Homework 5

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Chapter 15: Lebesgue Measure

25. Suppose that $m^*(E) > 0$. Given $0 < \alpha < 1$, show that there exists an open interval I such that $m^*(E \cap I) > \alpha m^*(I)$. (Hint: It is enough to consider the case $m^*(E) < \infty$. Now suppose that the conclusion fails.)

    Proof. Consider the case when $m^*(E) < \infty$. Fix α and $\varepsilon > 0$, and suppose that $m^*(E \cap I) \leq \alpha m^*(I)$ for any open interval I. Then there exists an open set $G \supset E$ such that $m^*(G) < m^*(E) + \varepsilon$. Then $G = \bigcup_{n=1}^\infty I_n$ for disjoint, open intervals I_n, and we have $E = \bigcup_{n=1}^\infty E \cap I_n$. Now,

$$m^*(E) = m^*\left(\bigcup_{n=1}^\infty (E \cap I_n)\right) \leq \sum_{n=1}^\infty m^*(E \cap I_n) \leq \sum_{n=1}^\infty \alpha m^*(I_n) = \alpha \sum_{n=1}^\infty m^*(I_n) < \sum_{n=1}^\infty m^*(I_n) = m^*(G)$$

    This is a contradiction, so there must exist an open interval I such that $m^*(E \cap I) > \alpha m^*(I)$. □

28. Fix α with $0 < \alpha < 1$ and repeat our "middle thirds" construction for the Cantor set except that now, at the nth stage, each of the 2^{n-1} open intervals we discard from $[0, 1]$ is to have length $(1-\alpha)3^{-n}$. Check that $m^*(\Delta_\alpha) = \alpha$. (Hint: You only need upper estimates for $m^*(\Delta_\alpha)$ and $m^*(\Delta_\alpha^c)$.)

    Proof. For Δ_α^c, this is the union of all the middle third intervals, which are all disjoint, bounded, open intervals, and thus Δ_α^c is measurable with

$$m^*(\Delta_\alpha^c) = m(\Delta_\alpha^c) = \sum_{n=1}^\infty 2^{n-1}(1-\alpha)3^{-n} = \frac{1-\alpha}{2}\sum_{n=1}^\infty \left(\frac{2}{3}\right)^n = \frac{1-\alpha}{2} \cdot \frac{2/3}{1-\frac{2}{3}} = 1-\alpha$$

    so $m^*(\Delta_\alpha) = 1 - (1-\alpha) = \alpha$. □

38. Prove that E is measurable if and only if $E \cap K$ is measurable for every compact set K.

    Proof. $(\implies)$: Since compact sets are measurable, and intersections of measurable sets are measurable, it follows that $E \cap K$ is measurable.

    $(\impliedby)$: Let $E_n = E \cap [-n, n]$. Then since $[-n, n]$ is compact, E_n is measurable for all n, and $E = \bigcup_{n=1}^\infty E_n$, so E is measurable. □

40. If A and B are measurable sets, show that $m(A \cup B) + m(A \cap B) = m(A) + m(B)$.

    Proof. Define the sets

$$\begin{aligned} E_1 &= A \cap B \\ E_2 &= A \setminus (A \cap B) \\ E_3 &= B \setminus (A \cap B) \end{aligned}$$

Then E_1, E_2, E_3 are pairwise disjoint sets, and we have

$$\begin{aligned} E_1 \cup E_2 &= A \\ E_1 \cup E_3 &= B \\ E_1 \cup E_2 \cup E_3 &= A \cup B \\ \implies m(A \cup B) + m(A \cap B) &= m(E_1 \cup E_2 \cup E_3) + m(E_1) \\ &= m(E_1) + m(E_2) + m(E_3) + m(E_1) \\ &= m(E_1 \cup E_2) + m(E_1 \cup E_3) = m(A) + m(B) \end{aligned}$$

as desired. □

45. Let $f : X \to Y$ be any function.

(a) If $\mathcal{B}$ is a σ-algebra of subsets of Y, show that $\mathcal{A} = \left\{f^{-1}(B) : B \in \mathcal{B}\right\}$ is a σ-algebra of subsets of X.

Proof. We have $\varnothing \in \mathcal{A}$ since $\varnothing \in \mathcal{B}$ and $\varnothing = f^{-1}(\varnothing)$. Let $A \in \mathcal{A}$, so $A = f^{-1}(B) = \{a \in X : f(a) \in B\}$ for some $B \in \mathcal{B}$. Then

$$A^c = \{a \in X : f(a) \notin B\} = \{a \in X : f(a) \in B^c\}$$

Here $B^c \in \mathcal{B}$ since $\mathcal{B}$ is a σ-algebra, so $A^c \in \mathcal{A}$, so $\mathcal{A}$ is closed under complement.
Let (A_n) be a sequence of subsets of X in $\mathcal{A}$. Then let (B_n) be a sequence of subsets in $\mathcal{B}$ such that $A_i = f^{-1}(B_i)$ for all i. We have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \{a \in X : f(a) \in B_n\} = \left\{a \in X : f(a) \in \bigcup_{n=1}^{\infty} B_n\right\}$$

and since $\mathcal{B}$ is closed under countable union, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$, so it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, so $\mathcal{A}$ is closed under countable union. Similarly,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \{a \in X : f(a) \in B_n\} = \left\{a \in X : f(a) \in \bigcap_{n=1}^{\infty} B_n\right\}$$

and since $\mathcal{B}$ is closed under countable intersection, it follows that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$, so $\mathcal{A}$ is closed under countable intersection. Thus, $\mathcal{A}$ is a σ-algebra of subsets of X. □

(b) If $\mathcal{A}$ is a σ-algebra of subsets of X, show that $\mathcal{B} = \left\{B : f^{-1}(B) \in \mathcal{A}\right\}$ is a σ-algebra of subsets of Y.

