Homework 9

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April 27, 2017

1. Create an intersection graph for each collection of sets.

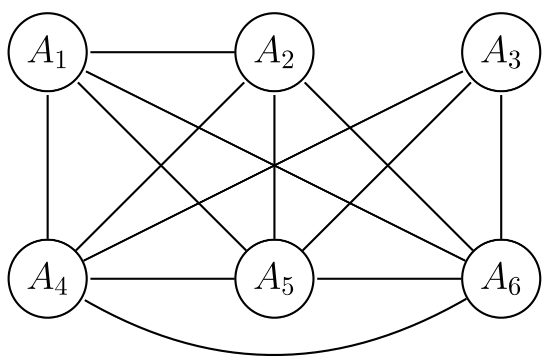
   (a)

$$
\begin{aligned}
A_1 &= \{\, x \in \mathbb{R} \mid x < 0 \,\} \\
A_2 &= \{\, x \in \mathbb{R} \mid -1 < x < 0 \,\} \\
A_3 &= \{\, x \in \mathbb{R} \mid 0 < x < 1 \,\} \\
A_4 &= \{\, x \in \mathbb{R} \mid -1 < x < 1 \,\} \\
A_5 &= \{\, x \in \mathbb{R} \mid x > -1 \,\} \\
A_6 &= \mathbb{R}
\end{aligned}
$$

Solution. The non-empty intersections are

$$
\begin{array}{cccc}
A_1 \cap A_2 & A_1 \cap A_4 & A_1 \cap A_5 & A_1 \cap A_6 \\
 & A_2 \cap A_4 & A_2 \cap A_5 & A_2 \cap A_6 \\
 & A_3 \cap A_4 & A_3 \cap A_5 & A_3 \cap A_6 \\
 & & A_4 \cap A_5 & A_4 \cap A_6 \\
 & & & A_5 \cap A_6
\end{array}
$$

The intersection graph is given by



$\square$

(b)

$$\begin{aligned}
A_1 &= \{\, x \in \mathbb{Z} \mid -4 \le x \le 0 \,\} \\
A_2 &= \{\, x \in \mathbb{Z} \mid \exists n \in \mathbb{N} \text{ such that } x = 2^n \text{ or } x = -(2)^n \,\} \\
A_3 &= \{\, x \in \mathbb{Z} \mid 2 \mid x \,\} \\
A_4 &= \{\, x \in \mathbb{Z} \mid x \text{ is odd} \,\} \\
A_5 &= \{\, x \in \mathbb{Z} \mid 3 \mid x \,\}
\end{aligned}$$

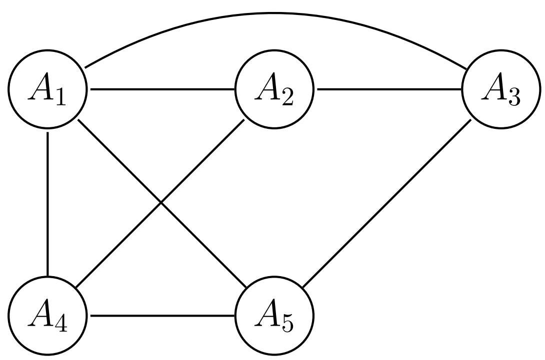
Solution. We have

$$\begin{aligned}
A_1 &= \{-4, -3, -2, -1, 0\} \\
A_2 &= \{\cdots, -8, -4, -2, -1, 1, 2, 4, 8, \cdots\} \\
A_3 &= \{\cdots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \cdots\} \\
A_4 &= \{\cdots, -7, -5, -3, -1, 1, 3, 5, 7, \cdots\} \\
A_5 &= \{\cdots, -6, -3, 0, 3, 6, \cdots\}
\end{aligned}$$

and the non-empty intersections are

$$\begin{array}{cccc}
A_1 \cap A_2 & A_1 \cap A_3 & A_1 \cap A_4 & A_1 \cap A_5 \\
 & & A_2 \cap A_3 & A_2 \cap A_4 \\
 & & & A_3 \cap A_5 \\
 & & & A_4 \cap A_5
\end{array}$$

