Homework 3

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Section 2.3

4. Using Definition 4, show that each of the following functions is nowhere differentiable.

   (a) $\operatorname{Re} z$

   Proof. Suppose $\operatorname{Re} z$ was differentiable at $z_0 = a_0 + b_0 i$. Then

$$\frac{d(\operatorname{Re} z)}{dz}(z_0) = \lim_{h\to 0} \frac{\operatorname{Re}(z_0+h) - \operatorname{Re} z}{h} = \lim_{a+bi\to 0} \frac{\operatorname{Re}\left[(a_0+b_0i)+(a+bi)\right] - \operatorname{Re}(a_0+b_0i)}{a+bi}$$
$$= \lim_{a+bi\to 0} \frac{(a_0+a)-a_0}{a+bi} = \lim_{a+bi\to 0} \frac{a}{a+bi}$$

   This limit does not exist because if we go along the real axis, the limit is 1, but if we go along the imaginary axis, the limit is 0. Thus, $\operatorname{Re} z$ is not differentiable at any point. □

   (c) $|z|$

   Proof. Suppose $|z|$ was differentiable at $z_0 = a_0 + b_0 i$. Then

$$\frac{d(|z|)}{dz}(z_0) = \lim_{h\to 0} \frac{|z_0+h| - |z_0|}{h} = \lim_{a+bi\to 0} \frac{|(a_0+b_0i)+(a+bi)| - |a_0+b_0i|}{a+bi}$$
$$= \lim_{a+bi\to 0} \frac{\sqrt{(a_0+a)^2+(b_0+b)^2} - \sqrt{a_0^2+b_0^2}}{a+bi}$$

   If we approach along the real axis, $b = 0$, so the limit is

$$\lim_{a\to 0} \frac{\sqrt{(a_0+a)^2+b_0^2} - \sqrt{a_0^2}}{a} \to \infty$$

   so the limit does not exist. □

8. Suppose that f is analytic at z_0 and $f'(z_0) \neq 0$. Show that

$$\lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|} = |f'(z_0)|$$

   and

$$\lim_{z\to z_0} \{\arg\left[f(z)-f(z_0)\right] - \arg(z-z_0)\} = \arg f'(z_0)$$

   Proof. If f is analytic at z_0, then using the substitution $z = z_0 + h \implies h = z - z_0$ we have

$$|f'(z_0)| = \left|\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}\right| = \lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}$$

as desired. Since $\arg z_1 - \arg z_2 = \arg \frac{z_1}{z_2}$, we have

$$\begin{aligned}\lim_{z\to z_0}\{\arg[f(z)-f(z_0)] - \arg(z-z_0)\} &= \lim_{z\to z_0}\arg\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\\ &= \arg\left(\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\right) = \arg f'(z_0)\end{aligned}$$

by the same substitution. □

11. Discuss the analyticity of each of the following functions.

(b) $\frac{z}{\overline{z}+2}$

Solution. We have the derivative at point z given by

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{\frac{z+h}{\overline{z+h}+2}-\frac{z}{\overline{z}+2}}{h} = \lim_{h\to 0}\frac{(z+h)(\overline{z}+2)-z(\overline{z}+\overline{h}+2)}{h(\overline{z}+\overline{h}+2)(\overline{z}+2)}\\ &= \lim_{h\to 0}\frac{z\overline{z}+2z+h\overline{z}+2h-z\overline{z}-z\overline{h}-2z}{h(\overline{z}+\overline{h}+2)(\overline{z}+2)} = \lim_{h\to 0}\frac{h\overline{z}+2h-z\overline{h}}{h(\overline{z}+\overline{h}+2)(\overline{z}+2)}\end{aligned}$$

At $z=0$, the limit is

$$\lim_{h\to 0}\frac{2h}{h(\overline{h}+2)(2)} = \lim_{h\to 0}\frac{1}{\overline{h}+2} = \frac{1}{2}$$

Otherwise if $z\neq 0$, then if we approach along the real axis, $\overline{h}=h$, so this limit is

$$\lim_{h\to 0}\frac{h(\overline{z}+2-z)}{h(\overline{z}+\overline{h}+2)(\overline{z}+2)} = \lim_{h\to 0}\frac{\overline{z}+2-z}{(\overline{z}+h+2)(\overline{z}+2)} = \frac{\overline{z}+2-z}{(\overline{z}+2)^2}$$

but if we approach along the imaginary axis, $\overline{h}=-h$, so this limit is

$$\lim_{h\to 0}\frac{h(\overline{z}+2+z)}{h(\overline{z}-h+2)(\overline{z}+2)} = \frac{\overline{z}+2+z}{(\overline{z}+2)^2}$$

These two limits are not equal as long as $z\neq 0$, so this function is not differentiable except at 0. Since 0 is not an open set in $\mathbb{C}$, this function is nowhere analytic. □

(f) $\left(x+\frac{x}{x^2+y^2}\right)+i\left(y-\frac{y}{x^2+y^2}\right)$

Solution. If $z=x+yi$, then $z+\frac{1}{z}=(x+yi)+\frac{x-yi}{x^2+y^2}=\left(x+\frac{x}{x^2+y^2}\right)+i\left(y-\frac{y}{x^2+y^2}\right)$. Since z is analytic everywhere and $1/z$ is analytic everywhere except 0, this is analytic everywhere but 0. □

(g) $|z|^2+2z$

Solution. Since $|z|$ is nowhere analytic, this is also nowhere analytic. □

Section 2.4

3. Use Theorem 5 to show that $g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is entire. Write this function in terms of z.

