

Homework 1

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- 22.12 Prove: For every positive integer n , the Tower of Hanoi puzzle with n disks can be solved in $2^n - 1$ moves.

Proof. $n = 1$: If there is only 1 disk, it can be moved to the correct position in $1 = 2^1 - 1$ moves, so the base case is satisfied. Suppose a puzzle with k disks is solved in $2^k - 1$ moves. Then for a puzzle with $k + 1$ disks, we can equivalently move the first k disks to the middle position in $2^k - 1$ moves. Then we move the bottom disk to the correct position, and then move the k middle disks to the correct position in $2^k - 1$ moves. The total number of moves is $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$, so the formula holds for $k + 1$, and the statement is proved by induction. \square

- 22.16 (e) Let $e_0 = 1, e_1 = 4$, and for $n > 1$, let $e_n = 4(e_{n-1} - e_{n-2})$. What are the first five terms of the sequence e_0, e_1, e_2, \dots ? Prove $e_n = (n + 1)2^n$.

Proof. we have

$$\begin{aligned} e_0 &= 1 \\ e_1 &= 4 \\ e_2 &= 4(4 - 1) = 12 \\ e_3 &= 4(12 - 4) = 32 \\ e_4 &= 4(32 - 12) = 80 \end{aligned}$$

Now proceed by strong induction. For $n = 0$, we have $e_0 = 1 = (0 + 1)2^0$, so the base case is satisfied. Then suppose that the formula holds for all of 0 to k . That means $e_k = (k + 1)2^k$ and $e_{k-1} = k2^{k-1}$. Then

$$\begin{aligned} e_{k+1} &= 4(e_k - e_{k-1}) = 4[(k + 1)2^k - k2^{k-1}] \\ &= 4k2^k + 4 \cdot 2^k - 4k2^{k-1} = k2^{k+2} + 2^{k+2} - k2^{k+1} \\ &= 2^{k+1}(2k + 2 - k) = [(k + 1) + 1]2^{k+1} \end{aligned}$$

so the formula holds for $k + 1$ and the statement is proved by strong induction. \square

3. Let n be a positive integer. Use induction to prove that

$$\sum_{j=1}^n j^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

Proof. $n = 1$: The base case is satisfied because

$$1^4 = 1 = \frac{6 + 15 + 10 - 1}{30}$$

Now suppose the formula holds for arbitrary k . Then we have

$$\begin{aligned}
 \sum_{j=1}^{k+1} j^4 &= \sum_{j=1}^k j^4 + (k+1)^4 = \frac{6k^5 + 15k^4 + 10k^3 - k}{30} + (k+1)^4 \\
 &= \frac{(6k^5 + 15k^4 + 10k^3 - k) + 30(k^4 + 4k^3 + 6k^2 + 4k + 1)}{30} \\
 &= \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30} \\
 &= \frac{6(k+1)^5 + 15(k+1)^4 + 10(k+1)^3 - (k+1)}{30}
 \end{aligned}$$

so the formula holds for $k+1$ and the statement is proved by induction. \square

4. Consider the following nonlinear recurrence relation defined for $n \in \mathbb{N}$:

$$a_0 = 1, \quad a_n = na_0 + (n-1)a_1 + (n-2)a_2 + \cdots + 2a_{n-2} + 1a_{n-1}$$

(a) Calculate a_1, a_2, a_3, a_4

Solution.

$$\begin{aligned}
 a_1 &= 1a_0 = 1 \\
 a_2 &= 2a_0 + 1a_1 = 3 \\
 a_3 &= 3a_0 + 2a_1 + 1a_2 = 8 \\
 a_4 &= 4a_0 + 3a_1 + 2a_2 + 1a_3 = 21
 \end{aligned}$$

\square

(b) Use induction to prove for all positive integers n that

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right]$$

Proof. $n = 1$: The base case is satisfied because

$$1 = a_1 = \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^1 - \left(\frac{3-\sqrt{5}}{2} \right)^1 \right] = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2}$$

Now suppose the formula holds for arbitrary k . Note that

$$\begin{aligned}
 a_k &= ka_0 + (k-1)a_1 + (k-2)a_2 + \cdots + 2a_{k-2} + 1a_{k-1} \\
 a_{k+1} &= (k+1)a_0 + ka_1 + (k-1)a_2 + \cdots + 3a_{k-2} + 2a_{k-1} + 1a_k \\
 \implies a_{k+1} - a_k &= a_0 + a_1 + a_2 + \cdots + a_{k-2} + a_{k-1} + a_k
 \end{aligned}$$

The RHS is given by

$$\sum_{i=0}^k \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2} \right)^i - \left(\frac{3-\sqrt{5}}{2} \right)^i \right] = \frac{1}{\sqrt{5}} \left[\sum_{i=0}^k \left(\frac{3+\sqrt{5}}{2} \right)^i - \sum_{i=0}^k \left(\frac{3-\sqrt{5}}{2} \right)^i \right]$$

