

Homework 4

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Chapter 3: Metrics and Norms

6. If d is any metric on M , show that $\rho(x, y) = \sqrt{d(x, y)}$, $\sigma(x, y) = \frac{d(x, y)}{1+d(x, y)}$, and $\tau(x, y) = \min\{d(x, y), 1\}$ are also metrics on M .

Proof. ρ : Clearly ρ is non-negative since d is non-negative by being a metric, and

$$\rho(x, y) = 0 = \sqrt{d(x, y)} \iff d(x, y) = 0 \iff x = y$$

It is also symmetric because d is symmetric, and finally

$$\begin{aligned} \rho(x, y) + \rho(y, z) &= \sqrt{d(x, y)} + \sqrt{d(y, z)} \\ \implies [\rho(x, y) + \rho(y, z)]^2 &= d(x, y) + d(y, z) + 2\sqrt{d(x, y)d(y, z)} \\ &\geq d(x, z) + 2\sqrt{d(x, y)d(y, z)} \geq d(x, z) \\ \implies \rho(x, y) + \rho(y, z) &\geq \sqrt{d(x, z)} = \rho(x, z) \end{aligned}$$

σ : Clearly σ is non-negative since d is non-negative, and

$$\sigma(x, y) = 0 = \frac{d(x, y)}{1+d(x, y)} \iff d(x, y) = 0 \iff x = y$$

It is also symmetric because d is symmetric. Now, define $F(t) = \frac{t}{1+t}$. Then $F'(t) = \frac{1}{(1+t)^2} > 0$ so F is increasing, and we have

$$\begin{aligned} F(t) + F(s) &= \frac{t}{1+t} + \frac{s}{1+s} = \frac{t+ts+s+st}{(1+t)(1+s)} = \frac{s+t+2st}{1+s+t+st} \\ &= \frac{s+t+st}{1+s+t+st} + \frac{st}{1+s+t+st} = F(s+t+st) + \frac{st}{1+s+t+st} \\ &\geq F(s+t) \end{aligned}$$

since F is increasing since $F'(t) = (1+t)^{-2} > 0$. Thus,

$$\begin{aligned} \sigma(x, y) + \sigma(y, z) &= F(d(x, y)) + F(d(y, z)) \geq F(d(x, y) + d(y, z)) \\ &\geq F(d(x, z)) = \sigma(x, z) \end{aligned}$$

τ : Clearly τ is non-negative since d and 1 are non-negative, and

$$\tau(x, y) = 0 = \min\{d(x, y), 1\} \iff d(x, y) = 0 \iff x = y$$

It is also symmetric because d is symmetric. Suppose that

$$\begin{aligned} \tau(x, y) + \tau(y, z) &< \tau(x, z) \\ \min\{d(x, y), 1\} + \min\{d(y, z), 1\} &= m_1 + m_2 < \min\{d(x, z), 1\} \\ \implies m_1 + m_2 &< 1, \quad m_1 + m_2 < d(x, z) \end{aligned}$$

If $m_1 + m_2 < 1$, then we must have $m_1 = d(x, y)$ and $m_2 = d(y, z)$, but since d is a metric, $m_1 + m_2 = d(x, y) + d(y, z) \geq d(x, z)$, so it is impossible for both conditions to be true. Contradiction, so $\tau(x, y) + \tau(y, z) \geq \tau(x, z)$, and τ is a metric. \square

15. We define the diameter of a nonempty subset A of M by $\text{diam}(A) = \sup \{d(a, b) : a, b \in A\}$. Show that A is bounded if and only if $\text{diam}(A)$ is finite.

Proof. (\implies) : If A is bounded, then $\exists x_0 \in M$ and $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. Then

$$\text{diam}(A) = \sup \{d(a, b) : a, b \in A\} \leq \sup \{d(a, x_0) + d(x_0, b) : a, b \in A\} \leq 2C < \infty$$

(\impliedby) : If $\text{diam}(A)$ is finite, say $s = \text{diam}(A)$. Then take any $x_0 \in A \subset M$, and take $C = s$. Since s is the supremum, it follows that

$$C = s = \sup \{d(a, b) : a, b \in A\} \geq d(a, x_0)$$

for any $a \in A$, so A is bounded, as desired. \square

22. Show that $\|x\|_\infty \leq \|x\|_2$ for any $x \in \ell_2$, and that $\|x\|_2 \leq \|x\|_1$ for any $x \in \ell_1$.

