Homework 5

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1. Let $X_1, X_2, X_3, \cdots$ be iid random variables. Let $M(t) = E\left[e^{tX_1}\right]$ be the MGF of X_1 (and thus of each X_i). Fix t and assume that $M(t) < \infty$. Define the partial sum process by letting $S_0 = 0$ and for $n > 0$,
$$S_n = X_1 + \cdots + X_n.$$
Let
$$Z_n = \frac{e^{tS_n}}{M(t)^n}$$
Show that $\{Z_n\}_{n=0}^\infty$ is a martingale with respect to $\{X_n\}_{n=0}^\infty$.

Proof. We have
$$\begin{aligned} E\left[|Z_n|\right] &= E\left[\left|\frac{e^{tS_n}}{M(t)^n}\right|\right] = \frac{1}{M(t)^n} E[e^{t(X_1+\cdots+X_n)}] = \frac{1}{M(t)^n} \prod_{i=1}^n E[e^{tX_i}] \\ &= \frac{1}{M(t)^n} \prod_{i=1}^n M(t) = 1 < \infty \end{aligned}$$
and
$$\begin{aligned} E[Z_{n+1} \mid Z_1, \cdots, Z_n] &= E[Z_{n+1} \mid X_1, \cdots, X_n] \\ &= E\left[\frac{e^{tS_{n+1}}}{M(t)^{n+1}} \mid X_1, \cdots, X_n\right] \\ &= \frac{1}{M(t)^{n+1}} E\left[e^{tS_n} \cdot e^{tX_{n+1}} \mid X_1, \cdots, X_n\right] \\ &= \frac{1}{M(t)^{n+1}} E[e^{tS_n} \mid X_1, \cdots, X_n] E[e^{tX_{n+1}} \mid X_1, \cdots, X_n] \\ &= \frac{1}{M(t)^{n+1}} \cdot e^{tS_n} E[e^{tX_{n+1}}] \\ &= \frac{1}{M(t)^{n+1}} e^{tS_n} M(t) = \frac{e^{tS_n}}{M(t)^n} = Z_n \end{aligned}$$
so this is indeed a martingale. □

2. Consider a Markov chain $\{X_n, n \geq 0\}$ with state space consisting of $N+1$ states which are real numbers $x_0 < x_1 < x_2 < \cdots < x_N$, and with transition matrix $P(i,j) = P[X_{n+1} = x_j \mid X_n = x_i]$ for $0 \leq i, j \leq N$. Suppose that $\{X_n, n \geq 0\}$ is also a martingale. Show that the states x_0 and x_N are absorbing states.

Proof. Since this is a martingale, we have $E[X_{n+1} \mid X_n = x_0] = x_0$. This expectation is also
$$\begin{aligned} E[X_{n+1} \mid X_n = x_0] &= \sum_{j=0}^N x_j P(0,j) \\ &= x_0 P(0,0) + \sum_{j=1}^N x_j P(0,j) \end{aligned}$$

If $\exists j > 0$ such that $P(0,j) > 0$, then since $x_j > x_0$ for all $j > 0$, we have the strict inequality

$$x_0 P(0,0) + \sum_{j=1}^{N} x_j P(0,j) > x_0 P(0,0) + \sum_{j=1}^{N} x_0 P(0,j) = x_0 \left(P(0,0) + \sum_{j=1}^{N} P(0,j) \right) = x_0$$

This is a contradiction, since the expectation is exactly equal to x_0, so it follows that $P(0,j) = 0$ for all $j > 0$, so x_0 is an absorbing state.

Similarly, we have

$$\begin{aligned} x_N = E[X_{n+1} \mid X_n = x_N] &= \sum_{j=0}^{N} x_j P(0,j) \\ &= x_N P(0,N) + \sum_{j=0}^{N-1} x_j P(0,j) \end{aligned}$$

and if $\exists j < N$ such that $P(0,j) > 0$, then we have the strict inequality

$$x_N P(0,N) + \sum_{j=0}^{N-1} x_j P(0,j) < x_N P(0,N) + \sum_{j=0}^{N-1} x_N P(0,j) = x_N \left(P(0,N) + \sum_{j=0}^{N-1} P(0,j) \right) = x_N$$

which is a contradiction, so $P(0,j) = 0$ for all $j < N$, so x_N is absorbing, as desired. □

3. Calculate the PGF for a random variable X which has

   (a) a Geometric($\frac{1}{2}$) distribution.

   Solution. We have

$$\begin{aligned} G(s) = E[s^X] &= \sum_{i=1}^{\infty} P[X = i] s^i \\ &= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i s^i = \sum_{i=1}^{\infty} \left(\frac{s}{2}\right)^i \\ &= \frac{\frac{s}{2}}{1 - \frac{s}{2}} = \frac{s}{2-s} \end{aligned}$$

□

   (b) a Poisson(λ) distribution.

   Solution. We have

$$\begin{aligned} G(s) = E[s^X] &= \sum_{i=0}^{\infty} P[X = i] s^i \\ &= \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \cdot s^i \\ &= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda s)^i}{i!} \\ &= e^{-\lambda} \cdot e^{\lambda s} = e^{\lambda(s-1)} \end{aligned}$$

□

4. Let $\{X_1, X_2, X_3, \cdots\}$ be a sequence of iid random variables with mean μ and variance σ^2. Let $S_n = X_1 + X_2 + \cdots + X_n$ for each integer $n \geq 1$. Let N be a positive integer random variable which is independent of the $\{X_i\}_{i\geq 1}$, and has mean ν and variance τ^2. Calculate the variance of S_N.

