Homework 7

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1. Suppose an iid vector of data $\underline{X} = (X_1, \cdots, X_n)$ can belong to one of two classes Y, where $Y = 0$ or $Y = 1$. A *decision rule* or *classifier* g is a function $g : \mathbb{R}^n \to \mathbb{R}$ that assigns to any n-tuple of data a value 0 or 1. Suppose that the so-called *class conditional* densities f_0 and f_1 of $\underline{X}$ are given,

$$f_j(x_1, \cdots, x_n) = f(x_1, \cdots, x_n \mid Y = j), \quad j = 0, 1$$

are given. Define $L^0(g)$ and $L^1(g)$ as follows:

$$L^0(g) = P(g(\underline{X}) = 1 \mid Y = 0), \quad L^1(g) = P(g(\underline{X}) = 0 \mid Y = 1)$$

For $c > 0$, define the decision rule

$$g_c(x_1, \cdots, x_n) = \begin{cases} 1 \text{ if } cf_1(x_1, \cdots, x_n) > f_0(x_1, \cdots, x_n) \\ 0 \text{ otherwise} \end{cases}$$

Prove that for any classifier g, if $L^0(g) < L^0(g_c)$, then $L^1(g) > L^1(g_c)$. In other words, if L^0 is required to be kept under a certain level, then the decision rule minimizing L^1 has the form g_c for some c.

Proof. Since $g_c(\underline{X})$ and $g(\underline{X})$ are Bernoulli random variables, the condition $L^0(g) < L^0(g_c)$ means that

$$\begin{aligned} P_0(g(X) = 1) &< P_0(g_c(X) = 1) \\ E_0[g(X)] &< E_0[g_c(X)] \end{aligned} \tag{1}$$

Consider the inequality

$$g_c(X)\left[cf_1(X) - f_0(X)\right] \geq g(X)\left[cf_1(X) - f_0(X)\right]$$

If $g_c(X) = 1$, then $cf_1(X) > f_0(X)$ so the LHS is positive. If $g(X) = 1$ then the two sides are equal, and if $g(X) = 0$, then the LHS is greater. If $g_c(X) = 0$, then $cf_1(X) \leq f_0(X)$, so the RHS is either 0 or non-positive, while the LHS is 0.

If we integrate both sides over all possible X, using the fact that $\int g_c(X)f_1(X) = E[g_c(X) \mid Y = 1] = E_1[g_c(X)]$ and etc, we have

$$\begin{aligned} cE_1[g_c(X)] - E_0[g_c(X)] &\geq cE_1[g(X)] - E_0[g(X)] \\ E_0[g(X)] - E_0[g_c(X)] &\geq c\left(E_1[g(X)] - E_1[g_c(X)]\right) \\ E_0[g_c(X)] - E_0[g(X)] &\leq c\left(E_1[g_c(X)] - E_1[g(X)]\right) \end{aligned}$$

From (1), the LHS is positive, so the RHS is positive as well. Thus,

$$E_1[g_c(X)] > E_1[g(X)]$$

and since these are Bernoulli random variables, we have

$$\begin{aligned} P_1(g_c(X) = 1) &> P_1(g(X) = 1) \\ 1 - P_1(g_c(X) = 0) &> 1 - P_1(g(X) = 0) \\ P_1(g(X) = 0) &> P_1(g_c(X) = 0) \\ L^1(g) &> L^1(g_c) \end{aligned}$$

as desired.

□

2. Suppose we consider a Bayesian framework for hypothesis testing, in which we consider testing a simple null vs a simple alternative:
$$H_0 : \mu = \mu_0, \quad H_a : \mu = \mu_a$$
Suppose we have the probabilities $P(\mu = \mu_0)$ and $P(\mu = \mu_a)$ as the *prior probabilities* that the null or alternative are true. Suppose we are given a distribution of the data under both null and alternative, so that
$$f(x_1, \cdots, x_n \mid \mu = \mu_0), \quad f(x_1, \cdots, x_n \mid \mu = \mu_a)$$
are given. How would you use the prior and likelihood to construct a test of hypothesis. Be completely specific about your test statistic and how it is computed.

