Homework 3

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Section 1.4: Permutations

6. If σ and τ fix k, show that $\sigma\tau$ and σ^{-1} both fix k.

12. Let $\sigma = (1 \quad 2 \quad 3)$ and $\tau = (1 \quad 2)$ in S_3.
    (a) Show that $S_3 = \{\varepsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$ and that $\sigma^3 = \varepsilon = \tau^2$ and $\sigma\tau = \tau\sigma^2$.
    (b) Use (a) to fill in the multiplication table for S_3.

16. If $\sigma = (1 \quad 2 \quad 3 \quad \cdots \quad n)$, show that $\sigma^n = \varepsilon$ and that n is the smallest positive integer with this property.

Section 2.1: Binary Operations

1. In each case a binary operation $*$ is given on a set M. Decide whether it is commutative or associative, whether a unity exists, and find the units (if there is a unity).
    (c) $M = \mathbb{R}; a * b = a + b - ab$
    (g) $M = \mathbb{N}^+; a * b = \gcd(a, b)$

5. Given an alphabet A, call an n-tuple $(a_1, a_2, \cdots, a_n)$ with $a_i \in A$ a word of length n from A and write it as $a_1 a_2 \cdots a_n$. Multiply two words by $(a_1 a_2 \cdots a_n) \cdot (b_1 b_2 \cdots b_m) = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$, and call this product juxtaposition. We decree the existence of an empty word λ with no letters. Show that the set W of all words from A is a monoid, noncommutative if $|A| > 1$, and find the units.

11. An element e is called a left unity for an operation if $ex = x$ for all x. If an operation has two left unities, show that it has no right unity.

Section 2.2: Groups

7. Show that the set
$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$
is a group under matrix multiplication.

16. If $fgh = 1$ in a group G, show that $ghf = 1$. Must $gfh = 1$?

20. Show that a group G is abelian if $g^2 = 1$ for all $g \in G$. Give an example showing that the converse is false.

28. Let a and b be elements of a group G. If $a^n = b^n$ and $a^m = b^m$ where $\gcd(m, n) = 1$, show that $a = b$. (Hint: Theorem 1.2.4)