Homework 8

Aleck Zhao

April 16, 2017

Problems on Expected Time Until Hitting a State

1. Two possible infinitesimal generators for a 4-state Markov Process are given below. For each generator, find the expected time until the process hits state 4 if it starts in state 1.

   (a) $\begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

   Solution. The transition matrix for the embedded Markov chain is

   $$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   For notational convenience, let E_i represent the expected time to reach state 4 starting from state i, and let T_i represent the expected holding time at state i, which is just $-1/v_{ii}$ in the infinitesimal generator. Then we have

   $$\begin{aligned} E_1 &= T_1 + \frac{1}{2}(E_2 + E_3) = \frac{1}{2} + \frac{1}{2}(E_2 + E_3) \\ E_2 &= T_2 + E_3 = 1 + E_3 \\ E_3 &= T_3 + \frac{1}{3}(E_1 + E_2 + E_4) = \frac{1}{3} + \frac{1}{3}(E_1 + E_2 + E_4) \\ E_4 &= 0 \end{aligned}$$

   and solving, we find that $E_1 = 4$. □

   (b) $\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

   Solution. The transition matrix for the embedded Markov chain is

   $$\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   Using the same notation from part (a), we have

   $$\begin{aligned} E_1 &= T_1 + \frac{1}{3}(E_2 + E_3 + E_4) = \frac{1}{3} + \frac{1}{3}(E_2 + E_3 + E_4) \\ E_2 &= T_2 + \frac{2}{3}E_3 + \frac{1}{3}E_4 = \frac{1}{3} + \frac{2}{3}E_3 + \frac{1}{3}E_4 \\ E_3 &= T_3 + \frac{1}{4}E_1 + \frac{1}{2}E_2 + \frac{1}{4}E_4 = \frac{1}{4} + \frac{1}{4}E_1 + \frac{1}{2}E_2 + \frac{1}{4}E_4 \\ E_4 &= 0 \end{aligned}$$

   and solving, we find that $E_1 = 1$. □

Chapter 6: Continuous-Time Markov Chains

8. Consider two machines, both of which have an exponential lifetime with mean $1/\lambda$. There is a single repairman that can service machines at an exponential rate μ. Set up the Kolmogorov backward equations; you do not need to solve them.

   Solution. If the states $\{0, 1, 2\}$ are the number of broken machines, this is a birth and death process

   $$\begin{aligned} \lambda_0 &= 2\lambda \\ \lambda_1 &= \lambda \\ \mu_1 &= \mu_2 = \mu \end{aligned}$$

   Using this, we can construct A and the Kolmogorov backward equation:

   $$A = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$
   $$P_t' = AP_t = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & \mu & -\mu \end{bmatrix} P_t$$

   □

12. Each individual in a biological population is assume to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

    (a) Set this up as a birth and death model.

    Solution. If n is the number of people, then

    $$\lambda_n = \begin{cases} n\lambda + \theta & 0 \leq n < N \\ n\lambda & n \geq N \end{cases}$$
    $$\mu_n = n\mu \qquad n \geq 1$$

    □

    (b) If $N = 3, \lambda = \theta = 1, \mu = 2$, determine the proportion of time that immigration is restricted.

    Solution. If π_i represents the proportion of time spent in state i (which means i individuals in the population), then we have

    $$\mu_{n+1}\pi_{n+1} = \lambda_n\pi_n \implies (n+1)\mu\pi_{n+1} = \lambda_n\pi_n \implies \pi_{n+1} = \frac{\lambda_n}{(n+1)\mu}\pi_n$$
    $$\pi_1 = \frac{\lambda_0}{\mu}\pi_0 = \frac{\theta}{\mu}\pi_0 = \frac{1}{2}\pi_0$$
    $$\pi_2 = \frac{\lambda_1}{2\mu}\pi_1 = \frac{\lambda+\theta}{2\mu}\pi_1 = \frac{1+1}{2\cdot 2}\cdot\frac{1}{2}\pi_0 = \frac{1}{4}\pi_0$$
    $$\pi_3 = \frac{\lambda_2}{3\mu}\pi_2 = \frac{2\lambda+\theta}{3\mu}\pi_2 = \frac{2\cdot 1+1}{3\cdot 2}\cdot\frac{1}{4}\pi_0 = \frac{1}{8}\pi_0$$