The intersection graph is given by



∎

47.16 Prove that in any graph with two or more vertices, there must be two vertices of the same degree.

Proof. If the graph G has two or more 0-degree vertices, then we are done. Otherwise, consider the graph G' which is obtained by removing all 0-degree vertices from G. If G' has n vertices, then the possible values of degrees of vertices is $\{1, 2, \cdots, n-1\}$, but since there are n vertices, by the Pigeonhole principle there must be two vertices of the same degree. ∎

47.21 Let G and H be graphs. We say that G is isomorphic to H provided there is a bijection $f : V(G) \to V(H)$ such that for all $a, b \in V(G)$, we have $a \sim b$ in G if and only if $f(a) \sim f(b)$ in H.

(a) Prove that isomorphic graphs have the same number of vertices.

Proof. Since f is a bijection between finite sets $V(G)$ and $V(H)$, the cardinalities of the two sets must be equal, so G and H must have the same number of vertices. ∎

(b) Prove that if $f : V(G) \to V(H)$ is an isomorphism of graphs G and H and if $v \in V(G)$, then he degree of v in G equals the degree of $f(v)$ in H.

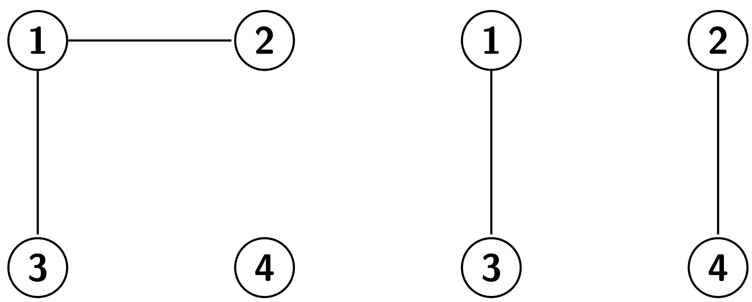
Proof. Suppose $d_G(v) = n$, and the neighbors of v are $\{u_1, \cdots, u_n\} \in V(G)$. Then since $u_i \sim v, \forall i$ in G, it follows that $f(u_i) \sim f(v), \forall i$ in H since f is an isomorphism. Furthermore, this condition is bijective, so the neighbors of $f(v)$ are exactly $\{f(u_1), \cdots, f(u_n)\}$ so $d_H(f(v)) = n$. $\square$

(c) Prove that isomorphic graphs have the same number of edges.

Proof. If an edge (a, b) is in $E(G)$, then $a \sim b \iff f(a) \sim f(b)$ so the edge $(f(a), f(b))$ is in $E(H)$, and conversely as well. Thus isomorphic graphs have the same number of edges. $\square$

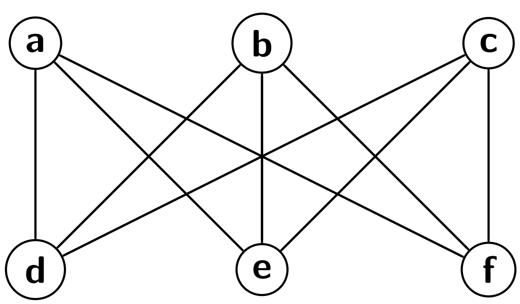
(d) Give an example of two graphs that have the same number of vertices and the same number of edges but that are not isomorphic.

Solution. Here are two 4-vertex, 2-edge graphs that are not isomorphic.

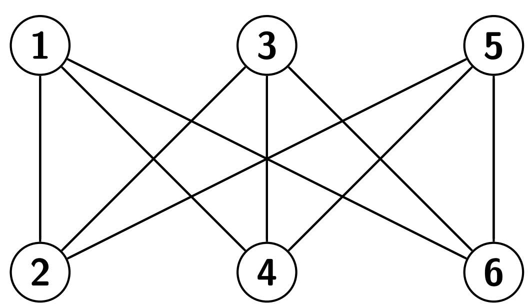


$\square$

(e) Let G be the graph whose vertex set is $\{1, 2, 3, 4, 5, 6\}$. In this graph, there is an edge from v to w if and only if $v - w$ is odd. Let H be the graph in the figure. Find an isomorphism $f : V(G) \to V(H)$.



