Homework 3

Aleck Zhao

February 14, 2017

Section 5.2: Principal Ideal Domains

13. (a) Show that $\mathbb{Z}[\sqrt{-2}]$ is euclidean with $\delta(a) = |N(a|$.
    (b) If $a = 4 + 3\sqrt{-2}$ and $b = 3 - \sqrt{-2}$, write $a = qb + r$, where $r = 0$ or $\delta(r) < \delta(b)$.

Section 6.1: Vector Spaces

21. If $u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ in a vector space V, and if $a_1 \neq 0$, show that $\operatorname{span}\{v_1, v_2, \cdots, v_n\} = \operatorname{span}\{u, v_2, \cdots, v_n\}$.

23. (a) Show that an independent set $\{v_1, \cdots, v_n\}$ in ${}_F V$ with n maximal basis is a basis.
    (b) Show that a spanning set $\{v_1, \cdots, v_n\}$ of ${}_F V$ with n minimal basis is a basis.

31. A linear transformation $\varphi : {}_F V \to {}_F W$ is a map such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(av) = a\varphi(v)$ for all $a \in F$ and all $v, w \in V$.
    (a) Show that $\ker \varphi$ and $\operatorname{im} \varphi$ are subspaces of V and W, respectively.
    (b) If V is finite dimensional, show that $\operatorname{im} \varphi$ is also finite dimensional.
    (c) If V is finite dimensional, show that $\dim V = \dim(\ker \varphi) + \dim(\operatorname{im} \varphi)$.

32. Vector spaces ${}_F V$ and ${}_F W$ are called isomorphic if a one-to-one, onto linear transformation $V \to W$ exists. If ${}_F V$ has dimension n, show that $V \cong F^n$.