Homework 8

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Section 8.2: Cauchy's Theorem

2. Partition D_n into conjugacy classes where n is odd.

Solution. Let $D_n = \langle r, s \mid r^n = s^2 = srsr = 1 \rangle$. There is the trivial conjugacy class $cl(1) = \{1\}$. Next, consider an arbitrary element r^k. We can conjugate it by r^i which gives

$$r^i r^k r^{-i} = r^k$$

and by sr^i which gives

$$(sr^i)r^k(sr^i)^{-1} = sr^i r^k r^{-i} s^{-1} = sr^k s^{-1} = r^{-k}$$

Thus, $cl(r^k) = \{r^k, r^{n-k}\}$ for all r^k. Finally, consider the element sr^k. Conjugating by r^i we have

$$r^i sr^k r^{-i} = sr^{-i} r^k r^{-i} = sr^{k-2i}$$

Conjugating by sr^i we have

$$(sr^i)sr^k(sr^i)^{-1} = sr^i sr^k r^{-i} s^{-1} = ssr^{-i} r^k r^{-i} s = sr^{2i-k}$$

Thus, the conjugacy class of sr^k is exactly

$$\{sr^{k-2i} \mid i \in \mathbb{Z}\}$$

Since n is odd, this set cycles through all exponents of r, so $cl(sr^k) = \{s, sr, \cdots, sr^{k-1}\}$.

Thus,

$$D_n = \{1\} \cup \coprod \{r^i, r^{n-i}\} \cup \{s, sr, \cdots, sr^{n-1}\}$$

□

Section 8.3: Group Actions

2. If $|G| = 24$ and G has a subgroup of order 8, show that G is not simple.

Proof. Let H be this subgroup, so $|G : H| = 3$. Thus by the Extended Cayley Theorem, there exists a homomorphism $\theta : G \to S_3$ where $\ker\theta$ is a normal subgroup in G. If $\ker\theta = \{1\}$, then $|\theta``(G)| \leq |S_3| = 6$, but $|G| = 36$, so this is a contradiction. Thus, $\ker\theta \neq \{1\}$ so G has a non-trivial normal subgroup, and is not simple.

□

4. Show that every group of order 15 is cyclic.

Proof. Suppose the group is G. By Cauchy's Theorem, the primes 3 and 5 divide the order of the group, so there exist elements of order 3 and order 5. Suppose $o(a) = 5$ and $o(b) = 3$. Since $A = \langle a \rangle$ has index 3, it is normal in G by Corollary 1. Thus, $gag^{-1} \in A$ for any $g \in G$, so we must have

$$bab^{-1} \in A \implies bab^{-1} = a^k$$

for some $1 \leq k \leq 4$. Note that $k \neq 0$ because otherwise $bab^{-1} = 1 \implies a = 1$.

Now, we claim $b^n ab^{-n} = a^{k^n}$ for all $n \geq 1$. Proceed by induction. The base case is trivially $bab^{-1} = a^k$ as we established earlier. Suppose $b^i ab^{-i} = a^{k^i}$ for some i. Then raise each side to the k power:

$$\begin{aligned}(b^i ab^{-i})^k &= (b^i ab^{-i})(b^i ab^{-i}) \cdots (b^i ab^{-i}) \\ &= b^i a^k b^{-i} \\ &= b^i (bab^{-1}) b^{-i} \\ &= b^{i+1} ab^{-(i+1)} \\ &= (a^{k^i})^k = a^{k^{i+1}}\end{aligned}$$

Thus, $b^{i+1} ab^{-(i+1)} = a^{k^{i+1}}$ so the claim is proven.

We have

$$\begin{aligned} a &= b^{-n} a^{k^n} b^n = b^{-(n+3)} a^{k^{n+3}} b^{n+3} \\ \implies b^3 a^{k^n} &= a^{k^{n+3}} b^3 \\ \implies a^{k^n} &= a^{k^{n+3}} \\ \implies a^{k^{n+3} - k^n} &= 1 \\ \implies a^{k^n (k^3 - 1)} &= 1 \end{aligned}$$

Now, since $o(a) = 5$, we must have $5 \mid k^n(k^3 - 1)$ for all n where $1 \leq k \leq 4$. Obviously $5 \nmid k^n$ so $5 \mid (k^3 - 1)$, and it's easy to check that this holds only for $k = 1$. Thus, $bab^{-1} = a$. Thus, $ba = ab$, so a and b commute. Consider the order of $o(ab) = n$. Then $(ab)^n = a^n b^n = 1$, and the smallest value this can happen for is 15. Thus, G contains an element of order 15, so it is cyclic, as desired.

