

## Homework 3

ALECK ZHAO

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### Section 2.3

4. Using Definition 4, show that each of the following functions is nowhere differentiable.

(a)  $\operatorname{Re} z$

*Proof.* Suppose  $\operatorname{Re} z$  was differentiable at  $z_0 = a_0 + b_0i$ . Then

$$\begin{aligned} \frac{d(\operatorname{Re} z)}{dz}(z_0) &= \lim_{h \rightarrow 0} \frac{\operatorname{Re}(z_0 + h) - \operatorname{Re} z_0}{h} = \lim_{a+bi \rightarrow 0} \frac{\operatorname{Re}[(a_0 + b_0i) + (a + bi)] - \operatorname{Re}(a_0 + b_0i)}{a + bi} \\ &= \lim_{a+bi \rightarrow 0} \frac{(a_0 + a) - a_0}{a + bi} = \lim_{a+bi \rightarrow 0} \frac{a}{a + bi} \end{aligned}$$

This limit does not exist because if we go along the real axis, the limit is 1, but if we go along the imaginary axis, the limit is 0. Thus,  $\operatorname{Re} z$  is not differentiable at any point.  $\square$

(c)  $|z|$

*Proof.* Suppose  $|z|$  was differentiable at  $z_0 = a_0 + b_0i$ . Then

$$\begin{aligned} \frac{d(|z|)}{dz}(z_0) &= \lim_{h \rightarrow 0} \frac{|z_0 + h| - |z_0|}{h} = \lim_{a+bi \rightarrow 0} \frac{|(a_0 + b_0i) + (a + bi)| - |a_0 + b_0i|}{a + bi} \\ &= \lim_{a+bi \rightarrow 0} \frac{\sqrt{(a_0 + a)^2 + (b_0 + b)^2} - \sqrt{a_0^2 + b_0^2}}{a + bi} \end{aligned}$$

If we approach along the real axis,  $b = 0$ , so the limit is

$$\lim_{a \rightarrow 0} \frac{\sqrt{(a_0 + a)^2 + b_0^2} - \sqrt{a_0^2 + b_0^2}}{a} \rightarrow \infty$$

so the limit does not exist.  $\square$

8. Suppose that  $f$  is analytic at  $z_0$  and  $f'(z_0) \neq 0$ . Show that

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

and

$$\lim_{z \rightarrow z_0} \{\arg[f(z) - f(z_0)] - \arg(z - z_0)\} = \arg f'(z_0)$$

11. Discuss the analyticity of each of the following functions.

(b)  $\frac{z}{\bar{z}+2}$

*Solution.* We have the derivative at point  $z$  given by

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{z+h}{z+h+2} - \frac{z}{\bar{z}+2}}{h} = \lim_{h \rightarrow 0} \frac{(z+h)(\bar{z}+2) - z(\bar{z}+\bar{h}+2)}{h(\bar{z}+\bar{h}+2)(\bar{z}+2)} \\ &= \lim_{h \rightarrow 0} \frac{z\bar{z} + 2z + h\bar{z} + 2h - z\bar{z} - z\bar{h} - 2z}{h(\bar{z}+\bar{h}+2)(\bar{z}+2)} = \lim_{h \rightarrow 0} \frac{h\bar{z} + 2h - z\bar{h}}{h(\bar{z}+\bar{h}+2)(\bar{z}+2)}\end{aligned}$$

At  $z = 0$ , the limit is

$$\lim_{h \rightarrow 0} \frac{2h}{h(\bar{h}+2)(2)} = \lim_{h \rightarrow 0} \frac{1}{\bar{h}+2} = \frac{1}{2}$$

Otherwise if  $z \neq 0$ , then if we approach along the real axis,  $\bar{h} = h$ , so this limit is

$$\lim_{h \rightarrow 0} \frac{h(\bar{z}+2-z)}{h(\bar{z}+\bar{h}+2)(\bar{z}+2)} = \lim_{h \rightarrow 0} \frac{\bar{z}+2-z}{(\bar{z}+h+2)(\bar{z}+2)} = \frac{\bar{z}+2-z}{(\bar{z}+2)^2}$$

but if we approach along the imaginary axis,  $\bar{h} = -h$ , so this limit is

$$\lim_{h \rightarrow 0} \frac{h(\bar{z}+2+z)}{h(\bar{z}-h+2)(\bar{z}+2)} = \frac{\bar{z}+2+z}{(\bar{z}+2)^2}$$

