Homework 1

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1. Recall that a random variable is exponentially distributed with parameter λ if its density function $f(x)$ is given by
$$f(x) = \lambda e^{-\lambda x} \text{ for } x \text{ positive, and 0 otherwise.}$$

   (a) Suppose a random variable T has a Weibull (λ, α) distribution, i.e. that T has density
$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}}$$
for $t > 0, \lambda > 0, \alpha > 0$. Show that T^a has exponential (λ) distribution.

Proof. We use the fact that
$$\frac{d}{dx} F_X(x) = f_X(x)$$
where $F_X(x)$ is the cumulative distribution function, that is,
$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$
Thus to determine the distribution of T^α we may compute the CDF of T^α and take the derivative with respect to x. The probability we desire is
$$P(T^\alpha \leq x) = P(T \leq x^{1/\alpha})$$
where we used the fact that T^α is convex for $\alpha > 0$. We know the distribution of T, so this probability is
$$F_T(x^{1/\alpha}) = \int_{-\infty}^{x^{1/\alpha}} \lambda \alpha t^{\alpha-1} e^{-\lambda t^\alpha}\, dt.$$
Taking the derivative of both sides with respect to x and invoking the chain rule, we have
$$\begin{aligned} \frac{d}{dx} F_T(x^{1/\alpha}) &= \left(\frac{d}{dx} x^{1/\alpha} \right) \lambda \alpha (x^{1/\alpha})^{\alpha-1} e^{-\lambda (x^{1/\alpha})^\alpha} \\ &= \left(\frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} \right) \lambda \alpha x^{\frac{\alpha-1}{\alpha}} e^{-\lambda x} \\ &= \lambda e^{-\lambda x}. \end{aligned}$$
Thus the distribution for T^α is the exponential distribution with parameter λ, as desired. $\square$

(b) Show that if U is a uniform (0, 1) random variable, then $T = (-\lambda^{-1}\log U)^{1/\alpha}$ has a Weibull (λ, α) distribution.

Proof. We proceed with the same approach as the previous part, where we calculate the CDF of T, then take its first derivative to determine its PDF.

$$\begin{aligned} P(T \le t) &= P\left[\left(-\frac{1}{\lambda}\log U\right) \le t\right] \\ &= P\left(-\frac{1}{\lambda}\log U \le t^\alpha\right) \\ &= P\left(\log U \ge -\lambda t^\alpha\right) \\ &= P\left(U \ge e^{-\lambda t^\alpha}\right) \end{aligned}$$

Since U is a uniform (0, 1) variable, this probability is equal to $1 - e^{-\lambda t^\alpha}$. To finish, note that

$$f_T(t) = \frac{d}{dt}P(T \le t) = \frac{d}{dt}\left(1 - e^{\lambda t^\alpha}\right) = \lambda\alpha t^{\alpha-1}e^{-\lambda t^\alpha},$$

which is the desired distribution.

□

(c) Let $X_1, X_2, \cdots, X_n$ be i.i.d exponential λ random variables. Calculate the distribution of their sum $T_n = \sum_{i=1}^n X_i$.

Proof. We will show that the density distribution of T_n is given by

$$g_n(t) = \frac{(\lambda t)^{n-1}}{(n-1)!}\lambda e^{-\lambda t}.$$

We proceed by induction. The base case $n = 1$ is trivial,

$$g_1(t) = \frac{(\lambda t)^{1-1}}{(1-1)!}\lambda e^{\lambda t} = \lambda e^{-\lambda t},$$

which is the distribution for X_1. Next, assume that

$$g_k(t) = \frac{(\lambda t)^{k-1}}{(k-1)!}\lambda e^{-\lambda t}$$

for arbitrary $k > 1$. We wish to determine the density $h(t)$ for $T_{k+1} = T_k + X_{k+1}$, which we can do via convolution of the two random variables. Let $f(x)$ be the density for X_{k+1}, which is an exponential variable. We have

$$\begin{aligned} h(t) &= \int_0^t g_k(x)f(t-x)\,dx \\ &= \int_0^t \frac{(\lambda x)^{k-1}}{(k-1)!}\lambda e^{-\lambda x}\cdot\lambda e^{-\lambda(t-x)}\,dx \\ &= \frac{\lambda^{k+1}e^{-\lambda t}}{(k-1)!}\int_0^t x^{k-1}\,dx \\ &= \frac{\lambda^{k+1}e^{-\lambda t}}{(k-1)!}\left(\frac{1}{k}x^k\right)\Big|_0^t \\ &= \frac{\lambda^k t^k}{k!}\lambda e^{-\lambda t} = g_{k+1}(t), \end{aligned}$$

thus $h(t) = g_{k+1}(t)$, so the inductive hypothesis is true and we are done.

□

(d) Show that this is a special case of the more general *gamma distribution with parameters* α *and* β, whose density $f(x)$ is given by

$$f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad \text{for } x > 0$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$.

Proof. Let $\alpha = k$ and $\beta = 1/\lambda$. If $k \in \mathbb{Z}^+$, we have $\Gamma(k) = (k-1)!$ (this is true in our special case, since k indexes the X_i). Thus, we have

$$f(x) = \frac{x^{k-1}e^{-\lambda x}}{(1/\lambda)^k (k-1)!} = \frac{(\lambda x)^{k-1}}{(k-1)!}\lambda e^{-\lambda x},$$

as desired. □

2. Let X_i be i.i.d uniform [1, 2] random variables.

   (a) Compute the expectation μ and the variance σ^2 of the random variables $R_i = \ln X_i$.

