Homework 3

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September 19, 2017

Chapter 2: Countable and Uncountable Sets

22. Show that Δ contains no nonempty open intervals. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $x \in [0,1] \setminus \Delta$ with $x < z < y$.

23. The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in Δ of the form $a/3^n$ for some integers n and $0 \le a \le 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as $0.a_1a_2\cdots a_{n+1}$ (base 3), where each a_k is 0 or 2, except a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2\cdots a_n1, 0.a_1a_2\cdots a_n2)$, where both entries are points of Δ written in base 3.

26. Let $f : \Delta \to [0,1]$ be the Cantor function and let $x, y \in \Delta$ with $x < y$. Show that $f(x) \le f(y)$. If $f(x) = f(y)$, show that x has two distinct binary expansions. Finally show that $f(x) = f(y)$ if and only if x and y are "consecutive" endpoints of the form $x = 0.a_1a_2\cdots a_n1$ and $y = 0.a_1a_2\cdots a_n2$ (base 3).

29. Prove that the extended Cantor function $f : [0,1] \to [0,1]$ is increasing.

30. Check that the construction of the generalized Cantor set with parameter α, as described above, leads to a set of measure $1 - \alpha$; that is, check that the discarded intervals now have total length α.

32. Deduce from Theorem 2.17 that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has points of continuity in every open interval.

33. Let $f : [a,b] \to \mathbb{R}$ be monotone. Given n distinct points $a < x_1 < x_2 < \cdots < x_n < b$, show that $\sum_{i=1}^n |f(x_i+) - f(x_i-)| \le |f(b) - f(b)|$. Use this to give anothe rproof that f has at most countably many (jump) discontinuities.