

## Homework 6

ALECK ZHAO

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1. Recall that by strong duality there are exactly four possibilities for any linear program LP and its dual DP. Namely,

- i) LP and DP are both feasible and have equal optimal objective function values
- ii) LP is unbounded and DP is infeasible
- iii) LP is infeasible and DP is unbounded
- iv) LP and DP are both infeasible

Consider the linear program (LP)  $\min c^T x$  such that  $Ax = b$  and  $x \geq 0$  where  $A$  happens to be a matrix of all zeros. Determine and show which of the four above scenarios occurs in (each possible) different choices of  $b$  and  $c$ .

*Solution.* The dual program (DP) is given by  $\max y^T b$  such that  $A^T y \leq c, y$  unrestricted.

- i) In this case, since  $A$  is a matrix of all 0, we must have  $b = \vec{0}$  in order for (LP) to be feasible. Then  $A^T y = 0 \leq c$  so we must have  $c \geq 0$  in order for (DP) to be feasible. Then the optimal objective function value is exactly 0.
- ii) In this case, if (LP) is feasible, then we must have  $b = \vec{0}$  again. Then if any entry of  $c^T$  is negative, the objective function value is unbounded since any  $x \geq 0$  satisfies  $Ax = 0$ . This also corresponds to  $A^T y = 0 \leq c$  not being feasible if  $c$  is not non-negative.
- iii) In this case, if (DP) is feasible, then we must have  $A^T y = 0 \leq c$ . Then if any entry of  $b$  is non-zero, then (LP) is infeasible, which also corresponds to the (DP) being unbounded since  $y$  can be anything.
- iv) In this case, if  $b \neq 0$  and  $c \not\geq 0$ , both (LP) and (DP) are infeasible.

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2. Consider the following LP:

$$\begin{aligned}
 \max \quad & 4x_1 - 6x_2 \\
 \text{s.t.} \quad & x_1 + 3x_2 \geq 3 \\
 & 5x_1 - x_2 \leq 7 \\
 & -2x_1 + 3x_2 = 9 \\
 & x_1 \geq 0 \quad x_2 \text{ unrestricted}
 \end{aligned}$$

- a) Write LP in standard form, then write its standard form dual.

*Solution.* Let  $x_2 = a - b$  where  $a, b \geq 0$ . Maximizing the objective function is equivalent to minimizing its negative. Substitute this into the constraints and the objective function:

$$\begin{aligned}
 \min \quad & -4x_1 + 6a - 6b \\
 \text{s.t.} \quad & x_1 + 3a - 3b \geq 3 \\
 & 5x_1 - a + b \leq 7 \\
 & -2x_1 + 3a - 3b = 9 \\
 & x_1, a, b \geq 0
 \end{aligned}$$

Add slack variables  $c, d \geq 0$  the first and second constraints become

$$\begin{aligned}x_1 + 3a - 3b - c &= 3 \\ 5x_1 - a + b + d &= 7\end{aligned}$$

so finally the standard form is given by  $\min c^T x$  where  $Ax = b, x \geq 0$  where

$$A = \begin{bmatrix} 1 & 3 & -3 & -1 & 0 \\ 5 & -1 & 1 & 0 & 1 \\ -2 & 3 & -3 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix} \quad c = [-4 \quad 6 \quad -6 \quad 0 \quad 0]^T$$

The corresponding dual is given by

$$\begin{aligned}\max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \text{ unrestricted}\end{aligned}$$

or

$$\begin{aligned}\max \quad & 3y_1 + 7y_2 + 9y_3 \\ \text{s.t.} \quad & y_1 + 5y_2 - 2y_3 \leq -4 \\ & 3y_1 - y_2 + 3y_3 \leq 6 \\ & -3y_1 + y_2 - 3y_3 \leq -6 \\ & -y_1 \leq 0 \\ & y_2 \leq 0 \\ & y_1, y_2, y_3 \text{ unrestricted}\end{aligned}$$

□

b) Write LP in canonical form, then write its canonical form dual.

