Homework 3

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Chapter 14: The Riemann-Stieltjes Integral

29. Show that $|S_\alpha(f,P,T)| \le \|f\|_\infty V(\alpha,P)$.

Proof. We have $V(\alpha,P) \ge |\alpha(b)-\alpha(a)| \ge \alpha(b)-\alpha(a)$ for any partition P. Thus,

$$\begin{aligned} S_\alpha(f,P,T) &= \sum_{i=1}^n f(t_i)\left[\alpha(x_i)-\alpha(x_{i-1}\right] \le \sum_{i=1}^n \|f\|_\infty \left[\alpha(x_i)-\alpha(x_{i-1})\right] \\ &= \|f\|_\infty \sum_{i=1}^n \left[\alpha(x_i)-\alpha(x_{i-1})\right] = \|f\|_\infty \left[\alpha(b)-\alpha(a)\right] \\ &\le \|f\|_\infty V(\alpha,P) \end{aligned}$$

□

31. Let $a<c<b$, and suppose that $f\in\mathcal{R}_\alpha[a,c]\cap\mathcal{R}_\alpha[c,b]$. Show that $f\in\mathcal{R}_\alpha[a,b]$ and that $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. In fact, if any two of these integrals exist, then so does the third and the equation above still holds.

Proof. Since $f\in\mathcal{R}_\alpha[a,c]$ and $f\in\mathcal{R}_\alpha[c,b]$, let I_1 and I_2 be $\int_a^c f\,d\alpha$ and $\int_c^b f\,d\alpha$, respectively. Let $\varepsilon>0$. There exists partitions P^* and Q^* of $[a,c]$ and $[c,b]$ such that

$$\begin{aligned} |S_\alpha(f,P,T_1)-I_1| &< \frac{\varepsilon}{2} \\ |S_\alpha(f,Q,T_2)-I_2| &< \frac{\varepsilon}{2} \end{aligned}$$

for all $P\supset P^*$ and $Q\supset Q^*$ and all choices T_1 and T_2. Then let $R^*=P^*\cup Q^*$ be a partition of $[a,b]$. Then for any $R\supset R^*$, we have $R=A\cup B$ for $A\supset P^*$ and $B\supset Q^*$, so

$$\begin{aligned} |S_\alpha(f,R,T_3)-(I_1+I_2)| &= \left|\left[S_\alpha(f,A,T_1)+S_\alpha(f,B,T_2)\right]-(I_1+I_2)\right| \\ &\le |S_\alpha(f,A,T_1)-I_1| + |S_\alpha(f,B,T_2)-I_2| \\ &< \varepsilon \end{aligned}$$

so the integral is equal to I_1+I_2, as desired. □

36. If $\alpha\in BV[a,b]$ and $f\in\mathcal{R}_\alpha[a,b]$, show that $f\in\mathcal{R}_\alpha[c,d]$ for every subinterval $[c,d]\subset[a,b]$.

Proof. Let $\beta(x)=V_a^x\alpha$, so β and $\beta-\alpha$ are increasing functions. Then since $f\in\mathcal{R}_\alpha[a,b]$, it follows that

$$f\in\mathcal{R}_\beta[a,b]\cap\mathcal{R}_{\beta-\alpha}[a,b] \implies f\in\mathcal{R}_\beta[a,b] \text{ and } f\in\mathcal{R}_{\beta-\alpha}[a,b]$$

From HW2, since β and $\beta-\alpha$ are increasing, it follows that $f\in\mathcal{R}_\beta[c,d]$ and $f\in\mathcal{R}_{\beta-\alpha}[c,d]$, so $f\in\mathcal{R}_\beta[c,d]\cap\mathcal{R}_{\beta-\alpha}[c,d]=R_\alpha[c,d]$, as desired. □

39. Given $\alpha \in BV[a,b]$, let p and n be the positive and negative variations of α. Show that $\mathcal{R}_\alpha = \mathcal{R}_p \cap \mathcal{R}_n$ and that $\int_a^b f\,d\alpha = \int_a^b f\,dp - \int_a^b f\,dn$ for any $f \in \mathcal{R}_\alpha$.

Proof. Since $\alpha(x) = \alpha(a) + p(x) - n(x)$, we have

$$\mathcal{R}_\alpha = \mathcal{R}_{\alpha(a)+p-n} \supset \mathcal{R}_p \cap \mathcal{R}_n$$

We wish to show the reverse inclusion. Let $f \in \mathcal{R}_\alpha$ and fix $\varepsilon > 0$. Let $I = \int_a^b f\,d\alpha$. Then there exists a partition P^* such that

$$\begin{aligned}
|S_\alpha(f,P,T) - I| &= \left| \sum_{i=1}^m f(t_i)\,[\alpha(x_i) - \alpha(_{i-1})] - I \right| \\
&= \left| \sum_{i=1}^m f(t_i) \Big([\alpha(a) + p(x_i) - n(x_i)] - [\alpha(a) + p(x_{i-1}) - n(x_{i-1})] \Big) - I \right| \\
&= \left| \sum_{i=1}^m f(t_i)\,[p(x_i) - p(x_{i-1})] - \sum_{i=1}^m f(t_i)\,[n(x_i) - n(x_{i-1})] - I \right| < \varepsilon
\end{aligned}$$

□

41. Suppose that (α_n) is a sequence in $BV[a,b]$ and that $V_a^b(\alpha_n - \alpha) \to 0$. Show that $\int_a^b f\,d\alpha_n \to \int_a^b f\,d\alpha$ for all $f \in C[a,b]$.

