Homework 7

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Section 4.2

5. Utilize Example 2 to evaluate

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2\right] dz$$

where C is the circle $|z-i| = 4$ traversed once counterclockwise.

Solution. From the result of example 2, since this is a circle centered at $z_0 = i$, we have

$$\begin{aligned}
\int_C (z-i)^n \, dz &= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases} \\
&\implies \int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3\left(z-i\right)^2\right] dz \\
&= 6\int_C (z-i)^{-2} \, dz + 2\int_C (z-i)^{-1} \, dz + \int_C (z-i)^0 \, dz - 3\int_C (z-i)^2 \, dz \\
&= 6 \cdot 0 + 2 \cdot 2\pi i + 0 - 3 \cdot 0 = 4\pi i
\end{aligned}$$

□

Section 4.3

1. Calculate each of the following integrals along the indicated contours.

   (b) $\int_\Gamma e^z \, dz$ along the upper half of the circle $|z| = 1$ from $z = 1$ to $z = -1$.

   Solution. Since e^z is entire, by the FTC, we have

   $$\int_\Gamma e^z \, dz = e^z \Big|_1^{-1} = e^{-1} - e$$

   □

   (e) $\int_\Gamma \sin^2 z \cos z \, dz$ along the contour in Fig 4.24.

   Solution. Since $\sin^2 z \cos z$ is entire, by the FTC, we have

   $$\int_\Gamma \sin^2 z \cos z \, dz = \left[\frac{1}{3}\sin^3 z\right]\Bigg|_\pi^i = \frac{1}{3}\sin^3 i - \frac{1}{3}\sin^3 \pi = \frac{1}{3} \cdot \frac{e^{i \cdot i} - e^{-i \cdot i}}{2i} = \frac{e^{-1} - e}{6i}$$

   □

   (g) $\int_\Gamma z^{1/2} \, dz$ for the principal branch of $z^{1/2}$ along the contour in Fig 4.24.

Solution. By the FTC, we have

$$\int_\Gamma z^{1/2}\,dz = \left[\frac{2}{3}z^{3/2}\right]\Bigg|_\pi^i = \frac{2}{3}i^{3/2} - \frac{2}{3}\pi^{3/2} = \frac{2}{3}(e^{\pi i/2})^{3/2} - \frac{2}{3}\pi^{3/2} = \frac{2}{3}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) - \frac{2}{3}\pi^{3/2}$$
$$= \left(-\frac{\sqrt{2}}{3} - \frac{2}{3}\pi^{3/2}\right) + i\frac{\sqrt{2}}{3}$$

□

(h) $\int_\Gamma (\operatorname{Log} z)^2\,dz$ along the line segment from $z = 1$ to $z = i$.

Solution. By the FTC, we have

$$\int_\Gamma (\operatorname{Log} z)^2\,dz = \left[z\operatorname{Log}^2 z - 2z\operatorname{Log} z + 2z\right]\Big|_1^i = \left(i\operatorname{Log}^2 i - 2i\operatorname{Log} i + 2i\right) - \left(1\operatorname{Log}^2 1 - 2\operatorname{Log} 1 + 2\right)$$
$$= \left[i\cdot\left(\frac{\pi i}{2}\right)^2 - 2i\cdot\frac{\pi i}{2} + 2i\right] - 2 = \pi - 2 + i\left(2 - \frac{\pi^2}{4}\right)$$

□

(i) $\int_\Gamma 1/(1+z^2)\,dz$ along the line segment from $z = 1$ to $z = 1+i$.

Solution. By the FTC, we have

$$\int_\Gamma \frac{1}{1+z^2}\,dz = \tan^{-1} z\Big|_1^{1+i} = \tan^{-1}(1+i) - \tan^{-1} 1 = \tan^{-1}(1+i) - \frac{\pi}{4}$$