□

14. Let $X = \mathbb{R}[x_1, \cdots, x_n]$, the polynomial ring in the indeterminates $x_1, \cdots, x_n$. Given $\sigma \in S_n$ and $f = f(x_1, \cdots, x_n) \in X$, define $\sigma \cdot f = f(x_{\sigma 1}, x_{\sigma 2}, \cdots, x_{\sigma n})$. Show that this is an action and describe the fixer. If $n = 3$, give three polynomials in the fixer and compute $S_3 \cdot g$ and $S(g)$, where $g(x_1, x_2, x_3) = x_1 + x_2$.

Proof. If ε is the identity permutation, we have

$$\varepsilon \cdot f = f(x_{\varepsilon 1}, \cdots, x_{\varepsilon n}) = f(x_1, \cdots, x_n) = f$$

Next, if $\sigma, \tau \in S_n$, we have

$$\begin{aligned}\sigma \cdot (\tau \cdot f) &= \sigma \cdot f(x_{\tau 1} \cdots, x_{\tau n}) \\ &= f(x_{\sigma(\tau 1)}, \cdots, x_{\sigma(\tau n)}) \\ &= f(x_{(\sigma\tau)1}, \cdots, x_{(\sigma\tau)n}) \\ &= (\sigma\tau) \cdot f\end{aligned}$$

Thus, this is a group action, as desired.

The fixer is the set $\{ \sigma \in S_n \mid \sigma \cdot f = f \}$. This means

$$\sigma \cdot f = f(x_{\sigma 1}, \cdots, x_{\sigma n}) = f(x_1, \cdots, x_n)$$

for all $f \in X$, so the only element in the fixer is ε.

In the case $n = 3$, three polynomials that are always fixed are

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\ f_2(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 \\ f_3(x_1, x_2, x_3) &= 2x_1 + 2x_2 + 2x_3 \end{aligned}$$

If $g(x_1, x_2, x_3) = x_1 + x_2$ and $S_3 = \{\varepsilon, (123), (132), (12), (13), (23)\}$ then we have

$$\begin{aligned} \varepsilon \cdot g(x_1, x_2, x_3) &= x_1 + x_2 \\ (123) \cdot g(x_1, x_2, x_3) &= g(x_2, x_3, x_1) = x_2 + x_3 \\ (132) \cdot g(x_1, x_2, x_3) &= g(x_3, x_1, x_2) = x_3 + x_1 \\ (12) \cdot g(x_1, x_2, x_3) &= g(x_2, x_1, x_3) = x_2 + x_1 \\ (13) \cdot g(x_1, x_2, x_3) &= g(x_3, x_2, x_1) = x_3 + x_2 \\ (23) \cdot g(x_1, x_2, x_3) &= g(x_1, x_3, x_2) = x_1 + x_3 \end{aligned}$$

So let $g_1 = x_1 + x_3$ and $g_2 = x_2 + x_3$. Then $S_3 \cdot g = \{g, g_1, g_2\}$.

From above, we see elements that fix g are just ε and (12), so $S(g) = \{\varepsilon, (12)\}$.

□

26. Let G be a finite p-group. If $\{1\} \neq H \trianglelefteq G$, show that $H \cap Z(G) \neq \{1\}$.

Proof. Every normal subgroup contains the center because $ghg^{-1} = h$ whenever $h \in Z(G)$. Since G is a p-group, its center is non-trivial. Thus, $H \cap Z(G) \neq \{1\}$, as desired.

□

Section 8.4: The Sylow Theorems

1. Find all Sylow 3-subgroups of S_4, and show explicitly that all are conjugate.

Solution. The Sylow 3-subgroups of S_4 have order 3. They are

$$\langle (123) \rangle, \langle (124) \rangle, \langle 234 \rangle, \langle (134) \rangle$$

We have

$$\begin{aligned} (34)(123)(34)^{-1} &= (124) \\ (234)(123)(234)^{-1} &= (134) \\ (1234)(123)(1234)^{-1} &= (234) \end{aligned}$$

so these groups are all conjugate, as desired.

