Homework 3

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1. Consider the linear program (LP) min $c^T x$ such that $Ax = b, x \geq 0$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad c = \begin{bmatrix} 3 & -4 & 5 \end{bmatrix}$$

a) Find all basic feasible solutions. (There are three possibilities, two of which are basic feasible solutions and one isn't. Be clear why the third possibility fails to be a BFS.)

Solution. Suppose $x_3 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \implies B^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$
$$x_B = B^{-1}b = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$
$$x = \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \end{bmatrix}$$

is a BFS.

Now suppose $x_2 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \implies B^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$
$$x_B = B^{-1}b = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is a BFS.

Now suppose $x_1 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$x_B = B^{-1}b = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$$

and this is not a BFS because $x_B \ngeq 0$.

□

b) Evaluate r_N for each of the two basic feasible solutions.

Solution. We have

$$r_N^T = c_N^T - c_B^T B^{-1} N$$

as the expression for r_n^T.

For $x_B = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$, we have $c_N^T = [5]$ and $c_B^T = \begin{bmatrix} 3 & -4 \end{bmatrix}$ and $N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$$\begin{aligned} r_N^T &= [5] - \begin{bmatrix} 3 & -4 \end{bmatrix} \left(-\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [5] - \begin{bmatrix} 3 & -4 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \\ &= [5] - [2/3] = [13/3] \end{aligned}$$

For $x_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $c_N^T = [-4]$ and $c_B^T = \begin{bmatrix} 3 & 5 \end{bmatrix}$ and $N = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so

$$\begin{aligned} r_N^T &= [-4] - \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= [-4] - \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ &= [-4] - [9] = [-13] \end{aligned}$$

□

c) Identify the optimal solution to the (LP) and explain why it is the optimal solution.

Solution. Since we have

$$c^T x = c_B^T x_B + r_N^T x_N,$$

the optimal solution corresponds with the r_N^T value that is non-negative, since any increases to x_N will increase the objective function value. This occurs when $r_N^T = [13/3]$, and the solution that corresponds with that is

$$\boxed{x = \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \end{bmatrix}}$$

□

2. Consider the linear program (LP) min $c^T x$ such that $Ax = b, x \geq 0$ where

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

a) Compute the BFS that comes from using the second, third, and fourth columns of A as the basis, and compute the associated vector r_N.

Solution. If we use the second, third, and fourth columns as a basis, we have

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the inverse B^{-1} was calculated with MATLAB.
Now, we have

$$c_N^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad c_B^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so the vector r_N^T is given by

$$\begin{aligned} r_N^T &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left(\frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \right) \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 7 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

□

b) By examining the vector r_N from part a, explain how changing the different nonbasic variables will affect the objective function.

Solution. A unit increase of x_1 will decrease the objective function value by 4 units (and increase by 4 if x_1 is decreased by 1 unit). Changing the other non-basic variables x_5, x_6, x_7 will not affect the objective function value.

□

c) Now compute the BFS that comes from using the first three columns of A as the basis, and compute the associated vector r_N.

Solution. Now, we have

$$B = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix} \implies B^{-1} = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$x_B = B^{-1}b = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix}$$

where the inverse B^{-1} was calculated with MATLAB.
Now, we have

$$c_N^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad c_B^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so the vector r_N^T is given by

$$\begin{aligned}
r_N^T &= \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 1 & 1\end{bmatrix}\left(\frac{1}{28}\begin{bmatrix}6 & -1 & -1\\ -1 & 6 & -1\\ -1 & -1 & 6\end{bmatrix}\right)\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 1 & 1\end{bmatrix}\left(\frac{1}{7}\right)\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} - \frac{3}{7}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\\
&= \begin{bmatrix}4/7 & 4/7 & 4/7 & 4/7\end{bmatrix}
\end{aligned}$$

□

d) Find the optimal solution for (LP).

Solution. Since r_N^T corresponding to the basis being the first 3 columns is entirely non-negative, the optimal solution occurs when

$$x_B = \begin{bmatrix}1/7\\ 1/7\\ 1/7\end{bmatrix} \implies x = \begin{bmatrix}1/7\\ 1/7\\ 1/7\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

and the objective function value is 3/7.

□

3. Suppose you are handed a linear program which is given to you in the form min $c^T x$ such that $Ax \leq b, x \geq 0$, and suppose that the vector b happens to be non-negative. This linear program is currently not in standard form; explain how to then immediately identify a BFS once it is converted to standard form. Then explain what r_N will be for this BFS.

Solution. We add a dummy variable z, so that

$$[A \,|\, I]\begin{bmatrix}x\\ z\end{bmatrix} = b$$

where I is the identity matrix. Then if we simply let I be the basis, we have $Iz = b$ so then $\begin{bmatrix}\vec{0}\\ z\end{bmatrix}$ will be a BFS.

Then the objective function will be $\begin{bmatrix}c^T\\ \vec{0}\end{bmatrix}^T \begin{bmatrix}x\\ z\end{bmatrix}$. Let the combined vector $\begin{bmatrix}c^T\\ \vec{0}\end{bmatrix} = d^T$. Here,

$$B = I \implies B^{-1} = I,$$

and we have

$$d_N^T = c^T, \quad d_B^T = \vec{0}, \quad N = A$$

so r_N^T is given by

$$\begin{aligned}
r_N^T &= d_N^T - d_B^T B^{-1} N\\
&= c^T - \vec{0} I A\\
&= c^T.
\end{aligned}$$

□