These two limits are not equal as long as  $z \neq 0$ , so this function is not differentiable except at 0. Since 0 is not an open set in  $\mathbb{C}$ , this function is nowhere analytic.  $\square$

(f)  $\left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$

*Solution.* If  $z = x + yi$ , then  $z + \frac{1}{z} = (x + yi) + \frac{x-yi}{x^2+y^2} = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$ . Since  $z$  is analytic everywhere and  $1/z$  is analytic everywhere except 0, this is analytic everywhere but 0.  $\square$

(g)  $|z|^2 + 2z$

*Solution.* Since  $|z|$  is nowhere analytic, this is also nowhere analytic.  $\square$

## Section 2.4

3. Use Theorem 5 to show that  $g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$  is entire. Write this function in terms of  $z$ .

*Proof.* Here,  $u = 3x^2 + 2x - 3y^2 - 1$  and  $v = 6xy + 2y$ . We have

$$\begin{aligned}\frac{\partial u}{\partial x} &= 6x + 2 \\ \frac{\partial v}{\partial y} &= 6x + 2 \\ \frac{\partial u}{\partial y} &= -6y \\ \frac{\partial v}{\partial x} &= 6y\end{aligned}$$

so the Cauchy-Riemann equations are satisfied, and they are satisfied at all points in  $\mathbb{C}$ . The first partials are also all continuous, so  $g$  is entire.

Using the identities  $x = \frac{z+\bar{z}}{2}$  and  $y = \frac{z-\bar{z}}{2i}$ , we have

$$\begin{aligned} g(z) &= \left[ 3 \left( \frac{z+\bar{z}}{2} \right)^2 + 2 \left( \frac{z+\bar{z}}{2} \right) - 3 \left( \frac{z-\bar{z}}{2i} \right)^2 - 1 \right] + i \left[ 6 \left( \frac{z+\bar{z}}{2} \right) \left( \frac{z-\bar{z}}{2i} \right) + 2 \left( \frac{z-\bar{z}}{2i} \right) \right] \\ &= \left[ 3 \left( \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} \right) + (z + \bar{z}) - 3 \left( \frac{z^2 - 2z\bar{z} + \bar{z}^2}{-4} \right) - 1 \right] + \frac{3}{2}(z^2 - \bar{z}^2) + (z - \bar{z}) \\ &= \frac{3}{2}z^2 + \frac{3}{2}\bar{z}^2 + z + \bar{z} - 1 + \frac{3}{2}z^2 - \frac{3}{2}\bar{z}^2 + z - \bar{z} \\ &= 3z^2 + 2z - 1 \end{aligned}$$

□

5. Show that the function  $f(z) = e^{x^2-y^2} [\cos(2xy) + i \sin(2xy)]$  is entire, and find its derivative.

*Proof.* Here,  $u = e^{x^2-y^2} \cos(2xy)$  and  $v = e^{x^2-y^2} \sin(2xy)$ . We have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2xe^{x^2-y^2} \cos(2xy) - 2ye^{x^2-y^2} \sin(2xy) \\ \frac{\partial v}{\partial y} &= -2ye^{x^2-y^2} \sin(2xy) + 2xe^{x^2-y^2} \cos(2xy) \\ \frac{\partial u}{\partial y} &= -2ye^{x^2-y^2} \cos(2xy) - 2xe^{x^2-y^2} \sin(2xy) \\ \frac{\partial v}{\partial x} &= 2xe^{x^2-y^2} \sin(2xy) + 2ye^{x^2-y^2} \cos(2xy) \end{aligned}$$

so the Cauchy-Riemann equations are satisfied, the first partials are continuous, and thus  $f$  is analytic at every point in  $\mathbb{C}$ , so  $f$  is also entire. By De Moivre's theorem, the derivative is

$$\begin{aligned} f(z) &= e^{x^2-y^2} [\cos(2xy) + i \sin(2xy)] = e^{x^2-y^2} e^{2xyi} \\ &= e^{x^2+2xyi-y^2} = e^{(x+yi)^2} = e^{z^2} \\ \implies f'(z) &= 2ze^{z^2} \end{aligned}$$

□

8. Show that if  $f$  is analytic in a domain  $D$  and either  $\operatorname{Re} f(x)$  or  $\operatorname{Im} f(x)$  is constant in  $D$ , then  $f(z)$  must be constant in  $D$ .

