Homework 8

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Chapter 13: Binomial Trees

1. A stock price is currently \$40. It is known that at the end of 1 month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-month European call option with a strike price of \$39?

   Solution. After 1 month, if the stock price is 42, the value of the option is 3, and if the price is 38, the value of the option is 0. If we have a portfolio of long Δ shares and short 1 option, after 1 month, we have

   $$42\Delta - 3 = 38\Delta \implies \Delta = 0.75$$

   so the riskless portfolio has 0.75 shares of the stock and short 1 option. Then the value of this portfolio after 1 month is $42(0.75) - 3 = 28.5$. If f is the option price today, the value of the portfolio is $42(0.75) - f = 31.5 - f$, so we have

   $$(31.5 - f)e^{0.08 \cdot \frac{1}{12}} = 28.5 \implies f = \boxed{2.689}$$

   □

5. A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100?

   Solution. After 6 months, the price of the stock is either 110 or 90. If it is 90, then the option price at that time is 0 since the stock price can only go to either 99 or 81. Otherwise, if it goes to 110, then afterwards it can go to either 121 or 99. Then $f_{uu} = 21$ and $f_{ud} = 0$. Using $u = 1.1, d = 0.9, r = 0.08$, and $T = 0.5$, we have

   $$f_u = e^{-rT}[pf_{uu} + (1-p)f_{ud}] = e^{-0.08 \cdot 0.5} \cdot \frac{e^{0.08 \cdot 0.5} - 0.9}{1.1 - 0.9} \cdot 21 = 14.205$$

   Now, from the present till 6 months, we have $u = 1.1, d = 0.9, f_u = 14.205, f_d = 0$, so we have

   $$f = e^{-rT}[pf_u + (1-p)f_d] = e^{-0.08 \cdot 0.5} \cdot \frac{e^{0.08 \cdot 0.5} - 0.9}{1.1 - 0.9} \cdot 14.205 = \boxed{9.609}$$

   □

6. For the situation considered in Problem 13.5, what is the value of a 1-year European put option with a strike price of \$100? Verify that the European call and the European put prices satisfy put-call parity.

Solution. After 6 months, the price of the stock is either 110 or 90. If it is 90, then it can go to either 99 or 81. In this case, $f_{du} = 1$ and $f_{dd} = 19$. Using $u = 1.1, d = 0.9, r = 0.08$, and $T = 0.5$, we have

$$\begin{aligned} f_d = e^{-rT}[pf_{du} + (1-p)f_{dd}] &= e^{-0.08\cdot 0.5}\left[\frac{e^{0.08\cdot 0.5} - 0.9}{1.1 - 0.9}\cdot 1 + \left(1 - \frac{e^{0.08\cdot 0.5} - 0.9}{1.1 - 0.9}\right)\cdot 19\right] \\ &= 6.079 \end{aligned}$$

If, after 6 months, it is 110, then it can go to either 121 or 99. In this case, $f_{uu} = 0$ and $f_{ud} = 1$. We have

$$f_u = e^{-rT}[pf_{uu} + (1-p)f_{ud}] = e^{-0.08\cdot 0.5}\cdot\left(1 - \frac{e^{0.08\cdot 0.5} - 0.9}{1.1 - 0.9}\right)\cdot 1 = 0.284$$

Now, from the present till 6 months, we have $u = 1.1, d = 0.9, f_u = 0.284, f_d = 6.079$, so we have

$$\begin{aligned} f = e^{-rT}[pf_u + (1-p)f_d] &= e^{-0.08\cdot 0.5}\left[\frac{e^{0.08\cdot 0.5} - 0.9}{1.1 - 0.9}\cdot 0.284 + \left(1 - \frac{e^{0.08\cdot 0.5} - 0.9}{1.1 - 0.9}\right)\cdot 6.079\right] \\ &= \boxed{1.921} \end{aligned}$$

We have

$$\begin{aligned} c + Ke^{-rT} &= 9.609 + 100e^{-0.08\cdot 1} = 101.921 \\ &= 1.921 + 100 = p + S_0 \end{aligned}$$

so put-call parity is satisfied. □

11. A stock price is currently \$40. It is known that at the end of 3 months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a 3-month European put option on the stock with an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

Solution. If the stock goes up to 45, then $f_u = 0$ and if it goes down to 35, $f_d = 5$. We have $u = 45/40 = 1.125, d = 35/40 = 0.875, r = 0.08$, and $T = 0.25$, so

$$f = e^{-rT}[pf_u + (1-p)f_d] = e^{-0.08\cdot 0.25}\left(1 - \frac{e^{0.08\cdot 0.25} - 0.875}{1.125 - 0.875}\right)\cdot 5 = \boxed{2.054}$$

For the risk-neutral valuation, if p is the probability of an upward movement, then we have

$$45p + 35(1-p) = 40e^{0.08\cdot 0.25} \implies p = 0.581$$

so the option has probability p of being worth 0 and probability $1-p$ of being worth 5, so its expected value is $(1-p)\cdot 5 = 2.096$. Discounting this at the risk-free rate, the value of the option today is $2.096e^{-0.08\cdot 0.25} = 2.054$, so the risk-neutral and no-arbitrage valuations are the same. □

25. Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months.

    (a) Calculate u, d, and p for a two-step tree.

    Solution. We have

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{0.25}} = 1.162 \\ d &= e^{-\sigma\sqrt{\Delta t}} = e^{-0.3\sqrt{0.25}} = 0.861 \\ p &= \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.04\cdot 0.25} - 0.861}{1.162 - 0.861} = 0.495 \end{aligned}$$

□

(b) Value the option using a two-step tree.

Solution. After 3 months, the stock can go to either $40 \cdot 1.162 = 46.48$ or $40 \cdot 0.861 = 34.44$. We have $f_{ud} = f_{du} = 0$ since the price would go back to 40, and $f_{dd} = 0$, so $f_d = 0$. After 6 months, we have $f_{uu} = 40 \cdot 1.162^2 - 40 = 14.01$. Thus,

$$\begin{aligned} f_u &= e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}] = e^{-0.04 \cdot 0.25} \cdot 0.495 \cdot 14.01 = 6.88 \\ \implies f &= e^{-r\Delta t}[pf_u + (1-p)f_d] = e^{-0.04 \cdot 0.25} \cdot 0.495 \cdot 6.88 = \boxed{3.378} \end{aligned}$$

□

(c) Verify that DerivaGem gives the same answer.

Answer. DerivaGem gives the same answer.

(d) Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

Solution. The values for 5, 50, 100, and 500 steps are

$$\begin{aligned} f_5 &= 3.923 \\ f_{50} &= 3.739 \\ f_{100} &= 3.748 \\ f_{500} &= 3.754 \end{aligned}$$

□