   Solution. Since X_i are uniform [1, 2] random variables, the density distribution is identically 1 on its domain. Then

$$E[R_i] = \int_1^2 \ln x \, dx = (x \ln x - x)\Big|_1^2 = \boxed{2\ln 2 - 1,}$$

   where we integrated by parts to determine the integral of $\ln x$. Next, $var(R_i) = E[R_i^2] - (E[R_i])^2$, so we are interested in $E[R_i^2]$. This is

$$\int_1^2 (\ln x)^2 \, dx.$$

   We integrate by parts using $u = \ln x, du = 1/x, dv = \ln x \, dx, v = x \ln x - x$, and the integral evaluates to

$$\begin{aligned}
\int_1^2 (\ln x)^2 \, dx &= \big(x(\ln x)^2 - x \ln x\big)\Big|_1^2 - \int_1^2 \ln x - 1 \, dx \\
&= \big(x(\ln x)^2 - x\ln x - (x \ln x - x - x)\big)\Big|_1^2 \\
&= \big(x(\ln x)^2 - 2x \ln x + 2x\big)\Big|_1^2 \\
&= 2(\ln 2)^2 - 4 \ln 2 + 2
\end{aligned}$$

   Thus

$$\begin{aligned}
var(R_i) &= E[R_i^2] - (E[R_i])^2 \\
&= 2(\ln 2)^2 - 4\ln 2 + 2 - (2\ln 2 - 1)^2 \\
&= \boxed{1 - 2(\ln 2)^2.}
\end{aligned}$$

   □

   (b) Let Y_n be the random variable

$$Y_n = (X_1 X_2 \cdots X_n)^{1/n}.$$

   Determine what happens of Y_n as $n \to \infty$; if it converges, specify its limiting distribution; and if not, explain why.

   Solution. To begin, note that Y_n is bounded between the values 1 and 2, thus it converges. Consider the random variable

$$Z_n = \ln Y_n = \ln \left(\prod_{i=1}^n X_i \right)^{1/n} = \frac{1}{n} \sum_{i=1}^n \ln X_i.$$

   Thus as $n \to \infty$, the distribution of Z_n approaches a normal distribution with

$$\begin{aligned}
\mu &= E[\ln X_i] = 2\ln 2 - 1 \\
\sigma^2 &= \frac{var(\ln X_i)}{n} = \frac{1 - 2(\ln 2)^2}{n}.
\end{aligned}$$

   Thus

$$\ln Y_n \sim N\left(2\ln 2 - 1, \frac{1 - 2(\ln 2)^2}{n}\right)$$

   so Y_n is limited by the log-normal distribution with the above parameters.

   □

3. Suppose the pair (X_1, X_2) has a bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where $|\rho| < 1$; that is, the bivariate density is given by

$$f_{X_1,X_2}(x_1, x_2) = \frac{\exp\left(-A(x_1, x_2)/2\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

and $A(x_1, x_2)$ is defined as

$$A(x_1, x_2) = \left[\left(\frac{1}{1-\rho^2}\right)\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right].$$

(a) Let $\vec{x}$ denote the row vector $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, and let μ denote the row vector of means $\mu = \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}$. Let Σ denote the matrix of covariances, so $\Sigma_{ij} = cov(X_i, X_j)$. Show that the bivariate density above can be equivalently expressed as

$$f_{X_1,X_2}(x_1, x_2) = \frac{\exp\left(-A(x_1, x_2)/2\right)}{C}$$

where C is a constant, and $A(x_1, x_2)$ is the *quadratic form* defined by

$$A(x_1, x_2) = (\vec{x} - \mu)\Sigma^{-1}(\vec{x} - \mu)^T$$

and

$$\rho = \frac{cov(X_1, X_2)}{\sigma_1\sigma_2} \text{ is the so-called } \textit{correlation between } X_1 \text{ and } X_2.$$

Express the constant C in terms of the determinant of Σ.

Proof. We have

$$\begin{aligned}
\Sigma &= \begin{bmatrix} cov(X_1, X_1) & cov(X_1, X_2) \\ cov(X_2, X_1) & cov(X_2, X_2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \\
\det\Sigma &= \sigma_1^2\sigma_2^2(1-\rho^2) \\
\implies \Sigma^{-1} &= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}
\end{aligned}$$

Then

$$(\vec{x} - \mu) = \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix}, \qquad (\vec{x} - \mu)^T = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

so

$$\begin{aligned}
(\vec{x} - \mu)\Sigma^{-1}(\vec{x} - \mu)^T &= \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \cdot \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \sigma_2^2(x_1-\mu_1) - \rho\sigma_1\sigma_2(x_2-\mu_2) \\ \sigma_1^2(x_2-\mu_2) - \rho\sigma_1\sigma_2(x_1-\mu_1) \end{bmatrix} \\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \left[\sigma_2^2(x_1-\mu_1)^2 - 2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\right] \\
&= \left(\frac{1}{1-\rho^2}\right)\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right) \\
&= A(x_1, x_2)
\end{aligned}$$

Thus the two representations of $A(x_1, x_2)$ are equivalent, and $\boxed{C = 2\pi\sqrt{\det\Sigma}.}$

□

(b) Compute the conditional density $f_{X_1|X_2}(x_1|x_2)$ of X_1 given X_2.

Solution. We have the relation

$$f_{X_1|X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}$$

so wish we compute the marginal distribution

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1.$$

Rearranging $A(x_1,x_2)$ and completing the square, we have

$$\begin{aligned}
A(x_1,x_2) &= \left(\frac{1}{1-\rho^2}\right)\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right) \\
&= \frac{1}{\sigma_1^2(1-\rho^2)}\cdot\left((x_1-\mu_1)^2 - 2\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)(x_1-\mu_1)\right) + \frac{1}{1-\rho^2}\cdot\frac{(x_2-\mu_2)^2}{\sigma_2^2} \\
&= \frac{1}{\sigma_1^2(1-\rho^2)}\cdot\left((x_1-\mu_1) - \frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)\right)^2 + \frac{1}{1-\rho^2}\cdot\left(\frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{\rho^2(x_2-\mu_2)^2}{\sigma_2^2}\right) \\
&= \frac{1}{\sigma_1^2(1-\rho^2)}\cdot\left((x_1-\mu_1) - \frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)\right)^2 + \frac{(x_2-\mu_2)^2}{\sigma_2^2}.
\end{aligned}$$

Thus the integral is

$$\begin{aligned}
f_{X_2}(x_2) &= \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{1}{2}\left[\frac{1}{\sigma_1^2(1-\rho^2)}\cdot\left((x_1-\mu_1)-\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)\right)^2 + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\,dx_1 \\
&= \frac{\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\cdot\frac{(x_1-(\mu_1+\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)))^2}{\sigma_1^2(1-\rho^2)}\right)dx_1 \\
&= \frac{\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)}{\sigma_2\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\cdot\frac{(x_1-(\mu_1+\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)))^2}{\sigma_1^2(1-\rho^2)}\right)dx_1
\end{aligned}$$

