Homework 7

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Chapter 6: Continuous-Time Markov Chains

2. Suppose that a one-celled organism can be in one of two states - either A or B. An individual in state A will change to state B at an exponential rate α; an individual in state B divides into two new individuals of type A at an exponential rate β. Define an appropriate continuous-tine Markov chain for a population of such organisms and determine the appropriate parameters for this model.

   Solution. Let $N_A(t)$ and $N_B(t)$ represent the number of individuals in states A and B, respectively. Then $\{(N_A(t), N_B(t))\}$ is a Markov process. If there are a in state A and b in state B, then the total transition rate is $v_{(a,b)} = a\alpha + b\beta$. The individual transition probabilities are

$$P_{(a,b),(a-1,b+1)} = \frac{a\alpha}{a\alpha + b\beta}$$
$$P_{(a,b),(a+2,b-1)} = \frac{b\beta}{a\alpha + b\beta}$$

□

5. There are N individuals in a population, some of whom have a certain infection that spreads as follows. Contacts between two members of this population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and non-infected individual, then with probability p the non-infected individual becomes infected. Once infected, an individual remains infected throughout. Let $X(t)$ denote the number of infected members of the population at time t.

   (a) Is $\{X(t), t \geq 0\}$ a continuous-time Markov chain?

   Answer. If the states are $\{0, 1, \cdots, N\}$ representing the number of infected people, then since contacts are a Poisson process, it follows that the transition rates are exponentially distributed, so the process is memory-less. Thus, it is a Markov process.

   (b) Specify its type.

   Answer. This is a pure birth process, since individuals cannot become non-infected.

   (c) Starting with a single infected individual, what is the expected time until all members are infected?

   Solution. If there are i infected individuals, there are $N - i$ non-infected individuals, so $i(N - i)$ contacts between them. There are a total of $\binom{N}{2}$ possible contacts, so the probability of a contact between infected and non-infected is $i(N-i)/\binom{N}{2}$, and the probability of infection is $pi(N-i)/\binom{N}{2}$, so the birth rates are

$$\lambda_i = \frac{\lambda pi(N-i)}{\binom{N}{2}}, \quad i = 1, 2, \cdots, N-1$$

   If T_i represents the time to transition from i infected individuals to $i + 1$, then we seek

$$E\left[\sum_{i=1}^{N-1} T_i\right] = \sum_{i=1}^{N-1} E[T_i] = \sum_{i=1}^{N-1} \frac{1}{\lambda_i} = \sum_{i=1}^{N-1} \frac{\binom{N}{2}}{\lambda pi(N-i)} = \frac{N(N-1)}{2\lambda p} \sum_{i=1}^{N-1} \frac{1}{i(N-i)}$$

□

6. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda, i \geq 0$, and death rates $\mu_i = i\mu, i \geq 0$.

(a) Determine the expected time to go from state 0 to state 4.

Solution. We use the recursive formula

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}] = \frac{1}{(i+1)\lambda} \left(1 + i\mu E[T_{i-1}]\right)$$

Starting with $E[T_0] = 1/\lambda_0 = 1/\lambda$, we have

$$\begin{aligned} E[T_1] &= \frac{1}{2\lambda}\left(1 + \mu \cdot \frac{1}{\lambda}\right) = \frac{\lambda+\mu}{2\lambda^2} \\ E[T_2] &= \frac{1}{3\lambda}\left(1 + 2\mu \cdot \frac{\lambda+\mu}{2\lambda^2}\right) = \frac{\lambda^2+\lambda\mu+\mu^2}{3\lambda^3} \\ E[T_3] &= \frac{1}{4\lambda}\left(1 + 3\mu \cdot \frac{\lambda^2+\mu(\lambda+\mu)}{3\lambda^3}\right) = \frac{\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3}{4\lambda^4} \end{aligned}$$

The expected time to go from state 0 to state 4 is thus

$$\begin{aligned} E[T_0+T_1+T_2+T_3] &= E[T_0]+E[T_1]+E[T_2]+E[T_3] \\ &= \frac{1}{\lambda} + \frac{\lambda+\mu}{2\lambda^2} + \frac{\lambda^2+\lambda\mu+\mu^2}{3\lambda^3} + \frac{\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3}{4\lambda^4} \end{aligned}$$

□

(b) Determine the expected time to go from state 2 to state 5.

