Homework 11

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Chapter 11: The Space of Continuous Functions

1. For each n, let Q_n be the set of all polygonal functions that have nodes at $k/n, k = 0, \cdots, n.$, and that take on only rational values at those points. Check that Q_n is a countable set, and hence that the union of the Q_n's is a countable dense set in $C[0, 1]$.

   Proof. If $f \in Q_n$, then f has $n + 1$ nodes. These nodes uniquely define f and each node can take on any value in $\mathbb{Q}$, so there is a natural bijection between Q_n and $\mathbb{Q}^{n+1}$. Since $\mathbb{Q}^{n+1}$ is countable, it follows that Q_n is countable.

   Then $Q = \bigcup_{n=0}^{\infty} Q_n$ is a countable union of countable sets, and thus countable. □

7. If p is a polynomial and $\varepsilon > 0$, prove that there is a polynomial q with rational coefficients such that $\|p - q\|_\infty < \varepsilon$ on $[0, 1]$.

   Proof. Suppose $p = a_0 + a_1x + \cdots + a_nx^n$ where $a_i \in \mathbb{R}$ for each i. Let $\varepsilon > 0$. Then since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $b_0, b_1, \cdots, b_n \in \mathbb{Q}$ such that $0 < a_i - b_i < \frac{\varepsilon}{n+1}$ for each i. Let $q = b_0 + b_1x + \cdots + b_nx^n$. Since $x \in [0, 1]$, the max of the polynomial occurs at $x = 1$, so

$$\begin{aligned} \|p - q\|_\infty &= \|(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n\|_\infty \\ &= (a_0 - b_0) + (a_1 - b_1) + \cdots + (a_n - b_n) \\ &< (n + 1) \cdot \frac{\varepsilon}{n + 1} = \varepsilon \end{aligned}$$

   as desired. □

9. Let $\mathcal{P}_n$ denote the set of polynomials of degree at most n, considered as a subset of $C[a, b]$). Clearly $\mathcal{P}_n$ is a subspace of $C[a, b]$ of dimension $n + 1$. Also, $\mathcal{P}_n$ is closed in $C[a, b]$. How do you know that $\mathcal{P}$, the union of all of the P_n, is not all of $C[a, b]$? That is, why are there necessarily non-polynomial elements in $C[a, b]$?

   Solution. If $\mathcal{P}$ was all of $C[a, b]$, then there are no continuous functions that aren't polynomials. However, $\sin x$ cannot be represented as a polynomial. If it could, then it would have finitely many roots, since polynomials are finite degree, but the roots of $\sin x$ are $2\pi k$ for $k \in \mathbb{Z}$. Thus, $\mathcal{P}$ is not all of $C[a, b]$. □

12. Let p_n be a polynomial of degree m_n, and suppose that $p_n \rightrightarrows f$ on $[a, b]$, where f is not a polynomial. Show that $m_n \to \infty$.

14. Let $f \in C[a, b]$ be continuously differentiable, and let $\varepsilon > 0$. Show that there is a polynomial p such that $\|f - p\|_\infty < \varepsilon$ and $\|f' - p'\|_\infty < \varepsilon$. Conclude that $C^{(1)}[a, b]$ is separable.

Proof. Given f, there exists a polynomial q such that $\|f' - q\|_\infty < \varepsilon/(b-a)$ by WAT. Let p be the anti-derivative of q, with $p(a) = q(a)$. Then

$$\begin{aligned}
\|f - p\|_\infty &= \sup_{x\in[a,b]} |f(x) - p(x)| = \sup_{x\in[a,b]} \left| \int_a^x (f'(t) - q(t))\, dt \right| \\
&\le \sup_{x\in[a,b]} \int_a^x |f'(t) - q(t)|\, dt \le (b-a)\,\|f' - q\|_\infty \\
&< (b-a) \cdot \frac{\varepsilon}{b-a} = \varepsilon
\end{aligned}$$

as desired. □

27. Let T be a trig polynomial. Prove:

(a) If T is an even function, then T can be written using only cosines.

Proof. If T is even, then we have

$$\begin{aligned}
T(x) &= a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \\
T(-x) &= a_0 + \sum_{k=1}^n (a_k \cos(-kx) + b_k \sin(-kx)) = a_0 + \sum_{k=1}^n (a_k \cos kx - b_k \sin kx) \\
T(x) = T(-x) &\implies \sum_{k=1}^n b_k \sin kx = -\sum_{k=1}^n b_k \sin kx \implies \sum_{k=1}^n b_k \sin kx = 0 \\
\implies T(x) &= a_0 + \sum_{k=1}^n a_k \cos kx
\end{aligned}$$

Thus T can be written using only cosines. □

(b) If T is an odd function, then T can be written using only sines.

Proof. If T is even, then we have

$$\begin{aligned}
T(x) &= a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \\
-T(-x) &= -a_0 - \sum_{k=1}^n (a_k \cos(-kx) + b_k \sin(-kx)) = -a_0 + \sum_{k=1}^n (-a_k \cos kx + b_k \sin kx) \\
T(x) = -T(x) &\implies a_0 + \sum_{k=1}^n a_k \cos kx = -\left(a_0 + \sum_{k=1}^n a_k \cos kx\right) \implies a_0 + \sum_{k=1}^n a_k \cos kx = 0 \\
\implies T(x) &= \sum_{k=1}^n b_k \sin kx
\end{aligned}$$

Thus T can be written using only sines. □