For $n \geq 3$, immigration is restricted, so $\lambda_k = k\lambda$ for $k \geq 3$. Then

$$\begin{aligned}
\pi_{k+1} &= \frac{\lambda_k}{(k+1)\mu}\pi_k = \frac{k\lambda}{(k+1)\mu} \cdot \frac{\lambda_{k-1}}{k\mu}\pi_{k-1} = \frac{k\lambda}{(k+1)\mu} \cdot \frac{(k-1)\lambda}{k\mu} \cdot \frac{(k-2)\lambda}{(k-1)\mu}\pi_{k-2} = \cdots \\
&= \frac{k\lambda}{(k+1)\mu} \cdot \frac{(k-1)\lambda}{k\mu} \cdot \frac{(k-2)\lambda}{(k-1)\mu} \cdots \frac{3\lambda}{4\mu}\pi_3 \\
&= \frac{3}{k+1}\left(\frac{\lambda}{\mu}\right)^{k-2} \cdot \frac{1}{8}\pi_0 \\
\implies \pi_k &= \frac{3}{8k}\left(\frac{\lambda}{\mu}\right)^{k-3}\pi_0 = \frac{3}{8k}\left(\frac{1}{2}\right)^{k-3}\pi_0 = \frac{3}{k}\left(\frac{1}{2}\right)^{k}\pi_0
\end{aligned}$$

Now, since these are limiting probabilities, we have

$$\begin{aligned}
1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \sum_{k=4}^{\infty} \pi_k \\
&= \pi_0 + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{8}\pi_0 + \sum_{k=4}^{\infty} \frac{3}{k}\left(\frac{1}{2}\right)^k \pi_0 \\
&= \pi_0 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 3\sum_{k=4}^{\infty}\frac{1}{k2^k}\right]
\end{aligned}$$

Now, use the Taylor expansion

$$\ln\left(\frac{x}{x-1}\right) = \sum_{k=1}^{\infty}\frac{1}{kx^k}$$

where $x = 2$ to get

$$\sum_{k=4}^{\infty}\frac{1}{k2^k} = \ln\left(\frac{2}{2-1}\right) - \frac{1}{1\cdot 2^1} - \frac{1}{2\cdot 2^2} - \frac{1}{3\cdot 2^3} = \ln 2 - \frac{2}{3}$$

and substituting back above, we get

$$\begin{aligned}
1 &= \pi_0\left[\frac{15}{8} + 3\left(\ln 2 - \frac{2}{3}\right)\right] = \pi_0\left(3\ln 2 - \frac{1}{8}\right) \\
\implies \pi_0 &= \frac{1}{3\ln 2 - \frac{1}{8}} = \frac{8}{24\ln 2 - 1}
\end{aligned}$$

Now, the proportion of time that immigration is restricted is the complement of the proportion of time we spend in states 0, 1, and 2, which is

$$1 - \pi_0 - \pi_1 - \pi_2 = 1 - \frac{8}{24\ln 2 - 1} - \frac{4}{24\ln 2 - 1} - \frac{2}{24\ln 2 - 1} = 1 - \frac{14}{24\ln 2 - 1}$$

$\square$

13. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variable with mean 1/4 hour.

   (a) What is the average number of customers in the shop?

Solution. The states are $\{0, 1, 2\}$ for the number of customers in the shop. The rates are

$$\lambda_0 = \lambda_1 = 3$$
$$\mu_1 = \mu_2 = 4$$

If π_i is the long-run proportion of time we are in state i, then we have

$$\mu_1\pi_1 = \lambda_0\pi_0 \implies \pi_1 = \frac{3}{4}\pi_0$$
$$\mu_2\pi_2 = \lambda_1\pi_1 \implies \pi_2 = \frac{3}{4}\pi_1 = \frac{9}{16}\pi_0$$

Since $\pi_0 + \pi_1 + \pi_2 = 1$, we have

$$1 = \pi_0 + \pi_1 + \pi_2 = \pi_0 + \frac{3}{4}\pi_0 + \frac{9}{16}\pi_0 = \pi_0\left(1 + \frac{3}{4} + \frac{9}{16}\right) = \pi_0 \cdot \frac{37}{16}$$
$$\implies \pi_0 = \frac{16}{37} \implies \pi_1 = \frac{12}{37} \implies \pi_2 = \frac{9}{37}$$

Thus, the average number of customers is

$$0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 = \frac{12}{37} + 2 \cdot \frac{9}{37} = \frac{30}{37}$$

□

(b) What is the proportion of customers that enter the shop?

