Homework 5

Aleck Zhao

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1. Suppose that $R = \prod_{i=1}^m R_i$ is a product of rings. If M_i is an R_i module for each i, then $\bigoplus_{i=1}^m M_i$ is naturally an R-module, via the rule

$$(r_1, \cdots, r_m) \cdot (x_1, \cdots, x_m) = (r_1 x_1, \cdots, r_m x_m)$$

For $i = 1, \cdots, m$, let $e_i \in R$ be the tuple whose ith entry is 1_R, and whose other entries are all 0. Define the submodule $M_i := e_i M$. Show that M_i is naturally an R_i-module, and that $M = \bigoplus_{i=1}^m M_i$.

2. Let R be a PID, let $d \in R$ be a nonzero nonunit, and let $d \sim p_1^{k_1} \cdots p_m^{k_m}$ be a prime factorization of d, where $p_1, \cdots, p_k$ are pairwise non-associated prime elements and $k_i > 0$ for all i. Show that the canonical homomorphism

$$\begin{aligned} R &\to \prod_{i=1}^m R/\left\langle p_i^{k_i} \right\rangle \\ r &\mapsto \left(r + \left\langle p_1^{k_1} \right\rangle, \cdots, r + \left\langle p_m^{k_m} \right\rangle\right) \end{aligned}$$

induces an isomorphism $R/\langle d \rangle \cong \prod_{i=1}^m R/\left\langle p_i^{k_i} \right\rangle$.

3. Keep the notation of Problem 2. Let M be an R-module such that $dM = 0$. By the paragraph preceding Theorem 7, Section 7.1, $M/dM \cong M$ is naturally an $R/\langle d \rangle$-module. Hence by Problem 2, M is naturally an $R/\left\langle p_1^{k_1} \right\rangle \times \cdots \times R/\langle p_m^{k_m} \rangle$-module. Let $M = \bigoplus_{i=1}^m M_i$ be the corresponding direct sum decomposition obtained from Problem 1.

   (a) Show that $M_i = p_i M$ as submodules of M for all i.

   (b) Show that if $\langle d \rangle = \mathrm{ann}(M)$, then $p_i M \neq 0$ for all i.