Homework 1

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1 Asymptotic Notation

For each of the following statements explain if it is true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

(a) $100(n\log^4 n + \frac{1}{2}n^2) = \Theta(n^2)$

Proof. This is true. We wish to show that there exists $c_1, c_2, n_0 > 0$ such that

$$c_1 n^2 \leq n\log^4 n + \frac{1}{2}n^2 \leq c_2 n^2$$

Clearly, $c_1 = \frac{1}{2}$ satisfies the left hand inequality. Now, suppose $c_2 = 1$, and suppose $n_0 = 2^k$ for some k. Then we must have

$$\begin{aligned} n\log^4 n + \frac{1}{2}n^2 = 2^k \cdot k^4 + 2^{2k-1} &\leq 2^{2k} = n^2 \\ \implies k^4 &\leq 2^{k-1} \end{aligned}$$

Now, k^4 grows slower than 2^{k-1}, so take a large enough value for k (say, 100), and we will be done. Thus $n_0 = 2^{100}$ and $c = 1$ will satisfy the inequality, so $100\left(n\log^4 n + \frac{1}{2}n^2\right) = \Theta(n^2)$, as desired. $\square$

(b) $2^n = \Omega(2^{n/2})$

Proof. This is true. We have

$$2^n \geq c(2^{n/2}) \implies 2^{n/2} \geq c$$

which is true for $c = 1, n_0 = 1$, so $2^n = \Omega(2^{n/2})$, as desired. $\square$

(c) $\log(n^{6\log n}) = \Theta\left(\left(\log n^{1/3}\right)^2\right)$

Proof. This is true. Simplifying the logarithms, we have

$$\log\left(n^{6\log n}\right) = 6\log n \log n = 6\log^2 n$$

$$\left(\log n^{1/3}\right)^2 = \left(\frac{1}{3}\log n\right)^2 = \frac{1}{9}\log^2 n$$

Thus, taking $c_1 = 54$ and $c_2 = 54$, we have

$$\begin{aligned} c_1 \cdot \left(\frac{1}{9}\log^2 n\right) \leq 6\log^2 n &\leq c_2\left(\frac{1}{9}\log^2 n\right) \\ \implies \log\left(n^{6\log n}\right) &= \Theta\left(\left(\log n^{1/3}\right)^2\right) \end{aligned}$$

as desired. $\square$

(d) $3^n = \Theta\left(3.1^n\right)$

Proof. This is false. Suppose $c, n_0 > 0$ satisfy $3^n \geq c(3.1)^n$ for all $n > n_0$. Then $(3/3.1)^n \geq c$, so

$$\begin{aligned} \log\left(\frac{3}{3.1}\right)^n &\geq \log c \\ n\log\left(\frac{3}{3.1}\right) &\geq \log c \\ n &\leq \frac{\log c}{\log 3 - \log 3.1} \end{aligned}$$

So if we take some $N > \frac{\log c}{\log 3 - \log 3.1}$, then the inequality fails, so $3^n \neq \Omega(3.1^n)$, and thus $3^n \neq \Theta(3.1^n)$, as desired. □

(e) $\sqrt{n + \cos n} = O(\sqrt{n})$

Proof. This is true. We have $-1 \leq \cos n \leq 1$, so $\sqrt{n+\cos n} \leq \sqrt{n+1}$. Take $c = 2, n_0 = 1$. Then we have $\sqrt{n+1} \leq 2\sqrt{n} = \sqrt{4n}$ which holds for all $n > 1$, so $\sqrt{n+\cos n} = O\left(\sqrt{n}\right)$, as desired. □

(f) Let f, g be positive functions. Then $f(n) + g(n) = O(\max\{f(n), g(n)\})$.

