Homework 1

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1. Show that for any three sets A, B, C, we have that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Proof. $(\subset)$: Let $x \in (A \cap B) \cup C$. Then $x \in (A \cap B)$ or $x \in C$. If $x \in (A \cap B)$, then $x \in A$ and $x \in B$, so $x \in (A \cup C)$ and $x \in (B \cup C)$, so $x \in (A \cup C) \cap (B \cup C)$, as desired. Otherwise, if $x \in C$, it follows that $x \in (A \cup C)$ and $x \in (B \cup C)$, and the conclusion follows.

$(\supset)$: If $x \in (A \cup C) \cap (B \cup C)$, then $x \in (A \cup C)$ and $x \in (B \cup C)$. Thus $x \in A$ or $x \in C$, and $x \in B$ or $x \in C$. If $x \in C$, then $x \in (A \cap B) \cup C$, as desired. Otherwise, if $x \notin C$, then we must have $x \in A$ and $x \in B$, so $x \in (A \cap B)$, and thus $x \in (A \cap B) \cup C$, as desired.

Thus, the two sets are equal. $\square$

2. Show that every undirected graph with 2 or more nodes contains two nodes with the same degree.

Proof. Suppose the graph has n nodes of all different degrees. The maximum possible degree is $n - 1$, so the degrees of the nodes are $0, 1, \cdots, n-1$. Then consider the graph obtained by removing the vertex of degree 0. We now have a graph with $n - 1$ nodes, and one node having degree $n - 1$, which is a contradiction. Thus, the nodes cannot all have different degree, so there must exist two nodes with the same degree. $\square$

3. Show that there exist no integers x, y, z such that $x^2 + y^2 = 3z^2$, except $x = y = z = 0$.

Proof. Clearly $x = y = z = 0$ is a solution. WLOG $x \neq 0$. Let $g = \gcd(x, y)$, and let $x = ga$ and $y = gb$. Then $x^2 + y^2 = g^2(a^2 + b^2) = 3z^2$. Since g^2 divides the LHS, it must divide the RHS, so $g \mid z$, and let $z = gc$.

Then $g^2(a^2 + b^2) = 3g^2c^2 \implies a^2 + b^2 = 3c^2$. Now, squares modulo 4 have residues 0 and 1 (since every integer is either $2k$ or $2k + 1$ for some $k \in \mathbb{Z}$). We have $3c^2 \equiv 0$ or $3c^2 \equiv 3$ modulo 4, but only the former has a possible solution for a and b, in which case $a^2 \equiv b^2 \equiv 0 \pmod 4$. This means a and b are both even, but from above, we assumed g was the GCD of x and y, so $\gcd(a, b) = 1$. Contradiction, so there are no other solutions. $\square$

4. Let r be a number such that $r + 1/r$ is an integer. Use induction to show that for every positive integer $n, r^n + 1/r^n$ is an integer.

Proof. The base case is $n = 1$, and $r^1 + 1/r^1$ is an integer by the premise. Suppose $r^k + 1/r^k$ is an integer for all integers up to arbitrary k. Then

$$\begin{aligned}\left(r + \frac{1}{r}\right)\left(r^k + \frac{1}{r^k}\right) &= r^{k+1} + \frac{1}{r^{k-1}} + r^{k-1} + \frac{1}{r^{k+1}} \\ &= \left(r^{k+1} + \frac{1}{r^{k+1}}\right) + \left(r^{k-1} + \frac{1}{r^{k-1}}\right)\end{aligned}$$

Since $r + 1/r$ and $r^k + 1/r^k$ are both integers by assumption, their product is also an integer. Since $r^{k-1} + 1/r^{k-1}$ is also an integer by assumption, it follows that $r^{k+1} + 1/r^{k+1}$ is also an integer, as desired. $\square$