Suppose $\tan z = 1 + i$. Then we have

$$\begin{aligned}
\tan z &= \frac{e^{2iz}-1}{i(e^{2iz}+1)} = 1+i \\
\implies e^{2iz} - 1 &= i(1+i)\left(e^{2iz}+1\right) \\
\implies e^{2iz} - 1 &= (i-1)e^{2iz} + (i-1) \\
\implies e^{2iz}\left[1-(i-1)\right] &= e^{2iz}(2-i) = e^{2iz}e^{\operatorname{Log}(2-i)} = e^{\pi i/2} \\
\implies e^{2iz} &= e^{\pi i/2 - \operatorname{Log}(2-i)} \\
\implies 2iz &= \frac{\pi i}{2} - (\operatorname{Log}|2-i| + i\operatorname{Arg}(2-i)) \\
\implies z &= \frac{1}{2i}\left(\frac{\pi i}{2} - \operatorname{Log}\sqrt{5} - i\operatorname{Arg}(2-i)\right) \\
&= \frac{\pi}{4} - \frac{1}{4i}\operatorname{Log} 5 + \frac{1}{2}\tan^{-1}\frac{1}{2} = \frac{\pi}{4} + \frac{i}{4}\operatorname{Log} 5 + \frac{1}{2}\tan^{-1}\frac{1}{2} \\
\implies \tan^{-1}(1+i) - \frac{\pi}{4} &= \frac{i}{4}\operatorname{Log} 5 + \frac{1}{2}\tan^{-1}\frac{1}{2}
\end{aligned}$$

□

2. If $P(z)$ is a polynomial and Γ is any closed contour, explain why $\int_\Gamma P(z)\,dz = 0$.

   Answer. Since polynomials are entire and continuous on all of $\mathbb{C}$, the integral is always 0 for any closed contour since closed contours can be decomposed into closed loops.

4. True or false: If f is analytic at each point of a closed contour Γ, then $\int_\Gamma f(z)\,dz = 0$.

Answer. This is false. If $f(z) = 1/z$ and if Γ is the unit circle centered at the origin oriented counterclockwise, then f is analytic on all of Γ, but $\int_\Gamma f(z)\,dz = 2\pi i$.

6. Apply Theorem 6 to compute the integral along the portion of C from α to β as indicated in Fig 4.25. Now let α and β approach the point τ on the cut to evaluate the given integral over all of C.

Solution. By Theorem 6, since $1/z$ has continuous anti-derivative $\operatorname{Log}(z - z_0)$, along the curve γ from α to β, we have

$$\begin{aligned}\int_\gamma \frac{1}{z - z_0}\,dz &= \operatorname{Log}(\beta - z_0) - \operatorname{Log}(\alpha - z_0) \\ &= \operatorname{Log}|\beta - z_0| + i\operatorname{Arg}(\beta - z_0) - \operatorname{Log}|\alpha - z_0| - i\operatorname{Arg}(\alpha - z_0)\end{aligned}$$

As $\alpha, \beta \to \tau$, we have

$$\begin{aligned}\lim_{\alpha,\beta\to\tau}\Big(\operatorname{Log}|\beta - z_0| + i\operatorname{Arg}(\beta - z_0) - \operatorname{Log}|\alpha - z_0| - i\operatorname{Arg}(\alpha - z_0)\Big) &= i\operatorname{Arg}(\tau - z_0) - i\operatorname{Arg}(\tau - z_0) \\ &= i\pi - (-i\pi) = 2\pi i\end{aligned}$$

□

7. Show that if C is a positively oriented circle and z_0 lies outside C, then

$$\int_C \frac{dz}{z - z_0} = 0$$

Proof. If z_0 lies outside C, then we can choose a domain D containing C and not containing z_0, where $1/(z - z_0)$ is analytic on all of D. Then $\int dz/(z - z_0) = \operatorname{Log}(z - z_0)$, and by the FTC, this is equal to 0 because we are integrating over a closed loop. □

Section 4.4

13. Evaluate $\int 1/(z^2 + 1)\,dz$ along the three closed contours $\Gamma_1, \Gamma_2, \Gamma_3$ in Fig 4.47.

(a) *Solution.* We first find the partial fraction decomposition of the integrand as

$$\begin{aligned}\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} &= \frac{A}{z-i} + \frac{B}{z+i} \\ \implies A(z+i) + B(z-i) = 1\end{aligned}$$

Letting $z = i$ and $z = -i$, we have

$$\begin{aligned}A(i+i) = 1 &\implies A = \frac{1}{2i} = -\frac{i}{2} \\ B(-i-i) = 1 &\implies B = \frac{1}{-2i} = \frac{i}{2}\end{aligned}$$

