Homework 2

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(1) In a simple symmetric random walk, let T denote the time of the first return to the origin. Use the tail probability representation of the expectation to show that $E[T] = +\infty$.

Proof. From last time, we have

$$E[T] = \sum_{n=0}^{\infty} P[T > n]$$

Note that $P[T > 2k] = P[T > 2k+1]$ for all k, and $P[T > 0] = P[T > 1] = 1$, so

$$\begin{aligned} E[T] &= 2\sum_{k=0}^{\infty} P[T > 2k] = 2(1+1) + 2\sum_{k=1}^{\infty} P[S_1 \neq 0, S_2 \neq 0, \cdots, S_{2k} \neq 0] \\ &= 4 + 2\sum_{k=1}^{\infty} u_{2k} = 4 + 2\sum_{k=1}^{\infty} \binom{2k}{k} 2^{-2k} = 4 + 2\sum_{k=1}^{\infty} \frac{(2k)!}{k!k!} 2^{-2k} \end{aligned}$$

By Stirling's Formula, this is asymptotic to

$$4 + 2\sum_{k=1}^{\infty} \frac{\sqrt{2\pi(2k)}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{k}} \cdot 2^{-2k} = 4 + 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi k}} \to \infty$$

as desired. □

(2) Let X denote a random variable which has the arc sine distribution.

(a) Calculate $P\left[\frac{1}{4} < X < \frac{3}{4}\right]$.

Solution. The CDF for a variable with the arc sine distribution is

$$F(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$$

so the desired probability is

$$F(3/4) - F(1/4) = \frac{2}{\pi}\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{2}{\pi}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{3}$$

□

(b) Calculate $E[X]$.

Solution. The distribution for X is given by

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 < x < 1$$

so the expectation is

$$\begin{aligned}\int_0^1 x \cdot \frac{1}{\pi\sqrt{x(1-x)}}\,dx = \frac{1}{2\pi}\int_0^1 \frac{2x}{\sqrt{x-x^2}}\,dx &= \frac{1}{2\pi}\int_0^1 \left(\frac{2x-1}{\sqrt{x-x^2}} + \frac{1}{\sqrt{x(1-x)}}\right) dx \\ &= \frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2}\int_0^1 \frac{1}{\pi\sqrt{x(1-x)}}\,dx \\ &= \frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2}\end{aligned}$$

Using the substitution

$$u = x - x^2 \implies du = 1 - 2x\,dx \implies -du = 2x - 1\,dx$$

the expectation becomes

$$\frac{1}{2\pi}\int_0^1 \frac{2x-1}{\sqrt{x-x^2}}\,dx + \frac{1}{2} = -\frac{1}{2\pi}\int_0^0 \frac{1}{\sqrt{u}}\,du + \frac{1}{2} = \boxed{\frac{1}{2}}$$

□

(c) Calculate $\mathrm{Var}(X)$.

Solution. We have the relation $\mathrm{Var}(X) = E[X^2] - (E[X])^2$. For $E[X^2]$, we have

$$\begin{aligned}E[X^2] &= \int_0^1 x^2 \cdot \frac{1}{\pi\sqrt{x(1-x)}}\,dx = \frac{1}{\pi}\int_0^1 \frac{x^2-x}{\sqrt{x-x^2}}\,dx + \int_0^1 \frac{x}{\pi\sqrt{x(1-x)}}\,dx \\ &= -\frac{1}{\pi}\int_0^1 \sqrt{x(1-x)}\,dx + \frac{1}{2}\end{aligned}$$

Completing the square, we have

$$\sqrt{x-x^2} = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}$$

so using the substitution

$$x - \frac{1}{2} = \frac{1}{2}\cos\theta \implies dx = -\frac{1}{2}\sin\theta\,d\theta$$

the integral becomes

$$\begin{aligned}-\frac{1}{\pi}\int_0^1 \sqrt{x(1-x)}\,dx = -\frac{1}{\pi}\int_\pi^0 \frac{1}{2}\sin\theta\left(-\frac{1}{2}\sin\theta\,d\theta\right) &= \frac{1}{4\pi}\int_\pi^0 \sin^2\theta\,d\theta \\ &= \frac{1}{4\pi}\int_\pi^0 \left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta = \frac{1}{4\pi}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Bigg|_\pi^0 \\ &= -\frac{1}{8}\end{aligned}$$

Thus, we have

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \left(\frac{1}{2} - \frac{1}{8}\right) - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{8}}$$

□

(3) Consider a simple symmetric random walk of length 12. Let L_{12} denote the amount of time that the random walk is positive.

(a) Use the formula given in class to calculate the values of the frequency function of L_{12} to three decimal places.

