Homework 1

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Chapter 13: Functions of Bounded Variation

1. Show that $V_a^b(\chi_{\mathbb{Q}}) = +\infty$ on any interval $[a, b]$.

   Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition. If there exists a pair x_k, x_{k+1} that are both rational, we can refine P by adding $x_k < y < x_{k+1}$ to the partition, with y irrational. This is possible because $\mathbb{Q}$ is dense in $\mathbb{R}$. Similarly if there exists a consecutive pair of irrational numbers, we can refine by inserting a rational between them.

   Iterating this process, we end up with a partition of alternating rational and irrational numbers, say $Q = \{a = y_0 < y_1 < \cdots < y_m = b\}$. Then $|\chi_{\mathbb{Q}}(x_i) - \chi_{\mathbb{Q}}(x_{i-1})| = 1$ for all i, so

   $$V_a^b(\chi_{\mathbb{Q}}) \geq V(\chi_{\mathbb{Q}}, Q) = \sum_{i=1}^m |\chi_{\mathbb{Q}}(x_i) - \chi_{\mathbb{Q}}(x_{i-1})| = m$$

   However, given Q, we can always refine Q by inserting a pair of rational and irrational numbers between any consecutive terms, which increases the variation by 1. Thus, the variation is arbitrarily large. □

3. If f has a bounded derivative on $[a, b]$, show that $V_a^b f \leq \|f'\|_\infty (b - a)$.

   Proof. For any $x < y \in [a, b]$, we have $\frac{f(y)-f(x)}{y-x} = f'(c)$ for some $c \in [a, b]$ by the mean value theorem. Since f has a bounded derivative, it follows that

   $$\left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)| \leq \|f'\|_\infty \implies |f(y) - f(x)| \leq \|f'\|_\infty |y - x|$$

   for any $x, y \in [a, b]$. Thus, given any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

   $$\begin{aligned} V(f, P) &= \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^n \|f'\|_\infty |x_i - x_{i-1}| \\ &= \|f'\|_\infty \sum_{i=1}^n (x_i - x_{i-1}) = \|f'\|_\infty (b - a) \end{aligned}$$

   Since this is true for all P, it holds that $V_a^b f = \sup_P V(f, P) \leq \|f'\|_\infty (b - a)$. □

5. Complete the proof of Lemma 13.3.

   (i) $V_a^b f = 0$ if and only if f is constant

   Proof. $(\implies)$: If $V_a^b f = 0$, then for any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

   $$V(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = 0$$

   so $f(x_i) = f(x_{i-1})$ for all i, and thus $f(x_i) = f(a)$ for all i. Any refinement will keep the total variation at 0, so it follows that $f(x) = f(a)$ for all $x \in [a, b]$.
   $(\impliedby)$: If f is constant, then its variation is trivially 0. □

(ii) $V_a^b(cf) = |c|\, V_a^b f$

Proof. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

$$V(cf, P) = \sum_{i=1}^n |cf(x_i) - cf(x_{i-1})| = c \cdot \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = cV(f, P)$$

so taking supremums over P, it follows that $V_a^b(cf) = cV_a^b f$. $\square$

(iv) $V_a^b(fg) \le \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$

Proof. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

$$\begin{aligned}
V(fg, P) &= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(x_{i-1}) + f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})| \\
&\le \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| + \sum_{i=1}^n |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)|\, |g(x_i) - g(x_{i-1})| + \sum_{i=1}^n |g(x_{i-1})|\, |f(x_i) - f(x_{i-1})| \\
&\le \sum_{i=1}^n \|f\|_\infty |g(x_i) - g(x_{i-1})| + \sum_{i=1}^n \|g\|_\infty |f(x_i) - f(x_{i-1})| \\
&= \|f\|_\infty V(g, P) + \|g\|_\infty V(f, P)
\end{aligned}$$

so taking supremums over P, the result follows. $\square$

(v) $V_a^b |f| \le V_a^b f$

Proof. For $x, y \in \mathbb{R}$, it holds that $|x| - |y| \le x - y$. Check by casework on the signs of x, y. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we have

$$V(|f|, P) = \sum_{i=1}^n ||f(x_i)| - |f(x_{i-1})|| \le \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = V(f, P)$$

so taking supremums of P, the result follows. $\square$

6. We can test several of the inclusions explicit in our discussion up to this point by means of a single family of functions. For $\alpha \in \mathbb{R}$, and $\beta > 0$, set $f(x) = x^\alpha \sin(x^{-\beta})$, for $0 < x \le 1$, and $f(0) = 0$. Show that

(a) f is bounded if and only if $\alpha \ge 0$

Proof. $(\implies)$: If f is bounded, then we must have

$$\left|x^\alpha \sin(x^{-\beta})\right| \le |x^\alpha| \le K$$

for some K. If $\alpha < 0$, then this grows arbitrarily large as $x \to 0$, so it must be that $\alpha \ge 0$.
$(\impliedby)$: If $\alpha \ge 0$, then $\left|x^\alpha \sin(x^{-\beta})\right| \le 1$, so f is bounded. $\square$