Solution. We have the posterior ratio
$$\frac{P(H_0 \mid \underline{X})}{P(H_a \mid \underline{X})} = \frac{P(H_0)}{P(H_a)} \cdot \frac{f(\underline{X} \mid H_0)}{f(\underline{X} \mid H_a)}$$
If this ratio is greater than 1, we choose H_0, and otherwise, we choose H_a. Thus, we have
$$\frac{f(\underline{X} \mid H_0)}{f(\underline{X} \mid H_a)} > \frac{P(H_a)}{P(H_0)}$$
as the likelihood ratio test. We know the prior probabilities, so this is a valid test.

□

11. Suppose $X_1, \cdots, X_n$ are iid standard normal data. Show that the vector of random variables given by $(X_1 - \bar{X}, \cdots, X_n - \bar{X})$ is independent of $\bar{X}$ for the case $n = 2$. Use this independence for more general n to show that for any sample size n, the scaled sample variance $(n-1)s^2$ is the sum of squares of independent standard normal random variables.

Proof. Consider the joint density of $(X_1 - \bar{X}, \bar{X})$. Since X_i are normal random variables, these are just linear combinations of normal random variables, thus they are both normal. Then the covariance is
$$\begin{aligned}
\operatorname{Cov}(X_1 - \bar{X}, \bar{X}) &= E[(X_1 - \bar{X})\bar{X}] - E[X_1 - \bar{X}]E[\bar{X}] \\
&= E[X_1\bar{X}] - E[\bar{X}^2] = E\left[X_1 \cdot \frac{1}{n}\sum_{i=1}^n X_i\right] - \left(E[\bar{X}^2] - (E[\bar{X}]^2)\right) - (E[\bar{X}])^2 \\
&= \frac{1}{n}\sum_{i=1}^n E[X_1 X_i] - \operatorname{Var}(\bar{X}) \\
&= \frac{1}{n}\left(E[X_1^2] + \sum_{i=2}^n E[X_1]E[X_i]\right) - \frac{1}{n} \\
&= \frac{1}{n}\left(\operatorname{Var}(X_1) + (E[X_1])^2\right) - \frac{1}{n} = \frac{1}{n}(1) - \frac{1}{n} = 0
\end{aligned}$$
Since this is a pair of bivariate normal random variables, and their covariance is zero, it follows that they are independent. Thus, by extension, the entire vector is independent of $\bar{X}$, as desired.

We have the scaled sample variance
$$(n-1)s^2 = \sum_{i=1}^n (X_i - \bar{X})^2$$
Each of $X_i - \bar{X}$ is a standard normal random variable, and they are each independent, so we are done.

□

Chapter 9: Hypothesis Testing and Assessing Goodness of Fit

10. Suppose that $X_1, \cdots, X_n$ form a random sample from a density function, $f(x \mid \theta)$, for which T is a sufficient statistic for θ. Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ vs $H_A : \theta = \theta_1$ is a function of T. Explain how, if the distribution of T is known under H_0, the rejection region of the test may be chosen so that the test has the level α.

    Proof. Since T is a sufficient statistics, we can factor the likelihood as

    $$f(X_1, \cdots, X_n \mid \theta) = g(T, \theta)h(X_1, \cdots, X_n)$$

    Thus, the likelihood ratio test is given by

    $$\begin{aligned}\Lambda &= \frac{f(X_1, \cdots, X_n \mid \theta = \theta_0)}{f(X_1, \cdots, X_n \mid \theta = \theta_1)} = \frac{g(T, \theta_0)h(X_1, \cdots, X_n)}{g(T, \theta_1)h(X_1, \cdots, X_n)} \\ &= \frac{g(T, \theta_0)}{g(T, \theta_1)} = h(T)\end{aligned}$$

    since θ_0 and θ_1 are just constant values, this ratio is only a function of T, as desired.

    Λ is just some function of T, and we can find the probability

    $$P(\Lambda \leq c \mid H_0) = P(h(T) \leq c \mid H_0) = \alpha$$

    This probability only depends on the null distribution of T supposing we can invert h (or even if it is not invertible). Thus, if we are given α, we can solve for the value of c, and thus the rejection region corresponds with $\Lambda = h(T) \leq c$ for which we can solve for values of T that we would reject.