Note that the integrand in this final expression is the density function for a normal random variable with $\mu = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)$ and $\sigma^2 = \sigma_1^2(1-\rho^2)$, so it evaluates to 1. Thus, we have

$$f_{X_2}(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}}\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

which is the distribution for a normal random variable $\mu = \mu_2$ and $\sigma = \sigma_2$. Finally, the distribution for the conditional density is

$$\begin{aligned}
f_{X_1|X_2}(x_1,x_2) &= \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} \\
&= \frac{\frac{\exp\left(-\frac{1}{2}\left[\frac{1}{\sigma_1^2(1-\rho^2)}\cdot\left((x_1-\mu_1)-\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)\right)^2+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}}{\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)} \\
&= \boxed{\frac{1}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\cdot\frac{(x_1-(\mu_1+\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)))^2}{\sigma_1^2(1-\rho^2)}\right)}.
\end{aligned}$$

□

(c) Suppose the covariance of X_1 and X_2 is zero. Show that X_1 and X_2 are independent.

Proof. Two random variables A and B are independent if and only if $f_{A,B}(a,b) = f_A(a)f_B(b)$ for all $a \in A, b \in B$. Since

$$cov(X_1, X_2) = 0 \implies \rho = \frac{cov(X_1, X_2)}{\sigma_1\sigma_2} = 0,$$

we have

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{1}{2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right).$$

We also have the marginal distributions of X_1 and X_2, and their product is

$$\begin{aligned} f_{X_1}(x_1)f_{X_2}(x_2) &= \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right)\cdot\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right) \\ &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{1}{2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right) \\ &= f_{X_1,X_2}(x_1,x_2), \end{aligned}$$

thus X_1 and X_2 are independent, as desired.

□

(d) Let $U = a_1X_1 + a_2X_2$ and let $V = b_1X_2 + b_2X_2$. Explicitly determine the joint density of (U, V).

Solution. Let $U = u(X_1, X_2)$ and $V = v(X_1, X_2)$. Then

$$f_{U,V}(u,v) = \frac{f_{X_1,X_2}(x_1,x_2)}{\left|\frac{\partial(u,v)}{\partial(x_1,x_2)}\right|}.$$

The determinant is evaluated as

$$\begin{aligned} \left|\frac{\partial(u,v)}{\partial(x_1,x_2)}\right| &= \det\begin{bmatrix}\partial u/\partial x_1 & \partial u/\partial x_2 \\ \partial v/\partial x_1 & \partial v/\partial x_2\end{bmatrix} \\ &= \det\begin{bmatrix}a_1 & a_2 \\ b_1 & b_2\end{bmatrix} = a_1b_2 - a_2b_1 \end{aligned}$$

thus the joint density of (U, V) is

$$f_{U,V}(u,v) = \frac{1}{a_1b_2 - a_2b_1}f_{X_1,X_2}(x_1,x_2)$$

We may solve for X_1 and X_2 in terms of V and U, and we get

$$\begin{aligned} X_1 &= \frac{b_2U - a_2V}{a_1b_2 - a_2b_1} \\ X_2 &= \frac{-b_1U + a_1V}{a_1b_2 - a_2b_1} \end{aligned}$$

so the resulting distribution is

$$f_{U,V}(u,v) = \boxed{\frac{1}{a_1b_2 - a_2b_1}f_{X_1,X_2}\left(\frac{b_2u - a_2v}{a_1b_2 - a_2b_1}, \frac{-b_1u + a_1v}{a_1b_2 - a_2b_1}\right)}$$

(expansion of f_{X_1,X_2} omitted.)

□

(e) Compute the conditional expectation $E[U|V]$ for the case when $a_1 = 1, b_1 = 4, a_2 = -2, b_2 = 2$. Assume X_1 and X_2 have equal variances.

Solution. We have $a_1 b_2 - a_2 b_1 = 1 \cdot 2 - (-2) \cdot 4 = 10$. Then solving for X_1 and X_2, we get

$$\begin{aligned} X_1 &= \frac{U+V}{5} \\ X_2 &= \frac{-4U+V}{10} \end{aligned}$$

Thus the joint distribution for (U, V) is

$$f_{U,V}(u,v) = \frac{1}{10} f_{X_1,X_2}\left(\frac{u+v}{5}, \frac{-4u+v}{10}\right) = \frac{1}{10} f_{X_1}\left(\frac{u+v}{5}\right) f_{X_2}\left(\frac{-4u+v}{10}\right).$$

$V = 4X_1 + 2X_2$ is a sum of independent normal variables so itself is a normal random variable, where $E[V] = E[4X_1 + 2X_2] = 4\mu_1 + 2\mu_2$ and $var(V) = var(4X_1 + 2X_2) = 4^2\sigma^2 + 2^2\sigma^2 = 20\sigma^2$. Using these, we can compute the density of V to be

$$f_V(v) = \frac{1}{\sqrt{2\pi}\sqrt{20\sigma^2}} \exp\left(-\frac{(v - (4\mu_1 + 2\mu_2))^2}{2(20\sigma^2)}\right).$$

Finally, we compute the conditional expectation:

$$\begin{aligned} E[U|V] &= \int_{-\infty}^{\infty} u f_{U|V}(u,v)\, du \\ &= \int_{-\infty}^{\infty} u \frac{f_{U,V}(u,v)}{f_V(v)}\, du \end{aligned}$$

I have discovered a truly marvelous solution which this margin is too small to contain.

□

4. Let X be the minimum and Y be the maximum of two independent random variables S and T with common continuous density f. Let Z denote the indicator function of the event $(S > T)$.

   (a) What is the distribution of Z?

   Solution. Since $P(S > T) = P(S \leq T)$ by symmetry, we have $P(S > T) = P(S \leq T) = 1/2$, thus Z follows a Bernoulli distribution with $p = 1/2$. □

   (b) Are X and Z independent? Are Y and Z independent? Are (X, Y) and Z independent?

   i. X and Z.

