Homework 6

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Section 4.1

1. For each of the following smooth curves give an admissible parametrization that is consistent with the indicated direction.

   b. the circle $|z-2i|=4$ traversed once in the clockwise direction starting from the point $z=4+2i$.

   Solution. This is a circle radius 4 centered at $2i$, so we have $z(t)=4e^{-it}+2i, 0\le t\le 2\pi$. □

   d. the segment of the parabola $y=x^2$ from the point $(1,1)$ to the point $(3,9)$.

   Solution. Each point on this parabola is of the form (x,ix^2), so we have $z(t)=t+it^2, 1\le t\le 3$. □

3. Show that the ellipse $x^2/a^2+y^2/b^2=1$ is a smooth curve by producing an admissible parametrization.

   Solution. The typical parametrization of an ellipse into polar coordinates is $x(t)=a\cos t$ and $y(t)=b\sin t$, so that $x^2/a^2+y^2/b^2=\cos^2 t+\sin^2 t=1$. In the complex plane, we can just parametrize it as

$$z(t)=a\cos t+ib\sin t,\quad 0\le t\le 2\pi$$

□

4. Show that the range of the function $z(t)=t^3+it^6, -1\le t\le 1$, is a smooth curve even though the given parametrization is not admissible.

   Solution. We have $\operatorname{Im} z=(\operatorname{Re} z)^2$, so this is the curve $y=x^2$, with an admissible parametrization of $z_2(t)=t+it^2, -1\le t\le 1$. The given parametrization is not admissible because $z'(0)=0$. □

Section 4.2

3. Evaluate each of the following integrals.

   b. $\int_{-2}^{0}(1+i)\cos(it)\,dt$

   Solution. This is

$$\begin{aligned}\int_{-2}^{0}(1+i)\cos(it)\,dt&=\left[(1+i)\cdot\frac{1}{i}\sin(it)\right]\Big|_{-2}^{0}=\frac{1+i}{i}\sin 0-\frac{1+i}{i}\sin(-2i)\\&=\frac{1+i}{i}\frac{e^{i(-2i)}-e^{-i(-2i)}}{2i}=\frac{1+i}{2}\left(e^2-e^{-2}\right)\end{aligned}$$

□

d. $\int_0^2 \frac{t}{(t^2+i)^2}\, dt$

Solution. Using the substitution $u = t^2 + i \implies du = 2t\, dt$, we have

$$\int_0^2 \frac{t}{(t^2+i)^2}\, dt = \int_i^{4+i} \frac{1/2}{u^2}\, du = \left[-\frac{1}{2}u^{-1}\right]_i^{4+i} = -\frac{1}{2}\cdot\frac{1}{4+i} + \frac{1}{2}\cdot\frac{1}{i}$$
$$= -\frac{4-i}{2(4^2+1^2)} - \frac{i}{2} = -\frac{2}{17} - \frac{8}{17}i$$

□

6. Compute $\int_\Gamma \bar{z}\, dz$ where

(a) Γ is the circle $|z| = 2$ traversed once counterclockwise.

Solution. Let $\gamma(t) = 2e^{it}, 0 \le t \le 2\pi$ be a parametrization. Then $\gamma'(t) = 2ie^{it}$, so

$$\int_\Gamma \bar{z}\, dz = \int_0^{2\pi} \overline{2e^{it}} \cdot 2ie^{it}\, dt = \int_0^{2\pi} 2e^{-it}\cdot 2ie^{it}\, dt = \int_0^{2\pi} 4i\, dt = 8\pi i$$

□

(b) Γ is the circle $|z| = 2$ traversed once clockwise.

Solution. Let $\varphi(t) = 2e^{-it}, 0 \le t \le 2\pi$ be a parametrization. Then $\varphi'(t) = -2ie^{-it}$, so

$$\int_\Gamma \bar{z}\, dz = \int_0^{2\pi} \overline{2e^{-it}} \cdot -2ie^{-it}\, dt = \int_0^{2\pi} 2e^{it}\cdot -2ie^{-it}\, dt = \int_0^{2\pi} -4i\, dt = -8\pi i$$

□

(c) Γ is the circle $|z| = 2$ traversed three times clockwise.

Solution. Let $\theta(t) = 2e^{-it}, 0 \le t \le 6\pi$ be a parametrization. Then $\theta'(t) = -2ie^{-it}$, so

$$\int_\Gamma \bar{z}\, dz = \int_0^{6\pi} -4i\, dt = -24\pi i$$

using the result of part b. □

7. Compute $\int_\Gamma \operatorname{Re} z\, dz$ along the directed line segment from $z = 0$ to $z = 1 + 2i$.

Solution. Let $z(t) = (1+2i)t, 0 \le t \le 1$ be a parametrization. Then $z'(t) = 1 + 2i$, so

$$\int_\Gamma \operatorname{Re} z\, dz = \int_0^1 \operatorname{Re}[(1+2i)t]\cdot(1+2i)\, dt = \int_0^1 t(1+2i)\, dt = \left[\frac{1}{2}(1+2i)t^2\right]\Bigg|_0^1 = \frac{1}{2} + i$$

