

Homework 11

ALECK ZHAO

April 23, 2018

Section 6.1

3. Evaluate each of the following integrals by means of the Cauchy residue theorem.

(a) $\oint_{|z|=5} \frac{\sin z}{z^2-4} dz$

Solution. The integrand has simple poles at $z = 2, z = -2$, which are both contained inside $|z| = 5$. Then we calculate the residues as

$$\begin{aligned} \frac{\sin z}{(z-2)(z+2)} &= a_{-1}(z-2)^{-1} + a_0 + \cdots \implies \frac{\sin z}{z+2} = a_{-1} + a_0(z-2) + \cdots \\ \implies \operatorname{Res}(2) &= a_{-1} = \left. \frac{\sin z}{z+2} \right|_2 = \frac{\sin 2}{4} \\ \frac{\sin z}{(z-2)(z+2)} &= b_{-1}(z+2)^{-1} + b_0 + \cdots \implies \frac{\sin z}{z-2} = b_{-1} + b_0(z+2) + \cdots \\ \implies \operatorname{Res}(-2) &= b_{-1} = \left. \frac{\sin z}{z-2} \right|_{-2} = \frac{\sin(-2)}{-4} = \frac{\sin 2}{4} \end{aligned}$$

so by the Residue theorem, the integral is equal to

$$2\pi i (\operatorname{Res}(2) + \operatorname{Res}(-2)) = 2\pi i \left(\frac{\sin 2}{4} + \frac{\sin 2}{4} \right) = \pi i \sin 2$$

□

(d) $\oint_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)} dz$

Solution. The integrand has simple poles at 2 and $-5i$, and a pole of order 2 at 0. Only the poles at 2 and 0 are contained inside $|z| = 3$, so we calculate the residues as

$$\begin{aligned} \frac{e^{iz}}{z^2(z-2)(z+5i)} &= a_{-1}(z-2)^{-1} + a_0 + \cdots \implies \frac{e^{iz}}{z^2(z+5i)} = a_{-1} + a_0(z-2) + \cdots \\ \implies \operatorname{Res}(2) &= a_{-1} = \left. \frac{e^{iz}}{z^2(z+5i)} \right|_2 = \frac{e^{2i}}{2^2(2+5i)} = \frac{e^{2i}(2-5i)}{116} \\ \frac{e^{iz}}{z^2(z-2)(z+5i)} &= b_{-2}z^{-2} + b_{-1}z^{-1} + b_0 + \cdots \implies \frac{e^{iz}}{(z-2)(z+5i)} = b_{-2} + b_{-1}z + b_0 + \cdots \\ \implies \operatorname{Res}(0) &= b_{-1} = \left. \frac{d}{dz} \left[\frac{e^{iz}}{(z-2)(z+5i)} \right] \right|_0 = \frac{12-5i}{-100} \end{aligned}$$

so by the Residue theorem, the integral is equal to

$$2\pi i (\operatorname{Res}(2) + \operatorname{Res}(0)) = 2\pi i \left(\frac{e^{2i}(2-5i)}{116} - \frac{12-5i}{100} \right)$$

□

(e) $\oint_{|z|=1} \frac{1}{z^2 \sin z} dz$

Solution. The integrand has a pole of order 2 at 0, and simple poles at $k\pi$ for $k \in \mathbb{Z}$, so the only one inside $|z| = 1$ is the pole $z = 0$ of order 3. Then we calculate the residue as

$$\begin{aligned} \frac{1}{z^2 \sin z} &= a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \cdots \implies \frac{z}{\sin z} = a_{-3} + a_{-2}z + a_{-1}z^2 + a_0z^3 + \cdots \\ \implies \text{Res}(0) &= a_{-1} = \lim_{z \rightarrow 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{z}{\sin z} \right] = \frac{1}{6} \end{aligned}$$

