Homework 12

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Section 6.3

1. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}\,dx = \pi$

   Solution. This satisfies the required form where $2+\deg P \le \deg Q$. The poles are at the roots $z=-1\pm i$, so taking the loop around the pole in the upper half plane, we have the residue at $-1+i$ to be

   $$\left.\frac{1}{x-(-1-i)}\right|_{-1+i} = \frac{1}{2i}$$

   so by the residue theorem, the integral is $\frac{1}{2i}\cdot 2\pi i = \pi$. □

2. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2}\,dx = \frac{\pi}{6}$

   Solution. This satisfies the required form where $2+\deg P \le \deg Q$. The poles are at the roots $z=\pm 3i$, so taking the loop around the pole in the upper half plane, we have the residue at $3i$ to be

   $$\frac{d}{dz}\left.\left[\frac{z^2}{(z+3i)^2}\right]\right|_{3i} = \left.\frac{6iz}{(z+3i)^3}\right|_{3i} = \frac{6i(3i)}{(6i)^3} = \frac{1}{12i}$$

   so by the residue theorem, the integral is $\frac{1}{12i}\cdot 2\pi i = \frac{\pi}{6}$. □

6. $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$

   Solution. This satisfies the required form. We have $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$, so consider the second improper integral. The poles are at the roots $z=\pm i$ and $z=\pm 2i$, so taking the loop around the poles in the upper half plane, we have the residues at i and $2i$ to be

   $$\left.\frac{z^2}{(z+i)(z^2+4)}\right|_{i} = -\frac{1}{6i}$$
   $$\left.\frac{z^2}{(z^2+1)(z+2i)}\right|_{2i} = \frac{1}{3i}$$

   so by the residue theorem, the integral is $\frac{1}{2}\left(\frac{1}{3i}-\frac{1}{6i}\right)\cdot 2\pi i = \frac{\pi}{6}$. □

11. Show that

    $$\int_0^{\infty} \frac{dx}{x^3+1}\,dz = \frac{2\pi\sqrt{3}}{9}$$

    by integrating $1/(z^3+1)$ around the boundary of the circular sector $S_p = \left\{z=re^{i\theta} : 0\le\theta\le 2\pi/3, 0\le r\le p\right\}$ and letting $p\to\infty$.

Solution. Let γ_1 be the segment from 0 to p on the real axis, γ_2 be the arc, and γ_3 be the segment back to the origin. We have the parametrizations

$$\begin{aligned}
\gamma_1(t) &= t, \quad 0 \le t \le p \\
\gamma_2(t) &= pe^{it}, \quad 0 \le t \le \frac{2\pi}{3} \\
\gamma_3(t) &= e^{2\pi i/3}(p-t), \quad 0 \le t \le p
\end{aligned}$$

and thus the integral over S_p is given by

$$\begin{aligned}
\int_{\gamma_1} \frac{dz}{z^3+1} + \int_{\gamma_2} \frac{dz}{z^3+1} + \int_{\gamma_3} \frac{dz}{z^3+1} &= \int_0^p \frac{1}{t^3+1}\,dt + \int_0^{2\pi/3} \frac{pie^{it}}{p^3e^{3it}+1}\,dt + \int_0^p \frac{-e^{2\pi i/3}}{(p-t)^3+1}\,dt \\
&= \left(1 - e^{2\pi i/3}\right) \int_0^p \frac{1}{t^3+1}\,dt + \int_0^{2\pi/3} \frac{pie^{it}}{p^3e^{3it}+1}\,dt
\end{aligned}$$

Now, taking the limit as $p \to \infty$, we have

$$\left| \int_0^{2\pi/3} \frac{pie^{it}}{p^3e^{3it}+1}\,dt \right| \le \int_0^{2\pi/3} \frac{\left|pie^{it}\right|}{\left|p^3e^{3it}+1\right|}\,dt \le \int_0^{2\pi/3} \frac{p}{p^3-1} = \frac{2\pi}{3} \cdot \frac{p}{p^3-1} \to 0$$

Now, the integrand has a pole at $e^{\pi i/3}$ in S_p, where the residue is

$$\frac{1}{\frac{d}{dz}\left[z^3+1\right]\Big|_{e^{\pi i/3}}} = \frac{1}{3e^{2\pi i/3}}$$

so by the residue theorem the integral around S_p is $2\pi i \cdot \frac{1}{3e^{2\pi i/3}} = \frac{2\pi i}{3} e^{-2\pi i/3}$. Equating the two, we have

$$\begin{aligned}
\frac{2\pi i}{3} e^{-2\pi i/3} = \lim_{p\to\infty} \left(1 - e^{2\pi i/3}\right) \int_0^p \frac{1}{t^3+1}\,dt \implies \int_0^\infty \frac{1}{t^3+1}\,dt = \frac{\frac{2\pi i}{3} e^{-2\pi i/3}}{1 - e^{2\pi i/3}} \\
= \frac{\frac{2\pi i}{3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}{1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} = \frac{2\pi\sqrt{3}}{9}
\end{aligned}$$