    □

12. Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with the density function $f(x \mid \theta) = \theta \exp(-\theta x)$. Derive a likelihood ratio test of $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$, and show that the rejection region is of the form $\left\{\bar{X} \exp(-\theta_0 \bar{X}) \leq c\right\}$.

    Solution. The MLE of the exponential distribution is given by $\hat{\theta} = 1/\bar{X}$. The GLR is given by

    $$\begin{aligned}\Lambda &= \frac{f(X_1, \cdots, X_n \mid \theta = \theta_0)}{f(X_1, \cdots, X_n \mid \theta = 1/\bar{X})} = \frac{\prod_{i=1}^n \theta_0 e^{-\theta_0 X_i}}{\prod_{i=1}^n \frac{1}{\bar{X}} e^{-X_i/\bar{X}}} \\ &= (\theta_0 \bar{X})^n \exp\left[-\left(\theta_0 - \frac{1}{\bar{X}}\right)\sum_{i=1}^n X_i\right] \\ &= (\theta_0 \bar{X})^n \exp\left[-\left(\theta_0 - \frac{1}{\bar{X}}\right)n\bar{X}\right] \\ &= (\theta_0 \bar{X})^n \exp\left[n - n\theta_0 \bar{X}\right] \\ &= \theta_0^n e^n \left(\bar{X} \exp\left(-\theta_0 \bar{X}\right)\right)^n\end{aligned}$$

    Since we reject this for small Λ, the rejection region is defined by the set

    $$\begin{aligned}\{\, \underline{X} \mid \Lambda \leq c \,\} &= \{\, \underline{X} \mid \theta_0^n e^n \left(\bar{X} \exp\left(-\theta_0 \bar{X}\right)\right)^n \leq c \,\} \\ &= \left\{ \underline{X} \mid \bar{X} \exp(-\theta_0 \bar{X}) \leq \frac{c^{1/n}}{\theta_0 e} \right\}\end{aligned}$$

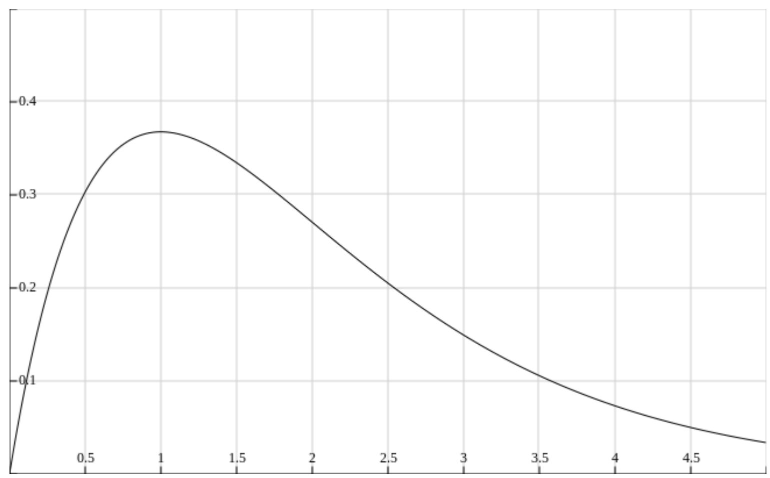
    as desired.

    □

13. Suppose, to be specific, that in Problem 12, $\theta_0 = 1, n = 10$, and that $\alpha = 0.05$. In order to use the test, we must find the appropriate value of c.

   a. Show that the rejection region is of the form $\left\{\bar{X} \le x_0\right\} \cup \left\{\bar{X} \ge x_1\right\}$, where x_0 and x_1 are determined by c.

   Proof. We have
$$\Lambda = e^n(\bar{X}\exp(-\bar{X}))^n \le c$$
so
$$\bar{X}\exp(-\bar{X}) \le \frac{c^{1/10}}{e}$$
is the rejection region. The plot of the LHS is shown below:



which shows that any horizontal line through this graph (a threshold) intersects twice, and the solution region is a union of two intervals of the form desired. The endpoints of these intervals depends on the horizontal line chosen, which corresponds to whatever $c^{1/10}/e$ is.

