Homework 1

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Chapter 1: Calculus Review

3. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $s = \sup A$ if and only if

   (i) s is an upper bound for A

   (ii) for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$.

   State and prove the corresponding result for the infimum of a nonempty subset of $\mathbb{R}$ that is bounded below.

   Proof. $(\implies)$: By definition, (i) is true. Then suppose for some ε, there is no $a \in A$ such that $a > s-\varepsilon$. Thus, $s - \varepsilon$ is an upper bound since $a \leq s - \varepsilon, \forall a \in A$, but $s - \varepsilon < s$, contradicting the minimality of s. Thus, such an a must exist.

   $(\impliedby)$: Suppose there exists an upper bound b for A such that $b < s$. Then let $\varepsilon = s - b > 0$. Then $s - \varepsilon = s - (s - b) = b$, but since b is an upper bound for A, there cannot exist $a \in A$ such that $a > b$, contradicting (ii). Thus, b does not exist, so $s \leq b$ for all upper bounds b, and thus $s = \sup A$.

   The corresponding result for the infimum: Prove that $m = \inf A$ if and only if

   (i) m is a lower bound for A

   (ii) for every $\varepsilon > 0$, there is an $a \in A$ such that $a < m + \varepsilon$.

   Proof. $(\implies)$: By definition, (i) is true. Then suppose for some ε, there is no $a \in A$ such that $a < m + \varepsilon$. Thus $m + \varepsilon$ is a lower bound since $a \geq m + \varepsilon, \forall a \in A$, but $m + \varepsilon > m$, contradicting the maximality of m. Thus, such a a must exist.

   $(\impliedby)$: Suppose there exists a lower bound b for A such that $b > m$. Then let $\varepsilon = b - m > 0$. Then $m + \varepsilon = m + (b - m) = b$, but since b is a lower bound for A, there cannot exist $a \in A$ such that $a < b$, contradicting (ii). Thus, b does not exist, so $m \geq b$ for all lower bounds b, and thus $m = \inf A$. ■

   □

7. If $a < b$, then there is also an irrational $x \in \mathbb{R} \setminus \mathbb{Q}$ with $a < x < b$.

   Proof. If $a < b$ then $a/\sqrt{2} < b/\sqrt{2}$, so by Theorem 1.3, there exists a rational $p/q \in \mathbb{Q}$ such that $a/\sqrt{2} < p/q < b/\sqrt{2}$. Then $a < \frac{p\sqrt{2}}{q} < b$, and $\frac{p\sqrt{2}}{q}$ is irrational, as desired. □

15. Show that a Cauchy sequence with a convergent subsequence actually converges.

    Proof. Suppose (x_n) is a sequence with a convergent subsequence $(x_{k_j}) \to y$. Let $\varepsilon > 0$. Since (x_n) is Cauchy, choose $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon/2$ for all $n, m \geq N$. Next, since $(x_{k_j}) \to y$, choose M such that $\left|x_{k_j} - y\right| < \varepsilon/2$ for all $k_j \geq M$. Take $K = \max\{N, M\}$, so that $\left|x_n - x_{k_j}\right| < \varepsilon/2$ and $\left|x_{k_j} - y\right| < \varepsilon/2$ for all $n, k_j \geq K$. By the triangle inequality, we have

    $$|x_n - y| \leq \left|x_n - x_{k_j}\right| + \left|x_{k_j} - y\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

    for all $n \geq K$, as desired. □

17. Given real numbers a and b, establish the following formulas:

(a) $|a+b| \leq |a| + |b|$

Proof. Using the fact that

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

we have

$$\begin{aligned} a, b \geq 0 &\implies |a+b| = a+b \leq a+b = |a|+|b| \\ a, b < 0 &\implies |a+b| = -(a+b) \leq -a-b = |a|+|b| \\ a \geq 0, b < 0, a+b \geq 0 &\implies |a+b| = a+b \leq a-b = |a|+|b| \\ a \geq 0, b < 0, a+b < 0 &\implies |a+b| = -(a+b) \leq a-b = |a|+|b| \end{aligned}$$

The case where $a < 0, b \geq 0$ is identical to the third and fourth inequalities. $\square$