Proof. Since $f \in C[a,b]$, it is integrable, $f \in \mathcal{R}_\alpha \cap \mathcal{R}_{\alpha_n}$, so

$$\left| \int_a^b f\,d\alpha_n - \int_a^b f\,d\alpha \right| = \left| \int_a^b f\,d(\alpha_n - \alpha) \right|$$

From the result of Problem 29, we have

$$|S_{\alpha_n - \alpha}(f,P,T)| \le \|f\|_\infty V(\alpha_n - \alpha, P) \le \|f\|_\infty V_a^b(\alpha_n - \alpha) \to 0$$

Thus, $\left| \int_a^b f\,d(\alpha_n - \alpha) \right| \to 0$, so $\left| \int_a^b f\,d\alpha_n - \int_a^b f\,d\alpha \right| \to 0$. □

42. Suppose that φ is a strictly increasing continuous function from $[c,d]$ onto $[a,b]$. Given $\alpha \in BV[a,b]$ and $f \in \mathcal{R}_\alpha[a,b]$, show that $\beta = \alpha \circ \varphi \in BV[c,d]$ and that $g = f \circ \varphi \in \mathcal{R}_\beta[c,d]$. Moreover, $\int_c^d g\,d\beta = \int_a^b f\,d\alpha$.

Proof. Let $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ be a partition of $[c,d]$. Then since φ is strictly increasing and onto $[a,b]$, it follows that $Q = \{a = \varphi(x_0) < \varphi(x_1) < \cdots < \varphi(x_n) = b\}$ is a partition of $[a,b]$. Then

$$V(\beta, P) = \sum_{i=1}^n |\alpha \circ \varphi \circ (x_i) - \alpha \circ \varphi(x_{i-1})| = V(\alpha, Q)$$

and since $\alpha \in BV[a,b]$, it follows that $\beta \in BV[c,d]$ since P was arbitrary.

Let $I = \int_a^b f\,d\alpha$ and let $\varepsilon > 0$. Since $f \in \mathcal{R}_\alpha[a,b]$, there exists a partition $P^* = \{a = x_0 < \cdots < x_n = b\}$ such that

$$|S_\alpha(f,P,T) - I| < \varepsilon$$

for all $P \supset P^*$ and all choices of points T. Then $Q^* = \left\{c = \varphi^{-1}(x_0) < \cdots < \varphi^{-1}(x_n) = d\right\}$ is a partition of $[c, d]$. Let $Q = \{c = y_0 < \cdots < y_m = d\} \supset Q^*$ and let T be an arbitrary selection of points under Q. We have

$$|S_\beta(g, Q, T_1) - I| = \left|\sum_{i=1}^{m} g(t_i)\,[\beta(y_i) - \beta(y_{i-1})] - I\right| = \left|\sum_{i=1}^{m} \alpha(\varphi(t_i))\,[\alpha(\varphi(y_i)) - \alpha(\varphi(y_{i-1}))] - I\right|$$

Now, since Q is a partition of $[c, d]$ containing Q^*, it follows that $\varphi(Q) = \{\varphi(y) : y \in Q\}$ is a partition of $[a, b]$ containing P^*, and $\varphi(T) = \{\varphi(t) : t \in T\}$ is a selection of points, so from above, we have

$$|S_\alpha(f, \varphi(Q), \varphi(T)) - I| = \left|\sum_{i=1}^{m} \alpha(\varphi(t_i))\,[\alpha(\varphi(y_i))\alpha(\varphi(y_{i-1}))] - I\right| < \varepsilon$$

and thus $g \in \mathcal{R}_\beta[c, d]$ and $\int_c^d g\, d\beta = I = \int_a^b f\, d\alpha$. □

50. If f is continuous on $[a, b]$, and if $\int_a^b |f(x)|\, dx = 0$, show that $f = 0$.

Proof. Suppose $f \not\equiv 0$, so there exists $c \in (a, b)$ such that $|f(c)| > 0$. Then since f is continuous, it follows that $|f|$ is continuous by the $\varepsilon - \delta$ definition since $||f(x)| - |f(y)|| \le |f(x) - f(y)|$. Thus for a fixed $\varepsilon = \frac{f(c)}{2}$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{f(c)}{2} \implies f(x) > \frac{f(c)}{2}$$

Thus, $f(x) > f(c)/2$ on the interval $[c - \delta, c + \delta] \subset [a, b]$, so $\int_a^b f\, dx \ge 2\delta \cdot \frac{f(c)}{2} > 0$, a contradiction. □