Proof. This is true. We have

$$\begin{aligned} \max\{f(n), g(n)\} &\geq \frac{1}{2}\left[f(n) + g(n)\right] \\ \implies 2 \cdot \max\{f(n), g(n)\} &\geq f(n) + g(n) \end{aligned}$$

so taking $c = 2, n_0 = 1$, we get $f(n) + g(n) = O\left(\max\{f(n), g(n)\}\right)$, as desired. □

(g) Let f, g be positive functions, and let $g(n) = \omega(f(n))$. Then $f(n) + g(n) = \Theta(g(n))$.

Proof. This is true. Since $g(n) = \omega(f(n))$, for all $c > 0$, there exists $n_0 > 0$ such that

$$g(n) > cf(n) \implies \frac{f(n)}{g(n)} < \frac{1}{c} \tag{1}$$

for all $n > n_0$. We wish to show that there exists $c_1, c_2, n_0 > 0$ such that

$$c_1 g(n) \leq f(n) + g(n) \leq c_2 g(n)$$

Clearly $c_1 = 1$ works for the left-hand inequality, since $f(n)$ is positive. Next, fix d and m_0 such that (1) is true. Then

$$\frac{f(n)}{g(n)} + 1 = \frac{f(n) + g(n)}{g(n)} < \frac{1}{d} + 1 \leq \frac{1}{d} + 2$$

for all $n > m_0$. Now let $c_2 = \frac{1}{d} + 2$ and $n_0 = m_0$. Then

$$\frac{f(n) + g(n)}{g(n)} \leq c_2 \implies f(n) + g(n) \leq c_2 g(n)$$

for all $n > n_0$, so $f(n) + g(n) = \Theta(g(n))$. □

(h) $2^{\frac{\log n}{2}} = \Theta(n)$.

Proof. This is false. Simplifying, we have $2^{\frac{\log n}{2}} = n^{1/2}$. Suppose $n^{1/2} = \Omega(n)$. Fix some c, and suppose there exists n_0 such that $n^{1/2} \geq cn$ for all $n > n_0$. Then

$$\frac{1}{c} \geq n^{1/2} \implies n \leq \frac{1}{c^2}$$

However, if we take $N > \frac{1}{c^2}$, the inequality fails. Thus, $n^{1/2} \neq \Omega(n)$, so $n^{1/2} = 2^{\frac{\log 2}{2}} \neq \Theta(n)$. □

2 Recurrences

Solve the following recurrences, giving your answer in Θ notation. For each of them you may assume $T(x) = 1$ for $x \leq 5$ (or if it makes the base case easier you may assume $T(x)$ is any other constant for $x \leq 5$). Justify.

(a) $T(n) = 3T(n-2)$

Solution. We have

$$T(n) = 3T(n-2) = 3^2T(n-4) = \cdots = 3^kT(n-2k)$$

Supposing $n - 2k = 5$, we have $k = \frac{n-5}{2}$, so

$$T(n) = 3^{\frac{n-5}{2}}T(5) = 3^{\frac{n-5}{2}} = 3^{-5/2}3^{n/2} = \Theta(3^{n/2})$$

□

(b) $T(n) = n^{1/3}T(n^{2/3}) + n$

Solution. Claim: $T(n) = kn + n^{1-(2/3)^k}T\left(n^{(2/3)^k}\right)$ for $k \geq 1$. The base case is trivial. Suppose the claim holds for all $1 \leq k \leq m$ for arbitrary m. Then

$$\begin{aligned} T(n) = mn + n^{1-(2/3)^m}T\left(n^{(2/3)^m}\right) &= mn + n^{1-(2/3)^m}\left[\left(n^{(2/3)^m}\right)^{1/3}T\left(n^{(2/3)^{m+1}}\right) + n^{(2/3)^m}\right] \\ &= mn + n^{1-\left(\frac{2}{3}\right)^m + \frac{1}{3}\cdot\left(\frac{2}{3}\right)^m}T\left(n^{(2/3)^{m+1}}\right) + n \\ &= (m+1)n + n^{1-(2/3)^{m+1}}T\left(n^{(2/3)^{m+1}}\right) \end{aligned}$$

