Homework 8

Aleck Zhao

November 9, 2017

1 Matroids (40 points)

(a) Let U be a finite set, let $k \geq 0$ be an integer, and let $\mathcal{I} = \{S \subseteq U : |S| \leq k\}$. Prove that $(U, \mathcal{I})$ is a matroid.

Proof. We show that $(U, \mathcal{I})$ satisfies the 3 properties of matroids.

(i) Clearly $\varnothing \in \mathcal{I}$ because $|\varnothing| = 0 \leq k$.

(ii) If $F \in \mathcal{I}$, then $|F| \leq k$. If $F' \subseteq F$, then $|F'| \leq |F| \leq k$, so $F' \in \mathcal{I}$ as well.

(iii) If $F_1, F_2 \in \mathcal{I}$ with $|F_2| > |F_1|$, then we have

$$|F_2| \leq k \implies |F_1| \leq k - 1$$

Since $F_2 \setminus F_1 \neq \varnothing$, take $e \in F_2 \setminus F_1$, so

$$\begin{aligned} |F_1 \cup \{e\}| &= |F_1| + 1 \leq k \\ \implies F_1 \cup \{e\} &\in \mathcal{I} \end{aligned}$$

Thus, $(U, \mathcal{I})$ is a matroid. □

(b) Let U be a finite set and let $U_1, U_2, \ldots, U_k$ be a partition of U into nonempty disjoint subsets (where $k \geq 2$). Let $r_1, r_2, \ldots r_k$ be positive integers. Let $\mathcal{I} = \{S \subseteq U : |S \cap U_i| \leq r_i\}$ for all $i \in \{1, 2, \ldots k\}$. Prove that $(U, \mathcal{I})$ is a matroid.

Proof. We show that $(U, \mathcal{I})$ satisfies the 3 properties of matroids.

(i) Clearly $\varnothing \in \mathcal{I}$ because $|\varnothing \cap U| = |\varnothing| = 0 \leq r_i$ for all i since r_i are positive.

(ii) If $F \in \mathcal{I}$, then $|F \cap U_i| \leq r_i$ for all i. If $F' \subseteq F$, then $F' \cap U_i \subseteq F \cap U_i$ because if $x \in F' \cap U_i$, we have $x \in F' \implies x \in F$ and $x \in U_i$, so $x \in F \cap U_i$. Thus $|F' \cap U_i| \leq |F \cap U_i| \leq r_i$, so $F' \in \mathcal{I}$.

(iii) Let $F_1, F_2 \in \mathcal{I}$ with $|F_2| > |F_1|$. Since the U_i partition U, we have

$$\sum_{i=1}^{n} |F_2 \cap U_i| = |F_2| > |F_1| = \sum_{i=1}^{n} |F_1 \cap U_i|$$

If $|F_2 \cap U_i| \leq |F_1 \cap U_i|$ for all i, then $|F_2| \leq |F_1|$, a contradiction, so there exists some k such that $|F_2 \cap U_k| > |F_1 \cap U_k|$. Then $(F_2 \cap U_k) \setminus (F_1 \cap U_k) \neq \varnothing$, so take some $e \in (F_2 \cap U_k) \setminus (F_1 \cap U_k)$. Because the U_i are disjoint, $e \in U_k$ and $e \notin U_i$ for all $i \neq k$. We have

$$\begin{aligned} |(F_1 \cup \{e\}) \cap U_i| &= |(F_1 \cap U_i) \cup (\{e\} \cap U_i)| \\ &= \begin{cases} |(F_1 \cap U_i) \cup \varnothing| = |F_1 \cap U_i| \leq r_i & \text{if } i \neq k \\ |(F_1 \cap U_k) \cup \{e\}| \leq |F_2 \cap U_k| \leq r_k & \text{if } i = k \end{cases} \\ \implies |(F_1 \cup \{e\}) \cap U_i| &\leq r_i, \forall i \\ \implies F_1 \cup \{e\} &\in \mathcal{I} \end{aligned}$$

Thus, $(U, \mathcal{I})$ is a matroid. □

2 Cuts and Flows (60 points)

(a) Suppose you are given a directed graph $G = (V, E)$, two vertices s and t, a capacity function $c : E \to \mathbb{R}^+$, and a second function $f : E \to \mathbb{R}$. Give an $O(m + n)$-time algorithm to determine whether f is a maximum (s,t)-flow in G. As always, prove running time and correctness.

Solution. First, check the capacity constraint, so we must have $f(e) \le c(e)$ for all $e \in E$. Next, construct the residual graph G_f. Then f is a max (s,t)-flow in G if an only if there is no (s,t)-path in G_f, so perform a DFS from s in G_f. If t is found, then f not a max flow. If t is not found, then f is a max flow.

Checking the capacity takes $O(m)$ since we just need to iterate over all edges. Then the residual graph has at most $2m$ edges, one for each direction, so constructing it takes $O(n + 2m) = O(n + m)$. Finally, DFS takes $O(n + 2m) = O(n + m)$, so the total running time is $O(n + m)$. The algorithm is correct because a flow f is a max (s,t)-flow if and only if there is no (s,t)-path in G_f. $\square$

Cuts are sometimes defined as subsets of the edges of the graph, instead of as partitions of its vertices. We will sometimes go back and forth between these two definitions without making much of a distinction, so in this problem you will prove that these two definitions are almost equivalent.

Let $G = (V, E)$ be a directed graph, and let $s, t \in V$. We say that a set of edges $X \subseteq E$ *separates* s and t if every directed path from s to t contains at least one edge in X. For any subset S of vertices, let $\delta(S)$ denote the set of edges leaving S, i.e., let $\delta(S) = \{(u, v) \in E : u \in S, v \notin S\}$.

(b) Let $(S, \bar{S})$ be an (s,t) cut (i.e., $s \in S$ and $t \in \bar{S}$, where $\bar{S} = V \setminus S$). Prove that $\delta(S)$ separates s and t.

Proof. Suppose there exists a path from s to t with no edge in $\delta(S)$. Then such a path must stay entirely within S since it has no edges leaving S, but $t \in \bar{S} = V \setminus S$, which is a contradiction. Thus, every path must contain an edge in $\delta(S)$. $\square$

(c) Let X be an arbitrary subset of edges that separates s and t. Prove that there is an (s,t)-cut $(S, \bar{S})$ such that $\delta(S) \subseteq X$.

Proof. Consider $G' = (V, E \setminus X)$. Then G' is disconnected, since any (s,t)-path in G had an edge in X. In particular, s and t are in separate connected components. Let S be the component containing s. Then if $e \in \delta(S)$, it is an edge in G leaving S, so it can't be in G', since S is a connected component, so $e \in X$, and thus $\delta(S) \subset X$. $\square$

(d) Let X be a *minimal* subset of edges that separates s and t (so for all $X' \subsetneq X$, we know that X' does not separate s and t). Prove that there is an (s,t)-cut $(S, \bar{S})$ such that $\delta(S) = X$.

Proof. Consider $G' = (V, E \setminus X)$. Then G' is disconnected, and in particular must have exactly 2 connected components. If it has 3 or more, if we add an edge e connecting any two of them, G' is still disconnected since it has at least 2 connected components, so $(V, (E \setminus X) \cup \{e\}) = (V, E \setminus (X \setminus \{e\}))$ is disconnected, so $X \setminus \{e\}$ is a separating set, but we assumed that X was minimal. Contradiction, so there are exactly 2 connected components. s and t are in different components, so let $s \in S$ and $t \in V \setminus S = \bar{S}$. Then X is exactly the set of edges that start in S and end in $\bar{S}$, so $\delta(S) = X$. $\square$