Homework 1

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Chapter 13: Functions of Bounded Variation

1. Show that $V_a^b(\chi_{\mathbb{Q}}) = +\infty$ on any interval $[a, b]$.

3. If f has a bounded derivative on $[a, b]$, show that $V_a^b f \leq \|f'\|_\infty (b - a)$.

5. Complete the proof of Lemma 13.3.

6. We can test several of the inclusions explicit in our discussion up to this point by means of a single family of functions. For $\alpha \in \mathbb{R}$, and $\beta > 0$, set $f(x) = x^\alpha \sin(x^{-\beta})$, for $0 < x \leq 1$, and $f(0) = 0$. Show that .
   (a) f is bounded if and only if $\alpha \geq 0$
   (b) f is continuous if and only if $\alpha > 0$
   (c) $f'(0)$ exists if and only if $\alpha > 1$
   (d) f' is bounded if and only if $\alpha \geq 1 + \beta$
   (e) If $\alpha > 0$, then $f \in BV[0, 1]$ for $0 < \beta < \alpha$ and $f \notin BV[0, 1]$ for $\beta \geq \alpha$. (Hint: Try a few easy cases first, say $\alpha = \beta = 2$.)

11. If $f_n \to f$ pointwise on $[a, b]$, show that $V(f_n, P) \to V(f, P)$ for any partition P of $[a, b]$. In particular, if we also have $V_a^b f_n \leq K$ for all n, then $V_a^b f \leq K$ too.

14. Let $I(x) = 0$ if $x < 0$ and $I(x) = 1$ if $x \geq 0$. Given a sequence of scalars (c_n) with $\sum_{n=1}^\infty |c_n| < \infty$ and a sequence of distinct points (x_n) in $(a, b]$, define $f(x) = \sum_{n=1}^\infty c_n I(x - x_n)$ for $x \in [a, b]$. Show that $f \in BV[a, b]$ and that $V_a^b f = \sum_{n=1}^\infty |c_n|$.

15. Show that $f \in C[a, b] \cap BV[a, b]$ if and only if f can be written as the difference of two strictly increasing continuous functions.