Homework 8

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Section 4.4

5. Write down a function $z(s,t)$ deforming Γ_0 to Γ_1 in the domain D, where Γ_0 is the ellipse $x^2/4+y^2/9=1$ traversed once counterclockwise starting from $(2,0)$, and Γ_1 is the circle $|z|=1$ traversed once counterclockwise starting from $(1,0)$, and D is the annulus $1/2<|z|<4$.

   Solution. We start with the parametrization $x(t)=2\cos 2\pi t, y(t)=3\sin 2\pi t, 0\le t\le 1$ for Γ_0. We wish to deform this to the parametrization $x'(t)=\cos 2\pi i t, y(t)=\sin 2\pi i, 0\le t\le 1$ for Γ_1, which can be accomplished with the function

   $$z(s,t)=(2-s)\cos 2\pi t+i(3-2s)\sin 2\pi t, \quad 0\le 1\le 1, 0\le s\le 1$$

   □

9. Which of the following domains are simply connected?

   (a) the horizontal strip $|\operatorname{Im} z|<1$

   (b) the annulus $1<|z|<2$

   (c) the set of all points in the plane except those on the non-positive x-axis

   (d) the interior of the ellipse $4x^2+y^2=1$

   (e) the exterior of the ellipse $4x^2+y^2=1$

   (f) the domain D in Fig 4.46.

   Answer. The domains in (a), (c), (d), and (f) are simply connected.

18. Let

   $$I := \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}$$

   Below is an outline of a proof that $I=0$. Justify each step.

   (a) For every $R>2, I=I(R)$, where

   $$I(R) := \oint_{|z|=R} \frac{1}{z^2(z-1)^3}\,dz$$

   Answer. The poles of the integrand are 0 and 1, so if $R>2$, there exists a continuous deformation from the circle $|z|=R$ to the circle $|z|=2$, so the two integrals are equal.

   (b) $|I(R)|\le \frac{2\pi}{R(R-1)^3}$ for $R>2$.

Answer. We have

$$|I(R)| = \left|\oint_{|z|=R} \frac{1}{z^2(z-1)^3}\,dz\right| \leq \oint_{|z|=R} \left|\frac{1}{z^2(z-1)^3}\right|\,dz = \oint_{|z|=R} \frac{1}{|z|^2\,|z-1|^3}\,dz$$

By the triangle inequality, we have

$$|z| \leq |z-1| + |1| \implies |z| - 1 \leq |z-1| \implies \frac{1}{|z-1|} \leq \frac{1}{|z|-1}$$

so the integral along the contour $|z| = R$ is

$$\oint_{|z|=R} \frac{1}{|z|^2\,|z-1|^3}\,dz \leq \oint_{|z|=R} \frac{1}{|z|^2\,(|z|-1)^3}\,dz = \frac{2\pi R}{R^2(R-1)^3} = \frac{2\pi}{R(R-1)^3}$$

(c) $\lim_{R\to+\infty} I(R) = 0$

Answer. Since

$$\lim_{R\to+\infty} |I(R)| = \lim_{R\to+\infty} \frac{2\pi}{R(R-1)^3} = 0$$

it must be that $I(R)$ tends to the origin as $R \to \infty$.

(d) $I = 0$.

Answer. Since $I(R)$ is arbitrarily small as $R \to \infty$ and $I = I(R)$ for $R > 2$, it follows that $I = I(R) = 0$.

19. Using the method of proof in Prob 18, establish the following theorem. If P is a polynomial of degree at least 2 and P has all its zeros inside the circle $|z| = r$, then

$$\oint_{|z|=r} \frac{1}{P(z)}\,dz = 0$$

Proof. Let I be the value of this integral. If P is a polynomial of degree at least 2 with roots $r_1, \cdots, r_n$ all inside the circle $|z| = r$, then P factorizes as

$$\begin{aligned} P(z) &= c(z-r_1)\cdots(z-r_n) \\ \implies \frac{1}{P(z)} &= \frac{1}{c(z-r_1)\cdots(z-r_n)} \end{aligned}$$

Now, for $R > r$, all poles will lie inside the contour $|z| = R$, so there exists a continuous deformation from the circle $|z| = r$ to $|z| = R$, so thus

$$I(R) := \oint_{|z|=R} \frac{1}{P(z)}\,dz = \oint_{|z|=r} \frac{1}{P(z)}\,dz$$

Now, we have

$$\begin{aligned} |I(R)| &= \left|\oint_{|z|=R} \frac{1}{c(z-r_1)\cdots(z-r_n)}\,dz\right| \leq \oint_{|z|=R} \left|\frac{1}{c(z-r_1)\cdots(z-r_n)}\right|\,dz \\ &= \frac{1}{|c|}\oint_{|z|=R} \frac{1}{|z-r_1|\cdots|z-r_n|}\,dz \leq \frac{1}{|c|}\oint_{|z|=R} \frac{1}{(|z|-r_1)\cdots(|z|-r_n)}\,dz \\ &= \frac{1}{|c|}\frac{1}{(R-r_1)\cdots(R-r_n)} \\ \implies \lim_{R\to\infty} |I(R)| &= 0 \\ \implies I = I(R) &= 0 \end{aligned}$$

$\square$

20. Let Γ denote the four-leaf clover path traversed once as shown in Fig 4.50. Show that

$$\int_\Gamma \frac{1}{z^4-1}\,dz = 0$$

in two ways; first, by using partial fractions, and second, by using the result of Prob 19.