Thus the integral is

$$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = -\frac{i}{2}\int_{\Gamma_1}\frac{1}{z-i}\,dz + \frac{i}{2}\int_{\Gamma_1}\frac{1}{z+i}\,dz$$

Since Γ_1 is a counterclockwise loop around i not containing $-i$, the integral $\int_{\Gamma_1} \frac{1}{z+i}\,dz$ vanishes, so we have

$$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = -\frac{i}{2}\int_{\Gamma_1}\frac{1}{z-i}\,dz = -\frac{i}{2}\cdot 2\pi i = \pi$$

□

(b) *Solution.* Using the same partial fraction decomposition, we have

$$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = -\frac{i}{2}\int_{\Gamma_2} \frac{1}{z-i}\,dz + \frac{i}{2}\int_{\Gamma_2} \frac{1}{z+i}\,dz$$

and since Γ_2 is the counterclockwise loop containing both i and $-i$, both integrals evaluate to $2\pi i$, so the result is 0. □

(c) *Solution.* Using the same partial fraction decomposition, we have

$$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = -\frac{i}{2}\int_{\Gamma_2} \frac{1}{z-i}\,dz + \frac{i}{2}\int_{\Gamma_2} \frac{1}{z+i}\,dz$$

and since Γ_3 contains a counterclockwise loop around $-i$ and a clockwise loop not containing i, the integral $\int_{\Gamma_3} \frac{1}{z-i}\,dz$ vanishes, so we have

$$\int_{\Gamma_3} \frac{1}{z^2+1} = \frac{i}{2}\int_{\Gamma_3} \frac{1}{z+i}\,dz = \frac{i}{2}\cdot 2\pi i = -\pi$$

□

15. Evaluate

$$\int_\Gamma \frac{z}{(z+2)(z-1)}\,dz$$

where Γ is the circle $|z| = 4$ traversed twice in the clockwise direction.

Solution. We first find the partial fraction decomposition of the integrand as

$$\begin{aligned}\frac{z}{(z+2)(z-1)} &= \frac{A}{z+2} + \frac{B}{z-1}\\ &\implies z = A(z-1) + B(z+2)\end{aligned}$$

Letting $z = 1$ and $z = -2$, we have

$$\begin{aligned}1 = B(1+2) &\implies B = \frac{1}{3}\\ -2 = A(-2-1) &\implies A = \frac{2}{3}\\ \implies \int_\Gamma \frac{z}{(z+2)(z-1)}\,dz &= \frac{2}{3}\int_\Gamma \frac{1}{z+2}\,dz + \frac{1}{3}\int_\Gamma \frac{1}{z-1}\,dz\end{aligned}$$

This circle contains both 1 and -2, and since it goes around twice, we have

$$\frac{2}{3}\int_\Gamma \frac{1}{z+2}\,dz + \frac{1}{3}\int_\Gamma \frac{1}{z-1}\,dz = \frac{2}{3}\cdot 2 \cdot -2\pi i + \frac{1}{3}\cdot 2\cdot -2\pi i = -4\pi i$$

□

17. Evaluate

$$\int_\Gamma \frac{2z^2 - z + 1}{(z-1)^2(z+1)}\,dz$$

where Γ is the figure-eight contour traversed once as shown in Fig 4.49.

Solution. We first find the partial fraction decomposition of the integrand as

$$\frac{2z^2 - z + 1}{(z-1)^2(z+1)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z+1}$$
$$\implies 2z^2 - z + 1 = A(z+1) + B(z-1)(z+1) + C(z-1)^2$$

Letting $z = 1$ and $z = -1$, we have

$$2(1)^2 - 1 + 1 = A(1+1) \implies A = 1$$
$$2(-1)^2 - (-1) + 1 = C(-1-1)^2 \implies C = 1$$

and finally

$$2z^2 - z + 1 = (z+1) + B(z^2 - 1) + (z^2 - 2z + 1) \implies B = 1$$

Since $\frac{1}{(z-1)^2}$ has an anti-derivative, the integral vanishes. If γ_1 is the clockwise contour around 1 and γ_2 is the counterclockwise contour around -1, then we have

$$\int_\Gamma \left(\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{z+1} \right) dz = \int_{\gamma_1} \frac{1}{z-1}\, dz + \int_{\gamma_1} \frac{1}{z+1}\, dz + \int_{\gamma_2} \frac{1}{z-1}\, dz + \int_{\gamma_2} \frac{1}{z+1}\, dz$$
$$= -2\pi i + 0 + 0 + 2\pi i = 0$$

