Homework 8

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Chapter 8: Compactness

48. Prove that a uniformly continuous map sends Cauchy sequences into Cauchy sequences.

    Proof. Let $f : (M, d) \to (N, \rho)$ be uniformly continuous, let $(x_n) \subset M$ be Cauchy, and let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$ for $x, y \in M$. Then since (x_n) is Cauchy, $\exists N$ such that $d(x_n, x_m) < \delta$ whenever $n, m \geq N$, which means $\rho(f(x_n), f(x_m)) < \varepsilon$ for all $n, m \geq N$, by uniform continuity of f, so $f(x_n) \subset N$ is Cauchy. ☐

77. Fix $k \geq 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x_k$. Show that f is linear and has $\|f\| = 1$.

    Proof. Let $x, y \in \ell_\infty$ and $\alpha, \beta \in \mathbb{R}$. Then $\alpha x + \beta y = (\alpha x_i + \beta y_i)_{i=1}^\infty$, so

    $$\begin{aligned} f(\alpha x + \beta y) &= f\left[(\alpha x_i + \beta y_i)_{i=1}^\infty\right] = \alpha x_k + \beta y_k \\ &= \alpha f(x) + \beta f(y) \end{aligned}$$

    so f is linear. We have $|x_k| \leq \sup_i |x_i|$ for $x \in \ell_\infty$, with equality if x_k is the maximal element, so

    $$\|f\| = \inf\left\{C : |f(x)| = x_k \leq C \sup_i |x_i|\right\} = 1$$

    ☐

78. Define a linear map $f : \ell_2 \to \ell_2$ by $f(x) = (x_n/n)_{n=1}^\infty$. Is f bounded? If so, what is $\|f\|$?

    Proof. We claim f is bounded. Let $x \in \ell_2$. Then since $|x_n/n| \leq |x_n/1| = |x_n|$, we have

    $$\|f(x)\|_2 = \left(\sum_{n=1}^\infty \left|\frac{x_n}{n}\right|^2\right)^{1/2} \leq 1 \cdot \left(\sum_{n=1}^\infty |x_n|^2\right)^{1/2} = \|x\|_2$$

    as desired. Equality occurs when $x_1 \neq 0$ and $x_i = 0, \forall i \geq 2$, so no tighter bound exists, so $\|f\| = 1$. ☐

80. Show that the definite integral $I(f) = \int_a^b f(t)\,dt$ is continuous from $C[a, b]$ into $\mathbb{R}$. What is $\|I\|$?

    Proof. Let $\varepsilon > 0$, and let $f, g \in C[a, b]$. If we take $\|f\| = \int_a^b |f(t)|\,dt$, then let $\delta = \varepsilon$. We have

    $$\begin{aligned} \left|\int_a^b (f(t) - g(t))\,dt\right| &\leq \int_a^b |f(t) - g(t)|\,dt \\ \implies \left|\int_a^b (f(t) - g(t))\,dt\right| &< \varepsilon \text{ whenever } \int_a^b |f(t) - g(t)|\,dt < \varepsilon \\ \implies |I(f) - I(g)| &< \varepsilon \text{ whenever } \|f - g\| < \delta \end{aligned}$$

    as desired. Equality occurs when $f(t) > g(t)$ over $[a, b]$, so no tighter bound exists, so $\|I\| = 1$. ☐

81. Prove that the indefinite integral, defined by $T(f)(x) = \int_a^x f(t)\,dt$, is continuous as a map from $C[a,b]$ into $C[a,b]$. Estimate $\|T\|$.

Proof. Let $\varepsilon > 0$, and let $f, g \in C[a,b]$. If we take $\|f\| = \int_a^b |f(t)|\,dt$, then for $t \in [a,b]$, we have

$$\left|\int_a^t (f(s) - g(s))\,ds\right| \le \int_a^t |f(s) - g(s)|\,ds \le \int_a^b |f(s) - g(s)|\,ds$$

so if we let $\delta = \varepsilon/(b-a)$, then if

$$\|f - g\| = \int_a^b |f(t) - g(t)|\,dt < \delta = \frac{\varepsilon}{b-a}$$

we have

$$\begin{aligned}
\|T(f) - T(g)\| &= \int_a^b \left|\int_a^t (f(s) - g(s))\,ds\right| dt \\
&\le \int_a^b \left(\int_a^b |f(s) - g(s)|\,ds\right) dt \\
&= (b-a)\int_a^b |f(s) - g(s)|\,ds \\
&< (b-a)\cdot\frac{\varepsilon}{b-a} = \varepsilon
\end{aligned}$$