so by the Residue theorem, the integral is equal to

$$2\pi i \text{Res}(0) = \frac{\pi i}{3}$$

□

(f) $\oint_{|z|=3} \frac{3z+1}{z^4+1} dz$

Solution. The integrand has simple poles at the solutions to

$$z^4 + 1 = 0 \implies z^4 = -1 = e^{\pi i} \implies z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4} = r_1, r_2, r_3, r_4$$

where we have

$$\begin{aligned} \text{Res}\left(e^{\pi i/4}\right) &= \frac{3z+2}{(z-r_2)(z-r_3)(z-r_4)} \Big|_{r_1} = \frac{3r_1+2}{(r_1-r_2)(r_1-r_3)(r_1-r_4)} \\ \text{Res}\left(e^{3\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_3)(z-r_4)} \Big|_{r_2} = \frac{3r_2+2}{(r_2-r_1)(r_2-r_3)(r_2-r_4)} \\ \text{Res}\left(e^{5\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_2)(z-r_4)} \Big|_{r_3} = \frac{3r_3+2}{(r_3-r_1)(r_3-r_2)(r_3-r_4)} \\ \text{Res}\left(e^{7\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_2)(z-r_3)} \Big|_{r_4} = \frac{3r_4+2}{(r_4-r_1)(r_4-r_2)(r_4-r_3)} \end{aligned}$$

so by the Residue theorem, the integral is equal to

$$\begin{aligned} &2\pi i \left(\text{Res}\left(e^{\pi i/4}\right) + \text{Res}\left(e^{3\pi i/4}\right) + \text{Res}\left(e^{5\pi i/4}\right) + \text{Res}\left(e^{7\pi i/4}\right) \right) \\ &= 2\pi i \left(\frac{(3r_1+2)(r_2-r_3)(r_2-r_4)(r_3-r_4) - (3r_2+2)(r_1-r_3)(r_1-r_4)(r_3-r_4)}{(r_1-r_2)(r_1-r_3)(r_1-r_4)(r_2-r_3)(r_2-r_4)(r_3-r_4)} \right) \\ &+ 2\pi i \left(\frac{(3r_3+2)(r_1-r_2)(r_1-r_4)(r_2-r_4) - (3r_4+2)(r_1-r_2)(r_1-r_3)(r_2-r_3)}{(r_1-r_2)(r_1-r_3)(r_1-r_4)(r_2-r_3)(r_2-r_4)(r_3-r_4)} \right) \\ &= 0 \end{aligned}$$

□

5. Is there a function f having a simple pole at z_0 with $\text{Res}(f; z_0) = 0$? How about a function with a pole of order 2 at z_0 and $\text{Res}(f; z_0) = 0$?

Answer. The first scenario is impossible, because since poles are isolated, there exists a sufficiently small ε such that z_0 is the only pole contained in $B_\varepsilon(z_0)$, and the integral around this circle would be 0 because $\text{Res}(f; z_0) = 0$, but it should be $2\pi i$.

For the second scenario, we can take $f(z) = 1/z^2$, which has a pole of order 2 at 0 but $\text{Res}(f; 0) = 0$.

7. Evaluate

$$\oint_{|z|=1} e^{1/z} \sin(1/z) dz$$

Solution. The integrand has a pole at 0. We have the Taylor series

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \dots \\ \sin(1/z) &= \frac{1}{z} - \frac{(1/z)^3}{3!} + \dots \\ \implies e^{1/z} \sin(1/z) &= \left(1 + \frac{1}{z} + \dots\right) \left(\frac{1}{z} - \dots\right) = \frac{1}{z} + \dots \\ \implies \operatorname{Res}(0) &= 1 \end{aligned}$$

so the integral is $2\pi i \operatorname{Res}(0) = 2\pi i$. □

Section 6.2

1. $\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \frac{2\pi}{\sqrt{3}}$

Solution. Using the substitution $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z}\right)$ along the parametrization $e^{i\theta}$ of the unit circle,

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \oint_{|z|=1} \frac{1}{2 + \frac{1}{2i} \left(z - \frac{1}{z}\right)} \cdot \frac{1}{iz} dz = \oint_{|z|=1} \frac{2}{(z^2 + 4iz - 1)}$$

