Homework 9

Aleck Zhao

April 12, 2018

Chapter 18: The Lebesgue Integral

4. Find a sequence (f_n) of non-negative measurable functions such that $\lim_{n\to\infty} f_n = 0$, but $\lim_{n\to\infty} \int f_n = 1$. In fact, show that (f_n) can be chosen to converge uniformly to 0.

   Solution. Let $f_n = \chi_{[n,n+1]}$. Then $\lim_{n\to\infty} f_n = 0$, but $\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} 1 = 1$. □

6. Suppose that f and (f_n) are non-negative measurable functions, that (f_n) decreases pointwise to f, and that $\int f_k < \infty$ for some k. Prove that $\int f = \lim_{n\to\infty} \int f_n$. (Hint: Consider $(f_k - f_n)$ for $n > k$.) Give an example showing that this fails without the assumption that $\int f_k < \infty$ for some k.

   Proof. Let $g_n = f_k - f_n$ for $n > k$. Then since (f_n) is decreasing, we have $0 \le g_n \le g_{n+1} \le f_k - f$, so by the monotone convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int g_n &= \int \lim_{n\to\infty} g_n \\ \implies \lim_{n\to\infty} \int (f_k - f_n) = \int f_k - \lim_{n\to\infty} \int f_n &= \int \lim_{n\to\infty} (f_k - f_n) = \int (f_k - f) = \int f_k - \int f \\ \implies \lim_{n\to\infty} \int f_n &= \int f \end{aligned}$$

   If we take $f_n = \frac{1}{x+n} \cdot \chi_{[1,\infty)}$, then (f_n) decreases pointwise to 0, but $\int f_n = \infty$ for all n. □

10. If f is non-negative and measurable, show that $\int_{-\infty}^{\infty} f = \lim_{n\to\infty} \int_{-n}^{n} f = \lim_{n\to\infty} \int_{\{f \ge (1/n)\}} f$.

   Proof. If f is non-negative and measurable, then if $g_n = f \cdot \chi_{[-n,n]}$, we have $0 \le g_n \le g_{n+1} \le f$, so by the monotone convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int g_n &= \int \lim_{n\to\infty} g_n \\ \implies \lim_{n\to\infty} \int f \cdot \chi_{[-n,n]} = \lim_{n\to\infty} \int_{-n}^{n} f &= \int \lim_{n\to\infty} f \cdot \chi_{[-n,n]} = \int f \cdot \chi_{\mathbb{R}} = \int_{-\infty}^{\infty} f \end{aligned}$$

   which establishes the first equality. Similarly, we can let $h_n = f \cdot \chi_{\{f \ge (1/n)\}}$, so $0 \le h_n \le h_{n+1} \le f$, so by the monotone convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int h_n &= \int \lim_{n\to\infty} h_n \\ \implies \lim_{n\to\infty} \int f \cdot \chi_{\{f \ge (1/n)\}} = \lim_{n\to\infty} \int_{\{f \ge (1/n)\}} f &= \int \lim_{n\to\infty} f \cdot \chi_{\{f \ge (1/n)\}} = \int f \cdot \chi_{\{f \ge 0\}} \end{aligned}$$

   and since f is non-negative, we have $\chi_{\{f \ge 0\}} = \chi_{\mathbb{R}}$, and the second equality follows. □

15. Let f be non-negative and measurable. Prove that $\int f < \infty$ if and only if $\sum_{k=-\infty}^{\infty} 2^k m\left\{f > 2^k\right\} < \infty$.

Proof. Let $E_n := \{f > 2^n\}$ and let $F_n := \left\{1 \geq f > 2^{-n+1}\right\}$ for $n \geq 0$. Then we have

$$\bigcup_{n=0}^{\infty} E_n \setminus E_{n+1} = \{f > 1\}$$
$$\bigcup_{n=0}^{\infty} F_n \setminus F_{n+1} = \{1 \geq f \geq 0\}$$

where the sets $E_n \setminus E_{n+1}$ are pairwise disjoint, and likewise for $F_n \setminus F_{n+1}$. Then let

$$g = \sum_{n=0}^{\infty} 2^{n+1}\chi_{E_n\setminus E_{n+1}} + \sum_{n=0}^{\infty} 2^{-n+1}\chi_{F_n\setminus F_{n+1}}$$

□

23. If (f_n) is a sequence of Lebesgue integrable functions on $[a,b]$, and $f_n \implies f$ on $[a,b]$, prove that f is integrable and that $\int_a^b |f_n - f| \to 0$.