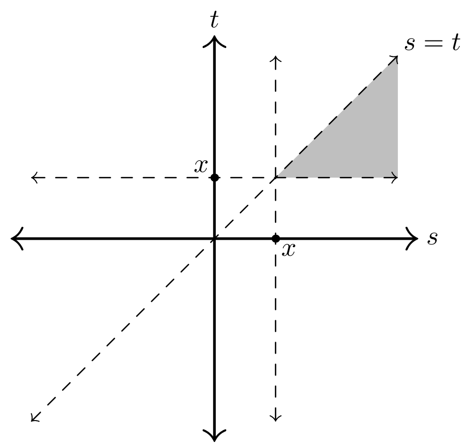
   Solution. We claim $\boxed{X \text{ and } Z \text{ are independent.}}$ Random variables X and Z are independent if and only if $f_{X|Z}(x|z) = f_X(x)$. Consider the probability $P(X \leq x|Z = z)$, so that its derivative is the conditional density. This is equivalent to $P(\min(S, T) \leq x|Z = z) = 1 - P(\min(S, T) > x|Z = z)$. If the minimum of S and T is greater than x, it must be the case that they are both individually greater than x, so the probability is

$$1 - P(S > x, T > x|Z = z) = 1 - \frac{P(S > x, T > x, Z = z)}{P(Z = z)}.$$

   We split into two cases: $Z = 1$ and $Z = 0$.
   If $Z = 1$, then $S > T$. Then the probability from before is

$$1 - \frac{P(S > x, T > x, S > T)}{P(S > T)}.$$

   The probability in the numerator is conceptually equal to the integral of the joint density over the defined region. The region described is shown below:



   Since we know $P(S > T) = 1/2$, we may write the probability as a double integral:

$$1 - 2\int_x^\infty \int_x^s f_{S,T}(s, t)\, dt\, ds = 1 - 2\int_x^\infty \int_x^s f(s)f(t)\, dt\, ds = 1 - 2\int_x^\infty \left(f(x)\int_x^s f(t)\, dt\right) ds.$$

   where the joint distribution is the product of marginals since S and T are independent. Rewrite the inner integral and the limits of the outer integral:

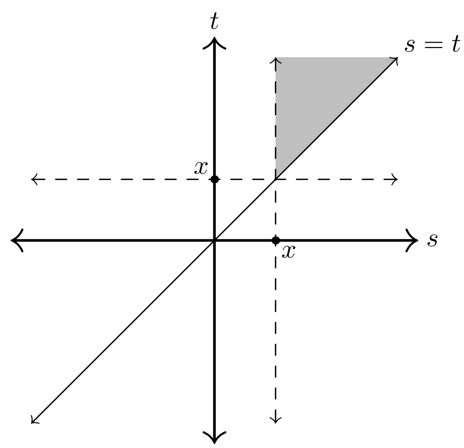
$$1 + 2\int_\infty^x f(s)(F(s) - F(x))\, ds = 1 + 2\int_\infty^x f(s)F(s)\, ds - 2\int_\infty^x f(s)F(x)\, ds,$$

   where we let F be the anti-derivative of f, the evaluated the integral at the limits x and s.

Now, taking the derivative of this expression, and applying the fundamental theorem of calculus, we have

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{d}{dx}\left[1 + 2\int_{\infty}^{x} f(s)F(s)\,ds - 2\int_{\infty}^{x} f(s)F(x)\,ds\right] \\ &= 2f(x)F(x) - 2\frac{d}{dx}\left[F(x)\int_{\infty}^{x} f(s)\,ds\right] \\ &= 2f(x)F(x) - 2\left(f(x)\int_{\infty}^{x} f(s)\,ds + F(x)f(x)\right) \\ &= 2f(x)\int_{x}^{\infty} f(s)\,ds \\ &= 2f(x)\left(1 - \int_{-\infty}^{x} f(s)\,ds\right). \end{aligned}$$

Next, we consider the case when $Z = 0$, or when $S \leq T$. Similar to the last part, the region is shown below:



and the probability is given by the expression

$$1 - 2\int_{x}^{\infty}\int_{x}^{t} f(s)f(t)\,ds\,dt$$

which we note is the same as the expression for when $S > T$, except that s and t are switched. Since they are identical random variables, the integrals are in fact equal, and the density of $f_{X|Z}(x|z)$ is given by

$$f_{X|Z}(x|z) = 2f(x)\left(1 - \int_{-\infty}^{x} f(s)\,ds\right).$$

Next, we compute the density of X, which is $f_X(x)$. Consider the probability

$$\begin{aligned} P(X \leq x) = 1 - P(X > x) = 1 - P(\min{(S,T)} > x) = 1 - P(S > x)P(T > x) \\ = 1 - (1 - P(S \leq x))(1 - P(T \leq x)) = 1 - \left(1 - \int_{-\infty}^{x} f(s)\,ds\right)\left(1 - \int_{-\infty}^{x} f(t)\,dt\right) \\ = 1 - \left(1 - \int_{-\infty}^{x} f(k)\,dk\right)^2 \end{aligned}$$

and take its derivative with respect to x :

$$\begin{aligned} f_X(s) = \frac{d}{dx}P(X \le x) = \frac{d}{dx}\left[1 - \left(1 - \int_{-\infty}^{x} f(k)\,dk\right)^2\right] \\ = -2\left(1 - \int_{-\infty}^{x} f(k)\,dk\right) \cdot \frac{d}{dx}\left[1 - \int_{-\infty}^{x} f(k)\,dk\right] \\ = 2f(x)\left(1 - \int_{-\infty}^{x} f(k)\,dk\right) \end{aligned}$$

Thus, $f_{X|Z}(x|z) = f_X(x)$, so X and Z are independent, as desired.

□

ii. Y and Z.

