Homework 11

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Section 4.1: Polynomials

7. a. Let f and g be nonzero polynomials in $R[x]$ and assume that the leading coefficient of one of them is a unit. Show that $fg \neq 0$ and that $\deg(fg) = \deg f + \deg g$.

Proof. WLOG, the leading coefficient of f is $r \in R$ where r is a unit. We may write f and g as

$$f = rx^n + a_{n-1}x^{n-1} + \cdots + a_0$$
$$g = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

where $b_m \neq 0$. The coefficient of x^{m+n} in the product fg is given by rb_m. Suppose $rb_m = 0$, then multiplying by $1/r$ on both sides (which exists because r is a unit) we have $b_m = 0$, a contradiction. Thus, $rb_m \neq 0$, so $fg \neq 0$, and the term of maximal degree in fg is $rb_m x^{m+n}$, so

$$\deg(fg) = m + n = \deg f + \deg g$$

as desired. □

b. If R is not a domain, show that linear polynomials f and g exist in $R[x]$ such that $\deg(fg) < \deg f + \deg g$.

Proof. Since R is not a domain, there exist $a, b \in \mathbb{R}$ such that $ab = 0$ and $a, b \neq 0$. Consider the polynomials $f = ax$ and $g = bx$. Then $fg = (ax)(bx) = (ab)x = 0$. Here,

$$\deg(fg) = 0 < 1 + 1 = \deg f + \deg g$$

as desired. □

13. Divide $x^3 - 4x + 5$ by $2x + 1$ in $\mathbb{Q}[x]$. Why is it impossible in $\mathbb{Z}[x]$?

Solution. We have

$$x^3 - 4x + 5 = \left(\frac{1}{2}x^2 - \frac{1}{4}x - \frac{15}{8}\right) \cdot (2x+1) + \frac{55}{8}$$

The division is impossible in $\mathbb{Z}[x]$ because $2x + 1$ is not monic, and quotients don't make sense in $\mathbb{Z}$. □

26. Show that $\sqrt[n]{m}$ is not rational unless $m = k^n$ for some integer k.

Proof. The problem statement appears to be wrong. If $m = (a/b)^n \in \mathbb{Q}$ for $a, b \in \mathbb{Z}$, then we also have $\sqrt[n]{m} = a/b \in \mathbb{Q}$, so we do not require that $m = k^n$ for some integer k. We may assume that $m \in \mathbb{Z}$ to make this question somewhat interesting.

Suppose $\sqrt[n]{m} = k/b$ where $k, b \in \mathbb{Z}$ are relatively prime. Then $k^n = mb^n$, so $b^n \mid k^n \implies b \mid k$. Since $b \mid k$ it must be that $b = 1$ since $\gcd(k, b) = 1$. Thus, it must be that $\sqrt[n]{m} = k \implies m = k^n$, as desired. □

Section 4.2: Factorization of Polynomials over a Field

20. Factor $x^5 + x^2 - x + 1$ as a product of irreducible polynomials in $\mathbb{Z}_3[x]$.

Solution. Let $f(x) = x^5 + x^2 - x + 1$. We have

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 35 \equiv 2$$

so f has no roots in $\mathbb{Z}_3$, thus has no degree 1 divisors. Thus, f must factorize as the product of an irreducible quadratic and cubic. Let

$$f = (x^3 + ax^2 + bx + c)(x^2 + dx + e) = x^5 + (a+d)x^4 + (b+ad+e)x^3 + (ae+bd+c)x^2 + (be+cd)x + ce$$

Equating coefficients, we have the system

$$\begin{aligned} a + d &= 0 \\ b + ad + e &= 0 \\ ae + bd + c &= 1 \\ be + cd &= -1 \\ ce &= 1 \end{aligned}$$

From the last equation, we can have $c = e = 1$. Since $a = -d$, the system becomes

$$\begin{aligned} b - a^2 + 1 &= 0 \\ a - ab + 1 &= 1 \\ b - a &= -1 \end{aligned}$$

The second equation becomes $a(1 - b) = 0$, so the possibilities are $a = 0$ or $b = 1$ since $\mathbb{Z}_3$ is an integral domain. If $b = 1$, there is no solution for a, but if $a = 0$, then $b = 2$ is a solution. Thus, $d = -a = 0$, so we have the factorization

$$x^5 + x^2 - x + 1 = (x^3 + 2x + 1)(x^2 + 1)$$

Factorizations are always unique, but we may also check that $c = e = 2$ in the last equation does not have any solutions. □

24. Show that $f = x^4 + 4x^3 + 4x^2 + 4x + 5$ is irreducible over $\mathbb{Q}$ by considering $f(x - 1)$.

Proof. We have

$$\begin{aligned} f(x-1) &= (x-1)^4 + 4(x-1)^3 + 4(x-1)^2 + 4(x-1) + 5 \\ &= (x^4 - 4x^3 + 6x^2 - 4x + 1) + 4(x^3 - 3x^2 + 3x - 1) + 4(x^2 - 2x + 1) + 4(x-1) + 5 \\ &= x^4 - 2x^2 + 4x + 2 \end{aligned}$$

The possible rational roots of this polynomial are $\pm 1, \pm 2$, none of which evaluate to 0. Thus, if this polynomial were to be reducible over $\mathbb{Q}$, it must factor as two irreducible quadratics. Let

$$f(x-1) = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a+c)x^3 + (b+d+ac)x^2 + (ad+bc)x + bd = x^4 - 2x^2 + 4x + 2$$

Then equating coefficients, we have the system

$$\begin{aligned} a + c &= 0 \\ b + d + ac &= -2 \\ ad + bc &= 4 \\ bd &= 2 \end{aligned}$$

Since $a, b, c, d \in \mathbb{Z}$, suppose $b = 1, d = 2$ to satisfy equation 4. Then since $a = -c$ from equation 1, we have

$$1 + 2 - a^2 = -2 \implies a^2 = 5$$

which has no solutions for $a \in \mathbb{Z}$. Otherwise, suppose $b = -1, d = -2$, so

$$-1 - 2 - a^2 = -2 \implies a^2 = -1$$

which also has no solutions for a. The other situations are $b = 2, d = 1$ and $b = -2, d = -1$ which are identical to this case. Thus, since $f(x-1)$ has no proper factorization in $\mathbb{Z}$, we conclude that f is irreducible over $\mathbb{Q}$, as desired.

$\square$