Solution. G is given by



So an isomorphism f is

$$f : \begin{cases} 1 & \mapsto a \\ 2 & \mapsto d \\ 3 & \mapsto b \\ 4 & \mapsto e \\ 5 & \mapsto c \\ 6 & \mapsto f \end{cases}$$

□

48.11 Recall the definition of graph isomorphism from Exercise 47.21. We call a graph G self-complementary if G is isomorphic to $\overline{G}$.

(a) Show that the graph $G = (\{a, b, c, d\}, \{ab, bc, cd\})$ is self-complementary.

Proof. We have

$$V(\overline{G}) = \{a, b, c, d\}$$
$$E(\overline{G}) = \{ac, ad, bd\}$$

and we have an isomorphism

$$f : \begin{cases} a & \mapsto c \\ b & \mapsto a \\ c & \mapsto d \\ d & \mapsto b \end{cases}$$

This is an isomorphism because

$$a \sim_G b \iff f(a) = c \sim_H a = f(b)$$
$$b \sim_G c \iff f(b) = a \sim_H d = f(c)$$
$$c \sim_G d \iff f(c) = d \sim_H b = f(d)$$

□

(b) Find a self-complementary graph with five vertices.

Solution. If $G = (\{a, b, c, d, e\}, \{ab, bc, cd, de, ae\})$, then G is self-complementary because $\overline{G} = (\{a, b, c, d, e\}, \{ac, ad, bd, be, ce\})$ and we have an isomorphism (easy to check...)

$$f : \begin{cases} a & \mapsto a \\ b & \mapsto c \\ c & \mapsto e \\ d & \mapsto b \\ e & \mapsto d \end{cases}$$

□

(c) Prove that if a self-complementary graph has n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Proof. There are a total of $\binom{n}{2} = \frac{n(n-1)}{2}$ possible edges. If a graph G is self-complementary, then its complement $\overline{G}$ must have the same number of edges, and combined these exhaust all $\binom{n}{2}$ edges. Thus $\frac{n(n-1)}{2}$ must be even.

Suppose $n \equiv 2 \pmod 4$. Then $n = 4k+2$ for some $k \in \mathbb{N}$, and

$$\frac{n(n-1)}{2} = \frac{(4k+2)(4k+1)}{2} = (2k+1)(4k+1)$$

which is odd, which can't work. Now if $n \equiv 3 \pmod 4$, then $n = 4k+3$ for some $k \in \mathbb{N}$ and

$$\frac{n(n-1)}{2} = \frac{(4k+3)(4k+2)}{2} = (4k+3)(2k+1)$$

which is odd, which can't work. Thus, we must have either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, and as demonstrated from parts (a) and (b), graphs that satisfy these conditions ($n = 4$ and $n = 5$) exist. $\square$

49.4 Let $n \geq 2$ be an integer. Form a graph G_n, whose vertices are all the two-element subsets of $\{1, 2, \cdots, n\}$. In this graph we have an edge between distinct vertices $\{a, b\}$ and $\{c, d\}$ exactly when $\{a, b\} \cap \{c, d\} = \varnothing$.

(a) How many vertices does G_n have?

Solution. The number of vertices is the number of 2-element subsets of $\{1, 2, \cdots, n\}$, or $\binom{n}{2}$. $\square$

(b) How many edges does G_n have?

Solution. For a vertex $\{a, b\}$, the number of disjoint 2-element subsets of $\{1, 2, \cdots, n\}$ is just the number of 2-element subsets of $\{1, 2, \cdots, n\} \setminus \{a, b\}$, which is $\binom{n-2}{2}$. There are $\binom{n}{2}$ total vertices, but this double counts every edge, so the total number of edges is $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$. $\square$

(c) For which values of $n \geq 2$ is G_n connected? Prove your answer.