Solution. The proportion of customers that enter the shop is the complement of the proportion of those who don't. Customers don't enter only when there are already 2 customers in the shop, which occurs with long-term probability $\pi_2 = \frac{9}{37}$, so the proportion of customers that enter the shop is $1 - \frac{9}{37} = \frac{28}{37}$. □

(c) If the barber could work twice as fast, how much more business would he do?

Solution. If the barber could work twice as fast, then $\mu_1 = \mu_2 = 8$. Then since λ_0, λ_1 are unchanged we have the equations

$$\mu_1\pi_1 = \lambda_0\pi_0 \implies \pi_1 = \frac{3}{8}\pi_0$$
$$\mu_2\pi_2 = \lambda_1\pi_1 \implies \pi_2 = \frac{3}{8}\pi_1 = \frac{9}{64}\pi_0$$
$$\implies 1 = \pi_0 + \pi_1 + \pi_2 = \pi_0\left(1 + \frac{3}{8} + \frac{9}{64}\right) = \pi_0 \cdot \frac{97}{64}$$
$$\implies \pi_0 = \frac{64}{97} \implies \pi_1 = \frac{24}{97} \implies \pi_2 = \frac{9}{97}$$

In the original case, $\frac{28}{37}$ of the customers enter. Now, $1 - \frac{9}{97} = \frac{88}{97}$ of the customers enter. Thus, since 3 customers enter per hour on average, the barber is getting $2\left(\frac{88}{97} - \frac{28}{37}\right) \approx 0.45$ more customers per hour. □

14. Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than 2 cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

(a) What fraction of the attendant's time will be spent servicing cars?

Solution. The states are $\{0, 1, 2\}$ for the number of cars in the station. The rates (per hour) are

$$\begin{aligned}\lambda_0 = \lambda_1 &= 20\\ \mu_1 = \mu_2 &= 12\end{aligned}$$

If π_i is the long-run proportion of time we are in state i, then we have

$$\begin{aligned}\mu_1\pi_1 = \lambda_0\pi_0 &\implies \pi_1 = \frac{5}{3}\pi_0\\ \mu_2\pi_2 = \lambda_1\pi_1 &\implies \pi_2 = \frac{5}{3}\pi_1 = \frac{25}{9}\pi_0\end{aligned}$$

Since $\pi_0 + \pi_1 + \pi_2 = 1$, we have

$$\begin{aligned}1 = \pi_0 + \pi_1 + \pi_2 = \pi_0 + \frac{5}{3}\pi_0 + \frac{25}{9}\pi_0 = \pi_0\left(1 + \frac{5}{3} + \frac{25}{9}\right) = \pi_0 \cdot \frac{49}{9}\\ \implies \pi_0 = \frac{9}{49} \implies \pi_1 = \frac{15}{49} \implies \pi_2 = \frac{25}{49}\end{aligned}$$

Thus, the fraction of time the attendant will be servicing cars is $\pi_1 + \pi_2 = \frac{40}{49}$. □

(b) What fraction of potential customers are lost?

Solution. The fraction of potential customers lost is the fraction of time the process is in state 2, or $\pi_2 = \frac{25}{49}$. □

22. Customers arrive at a single-server queue in accordance with a Poisson process having rate λ. However, an arrival that finds n customers already in the system will only join the system with probability $1/(n+1)$. Show that the limiting distribution of the number of customers in the system is Poisson with mean λ/μ.

