Homework 1

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Let K be a function defined on the unit square $Q := [0,1] \times [0,1]$. Define the map $f \mapsto Kf$ defined, for functions defined on $[0,1]$, by $(Kf)(x) = \int_{[0,1]\times[0,1]} K(x,y)f(y)\,dy$, for $x \in [0,1]$, so that Kf is a function on $[0,1]$.

- Assume $K \in \mathcal{C}(Q)$, and consider the map above for $f \in \mathcal{C}([0,1])$. Is the map well-defined?

  Answer. Yes. If $f, g \in \mathcal{C}([0,1])$ with $f = g$ then for $x \in [0,1]$, we have

$$\begin{aligned} (Kf)(x) - (Kg)(x) &= \int_{[0,1]} K(x,y)f(y)\,dy - \int_{[0,1]} K(x,y)g(y)\,dy \\ &= \int_{[0,1]} K(x,y)\left[f(y) - g(y)\right]\,dy = 0 \\ \implies (Kf)(x) &= (Kg)(x) \implies Kf = Kg \end{aligned}$$

  so the map is well-defined.

- To which space does Kf belong?

  Answer. From below, $Kf \in \mathcal{C}([0,1])$.

- If yes, does it define a linear operator?

  Answer. Yes. Let $f, g \in \mathcal{C}([0,1])$ and $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} K(f+g) &= \int_{[0,1]} K(x,y)\left[(f+g)(y)\right]\,dy = \int_{[0,1]} K(x,y)f(y)\,dy + \int_{[0,1]} K(x,y)g(y)\,dy = Kf + Kg \\ K(\alpha \cdot f) &= \int_{[0,1]} K(x,y)(\alpha \cdot f)(y)\,dy = \alpha \cdot \int_{[0,1]} K(x,y)f(y)\,dy = \alpha \cdot Kf \end{aligned}$$

  so the map is a linear operator.

- If yes, is such a linear operator bounded/continuous/Lipschitz from $\mathcal{C}\left([0,1], \|\cdot\|_\infty\right)$ onto itself?

  Answer. Since Q is compact and $K \in \mathcal{C}(Q)$, it must be that $\sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)|\,dy = I < \infty$. Let $\varepsilon > 0$ and take $f, g \in \mathcal{C}([0,1])$ with $\|f - g\|_\infty < \varepsilon / I$. Then

$$\begin{aligned} \|Kf - Kg\|_\infty &= \sup_{x\in[0,1]} \left| \int_{[0,1]} K(x,y)f(y)\,dy - \int_{[0,1]} K(x,y)g(y)\,dy \right| = \sup_{x\in[0,1]} \left| \int_{[0,1]} K(x,y)\left[f(y) - g(y)\right]\,dy \right| \\ &\le \sup_{x\in[0,1]} \int_{[0,1]} \left|K(x,y)\left[f(y) - g(y)\right]\right|\,dy \le \sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)|\, \|f-g\|_\infty\,dy \\ &= \|f - g\|_\infty \sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)|\,dy < \varepsilon \end{aligned}$$

- If yes, can you find an upper bound on the norm of such an operator; if no make sure you show why it is not bounded/continuous/Lipschitz.

  Answer. We have

$$\|Kf\|_\infty = \sup_{x\in[0,1]} \left| \int_{[0,1]} K(x,y)f(y)\,dy \right| \leq \sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)f(y)|\,dy \leq \sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)|\,\|f\|_\infty\,dy$$

  so I is an upper bound on the value of the operator norm.

- If it is bounded, can you find the exact norm of the operator?

  Answer. Consider the value x_m that maximizes $\int_0^1 |K(x,y)|\,dy$. Then define

$$f_m(y) = \begin{cases} 1 & K(x_m,y) \geq 0 \\ -1 & K(x_m,y) < 0 \end{cases}$$

  so that $\|f\|_\infty = 1$. Then we have

$$\sup_{x\in[0,1]} \left| \int_0^1 K(x,y)f_m(y)\,dy \right| \geq \left| \int_0^1 K(x_m,y)f_m(y)\,dy \right| = \left| \int_0^1 |K(x_m,y)|\,dy \right| = \int_0^1 |K(x_m,y)|\,dy$$
$$\implies \|Kf\|_\infty \geq \|f\|_\infty \int_0^1 |K(x_m,y)|\,dy = \|f\|_\infty \sup_{x\in[0,1]} \int_0^1 |K(x,y)|\,dy$$

  and thus since we have the same inequality in the reverse direction from earlier, it follows that we have equality. Although f is a step-wise function, we can approximate it arbitrarily close with continuous functions, and thus the we can't improve the upper bound, so it is the true norm of the operator.

- Same questions as above, but for $K \in L^2(Q), f \in L^2([0,1])$, and the linear operator viewed from $L^2\left([0,1], \|\cdot\|_2\right)$ onto itself. Give an example of a $K \in L^2(Q)$ that is not in $\mathcal{C}(Q)$.

  Solution. The map is well-defined and it is also a linear operator for the same reasons as above (neither of these two properties requires the use of the specific norm, only properties of the integral).

  By the Cauchy-Schwarz inequality, We have

$$(Kf)^2(x) = \left[\int_{[0,1]} K(x,y)f(y)\,dy\right]^2 \leq \int_{[0,1]} K^2(x,y)\,dy \int_{[0,1]} f^2(y)\,dy$$

  and thus

$$\|Kf\|_2 = \left(\int_0^1 \left[\int_{[0,1]} K(x,y)f(y)\,dy\right]^2 dx\right)^{1/2} \leq \left(\int_0^1 \left[\int_{[0,1]} K^2(x,y)\,dy \int_{[0,1]} f^2(y)\,dy\right] dx\right)^{1/2}$$
$$= \left(\int_{[0,1]} f^2(y)\,dy\right)^{1/2} \left(\int_0^1 \int_{[0,1]} K^2(x,y)\,dy\,dx\right)^{1/2} = \|f\|_2 \left(\int_0^1 \int_0^1 K^2(x,y)\,dy\,dx\right)^{1/2}$$

  so the operator is bounded, and an upper bound for the operator norm is $\left(\int_Q K^2(x,y)\right)^{1/2}$. At the moment I am unable to prove that this upper bound is indeed the best possible bound.

  We can simply use a discontinuous piece-wise function on Q, which will be square-integrable but not continuous. $\square$