Proof. We have

$$|x_k| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}$$

for all k (obvious once we square both sides), so it follows that

$$\|x\|_\infty = \sup |x_k| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} = \|x\|_2$$

By Cauchy, we have

$$\begin{aligned} \sum_{i=1}^{\infty} |x_i \cdot x_i| &\leq \sum_{i=1}^{\infty} |x_i| \sum_{i=1}^{\infty} |x_i| \\ \implies \|x\|_2 &= \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \leq \sum_{i=1}^{\infty} |x_i| = \|x\|_1 \end{aligned}$$

\square

23. The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0 . Note that c_0 is actually a linear subspace of ℓ_∞ ; thus c_0 is also a normed vector space under $\|\cdot\|_\infty$. Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

Proof. Suppose $x \in \ell_1$. Then $\sum_{i=1}^{\infty} |x_i| < \infty$. By Cauchy, we have

$$\begin{aligned} \sum_{i=1}^{\infty} |x_i \cdot x_i| &\leq \sum_{i=1}^{\infty} |x_i| \sum_{i=1}^{\infty} |x_i| \\ \implies \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} &\leq \sum_{i=1}^{\infty} |x_i| < \infty \end{aligned}$$

so $\ell_1 \subset \ell_2$. Then the reverse inclusion does not hold because for the sequence $x_n = \frac{1}{n}$, we have

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi}{6} \quad \text{but} \quad \sum_{i=1}^{\infty} \frac{1}{i} \rightarrow \infty$$

Suppose $y \in \ell_2$, but y_n does not converge to 0. Clearly y must be bounded, otherwise it would not be in ℓ_2 . Then there exists some $\varepsilon > 0$ such that $|y_n| \geq \varepsilon$ infinitely often. Then $\sum_{i=1}^{\infty} |y_i|^2 \rightarrow \infty$, so $y \notin \ell_2$, contradiction. Thus $y_n \rightarrow 0 \implies y \in c_0$, as desired. The reverse inclusion does not hold for the sequence $y_n = \frac{1}{\sqrt{n}}$, since $y_n \rightarrow 0$ but $\sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{i}}\right)^2 \rightarrow \infty$.

Finally, c_0 is a linear subspace of ℓ_{∞} , and the reverse inclusion does not hold if we consider the sequence $(1, 0, 1, 0, 1, 0, \dots)$. This is bounded, but does not converge to 0. \square

25. The same techniques can be used to show that $\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$ defines a norm on $C([0, 1])$ for any $1 < p < \infty$. State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for $p = 1$ and $q = \infty$?)

Lemma 3.7': Let $1 < p < \infty$ and let q be defined by $1/p + 1/q = 1$. Given $f, g \in C([0, 1])$, we have $\int_0^1 |f(x)g(x)| dx \leq \|f\|_p \|g\|_q$.

Proof. From Young's inequality, we have

$$\begin{aligned} \left| \frac{f(x)}{\|f\|_p} \right| \cdot \left| \frac{g(x)}{\|g\|_q} \right| &\leq \frac{1}{p} \left| \frac{f(x)}{\|f\|_p} \right|^p + \frac{1}{q} \left| \frac{g(x)}{\|g\|_q} \right|^q \\ \implies \int_0^1 \left| \frac{f(x)g(x)}{\|f\|_p \|g\|_q} \right| dx &\leq \int_0^1 \frac{1}{p} \left| \frac{f(x)}{\|f\|_p} \right|^p dx + \int_0^1 \frac{1}{q} \left| \frac{g(x)}{\|g\|_q} \right|^q dx \\ &= \frac{1}{p} + \frac{1}{q} = 1 \\ \implies \int_0^1 |f(x)g(x)| dx &\leq \|f\|_p \|g\|_q \end{aligned}$$