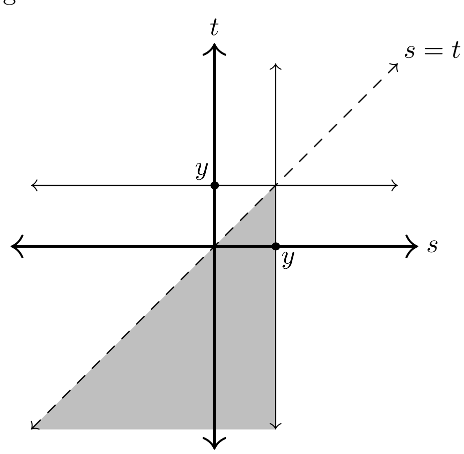
Solution. We claim $\boxed{Y \text{ and } Z \text{ are independent.}}$ We handle this similarly to the above case, the only difference is that Y is defined as $\max(S,T)$.
Consider $P(Y \le y|Z = z) = P(\max(S,T) \le y|Z = z)$. If the max of two numbers is less than or equal to y, it follows that both numbers are less than or equal to y, so this probability is

$$P(S \le y, T \le y \,|\, Z = z).$$

Now we consider the cases when $Z = 1$ and $Z = 0$.
When $Z = 1$, then $S > T$, and the probability becomes

$$\frac{P(S \le y, T \le y, S > T)}{P(S > T)} = 2P(S \le y, T \le y, S > T).$$

The graph of the above region is shown below:



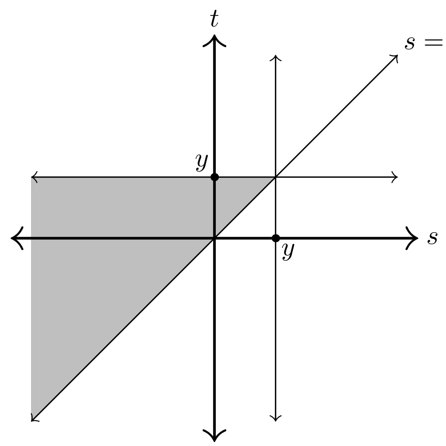
so the probability is the double integral

$$\begin{aligned} 2\int_{-\infty}^{y}\int_{t}^{y} f(s)f(t)\,ds\,dt = 2\int_{-\infty}^{y}\left(f(t)\int_{t}^{y} f(s)\,ds\right)\,dt \\ = 2\int_{-\infty}^{y} f(t)(F(y) - F(t))\,dt \\ = 2\int_{-\infty}^{y} f(t)F(y)\,dt - 2\int_{-\infty}^{y} f(t)F(t)\,dt \end{aligned}$$

where F is the anti-derivative of f, and taking the derivative with respect to y gives

$$\begin{aligned} f_{Y|Z}(y|z=1) &= \frac{d}{dy}\left[2F(y)\int_{-\infty}^{y} f(t)\,dt\right] - \frac{d}{dy}\left[2\int_{-\infty}^{y} f(t)F(t)\,dt\right] \\ &= 2\left(f(y)\int_{-\infty}^{y} f(t)\,dt + F(y)f(y)\right) - 2f(y)F(y) \\ &= 2f(y)\int_{-\infty}^{y} f(t)\,dt. \end{aligned}$$

In the case of $Z=0$, when $S \le T$, the region becomes



and the probability is expressed as

$$P(Y \le y|Z=0) = \frac{P(S \le y, T \le y, S \le T)}{P(S \le T)} = 2\int_{-\infty}^{y}\int_{s}^{y} f(t)f(s)\,dt\,ds.$$

Note that this is identical to the double integral from above, so we conclude that the conditional density of $Y|Z$ is

$$f_{Y|Z}(y|z) = 2f(y)\int_{-\infty}^{y} f(t)\,dt.$$

Next, we compute the density of Y. Consider the probability

$$\begin{aligned} P(Y \le y) = P(\max(S,T) \le y) &= P(S \le y, T \le y) = P(S \le y)P(T \le y) \\ &= \left(\int_{-\infty}^{y} f(s)\,ds\right)\left(\int_{-\infty}^{y} f(t)\,dt\right) \\ &= \left(\int_{-\infty}^{y} f(k)\,dk\right)^2 \end{aligned}$$

where we were able to split the probabilities because S and T are independent. Then to compute the density we take the derivative with respect to y, which is

$$\begin{aligned} \frac{d}{dy}\left[\left(\int_{-\infty}^{y} f(k)\,dk\right)^2\right] &= 2\left(\int_{-\infty}^{y} f(k)\,dk\right)\cdot\frac{d}{dy}\left[\int_{-\infty}^{y} f(k)\,dk\right] \\ &= 2f(y)\left(\int_{-\infty}^{y} f(k)\,dk\right). \end{aligned}$$

Thus, $f_{Y|Z}(y|z) = f_Y(y)$, so Y and Z are independent, as desired.

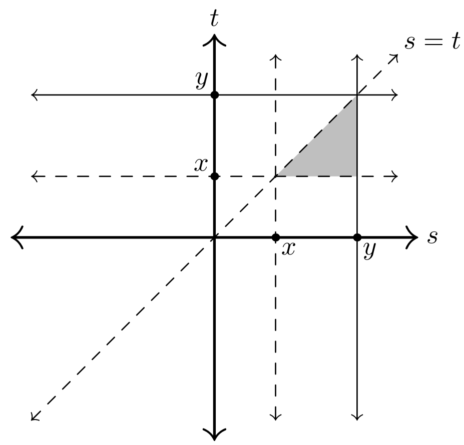
$\square$

iii. (X, Y) and Z.

Solution. We claim $\boxed{(X, Y) \text{ and } Z \text{ are independent.}}$ We have the relation $f_{A,B}(a, b) = \frac{\partial^2}{\partial a\, \partial b} F_{A,B}(a, b)$ where F is the joint CDF of A and B, for any random variables A and B. Consider the probability

$$\begin{aligned}
P(X \le x, Y \le y \mid Z) &= 1 - P(X > x, Y \le y \mid Z) \\
&= 1 - P(\min(S, T) > x, \max(S, T) \le y \mid Z) \\
&= 1 - P(S > x, T > x, S \le y, T \le y \mid Z) \\
&= 1 - \frac{P(S > x, T > x, S \le y, T \le y, Z)}{P(Z)}
\end{aligned}$$

whose mixed partial derivatives give the joint density.
If $Z = 1$, then $S > T$, and the region in the numerator is shown below:



