Homework 7

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Chapter 7: Completeness

35. Prove that a normed vector space X is complete if and only if its closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is complete.

Proof. ($\Longrightarrow$) : Let y_n be a sequence in B with $\sum_{n=1}^{\infty} \|y_n\| < \infty$. Then since X is complete and y_n is also a sequence in X, it follows that $\sum_{n=1}^{\infty} y_n$ converges in X. $\square$

40. Extend the result in Example 7.15 as follows: Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable in (a, b), and satisfies $F(a) < 0, F(b) > 0$, and $0 < K_1 \leq F'(x) \leq K_2$. Show that there is a unique solution to the equation $F(x) = 0$. (Hint: Consider the equation $f(x) = x$, where $f(x) = x - \lambda F(x)$ for some suitably chosen λ.)

47. A function $f : (M, d) \to (n, \rho)$ is said to be uniformly continuous if f is continuous and if, given $\varepsilon > 0$, there is always a single $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ for any $x, y \in M$ with $d(x, y) < \delta$. That is, δ is allowed to depend on f and ε but not on x or y. Prove that any Lipschitz map is uniformly continuous.

Proof. If f is Lipschitz, then there exists $K > 0$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$. Given $\varepsilon > 0$, take $\delta = \varepsilon/K$. Then for all $x, y \in M$ where $d(x, y) < \delta = \varepsilon/K$, we have

$$\rho(f(x), f(y)) \leq Kd(x, y) < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Since all Lipschitz maps are continuous, it follows that f is uniformly continuous. $\square$

Chapter 8: Compactness

2. Let $E = \left\{x \in \mathbb{Q} : 2 < x^2 < 3\right\}$, considered as a subset of $\mathbb{Q}$ (with its usual metric). Show that E is closed and bounded but not compact.

Proof. Consider the sequence $1, 1.7, 1.73, 1.732, \cdots$ of rationals converging to $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$. This is a sequence in E, but any subsequence also converges to $\sqrt{3}$ and thus fails to converge in E. Thus, E is not compact. $\square$

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8. Prove that the set $\{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $\mathbb{R}^n$ under the Euclidean norm.

Proof. Consider $B = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$. Then B is a closed, bounded subset of $\mathbb{R}^n$, and therefore compact. Then $S = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} \subset B$ is closed in B because $S^c = \{x \in \mathbb{R}^n : \|x\|_1 < 1\}$ is open. Thus, since B is compact and S is closed in B, it follows that S is compact. $\square$

10. Show that the Heine-Borel theorem (closed, bounded sets in $\mathbb{R}$ are compact) implies the Bolzano-Weierstrass theorem. Conclude that the Heine-Borel theorem is equivalent to the completeness of $\mathbb{R}$.

37. A real-valued function f on a metric space M is called lower semi-continuous if, for each real α, the set $\{x \in M : f(x) > \alpha\}$ is open in M. prove that f is lower semi=continuous if and only if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x$ in M.

40. Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto. (Hint: If $B_\varepsilon(x) \cap f(M) = \varnothing$, consider the sequence $f^n(x)$.)

    Proof. Since f is an isometry, it is continuous, so $f(M) \subset M$ is compact, and therefore closed. Suppose f is not onto. Then there exists $x \in M \setminus f(M)$, so $B_\varepsilon(x) \cap f(M) = \varnothing$ for some $\varepsilon > 0$. Now, consider the sequence $(f(x), f(f(x)), f(f(f(x))), \cdots) = (f^n(x))$ in $f(M)$. Such a sequence cannot have a Cauchy subsequence because for any $n > m$, we have

$$d(f^n(x), f^m(x)) = d(f^{n-m}(x), x) \geq \varepsilon$$

    since $B_\varepsilon(x) \cap f(M) = \varnothing$. Thus, $f(M)$ is not totally bounded, and therefore not compact. Contradiction, so f must be onto. □