These are the sums of two geometric series, and the closed form is

$$\begin{aligned}
 \frac{1}{\sqrt{5}} \left[\frac{\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - 1}{\frac{3+\sqrt{5}}{2} - 1} - \frac{\left(\frac{3-\sqrt{5}}{2}\right)^{k+1} - 1}{\frac{3-\sqrt{5}}{2} - 1} \right] &= \frac{1}{\sqrt{5}} \left[\frac{\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - 1}{\frac{1+\sqrt{5}}{2}} - \frac{\left(\frac{3-\sqrt{5}}{2}\right)^{k+1} - 1}{\frac{1-\sqrt{5}}{2}} \right] \\
 &= \frac{1}{\sqrt{5} \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)} \left(\left(\frac{1-\sqrt{5}}{2}\right) \left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - 1 \right] - \left(\frac{1+\sqrt{5}}{2}\right) \left[\left(\frac{3-\sqrt{5}}{2}\right)^{k+1} - 1 \right] \right) \\
 &= -\frac{1}{\sqrt{5}} \left[\left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^{k+1} + \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \sqrt{5} \right]
 \end{aligned}$$

Then a_{k+1} is obtained by adding a_k to the result above, which is

$$\begin{aligned}
 \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \sqrt{5} \right] &+ \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2}\right)^k - \left(\frac{3-\sqrt{5}}{2}\right)^k \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \cdot \frac{3-\sqrt{5}}{2} - 1\right) \left(\frac{3-\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2} \cdot \frac{3+\sqrt{5}}{2} - 1\right) \left(\frac{3+\sqrt{5}}{2}\right)^k - \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5}-5}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^k - \left(\frac{-\sqrt{5}-5}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^k - \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{5+\sqrt{5}}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^k - \left(\frac{5-\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^k - \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(1 + \frac{3+\sqrt{5}}{2}\right) \left(\frac{3+\sqrt{5}}{2}\right)^k - \left(1 + \frac{3-\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right)^k - \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{k+1} + \frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2} - \sqrt{5} \right] \\
 &= \frac{1}{\sqrt{5}} \left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{k+1} \right]
 \end{aligned}$$

Thus, the formula holds for $k+1$, so the statement is proved by induction. □

24.1 For each of the following relations, please answer these questions:

- (1) Is it a function? If not, explain why and stop.
 - (2) What are its domain and image?
 - (3) Is the function one-to-one? If not, explain why and stop.
 - (4) What is its inverse function?
- (a) $\{(1, 2), (3, 4)\}$

Answer. This is a function. Its domain is $\{1, 3\}$ and its range is $\{2, 4\}$. The function is one-to-one. The inverse function is $\{(2, 1), (4, 3)\}$.

(b) $\{(x, y) \mid x, y \in \mathbb{Z}, y = 2x\}$

Answer. This is a function. Its domain is \mathbb{Z} and its image is $2\mathbb{Z}$. The function is one-to-one. The inverse function is $\{(x, y) \mid x, y \in \mathbb{Z}, x = 2y\}$.

(c) $\{(x, y) \mid x, y \in \mathbb{Z}, x + y = 0\}$

Answer. This is a function. Its domain is \mathbb{Z} and its image is \mathbb{Z} . The function is one-to-one. The inverse function is $\{(x, y) \mid x, y \in \mathbb{Z}, x + y = 0\}$.

(d) $\{(x, y) \mid x, y \in \mathbb{Z}, xy = 0\}$

Answer. This is not a function. When $x = 0$, then $(0, y)$ satisfies the relation for all $y \in \mathbb{Z}$, so x is mapped to more than a single value.

(e) $\{(x, y) \mid x, y \in \mathbb{Z}, y = x^2\}$

Answer. This is a function. Its domain is \mathbb{Z} and its image is $\mathbb{Z}_{\geq 0}$. The function is not one-to-one because $(2, 4)$ and $(-2, 4)$ are both in the relation, but $2 \neq -2$.

(f) \emptyset

Answer. This is a function. Its domain is \emptyset and its image is \emptyset . The function is one-to-one. The inverse function is \emptyset .

(g) $\{(x, y) \mid x, y \in \mathbb{Q}, x^2 + y^2 = 1\}$

Answer. This is not a function. The pairs $(0.6, 0.8)$ and $(0.6, -0.8)$ are both in the relation, so 0.6 is mapped to more than a single value.

(h) $\{(x, y) \mid x, y \in \mathbb{Z}, x \mid y\}$

Answer. This is not a function. The pairs $(1, 2)$ and $(1, 3)$ are both in the relation, so 1 is mapped to more than a single value.