□

   b. Explain why c should be chosen so that $P(\bar{X}\exp(-\bar{X}) \le c) = 0.05$ when $\theta_0 = 1$.

   Answer. We are in the significance level $\alpha = 0.05$, so c should be chosen so the probability of type I error is 0.05.

   c. Explain why $\sum_{i=1}^{10} X_i$ and hence $\bar{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?

   Answer. The sum of independent exponential random variables follows a Gamma distribution. Thus, we can explicitly determine the distribution of $Y = \bar{X}\exp(-\bar{X})$ (although this isn't nicely solvable, theoretically it would be) so we can figure out c so that
$$P(Y \le c) = 0.05$$

   d. Suppose that you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer.

   Answer. Generate 10 exponential random variables with $\theta = 1$, compute $\bar{X}\exp(-\bar{X})$. Run this simulation many times, storing each trial. The mean of all the trials should be approximately normal by the Central Limit Theorem, so $P(\bar{X}\exp(-\bar{X}) \le c) = \alpha$ and we must find the value of c such that this is 0.05, which is easy if we standardize.

14. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = 2P(H_1)$. The hypothesis H_0 will be chosen if $P(H_0 \mid x) > P(H_1 \mid x)$. For $\sigma^2 = 0.1, 0.5, 1.0, 5.0$:

   a. For what values of X will H_0 be chosen?

   Solution. We choose H_0 if

$$\begin{aligned} \frac{P(H_0 \mid x)}{P(H_1 \mid x)} &= \frac{P(H_0)}{P(H_1)} \cdot \frac{f(x \mid H_0)}{f(x \mid H_1)} = 2 \cdot \frac{f(x \mid H_0)}{f(x \mid H_1)} > 1 \\ \implies \frac{f(x \mid H_0)}{f(x \mid H_1)} &> \frac{1}{2} \end{aligned}$$

   We have

$$\begin{aligned} \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-1)^2}{2\sigma^2}\right)} &= \exp\left(-\frac{1}{2\sigma^2}\left[x^2 - (x-1)^2\right]\right) \\ &= \exp\left(-\frac{2x-1}{2\sigma^2}\right) > \frac{1}{2} \end{aligned}$$

   Solving for x, we find

$$x < \frac{1}{2} + \sigma^2 \ln 2$$

   are the values of x for which we choose H_0. To find the values of X for which we choose H_0 at various values of σ^2, just substitute in.

   □

   b. In the long run, what proportion of the time will H_0 be chosen if H_0 is true 2/3 of the time?

   Answer. In the long run, we expect to choose H_0 approximately 2/3 of the time.

18. Let $X_1, \cdots, X_n$ be iid random variables from a double exponential distribution with density

$$f(x) = \frac{1}{2}\lambda \exp(-\lambda |x|).$$

   Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda = \lambda_1$ where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?

   Solution. We have the likelihood ratio

$$\begin{aligned} \Lambda = \frac{f(X_1, \cdots, X_n \mid \lambda = \lambda_0)}{f(X_1, \cdots, X_n \mid \lambda = \lambda_1)} &= \frac{\prod_{i=1}^{n} \frac{1}{2}\lambda_0 \exp(-\lambda_0 |X_i|)}{\prod_{i=1}^{n} \frac{1}{2}\lambda_1 \exp(-\lambda_1 |X_i|)} \\ &= \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\left[-(\lambda_0 - \lambda_1)\sum_{i=1}^{n} |X_i|\right] \le c \\ \sum_{i=1}^{n} |X_i| \le \frac{1}{\lambda_1 - \lambda_0} \log\left[c\left(\frac{\lambda_1}{\lambda_0}\right)^n\right] \end{aligned}$$

   Since H_0 and H_1 are simple hypotheses, the Neyman-Pearson Lemma guarantees this will be most powerful. For the alternative $H_1 : \lambda > \lambda_0$, since $\lambda_1 > \lambda_0$ for any simple alternative in this case, and this likelihood ratio is most powerful for each, it is uniformly most powerful against $H_1 : \lambda > \lambda_0$.

