Homework 5

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1. Suppose that $R = \prod_{i=1}^m R_i$ is a product of rings. If M_i is an R_i module for each i, then $\bigoplus_{i=1}^m M_i$ is naturally an R-module, via the rule

$$(r_1, \cdots, r_m) \cdot (x_1, \cdots, x_m) = (r_1 x_1, \cdots, r_m x_m)$$

For $i = 1, \cdots, m$, let $e_i \in R$ be the tuple whose ith entry is 1_{R_i}, and whose other entries are all 0. Let M be an R-module and define the submodule $M_i := e_i M$. Show that M_i is naturally an R_i-module, and that $M = \bigoplus_{i=1}^m M_i$.

Proof. WLOG, let $i = 1$. The argument is the same for any i. Then we have

$$e_1 M = \{ (1_{R_1}, 0, \cdots, 0) \cdot x \mid x \in M \}$$

so if $r_1 \in R_1$, we can write

$$r_1 [(1_{R_1}, 0, \cdots, 0) \cdot x] = (r_1, 0, \cdots, 0) \cdot x$$

so $e_1 M$ is naturally an R_1-module. By extension, $e_i M$ is an R_i-module for all i.

For any $x \in M$, since $(1_{R_1}, \cdots, 1_{R_m})$ is the identity element in R and since M is an R-module, we have

$$(1_{R_1}, \cdots, 1_{R_m}) \cdot x = (e_1 + \cdots + e_m) \cdot x = e_1 x + \cdots + e_m x = x$$

where $e_i x \in M_i$ for all i, so $M = M_1 + \cdots + M_m$. Suppose $y \in M_i \cap M_j$. Then

$$\begin{aligned} y &= e_i x_i = e_j x_j \\ \implies e_i(e_i x_i) &= (e_i e_i) x_i = e_i x_i \\ &= e_i(e_j x_j) = (e_i e_j) x_j = 0 \end{aligned}$$

so $y = e_i x_i = 0$, and thus the intersection between M_i and M_j is trivial, so this is a direct sum. □

2. Let R be a PID, let $d \in R$ be a nonzero nonunit, and let $d \sim p_1^{k_1} \cdots p_m^{k_m}$ be a prime factorization of d, where $p_1, \cdots, p_k$ are pairwise non-associated prime elements and $k_i > 0$ for all i. Show that the canonical homomorphism

$$\begin{aligned} R &\to \prod_{i=1}^m R/\left\langle p_i^{k_i} \right\rangle \\ r &\mapsto \left(r + \left\langle p_1^{k_1} \right\rangle, \cdots, r + \left\langle p_m^{k_m} \right\rangle \right) \end{aligned}$$

induces an isomorphism $R/\langle d \rangle \cong \prod_{i=1}^m R/\left\langle p_i^{k_i} \right\rangle$.

Proof. By the Chinese Remainder Theorem, we have

$$R/\left\langle p_1^{k_1} \right\rangle \times R/\left\langle p_2^{k_2} \right\rangle \cong R/\left\langle p_1^{k_1} p_2^{k_2} \right\rangle$$

since $\gcd\left(p_1^{k_1}, p_2^{k_2}\right) \sim 1$. Then since $\gcd\left(p_1^{k_1} p_2^{k_2}, p_3^{k_3}\right) \sim 1$, we can continue by induction to get

$$R/\left\langle p_1^{k_1} \right\rangle \times \cdots \times R/\left\langle p_m^{k_m} \right\rangle \cong R/\left\langle p_1^{k_1} \cdots p_m^{k_m} \right\rangle \cong R/\langle d \rangle$$

as desired. □

3. Keep the notation of Problem 2. Let M be an R-module such that $dM = 0$. By the paragraph preceding Theorem 7, Section 7.1, $M/dM \cong M$ is naturally an $R/\langle d\rangle$-module. Hence by Problem 2, M is naturally an $R/\left\langle p_1^{k_1}\right\rangle \times \cdots \times R/\left\langle p_m^{k_m}\right\rangle$-module. Let $M = \bigoplus_{i=1}^m M_i$ be the corresponding direct sum decomposition obtained from Problem 1.

   (a) Show that $M_i = M(p_i)$ as submodules of M for all i.

   Proof. Since M has the direct sum decomposition, we can write

$$M \ni x = x_1 + \cdots + x_m$$

   where $x_i \in M_i$. Then WLOG take $i = 1$, and the argument is the same for all i. Then we have

$$M(p_1) = \{\, x_1 + \cdots + x_m \mid p_1^n(x_1 + \cdots + x_m) = 0 \text{ for some } n \,\}$$

   Define $P_i := \left\langle p_i^{k_i}\right\rangle$ for all i. Then $M = \bigoplus_{i=1}^m M_i$ is an $R/P_1 \times \cdots \times R/P_m$-module by the action

$$(r_1 + P_1, \cdots, r_m + P_m) \cdot (x_1 + \cdots + x_m) = r_1x_1 + \cdots + r_mx_m$$

   so the condition in M_i is

$$\begin{aligned}
p_1^n(x_1 + \cdots + x_m) &= p_1^n x_1 + \cdots + p_1^n x_m \\
&= (p_1^n + P_1, p_1^n + P_2, \cdots, p_1^n + P_m) \cdot (x_1 + \cdots + x_m) \\
&= (0 + P_1, p_1^n + P_2, \cdots, p_1^n + P_m) \cdot (x_1 + \cdots + x_m) \\
&= p_1^n x_2 + \cdots + p_1^n x_m \\
&= p_1^n(x_2 + \cdots + x_m) \\
\implies p_1^n x_1 &= 0 \\
\implies p_1^n(x_2 + \cdots + x_m) &= 0 \\
\implies x_2 = \cdots = x_m &= 0
\end{aligned}$$