Solution. We have

$$P[L_{2n} = 2k] = u_{2k}u_{2n-2k} = \binom{2k}{k}2^{-2k}\binom{2n-2k}{n-k}2^{-2n+2k} = \binom{2k}{k}\binom{2n-2k}{n-k}2^{-2n}$$
$$P[L_{12} = 2k] = \binom{2k}{k}\binom{12-2k}{6-k}2^{-12}$$

Using $k = 0, 1, \cdots, 6$, we have

$$P[L_{12} = 0] = \binom{0}{0}\binom{12}{6}2^{-12} \approx 0.226$$
$$P[L_{12} = 2] = \binom{2}{1}\binom{10}{5}2^{-12} \approx 0.123$$
$$P[L_{12} = 4] = \binom{4}{2}\binom{8}{4}2^{-12} \approx 0.103$$
$$P[L_{12} = 6] = \binom{6}{3}\binom{6}{3}2^{-12} \approx 0.098$$
$$P[L_{12} = 8] = \binom{8}{4}\binom{4}{2}2^{-12} \approx 0.103$$
$$P[L_{12} = 10] = \binom{10}{5}\binom{2}{1}2^{-12} \approx 0.123$$
$$P[L_{12} = 12] = \binom{12}{6}\binom{0}{0}2^{-12} \approx 0.226$$

□

(b) To see how good the asymptotic approximation is, find the difference

$$\left| P\left[\frac{1}{4} < \frac{L_{12}}{12} < \frac{3}{4}\right] - P\left[\frac{1}{4} < X < \frac{3}{4}\right]\right|$$

where the latter value was calculated in problem 2a.

Solution. We have

$$P\left[\frac{1}{4} < \frac{L_{12}}{12} < \frac{3}{4}\right] = P\left[3 < L_{12} < 9\right] = P[L_{12} = 4] + P[L_{12} = 6] + P[L_{12} = 8]$$
$$\approx 0.103 + 0.098 + 0.103 = 0.304$$

From part 2a, we have $P[\frac{1}{4} < X < \frac{3}{4}] = \frac{1}{3}$, so the difference is

$$\left|0.304 - \frac{1}{3}\right| \approx \boxed{0.0293}$$

□

(4) Find the conditional probability that a simple symmetric random walk of length $2n$ is always positive, given that it ends at 0.

(a) Write an expression in terms of $S_1, S_2, S_3, \cdots, S_{2n}$ for the desired conditional probability, as a ratio of two unconditional probabilities, using the definition of conditional probability.

Solution. This probability is

$$P[S_1 > 0, S_2 > 0, \cdots, S_{2n-1} > 0 \mid S_{2n} = 0] = \frac{P[S_1 > 0, S_2 > 0, \cdots, S_{2n-1} > 0, S_{2n} = 0]}{P[S_{2n} = 0]}$$

□

(b) Write an exact formula for the denominator of the fraction in (a).

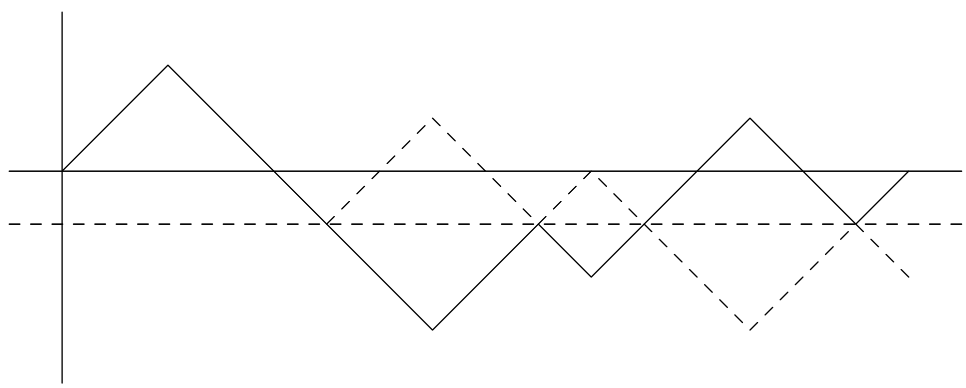
Solution. We have

$$P[S_{2n} = 0] = u_{2n} = \binom{2n}{n} 2^{-2n}$$

□

(c) To derive an expression for the numerator consider the (relative) complementary event that the random walk goes below the x-axis at some time but ends at 0.

Solution. Consider a path that goes below the x-axis but ends at 0:



If we reflect the path about $y = -1$ at the first time the path becomes negative, we get a path that ends at -2, which is guaranteed because the original path ended at 0. This is a 1-1 correspondence because anytime a path ends at -2, it must have passed through -1 at some point, so reflect the path after the first time that happened to get a path ending at 0. □

(d) Calculate an expression for the probability that a simple symmetric random walk of length $2n$ ends at height -2.

Solution. If the path of length $2n$ ends at -2, it must have had $n - 1$ positive and $n + 1$ negative results. Thus, the probability is

$$P[S_{2n} = -2] = \frac{(2n)!}{(n-1)!(n+1)!} 2^{-2n}$$

□

(e) Use parts (b), (c), and (d) to calculate the desired numerator.

Solution. The complement of the probability in the numerator is the sum of the probabilities that the path goes below the x-axis for all possible crossing times, while still ending at 0. We already showed that these paths have a 1-1 correspondence with paths that end at -2, so the numerator is

$$1 - \frac{(2n)!}{(n-1)!(n+1)!} 2^{-2n}$$

□

(f) Calculate the answer to the original question.

Solution. The answer to the original question is

$$
\begin{aligned}
P[S_1 > 0, S_2 > 0, \cdots, S_{2n-1} > 0 \mid S_{2n} = 0] &= \frac{P[S_1 > 0, S_2 > 0, \cdots, S_{2n-1} > 0, S_{2n} = 0}{P[S_{2n} = 0]} \\
&= \frac{1 - \frac{(2n)!}{(n-1)!(n+1)!} 2^{-2n}}{\frac{(2n)!}{n!n!} 2^{-2n}} \\
&= \frac{2^{2n}}{\binom{2n}{n}} - \frac{n}{n+1}
\end{aligned}
$$

□