□

2. Find all Sylow 2-subgroups of D_n, where n is odd, and show explicitly that all are conjugate.

Solution. Since $|D_n| = 2n$ and n is odd, the Sylow 2-subgroups have order 2. Thus, the Sylow 2-subgroups are given by

$$\{1, s\}, \{1, sr\}, \cdots, \{1, sr^{n-1}\}$$

Consider the subgroup $\{1, s\}$. Conjugating by sr^i we have

$$(sr^i)\{1, s\}(sr^i)^{-1} = \{1, r^i sr^{-i}\} = \{1, sr^{-2i}\} = \{1, sr^{n-2i}\}$$

Since n is odd, this will cycle through each possible exponent, and thus a suitable choice of i will allow us to generate any other Sylow 2-subgroup from $\{1, s\}$, so they are all conjugate.

□

10. Show that G has a cyclic normal subgroup of index 5 if

(a) $|G| = 385$

Proof. Since $385 = 5 \cdot 77$, a subgroup of index 5 has order 77. By Sylow's Third Theorem, we have $n_{77} \equiv 1 \pmod{77}$ and $n_{77} \mid 5$. Thus, n_{77} must be 1 or 5, but $5 \not\equiv 1 \pmod{77}$, so we must have $n_p = 1$. Thus, this subgroup is the only one of index 5, so it is unique, and thus normal.

Suppose this subgroup is H, where $|H| = 77 = 7 \cdot 11$, and consider Sylow p-subgroups of H. Then by Sylow's Theorem again, we have $n_7 \equiv 1 \pmod{7}$ and $n_5 \equiv \pmod{5}$ but since 5 and 7 are prime, it must be that $n_5 = n_7 = 1$. Thus, these subgroups are both unique in H are are both cyclic since they have prime order. Thus, by a theorem, $H \cong C_5 \times C_7$, which is cyclic, as desired.

□

(b) $|G| = 455$

Proof. Since $455 = 5 \cdot 91$, a subgroup of index 5 has order 91. Similarly to the previous part, we have $n_{91} \equiv 1 \pmod{91}$ so $n_{91} = 1$, so it is unique and normal. Then the subgroup of order 91 has $n_7 = n_{13} = 1$ subgroup of order 7 and 13, respectively, so these subgroups are normal. By a theorem, this subgroup is isomorphic to $C_7 \times C_{13}$, which is cyclic, as desired.

□

12. If $|G| = pq$ where $p < q$ are primes and p does not divide $q - 1$, show that G is cyclic.

Proof. We have $n_p \equiv 1 \pmod{p}$ and $n_p \mid q$, so since q is prime, $n_p = 1$ or $n_p = q$. If $n_p = q$, then $q \equiv 1 \pmod{p} \implies q - 1 \equiv 0 \pmod{p}$, but since p does not divide $q - 1$, this is a contradiction. Thus, $n_p = 1$, so the subgroup of order p is normal in G.

We also have $n_q \equiv 1 \pmod{q}$ and $n_q \mid p$, so $n_q = 1$ or $n_q = p$. If $n_q = p$, then $p \equiv 1 \pmod{q}$, which is impossible since we assumed $p < q$. Thus, $n_q = 1$, so the subgroup of order q is unique and normal in G.

By a theorem, we have $G \cong C_p \times C_q$, which is cyclic since p and q are prime, as desired.

□

16. Let $P \trianglelefteq H$ and $H \trianglelefteq G$. If P is a Sylow subgroup of G, show that $P \trianglelefteq G$.

Proof. Let P, P' be two Sylow subgroups of G with the same order. Thus they must be conjugate, so suppose $P = g_0 P' g_0^{-1}$ for some $g_0 \in G$. We know that $P \trianglelefteq H$ so $hPh^{-1} = P$ for all $h \in H$. Substituting, we have

$$\begin{aligned} h(g_0 P' g_0^{-1})h^{-1} &= g_0 P' g_0^{-1} \\ \implies (g_0^{-1} h g_0) P' (g_0^{-1} h g_0)^{-1} &= P' \end{aligned}$$

for all $h \in H$. Since $H \trianglelefteq G$, it follows that $g_0^{-1} h g_0 \in H$, so it follows that $h_1 P' h_1^{-1} = P'$ for all $h_1 \in H$. Thus, $P' \trianglelefteq H$. However, since P is a Sylow subgroup of H and is normal in H, it must be unique. Thus $P = P'$ so there is exactly one Sylow subgroup P in G. Thus, $P \trianglelefteq G$, as desired.

□