Homework 10

Aleck Zhao

April 19, 2018

Section 5.6

1. Find and classify the isolated singularities of each of the following functions.

(a) $\frac{z^3+1}{z^2(z+1)}$

Solution. We can simplify this as

$$\frac{(z+1)(z^2-z+1)}{z^2(z+1)} = \frac{z^2-z+1}{z^2}$$

which has a pole of order 2 at 0, and a removable singularity at -1. □

(b) $z^3 e^{1/z}$

Solution. This has an essential singularity at 0. □

(c) $\frac{\cos z}{z^2+1} + 4z$

Solution. This has poles of order 1 at i and $-i$. □

(d) $\frac{1}{e^z-1}$

Solution. This has poles whenever $e^z = 1 \implies z = 2k\pi i$ for $k \in \mathbb{Z}$. □

(e) $\tan z$

Solution. This has poles whenever $\cos z = 0 \implies z = \left(k + \frac{\pi}{2}\right) i$ for $k \in \mathbb{Z}$. □

(f) $\cos\left(1 - \frac{1}{z}\right)$

Solution. The Taylor series is

$$\cos\left(1 - \frac{1}{z}\right) = 1 - \frac{1}{2!}\left(1 - \frac{1}{z}\right)^2 + \frac{1}{4!}\left(1 - \frac{1}{z}\right)^4 - \cdots$$

so it has an essential singularity at 0. □

(g) $\frac{\sin 3z}{z^2} - \frac{3}{z}$

Solution. The Taylor series is

$$\begin{aligned}\frac{\sin 3z}{z^2} - \frac{3}{z} &= \frac{1}{z^2}\left(3z - \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!} - \cdots\right) - \frac{3}{z} = \left(\frac{3}{z} - \frac{3^3 z}{3!} + \frac{3^5 z^3}{5!} - \cdots\right) - \frac{3}{z} \\ &= -\frac{3^3 z}{3!} + \frac{3^5 z^3}{5!} - \cdots\end{aligned}$$

so it has a removable singularity at 0. □

(h) $\cot \frac{1}{z}$

Solution. This has poles whenever $\sin\frac{1}{z} = 0 \implies z = \frac{1}{k\pi i}$ for $k \in \mathbb{Z}$, and an essential singularity at 0. □

3. For each of the following, construct a function f, analytic in the plane except for isolated singularities, that satisfies the given conditions.

   (a) f has a zero of order 2 at $z = i$ and a pole of order 5 at $z = 2 - 3i$.

   Solution. We can let

$$f(z) = (z-i)^2 \cdot \frac{1}{(z-2+3i)^5}$$

□

   (b) f has a simple zero at $z = 0$ and an essential singularity at $z = 1$

   Solution. We can let

$$f(z) = z \exp\left\{\frac{1}{z-1}\right\}$$

□

   (c) f has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$, and an essential singularity at $z = i$.

   Solution. We can let

$$f(z) = \frac{\sin z}{z} \cdot \frac{1}{(z-1)^6} \cdot \exp\left\{\frac{1}{z-i}\right\}$$

□

   (d) f has a pole of order 2 at $z = 1 + i$ and essential singularities at $z = 0$ and $z = 1$.

   Solution. We can let

$$f(z) = \frac{1}{(z-1-i)^2} \exp\left\{\frac{1}{z(z-1)}\right\}$$

□

8. Verify Picard's theorem for the function $\cos(1/z)$ at $z_0 = 0$.

Solution. We have the Taylor series as

$$\cos\frac{1}{z} = 1 - \frac{(1/z)^2}{2!} + \frac{(1/z)^4}{4!} - \cdots$$

so $\cos(1/z)$ has an essential singularity at $z = 0$, so it should assume every complex number, with possibly one exception, in any neighborhood of this singularity. Indeed, if

$$a = \cos\frac{1}{z} = \frac{e^{i/z} + e^{-i/z}}{2} \implies 2ae^{i/z} = e^{2i/z} + 1$$

$$\implies e^{2i/z} - 2ae^{i/z} + 1 = 0 \implies e^{i/z} = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$$

$$\implies \frac{i}{z} = \operatorname{Log}\left(a \pm \sqrt{a^2-1}\right) \implies z = \frac{i}{\operatorname{Log}\left(a \pm \sqrt{a^2-1}\right)}$$

so every value of a can be achieved. □

Section 5.7

1. Classify the behavior at ∞ for each of the following functions (if a zero or pole, give its order):

   (a) e^z

   Solution. We have

   $$e^{1/z} = 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \cdots = 1 + z^{-1} + \frac{1}{2!}z^{-2} + \cdots$$

   so $e^{1/z}$ has an essential singularity at 0, and thus e^z has an essential singularity at ∞. □