Thus, the claim holds for $k = m+1$, so the claim is proven by induction. Now suppose that

$$\begin{aligned} n^{(2/3)^k} = 2 \implies \left(\frac{2}{3}\right)^k \log n = \log 2 = 1 \\ \implies \left(\frac{2}{3}\right)^k = \frac{1}{\log n} \implies k\log\frac{2}{3} = \log\left(\frac{1}{\log n}\right) = -\log(\log n) \\ \implies k = \frac{-\log(\log n)}{\log 2 - \log 3} = \frac{\log(\log n)}{\log 3 - 1} \end{aligned}$$

Using this value of k, we have

$$\begin{aligned} T(n) = kn + n^{1-(2/3)^k}T\left(n^{(2/3)^k}\right) = n \cdot \frac{\log(\log n)}{\log 3 - 1} + \frac{n}{n^{(2/3)^k}}T(2) \\ = \frac{1}{\log 3 - 1}n\log(\log n) + \frac{n}{2} = \frac{1}{\log 3 - 1}n\log(\log n) + \frac{n}{2} \end{aligned}$$

Claim: This is $\Theta\left(n\log(\log n)\right)$. We must find $c_1, c_2, n_0 > 0$ such that

$$c_1 \cdot n\log(\log n) \leq \frac{1}{\log 3 - 1}n\log(\log n) + \frac{n}{2} \leq c_2 \cdot n\log(\log n)$$

for all $n > n_0$. Clearly $c_1 = \frac{1}{\log 3 - 1}$ satisfies the left hand inequality. Suppose we fix a value for c_2. Then we must have

$$\frac{1}{\log 3 - 1}n\log(\log n) + \frac{n}{2} \leq c_2 \cdot n\log(\log n) \implies n \geq 10^{10^{\frac{1}{2\left(c_2 - \frac{1}{\log 3 - 1}\right)}}}$$

Thus, take any value n_0 greater than this to satisfy the inequality, and the claim is proven. □

(c) $T(n) = 8T(n/4) + n$

Solution. In the notation of the master theorem, $a = 8, b = 4, f(n) = n$. Then $\log_b a = \log_4 8 = 3/2$ and $f(n) = n = O\left(n^{\frac{3}{2}-\frac{1}{2}}\right) = O(n)$ where $\varepsilon = 1/2$. Thus, by the master theorem, $T(n) = \Theta(n^{3/2})$. $\square$

(d) $T(n) = T(n-3) + 5$

Solution. We have

$$T(n) = T(n-3) + 5 = T(n-6) + 10 = \cdots = T(n-3k) + 5k$$

Supposing $n - 3k = 5$, we have $k = \frac{n-5}{3}$, so

$$T(n) = T(5) + 5 \cdot \frac{n-5}{3} = 1 + \frac{5n-25}{3} = \Theta(n)$$

$\square$

(e) $T(n) = 3T(n/3) + n \log_3 n$

Solution. Using a recursive tree, at level 0 we have $n \log_3 n$, at level 1 we have $3 \cdot \frac{n}{3} \log_3 \frac{n}{3}$, and in general, at level i we have $3^i \cdot \frac{n}{3^i} \log_3 \frac{n}{3^i} = n(\log_3 n - i)$. The lowest level is $\log_3 n$, so

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_3 n} n\left(\log_3 n - i\right) = \sum_{i=0}^{\log_3 n} n \log_3 n - n \sum_{i=0}^{\log_3 n} i \\ &= n \log_3 n \left(\log_3 n + 1\right) - n \cdot \frac{\log_3 n(\log_3 n + 1)}{2} \\ &= \frac{1}{2} n \log_3^2 n + \frac{1}{2} n \log_3 n \end{aligned}$$

