Homework 7

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Chapter 17: Measurable Functions

1. Prove Corollary 17.2. Let $f : D \to \mathbb{R}$ where D is measurable. Then f is measurable if and only if $f^{-1}(U)$ is measurable for every open set $U \subset \mathbb{R}$.

   Proof. ($\Longrightarrow$) : If f is measurable, then $\{\alpha < f < \beta\}$ is measurable for every $\alpha < \beta$. Then since any open set $U \subset \mathbb{R}$ can be written as a countable union of open intervals, we have

$$f^{-1}(U) = \{x \in D : f(x) \in U\} = \left\{ x \in D : f(x) \in \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$
$$= \bigcup_{n=1}^{\infty} \{x \in D : f(x) \in (a_n, b_n)\} = \bigcup_{n=1}^{\infty} \{a_n < f < b_n\}$$

   and since $\{a_n < f < b_n\}$ are all measurable, this union is measurable by the σ-algebra property of $\mathcal{M}$.
   ($\Longleftarrow$) : If $f^{-1}(U)$ is measurable for every open set U, then since (α, ∞) is open for every α, it follows that

$$\{f > \alpha\} = \{x \in D : f(x) \in (\alpha, \infty)\} = f^{-1}((\alpha, \infty))$$

   which is measurable, and thus f is measurable. □

3. Let $f : D \to \mathbb{R}$ where D is measurable. Show that f is measurable if and only if the function $g : \mathbb{R} \to \mathbb{R}$ is measurable, where $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$.

   Proof. We have

$$\{g > \alpha\} = \{x \in \mathbb{R} : g(x) > \alpha\} = \{x \in D : g(x) > \alpha\} \cup \{x \in \mathbb{R} \setminus D : g(x) > \alpha\}$$

   Now, since f and g agree on D, we have $\{x \in D : g(x) > \alpha\} = \{x \in D : f(x) > \alpha\}$. Then since $g(x) = 0$ for all $x \in \mathbb{R} \setminus D$, we have

$$\{x \in \mathbb{R} \setminus D : g(x) > \alpha\} = \begin{cases} \varnothing & \alpha \geq 0 \\ \mathbb{R} \setminus D & \alpha < 0 \end{cases}$$

   which are both measurable since D is measurable. Now, if f is measurable, then $\{g > \alpha\} = \{f > \alpha\} \cup \{x \in \mathbb{R} \setminus D : g(x) > \alpha\}$ is measurable since $\{f > \alpha\}$ is measurable, and thus g is measurable. Likewise, if g is measurable, then it must be that $\{f > \alpha\}$ is measurable, and thus f is measurable. □

4. Prove that χ_E is measurable if and only if E is measurable.

Proof. We have

$$\{\chi_E > \alpha\} = \begin{cases} \varnothing & \alpha > 1 \\ E & 0 < \alpha \leq 1 \\ \mathbb{R} & \alpha \leq 0 \end{cases}$$

If χ_E is measurable, it follows that E is measurable, and if E is measurable, since $\varnothing$ and $\mathbb{R}$ are measurable, it follows that χ_E is also measurable. $\square$

5. Let N be a non-measurable subset of $(0, 1)$, and let $f(x) = x \cdot \chi_N(x)$. Show that f is non-measurable, but that each of the sets $\{f = \alpha\}$ is measurable.

Proof. Let $\alpha = 0$. Now, $f(x) = x \cdot \chi_N(x) = x$ whenever $x \in N$ and 0 otherwise. Thus,

$$\{f > 0\} = \{x \in \mathbb{R} : f(x) > 0\} = N$$

which is not measurable, since N is not measurable. Thus, f is not measurable.

Now, we have $f(x) = x$ whenever $x \in N$, and $f(x) = 0$ otherwise, so

$$\{f = \alpha\} = \begin{cases} \{\alpha\} & \alpha \in N \\ \varnothing & \alpha \in \mathbb{R} \setminus (N \cup \{0\}) \end{cases}$$

which are both measurable sets. However, it seems like the case when $\alpha = 0$ is not measurable... $\square$

8. Suppose that $D = A \cup B$, where A and B are measurable. Show that $f : D \to \mathbb{R}$ is measurable if and only if $f|_A$ and $f|_B$ are measurable.

Proof. We have

$$\begin{aligned} \{f > \alpha\} &= \{x \in D : f(x) > \alpha\} = \{x \in A : f(x) > \alpha\} \cup \{x \in B : f(x) > \alpha\} \\ &= \{f|_A > \alpha\} \cup \{f|_B > \alpha\} \end{aligned}$$

If $f|_A$ and $f|_B$ are both measurable, then the union of the two above sets is measurable, and thus f is also measurable, and likewise if f is measurable, this forces both $f|_A$ and $f|_B$ to be measurable. $\square$

12. If $f : [a, b] \to \mathbb{R}$ is Lipschitz with constant K, and if $E \subset [a, b]$, show that $m^*(f(E)) \leq K m^*(E)$. In particular, f maps null sets to null sets.

Proof. This was a question from HW4... If E has measure ∞, the inequality trivially holds. Otherwise, $m^*(E) < \infty$. Let $\varepsilon > 0$. Then there exists a sequence of intervals (a_n, b_n) such that $E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then we have $\sum_{n=1}^{\infty}(b_n - a_n) < m^*(E) + \frac{\varepsilon}{K}$. We also have that $f(E) \subset \bigcup_{n=1}^{\infty} f\left[(a_n, b_n)\right]$, where

$$f[(a_n, b_n)] = (f(c_n), f(d_n)), \quad c_n, d_n \in (a_n, b_n)$$

since Lipschitz functions are continuous. Then since $f[(a_n, b_n)]$ is a covering of $f(E)$, we have

$$\begin{aligned} m^*(f(E)) &\leq \sum_{n=1}^{\infty} \ell\Big(f[(a_n, b_n)]\Big) = \sum_{n=1}^{\infty}(f(d_n) - f(c_n)) \leq \sum_{n=1}^{\infty} K(d_n - c_n) \leq \sum_{n=1}^{\infty} K(b_n - a_n) \\ &< K m^*(E) + \varepsilon \\ \implies m^*(f(E)) &\leq K m^*(E) \end{aligned}$$

If E is a null set, then $m^*(f(E)) \leq K m^*(E) = 0$ so $m^*(f(E)) = 0$, and thus f maps null sets to null sets. $\square$