Solution. Assume different variables represent different values. Starting at $\{a, b\}$ we can travel to any $\{c, d\}$ because there is an edge between them. Suppose we wish to travel to $\{a, c\}$ from $\{a, b\}$. There is no edge because these are not disjoint. Thus we must find an intermediate edge disjoint from both of these: $\{d, e\}$, and then there is a path $\{a, b\} \to \{d, e\} \to \{a, c\}$. Thus, there must be distinct values a, b, c, d, e, so n must be at least 5. $\square$

6. Let A be the set of all simple undirected graphs. Let R be the relation on A defined as follows: $(G, H) \in R$ if there is an isomorphism $f : V(G) \to V(H)$. Prove R is an equivalence relation.

Proof. Reflexive: Clearly $(G, G) \in R$ because there is the identity isomorphism $f = \mathrm{id}_G$.

Symmetric: If $(G, H) \in R$, then there exists an isomorphism $f : V(G) \to V(H)$. Then its inverse $f^{-1} : V(H) \to V(G)$ is also an isomorphism, so $(H, G) \in R$.

Transitive: If $(G, H) \in R$ and $(H, K) \in R$, then there exists isomorphisms $f : V(G) \to V(H)$ and $g : V(H) \to V(K)$. Then the composition $g \circ f : V(G) \to V(K)$ is also an isomorphism, so $(G, K) \in R$ as well. $\square$

7. Let $k, n \in \mathbb{N}$ with $n > 0$. Derive a formula using n and k the number of subgraphs of K_n with exactly k vertices.

Solution. There are $\binom{n}{k}$ ways to choose k vertices to be in the sub-graph. Of all of the possible $\binom{k}{2}$ edges between these vertices (all of which are also in K_n), we choose a subset of these to be in the sub-graph, in $2^{\binom{k}{2}}$ ways. Thus the number of subgraphs of K_n with exactly k vertices is $\binom{n}{k}2^{\binom{k}{2}}$. $\square$

8. (a) Give one advantage and one disadvantage each for the star, ring, and hybrid LAN topologies.

Answer. Star:

- Every node is connected to every other node.
- High dependence on central node.

Ring:

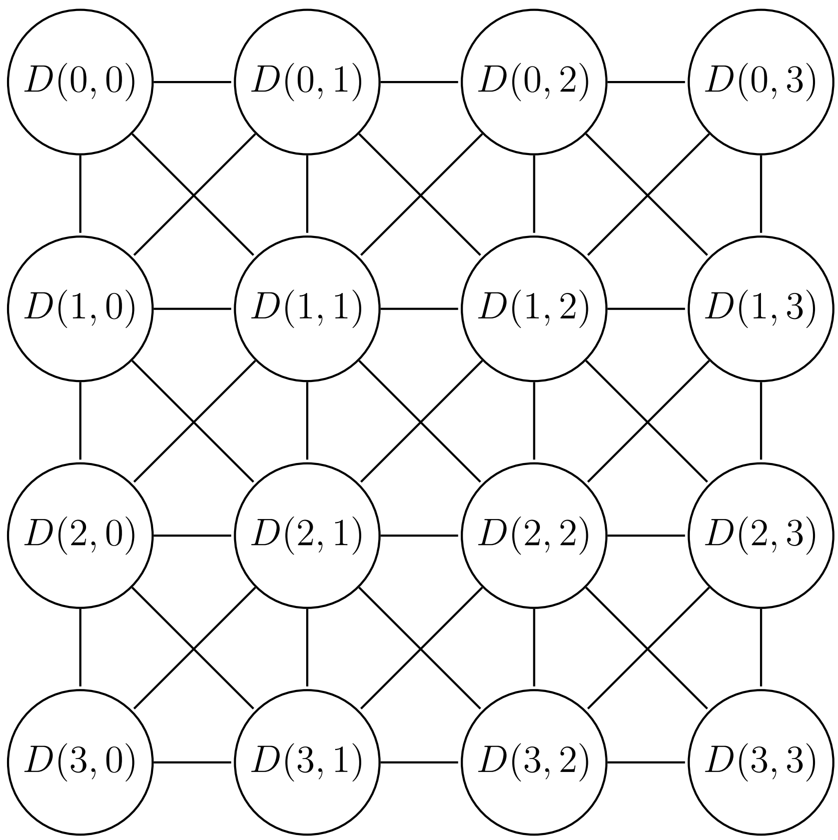
- Every node is connected to every other node.
- Connections are inefficient because most connections must pass through other nodes.