   (b) $\cosh z$

   Solution. We have

   $$\begin{aligned}\cosh\frac{1}{z} &= \frac{1}{2}\left(e^{1/z} + e^{-1/z}\right) = \frac{1}{2}\left[\left(1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \cdots\right) + \left(1 - \frac{1}{z} + \frac{(-1/z)^2}{2!} + \cdots\right)\right] \\ &= 1 + \frac{1}{2!}z^{-2} + \frac{1}{4!}z^{-4} + \cdots\end{aligned}$$

   so $\cosh\frac{1}{z}$ has an essential singularity at 0, and thus $\cosh z$ has an essential singularity at ∞. □

   (c) $\frac{z-1}{z+1}$

   Solution. We have

   $$\frac{\frac{1}{z} - 1}{\frac{1}{z} + 1} = \frac{\frac{1-z}{z}}{\frac{1+z}{z}} = \frac{1-z}{1+z}$$

   is analytic at $z = 0$, so $\frac{z-1}{z+1}$ is analytic at ∞. □

   (d) $\frac{z}{z^3+i}$

   Solution. We have

   $$\frac{1/z}{(1/z)^3 + i} = \frac{\frac{1}{z}}{\frac{1+iz^3}{z^3}} = \frac{z^2}{1+iz^3}$$

   has a root of order 2 at $z = 0$, so the original function has a root of order 2 at ∞. □

   (e) $\frac{z^3+i}{z}$

   Solution. We have

   $$\frac{(1/z)^3 + i}{1/z} = \frac{1+iz^3}{z^2}$$

   has a pole of order 2 at $z = 0$, so the original function has a pole of order 2 at ∞. □

   (f) $e^{\sinh z}$

   Solution. We have

   $$e^{\sinh(1/z)} = 1 + \sinh\frac{1}{z} + \frac{\sinh^2(1/z)}{2!} + \cdots$$

   which has an essential singularity at $z = 0$, so the original function has an essential singularity at ∞. □

(g) $\frac{\sin z}{z^2}$

Solution. We have

$$\frac{\sin\frac{1}{z}}{(1/z)^2} = z^2\sin\frac{1}{z} = z^2\left(\frac{1}{z} - \frac{(1/z)^3}{3!} + \cdots\right) = z - \frac{1}{3!z} + \frac{1}{5!z^3} - \cdots$$

has an essential singularity at $z = 0$, so the original function has an essential singularity at ∞. □

(h) $\frac{1}{\sin z}$

Solution. We have

$$\begin{aligned}\frac{1}{\sin\frac{1}{z}} &= \frac{1}{1-\left(1-\sin\frac{1}{z}\right)} = 1 + \left(1-\sin\frac{1}{z}\right) + \left(1-\sin\frac{1}{z}\right)^2 + \cdots \\ &= 1 + \left[1-\left(\frac{1}{z} - \frac{(1/z)^3}{3!} + \cdots\right)\right] + \left[1-\left(\frac{1}{z} - \frac{(1/z)^3}{3!} + \cdots\right)\right]^2 + \cdots\end{aligned}$$

has an essential singularity at $z = 0$, so the original function has an essential singularity at ∞. □

(i) $e^{\tan 1/z}$

Solution. We have

$$e^{\tan 1/(1/z)} = e^{\tan z} = 1 + \tan z + \frac{\tan^2 z}{2!} + \cdots$$

which is analytic at $z = 0$, so the original function is analytic at ∞. □

3. Construct the series mentioned in Prob 2 for the following functions.

(a) $\frac{z-1}{z+1}$

Solution. We have

$$\begin{aligned}\frac{\frac{1}{z}-1}{\frac{1}{z}+1} &= 1 - \frac{2}{1-\left(-\frac{1}{z}\right)} = 1 - 2\left(1 - \frac{1}{z} + \left(-\frac{1}{z}\right)^2 + \cdots\right) = -1 + \frac{2}{z} - \frac{2}{z^2} + \cdots \\ &= -1 + \sum_{j=1}^{\infty}\frac{2(-1)^{j-1}}{z^j}\end{aligned}$$