□

13. Show that

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dz = \frac{\pi(2n)!}{2^{2n}(n!)^2}, \quad n = 0, 1, 2, \cdots$$

Solution. We have a pole at i in the upper half plane, and the residue is given by

$$\frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{1}{(z+i)^{n+1}}\right]\Bigg|_i = \frac{1}{n!}(-1)^n(n+1)(n+2)\cdots(2n) \cdot \frac{1}{(2i)^{2n+1}} = \frac{(2n)!}{(n!)^2}(-1)^n \cdot \frac{1}{2^{2n+1}(-1)^n \cdot i}$$

so by the residue theorem, the integral is

$$2\pi i \cdot \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2^{2n+1} i} = \frac{\pi(2n)!}{2^{2n}(n!)^2}$$

□

Section 6.4

1. $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\, dx = \frac{\pi}{e^2}$

   Solution. We have

$$\int \frac{\cos 2z}{z^2+1}\, dz = \frac{1}{2}\int \frac{e^{2iz}}{z^2+1}\, dz + \frac{1}{2}\int \frac{e^{-2iz}}{z^2+1}$$

   These have poles at $\pm i$. Integrating the first along the upper half plane and the second along the lower half plane, we find that the residues at i and $-i$ are

$$\left.\frac{e^{2iz}}{z+i}\right|_{i} = \frac{e^{-2}}{2i}$$
$$\left.\frac{e^{-2iz}}{z-i}\right|_{-i} = \frac{e^{-2}}{-2i}$$

   so since the bottom half plane integral is clockwise, by the residue theorem the integral is

$$2\pi i \cdot \frac{1}{2}\left(\frac{e^{-2}}{2i} - \frac{e^{-2}}{-2i}\right) = \pi e^2$$

   □

2. $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10}\, dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$

   Solution. We have

$$\int \frac{z\sin z}{z^2-2z+10}\, dz = \frac{1}{2i}\int \frac{ze^{iz}}{z^2-2z+10}\, dz - \frac{1}{2i}\int \frac{ze^{-iz}}{z^2-2z+10}\, dz$$

   These have poles at $1 \pm 3i$. Integrating the first along the upper half plane and the second along the lower half plane, we find that the residues at $1+3i$ and $1-3i$ are

$$\left.\frac{ze^{iz}}{z-(1-3i)}\right|_{1+3i} = \frac{(1+3i)e^{-3+i}}{6i}$$
$$\left.\frac{ze^{-iz}}{z-(1+3i)}\right|_{1-3i} = \frac{(1-3i)e^{-3-i}}{-6i}$$

   so since the bottom half plane integral is clockwise, by the residue theorem the integral is

$$2\pi i \cdot \frac{1}{2i}\left(\frac{(1+3i)e^{-3+i}}{6i} + \frac{(1-3i)e^{-3-i}}{-6i}\right) = \pi e^{-3}\left(\frac{e^i - e^{-i}}{6i} + \frac{e^i + e^{-i}}{2}\right) = \pi e^{-3}\left(\frac{1}{3}\sin 1 + \cos 1\right)$$

   □

6. $\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+4}\, dx$

   Solution. The poles are at $\pm 2i$. Integrating along the lower half plane, we find that the residue is

$$\left.\frac{e^{-2iz}}{z-2i}\right|_{-2i} = \frac{e^{-4}}{-4i}$$

   so since the bottom half plane integral is clockwise, by the residue theorem the integral is

$$-2\pi i \cdot \frac{e^{-4}}{-4i} = \frac{\pi e^{-4}}{2}$$

   □

7. $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\, dx$

Solution. We have

$$\int \frac{\cos z}{(z^2+1)(z^2+4)}\, dz = \frac{1}{2} \int \frac{e^{iz}}{(z^2+1)(z^2+4)}\, dz + \frac{1}{2} \int \frac{e^{-iz}}{(z^2+1)(z^2+4)}\, dz$$

These have poles at $\pm i$ and $\pm 2i$. Integrating the first along the upper half plane and the second along the lower half plane, we find that the residues at $i, -i, 2i, -2i$ are

$$\left.\frac{e^{iz}}{(z+i)(z^2+4)}\right|_{i} = \frac{e^{-1}}{6i}$$
$$\left.\frac{e^{-iz}}{(z-i)(z^2+4)}\right|_{-i} = \frac{e^{-1}}{-6i}$$
$$\left.\frac{e^{iz}}{(z^2+1)(z+2i)}\right|_{2i} = \frac{e^{-2}}{-12i}$$
$$\left.\frac{e^{-iz}}{(z^2+1)(z-2i)}\right|_{-2i} = \frac{e^{-2}}{12i}$$

so since the bottom half plane integral is clockwise, by the residue theorem the integral is

$$2\pi i \cdot \frac{1}{2}\left[\left(\frac{e^{-1}}{6i} + \frac{e^{-2}}{-12i}\right) - \left(\frac{e^{-1}}{-6i} + \frac{e^{-2}}{12i}\right)\right] = \pi\left(\frac{e^{-1}}{3} - \frac{e^{-2}}{6}\right)$$

$\square$