(b) f is continuous if and only if $\alpha > 0$

Proof. ($\Longrightarrow$) : If $\alpha = 0$, then $f(x) = \sin(x^{-\beta})$ is discontinuous at 0, since f is oscillating between -1 and 1. If $\alpha < 0$, then $f(x) = \frac{\sin(x^{-\beta})}{x^{-\alpha}}$ diverges as $x \to 0$, so f would not be continuous. Thus we must have $\alpha > 0$.
($\Longleftarrow$) : If $\alpha > 0$, then $x^\alpha \sin(x^{-\beta}) \to 0$ as $x \to 0$ since $\sin(x^{-\beta}) \in [-1, 1]$ and $x^\alpha \to 0$. □

(c) $f'(0)$ exists if and only if $\alpha > 1$

Proof. We have the limit

$$f'(0) = \lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{x^\alpha \sin(x^{-\beta})}{x} = \lim_{x\to 0+} x^{\alpha-1} \sin(x^{-\beta})$$

must exist. From above, this limit exists and is continuous if and only if $\alpha - 1 > 0 \implies \alpha > 1$. □

(d) f' is bounded if and only if $\alpha \geq 1 + \beta$

Proof. We have

$$\begin{aligned} f'(x) &= x^\alpha \cos(x^{-\beta}) \cdot (-\beta x^{-\beta-1}) + \alpha x^{\alpha-1} \sin(x^{-\beta}) \\ &= -\beta x^{\alpha-\beta-1} \cos(x^{-\beta}) + \alpha x^{\alpha-1} \sin x^{-\beta} \end{aligned}$$

everywhere except 0. From (a), this is bounded if and only if the exponent of x is non-negative, so $\alpha - \beta - 1 \geq 0 \implies \alpha \geq 1 + \beta$. Then $f'(0)$ is bounded because it exists whenever $\alpha > 1$, which is true because $\beta > 0$. □

(e) If $\alpha > 0$, then $f \in BV[0, 1]$ for $0 < \beta < \alpha$ and $f \notin BV[0, 1]$ for $\beta \geq \alpha$. (Hint: Try a few easy cases first, say $\alpha = \beta = 2$.)

Proof. Consider the partition

$$\begin{aligned} t_0 &= 0 \\ t_k &= \left[\frac{2}{[2(n-k) - 1]\pi}\right]^{1/\beta}, k = 0, 1, \cdots, n-1 \\ t_n &= 1 \end{aligned}$$

Then we have

$$\begin{aligned} f(t_k) &= \left[\frac{2}{[2(n-k) - 1]\pi}\right]^{\alpha/\beta} \sin\left(\frac{[2(n-k) - 1]\pi}{2}\right) \\ f(t_{k-1}) &= \left[\frac{2}{[2(n-k) + 1]\pi}\right]^{\alpha/\beta} \sin\left(\frac{[2(n-k) + 1]\pi}{2}\right) \end{aligned}$$

which have opposite signs because the sine terms are ± 1. Then we have

$$\begin{aligned} |f(t_k) - f(t_{k-1})| &= \left(\frac{2}{\pi}\right)^{\alpha/\beta} \left[\left(\frac{1}{2(n-k) - 1}\right)^{\alpha/\beta} + \left(\frac{1}{2(n-k) + 1}\right)^{\alpha/\beta}\right] \\ &\geq 2 \cdot \left(\frac{2}{\pi}\right)^{\alpha/\beta} \left(\frac{1}{2(n-k) - 1}\right)^{\alpha/\beta} \end{aligned}$$

for $k = 2, \cdots, n-1$. Then we have

$$\begin{aligned} \sum_{k=2}^{n-1} |f(t_k) - f(t_{k-1})| &\geq 2 \cdot \left(\frac{2}{\pi}\right)^{\alpha/\beta} \sum_{k=2}^{n-1} \left(\frac{1}{2(n-k) - 1}\right)^{\alpha/\beta} \\ &= \frac{2}{\pi^{\alpha/\beta}} \sum_{i=1}^{n-2} \left(\frac{2}{2i - 1}\right)^{\alpha/\beta} \geq \frac{2}{\pi^{\alpha/\beta}} \sum_{i=1}^{n-2} \left(\frac{2}{2i}\right)^{\alpha/\beta} \end{aligned}$$

If $\beta \geq \alpha$, this is a divergent series as $n \to \infty$ since $\alpha/\beta \leq 1$. Thus, $f \notin BV[0,1]$ for $\beta \geq \alpha$. If $0 < \beta < \alpha$, this series converges, so $f \in BV[0,1]$ in that case (the differences $|f(t_1) - f(t_0)|$ and $|f(t_n) - f(t_{n-1})|$ are clearly finite). $\square$

11. If $f_n \to f$ pointwise on $[a,b]$, show that $V(f_n, P) \to V(f,P)$ for any partition P of $[a,b]$. In particular, if we also have $V_a^b f_n \leq K$ for all n, then $V_a^b f \leq K$ too.