(b) $||a| - |b|| \leq |a-b|$

Proof. If $a, b \geq 0$, then

$$\begin{aligned} a, b \geq 0 &\implies ||a|-|b|| = |a-b| \\ a, b < 0 &\implies ||a|-|b|| = |-a+b| = |a-b| \\ a \geq 0, b < 0 &\implies ||a|-|b|| = |a+b| \leq |a|+|b| = a-b = |a-b| \\ a < 0, b \geq 0 &\implies ||a|-|b|| = |-a-b| = |a+b| \leq |a|+|b| = -a+b = |a-b| \end{aligned}$$

where the third and fourth inequalities are from the result of (a). $\square$

(c) $\max\{a, b\} = \frac{1}{2}(a+b+|a-b|)$

Proof.

$$\begin{aligned} a \geq b &\implies \frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+(a-b)) = a = \max\{a,b\} \\ a < b &\implies \frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b-(a-b)) = b = \max\{a,b\} \end{aligned}$$

$\square$

(d) $\min\{a, b\} = \frac{1}{2}(a+b-|a-b|)$

Proof.

$$\begin{aligned} a \geq b &\implies \frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = b = \min\{a,b\} \\ a < b &\implies \frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b+(a-b)) = a = \min\{a,b\} \end{aligned}$$

$\square$

37. If (E_n) is a sequence of subsets of a fixed set S, we define

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$$
$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right)$$

Show that

(a) $\liminf\limits_{n\to\infty} E_n \subset \limsup\limits_{n\to\infty} E_n$

Proof. If $x \in \liminf E_n$ then $x \in \bigcap_{k=N}^{\infty} E_k$ for some N. It follows that $x \in E_k$ for all $k \geq N$, so $x \in \bigcup_{k=n}^{\infty} E_k$ for all n, and is thus in the intersection of these sets, so $x \in \limsup E_n$, and thus $\liminf E_n \subset \limsup E_n$. □

(b) $\liminf\limits_{n\to\infty}(E_n^c) = \left(\limsup\limits_{n\to\infty} E_n\right)^c$

Proof. Using the facts $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$, we have

$$\liminf(E_n^c) = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k^c\right) = \bigcup_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)^c = \left[\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right]^c = (\limsup E_n)^c$$

□

45. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(x) = 0$ whenever x is rational. Show that $f(x) = 0$ for every x in $[a, b]$.

Proof. Suppose $f(x') = y \neq 0$ for some $x' \in [a, b]$. Then consider a sequence of rationals $(x_n) \to x'$. Since f is continuous, we must have $f(x_n) \to f(x')$, but the sequence $(f(x_n))$ is entirely 0's since the x_i are rational, whereas $f(x') \neq 0$, contradiction. Thus, x does not exist, so $f(x) \equiv 0$ on $[a, b]$, as desired. □

46. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some open interval $(-a, a)$.

Proof. Suppose $f(0) = y > 0$. Then take $\varepsilon = y/2$. Now, since f is continuous at 0, there must exist $a > 0$ such that

$$\begin{aligned}
|x| < a &\implies |f(x) - y| < \frac{y}{2} \\
x \in (-a, a) &\implies -\frac{y}{2} < f(x) - y < \frac{y}{2} \\
&\implies 0 < \frac{y}{2} < f(x)
\end{aligned}$$

Here, $f(x) > 0$ for all $x \in (-a, a)$, as desired. □

(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with ≥ 0 replaced by > 0? Explain.

Proof. Suppose $f(x') = y < 0$ for some irrational x'. Then consider a sequence of rationals $(x_n) \to x'$. Since f is continuous, we must have $f(x_n) \to f(x')$, but the sequence $(f(x_n))$ is always non-negative since the x_i are rational, whereas $f(x) < 0$, contradiction. Thus, x' does not exist, so $f(x) \geq 0$ for all x, as desired.

If ≥ 0 is replaced by > 0, the statement does not hold. Suppose r is a fixed irrational number. Then let $f(x) = (r - x)^2$, which is continuous on $\mathbb{R}$, and positive for all $x \in \mathbb{Q}$ since r is irrational. However, $f(r) = 0$, so the statement is false. □