   Thus, we have

$$\begin{aligned}
M(p_1) &= \{\, x_1 + \cdots + x_m \mid p_1^n x_1 = 0, x_2 = \cdots = x_m = 0 \,\} \\
&= \{\, x_1 \in M_1 \mid p_1^n x_1 = 0 \,\}
\end{aligned}$$

   However, for all $x_1 \in M_1$, we have

$$p_1^n x_1 = (p_1^n + P_1) \cdot x_1 = (0 + P_1) \cdot x_1 = 0$$

   so it follows that

$$M(p_1) = M_1$$

   and by extension, $M(p_i) = M_i$ for all i. $\square$

   (b) Show that if $\langle d\rangle = \operatorname{ann}(M)$, then $M(p_i) \neq 0$ for all i.

   Proof. We have $d \sim p_1^{k_1} \cdots p_m^{k_m} \in \operatorname{ann}(M)$ so

$$(p_1^{k_1} \cdots p_m^{k_m}) \cdot x = p_1^{k_1}\left[(p_2^{k_2} \cdots p_m^{k_m}) \cdot x\right] = 0$$

   for all $x \in M$. However, since

$$(p_2^{k_2} \cdots p_m^{k_m}) \notin \left\langle p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}\right\rangle = \langle d\rangle = \operatorname{ann}(M)$$

   it follows that

$$(p_2^{k_2} \cdots p_m^{k_m}) \cdot x_1 \neq 0$$

   for some $x_1 \in M$. Then since $(p_2^{k_2} \cdots p_m^{k_m}) \cdot x_1$ is annihilated by $p_1^{k_1}$ and is nonzero, we must have $M(p_1) \neq 0$, and by a similar argument, $M(p_i) \neq 0$ for all i. $\square$

Section 7.2: Modules Over a PID

2. If p is a prime, determine all abelian groups of order:

   (a) p^4

   Solution. The abelian groups (up to isomorphism) are

$$\begin{array}{rl} \mathbb{Z}_{p^4} & \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} & \end{array}$$

   □

   (b) p^6

   Solution. The abelian groups (up to isomorphism) are

$$\begin{array}{rl} \mathbb{Z}_{p^6} & \mathbb{Z}_{p^5} \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \mathbb{Z}_{p^3} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_{p^2} & \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} \\ \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^3} & \end{array}$$

   □

13. If $K \subseteq M$ are modules, show that M is torsion if and only if both K and M/K are torsion.

   Proof. $(\implies)$: If M is torsion, then since $K \subseteq M$, it follows that K must also be torsion. Then for $m + K \in M/K$, where $m \in M$, suppose m is annihilated by nonzero $x \in R$, so that $xm = 0$. Then we have

$$x(m + K) = xm + K = 0 + K$$

   so $m + K$ is torsion since it is annihilated by a nonzero x.

   $(\impliedby)$: Let $m + K \in M/K$ be torsion and annihilated by nonzero $x \in R$, so

$$x(m + K) = xm + K = K \implies xm \in K$$

   Since K is torsion, it follows that xm is also torsion, which means that m is torsion. Thus, since $m \in M$ was arbitrary, M is also torsion. □

15. If $M = M_1 \oplus \cdots \oplus M_n$ are modules, show that $T(M) = T(M_1) \oplus \cdots \oplus T(M_n)$.

   Proof. All elements in M are of the form $(m_1, \cdots, m_n)$ where $m_i \in M_i$. Then if such an element is torsion suppose it is annihilated by nonzero $x \in R$, we have

$$x(m_1, \cdots, m_n) = (xm_1, \cdots, xm_n) = (0, \cdots, 0)$$

   so $xm_i = 0$ for all i. Thus, m_i is torsion for all i, so

$$T(M) \subset T(M_1) \oplus \cdots \oplus T(M_n)$$

   Let $m_i \in M_i$ be torsion and annihilated by nonzero $x_i \in R$ for all i. Then if $d = x_1 \cdots x_n$, it follows that

$$d(m_1, \cdots, m_n) = (0, \cdots, 0)$$

so $(m_1, \cdots, m_n) \in M$ is also torsion, and thus

$$T(M_1) \oplus \cdots \oplus T(M_n) \subset T(M)$$

so the two are equal. $\square$

24. Show that every submodule of a finitely generated module over a PID is again finitely generated.

Proof. Let M be finitely generated over a PID R by $x_1, \cdots, x_m$. Then there exists a surjective map $\varphi : R^m \twoheadrightarrow M$ where $\varphi(e_i) = x_i$ for $i = 1, \cdots, m$. Then if N is a submodule of M, since φ is surjective, we can find the inverse image of N in R^m, say it is $K \subset R^m$, which is a submodule of R^m. By the Submodule Theorem, since R^m is a free module, K has rank at most m, so it is finitely generated. Then we can write a surjective map $\theta : K \twoheadrightarrow N$ and since K is finitely generated, it follows that N must be as well. $\square$