Homework 5

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Section 3.2

7. Show that the formula $e^{iz} = \cos z + i \sin z$ holds for all complex numbers z.

Proof. We have

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \cdot \frac{e^{iz} - e^{-iz}}{2i} = e^{iz}$$

as desired. □

17. Find all numbers z (if any) such that

(a) $e^{4z} = 1$

Solution. We have $e^{4z} = 1 = e^0$ holds whenever $4z = 0 + 2k\pi i \implies z = k\pi i/2, k \in \mathbb{Z}$. □

(b) $e^{iz} = 3$

Solution. We have $e^{iz} = 3 = e^{\operatorname{Log} 3}$ holds whenever $iz = \operatorname{Log} 3 + 2k\pi i \implies z = -i \operatorname{Log} 3 + 2k\pi$ □

(c) $\cos z = i \sin z$

Solution. We have

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ i \sin z &= i \cdot \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} - e^{-iz}}{2} \\ \cos z = i \sin z &\implies \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2} \\ &\implies e^{-iz} = -e^{-iz} \implies e^{-iz} = 0 \end{aligned}$$

but this has no solution. □

20. Show that the function $w = e^z$ maps the shaded rectangle in Fig 3.2(a) one-to-one onto the semi-annulus in Fig 3.2(b).

Proof. The rectangle in fig 3.2(a) is the set $A = \{x + iy : -1 \le x \le 1, 0 \le y \le \pi\}$, and the semi-annulus in fig 3.2(b) is the set $B = \{z : e^{-1} \le |z| \le e, \operatorname{Im} z \ge 0\}$. Suppose $f(x_1 + iy_1) = f(x_2 + iy_2)$, so

$$\begin{aligned} e^{x_1 + iy_1} = e^{x_2 + iy_2} &\implies x_1 + iy_1 = x_2 + iy_2 + 2k\pi i \\ &\implies x_1 = x_2 \quad \text{and} \quad y_1 = y_2 + 2k\pi \end{aligned}$$

Since $0 \le y_1, y_2 \le \pi$, it follows that $k = 0$ and so $y_1 = y_2$, so $x_1 + iy_1 = x_2 + iy_2$, and thus f is injective.

Take some $z \in B$, so $z = re^{i\theta}$ where

$$e^{-1} \leq r \leq e \implies -1 \leq \operatorname{Log} r \leq 1$$

and θ lies in quadrants 1 and 2. Then let θ_0 be the argument of z in $[-\pi, \pi]$. Then we have

$$f(\operatorname{Log} r + i\theta_0) = re^{i\theta_0} = re^{i\theta}$$

where $\operatorname{Log} r + i\theta_0 \in A$, so f is surjective. □

21. (a) Show that the mapping $w = \sin z$ is one-to-one in the semi-infinite strip

$$S_1 = \{x + iy : -\pi < x < \pi, y > 0\}$$

and find the image of this strip. Hint: See prob 16.

Proof. Suppose $\sin z_1 = \sin z_2$ with $z_1, z_2 \in S_1$. By the result of exercise 16, we have

$$0 = \sin z_2 - \sin z_1 = 2\cos\left(\frac{z_2 + z_1}{2}\right)\sin\left(\frac{z_2 - z_1}{2}\right)$$

Thus, either $\cos\left(\frac{z_2+z_1}{2}\right) = 0$ or $\sin\left(\frac{z_2-z_1}{2}\right) = 0$. We know that $\cos z = 0$ if and only if $z = k\pi + \pi/2$ and $\sin z = 0$ if and only if $z = k\pi$ for $k \in \mathbb{Z}$. Thus we have either

$$\frac{z_2 + z_1}{2} = k\pi + \frac{\pi}{2} \implies z_2 + z_1 = \pi + 2k\pi$$
$$\frac{z_2 - z_1}{2} = k\pi \implies z_2 - z_1 = 2k\pi$$

The RHS of both sides is real, so the first option is not possible because $y_1, y_2 > 0$. Thus in the second case, we have $y_2 = y_1$, and $x_2 - x_1 = 2k\pi$. But since $-\pi < x_1, x_1 < \pi$, the only way this equality can hold is if $x_2 - x_1 = 0$, and thus $z_1 = x_1 + iy_1 = x_2 + iy_2 = z_2$, so this mapping is injective on this domain. □

(b) For $w = \sin z$, what is the image of the smaller semi-infinite strip

$$S_2 = \{x + iy : -\pi/2 < x < \pi/2, y > 0\}?$$

Solution. Let $z = x + iy \in S_2$, so

$$\begin{aligned}\sin z &= \sin(x + iy) = \sin x \cos(iy) + \sin(iy)\cos x \\ &= \sin x \cosh y + i \sinh y \cos x\end{aligned}$$