□

(b) $\frac{z^2}{z^2+1}$

Solution. We have

$$\begin{aligned}\frac{(1/z)^2}{(1/z)^2+1} &= 1 - \frac{1}{1-(-1/z^2)} = 1 - \left(1 - \frac{1}{z^2} + \left(-\frac{1}{z^2}\right)^2 + \cdots\right) = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots \\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{z^{2j}}\end{aligned}$$

□

(c) $\frac{1}{z^3-i}$

Solution. We have

$$\frac{1}{(1/z)^3 - i} = \frac{i}{1-(-i/z^3)} = i\left(1 - \frac{i}{z^3} + \left(-\frac{i}{z^3}\right)^2 + \cdots\right) = i + \frac{1}{z^3} - \frac{i}{z^6} + \cdots$$
$$= \sum_{j=1}^{\infty} \frac{i^j}{z^{3j-3}}$$

□

4. State Picard's theorem for functions with an essential singularity at ∞. Verify for e^z.

   Solution. If a function has an essential singularity at ∞, then it assumes the value of every complex number on a neighborhood $|z| > r$, except for possibly one.
   For e^z, we have
   $$e^{1/z} = 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \cdots$$
   which has an essential singularity at 0, so e^z has an essential singularity at ∞, and attains every value except 0 on a neighborhood $|z| > r$. □

5. What is the order of the zero at ∞ if $f(z)$ is a rational function of the form $\frac{P(z)}{Q(z)}$ with $\deg P < \deg Q$?

   Solution. Suppose
   $$P(z) = a_0 + a_1 z + \cdots + a_n x^m$$
   $$Q(z) = b_0 + b_1 z + \cdots + b_m z^m$$
   where $n < m$ and $a_n, b_m \neq 0$. Then we have
   $$f\left(\frac{1}{z}\right) = \frac{P(1/z)}{Q(1/z)} = \frac{a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}}{b_0 + \frac{b_1}{z} + \cdots + \frac{b_m}{z^m}} = \frac{\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{z^n}}{\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^m}} = z^{m-n} \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$$
   If we evaluate the rational part at $z = 0$, we obtain $\frac{a_n}{b_m} \neq 0$ since $a_n \neq 0$, so this has a zero of order $m - n = \deg Q - \deg P$ at 0, and thus the original function has a zero of order $\deg Q - \deg P$ at ∞. □

Extra

Determine the images of the following complex analytic functions:

1. $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = 12 + e^{z-1}$

   Solution. The function e^{z-1} cannot attain the value 0, so $12 + e^{z-1}$ cannot attain the value 12, so the image is $\mathbb{C} \setminus \{12\}$. □

2. $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z \sin z$

   Solution. We have
   $$f\left(\frac{1}{z}\right) = \frac{1}{z}\sin\left(\frac{1}{z}\right) = \frac{1}{z}\left(\frac{1}{z} - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!} - \cdots\right) = \frac{1}{z^2} - \frac{1}{3!z^4} + \frac{1}{5!z^6} - \cdots$$
   Here, $a_j \neq 0$ for an infinite number of negative values of j, so 0 is an essential singularity of $f(1/z)$, and thus ∞ is an essential singularity of $f(z)$, and thus $z \sin z$ attains all values in $\mathbb{C}$. □

3. $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = (z^3 + 1)e^z$

   Solution. We have an essential singularity at ∞, so $f(z)$ attains all values in $\mathbb{C}$. $\square$

4. $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^{z^2+1}$

   Solution. This is a composition of two functions e^z and $z^2 + 1$, which have images $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$, so the image of f is $\mathbb{C} \setminus \{0\}$. $\square$

5. $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^{2z} + e^z + 7$

   Solution. This is a composition of two functions $z^2 + z + 7$ and e^z, which have images $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$, so the image of f is $\mathbb{C}$, since we can attain $f(z) = 7$ with z being a solution to $e^{2z} + e^z = e^z\,(e^z + 1) = 0$, which can occur at $z = \pi i$. $\square$