Homework 2

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Section 1.6

(a) $|z - 1 + i| \leq 3$

(b) $|\arg z| < \pi/4$

(c) $0 < |z - 2| < 3$

(d) $-1 < \operatorname{Im} z \leq 1$

(e) $|z| \geq 2$

(f) $(\operatorname{Re} z)^2 > 1$

2. Sketch each of the given sets.

   Answer. Plots attached on last page. All generated by me in MATLAB.

3. Which of the given sets are open?

   Answer. (b), (c), (f) are open.

4. Which of the given sets are domains?

   Answer. (b), (c) are domains.

6. Describe the boundary of each of the given sets.

   Answer. (a) Solid boundary circle of radius 3 centered at $z = 1 - i$.
   (b) Two dotted rays from the origin, $\theta = \pi/4$ and $\theta = -\pi/4$.
   (c) Dotted boundary circle of radius 3 centered at $z = 2$, and a hole at $z = 2$.
   (d) Solid boundary line at $\operatorname{Im} z = 1$ and dotted boundary line at $\operatorname{Im} z = -1$.
   (e) Solid boundary circle of radius 2 centered at the origin.
   (f) Two dotted boundary lines at $\operatorname{Re} z = 1$ and $\operatorname{Re} z = -1$.

extra. Which of the sets are closed?

   Answer. (a), (e) are closed.

Section 2.1

3. Describe the range of each of the following functions.

   (a) $f(z) = z + 5$ for $\operatorname{Re} z > 0$

   Solution. The range is $\{z : \operatorname{Re} z > 5\}$, all complex numbers with real part greater than 5. □

   (b) $g(z) = z^2$ for z in the first quadrant, $\operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0$.

   Solution. Here, $z = re^{\theta i}$ with $0 \leq \theta \leq \pi/2$. Then $g(z) = z^2 = r^2 e^{2\theta i}$ where $0 \leq 2\theta \leq \pi/2$. Thus the range is the all of the first two quadrants. □

   (c) $h(z) = \frac{1}{z}$ for $0 < |z| \leq 1$

   Solution. The range is $\{z : |z| \geq 1\}$, all complex numbers at least 1 away from the origin. □

   (d) $p(z) = -2z^3$ for z in the quarter-disk $|z| < 1, 0 < \arg z < \pi/2$.

   Solution. Here, $z = re^{i\theta}$ with $0 < \theta < \pi/2$ and $r < 1$. Then $p(z) = -2z^3 = -2r^3 e^{3\theta i}$, where $0 < 3\theta < 3\pi/2$, and $\left|-2r^3 e^{3\theta i}\right| < 2$. Thus the range is the 3/4-disk of radius 2 centered at the origin, with $0 < \varphi < 3\pi/2$. □

5. (e) For the complex exponential function $f(z) = e^z$ defined in Sec 1.4, describe the image of the infinite strip $0 \leq \operatorname{Im} z \leq \pi/4$.

   Solution. For z in this strip, we have $z = a + bi$ where $0 \leq b \leq \pi/4$. Then $e^z = e^{a+bi} = e^a e^{bi}$, where e^{bi} lies between the rays $\theta = 0$ and $\theta = \pi/4$ in the complex plane. Then e^a can be any positive number, so the image is is all complex numbers with $0 \leq \theta \leq \pi/4$ excluding 0. □

6. The Joukowski mapping is defined by

$$w = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

   Show that

   (a) $J(z) = J(1/z)$

   Proof. We have

$$J\left(\frac{1}{z}\right) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{1/z}\right) = \frac{1}{2}\left(\frac{1}{z} + z\right) = J(z)$$

   □

   (b) J maps the unit circle $|z| = 1$ onto the real interval $[-1, 1]$.

   Proof. We have

$$J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{\overline{z}}{|z|^2}\right) = \frac{1}{2}(z + \overline{z}) = \operatorname{Re} z$$

   for z on the unit circle, which is the range $[-1, 1]$. □

   (c) J maps the circle $|z| = r \quad (r > 0, r \neq 1)$ onto the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1$$

   which has foci at ± 1.

Proof. Let $z = r(\cos\theta + i\sin\theta)$. Then

$$\begin{aligned} J(z) &= \frac{1}{2}\left[r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}\right] \\ &= \frac{1}{2}\left[r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)\right] \\ &= \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\cdot\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta \end{aligned}$$

If we treat this as a parametric equation in $\mathbb{R}^2$ where $u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta$ and $v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$, this traces out the ellipse

$$1 = \cos^2\theta + \sin^2\theta = \left(\frac{u}{\frac{1}{2}\left(r + \frac{1}{r}\right)}\right)^2 + \left(\frac{v}{\frac{1}{2}\left(r - \frac{1}{r}\right)}\right)^2$$

as desired. The foci c satisfy

$$\begin{aligned} c^2 &= \left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2 - \left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2 = \frac{1}{4}\left[\left(r^2 + \frac{1}{r^2} + 2\right) - \left(r^2 + \frac{1}{r^2} - 2\right)\right] = 1 \\ \implies c &= \pm 1 \end{aligned}$$

$\square$

Section 2.2

2. Sketch the first five terms of the sequence $(2i)^n, n = 1, 2, 3, \cdots$ and then describe the divergence of this sequence.

Solution. This graph was generated in MATLAB. The divergence goes to infinity.