□

Section 4.5

3. Let C be the circle $|z| = 2$ traversed once in the positive sense. Compute each of the following integrals.

   (d) $\int_C \frac{5z^2+2z+1}{(z-i)^3}\, dz$

   Solution. We first find the partial fraction decomposition of the integrand as

$$\frac{5z^2 + 2z + 1}{(z-i)^3} = \frac{A}{(z-i)^3} + \frac{B}{(z-i)^2} + \frac{C}{z-i}$$
$$\implies 5z^2 + 2z + 1 = A + B(z-i) + C(z-i)^2 = A + Bz - Bi + Cz^2 - 2Ciz - C$$
$$= Cz^2 + (B - Ci)z + (A - Bi - C) \implies C = 5$$

   Then the integrals of $1/(z-i)^2$ and $1/(z-i)^3$ vanish since these have anti-derivatives, and we have

$$\int_C \frac{5z^2 + 2z + 1}{(z-i)^3}\, dz = 5 \int_C \frac{1}{z-i}\, dz = 5 \cdot 2\pi i = 10\pi i$$

□

4. Compute

$$\int_C \frac{z+i}{z^3 + 2z^2}\, dz$$

   where C is

   (a) the circle $|z| = 1$ traversed once counterclockwise.

Solution. We first find the partial fraction decomposition of the integrand as

$$\begin{aligned}
&\frac{z+i}{z^3+2z^2} = \frac{z+i}{z^2(z+2)} = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+2} \\
\implies\ & z+i = A(z+2) + Bz(z+2) + Cz^2 = Az + 2A + Bz^2 + 2Bz + Cz^2 \\
&\qquad = (B+C)z^2 + (A+2B)z + 2A
\end{aligned}$$

so we have the equations

$$\begin{aligned}
&i = 2A \implies A = \frac{i}{2} \\
&1 = A + 2B = \frac{i}{2} + 2B \implies B = \frac{1}{2} - \frac{i}{4} \\
&0 = B + C \implies C = -\frac{1}{2} + \frac{i}{4}
\end{aligned}$$

and thus the integral is given by

$$\begin{aligned}
\int_C \frac{z+i}{z^3+2z^2}\,dz &= \frac{i}{2}\int_C \frac{1}{z^2}\,dz + \left(\frac{1}{2} - \frac{i}{4}\right)\int_C \frac{1}{z}\,dz + \left(-\frac{1}{2} + \frac{i}{4}\right)\int_C \frac{1}{z+2}\,dz \\
&= \left(\frac{1}{2} - \frac{i}{4}\right)\int_C \frac{1}{z}\,dz + \left(-\frac{1}{2} + \frac{i}{4}\right)\int_C \frac{1}{z+2}\,dz
\end{aligned}$$

since $1/z^2$ has an anti-derivative, so its integral vanishes on any closed loop. Now, C contains 0 and not -2, so this evaluates to

$$\left(\frac{1}{2} - \frac{i}{4}\right) \cdot 2\pi i = \frac{\pi}{2} + \pi i$$

□

(b) the circle $|z+2-i| = 2$ traversed once counterclockwise.

Solution. Using the decomposition from above, we have

$$\int_C \frac{z+i}{z^3+2z^2}\,dz = \left(\frac{1}{2} - \frac{i}{4}\right)\int_C \frac{1}{z} + \left(-\frac{1}{2} + \frac{i}{4}\right)\int_C \frac{1}{z+2}\,dz$$

Here, C contains -2 and not 0, so this evaluates to

$$\left(-\frac{1}{2} + \frac{i}{4}\right) \cdot 2\pi i = -\frac{\pi}{2} - \pi i$$

□

(c) the circle $|z-2i| = 1$ traversed once counterclockwise.

Solution. Using the same decomposition from above, we find that C does not contain either 0 or -2, so both integrals evaluate to 0. □