Homework 9

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- Let F be a field, and define projective n-space $\mathbb{P}^n(F)$ to be the set of 1-dimensional F-subspaces in F^{n+1}. Give a group G and a G-set X such that the set of orbits for the action is in natural bijection with $\mathbb{P}^n(F)$. When F is a finite field with q elements, deduce from this that

$$\#\mathbb{P}^n(F) = \frac{q^{n+1}-1}{q-1}$$

Section 10.1: Galois Groups and Separability

2. Prove: If $E \supseteq F$ are fields, $G = \mathrm{Aut}_F(E), u \in E$, and $\sigma \in G$, then
   (1) $\sigma[f(u)] = f[\sigma(u)]$ for all $f \in F[x]$.
   (2) In particular, if u is a root of f, then $\sigma(u)$ is also a root of f.
   (3) If u is algebraic over F, and $\sigma, \tau \in \mathrm{Aut}_F\left(F(u)\right)$, then $\sigma = \tau$ if an only if $\sigma(u) = \tau(u)$.

13. If $E = \mathbb{Q}(\sqrt[4]{2}, i)$, show that $\mathrm{Aut}_{\mathbb{Q}}(E) \cong D_4$.

20. Let $F = K(t)$ denote the field of rational forms over a field K in an indeterminate t. Show that $x^2 - t$ is irreducible over F but is not separable if $\mathrm{char}\, K = 2$.

22. (a) Show that the following are equivalent for a polynomial $f \in F[x;]$.
       (1) f has no repeated root in any extension field of f.
       (2) f has no repeated root in some splitting field over F.
       (3) f and f' are relatively prime in $F[x]$.
    (b) If f is as in (a), show that f is separable, but not conversely.

25. If $E \supseteq F$ and $f \in F[x]$ is separable over F, show that f is separable over E.

26. If $E \supseteq K \supseteq F$ and $E \supseteq F$ is a separable extension, show that both $E \supseteq K$ and $E \supseteq F$ are separable extensions.

27. Let F have characteristic p. If $f = x^p - a$ where $a \in F$, show that f is irreducible or a power of a linear polynomial. (Hint: Lemma 5 and Theorem 4)