Homework 4

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Section 2.2: Groups

13. If G is any group, define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that α is injective and surjective.

Proof. To show α is injective, consider g_1 and g_2 such that $\alpha(g_1) = \alpha(g_2)$. Then $g_1^{-1} = g_2^{-1}$, and left multiplying by g_2g_1, we have

$$\begin{aligned} g_2g_1g_1^{-1} &= g_2g_1g_2^{-1} \\ g_2 &= g_2g_1g_2^{-1} \\ g_2g_2 &= g_2g_1g_2^{-1}g_2 \\ g_2g_2 &= g_2g_1 \end{aligned}$$

and by the cancellation law, we have $g_2 = g_1$, so α is injective, as desired.

To show α is surjective, we must show that for all $g \in G$, there exists a $g_0 \in G$ such that $\alpha(g_0) = g$. Since $gg^{-1} = 1$ it follows that $(g^{-1})^{-1} = g$, so then $g_0 = g^{0-1}$ will satisfy this, and since G is a group, every element has an inverse, so α is surjective, as desired.

$\square$

Section 2.3: Subgroups

2. If H is a subset of a group G, show that H is a subgroup if and only if H is nonempty and $ab^{-1} \in H$ whenever $a \in H$ and $b \in H$.

5. (a) If G is an abelian group, show that $H = \left\{a \in G | a^2 = 1\right\}$ is a subgroup of G.
   (b) Give an example where H is not a subgroup.

8. If X is a nonempty subset of a group G, let $\langle X \rangle$ be the set of all products of powers of elements of X; more formally
$$\langle X \rangle = \left\{x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \mid m \geq 1, x_i \in X\right\}$$
   (a) Show that $\langle X \rangle$ is a subgroup of G that contains X.
   (b) Show that $\langle X \rangle \subseteq H$ for every subgroup H such that $X \subseteq H$. Thus, $\langle X \rangle$ is the *smallest* subgroup of G that contains X, and is called the **subgroup generated** by X.

13. (a) If G is a group, show that $\{(g, g) | g \in G\}$ is a subgroup of $G \times G$.
   (b) Determine the groups G such that $\left\{(g, g^{-1}) | g \in G\right\}$ is a subgroup of $G \times G$.

22. Find $Z[GL_2(\mathbb{R})]$.

Section 2.4: Cyclic Groups and the Order of an Element

6. If G is a group and $g \in G$, show that $\langle g \rangle = \langle g^{-1} \rangle$.

7. Let $o(g) = 20$ in a group G. Compute

   (a) $o(g^2)$

   Answer. Since $o(g) = 20$, that means $g^{20} = 1$. Then $(g^2)^{10} = g^{20} = 1$, so $o(g^2) = \boxed{10.}$

   (b) $o(g^8)$

   Answer. Since $o(g) = 20$, that means $g^{20} = 1$. Then $(g^8)^5 = g^{40} = (g^{20})^2 = 1$, so $o(g^8) = \boxed{5.}$.

   (c) $o(g^5)$

   Answer. Since $o(g) = 20$, that means $g^{20} = 1$. Then $(g^5)^4 = g^{20} = 1$, so $o(g^5) = \boxed{4.}$

   (d) $o(g^3)$

   Answer. Since $o(g) = 20$, that means $g^{20} = 1$. Then $(g^3)^{20} = (g^{20})^3 = 1$, so $o(g^3) = \boxed{20.}$

10. (a) If $gh = hg$ in a group and $o(g)$ and $o(h)$ are finite, show that $o(gh)$ is finite.

    (b) Show that (a) fails if $gh \neq hg$ by considering $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.

18. If $G = \langle g \rangle$ and $H = \langle h \rangle$, show that $G \times H = \langle (g, 1), (1, h) \rangle$