Proof. We have the partial fraction decomposition

$$\frac{1}{z^4-1} = \frac{1}{(z-1)(z+1)(z-i)(z+i)} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-i} + \frac{D}{z+i}$$
$$\implies 1 = A(z+1)(z-i)(z+i) + B(z-1)(z-i)(z+i) + C(z-1)(z+1)(z+i) + D(z-1)(z+1)(z-i)$$

and substituting $z = 1, -1, i, -i$, we have

$$1 = A(1+1)(1-i)(1+i) = 4A \implies A = \frac{1}{4}$$
$$1 = B(-1-1)(-1-i)(-1+i) = -4B \implies B = -\frac{1}{4}$$
$$1 = C(i-1)(i+1)(i+i) = 4iC \implies C = \frac{1}{4i} = -\frac{i}{4}$$
$$1 = D(-i-1)(-i+1)(-i-i) = -4iD \implies D = \frac{1}{-4i} = \frac{i}{4}$$

Since Γ forms counterclockwise loops around each of these poles, we have

$$\begin{aligned}\int_\Gamma \frac{1}{z^4-1}\,dz &= \frac{1}{4}\int_\Gamma \frac{1}{z-1}\,dz - \frac{1}{4}\int_\Gamma \frac{1}{z+1}\,dz - \frac{i}{4}\int_\Gamma \frac{1}{z-i} + \frac{i}{4}\int_\Gamma \frac{1}{z+i}\,dz \\ &= \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) - \frac{i}{4}(2\pi i) + \frac{i}{4}(2\pi i) = 0\end{aligned}$$

By the result of Prob 19, since the four-leaf clover can be continuously deformed to the circle $|z| = 2$ that contains all 4 roots of $P(z) = z^4 - 1$, it follows that

$$\oint_{|z|=2} \frac{1}{z^4-1}\,dz = 0$$

$\square$

Section 4.5

3. Let C be the circle $|z| = 2$ traversed once in the positive sense. Compute each of the following integrals.

   (b) $\int_C \frac{ze^z}{2z-3}\,dz$

   Solution. This is

$$\int_C \frac{ze^z/2}{z-3/2}\,dz$$

   where $3/2$ is contained in C and $f(z) = \frac{z}{2}e^z$ is analytic on and inside of C. Then

$$f\left(\frac{3}{2}\right) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-3/2}\,dz$$
$$\implies \int_C \frac{ze^z/2}{z-3/2}\,dz = 2\pi i f\left(\frac{3}{2}\right) = 2\pi i \cdot \left(\frac{3/2}{2}e^{3/2}\right) = \frac{3\pi i}{2}e^{3/2}$$

$\square$

(c) $\int_C \frac{\cos z}{z^3+9z}\, dz$

Solution. This is

$$\int_C \frac{\cos z}{z(z^2+9)}\, dz = \int_C \frac{\frac{\cos z}{z^2+9}}{z}\, dz$$

where 0 is contained in C and $f(z) = \frac{\cos z}{z^2+9}$ is analytic on and inside of C. Then

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\, dz$$
$$\implies \int_C \frac{\frac{\cos z}{z^2+9}}{z}\, dz = 2\pi i f(0) = 2\pi i \cdot \frac{\cos 0}{0^2+9} = \frac{2\pi i}{9}$$

□

(e) $\int_C \frac{e^{-z}}{(z+1)^2}\, dz$

Solution. Here, $f(z) = e^{-z} \implies f'(z) = -e^{-z}$ is analytic on and inside of C, and -1 is contained inside of C, so

$$f^{(1)}(-1) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z+1)^2}\, dz$$
$$\implies \int_C \frac{e^{-z}}{(z+1)^2}\, dz = 2\pi i f'(-1) = 2\pi i(-e^{-(-1)}) = -2\pi i e$$

□

5. Let C be the ellipse $x^2/4 + y^2/9 = 1$ traversed once in the positive direction, and define

$$G(z) := \int_C \frac{\zeta^2 - \zeta + 2}{\zeta - z}\, d\zeta \quad (z \text{ inside } C)$$

Find $G(1), G'(i)$, and $G''(-i)$.