Since, $P(Z = 1) = 1/2$, the probability can then be expressed as the double integral

$$1 - 2\int_x^y \int_x^s f(t) f(s)\, dt\, ds.$$

Let

$$g(s) = \int_x^s f(t) f(s)\, dt$$

then we can write the double integral as

$$1 - 2\int_x^y g(s)\, ds.$$

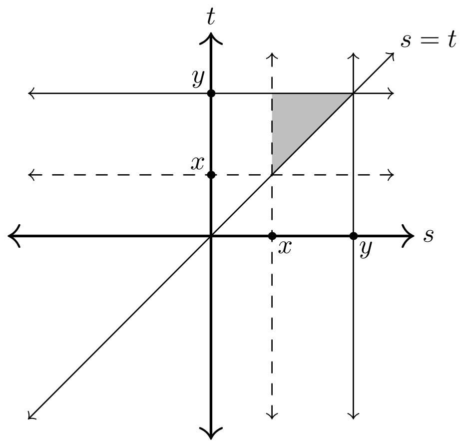
Taking the partial derivative with respect to y, we get

$$\begin{aligned}
\frac{\partial}{\partial y}\left[1 - 2\int_x^y g(s)\, ds\right] &= -2g(y) \\
&= -2\int_x^y f(t) f(y)\, dt \\
&= 2f(y)\int_y^x f(t)\, dt
\end{aligned}$$

then taking the partial with respect to x, we get

$$\frac{\partial}{\partial x}\left[2f(y)\int_y^x f(t)\, dt\right] = 2f(y)f(x) = f_{(X,Y)|Z}(x, y|z = 1).$$

Next, in the case when $Z = 0$, when $S \leq T$, the region looks like as below:



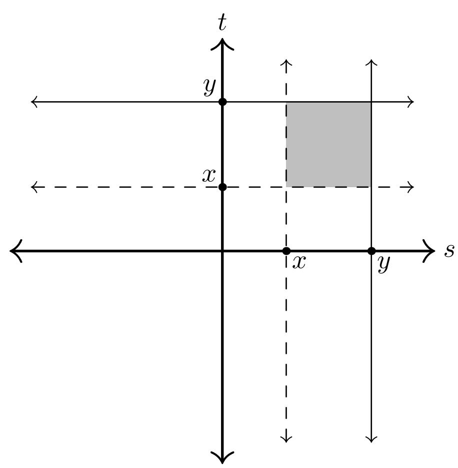
If we simply switch s and t in the integral limits, (as in the parts before this), we will see that this is exactly the same as the case when $Z = 1$. Thus the conditional distribution is

$$f_{(X,Y)|Z}(x, y|z) = 2f(x)f(y).$$

Now, we want to compute the joint distribution of X, Y. Consider again the joint CDF

$$P(X \leq x, Y \leq y) = 1 - P(X > x, Y \leq y) = 1 - P(S > x, T > x, S \leq y, T \leq y).$$

The region described by this is shown below:



and the probability is given by the double integral

$$\begin{aligned} 1 - \int_x^y \int_x^y f(s)f(t)\,dt\,ds &= 1 - \left(\int_x^y f(s)\,ds\right)\left(\int_x^y f(t)\,dt\right) \\ &= 1 - \left(\int_x^y f(k)\,dk\right)^2 \\ &= 1 - \left(\int_{-\infty}^y f(k)\,dk - \int_{-\infty}^x f(k)\,dk\right)^2 \end{aligned}$$

Now, take the mixed partial derivatives to determine the joint density:

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[1-\left(\int_{-\infty}^{y} f(k)\,dk - \int_{-\infty}^{x} f(k)\,dk\right)^2\right]\right] \\
&= \frac{\partial}{\partial y}\left[-2\left(\int_{x}^{y} f(k)\,dk\right)\cdot\frac{\partial}{\partial x}\left[\left(\int_{-\infty}^{y} f(k)\,dk - \int_{-\infty}^{x} f(k)\,dk\right)\right]\right] \\
&= \frac{\partial}{\partial y}\left[-2\left(\int_{x}^{y} f(k)\,dk\right)(-f(x))\right] \\
&= \frac{\partial}{\partial y}\left[2f(x)\left(\int_{-\infty}^{y} f(k)\,dk - \int_{-\infty}^{x} f(k)\,dk\right)\right] \\
&= 2f(x)\cdot\frac{\partial}{\partial y}\left[\left(\int_{-\infty}^{y} f(k)\,dk - \int_{-\infty}^{x} f(k)\,dk\right)\right] \\
&= 2f(x)f(y)
\end{aligned}
$$

Thus, $f_{(X,Y)|Z}(x,y|z) = f_{X,Y}(x,y)$, so (X,Y) and Z are independent, as desired. □

(c) How can these conclusions be extended to the order statistics of three or more random variables with common density f?

Solution. With more variables, the region that satisfies the constraints of the set in question only occupies a higher dimensional space $\mathbb{R}^n$. We can conclude that the minimum and maximum of any set of random variables is independent from their relative order. □

5. Suppose $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d random variables drawn from a common distribution F_θ, where $\theta \in \mathbb{R}$ is some unknown parameter. We say that the random variable $\hat{\theta}_n$ is an *estimator* for θ if $\hat{\theta}_n$ is a function of $\{X_1, \cdots, X_n\}$. We define the *bias* B of an estimator $\hat{\theta}$ by

$$B(\hat{\theta}) = E[\hat{\theta} - \theta].$$

The *mean square error MSE* of an estimator is defined by

$$MSE = E\left[(\hat{\theta} - \theta)^2\right].$$

(a) Show that $MSE(\hat{\theta}) = var(\hat{\theta}) + B^2$.

Proof. We have

$$\begin{aligned}
MSE(\hat{\theta}) &= E\left[(\hat{\theta} - \theta)^2\right] \\
&= E[\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2] \\
&= E[\hat{\theta}^2] - E[2\hat{\theta}\theta] + E[\theta^2] \\
&= E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2 \\
&= \left(E[\hat{\theta}^2] - (E[\hat{\theta}])^2\right) + \left((E[\hat{\theta}])^2 - 2\theta E[\hat{\theta}] + \theta^2\right) \\
&= var(\hat{\theta}) + (E[\hat{\theta}] - \theta)^2 \\
&= var(\hat{\theta}) + (E[\hat{\theta} - \theta])^2 \\
&= var(\hat{\theta}) + B^2
\end{aligned}$$

as desired. □

(b) We say that an estimator is *unbiased* if its bias is zero. Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are two unbiased estimators for θ, with variances σ_1^2 and σ_2^2, respectively. Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. Define a new estimator $\hat{\theta}_3$ by

$$\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2.$$

Determine the value of a that minimizes the variance of $\hat{\theta}_3$.