Solution. We have

$$\mathrm{Var}(S_N) = E[S_N^2] - (E[S_N])^2$$

Using the law of total probability, we have

$$\begin{aligned}
E[S_N] &= E\left[E[S_N \mid N]\right] = \sum_{n=1}^{\infty} E[S_N \mid N = n]P[N = n] \\
&= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{N} X_i \mid N = n\right] P[N = n] = \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} X_i \mid N = n\right] P[N = n] \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E[X_i]\right) P[N = n] = \sum_{n=1}^{\infty} (n\mu) P[N = n] \\
&= \mu \sum_{n=1}^{\infty} nP[N = n] = \mu\nu
\end{aligned}$$

and

$$\begin{aligned}
E[S_N^2] &= E\left[E[S_N^2 \mid N]\right] = \sum_{n=1}^{\infty} E[S_N^2 \mid N = n]P[N = n] \\
&= \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N = n\right] P[N = n] = \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{n} X_i\right)^2 \mid N = n\right] P[N = n] \\
&= \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] P[N = n] = \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{n} X_i^2\right) + \left(\sum_{j\neq k} X_j X_k\right)\right] P[N = n] \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E[X_i^2] + \sum_{j\neq k} E[X_j X_k]\right) P[N = n] \\
&= \sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} \left(E[X_i^2] - (E[X_i])^2 + (E[X_i])^2\right) + \sum_{j\neq k} E[X_j]E[X_k]\right] P[N = n] \\
&= \sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} (\sigma^2 + \mu^2) + \sum_{j\neq k} \mu^2\right] P[N = n] \\
&= \sum_{n=1}^{\infty} \left[n\sigma^2 + n\mu^2 + (n^2 - n)\mu^2\right] P[N = n] \\
&= \sum_{n=1}^{\infty} (n^2\mu^2 + n\sigma^2) P[N = n] = \mu^2 \sum_{n=1}^{\infty} n^2 P[N = n] + \sigma^2 \sum_{n=1}^{\infty} nP[N = n] \\
&= \mu^2 E[N^2] + \sigma^2\nu = \mu^2\left(E[N^2] - (E[N])^2 + (E[N^2])^2\right) + \sigma^2\nu \\
&= \mu^2(\tau^2 + \nu^2) + \sigma^2\nu = \mu^2\tau^2 + \mu^2\nu^2 + \sigma^2\nu
\end{aligned}$$

Combining these two, we have

$$\mathrm{Var}(S_N) = (\mu^2\tau^2 + \mu^2\nu^2 + \sigma^2\nu) - (\mu\nu)^2 = \mu^2\tau^2 + \sigma^2\nu$$

$\square$

5. Consider a branching process with offspring distribution given by the frequency function f, where $f(2) = a, f(1) = b$, and $f(0) = c$, with $a + b + c = 1$. Assume that the probability of extinction is $d, 0 < d < 1$. Express d in terms of a, b, c.

   Solution. The generating function for the offspring distribution is
$$G(s) = as^2 + bs + c$$
   and the extinction probability satisfies $d = G(d)$, so we have
$$\begin{aligned} d &= G(d) = ad^2 + bd + c \\ 0 &= ad^2 + (b-1)d + c \end{aligned}$$
   and solving for d we have
$$d = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$
   Since $0 < d < 1$ but 1 is a root of the quadratic, we must have the greater root be 1, so thus
$$d = \frac{1 - b - \sqrt{(b-1)^2 - 4ac}}{2a}$$
   □

6. Verify that if $\{Z_n\}$ is a branching process, then $\left\{\frac{Z_n}{\mu^n}\right\}$ is a martingale, where μ denotes the mean of the offspring distribution.

   Proof. We have
$$E\left[\left|\frac{Z_n}{\mu^n}\right|\right] = \frac{1}{\mu^n} E[Z_n] = \frac{1}{\mu^n}\left(\mu^n E[Z_0]\right) = E[Z_0] < \infty$$
   and
$$\begin{aligned} E\left[\frac{Z_{n+1}}{\mu^{n+1}} \mid \frac{Z_0}{\mu^0}, \frac{Z_1}{\mu}, \cdots, \frac{Z_n}{\mu^n}\right] &= \frac{1}{\mu^{n+1}} E[Z_{n+1} \mid Z_0, Z_1, \cdots, Z_n] \\ &= \frac{1}{\mu^{n+1}} E[Z_{n+1} \mid Z_n] \\ &= \frac{1}{\mu^{n+1}} (\mu Z_n) = \frac{Z_n}{\mu^n} \end{aligned}$$
   so this is indeed a martingale. □

7. A particle moves according to a Markov chain on $\{1, 2, \cdots, c+d\}$ where c and d are positive integers. Starting from any one of the first c states, the particle jumps in one transition to a state chosen uniformly from the last d states. Starting from any of the last d states, the particle jumps in one transition to a state chosen uniformly from the first c states.

   (a) Show that the chain is irreducible.

   Proof. Let C and D are sets of the first c and last d states, respectively. Then if $i \in C$ and $j \in D$, then i and j communicate because they can directly transition between themselves. If $i, j \in C$, then if $n \in D$, we can have $i \to n \to j$, so i and j communicate. By a similar argument, if $i, j \in D$, then i and j communicate. Thus, all states communicate, so the chain is irreducible. □

   (b) Find the invariant distribution.

   Solution. The chain is periodic, half the time we are in C and half the time we are in D. Since the individual states within C and D are indistinguishable in terms of their transition probabilities, they all have the same distribution. Thus, since there are c states in C and d states in D, the invariant distribution is $\frac{1}{2c}$ for the first c states, and $\frac{1}{2d}$ for the last d states. □