(i) $\{(x, y) \mid x, y \in \mathbb{N}, x \mid y, y \mid x\}$

Answer. This is a function since the condition is equivalent to $x = y$. The domain is \mathbb{N} and the image is \mathbb{N} . The function is one-to-one. The inverse function is $\{(x, y) \mid x, y \in \mathbb{N}, x = y\}$.

(j) $\{(x, y) \mid x, y \in \mathbb{N}, \binom{x}{y} = 1\}$

Answer. This is not a function. Since $\binom{2}{0} = \binom{2}{2} = 1$, the pairs $(2, 0)$ and $(2, 2)$ are in the relation, so 2 is mapped to more than a single value.

24.23 (a) Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x) = |x|$. If $X = \{-1, 0, 1, 2\}$, find $f(X)$.

Answer. We have $f(X) = \{f(-1), f(0), f(1), f(2)\} = \{0, 1, 2\}$.

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sin x$. If $X = [0, \pi]$, find $f(X)$.

Answer. The sin function takes on values from 0 to 1 inclusive over $[0, \pi]$, so $f(X) = [0, 1]$.

(c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 2^x$. If $X = [-1, 1]$, find $f(X)$.

Answer. Since 2^x is an increasing function, its minimum value over X is $2^{-1} = 1/2$ and its maximum value is $2^1 = 2$, so $f(X) = [\frac{1}{2}, 2]$.

(d) Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x) = 3x - 1$. What is $f(\{1\})$? Is it the same as $f(1)$?

Answer. We have $f(\{1\}) = \{f(1)\} = \{2\}$. It is not the same as $f(1) = 2$ because the former is a set, while the latter is a number.

(e) Let $f : A \rightarrow B$ be a function. What is $f(A)$?

Answer. Here, $f(A)$ is the image of f as a function.

- 24.24 (a) Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x) = |x|$. If $Y = \{1, 2, 3\}$ find $f^{-1}(Y)$.

Answer. Under absolute value, both 1 and -1 are mapped to 1, and similarly for 2 and 3. So $f^{-1}(Y) = \{-3, -2, -1, 1, 2, 3\}$.

- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^2$. If $Y = [1, 2]$, find $f^{-1}(Y)$.

Answer. Under f , the interval $[1, \sqrt{2}]$ maps to Y since $f(1) = 1$ and $f(\sqrt{2}) = 2$ and f is increasing over $[1, \sqrt{2}]$. The interval $[-\sqrt{2}, -1]$ also maps to Y since $f(-\sqrt{2}) = 2$ and $f(-1) = 1$ and f is decreasing over $[-\sqrt{2}, -1]$. Thus $f^{-1}(Y) = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

- (c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Find $f^{-1}(\{\frac{1}{2}\})$.

Answer. We have

$$\frac{1}{2} = \frac{1}{1 + 1^2} = \frac{1}{1 + (-1)^2}$$

so $f^{-1}(\{\frac{1}{2}\}) = \{-1, 1\}$.

- (d) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Find $f^{-1}(\{-\frac{1}{2}\})$.

Answer. Since f is strictly positive over \mathbb{R} , there are no values of x such that $f(x) = -1/2$, so $f^{-1}(\{-\frac{1}{2}\}) = \emptyset$.

26.1 For each pair of functions f and g please do the following:

- Determine which of $g \circ f$ and $f \circ g$ is defined.
- If one or both are defined, find the resulting function(s).
- If both are defined, determine whether $g \circ f = f \circ g$.

- (e) $f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$ and $g = \{(1, 1), (2, 1), (3, 4), (4, 4)\}$

Solution. $g \circ f$ is not defined because

$$\text{im } f = \{1, 2, 3, 4, 5\} \not\subseteq \{1, 2, 3, 4\} = \text{dom } g$$

On the other hand, $f \circ g$ is defined because $\text{im } g \subseteq \text{dom } f$. Then we have

$$f \circ g = \{(1, 2), (2, 2), (3, 5), (4, 5)\}$$

□

- (g) $f(x) = x + 3$ and $g(x) = x - 7$ (both for all $x \in \mathbb{Z}$)

Solution. We have $\text{im } f = \text{im } g = \text{dom } f = \text{dom } g = \mathbb{Z}$ so both $g \circ f$ and $f \circ g$ are defined. Then

$$(g \circ f)(x) = g(x + 3) = x - 4$$

$$(f \circ g)(x) = f(x - 7) = x - 4$$

and thus $g \circ f = f \circ g$.

□

- (i) $f(x) = \frac{1}{x}$ for $x \in \mathbb{Q}$ except $x = 0$ and $g(x) = x + 1$ for all $x \in \mathbb{Q}$

Solution. $g \circ f$ is defined because

$$\text{im } f = \mathbb{Q} \setminus \{0\} \subseteq \mathbb{Q} = \text{dom } g$$

However, $f \circ g$ is not defined because

$$\text{im } g = \mathbb{Q} \not\subseteq \mathbb{Q} \setminus \{0\}$$

Then we have

$$(g \circ f)(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1$$

□