Proof. This is a birth and death process with

$$\begin{aligned}\mu_n &= \mu, & n \geq 1\\ \lambda_n &= \frac{\lambda}{n+1}, & n \geq 0\end{aligned}$$

Since this is a birth and death process, it is irreducible, so it has a limiting distribution. If π_i is the long-term proportion of time spent in state i, then we have

$$\mu_{n+1}\pi_{n+1} = \lambda_n\pi_n \implies \mu\pi_{n+1} = \frac{\lambda}{n+1}\pi_n \implies \pi_{n+1} = \frac{\lambda}{(n+1)\mu}\pi_n$$

Using this, we have

$$\begin{aligned}\pi_1 &= \frac{\lambda}{\mu}\pi_0\\ \pi_2 &= \frac{\lambda}{2\mu}\pi_1 = \frac{\lambda^2}{2\mu^2}\pi_0\\ \pi_3 &= \frac{\lambda}{3\mu}\pi_2 = \frac{\lambda^3}{3 \cdot 2\mu^3}\pi_0\end{aligned}$$

and continuing by induction, the general form is

$$\pi_k = \frac{\lambda^k}{k!\mu^k}\pi_0 = \frac{(\lambda/\mu)^k}{k!}\pi_0$$

Now, we have

$$
\begin{aligned}
1 = \sum_{k=0}^{\infty} \pi_k = \sum_{k=0}^{\infty} \frac{(\lambda/\mu)^k}{k!}\pi_0 = \pi_0 \sum_{k=0}^{\infty} \frac{(\lambda/\mu)^k}{k!} = \pi_0 e^{\lambda/\mu} \\
\implies \pi_0 = e^{-\lambda/\mu} \\
\implies \pi_k = \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu}
\end{aligned}
$$

so the limiting distribution is Poisson with mean λ/μ, as desired. $\square$

Exploration on Multiplicative Functions

Suppose f is a real-valued function defined on $[0, \infty)$, which satisfies

$$f(t+s) = f(t)f(s)$$

for all $s, t \geq 0$.

(a) Show that either $f(t) = 0$ for all $t \geq 0$ or $f(t) > 0$ for all $t \geq 0$.

Proof. For any t, we have

$$f(t) = f\left(\frac{t}{2} + \frac{t}{2}\right) = f^2\left(\frac{1}{2}\right) \geq 0$$

Now, suppose $f(t_0) = 0$ for some t_0. Then

$$f(t_0 + s) = f(t_0)f(s) = 0$$

so $f(t) = 0$ for all $t \geq t_0$. Then we also have

$$f(t_0) = f\left(\frac{t_0}{2} + \frac{t_0}{2}\right) = f^2\left(\frac{t_0}{2}\right) \implies f\left(\frac{t_0}{2}\right) = 0$$

and continuing by induction it follows that $f\left(\frac{t_0}{2^k}\right) = 0$ for all $k \geq 0$. Now, for any $0 < a < t_0$, there exists some k such that $\frac{t_0}{2^k} \leq a$. Thus,

$$f(a) = f\left[\frac{t_0}{2^k} + \left(t_0 - \frac{t_0}{2^k}\right)\right] = f\left(\frac{t_0}{2^k}\right) f\left(t_0 - \frac{t_0}{2^k}\right) = 0$$

Thus, if there exists t_0 such that $f(t_0) = 0$, then $f(t) = 0$ for all $t > 0$. Otherwise, if there is no such t_0, then we know that $f(t) > 0$ for all $t \geq 0$. $\square$

In the remaining parts, assume that $f(t) > 0$ for all $t > 0$.

(b) Determine the value of $f(0)$.

Solution. If $s = 0$, then

$$
\begin{aligned}
f(t+s) = f(t)f(s) \\
\implies f(t) = f(t)f(0) \\
\implies f(t)\left[f(0) - 1\right] = 0
\end{aligned}
$$

Since $f(t) > 0$ for all $t \geq 0$, we must have $f(0) - 1 = 0 \implies f(0) = \boxed{1}$. $\square$

(c) Assume that f is differentiable on $(0, \infty)$ and has a right-hand derivative at 0. Show that f is an exponential function.

Proof. We have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(0)f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h}$$
$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{f(t)f(h) - f(t)}{h} = f(t) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

Thus, we have

$$f'(t) = f(t)f'(0) \implies \frac{f'(t)}{f(t)} = f'(0)$$

and solving this differential equation, we get

$$\int \frac{f'(t)}{f(t)}\, dt = \int f'(0)\, dt$$
$$\ln f(t) = f'(0)t + C$$
$$f(t) = e^{f'(0)t + C} = Ce^{f'(0)t}$$

so f is an exponential function, as desired. $\square$