Claim: $\frac{1}{2} n \log_3^2 n + \frac{1}{2} n \log_3 n = \Theta(n \log_3^2 n)$. We must find $c_1, c_2, n_0 > 0$ such that

$$c_1 n \log_3^2 n \le \frac{1}{2} n \log_3^2 n + \frac{1}{2} n \log_3 n \le c_2 n \log_3^2 n$$

for all $n > n_0$. Clearly, $c_1 = 1/2$ satisfies the left hand inequality. Fix some c_2. Then

$$\begin{aligned} \frac{1}{2} n \log_3^2 n + \frac{1}{2} n \log_3 n \le c_2 n \log_3^2 n &\implies \frac{1}{2} + \frac{1}{2\log_3 n} \le c_2 \\ &\implies c_2 - \frac{1}{2} \ge \frac{1}{2\log_3 n} \implies \log_3 n \ge \frac{1}{2c-1} \\ &\implies n \ge 3^{\frac{1}{2c-1}} \end{aligned}$$

Thus, as long as we choose $n_0 > 3^{\frac{1}{2c-1}}$, the inequality will be satisfied, and the claim is proven. $\square$

3 Basic Proofs

(a) Prove that $\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k} = \sum_{k=n+1}^{2n}\frac{1}{k}$ for all $n \geq 1$.

Proof. Base case: $n = 1$. We have

$$\sum_{k=1}^{2\cdot 1}(-1)^{k+1}\frac{1}{k} = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$
$$\sum_{k=1+1}^{2\cdot 1}\frac{1}{k} = \frac{1}{2}$$

Now, suppose the assertion holds for arbitrary m. Then

$$\sum_{k=1}^{2m}(-1)^{k+1}\frac{1}{k} = \sum_{k=m+1}^{2m}\frac{1}{k}$$

Then if we add

$$(-1)^{(2m+1)+1}\frac{1}{2m+1} + (-1)^{(2m+2)+1}\frac{1}{2m+2} = \frac{1}{2m+1} - \frac{1}{2m+2}$$

to both sides, we have

$$\begin{aligned}
\sum_{k=1}^{2(m+1)}(-1)^{k+1}\frac{1}{k} &= \sum_{k=m+1}^{2m}\frac{1}{k} + \frac{1}{2m+1} - \frac{1}{2m+2} \\
&= \left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}\right) + \frac{1}{2m+1} - \frac{1}{2}\cdot\frac{1}{m+1} \\
&= \frac{1}{m+2} + \cdots + \frac{1}{2m} + \frac{1}{2m+1} + \frac{1}{2m+2} = \sum_{k=(m+1)+1}^{2(m+1)}\frac{1}{k}
\end{aligned}$$

so the assertion is true for $m + 1$, completing the proof. □

(b) There are 9 course assistants for this class. Let us assume that 92 students submit their assignments for this problem set, and each submission is graded by one course assistant. Prove that there is some course assistant who grades at least 11 submissions.

Proof. Define n_i as the number of papers graded by course assistant i. Suppose $n_1, n_2, \cdots n_9$ are each at most 10. Then

$$n_1 + n_2 + \cdots + n_9 \leq 9 \cdot 10 = 90$$

but there are 92 papers total, contradiction. Thus, some n_i must be at least 11, as desired. Alternatively, trivialized by the Pigeonhole Principle. □

(c) Let $x_1, x_2, \cdots, x_n$ be real numbers. Prove that for any $1 \leq k \leq n$,

$$\sum_{i=k}^{n} x_i \leq n \cdot \max_{i=1}^{n} \{x_i\} - \sum_{j=1}^{k-1} x_j$$

Proof. Rearranging, we must show

$$\sum_{j=1}^{k-1} x_j + \sum_{i=k}^{n} x_i = \sum_{m=1}^{n} x_i \leq n \cdot \max_{i=1}^{n} \{x_i\}$$

WLOG, $\max_{i=1}^{n} \{x_i\} = x_1$. Then $x_2 \leq x_1, x_3 \leq x_1, \cdots, x_n \leq x_1$, so their sum is

$$\sum_{m=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n \leq x_1 + x_1 + x_1 + \cdots + x_1 = n \cdot x_1 = n \max_{i=1}^{n} \{x_i\}$$

as desired. □