n are relatively prime, show that $[F(u, v) : F] = mn$.
   (c) Is the converse to (b) true?

32. Let p and q in $\mathbb{Q}$ satisfy $\sqrt{p} \notin \mathbb{Q}$ and $\sqrt{q} \notin \mathbb{Q}(\sqrt{p})$.

   (a) Show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.
   (b) Use Theorem 5 to find a basis of $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ over $\mathbb{Q}$.
   (c) Deduce that $x^4 - 2(p + q)x^2 + (p - q)^2$ is the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over $\mathbb{Q}$.

Section 6.3: Splitting Fields

3. If $2 \neq 0$ in the field F, show that the splitting field E of $x^4 + 1$ over F is a simple extension of F and factors $x^4 + 1$ completely in $E[x]$. What happens if $2 = 0$ in F?

21. Show that the following conditions are equivalent for fields $E \supseteq F$:

    1. E is the splitting field of a polynomial in $F[x]$.
    2. $[E : F]$ is finite and every irreducible polynomial in $F[x]$ with a root in E splits completely in $E[x]$.