Here, $-1 \leq \sin x \leq 1$ and $\cosh y > 0$, so $\sin x \cosh y$ can be anything. Then $\sinh y > 0j$ and $0 \leq \cos x \leq 1$, so the image is the entire upper half plane, excluding the real axis. □

Section 3.5

3. Find the principal value of each of the following.

(a) $4^{1/2}$

Solution. This is $4^{1/2} = e^{\frac{1}{2}\operatorname{Log} 4} = e^{\operatorname{Log} 2} = 2$. □

(b) i^{2i}

Solution. This is

$$i^{2i} = e^{2i \operatorname{Log} i} = e^{2i(\operatorname{Log}|i| + i \operatorname{Arg}(i))} = e^{2i \cdot i\pi/2} = e^{-\pi}$$

□

(c) $(1+i)^{1+i}$

Solution. This is

$$\begin{aligned}
(1+i)^{1+i} &= e^{(1+i)\operatorname{Log}(1+i)} = e^{(1+i)[\operatorname{Log}|1+i| + i\operatorname{Arg}(1+i)]} \\
&= e^{(1+i)\left(\operatorname{Log}\sqrt{2} + i\pi/4\right)} = e^{\operatorname{Log}\sqrt{2} + i\pi/4 + i\operatorname{Log}\sqrt{2} - \pi/4} \\
&= e^{\operatorname{Log}\sqrt{2} - \pi/4} e^{i\left(\operatorname{Log}\sqrt{2} + \pi/4\right)} \\
&= \sqrt{2} e^{i\pi/4} e^{-\pi/4 + i\operatorname{Log}\sqrt{2}} = (1+i)\exp\left(-\frac{\pi}{4} + \frac{i}{2}\operatorname{Log} 2\right)
\end{aligned}$$

□

8. Show that all solutions of the equation $\sin z = 2$ are given by $\pi/2 + 2k\pi \pm i\operatorname{Log}(2+\sqrt{3})$, where $k = 0, \pm 1, \pm 2, \cdots$.

Proof. We have

$$\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{2i} = 2 \\
\implies e^{iz} - e^{-iz} &= 4i \implies e^{2iz} - 1 = 4ie^{iz} \\
\implies e^{2iz} - 4ie^{iz} - 1 = 0
\end{aligned}$$

so by the quadratic formula, we have

$$\begin{aligned}
e^{iz} &= \frac{4i \pm \sqrt{(-4i)^2 - 4(-1)}}{2} = \frac{4i \pm \sqrt{-12}}{2} = i\left(2 \pm \sqrt{3}\right) \\
&= e^{i\pi/2 + \operatorname{Log}\left(2 \pm \sqrt{3}\right)} \\
\implies iz &= \frac{i\pi}{2} + \operatorname{Log}\left(2 \pm \sqrt{3}\right) + 2k\pi i, \quad k \in \mathbb{Z}
\end{aligned}$$

Now, we have

$$\begin{aligned}
\frac{1}{2-\sqrt{3}} &= \frac{2+\sqrt{3}}{2^2 - 3} = 2 + \sqrt{3} \\
\implies \operatorname{Log}\left(2+\sqrt{3}\right) &= -\operatorname{Log}\left(2-\sqrt{3}\right)
\end{aligned}$$

so the solution is given by

$$z = \frac{\pi}{2} \pm i\operatorname{Log}\left(2+\sqrt{3}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

as desired. □

11. Find all solutions of the equation $\sin z = \cos z$.

Solution. We have

$$
\begin{aligned}
\sin z = \frac{e^{iz} - e^{-iz}}{2i} &= \frac{e^{iz} + e^{-iz}}{2} = \cos z \\
\implies e^{iz} - e^{-iz} &= ie^{iz} + ie^{-iz} \\
\implies e^{2iz} - 1 &= ie^{2iz} + i \\
\implies e^{2iz} &= \frac{1+i}{1-i} = \frac{(1+i)^2}{1^2+1^2} = \frac{1}{2}(1+i)^2 \\
\implies e^{2iz} &= \frac{1}{2}\left(\sqrt{2}e^{i\pi/4}\right)^2 = e^{i\pi/2} \\
\implies 2iz &= \frac{i\pi}{2} + 2k\pi i, \quad k \in \mathbb{Z} \\
\implies z &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}
\end{aligned}
$$

□