   □

20. Consider two PDFs on [0, 1]: $f_0(x) = 1$ and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

Solution. Since these are simple null and simple alternative, by the Neyman-Pearson Lemma, the likelihood ratio test is most powerful, so we use that. We have

$$\Lambda = \frac{f(X \mid X \sim f_0)}{f(X \mid X \sim f_1)} = \frac{1}{2X}$$

so the likelihood ratio test at significance level $\alpha = 0.10$ is

$$\begin{aligned} P(\Lambda \le c \mid H_0) &= \alpha = 0.10 \\ P\left(\frac{1}{2X} \le c \middle| X \sim f_0\right) &= P\left(X \ge \frac{1}{2c} \middle| X \sim f_0\right) \\ &= 1 - \frac{1}{2c} = 0.10 \\ \implies c &= \frac{5}{9} \end{aligned}$$

Thus, the power is given by

$$\begin{aligned} P(\Lambda \le c \mid H_1) &= P\left(\frac{1}{2X} \le \frac{5}{9} \middle| X \sim f_1\right) \\ &= P\left(X \ge \frac{9}{10} \middle| X \sim f_1\right) \\ &= \int_{9/10}^{1} \frac{1}{2x}\, dx = 1 - \left(\frac{9}{10}\right)^2 = 0.19 \end{aligned}$$

and this is maximal, as guaranteed by the Neyman-Pearson Lemma.

$\square$

24. Let X be a binomial random variable with n trials and probability p of success.

a. What is the GLR for testing $H_0 : p = 0.5$ vs $H_A : p \neq 0.5$?

Solution. For the MLE, we consider the log-likelihood and its derivative wrt p :

$$\begin{aligned} \log f(X = x \mid p) &= \log\left[\binom{n}{x} p^x (1-p)^{n-x}\right] \\ &= \log\binom{n}{x} + x \log p + (n-x)\log(1-p) \\ \frac{\partial}{\partial p} \log f(X = x \mid p) &= \frac{x}{p} - \frac{n-x}{1-p} = 0 \\ p &= \frac{x}{n} \end{aligned}$$

Thus, the GLR is given by

$$\begin{aligned} \Lambda &= \frac{f(X = x \mid p = 0.5)}{f(X = x \mid p = x/n)} = \frac{\binom{n}{x}\left(\frac{1}{2}\right)^n}{\binom{n}{x}\left(\frac{x}{n}\right)^x \left(1 - \frac{x}{n}\right)^{n-x}} \\ &= \left(\frac{1}{2}\right)^n \left(\frac{n}{x}\right)^x \left(\frac{n}{n-x}\right)^{n-x} = \left(\frac{n/2}{x}\right)^x \left(\frac{n/2}{n-x}\right)^{n-x} \end{aligned}$$

$\square$

b. Show that the test rejects for large values of $|X - n/2|$.

Proof. We have

$$\begin{aligned}\Lambda &= \left(\frac{n}{2x}\right)^x \left(\frac{n}{2(n-x)}\right)^{n-x} \\ &= \left(\frac{1}{\frac{2x}{n}}\right)^x \left(\frac{1}{2\left(1-\frac{x}{n}\right)}\right)^{n-x} \\ &= \left[\left(\frac{2x}{n}\right)^{-2x/n} \left(2-\frac{2x}{n}\right)^{-2+2x/n}\right]^{n/2}\end{aligned}$$

Let $y = 2x/n$, so this becomes

$$\Lambda = \left[y^{-y}\left(2-y\right)^{2-y}\right]^{n/2}$$

Let $z = 1 - y$. Then we have

$$\Lambda = \left[(1-z)^{-(1-z)}(1+z)^{-(1+z)}\right]^{n/2}$$

Note that this is symmetric about $z = 0$ (obvious, let $z' = -z$ and the two values are exactly equal) so this is symmetric about $2x/n = 1$. It also attains its maximum value at $z = 0$ (first and second derivative tests omitted). Thus, if we are far from $z = 0$, we reject the since then Λ is small. This is equivalent to a large value of

$$\left|\frac{2x}{n} - 1\right| = \frac{2}{n}\left|x - \frac{n}{2}\right|$$

so if $|x - n/2|$ is large, we reject the null, as desired.

□

c. Using the null distribution of X, show how the significance level corresponding to a rejection region $|X - n/2| > k$ can be determined.