as desired. If $p = 1$ and $q = \infty$, the statement still holds. We have

$$\begin{aligned} \|g\|_q &= \sup_{0 \leq t \leq 1} |g(t)| \\ \implies |f(x)| |g(x)| &\leq |f(x)| \cdot \sup_{0 \leq t \leq 1} |g(t)| \\ \implies \int_0^1 |f(x)g(x)| dx &\leq \int_0^1 |f(x)| \cdot \sup_{0 \leq t \leq 1} |g(t)| dx \\ &= \|g\|_{\infty} \|f\|_1 \end{aligned}$$

\square

Theorem 3.8': Let $1 < p < \infty$. If $f, g \in C([0, 1])$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. Let q be the conjugate of p . By the triangle inequality and Holder's inequality, we have

$$\begin{aligned} |f(x) + g(x)|^p &= |f(x) + g(x)| \cdot |f(x) + g(x)|^{p-1} \\ &\leq |f(x)| \cdot |f(x) + g(x)|^{p-1} + |g(x)| \cdot |f(x) + g(x)|^{p-1} \\ \implies \int_0^1 |f(x) + g(x)|^p dx &\leq \int_0^1 |f(x)| \cdot |f(x) + g(x)|^{p-1} dx + \int_0^1 |g(x)| \cdot |f(x) + g(x)|^{p-1} dx \\ &\leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q \end{aligned}$$

Now, since $1/p + 1/q = 1 \implies q = \frac{p}{p-1}$, we have

$$\begin{aligned} \|(f+g)^{p-1}\|_q &= \left(\int_0^1 |(f(x)+g(x))^{p-1}|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &= \left(\left(\int_0^1 |f(x)+g(x)|^p dx \right)^{1/p} \right)^{(p-1)} \\ &= \|f+g\|_p^{p-1} \end{aligned}$$

so it follows that

$$\begin{aligned} \int_0^1 |f(x)+g(x)|^p dx &= \|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p-1} \\ &\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p \end{aligned}$$

as desired. \square

31. Give an example where $\text{diam}(A \cup B) > \text{diam}(A) + \text{diam}(B)$. If $A \cap B \neq \emptyset$, show that $\text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B)$.

Proof. Let $A = \{0\}$ and $B = \{1\}$ under the discrete metric. Then $\text{diam}(A) = 0$ and $\text{diam}(B) = 0$, but $\text{diam}(A \cup B) = \text{diam}(\{0, 1\}) = 1$.

Take two points $x, y \in A \cup B$. If x and y are both in A , then $d(x, y) \leq \text{diam}(A)$, so

$$\text{diam}(A \cup B) = \sup \{d(a, b) : a, b \in A \cup B\} \leq \text{diam}(A)$$

and similarly if $x, y \in B$. WLOG $x \in A, y \in B$. Since $A \cap B \neq \emptyset$, take $z \in A \cap B \implies z \in A, z \in B$.

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(y, z) \leq \text{diam}(A) + \text{diam}(B) \\ \implies \text{diam}(A \cup B) &= \sup \{d(a, b) : a, b \in A \cup B\} \leq \text{diam}(A) + \text{diam}(B) \end{aligned}$$

as desired. \square

37. A Cauchy sequence with a convergent subsequence converges.

Proof. Suppose (x_n) is a sequence with a convergent subsequence $(x_{k_j}) \rightarrow y$. Let $\varepsilon > 0$. Since (x_n) is Cauchy, choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m \geq N$. Next, since $(x_{k_j}) \rightarrow y$, choose M such that $d(x_{k_j}, y) < \varepsilon/2$ for all $k_j \geq M$. Take $K = \max\{N, M\}$, so that $d(x_n, x_{k_j}) < \varepsilon/2$ and $d(x_{k_j}, y) < \varepsilon/2$ for all $n, k_j \geq K$. By the triangle inequality, we have

$$d(x_n, y) \leq d(x_n, x_{k_j}) + d(x_{k_j}, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq K$, as desired. \square