The poles are at the roots

$$r_1, r_2 = \frac{-4i \pm \sqrt{(4i)^2 + 4}}{2} = (-2 \pm \sqrt{3})i$$

where only the root $r_1 = (-2 + \sqrt{3})i$ lies inside the unit circle, so

$$\operatorname{Res}(r_1) = \frac{2}{z - r_2} \Big|_{r_1} = \frac{2}{(-2 + \sqrt{3})i - (-2 - \sqrt{3})i} = \frac{1}{i\sqrt{3}}$$

so the integral is $2\pi i \operatorname{Res}(r_1) = 2\pi i \cdot \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$. □

5. $\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1-a^2}}, \quad a^2 < 1$

Solution. Using the substitution $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z}\right)$ along the parametrization $e^{i\theta}$ of the unit circle,

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \oint_{|z|=1} \frac{1}{1 + a \cdot \frac{1}{2} \left(z + \frac{1}{z}\right)} \cdot \frac{1}{iz} dz = \oint_{|z|=1} \frac{2}{ai \left(z^2 + \frac{2}{a}z + 1\right)} dz$$

The poles are at the roots

$$r_1, r_2 = \frac{-\frac{2}{a} \pm \sqrt{\left(\frac{2}{a}\right)^2 - 4}}{2} = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

where only the root $r_1 = \frac{-1 + \sqrt{1-a^2}}{a}$ lies inside the unit circle, so

$$\operatorname{Res}(r_1) = \frac{2}{ai(z - r_2)} \Big|_{r_1} = \frac{2}{ai \left(\frac{-1 + \sqrt{1-a^2}}{a} - \frac{-1 - \sqrt{1-a^2}}{a}\right)} = \frac{1}{i\sqrt{1-a^2}}$$

so the integral is $2\pi i \operatorname{Res}(r_1) = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$. □

8. $\int_0^{2\pi} \frac{d\theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} = \frac{2\pi}{ab}, \quad a, b > 0$

Solution. Using the substitutions on the unit circle, we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} &= \oint \frac{1}{a^2 \cdot \left(\frac{z^2-1}{2iz}\right)^2 + b^2 \cdot \left(\frac{z^2+1}{2z}\right)} \cdot \frac{1}{iz} dz \\ &= \frac{1}{i} \oint \frac{4z}{(b^2 - a^2)z^4 + (2b^2 + 2a^2)z^2 + (b^2 - a^2)} dz \end{aligned}$$

We have

$$z^2 = \frac{-(2b^2 + 2a^2) \pm \sqrt{(2b^2 + 2a^2)^2 - 4(b^2 - a^2)^2}}{2(b^2 - a^2)} = \frac{b^2 \pm 2ab + a^2}{a^2 - b^2}$$

WLOG $a \geq b$, so the roots are

$$r_1 = \frac{a-b}{\sqrt{a^2-b^2}}, \quad r_2 = \frac{b-a}{\sqrt{a^2-b^2}}, \quad r_3 = \frac{b+a}{\sqrt{a^2-b^2}}, \quad r_4 = \frac{-b-a}{\sqrt{a^2-b^2}}$$

where r_1, r_2 lie within the unit circle. We have

$$\begin{aligned} \text{Res}(r_1) &= \frac{4z}{(b^2 - a^2)(z - r_2) \left(z^2 - \frac{(a+b)^2}{a^2-b^2}\right)} \Big|_{r_1} = \frac{4 \cdot \frac{a-b}{\sqrt{a^2-b^2}}}{(b^2 - a^2) \cdot 2 \cdot \frac{a-b}{\sqrt{a^2-b^2}} \left(\frac{(a-b)^2}{a^2-b^2} - \frac{(a+b)^2}{a^2-b^2}\right)} \\ &= \frac{2}{(a+b)^2 - (a-b)^2} = \frac{1}{2ab} \\ \text{Res}(r_2) &= \frac{4z}{(b^2 - a^2)(z - r_1) \left(z^2 - \frac{(a+b)^2}{a^2-b^2}\right)} \Big|_{r_2} = \frac{4 \cdot \frac{b-a}{\sqrt{a^2-b^2}}}{(b^2 - a^2) \cdot 2 \cdot \frac{b-a}{\sqrt{a^2-b^2}} \left(\frac{(b-a)^2}{a^2-b^2} - \frac{(a+b)^2}{a^2-b^2}\right)} \\ &= \frac{2}{(a+b)^2 - (b-a)^2} = \frac{1}{2ab} \end{aligned}$$