Hybrid:

- Every node is connected to every other node.
- It is a complicated system.

(b) In the basic mesh network, the number of devices, n, is a perfect square, so $n = m^2$ where $m \in \mathbb{N}$. The devices are labeled $D(i, j)$ where $0 \leq i, j \leq m - 1$. There is an edge joining $D(i, j)$ to any of the vertices $D(i \pm 1, j)$ and $D(i, j \pm 1)$ that exist. Draw the mesh network for $n = 16$.

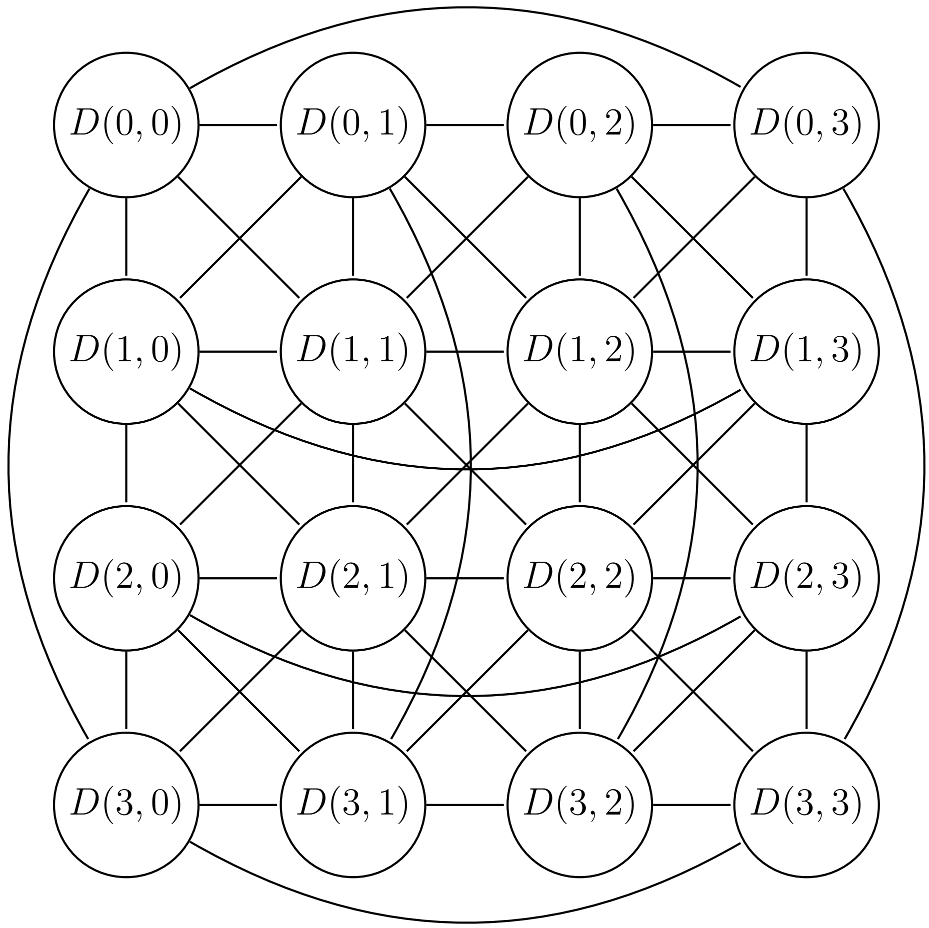
Solution. Here, $m = 4$. The mesh network is



□

(c) In a variant of the mesh network, there is an edge joining $D(i, j)$ to any of the vertices $D((i \pm 1) \mod m, j)$ and $D(i, (j \pm 1) \mod m)$. Draw this variant of the mesh network for $n = 16$.

Solution. Here, $m = 4$. This variant mesh network is



□

(d) Are there any advantages of a mesh network (either type) over the topologies discussed in part (a)? Explain.

Answer. In the variant mesh network, the maximum distance between any two nodes is 2 in the case $n = 16$, which is way better than the ring and star topologies for $n = 16$, and also better than the hybrid topology for $n = 16$, which is also worse than the regular mesh network.