□

8. Let C be the perimeter of the square with vertices at the points $z = 0, z = 1, z = 1 + i$, and $z = i$ traversed once in that order. Show that

$$\int_C e^z\, dz = 0$$

Solution. We can parametrize these segments by

$$
\begin{aligned}
&z_1(t) = t, 0 \le t \le 1 \implies z_1'(t) = 1\\
&z_2(t) = (1+i)t + (1-t) = 1 + it, 0 \le t \le 1 \implies z_2'(t) = i\\
&z_3(t) = it + (1-t)(1+i) = 1 + i - t, 0 \le t \le 1 \implies z_3'(t) = -1\\
&z_4(t) = (1-t)i, 0 \le t \le 1 \implies z_4'(t) = -i
\end{aligned}
$$

$$
\begin{aligned}
\implies \int_C e^z\,dz &= \int_0^1 e^t\,dt + \int_0^1 e^{1+it} \cdot i\,dt + \int_0^1 e^{1+i-t} \cdot -1\,dt + \int_0^1 e^{(1-t)i} \cdot -i\,dt\\
&= \int_0^1 \left(e^t + ie^{1+it} - e^{1+i-t} - ie^{(1-t)i}\right) dt\\
&= \left[e^t + e^{1+it} + e^{1+i-t} + e^{(1-t)i}\right]\Big|_0^1\\
&= \left(e^1 + e^{1+i} + e^i + e^0\right) - \left(e^0 + e^1 + e^{1+i} + e^i\right) = 0
\end{aligned}
$$

as desired. □

9. Evaluate $\int_\Gamma (x - 2xyi)\,dz$ over the contour $\Gamma : z = t + it^2, 0 \le t \le 1$, where $x = \operatorname{Re} z, y = \operatorname{Im} z$.

Solution. This integral is equal to $\int_\Gamma (\operatorname{Re} z - 2i \operatorname{Re} z \operatorname{Im} z)\,dz$, and using the given parametrization where $z'(t) = 1 + 2it$, we have

$$
\begin{aligned}
\int_\Gamma (\operatorname{Re} z - 2i \operatorname{Re} z \operatorname{Im} z)\ dz &= \int_0^1 \left[\operatorname{Re}(t+it^2) - 2i\operatorname{Re}(t+it^2)\operatorname{Im}(t+it^2)\right] \cdot (1+2it)\,dt\\
&= \int_0^1 \left(t - 2it \cdot t^2\right)(1+2it)\,dt = \int_0^1 \left(t - 2it^3 + 2it^2 + 4t^4\right)\ dt\\
&= \left[\frac{1}{2}t^2 - \frac{i}{2}t^4 + \frac{2i}{3}t^3 + \frac{4}{5}t^5\right]\Big|_0^1 = \frac{1}{2} - \frac{i}{2} + \frac{2i}{3} + \frac{4}{5} = \frac{13}{10} + \frac{1}{6}i
\end{aligned}
$$

□

10. Compute $\int_C \bar{z}^2\,dz$ along the perimeter of the square in Prob 8.

Solution. If $f(z) = \bar{z}^2$, using the parametrization from Prob 8, we have

$$
\begin{aligned}
\int_C f(z)\,dz &= \int_0^1 \Big[f(t) + f\left[(1+i)t\right] \cdot i + f(1+i-t) \cdot -1 + f\left[(1-t)i\right] \cdot -i\Big]\,dt\\
&= \int_0^1 \left[t^2 + (1-it)^2 i - (1-i-t)^2 - \left[-(1-t)i\right]^2 i\right]\ dt\\
&= \int_0^1 \left[t^2 + \left(i + 2t - it^2\right) - \left(t^2 - 2t + 2it - 2i\right) + \left(i - 2it + it^2\right)\right]\ dt\\
&= \int_0^1 \left[(4-4i)t + 4i\right]\ dt = \left[(2-2i)t^2 + 4it\right]\Big|_0^1 = (2-2i) + 4i = 2 + 2i
\end{aligned}
$$

□

13. Compute $\int_\Gamma \left(|z - 1 + i|^2 - z\right)\,dz$ along the semicircle $z = 1 - i + e^{it}, 0 \le t \le \pi$.

Solution. We have $\gamma(t) = 1 - i + e^{it}, 0 \leq t \leq \pi$ is a parametrization, where $\gamma'(t) = ie^{it}$, so

$$\begin{aligned}
\int_\Gamma \left(|z-1+i|^2 - z\right)\, dz &= \int_0^\pi \left(\left|(1-i+e^{it}) - 1 + i\right|^2 - (1-i+e^{it})\right) \cdot ie^{it}\, dt \\
&= \int_0^\pi \left(\left|e^{it}\right|^2 - 1 + i - e^{it}\right) \cdot ie^{it}\, dt = \int_0^\pi \left(1 - 1 + i - e^{it}\right) \cdot ie^{it}\, dt \\
&= \int_0^\pi \left(-e^{it} - ie^{2it}\right)\, dt = \left[-\frac{1}{i}e^{it} - \frac{i}{2i}e^{2it}\right]\Bigg|_0^\pi = \left[ie^{it} - \frac{1}{2}e^{2it}\right]\Bigg|_0^\pi \\
&= \left[ie^{\pi i} - \frac{1}{2}e^{2\pi i}\right] - \left[ie^0 - \frac{1}{2}e^0\right] = -2i
\end{aligned}$$

□