Thus, T is continuous, as desired. Let $h > 0$, and consider the function

$$f(t) = \begin{cases} -\frac{2(t-a-h)}{h^2} & \text{if } a \le t < a+h \\ 0 & \text{if } t > a+h \end{cases}$$

Then f is continuous on $[a,b]$, and defines a triangle of width h and height $2/h$, so $\|f\| = \int_a^b |f(t)|\,dt = 1$.

$$\begin{aligned}
\int_a^t f(s)\,ds &= \begin{cases} \frac{(t-a)(a+2h-t)}{h^2} & \text{if } a \le t < a+h \\ 1 & \text{if } t > a \end{cases} \\
\implies \|T(f)\| &= \int_a^b \left|\int_a^t f(s)\,ds\right| dt = \int_a^{a+h} \frac{(t-a)(a+2h-t)}{h^2}\,dt + \int_{a+h}^b 1\,dt \\
&= \frac{2h}{3} + (b - (a+h)) = b - a - \frac{h}{3}
\end{aligned}$$

By the result of #82, we have

$$\begin{aligned}
\|T\| &= \sup\{\|T(f)\| : \|f\| = 1\} \\
&\ge \sup\left\{b - a - \frac{h}{3} : h > 0\right\} = b - a
\end{aligned}$$

but on the other hand, from earlier, we had

$$\begin{aligned}
\|T(f) - T(g)\| &\le (b-a)\,\|f - g\| \\
\implies \|T\| &= \sup_{f \ne g} \frac{\|T(f) - T(g)\|}{\|f - g\|} \le b - a
\end{aligned}$$

so $\|T\| = b - a$. $\square$

82. For $T \in B(V,W)$, prove that $\|T\| = \sup\{\|\|Tx\|\| : \|x\| = 1\}$.

Proof. We have

$$\sup_{y\neq 0} \frac{\|\|Ty\|\|}{\|y\|} = \sup_{y\neq 0} \left\|\left\| \frac{1}{\|y\|} \cdot Ty \right\|\right\| = \sup_{x\neq 0} \left\|\left\| T\left(\frac{y}{\|y\|}\right)\right\|\right\|$$

Then if $x = y/\|y\|$, we have $\|x\| = 1$, so

$$\sup_{y\neq 0} \frac{\|\|Ty\|\|}{\|y\|} = \sup\{\|\|Tx\|\| : \|x\| = 1\}$$

□

84. Prove that $B(V,W)$ is complete whenever W is complete.

Proof. Let $(T_n) \subset B(V,W)$ be a sequence with $\sum_{n=1}^{\infty} \|T_n\| = C < \infty$. Then we have

$$C = \sum_{n=1}^{\infty} \|T_n\| = \sum_{n=1}^{\infty} \sup_{x\neq 0} \frac{\|T_n(x)\|_W}{\|x\|_V} \geq \sup_{x\neq 0} \sum_{n=1}^{\infty} \frac{\|T_n(x)\|_W}{\|x\|_V}$$
$$\implies C\|x\|_V \geq \sum_{n=1}^{\infty} \|T_n(x)\|_W$$

Here, $(T_n(x))$ is an absolutely summable sequence in W since it is bounded, and W is complete, so $\sum_{n=1}^{\infty} T_n(x)$ converges in W. Since the sum of linear maps is linear, we have

$$\begin{aligned}\lim_{N\to\infty} \left(\sum_{n=1}^{N} T_n\right)(\alpha x + \beta y) &= \lim_{N\to\infty} \left(\alpha \sum_{n=1}^{N} T_n(x) + \beta \sum_{n=1}^{N} T_n(y)\right) \\ &= \alpha \lim_{N\to\infty} \sum_{n=1}^{N} T_n(x) + \beta \lim_{N\to\infty} \sum_{n=1}^{N} T_n(y)\end{aligned}$$

which converges in W, so $\sum_{n=1}^{\infty} T_n$ is a linear map in $B(V,W)$, so $B(V,W)$ is complete. □