so by the Residue theorem, the integral is

$$2\pi i \cdot \frac{1}{i} (\text{Res}(r_1) + \text{Res}(r_2)) = 2\pi \left(\frac{1}{2ab} + \frac{1}{2ab} \right) = \frac{2\pi}{ab}$$

□

9. $\int_0^{2\pi} (\cos \theta)^{2n} d\theta = \frac{\pi \cdot (2n)!}{2^{2n-1} (n!)^2}, \quad n = 1, 2, \dots$

Solution. Using the substitutions on the unit circle, we have

$$\begin{aligned} \int_0^{2\pi} (\cos \theta)^{2n} d\theta &= \oint \left[\frac{1}{2} \left(z + \frac{1}{z} \right) \right]^{2n} \cdot \frac{1}{iz} dz \\ &= \frac{1}{2^{2n} i} \oint \frac{1}{z} \left[z^{2n} + \binom{2n}{1} z^{2n-2} + \dots + \binom{2n}{n} + \dots + \binom{2n}{2n-1} \frac{1}{z^{2n-2}} + \frac{1}{z^{2n}} \right] dz \\ &= \frac{1}{2^{2n} i} \oint \left[z^{2n-1} + \binom{2n}{1} z^{2n-3} + \dots + \binom{2n}{n} \frac{1}{z} + \dots + \binom{2n}{2n-1} \frac{1}{z^{2n-1}} + \frac{1}{z^{2n+1}} \right] dz \\ &= \frac{1}{2^{2n} i} \cdot 2\pi i \binom{2n}{n} = \frac{\pi (2n)!}{2^{2n-1} (n!)^2} \end{aligned}$$

□

10. $\int_0^{2\pi} e^{\cos \theta} \cos(n\theta - \sin \theta) d\theta = \frac{2\pi}{n!}, \quad n = 1, 2, \dots$

Solution. We have

$$\int_0^{2\pi} e^{\cos \theta} \cdot \frac{1}{2} \left[e^{i(n\theta - \sin \theta)} + e^{i(-n\theta + \sin \theta)} \right] d\theta = \frac{1}{2} \int_0^{2\pi} (e^{\cos \theta - i \sin \theta + in\theta} + e^{\cos \theta + i \sin \theta - in\theta}) d\theta$$

and using the substitution $z = e^{i\theta} = \cos \theta + i \sin \theta$ along the unit circle, this is

$$\begin{aligned} & \frac{1}{2} \oint \left(e^{1/z} z^n + e^z z^{-n} \right) \cdot \frac{1}{iz} dz = \frac{1}{2i} \left(\oint e^{1/z} z^n dz + \oint e^z z^{-n} dz \right) \\ &= \frac{1}{2i} \left[\oint z^{n-1} \left(1 + \frac{1}{z} + \frac{(1/z)^2}{2} + \dots + \frac{(1/z)^n}{n!} \right) dz + \oint z^{-n-1} \left(1 + z + \frac{z^2}{2} + \dots + \frac{z^n}{n!} + \dots \right) dz \right] \\ &= \frac{1}{2i} \left[\oint \left(\dots + \frac{1}{n!} \frac{1}{z} + \dots \right) dz + \oint \left(\dots + \frac{1}{n!} \frac{1}{z} + \dots \right) dz \right] = \frac{1}{2i} \cdot 2\pi i \cdot \frac{2}{n!} = \frac{2\pi}{n!} \end{aligned}$$

□