

Homework 6 Solutions

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1. (E&P 3.6.62) Prove this as follows. Given x_1, x_2 , and x_3 , define the cubic polynomial $P(y)$ to be

$$P(y) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & y & y^2 & y^3 \end{bmatrix} \quad (26)$$

Because $P(x_1) = P(x_2) = P(x_3) = 0$ (why?), the roots of $P(y)$ are x_1, x_2 , and x_3 . It follows that

$$P(y) = k(y - x_1)(y - x_2)(y - x_3)$$

where k is the coefficient of y^3 in $P(y)$. Finally, observe that expansion of the 4×4 determinant in (26) along its last row gives $k = V(x_1, x_2, x_3)$ and that $V(x_1, x_2, x_3, x_4) = P(x_4)$.

Proof. First, note that

$$P(x_1) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_1 & x_1^2 & x_1^3 \end{bmatrix}$$

Here, this matrix has two identical rows, so its determinant is 0, and thus $P(x_1) = 0$. Similarly, $P(x_2) = P(x_3) = 0$, so x_1, x_2 , and x_3 are the roots of P , so it factors in the form given. Now, expanding the determinant along the 4th row, the coefficient of y^3 is

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = V(x_1, x_2, x_3)$$

as desired, and if we substitute $y = x_4$, then $P(x_4)$ takes on the form of $V(x_1, x_2, x_3, x_4)$, as desired. \square

2. (E&P 4.7.26) Use the method of Example 8 to find a basis for the 2-dimensional solution space of the given differential equation. $y'' + 10y' = 0$.

Solution. Let $v(x) = y'(x)$. Then this equation says $v' + 10v = 0 \implies v' = -10v$, which has solution $v(x) = Ce^{-10x}$. Thus,

$$y(x) = \int y'(x) dx = \int v(x) dx = \int Ce^{-10x} dx = -\frac{1}{10}Ce^{-10x} + A$$

so the general solution is given by $y(x) = A + Be^{-10x}$, so a basis of this solution space is $\{1, e^{-10x}\}$. \square

3. (E&P 4.7.32) Let \mathbf{A} and \mathbf{B} be 4×4 (real) matrices partitioned into 2×2 submatrices or "block":

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

Then verify that \mathbf{AB} can be calculated in "blockwise" fashion:

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

Solution. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Consider only the top left block of \mathbf{AB} , since every other quadrant is computed similarly. We have

$$\begin{aligned} \mathbf{A}_{11} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, & \mathbf{B}_{11} &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ \mathbf{A}_{12} &= \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}, & \mathbf{B}_{21} &= \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \\ \Rightarrow \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} + a_{14}b_{41} & a_{13}b_{32} + a_{14}b_{42} \\ a_{23}b_{31} + a_{24}b_{41} & a_{23}b_{32} + a_{24}b_{42} \end{bmatrix} \end{aligned}$$

which is the same as the upper left 2×2 block we obtain if we multiply \mathbf{AB} directly. \square

4. Determine if $y_1 = x^3 - 1$, $y_2 = x + 1$, $y_3 = x^3 + x^2$, and $y_4 = x^2 + x$ form a basis for \mathcal{P}_4 (the space of all polynomials of degree less than 4.)

Solution. We claim this is a basis for \mathcal{P}_4 . We must show these span \mathcal{P}_4 and are linearly independent.

To show they are spanning, take $ax^3 + bx^2 + cx + d \in \mathcal{P}_4$ to be an arbitrary element of \mathcal{P}_4 . Now, let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$\begin{aligned} ax^3 + bx^2 + cx + d &= k_1(x^3 - 1) + k_2(x + 1) + k_3(x^3 + x^2) + k_4(x^2 + x) \\ &= (k_1 + k_3)x^3 + (k_3 + k_4)x^2 + (k_2 + k_4)x + (k_2 - k_1) \end{aligned}$$

so by matching coefficients, we have the 4 equations

$$\begin{aligned} a &= k_1 + k_3 \\ b &= k_3 + k_4 \\ c &= k_2 + k_4 \\ d &= k_2 - k_1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Now, by performing Gaussian elimination on this matrix, we find that it is non-singular, and thus a solution exists for k_1, k_2, k_3, k_4 , so any arbitrary element of \mathcal{P}_4 can be represented as a linear combination of the 4 given polynomials.

Now, because \mathcal{P}_4 has dimension 4, any 4 spanning elements must be linearly independent. Thus, $\{y_1, y_2, y_3, y_4\}$ form a basis for \mathcal{P}_4 , as desired. \square

5. MATLAB Practice Lesson 10

6. MATLAB Practice Lesson 11

7. Let \mathbf{A} be an $m \times n$ matrix with $m > n$. Consider the linear system $\mathbf{Ax} = \mathbf{b}$. This system is usually inconsistent so instead, we attempt to find a vector $\hat{\mathbf{x}}$ such that $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2$ is as small as possible. The vector $\hat{\mathbf{x}}$ is called the least squares solution for the system.

- We may find $\hat{\mathbf{x}}$ by solving

$$\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b}$$

- The orthogonal projection of \mathbf{b} onto the column space of \mathbf{A} is given by

$$\hat{\mathbf{b}} = \mathbf{A} \hat{\mathbf{x}}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

- (a) Find the least squares solution for the system $\mathbf{Ax} = \mathbf{b}$.

Solution. We have

$$\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b} \implies \hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

so the least squares solution can be solved as

$$\begin{aligned} \mathbf{A}^T \mathbf{A} &= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \\ \implies (\mathbf{A}^T \mathbf{A})^{-1} &= \frac{1}{3 \cdot 6 - (-2) \cdot (-2)} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \\ \implies \hat{\mathbf{x}} &= \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 15 \\ 49 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \end{bmatrix} \end{aligned}$$

□

- (b) Find the orthogonal projection of $\hat{\mathbf{b}}$ onto the column space of \mathbf{A} .

Solution. We have

$$\hat{\mathbf{b}} = \mathbf{A} \hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 5/2 \\ 6 \end{bmatrix}$$

□

8. (E&P 5.1.23) Determine whether the pair of functions are linearly independent or linearly dependent on the real line. $f(x) = xe^x, y(x) = |x|e^x$.

Solution. These functions are linearly dependent if they are constant multiples on the real line. Suppose

$$\frac{f(x)}{g(x)} = k \implies \frac{xe^x}{|x|e^x} = \frac{x}{|x|} = k$$

Now, if $x \geq 0$, then $|x| = x$, but if $x < 0$ then $|x| = -x$, so it is clear that there is no value of k that satisfies this equation, and thus f and g are linearly independent. □