Homework 5

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Chapter 4: Open Sets and Closed Sets

3. Some authors say that two metrics d and ρ on a set M are equivalent if they generate the same open sets. Prove this.

18. Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are elements of $\overline{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is closed.

33. Let A be a subset of M. A point $x \in M$ is called a limit point of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \varnothing$ for every $\varepsilon > 0$. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

41. Related to the notion of limit points and isolated points are boundary points. A point $x \in M$ is said to be a boundary point of A if each neighborhood of x hits both A and A^c. In symbols, x is a boundary point of A if and only if $B_\varepsilon(x) \cap A \neq \varnothing$ and $B_\varepsilon(x) \cap A^c \neq \varnothing$ for every $\varepsilon > 0$. Verify each of the following formulas, where $\partial(A)$ denotes the set of boundary points of A :

    (a) $\partial(A) = \partial(A^c)$

    (b) $\overline{A} = \partial(A) \cup A^\circ$

    (c) $M = A^\circ \cup \partial(A) \cup (A^c)^\circ$

48. A metric space is called separable if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}$, in $\mathbb{R}^2$, and in $\mathbb{R}^n$.

Chapter 5: Continuity

17. Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

42. Suppose that $f : \mathbb{Q} \to \mathbb{R}$ is Lipschitz. Show that f extends to a continuous function $h : \mathbb{R} \to \mathbb{R}$. Is h unique? Explain. (Hint: Given $x \in \mathbb{R}$, choose a sequence of rationals (r_n) converging to x and argue that $h(x) = \lim_{n\to\infty} f(r_n)$ exists and is actually independent of the sequence (r_n).)

46. Show that every metric space is homeomorphic to one of finite diameter. (Hint: Every metric is equivalent to a bounded metric.)

48. Prove that $\mathbb{R}$ is homeomorphic to $(0, 1)$ and that $(0, 1)$ is homeomorphic to $(0, \infty)$. Is $\mathbb{R}$ isometric to $(0, 1)$? to $(0, \infty)$? Explain.

56. Let $f : (M, d) \to (N, \rho)$.

    (i) We say that f is an open map if $f(U)$ is open in N whenever U is open in M; that is, f maps open sets to open sets. Give examples of a continuous map that is not open and an open map that is not continuous.

    (ii) Similarly, f is called closed if it maps closed sets to closed sets. Give examples of a continuous map that is not closed and a closed map that is not continuous.