

Homework 2

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1. Consider the 2π -period odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. NOTE: Discussed with Mauro, I am using the π -periodic function defined on just $[0, \pi]$. The results will not be identical but they should be okay.

- (a) Draw the graph of f
 (b) Compute the Fourier coefficients of f , and show that

$$f(\theta) = \frac{8}{\pi} \sum_{k \text{ odd} \geq 1} \frac{\sin k\theta}{k^3}$$

Is this function continuous on the circle? Continuously differentiable on the circle? In L^2 of the circle?

Solution. We have

$$\hat{f}(n) = \frac{1}{\pi} \int_0^\pi \theta(\pi - \theta) e^{-2in\theta} d\theta = \int_0^\pi \theta e^{-2in\theta} d\theta - \frac{1}{\pi} \int_0^\pi \theta^2 e^{-2in\theta} d\theta$$

Integrating by parts, we have

$$\begin{aligned} \int_0^\pi \theta e^{-2in\theta} d\theta &= -\frac{\theta}{2in} e^{-2in\theta} \Big|_0^\pi - \int_0^\pi -\frac{1}{2in} e^{-2in\theta} d\theta \\ &= -\frac{\pi}{2in} + \frac{1}{4n^2} e^{-2in\theta} \Big|_0^\pi = -\frac{\pi}{2in} \\ \int_0^\pi \theta^2 e^{-2in\theta} d\theta &= -\frac{\theta^2}{2in} e^{-2in\theta} \Big|_0^\pi - \int_0^\pi -\frac{1}{2in} e^{-2in\theta} \cdot 2\theta d\theta \\ &= -\frac{\pi^2}{2in} + \frac{1}{in} \int_0^\pi \theta e^{-2in\theta} d\theta = -\frac{\pi^2}{2in} + \frac{1}{in} \cdot \left(-\frac{\pi}{2in}\right) = -\frac{\pi^2}{2in} + \frac{\pi}{2n^2} \\ \implies \hat{f}(n) &= -\frac{\pi}{2in} - \frac{1}{\pi} \left(-\frac{\pi^2}{2in} + \frac{\pi}{2n^2}\right) = -\frac{1}{2n^2} \end{aligned}$$

Given a fixed $n \neq 0$, we have

$$\begin{aligned} \hat{g}(n, \theta) &= \hat{f}(n) e^{2in\theta} - \hat{f}(-n) e^{-2in\theta} = -\frac{1}{2n^2} (\cos 2n\theta + i \sin 2n\theta) + \frac{1}{2(-n)^2} (\cos(-2n\theta) + i \sin(-2n\theta)) \\ &= -\frac{1}{n^2} \cos 2n\theta \end{aligned}$$

due to the even and odd properties of \cos and \sin , respectively. We also have

$$\hat{f}(0) = \frac{1}{\pi} \int_0^\pi \theta(\pi - \theta) d\theta = \frac{1}{\pi} \left(\frac{\theta^2 \pi}{2} - \frac{\theta^3}{3} \right) \Big|_0^\pi = \frac{1}{\pi} \left(\frac{\pi^3}{2} - \frac{\pi^3}{3} \right) = \frac{\pi^2}{6}$$

and thus the Fourier series for f is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2in\theta} = \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \hat{g}(k, \theta) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{n^2} \cos 2n\theta$$

Now, since f is continuous, we have

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| = 2 \sum_{k=1}^{\infty} \left| \frac{1}{n^2} \right| + |\hat{f}(0)| = 2 \cdot \frac{\pi^2}{6} + \frac{\pi^2}{6} < \infty$$

and thus the Fourier series converges uniformly to f .

It is not continuously differentiable, it has a cusp at $k\pi$ for $k \in \mathbb{Z}$. It is also in L^2 (trivial to show, just integrating a quartic polynomial). \square

2. Prove that if f is an even 2π -period function, i.e. $f(\theta) = f(-\theta)$ for all $\theta \in [-\pi, \pi]$, then the Fourier series can be written as a cosine series.

Proof. We have

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi}$$

The Fourier series of f can be written as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$$

\square