Solution. We have

$$\begin{aligned}
var(\hat{\theta}_3) &= var(a\hat{\theta}_1 + (1-a)\hat{\theta}_2) \\
&= var(a\hat{\theta}_1) + var((1-a)\hat{\theta}_2) \\
&= a^2\sigma_1^2 + (1-a)^2\sigma_2^2 \\
&= (a^2\sigma_1^2 + (1-a)^2\sigma_2^2 - 2a(1-a)\sigma_1\sigma_2) + 2a(1-a)\sigma_1\sigma_2 \\
&= (a\sigma_1 - (1-a)\sigma_2)^2 + 2a(1-a)\sigma_1\sigma_2 \\
&\geq 2a(1-a)\sigma_1\sigma_2
\end{aligned}$$

where the first step is due to the independence of $\hat{\theta}_1$ and $\hat{\theta}_2$, and the last step is due to the Trivial Inequality. Equality (and therefore the minimum) is achieved when

$$\begin{aligned}
a\sigma_1 &= (1-a)\sigma_2 \\
\implies a &= \boxed{\frac{\sigma_2}{\sigma_1 + \sigma_2}}
\end{aligned}$$

□

(c) Suppose now that $\hat{\theta}_1$ and $\hat{\theta}_2$ are not independent. Assuming that the covariance of $\hat{\theta}_1$ and $\hat{\theta}_2$ is given by $cov(\hat{\theta}_1, \hat{\theta}_2) = c$, how should a be chosen to minimize the variance of $\hat{\theta}_3$?

Solution. If $\hat{\theta}_1$ and $\hat{\theta}_2$ are not independent, then $var(\hat{\theta}_3)$ is instead given by

$$\begin{aligned}
var(\hat{\theta}_3) &= var(a\hat{\theta}_1 + (1-a)\hat{\theta}_2) \\
&= var(a\hat{\theta}_1) + var((1-a)\hat{\theta}_2) + 2cov(a\hat{\theta}_1, (1-a)\hat{\theta}_2) \\
&= a^2\sigma_1^2 + (1-a)^2\sigma_2^2 + 2a(1-a)cov(\hat{\theta}_1, \hat{\theta}_2) \\
&= (a\sigma_1 - (1-a)\sigma_2)^2 + 2a(1-a)\sigma_1\sigma_2 + 2a(1-a)c \\
&\geq 2a(1-a)(c + \sigma_1\sigma_2)
\end{aligned}$$

Thus we see that the strategy for picking a does not change, so $a = \boxed{\dfrac{\sigma_2}{\sigma_1 + \sigma_2}}$ is the optimal value to minimize the variance of $\hat{\theta}_3$. □

6. Suppose $X_1, X_2, \cdots, X_n$ are i.i.d random variables with common mean μ and common variance σ^2. Recall that the sample mean $\bar{X}$ of these random variables is simply

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the sample variance s^2 is defined by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

(a) Show that s^2 is an unbiased estimator for σ^2.

Proof. If $E[s^2] = \sigma^2$ then s^2 is an unbiased estimator for σ^2. We have

$$\begin{aligned} E[s^2] &= E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \\ &= \frac{1}{n-1}\sum_{i=1}^{n} E[(X_i - \bar{X})^2] \\ &= \frac{1}{n-1}\sum_{i=1}^{n} E[X_i^2 - 2X_1\bar{X} + \bar{X}^2] \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n} E[X_i^2] - 2\sum_{i=1}^{n} E[\bar{X}X_i] + \sum_{i=1}^{n} E[\bar{X}^2]\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n}(\sigma^2 + \mu^2) - 2\sum_{i=1}^{n} E[\bar{X}X_i] + \sum_{i=1}^{n}\left(var(\bar{X}) + (E[\bar{X}])^2\right)\right), \end{aligned}$$

where

$$var(\bar{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} var(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

and

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\cdot n\mu = \mu.$$

We also have

$$\begin{aligned} E[\bar{X}X_i] &= E\left[\frac{X_i}{n}\sum_{j=1}^{n} X_j\right] = \frac{1}{n}E\left[\sum_{j=1}^{n} X_iX_j\right] \\ &= \frac{1}{n}\sum_{j=1}^{n} E[X_iX_j]. \end{aligned}$$

and since X_i is independent from X_j whenever $i \neq j$, this expression can be written as

$$\begin{aligned} \frac{1}{n}\left(E[X_i^2] + \sum_{j=1, j\neq i}^{n}(E[X_i]E[X_j])\right) &= \frac{1}{n}\left([\mu^2 + \sigma^2] + \sum_{j=1, j\neq i}^{n}\mu^2\right) \\ &= \mu^2 + \frac{\sigma^2}{n}. \end{aligned}$$

Substituting everything into our original expression, we get

$$\begin{aligned}
E[s^2] &= \frac{1}{n-1}\left(\sum_{i=1}^{n}(\sigma^2+\mu^2) - 2\sum_{i=1}^{n}\left(\mu^2+\frac{\sigma^2}{n}\right) + \sum_{i=1}^{n}\left(\frac{\sigma^2}{n}+\mu^2\right)\right) \\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n}(\sigma^2+\mu^2) - \sum_{i=1}^{n}\left(\mu^2+\frac{\sigma^2}{n}\right)\right] \\
&= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\mu^2+\frac{\sigma^2}{n}\right)\right] \\
&= \frac{1}{n-1}(n\sigma^2+n\mu^2-n\mu^2-\sigma^2) \\
&= \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2
\end{aligned}$$

thus s^2 is an unbiased estimator for σ^2, as desired. □

(b) Show that $n^{-1}s^2$ is an unbiased estimator for the variance $\sigma^2_{\bar{X}}$ of $\bar{X}$.

Proof. If $E\left[\frac{1}{n}s^2\right] = \sigma^2_{\bar{X}}$ then $n^{-1}s^2$ is an unbiased estimator for $\sigma^2_{\bar{X}}$. This equation is equivalent to $E[s^2] = n\sigma^2_{\bar{X}}$, and we know $E[s^2] = \sigma^2$, so it remains to show that $\sigma^2 = n\sigma^2_{\bar{X}}$. However, we already showed (above) that $var(\bar{X}) = \frac{\sigma^2}{n}$, so $n \cdot var(\bar{X}) = \sigma^2$, as desired. □

(c) Is s always an unbiased estimate for σ? Justify.