Proof. Let $\varepsilon > 0$. Then there exists N such that for all $x \in [a,b]$, we have $|f_n(x) - f(x)| < \varepsilon/(b-a)$ whenever $n \geq N$. Thus, taking n sufficiently large, we have

$$\int_a^b |f| \leq \int_a^b |f - f_n| + \int_a^b |f_n| < \int_a^b \frac{\varepsilon}{b-a} + \int_a^b |f_n| = \varepsilon + \int_a^b |f_n|$$

and since f_n is Lebesgue integrable, it follows that $|f_n|$ is LI, and thus $|f|$ is as well, so f is LI. We have

$$\int_a^b |f_n - f| < \int_a^b \frac{\varepsilon}{b-a} = \varepsilon$$

so $\int_a^b |f_n - f| \to 0$. □

24. Prove that $\int_0^\infty e^{-x}\,dx = \lim_{n\to\infty}\int_0^n (1-(x/n))^n\,dx = 1$. (Hint, for x fixed, $(1-(x/n))^n$ increases to e^{-x} as $n \to \infty$.)

Proof. Let $f_n(x) = (1-(x/n))^n \cdot \chi_{[0,n]}(x)$ Then $0 \leq f_n(x) \leq f_{n+1}(x) \leq e^{-x} \cdot \chi_{[0,\infty)}(x)$, so by the monotone convergence theorem, we have

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n$$
$$\implies \lim_{n\to\infty}\int \left(1-\frac{x}{n}\right)^n \cdot \chi_{[0,n]} = \lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n = \int \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n \cdot \chi_{[0,n]} = \int e^{-x}\cdot\chi_{[0,\infty)} = \int_0^\infty e^{-x}$$

We can evaluate this integral as

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n = \lim_{n\to\infty}\left[-\frac{n}{n+1}\left(1-\frac{x}{n}\right)^{n+1}\right]\Bigg|_0^n = 1$$

□

28. Suppose that f, g, and h are measurable and that $f \leq g \leq h$ a.e. If f and h are Lebesgue integrable, does it follow that g is Lebesgue integrable? Explain.

Solution. Yes, since

$$f \leq g \leq h \implies |g| \leq |f| + |h|$$

and since both $|f|$ and $|h|$ are integrable because f and h are, it follows that $|g|$ is as well. $\square$

37. Check that the operations $a[f] = [af]$ for $a \in \mathbb{R}, [f] + [g] = [f + g]$, and $[f] \leq [g]$ whenever $f \leq g$ a.e. are well defined, and that the collection of equivalence classes is a vector lattice when supplied with this arithmetic. What is $|[f]|$ in this lattice? Is it $[|f|]$?

Proof. We have

$$a[f] = a\{g : f \sim g\} = \{ag : f \sim g\}$$

so if $ag \in a[f]$, it follows that $ag \sim af$ since $g \sim f$, so $ag \in [af]$, and likewise if $h \in [af]$, we have $h \sim af \implies h \in a[f]$, so equality is well defined.

Then we have

$$[f] + [g] = \{h + j : h \sim f, j \sim g\}$$

so if $h + j \in [f] + [g]$, it follows that $h + j \sim f + g$ so $h + j \in [f + g]$, and likewise if $k \in [f + g]$, then $k \sim f + g$ where $f \sim f$ and $g \sim g$, so $k \in [f] + [g]$.

Then if $f \leq g$ a.e. it follows that if $h \sim f$ and $j \sim g$, we have $h \leq j$ a.e., so $[f] \leq [g]$. $\square$