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$. Fix $\varepsilon > 0$. Then since $f_n \to f$ pointwise, we have $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{2m}$ for $n \geq N_i, i = 0, 1, \cdots, m$. Taking $N = \max_i N_i$, we have

$$\begin{aligned}
|V(f_n,P) - V(f,P)| &= \left| \sum_{i=1}^{m} \bigg(|f_n(x_i) - f_n(x_{i-1})| - |f(x_i) - f(x_{i-1})| \bigg) \right| \\
&\leq \left| \sum_{i=1}^{m} |f_n(x_i) - f_n(x_{i-1}) - f(x_i) + f(x_{i-1})| \right| \\
&\leq \left| \sum_{i=1}^{m} |f_n(x_i) - f(x_i)| + \sum_{i=1}^{m} |f_n(x_i) - f(x_{i-1})| \right| \\
&< \left| \sum_{i=1}^{m} \frac{\varepsilon}{2m} + \sum_{i=1}^{m} \frac{\varepsilon}{2m} \right| \\
&= \varepsilon
\end{aligned}$$

whenever $n \geq N$. Thus, $V(f_n, P) \to V(f,P)$ for arbitrary P.

If $V_a^b f > K$, then $V(f,Q) > K$ for some Q. But $V(f_n, Q) \leq K$ for all n, so $V(f_n,Q) \not\to V(f,Q)$. $\square$

14. Let $I(x) = 0$ if $x < 0$ and $I(x) = 1$ if $x \geq 0$. Given a sequence of scalars (c_n) with $\sum_{n=1}^{\infty} |c_n| < \infty$ and a sequence of distinct points (x_n) in (a,b), define $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ for $x \in [a,b]$. Show that $f \in BV[a,b]$ and that $V_a^b f = \sum_{n=1}^{\infty} |c_n|$.

Proof. Let $P = \{a = y_0 < y_1 < \cdots < y_m = b\}$ be a partition. Then we have

$$\begin{aligned}
V(f,P) &= \sum_{i=1}^{m} \left| \sum_{n=1}^{\infty} c_n I(y_i - x_n) - \sum_{n=1}^{\infty} c_n I(y_{i-1} - x_n) \right| \\
&= \sum_{i=1}^{m} \left| \sum_{n=1}^{\infty} c_n \Big[I(y_i - x_n) - I(y_{i-1} - x_n) \Big] \right| \\
&\leq \sum_{i=1}^{m} \sum_{n=1}^{\infty} \left| c_n \Big[I(y_i - x_n) - I(y_{i-1} - x_n) \Big] \right| \\
&= \sum_{n=1}^{\infty} |c_n| \sum_{i=1}^{m} |I(y_i - x_n) - I(y_{i-1} - x_n)|
\end{aligned}$$

Now, for any n, the difference $I(y_i - x_n) - I(y_{i-1} - x_n)$ is non-zero only when $y_{i-1} < x_n \leq y_i$, where it equals 1. This can only happen once, since the sequence of y_i is increasing. Everywhere else the difference is 0. Thus, we have

$$V(f,P) \leq \sum_{n=1}^{\infty} |c_n| < \infty$$

Since this holds for arbitrary P, it follows that $f \in BV[a,b]$.

We have equality

$$\left|\sum_{n=1}^{\infty} c_n \left[I(y_i - x_n) - I(y_{i-1} - x_n)\right]\right| = \sum_{n=1}^{\infty} \left|c_n \left[I(y_i - x_n) - I(y_{i-1} - x_n)\right]\right|$$

if and only if all of the $c_n \left[I(y_i - x_n) - I(y_{i-1} - x_n)\right]$ have the same sign, which can happen if

$$I(y_i - x_n) - I(y_{i-1} - x_n) = \begin{cases} 0 & \text{if } c_n < 0 \\ 1 & \text{if } c_n \geq 0 \end{cases}$$

This construction is relatively easy to do, and produces equality of $V(f, P) = \sum_{n=1}^{\infty} |c_n|$, so since $V_a^b f \geq V(f, P)$ for all P, we have $V_a^b f = \sum_{n=1}^{\infty} |c_n|$. ☐

15. Show that $f \in C[a, b] \cap BV[a, b]$ if and only if f can be written as the difference of two strictly increasing continuous functions.

Proof. From Corollary 13.10, f can be written as the difference of two increasing continuous functions f_1 and f_2 as $f = f_1 - f_2$. Then let $g_1(x) = f_1(x) + x$ and $g_2(x) = f_2(x) + x$. Then g_1 and g_2 are both strictly continuous functions since x is strictly increasing. Thus $f = g_1 - g_2$ is the difference of two strictly increasing continuous functions. ☐