Solution. The null distribution is

$$P(X = x) = \binom{n}{x}\left(\frac{1}{2}\right)^n$$

Thus, we have

$$\begin{aligned}\alpha &= P(|X - n/2| > k \mid H_0) = P(X - n/2 > k) + P(n/2 - X > k) \\ &= P(X > k + n/2) + P(X < n/2 - k)\end{aligned}$$

To compute this, just sum the PMF from the null distribution over the values in the rejection region.

□

d. If $n = 10$ and $k = 2$, what is the significance level of the test?

Solution. From above, we have

$$\begin{aligned}\alpha &= P(X > 2 + 10/2) + P(X < 10/2 - 2) = P(X > 7) + P(X < 3) \\ &= \left(\frac{1}{2}\right)^{10}\left[\binom{10}{8} + \binom{10}{9} + \binom{10}{10} + \binom{10}{0} + \binom{10}{1} + \binom{10}{2}\right] \\ &= \frac{7}{64}\end{aligned}$$

□

e. Use the normal approximation to the binomial distribution to find the significance level if $n = 100$ and $k = 10$.

Solution. A binomial distribution with n trials and probability p is approximately

$$N(np, np(1-p)) = N\left(50, 5^2\right)$$

in this case. Thus, the significance level is

$$\begin{aligned}\alpha &= P(X > 10 + 100/2) + P(X < 100/2 - 10) = P(X > 60) + P(X < 400) \\ &= P\left(\frac{X-50}{5} > 2\right) + P\left(\frac{X-50}{5} < -2\right) \\ &\approx 2\Phi(-2) \approx 0.0455\end{aligned}$$

□

26. True or false:

a. The generalized likelihood ratio statistic Λ is always less than or equal to 1.

Answer. This is true. The likelihood in the denominator is the max over all possible values of a parameter, which is always greater than or equal to the max over a subset of the possibilities.

b. If the p-value is 0.03, the corresponding test will reject at the significance level 0.02.

Answer. This is false. The test will only reject if the p-values is less than the significance level.

c. If a test rejects at a significance level 0.06, then the p-value is less than or equal to 0.06.

Answer. This is true. We reject as p-values less than the significance level.

d. The p-value of a test is the probability that the null hypothesis is correct.

Answer. This is false. The p-value is the smallest significance level at which we reject the null hypothesis.

e. In testing a simple versus simple hypothesis via the likelihood ratio, the p-value equals the likelihood ratio.

Answer. This is false. The p-value is not the likelihood ratio, it is a probability.

f. If a chi-square test statistic with 4 degrees of freedom has a value of 8.5, the p-value is less than 0.05.

Answer. This is false. The p-value in this case is 0.07.

30. Suppose that the null hypothesis is true, that the distribution of the test statistic, T say, is continuous with CDF F and that the test rejects for large values of T. Let V denote the p-value of the test.

a. Show that $V = 1 - F(T)$.

Proof. Suppose we reject if $T > M$ for some M. The p-value is defined as

$$V = P(T \geq M \mid H_0) = 1 - P(T < M \mid H_0) = 1 - F(T)$$

as desired.

□

b. Conclude that the null distribution of V is uniform. (Hint: Prop C Section 2.3)

Solution. Since F is a CDF, it is increasing and invertible. Thus, we have

$$\begin{aligned} P(V \leq v) &= P(1 - F(T) \leq v) = P(F(T) \geq 1 - v) \\ &= P(T \geq F^{-1}(1 - v)) = 1 - P(T < F^{-1}(1 - v)) \\ &= 1 - F(F^{-1}(1 - v)) = 1 - (1 - v) = v \end{aligned}$$

So the density of V is the derivative of this wrt v, which is 1, so V is uniform on the interval [0, 1], as desired.

□

c. If the null hypothesis is true, what is the probability that the p-value is greater than 0.1?

Answer. Since V follows a uniform distribution, $P(V > 0.1) = 0.9$.

d. Show that the test that rejects if $V < \alpha$ has significance level α.

Proof. Since $V = P(T > M \mid H_0)$, we have $P(T > M \mid H_0) < \alpha$. At the significance level α, we reject H_0 if this probability is less than α, which is true in this case. Thus, we will reject H_0, as desired.

□