Solution. $f(\zeta) = \zeta^2 - \zeta + 2$ is analytic on and inside of C. We have the relation

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_C \frac{\zeta^2 - \zeta + 2}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}G(z)$$
$$\implies G(z) = 2\pi i f(z) \implies G(1) = 2\pi i \cdot (1^2 - 1 + 2) = 4\pi i$$
$$\implies G'(z) = 2\pi i f'(z) \implies G'(i) = 2\pi i(2i - 1) = -4\pi - 2\pi i$$
$$\implies G''(z) = 2\pi i f''(z) \implies G''(-i) = 2\pi i(2) = 4\pi i$$

□

6. Evaluate

$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\, dz$$

where Γ is the circle $|z| = 3$ traversed once counterclockwise.

Solution. Here, we can continuously deform Γ to enclose the two poles separately in the limit, so

$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\, dz = \int_{\Gamma_1} \frac{e^{iz}/(z+i)^2}{(z-i)^2}\, dz + \int_{\Gamma_2} \frac{e^{iz}/(z-i)^2}{(z+i)^2}\, dz$$

where Γ_1 and Γ_2 are circles of radius 1 enclosing i and $-i$, respectively. Then Γ_1 doesn't contain $-i$ and Γ_2 doesn't contain i, so $f(z) = e^{iz}/(z+i)^2$ and $g(z) = e^{iz}/(z-i)^2$ are analytic on and inside Γ_1 and Γ_2, respectively. We have

$$\begin{aligned} f'(z) &= \frac{(z+i)^2 ie^{iz} - 2(z+i)e^{iz}}{(z+i)^4} = \frac{e^{iz}(i(z+i)-2)}{(z+i)^3} \\ g'(z) &= \frac{(z-i)^2 ie^{iz} - 2(z-i)e^{iz}}{(z-i)^4} = \frac{e^{iz}(i(z-i)-2)}{(z-i)^3} \end{aligned}$$

so the integrals evaluate as

$$\begin{aligned} \int_{\Gamma_1} \frac{e^{iz}/(z+i)^2}{(z-i)^2}\,dz + \int_{\Gamma_2} \frac{e^{iz}/(z-i)^2}{(z+i)^2}\,dz &= 2\pi i f'(i) + 2\pi i g'(-i) \\ &= 2\pi i \left(\frac{e^{i^2}(i(i+i)-2)}{(i+i)^3} + \frac{e^{-i^2}(i(-i-i)-2)}{(-i-i)^3} \right) \\ &= 2\pi i \left(\frac{e^{-1}}{2i} \right) = \frac{\pi}{e} \end{aligned}$$

□

7, Compute

$$\int_\Gamma \frac{\cos z}{z^2(z-3)}\,dz$$

along the contour indicated in Fig 4.55.

Solution. This contour contains 0 and not 3, so we can write this integral as

$$\int_\Gamma \frac{\frac{\cos z}{z-3}}{z^2}\,dz$$

where $f(z) = \frac{\cos z}{z-3}$, which is analytic inside and on Γ. Then since Γ is counterclockwise,

$$\begin{aligned} f^{(1)}(0) &= \frac{1!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-0)^{1+1}}\,dz \\ \implies \int_\Gamma \frac{\frac{\cos z}{z-3}}{z^2}\,dz &= 2\pi i f'(0) \\ &= 2\pi i \left(\frac{(z-3)(-\sin z) - \cos z}{(z-3)^2} \right)\Bigg|_0 = -\frac{2\pi i}{9} \end{aligned}$$

□

9. Suppose that f is analytic inside and on the unit circle $|z| = 1$. Prove that if $|f(z)| \le M$ for $|z| = 1$, then $|f(0)| \le M$ and $|f'(0)| \le M$. What estimate can you give for $\left|f^{(n)}(0)\right|$?

Proof. If f is analytic inside and on the unit circle, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z - z_0}\,dz$$

for any z_0 inside the unit circle. In particular, if $z_0 = 0$, then

$$\begin{aligned}
f(0) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z}\, dz \\
\implies |f(0)| &= \left| \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z}\, dz \right| \leq \frac{1}{2\pi} \oint_{|z|=1} \left| \frac{f(z)}{z} \right| dz \\
&= \frac{1}{2\pi} \oint_{|z|=1} \frac{|f(z)|}{|z|}\, dz \leq \frac{1}{2\pi} \oint_{|z|=1} \frac{M}{|z|}\, dz = \frac{1}{2\pi}(M \cdot 2\pi \cdot 1) = M
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
f'(0) &= \frac{1!}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^2}\, dz \\
\implies |f'(0)| &\leq \frac{1}{2\pi} \oint_{|z|=1} \frac{|f(z)|}{|z|^2}\, dz \leq \frac{1}{2\pi}(M \cdot 2\pi \cdot 1) = M
\end{aligned}$$

and in general, we have

$$\begin{aligned}
f^{(n)}(0) &= \frac{n!}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}}\, dz \\
\implies \left| f^{(n)}(0) \right| &\leq \frac{n!}{2\pi} \oint_{|z|=1} \frac{|f(z)|}{|z|^{n+1}}\, dz \leq \frac{n!}{2\pi}(M \cdot 2\pi \cdot 1) = M \cdot n!
\end{aligned}$$

$\square$