Proof. Here, $u = 3x^2 + 2x - 3y^2 - 1$ and $v = 6xy + 2y$. We have

$$\frac{\partial u}{\partial x} = 6x + 2$$
$$\frac{\partial v}{\partial y} = 6x + 2$$
$$\frac{\partial u}{\partial y} = -6y$$
$$\frac{\partial v}{\partial x} = 6y$$

so the Cauchy-Riemann equations are satisfied, and they are satisfied at all points in $\mathbb{C}$. The first partials are also all continuous, so g is entire.

Using the identities $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we have

$$\begin{aligned} g(z) &= \left[3\left(\frac{z+\bar{z}}{2}\right)^2 + 2\left(\frac{z+\bar{z}}{2}\right) - 3\left(\frac{z-\bar{z}}{2i}\right)^2 - 1\right] + i\left[6\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) + 2\left(\frac{z-\bar{z}}{2i}\right)\right] \\ &= \left[3\left(\frac{z^2+2z\bar{z}+\bar{z}^2}{4}\right) + (z+\bar{z}) - 3\left(\frac{z^2-2z\bar{z}+\bar{z}^2}{-4}\right) - 1\right] + \frac{3}{2}(z^2-\bar{z}^2) + (z-\bar{z}) \\ &= \frac{3}{2}z^2 + \frac{3}{2}\bar{z}^2 + z + \bar{z} - 1 + \frac{3}{2}z^2 - \frac{3}{2}\bar{z}^2 + z - \bar{z} \\ &= 3z^2 + 2z - 1 \end{aligned}$$

□

5. Show that the function $f(z) = e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]$ is entire, and find its derivative.

Proof. Here, $u = e^{x^2-y^2}\cos(2xy)$ and $v = e^{x^2-y^2}\sin(2xy)$. We have

$$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$$
$$\frac{\partial v}{\partial y} = -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy)$$
$$\frac{\partial u}{\partial y} = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy)$$
$$\frac{\partial v}{\partial x} = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy)$$

so the Cauchy-Riemann equations are satisfied, the first partials are continuous, and thus f is analytic at every point in $\mathbb{C}$, so f is also entire. By De Moivre's theorem, the derivative is

$$\begin{aligned} f(z) &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)] = e^{x^2-y^2}e^{2xyi} \\ &= e^{x^2+2xyi-y^2} = e^{(x+yi)^2} = e^{z^2} \\ \implies f'(z) &= 2ze^{z^2} \end{aligned}$$

□

8. Show that if f is analytic in a domain D and either $\operatorname{Re} f(x)$ or $\operatorname{Im} f(x)$ is constant in D, then $f(z)$ must be constant in D.

Proof. Here, $u = \operatorname{Re} f(x)$ and $v = \operatorname{Im} f(x)$. If f is analytic, and $u \equiv c_1$ is constant, then u and v must satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \implies v \equiv c_2$$

Similarly, if $v \equiv c_2$, we get $u \equiv c_1$ in order to satisfy the Cauchy-Riemann equations, and thus $f(x) = c_1 + c_2 i$, which is a constant function. □

15. The Jacobian of a mapping

$$u = u(x, y), \quad v = v(x, y)$$

from the xy-plane to the uv-plane is defined to be the determinant

$$J(x_0, y_0) := \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

where the partial derivatives are all evaluated at (x_0, y_0). Show that if $f = u + iv$ is analytic at $z_0 = x_0 + iy_0$, then $J(x_0, y_0) = |f'(z_0)|^2$.

Proof. If $f = u + iv$ is analytic at (x_0, y_0) then by the Cauchy-Riemann equations, we must have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ where the partials are evaluated at (x_0, y_0). The Jacobian is

$$\begin{aligned} \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} &= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^2 = \left|\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}\right|^2 \\ &= |f'(z_0)|^2 \end{aligned}$$

as desired. □

Section 2.5

8. Suppose that the functions u and v are harmonic in a domain D.

(a) Is the sum $u + v$ necessarily harmonic in D?

Solution. We have

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(u+v) + \frac{\partial^2}{\partial y^2}(u+v) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \\ &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \end{aligned}$$

by the linearity of the differential operator, so $u + v$ is also harmonic in D. □

(b) Is the product uv necessarily harmonic in D?

Solution. This is not necessarily true. Take $u = y = xy$. Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ so u and v are both harmonic, but

$$\frac{\partial^2(uv)}{\partial x^2} + \frac{\partial^2(uv)}{\partial y^2} = \frac{\partial^2(x^2y^2)}{\partial x^2} + \frac{\partial^2(x^2y^2)}{\partial y^2} = 2y^2 + 2x^2 \neq 0$$

□

(c) Is $\partial u/\partial x$ harmonic in D?

Solution. We have

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial x}[0] = 0$$

because we can take partial derivatives in any order, so $\partial u/\partial x$ is also harmonic in D. □

12. Prove that if r and θ are polar coordinates, then the functions $r^n \cos n\theta$ and $r^n \sin n\theta$, where n is an integer, are harmonic as functions of x and y.

Proof. If $x = r\cos\theta$ and $y = r\sin\theta$, then we have

$$r^n \cos n\theta + i \cdot r^n \sin n\theta = \left(re^{i\theta}\right)^n = z^n$$

Thus, since $f(z) = z^n$ is analytic, it follows that $\operatorname{Re} f(z) = r^n \cos n\theta$ and $\operatorname{Im} f(z) = r^n \sin n\theta$ are both harmonic. □

13. Find a function harmonic inside the wedge bounded by the non-negative x-axis and the half-line $y = x \quad (x \geq 0)$ that goes to 0 on these sides but is not identically zero.

Solution. If $z = re^{i\theta}$, then $f(z) = \operatorname{Im} z^4 = \operatorname{Im}(r^4 e^{4\theta i}) = r^4 \sin 4\theta = 0$ exactly on the half lines $\theta = 0$ and $\theta = \pi/4$, but it is not identically zero. □