*Solution.* As before, we substitute  $x_2 = a - b$  and convert maximization to minimization:

$$\begin{aligned}\min \quad & -4x_1 + 6a - 6b \\ \text{s.t.} \quad & x_1 + 3a - 3b \geq 3 \\ & 5x_1 - a + b \leq 7 \\ & -2x_1 + 3a - 3b = 9 \\ & x_1, a, b \geq 0\end{aligned}$$

Now, flip the sign in the second constraint, and convert the third constraint into two inequalities:

$$\begin{aligned}-5x_1 + a - b &\geq -7 \\ -2x_1 + 3a - 3b &\geq 9 \\ 2x_1 - 3a + 3b &\geq -9\end{aligned}$$

so finally the canonical form is given by  $Ax \geq b, x \geq 0$  where

$$A = \begin{bmatrix} 1 & 3 & -3 \\ -5 & 1 & -1 \\ -2 & 3 & -3 \\ 2 & -3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -7 \\ 9 \\ -9 \end{bmatrix} \quad c = [-4 \quad 6 \quad -6]^T$$

The corresponding dual is given by

$$\begin{aligned} \max \quad & b^T z \\ \text{s.t.} \quad & A^T z \leq c \\ & z \geq 0 \end{aligned}$$

or

$$\begin{aligned} \max \quad & 3z_1 - 7z_2 + 9z_3 - 9z_4 \\ \text{s.t.} \quad & z_1 - 5z_2 - 2z_3 + 2z_4 \leq -4 \\ & 3z_1 + z_2 + 3z_3 - 3z_4 \leq 6 \\ & -3z_1 - z_2 - 3z_3 + 3z_4 \leq -6 \\ & z_1, z_2, z_3, z_4 \geq 0 \end{aligned}$$

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c) Show how the dual programs from part a, b, are equivalent.

*Proof.* In part a, the condition  $-y_1 \leq 0$  means that  $y_1 \geq 0$ , so let  $y_1 = z_1 \geq 0$ . Then we have  $y_2 \leq 0$  so let  $-y_2 = z_2 \geq 0$ . Then since  $y_3$  is unrestricted, let  $y_3 = z_3 - z_4$ , where  $z_3, z_4 \geq 0$ . Making these substitutions into the result from part a, we have

$$\begin{aligned} \max \quad & 3z_1 - 7z_2 + 9z_3 - 9z_4 \\ \text{s.t.} \quad & z_1 - 5z_2 - 2z_3 + 2z_4 \leq -4 \\ & 3z_3 + z_2 + 3z_3 - 3z_4 \leq 6 \\ & -3z_1 - z_2 - 3z_3 + 3z_4 \leq -6 \\ & z_1, z_2, z_3, z_4 \geq 0 \end{aligned}$$

which is exactly the problem we obtained in part b.

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3. Consider the linear program (LP)  $\min c^T x$  such that  $Ax = b, x \geq 0$  where

$$A = \begin{bmatrix} -6 & -5 & 25 & 3 & -85 & 4 & 30 \\ 24 & -2 & 28 & 6 & -55 & 1 & -9 \\ 9 & -5 & 11 & 2 & -55 & -1 & 10 \end{bmatrix} \quad b = \begin{bmatrix} 62 \\ 62 \\ 3 \end{bmatrix} \quad c = [23 \quad 1 \quad -17 \quad -1 \quad 52 \quad -6 \quad -12]^T$$

Write the dual program (DP) and then solve the dual problem.

*Solution.* The dual program (DP) is given by  $\max y^T b$  such that  $A^T y \leq c$ . That is,

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4. Concerning the specific LP discussed in the previous problem:

- Suppose you may change the value of  $b_1$  (currently 62) to anything you want. To what value should you set  $b_1$  in order to have the adjusted LP have optimal objective function value -70? Compute the optimal solution for the adjusted LP.
- Suppose you may change the value of  $b_2$  (currently 62) to anything you want. To what value should you set  $b_2$  in order to have the adjusted LP have optimal objective function value -68.5? Compute the optimal solution for the adjusted LP.

5. Consider the linear program (LP)  $\min c^T x$  such that  $Ax = b, x \geq 0$  where

$$A = \begin{bmatrix} 7 & 7 & 45 & -1 & 3 & -53 & -68 \\ 9 & -5 & 27 & -115 & 7 & -129 & 42 \\ 5 & -3 & 63 & -96 & 10 & -109 & 86 \end{bmatrix} \quad b = \begin{bmatrix} 26 \\ 18 \\ 34 \end{bmatrix} \quad c = [1 \quad 7 \quad -37 \quad 94 \quad -9 \quad 76 \quad -146]^T$$

Write the dual program (DP) and then solve the dual problem. Compute the optimal dual variables using one of the optimal bases for (LP), and then repeat this for the other optimal basis.