Proof. We claim that s is $\boxed{\text{not always an unbiased estimate for } \sigma.}$ We proceed by noting that $var(s) = E[s^2] - (E[s])^2$, so as long as $var(s) > 0$,

$$\begin{aligned}
E[s^2] - (E[s])^2 &> 0 \\
\implies E[s^2] &> (E[s])^2 \\
\implies \sigma^2 &> (E[s])^2 \\
\implies \sigma &> E[s]
\end{aligned}$$

thus $E[s] \neq \sigma$, which is the requirement for s being an unbiased estimator of σ, completing the proof. □

7. The Pareto (a, b) distribution has CDF

$$F(x) = 1 - (b/x)^a.$$

Determine the inverse probability transformation $F^{-1}(U)$ and use the inverse transform method to simulate, in R, a random sample from the Pareto (2, 2) distribution. Vary the sample size, in particular generating samples of size $n = 10, 50, 100$. For each of these samples, graph the density histogram of the sample with the Pareto (2, 2) density superimposed for comparison.

Solution. To determine the inverse probability transformation, we must find the smallest value x such that $F_X(x) = U$. The Pareto distribution has a CDF

$$F_X(x) = 1 - (b/x)^a,$$

so we seek to find the smallest value of x such that $1 - (b/x)^a = U$. Solving for x, this is

$$\begin{aligned} 1 - \left(\frac{b}{x}\right)^a &= U \\ (1-U)^{1/a} &= \frac{b}{x} \\ b(1-U)^{-1/a} &= x \end{aligned}$$

thus $F^{-1}(U) = \boxed{b(1-U)^{-1/a}.}$

In the case of the Pareto (2, 2) distribution, its inverse probability transform is $F^{-1}(U) = 2(1-U)^{-1/2}$. We are also interested in the density of the distribution, which is the derivative of the CDF with respect to x. This is

$$\begin{aligned} \frac{d}{dx}\left[1 - \left(\frac{b}{x}\right)^a\right] &= \frac{d}{dx}\left[1 - b^a x^{-a}\right] \\ &= ab^a x^{-a-1} \end{aligned}$$

so in the case of the Pareto (2, 2) distribution, its density is $f_X(x) = 2 \cdot 2^2 x^{-2-1} = 8x^{-3}$. Below are three histograms of the Pareto (2, 2) distribution, sampled 10, 50, and 100 times.

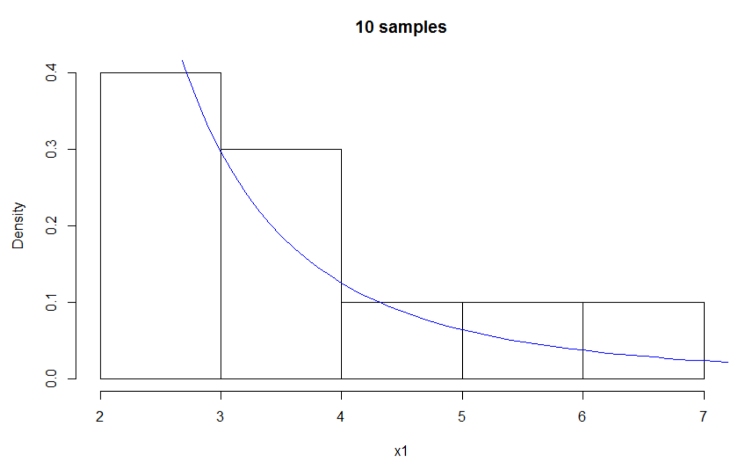


Figure 1: Pareto (2, 2) distribution, sampled 10 times

$\square$

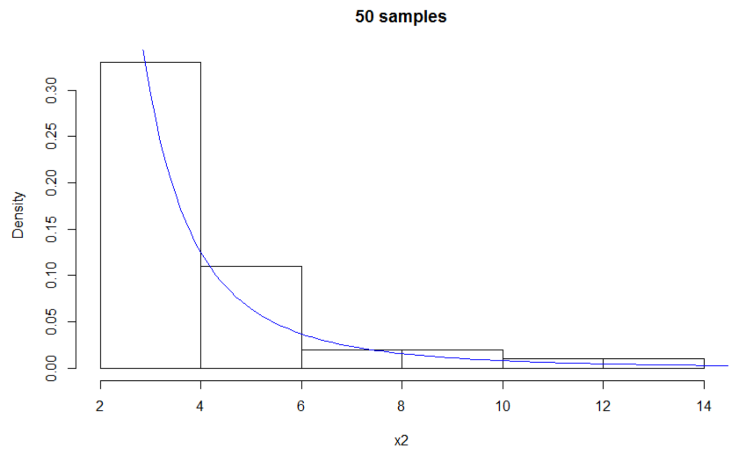


Figure 2: Pareto (2, 2) distribution, sampled 50 times

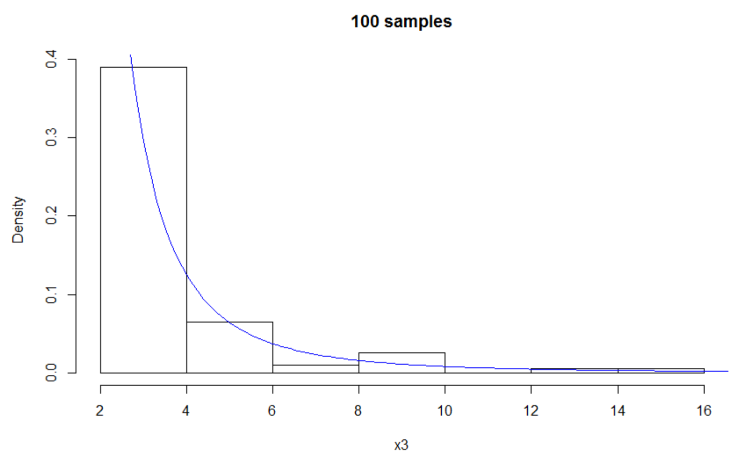


Figure 3: Pareto (2, 2) distribution, sampled 100 times