*Proof.* Here,  $u = \operatorname{Re} f(x)$  and  $v = \operatorname{Im} f(x)$ . If  $f$  is analytic, and  $u \equiv c_1$  is constant, then  $u$  and  $v$  must satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \implies v \equiv c_2$$

Similarly, if  $v \equiv c_2$ , we get  $u \equiv c_1$  in order to satisfy the Cauchy-Riemann equations, and thus  $f(x) = c_1 + c_2i$ , which is a constant function. □

15. The Jacobian of a mapping

$$u = u(x, y), \quad v = v(x, y)$$

from the  $xy$ -plane to the  $uv$ -plane is defined to be the determinant

$$J(x_0, y_0) := \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

where the partial derivatives are all evaluated at  $(x_0, y_0)$ . Show that if  $f = u + iv$  is analytic at  $z_0 = x_0 + iy_0$ , then  $J(x_0, y_0) = |f'(z_0)|^2$ .

*Proof.* If  $f = u + iv$  is analytic at  $(x_0, y_0)$  then by the Cauchy-Riemann equations, we must have  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  where the partials are evaluated at  $(x_0, y_0)$ . The Jacobian is

$$\begin{aligned} \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} &= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^2 = \left|\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}\right|^2 \\ &= |f'(z_0)|^2 \end{aligned}$$

as desired.  $\square$

## Section 2.5

8. Suppose that the functions  $u$  and  $v$  are harmonic in a domain  $D$ .

(a) Is the sum  $u + v$  necessarily harmonic in  $D$ ?

*Solution.* We have

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(u + v) + \frac{\partial^2}{\partial y^2}(u + v) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \\ &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \end{aligned}$$

by the linearity of the differential operator, so  $u + v$  is also harmonic in  $D$ .  $\square$

(b) Is the product  $uv$  necessarily harmonic in  $D$ ?

*Solution.* This is not necessarily true. Take  $u = v = xy$ . Then  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  so  $u$  and  $v$  are both harmonic, but

$$\frac{\partial^2(uv)}{\partial x^2} + \frac{\partial^2(uv)}{\partial y^2} = \frac{\partial^2(x^2y^2)}{\partial x^2} + \frac{\partial^2(x^2y^2)}{\partial y^2} = 2y^2 + 2x^2 \neq 0$$

$\square$

(c) Is  $\partial u / \partial x$  harmonic in  $D$ ?

*Solution.* We have

$$\frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial}{\partial x} [0] = 0$$

because we can take partial derivatives in any order, so  $\partial u / \partial x$  is also harmonic in  $D$ .  $\square$

12. Prove that if  $r$  and  $\theta$  are polar coordinates, then the functions  $r^n \cos n\theta$  and  $r^n \sin n\theta$ , where  $n$  is an integer, are harmonic as functions of  $x$  and  $y$ .

*Proof.* If  $x = r \cos \theta$  and  $y = r \sin \theta$ , then we have

$$r^n \cos n\theta + i \cdot r^n \sin n\theta = (re^{i\theta})^n = z^n$$

Thus, since  $f(z) = z^n$  is analytic, it follows that  $\operatorname{Re} f(z) = r^n \cos n\theta$  and  $\operatorname{Im} f(z) = r^n \sin n\theta$  are both harmonic.  $\square$

13. Find a function harmonic inside the wedge bounded by the non-negative  $x$ -axis and the half-line  $y = x$  ( $x \geq 0$ ) that goes to 0 on these sides but is not identically zero.

*Solution.* If  $z = re^{i\theta}$ , then  $f(z) = \operatorname{Im} z^4 = \operatorname{Im}(r^4 e^{4\theta i}) = r^4 \sin 4\theta = 0$  exactly on the half lines  $\theta = 0$  and  $\theta = \pi/4$ , but it is not identically zero.  $\square$