Homework 10

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May 4, 2017

Chapter 10: Brownian Motion and Stationary Processes

4. Show that

$$\begin{aligned} P[T_a < \infty] &= 1 \\ E[T_a] &= \infty, a \neq 0 \end{aligned}$$

Proof. By result 10.6 in the book, we have

$$\begin{aligned} P[T_a < \infty] = \lim_{t\to\infty} P[T_a \leq t] &= \lim_{t\to\infty} \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2}\, dy \\ &= 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = 2 \cdot \frac{1}{2} = 1 \end{aligned}$$

Using the tail probability formulation for expectation, we have

$$\begin{aligned} E[T_a] &= \int_0^\infty P[T_a > t]\, dt = \int_0^\infty (1 - P[T_a \leq t])\, dt \\ &= \int_0^\infty \left(1 - \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-y^2/2}\, dy\right) dt \end{aligned}$$

and somehow this integral diverges, but I don't know how to show it. ☐

5. What is $P[T_1 < T_{-1} < T_2]$?

Solution. This is the probability we hit 1 before -1 before 2. This is

$$\begin{aligned} P[T_1 < T_{-1}, T_{-1} < T_2] &= P[T_1 < T_{-1}] \cdot P[T_{-1} < T_2 \mid T_1 < T_{-1}] \\ &= \frac{1}{2} \cdot P[\text{down 2 before up 1}] \\ &= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \end{aligned}$$

☐

17. Show that standard Brownian motion is a Martingale.

Proof. We have

$$\begin{aligned}
E[|B(t)|] &= \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\
&= \int_{-\infty}^{0} (-x) \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx + \int_{0}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\
(u = x^2/2 \implies du = x\, dx) &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} -e^{-u}\, du + \int_{0}^{\infty} e^{-u}\, du \right) \\
&= \frac{1}{\sqrt{2\pi}} \left[(e^{-u}) \big|_{-\infty}^{0} + (-e^{-u}) \big|_{0}^{\infty} \right] \\
&= \frac{1}{\sqrt{2\pi}} (1+1) = \sqrt{\frac{2}{\pi}} < \infty
\end{aligned}$$

and for $s < t$,

$$\begin{aligned}
E[B(t) \mid B(u), 0 \le u \le s] &= E[B(t) - B(s) \mid B(u), 0 \le u \le s] + E[B(s) \mid B(u), 0 \le u \le s] \\
&= E[B(t-s)] + E[B(s) \mid B(u), 0 \le u \le s] \\
&= 0 + B(s) = B(s)
\end{aligned}$$

by independent and stationary increments. Thus standard Brownian motion is a Martingale. □

18. Show that $\{Y(t), t \ge 0\}$ is a Martingale when $Y(t) = B^2(t) - t$. What is $E[Y(t)]$? [Hint: First compute $E[Y(t) \mid B(u), 0 \le u \le s]$.]

Proof. We have

$$\begin{aligned}
B(t) &\sim N(0,t) = \sqrt{t} Z \\
\implies B^2(t) &\sim t Z^2 \\
\implies E\left[\left| B^2(t) - t \right| \right] &= E[\left| tZ^2 - t \right|] = t \cdot E\left[\left| Z^2 - 1 \right| \right] \\
&= t \left(E[Z^2 - 1, Z \ge 1] + E[1 - Z^2, 0 \le Z < 1] \right)
\end{aligned}$$

This is obviously bounded because each of these expectations is bounded.

For $s < t$, the conditional distribution of $B(t)$ given $B(s)$ is a normal random variable with mean $B(s)$ and variance $t - s$. Then using $E[X^2] = \mathrm{Var}(X) + (E[X])^2$, we have

$$\begin{aligned}
E[B^2(t) \mid B(u), 0 \le u \le s] &= (t-s) + B^2(s) \\
\implies E[B^2(t) - t \mid B(u), 0 \le u \le s] &= (t-s) + B^2(s) - t = B^2(s) - s
\end{aligned}$$

Thus $Y(t)$ is a Martingale. Then

$$E[Y(t)] = E[tZ^2 - t] = t(E[Z^2] - 1) = t(1-1) = 0$$

□

20. Let $T = \min\{t : B(t) = 2 - 4t\}$. Use the Martingale stopping theorem to find $E[T]$.

Solution. By the Martingale stopping theorem, we have

$$E[B(T)] = E[B(0)] = 0$$

Since $B(T) = 2 - 4T$, this is

$$\begin{aligned} &E[2-4T] = 0 \\ \implies\ &2 - 4E[T] = 0 \\ \implies\ &E[T] = \frac{1}{2} \end{aligned}$$

□

28. Compute the mean and variance of

(a) $\int_0^1 t\,dB(t)$

Solution. The mean is 0 using the result from the book. We have

$$\operatorname{Var}\left(\int_0^1 t\,dB(t)\right) = \int_0^1 t^2\,dt = \frac{1}{3}t^3\bigg|_0^1 = \frac{1}{3}$$

□

(b) $\int_0^1 t^2\,dB(t)$

Solution. The mean is 0 using the result from the book. We have

$$\operatorname{Var}\left(\int_0^1 t^2\,dB(t)\right) = \int_0^1 (t^2)^2\,dt = \frac{1}{5}t^5\bigg|_0^1 = \frac{1}{5}$$

□

31. For $s < t$, argue that $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.

Solution. We compute the covariance:

$$\begin{aligned} \operatorname{Cov}\left(B(s) - \frac{s}{t}B(t), B(t)\right) &= \operatorname{Cov}(B(s), B(t)) - \operatorname{Cov}\left(\frac{s}{t}B(t), B(t)\right) \\ &= E[B(s)B(t)] - E[B(s)]E[B(t)] - \frac{s}{t}\operatorname{Var}(B(t)) \\ &= s \wedge t - \frac{s}{t} \cdot t = s - s = 0 \end{aligned}$$

Thus these are independent. □