Solution. Using the same recursive formula, we have

$$E[T_4] = \frac{1}{5\lambda}\left(1 + 4\mu \cdot \frac{\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3}{4\lambda^4}\right) = \frac{\lambda^4+\lambda^3\mu+\lambda^2\mu^2+\lambda\mu^3+\mu^4}{5\lambda^5}$$

so the expected time to go from state 2 to state 5 is

$$\begin{aligned} E[T_2+T_3+T_4] &= E[T_2]+E[T_3]+E[T_4] \\ &= \frac{\lambda^2+\lambda\mu+\mu^2}{3\lambda^3} + \frac{\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3}{4\lambda^4} + \frac{\lambda^4+\lambda^3\mu+\lambda^2\mu^2+\lambda\mu^3+\mu^4}{5\lambda^5} \end{aligned}$$

□

(c) Determine the variances in parts (a) and (b).

Solution. We use the recursive formula

$$\begin{aligned} \mathrm{Var}(T_i) &= \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \frac{\mu_i}{\lambda_i}\mathrm{Var}(T_{i-1}) + \frac{\mu_i}{\lambda_i+\mu_i}\left(E[T_{i-1}]+E[T_i]\right)^2 \\ &= \frac{1}{(i+1)\lambda\left[(i+1)\lambda+i\mu\right]} + \frac{i\mu}{(i+1)\lambda}\mathrm{Var}(T_{i-1}) + \frac{i\mu}{(i+1)\lambda+i\mu}\left(E[T_{i-1}]+E[T_i]\right)^2 \end{aligned}$$

Starting with $\mathrm{Var}(T_0) = 1/\lambda_0^2 = 1/\lambda^2$, we have

$$\mathrm{Var}(T_1) = \frac{1}{2\lambda(2\lambda+\mu)} + \frac{\mu}{2\lambda}\cdot\frac{1}{\lambda^2} + \frac{\mu}{2\lambda+\mu}\left(\frac{1}{\lambda}+\frac{\lambda+\mu}{2\lambda^2}\right)^2$$

and etc... The algebra is pretty ugly and I think unnecessary, but the variances of parts (a) and (b), respectively, are

$$\begin{aligned} \mathrm{Var}(T_0+T_1+T_2+T_3) &= \mathrm{Var}(T_0)+\mathrm{Var}(T_1)+\mathrm{Var}(T_2)+\mathrm{Var}(T_3) \\ \mathrm{Var}(T_2+T_3+T_4) &= \mathrm{Var}(T_2)+\mathrm{Var}(T_3)+\mathrm{Var}(T_4) \end{aligned}$$

□

9. The birth and death process with parameters $\lambda_n = 0$ and $\mu_n = \mu, n > 0$ is called a pure death process. Find $P_{ij}(t)$.

Solution. The death rate is constant, so since the deaths arrive according to a Poisson process with rate μ, in any interval of length t, the probability we go from state i to state $j > 0$ ($i - j$ death arrivals) is

$$P_{ij}(t) = e^{-\mu t} \frac{(\mu t)^{i-j}}{(i-j)!}, \quad 0 < j \leq i$$

Then for $P_{i0}(t)$, this is the complement of the probability that there is still some positive number of individuals remaining in the population (i.e. there are fewer than i arrivals in a period of time t). Thus

$$P_{i0}(t) = 1 - \sum_{k=0}^{i-1} e^{-\mu t} \frac{(\mu t)^k}{k!} = \sum_{k=i}^{\infty} e^{-\mu t} \frac{(\mu t)^k}{k!}$$

□