Homework 9

Aleck Zhao

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Chapter 9: Category

5. If A is a subset of $\mathbb{R}$ and if x is in the interior of A, show that x is a point of continuity for χ_A (the characteristic function of A). Are there any other points of continuity?

   Proof. Since $x \in A^\circ$, there exists $\delta > 0$ such that $B_\delta(x) \subset A$. Then let $\varepsilon > 0$. Since $x \in A$ as well,

   $$\chi_A\left(B_\delta(x)\right) = \{1\} \subset B_\varepsilon(\chi_A(x)) = B_\varepsilon(1)$$

   so χ_A is continuous at x. Then χ_A is also continuous on A^c by a similar argument. □

9. If E is a closed set in $\mathbb{R}$, show that $E = D(f)$ for some bounded function f. (Hint: A sum of two characteristic functions will do the trick.)

   Proof. Let $f = \chi_{\partial E} + \chi_{E^\circ}$. Then the points of discontinuity are exactly ∂E and E°, and since E is closed, $E = \overline{E} = \partial E \cup E^\circ$. □

12. More generally, in any metric space, show that every open set is an F_σ and that every close set is a G_δ.

14. Prove that A has an empty interior in M if and only if A^c is dense in M.

    Proof. $(\implies)$: If $A^\circ = \varnothing$, then $(A^\circ)^c = M = \overline{A^c}$, so A^c is dense in M.
    $(\impliedby)$: If A^c is dense in M, then $\overline{A^c} = M = (A^\circ)^c$ so $A^\circ = \varnothing$. □

28. In a metric space M, show that any subset of a first category set is still first category, and that a countable union of first category sets is again first category.

    Proof. Let $A \subset M$ be first category, so $A = \bigcup_{n=1}^\infty E_n$ for nowhere dense sets $E_n \subset M$. Then if $B \subset A$,

    $$B = A \cap B = \left(\bigcup_{n=1}^\infty E_n\right) \cap B = \bigcup_{n=1}^\infty (E_n \cap B)$$

    Now, since E_n is nowhere dense, we have $\left(\overline{E_n}\right)^\circ = \varnothing$, so

    $$\left(\overline{E_n \cap B}\right)^\circ = \left(\overline{E_n} \cap \overline{B}\right)^\circ = \left(\overline{E_n}\right)^\circ \cap \overline{B}^\circ = \varnothing$$

    so $E_n \cap B$ is also nowhere dense, so B is a countable union of nowhere dense sets, thus first category.
    If $A_1, A_2, \cdots$ are first category sets, then write $A_i = \bigcup_{n=1}^\infty E_{in}$ for all i where E_{in} is nowhere dense. Then we have

    $$\bigcup_{m=1}^\infty A_m = \bigcup_{m=1}^\infty \left(\bigcup_{n=1}^\infty E_{nm}\right)$$

    is also a countable union of nowhere dense sets, so the union is first category as well. □

30. Show that $\mathbb{N}$ is first category in $\mathbb{R}$ but second category in itself.

Proof. For any point $n \in \mathbb{N}$, we have $\left(\overline{\{n\}}\right)^{\circ} = \varnothing$, so each point is a nowhere dense set in $\mathbb{R}$. Then we have $\mathbb{N} = \bigcup_{n=1}^{\infty} \{n\}$ is a countable union of nowhere dense sets in $\mathbb{R}$, so $\mathbb{N}$ is first category in $\mathbb{R}$.

Since every subset of $\mathbb{N}$ is open in $\mathbb{N}$, there are no nowhere dense subsets, since $(\overline{E})^{\circ} = \overline{E} \neq \varnothing$ for any $E \subset \mathbb{N}$. Thus, $\mathbb{N}$ is second category in itself. □

32. In $\mathbb{R}$, show that any open interval (and hence any nonempty, open set) is a second category set.

Proof. Let $a < b$ and suppose (a, b) is first category. Then since $\mathbb{R}$ is complete, $(a, b)^c$ must be dense, but it is clearly not, since it does not intersect (a, b). Thus, (a, b) is second category. □

47. Let $\mathcal{P}$ be the vector space of all polynomials supplied with the norm $\|p\| = \max_{0 \le i \le n} |a_i|$, where $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}$. Show that P is not complete.

Proof. Let $E_n \subset \mathcal{P}$ be the subset of all polynomials of degree at most n. Then E_n is closed since any sequence of polynomials in E has degree at most n, so if it converges in $\mathcal{P}$, the result must have degree at most n and thus be in E_n.

Now, suppose $E_n^{\circ} \ni p = a_0 + a_1 x + \cdots + a_k x^k$. Then that means $B_{\varepsilon}(p) \subset E_n$ for some ε. Let $q := a_0 + a_1 x + \cdots + a_k x^k + \frac{\varepsilon}{2} x^{n+k+1}$. Then

$$\begin{aligned} \|p - q\| &= \left\| -\frac{\varepsilon}{2} x^{n+k+1} \right\| = \frac{\varepsilon}{2} \\ &\implies q \in B_{\varepsilon}(p) \end{aligned}$$

but $q \notin E_n$ since it has degree $n + k + 1 > n$. Thus, $B_{\varepsilon}(p) \not\subset E_n$ for any ε, so $E_n^{\circ} = \varnothing$, and thus E_n is nowhere dense in $\mathcal{P}$.

Now, we have $\mathcal{P} = \bigcup_{n=0}^{\infty} E_n$, which is a countable union of nowhere dense sets, so $\mathcal{P}$ is first category. By the Baire Category theorem, all complete spaces are category two in themselves, so $\mathcal{P}$ is not complete. □