# Homework 4

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## Chapter 3: Metrics and Norms

6. If $d$ is any metric on $M$, show that $\rho(x,y) = \sqrt{d(x,y)}, \sigma(x,y) = \frac{d(x,y)}{1+d(x,y)}$, and $\tau(x,y) = \min\{d(x,y), 1\}$ are also metrics on $M$.

*Proof.* $\rho$ : Clearly $\rho$ is non-negative since $d$ is non-negative by being a metric, and

$$\rho(x,y) = 0 = \sqrt{d(x,y)} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because $d$ is symmetric, and finally

$$\begin{aligned}
\rho(x,y) + \rho(y,z) &= \sqrt{d(x,y)} + \sqrt{d(y,z)} \\
\implies [\rho(x,y) + \rho(y,z)]^2 &= d(x,y) + d(y,z) + 2\sqrt{d(x,y)d(y,z)} \\
&\geq d(x,z) + 2\sqrt{d(x,y)d(y,z)} \geq d(x,z) \\
\implies \rho(x,y) + \rho(y,z) &\geq \sqrt{d(x,z)} = \rho(x,z)
\end{aligned}$$

$\sigma$ : Clearly $\sigma$ is non-negative since $d$ is non-negative, and

$$\sigma(x,y) = 0 = \frac{d(x,y)}{1+d(x,y)} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because $d$ is symmetric. Now, define $F(t) = \frac{t}{1+t}$. Then $F'(t) = \frac{1}{(1+t)^2} > 0$ so $F$ is increasing, and we have

$$\begin{aligned}
F(t) + F(s) &= \frac{t}{1+t} + \frac{s}{1+s} = \frac{t+ts+s+st}{(1+t)(1+s)} = \frac{s+t+2st}{1+s+t+st} \\
&= \frac{s+t+st}{1+s+t+st} + \frac{st}{1+s+t+st} = F(s+t+st) + \frac{st}{1+s+t+st} \\
&\geq F(s+t)
\end{aligned}$$

since $F$ is increasing since $F'(t) = (1+t)^{-2} > 0$. Thus,

$$\begin{aligned}
\sigma(x,y) + \sigma(y,z) &= F(d(x,y)) + F(d(y,z)) \geq F\left(d(x,y) + d(y,z)\right) \\
&\geq F(d(x,z)) = \sigma(x,z)
\end{aligned}$$

$\tau$ : Clearly $\tau$ is non-negative since $d$ and 1 are non-negative, and

$$\tau(x,y) = 0 = \min\{d(x,y), 1\} \iff d(x,y) = 0 \iff x = y$$

It is also symmetric because $d$ is symmetric. Suppose that

$$\begin{aligned}
\tau(x,y) + \tau(y,z) &< \tau(x,z) \\
\min\{d(x,y), 1\} + \min\{d(y,z), 1\} = m_1 + m_2 &< \min\{d(x,z), 1\} \\
\implies m_1 + m_2 < 1, &\quad m_1 + m_2 < d(x,z)
\end{aligned}$$

If $m_1 + m_2 < 1$, then we must have $m_1 = d(x,y)$ and $m_2 = d(y,z)$, but since $d$ is a metric, $m_1 + m_2 = d(x,y) + d(y,z) \geq d(x,z)$, so it is impossible for both conditions to be true. Contradiction, so $\tau(x,y) + \tau(y,z) \geq \tau(x,z)$, and $\tau$ is a metric. □

15. We define the diameter of a nonempty subset $A$ of $M$ by $\operatorname{diam}(A) = \sup\{d(a,b) : a,b \in A\}$. Show that $A$ is bounded if and only if $\operatorname{diam}(A)$ is finite.

    *Proof.* ( $\Longrightarrow$ ) : If $A$ is bounded, then $\exists x_0 \in M$ and $C < \infty$ such that $d(a, x_0) \le C$ for all $a \in A$. Then

    $$\operatorname{diam}(A) = \sup\{d(a,b) : a,b \in A\} \le \sup\{d(a,x_0) + d(x_0,b) : a,b \in A\} \le 2C < \infty$$

    ( $\Longleftarrow$ ) : If $\operatorname{diam}(A)$ is finite, say $s = \operatorname{diam}(A)$. Then take any $x_0 \in A \subset M$, and take $C = s$. Since $s$ is the supremum, it follows that

    $$C = s = \sup\{d(a,b) : a,b \in A\} \ge d(a,x_0)$$

    for any $a \in A$, so $A$ is bounded, as desired. □

22. Show that $\|x\|_\infty \le \|x\|_2$ for any $x \in \ell_2$, and that $\|x\|_2 \le \|x\|_1$ for any $x \in \ell_1$.

23. The subset of $\ell_\infty$ consisting of all sequences that converge to 0 is denoted by $c_0$. (Note that $c_0$ is actually a linear subspace of $\ell_\infty$; thus $c_0$ is also a normed vector space under $\|\cdot\|_\infty$.) Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

25. The same techniques can be used to show that $\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p}$ defines a norm on $C([0,1])$ for any $1 < p < \infty$. State and prove the analogues of Lemma 3.7 and Theorem 3.8 in this case. (Does Lemma 3.7 still hold in this setting for $p = 1$ and $q = \infty$?)

31. Give an example where $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$. If $A \cap B \ne \varnothing$, show that $\operatorname{diam}(A \cup B) \le \operatorname{diam}(A) + \operatorname{diam}(B)$.

37. A Cauchy sequence with a convergent subsequence converges.