Homework 9

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Section 5.2

4. Let α be a complex number. Show that if $(1+z)^\alpha$ is taken as $e^{\alpha \operatorname{Log}(1+z)}$, then for $|z|<1$

$$(1+z)^\alpha = 1 + \frac{\alpha}{1}z + \frac{\alpha(\alpha-1)}{1\cdot 2}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}z^3 + \cdots$$

Proof. We have

$$\begin{aligned}
\frac{d}{dz}\left[(1+z)^\alpha\right] &= \alpha(1+z)^{\alpha-1} \\
\frac{d^2}{dz}\left[(1+z)^\alpha\right] &= \alpha(\alpha-1)(1+z)^{\alpha-2} \\
&\vdots \\
\frac{d^j}{dz^j}\left[(1+z)^\alpha\right] &= \alpha(\alpha-1)\cdots(\alpha-j+1)(1+z)^{\alpha-j}
\end{aligned}$$

and since $(1+z)^\alpha$ is analytic on the disc $|z|<1$, it is given by its Maclaurin series, which is

$$(1+z)^\alpha = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}z^j = \sum_{j=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}z^j$$

which is the form desired. $\square$

5. Find and state the convergence properties of the Taylor series for the following.

(a) $\frac{1}{1+z}$ around $z_0 = 0$

Solution. This is just $\frac{1}{1-(-z)} = \sum_{j=0}^{\infty}(-z)^j$, which converges for all $|z|<1$. $\square$

(c) $z^3 \sin 3z$ around $z_0 = 0$

Solution. f is analytic on the entire complex plane, so the Taylor series converges for all z. $\square$

(e) $\frac{1+z}{1-z}$ around $z_0 = i$

Solution. f is analytic on $\mathbb{C}\setminus\{1\}$, where $|1-i| = \sqrt{2}$, so the Taylor series converges on the largest open disc centered at i which does not intersect 1, which is $|z-i|<\sqrt{2}$. $\square$

(g) $\frac{z}{(1-z)^2}$ around $z_0 = 0$

Solution. f is analytic on $\mathbb{C}\setminus\{1,-1\}$, so the Taylor series converges on the largest open disc centered at 0 which does not intersect these points, which is $|z|<1$. $\square$

8. Use Taylor series to verify the following identities

(d) $e^{2z} = e^z \cdot e^z$

Solution. We have

$$
\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!} \\
\implies e^z \cdot e^z &= \sum_{j=0}^{\infty} c_j z^j
\end{aligned}
$$

where

$$
\begin{aligned}
c_j &= \sum_{\ell=0}^{j} a_{j-\ell} b_\ell = \sum_{\ell=0}^{j} \frac{1}{(j-\ell)!} \cdot \frac{1}{\ell!} = \frac{1}{j!} \sum_{\ell=0}^{j} \frac{j!}{(j-\ell)!\ell!} = \frac{1}{j!} \cdot 2^j \\
\implies e^z \cdot e^z &= \sum_{j=0}^{\infty} \frac{2^j}{j!} z^j = \sum_{j=0}^{\infty} \frac{(2z)^j}{j!} \\
&= e^{2z}
\end{aligned}
$$

□

11. Using Theorem 6 for computing the product of Taylor series, find the first three nonzero terms in the Maclaurin expansion of the following

(a) $e^z \cos z$

Solution. Let $f = e^z$ and $g = \cos z$, with Taylor expansions

$$
\begin{aligned}
f &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
g &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots
\end{aligned}
$$

so the Cauchy product of the two Taylor series is

$$
fg = \sum_{j=0}^{\infty} c_j z^j
$$

where

$$
\begin{aligned}
c_0 &= 1 \cdot 1 = 1 \\
c_1 &= 1 \cdot 1 = 1 \\
c_2 &= 1 \cdot \left(-\frac{1}{2!}\right) + \frac{1}{2!} \cdot 1 = 0 \\
c_3 &= 1 \cdot \left(-\frac{1}{2!}\right) + \frac{1}{3!} \cdot 1 = -\frac{1}{3}
\end{aligned}
$$

so the first three terms are

$$
fg = e^z \cos z = 1 + z - \frac{1}{3} z^3 + \cdots
$$

□

Section 5.4

10. The defining relations for the terms are

$$a_0 = a_1 = 1$$
$$a_n = a_{n-1} + a_{n-2} \quad (n \geq 2)$$

Show that

$$f(z) := a_0 + a_1 z + a_2 z^2 + \cdots$$

defines an analytic function satisfying the equation

$$f(z) = 1 + zf(z) + z^2 f(z)$$

Solve for $f(z)$ and compute the Maclaurin series to derive the expression

$$a_j = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{j+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+1}\right]$$