Proof. We have $\varnothing \in \mathcal{B}$ since $\varnothing \in \mathcal{A}$ and $f^{-1}(\varnothing) = \varnothing$. Let $B \in \mathcal{B}$, so $f^{-1}(B) = \{a \in X : f(a) \in B\} \in \mathcal{A}$. Then

$$f^{-1}(B^c) = \{a \in X : f(a) \in B^c\} = (\{a \in X : f(a) \in B\})^c$$

and since $\mathcal{A}$ is closed under complement, it follows that $f^{-1}(B^c) \in \mathcal{A}$, so $\mathcal{B}$ is closed under complement.
Let (B_n) be a sequence of subsets of Y in $\mathcal{B}$. Then it follows that $f^{-1}(B_i) \in \mathcal{A}$ for all i, so then

$$\bigcup_{n=1}^{\infty} f^{-1}(B_n) = \bigcup_{n=1}^{\infty} \{a \in X : f(a) \in B_n\} = \left\{a \in X : f(a) \in \bigcup_{n=1}^{\infty} B_n\right\} \in \mathcal{A}$$

so it follows that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$, so $\mathcal{B}$ is closed under countable unions. Similarly,

$$\bigcap_{n=1}^{\infty} f^{-1}(B_n) = \bigcap_{n=1}^{\infty} \{a \in X : f(a) \in B_n\} = \left\{a \in X : f(a) \in \bigcap_{n=1}^{\infty} B_n\right\} \in \mathcal{A}$$

so it follows that $\bigcap_{n=1}^{\infty} B_n \in \mathcal{B}$, so $\mathcal{B}$ is closed under countable intersections Thus, $\mathcal{B}$ is a σ-algebra of subsets of Y. $\square$

51. Let $\mathcal{A} = \{E \subset \mathbb{R} : \text{either } E \text{ or } E^c \text{ is finite}\}$. Is $\mathcal{A}$ an algebra? Is $\mathcal{A}$ a σ-algebra? Explain.

Solution. Clearly $\varnothing \in \mathcal{A}$ since it is empty and thus finite. Then if $E \in \mathcal{A}$, either E or E^c is finite, so $E^c \in \mathcal{A}$ as well. If $E_1, \cdots, E_N \in \mathcal{A}$, then either E_n or E_n^c is finite for all $1 \leq n \leq N$. If all the E_n are finite, then clearly $\bigcup_{n=1}^{N} E_n$ is finite. Otherwise, if E_k^c is finite for some k, then

$$\left(\bigcup_{n=1}^{N} E_n\right)^c = \bigcap_{n=1}^{N} E_n^c \subset E_k^c$$

which is finite. Thus, $\mathcal{A}$ is closed under finite union. Similarly, if E_k is finite for some k, then $\bigcap_{n=1}^{N} E_n \subset E_k$ which is finite. Otherwise, E_n^c is finite for all n, wo

$$\left(\bigcap_{n=1}^{N} E_n\right)^c = \bigcup_{n=1}^{N} E_n^c$$

is finite. Thus, $\mathcal{A}$ is closed under finite intersections.

If $E_n = \{n\}$, then each of the E_n is finite, but $\bigcup n = 1^{\infty} E_n = \mathbb{N}$ is not finite, and neither is $\mathbb{R} \setminus \mathbb{N}$. Thus, $\mathcal{A}$ is not closed under countable union, so it is not a σ-algebra. $\square$

61. Find a sequence of measurable sets (E_n) that decrease to $\varnothing$, but with $m(E_n) = \infty$ for all n.

Solution. Let $E_n = \bigcup_{k=1}^{\infty} \left(k, k + \frac{1}{n}\right)$. Then as $n \to \infty$, each of these sets goes to $\varnothing$, but $m(E_n) = \infty$ for all finite n. $\square$

66. In the notation of Exercise 65, define $d(E, F) = m(E\Delta F)$ for $E, F \in \mathcal{M}_1$. Prove that d defines a pseudometric on $\mathcal{M}_1$. (That is, d induces a metric on $\mathcal{M}_1/\sim$, the set of equivalence classes under equality a.e.)

Proof. Since measure is non-negative, we have $d(E, F) \geq 0$. Then

$$\begin{aligned} d(E, E) &= m(E\Delta E) = m(\varnothing) = 0 \\ d(E, F) &= m(E\Delta F) = m(F\Delta E) = d(F, E) \\ d(E, F) + d(F, G) &= m(E\Delta F) + m(F\Delta G) \geq m\left[(E\Delta F) \cup (F\Delta G)\right] \end{aligned}$$

Now, it is relatively easy to show $(E\Delta F) \cup (F\Delta G) = (E \cup F \cup G) \setminus (E \cap F \cap G)$. Then

$$\begin{aligned} & G\Delta E = (G \setminus E) \cup (E \setminus G) \subset (E \cup F \cup G) \setminus (E \cap F \cap G) \\ \implies & d(G, E) = m(G\Delta E) \leq m\left[(E \cup F \cup G) \setminus (E \cap F \cap G)\right] \\ & \qquad \leq m(E\Delta F) + m(F\Delta G) = d(E, F) + d(F, G) \end{aligned}$$

Thus, d defines a pseudometric on $\mathcal{M}_1$. $\square$