Homework 10

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1. Let E/F be a finite field extension, and let L/F be any field extension. Mimic the proof of § 10.2 Theorem 1 to show that
$$\#\{F\text{-embeddings } E \to L\} \leq [E : F].$$

Section 10.1: Galois Groups and Separability

30. Le $E \supseteq F$ be a finite extension, where $\operatorname{char} F = p$.
    (a) If $u \in E$ has a separable minimal polynomial q over F, show that $u \in F(u^p)$. [Hint: If m is the minimal polynomial of u over $F(u^p)$, show $m \mid q$ and $m \mid (x-u)^p$.]
    (b) Define $F(E^p) = \{ a_1u_1^p + \cdots + a_nu_n^p \mid a_i \in F, u_i \in E, n \geq 1 \}$. Show that $F(E^p)$ is a subfield of E.
    (c) If $E = F(E^p)$ and $\{w_1, \cdots, w_k\} \subseteq E$ is F-independent, show that $\{w_1^p, \cdots, w_k^p\}$ is F-independent. [Hint: Extend to a basis $\{w_1, \cdots, w_k, \cdots, w_n\}$ of E, show that $\{w_1^p, \cdots, w_k^p, \cdots, w_n^p\}$ span E, and apply Theorem 7 §6.1.]
    (d) Show that $E \supseteq F$ is separable if and only if $F(E^p) = E$. [Hint: If $E = F(E^p)$, use Theorem 4 §6.2 and (c)]

31. Let $E \supseteq K \supseteq F$ be fields with $[E : F]$ finite. Show that $E \supseteq F$ is separable if and only if both $E \supseteq K$ and $K \supseteq F$ are separable.

32. If $E \supseteq F$ is a finite extension, then $u \in E$ is called a separable element over F if its minimal polynomial in $F[x]$ is separable.
    (a) If $u \in E$ is separable over F and $E \supseteq K \supseteq F$, where K is a field, show that u is separable over K. [Hint: Exercise 30(d)]
    (b) Show that $u \in E$ is separable over F if and only if $F(u) \supseteq F$ is a separable extension.
    (c) Define $S = \{ u \in E \mid u \text{ is separable over F} \}$. Show that S is a subfield of E, that $S \supseteq F$ is separable, and that $E \supseteq K \supseteq F$, with $K \supseteq F$ separable, implies that $S \supseteq K$. [Hint: If $u, v \in S$, show that $F(u, v) \supseteq F$ is separable by (a) and Exercise 31.]

Section 10.2: The Main Theorem of Galois Theory

5. Let $E = F(t)$ be the field of rational forms over a field. In each case, compute $K = E_G$ and find the minimal polynomial $m \in K[x]$ of t over K.
    (a) $G = \langle \sigma \rangle$, where σ is that F-automorphism of E given by $\sigma(t) = -t$.
    (b) $G = \langle \sigma \rangle$, where σ is that F-automorphism of E gien by $\sigma(t) = 1 - t$.

10. Let $E \supseteq F$ be fields with $G = \operatorname{Gal}(E/F)$.

11. If $E \supseteq K \supseteq F$ are fields, show that $E \supseteq K$ is Galois if and only if K is closed as an intermediate field of $E \supseteq F$.