Proof. We have

$$\begin{aligned} zf(z) &= 0 + a_0 z + a_1 z^2 + a_2 z^3 + \cdots \\ z^2 f(z) &= 0 + 0z + a_0 z^2 + a_1 z^3 + \cdots \\ \implies 1 + zf(z) + z^2 f(z) &= 1 + a_0 z + (a_0 + a_1) z^2 + (a_1 + a_2) z^3 + \cdots \\ &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = f(z) \end{aligned}$$

by applying $a_0 = a_1 = 1$ and the recursive definition of a_n. We have

$$f(z) = 1 + zf(z) + z^2 f(z) \implies f(z) = \frac{1}{1 - z - z^2}$$

The denominator has roots $\frac{1\pm\sqrt{5}}{2}$, so let $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$. Then we have the partial fraction decomposition

$$\begin{aligned} \frac{-1}{(z-r_1)(z-r_2)} &= \frac{A}{r_1 - z} + \frac{B}{r_2 - z} \\ \implies -1 &= A(r_2 - z) + B(r_1 - z) \end{aligned}$$

and substituting $z = r_1, r_2$, we have the equations

$$\begin{aligned} -1 &= A(r_2 - r_1) = A\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right) = A\left(-\sqrt{5}\right) \implies A = \frac{1}{\sqrt{5}} \\ -1 &= B(r_1 - r_2) = B\left(\sqrt{5}\right) \implies B = -\frac{1}{\sqrt{5}} \end{aligned}$$

and thus the partial fraction decomposition

$$f(z) = \frac{-1}{1 - z - z^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{r_1 - z} - \frac{1}{\sqrt{5}} \cdot \frac{1}{r_2 - z} = \frac{1}{\sqrt{5}} \cdot \frac{1/r_1}{1 - (z/r_1)} - \frac{1}{\sqrt{5}} \cdot \frac{1/r_2}{1 - (z/r_2)}$$

and using the Taylor expansion of $(1 - (z/r_1))^{-1}$ and $(1 - (z/r_2))^{-1}$, we have

$$f(z) = \frac{1}{\sqrt{5}r_1} \cdot \sum_{j=0}^{\infty} \left(\frac{z}{r_1}\right)^j - \frac{1}{\sqrt{5}r_2} \sum_{j=0}^{\infty} \left(\frac{z}{r_2}\right)^j = \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left(\frac{1}{r_1^{j+1}} - \frac{1}{r_2^{j+1}}\right) z^j = \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left(\frac{r_2^{j+1} - r_1^{j+1}}{r_1^{j+1} r_2^{j+1}}\right) z^j$$

where $r_1 r_2 = \frac{1+\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2} = -1$, so this is

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left((-r_2)^{j+1} + r_1^{j+1} \right) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} (-1)^{j+1} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{j+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{j+1} \right] z^j$$

$$\implies a_j = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{j+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{j+1} \right]$$

□

Section 5.6

2. What is the order of the pole of

$$f(z) = \frac{1}{(2\cos z - 2 + z^2)^2}$$

at $z = 0$? (Hint: Work with $1/f(z)$.)

Solution. Consider $g(z) = \sqrt{1/f(z)} = 2\cos z - 2 + z^2$. Then the order of the pole of $f(z)$ is twice the degree of the zero $z = 0$ of $g(z)$. We have

$$\begin{aligned} g(0) &= 2\cos 0 - 2 + 0^2 = 0 \\ \implies g'(z) &= -2\sin z + 2z \implies g'(0) = -2\sin 0 + 2 \cdot 0 = 0 \\ \implies g''(z) &= -2\cos z + 2 \implies g''(0) = -2\cos 0 + 2 = 0 \\ \implies g^{(3)}(z) &= 2\sin z \implies g^{(3)}(0) = 2\sin 0 = 0 \\ \implies g^{(4)}(z) &= -2\cos z \implies g^{(4)}(0) = -2\cos 0 = -2 \neq 0 \end{aligned}$$

so $z = 0$ is a zero of order 4 for g, and thus a pole of order 8 for $f(z)$. □

3. Construct a function f, analytic in the plane except for isolated singularities, that satisfies the given conditions.

   (a) f has a zero of order 2 at $z = i$ and a pole of order 5 at $z = 2 - 3i$.

   Solution. Let

$$f = \frac{(z-i)^2}{(z - 2 + 3i)^5}$$

□

5. For each of the following, determine whether the statement made is always true or sometimes false.

   (a) If f and g have a pole at z_0, then $f + g$ has a pole at z_0.

   Answer. This is sometimes false. Take $f = \frac{z}{z - z_0}$ and $g = \frac{-z_0}{z - z_0}$. Then both f and g have a pole at z_0, but $f + g = \frac{z - z_0}{z - z_0} = 1$ does not have any poles.

   (c) If $f(z)$ has a pole of order m at $z = 0$, then $f(z^2)$ has a pole of order $2m$ at $z = 0$.

   Answer. This is always true. Let $f(z) = z^m g(z)$, where $g(0) \neq 0$. Then $f(z^2) = z^{2m} g(z^2)$, where $g(0^2) = g(0) \neq 0$.

16. Sketch the graphs for $s = 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \cdots$ of the level curves $\left| e^{1/z} \right| = s$, and observe that they all converge at the essential singularity $z = 0$ of $e^{1/z}$. (Hint: the level curves are all circles.)