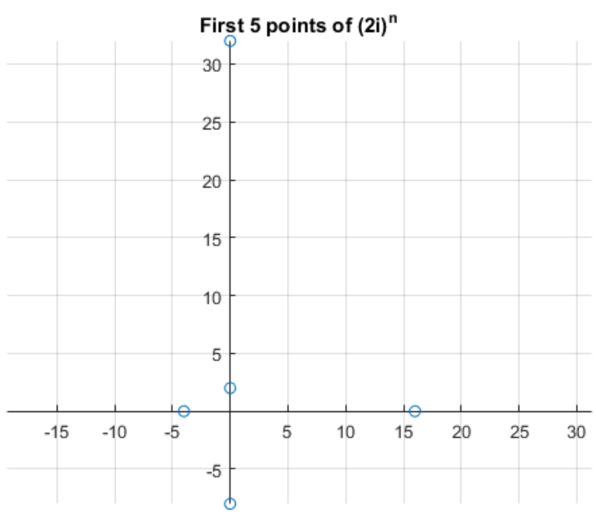


Figure 2: First 5 points of $(2i)^n$

$\square$

7. Decide whether each of the following sequences converges, and if so, find its limit.

   (a) $z_n = \frac{i}{n}$

   Solution. This sequence converges to 0. □

   (b) $z_n = i(-1)^n$

   Solution. This sequence does not converge because it oscillates between i and $-i$. □

   (c) $z_n = \arg\left(-1 + \frac{i}{n}\right)$

   Solution. We have

$$z_n = \arg\left(-1 + \frac{i}{n}\right) = \tan^{-1}\left(\frac{1}{n}\right) \to 0$$

   Since the complex number is in the second quadrant, $z_n \to \pi$. □

   (d) $z_n = \frac{n(2+i)}{n+i}$

   Solution. We have

$$\begin{aligned} z_n = \frac{n(2+i)}{n+i} &= \frac{(2n+ni)(n-i)}{n^2+1} = \frac{(2n^2+n)+(n^2-2n)i}{n^2+1} \\ &= \frac{2n^2+n}{n^2+1} + \frac{n^2-2n}{n^2+1}i \to 2+i \end{aligned}$$

   so the sequence converges to $2+i$. □

   (e) $z_n = \left(\frac{1-i}{4}\right)^n$

   Solution. We have

$$\begin{aligned} z_n &= \left(\frac{1}{4}\right)^n (1-i)^n = \left(\frac{\sqrt{2}}{4}\right)^n \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n \\ &= \left(\frac{\sqrt{2}}{4}\right)^n e^{-n\pi i/4} \to 0 \end{aligned}$$

   so this sequence converges to 0. □

   (f) $z_n = \exp\left(\frac{2n\pi i}{5}\right)$

   Solution. This sequence does not converge because it oscillates between 10 distinct values. □

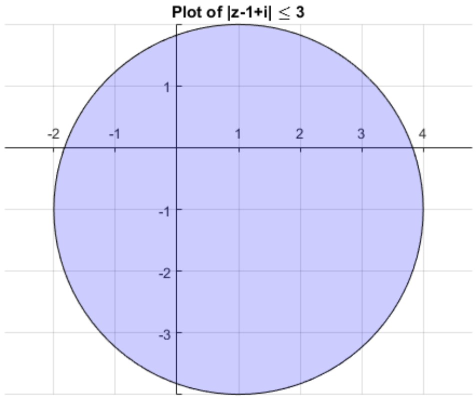
21. (d) Find the limit

$$\lim_{z\to -\pi i} \exp\left(\frac{z^2+\pi^2}{z+\pi i}\right)$$

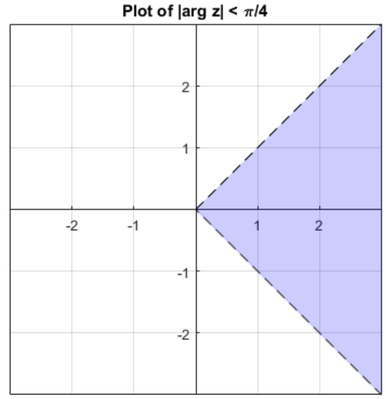
Solution. We have

$$\begin{aligned} \lim_{z\to -\pi i} \exp\left(\frac{z^2+\pi^2}{z+\pi i}\right) &= \lim_{z\to -\pi i} \exp\left(\frac{(z+\pi i)(z-\pi i)}{z+\pi i}\right) \\ &= \lim_{z\to -\pi i} e^{z-\pi i} = e^{-2\pi i} = 1 \end{aligned}$$

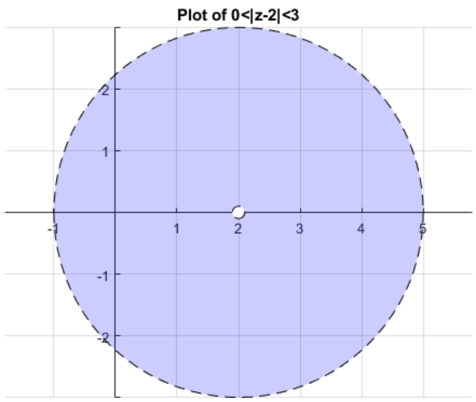
□



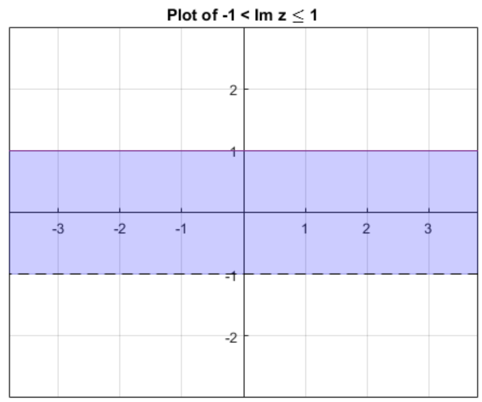
(a) Plot of $|z-1+i| \leq 3$



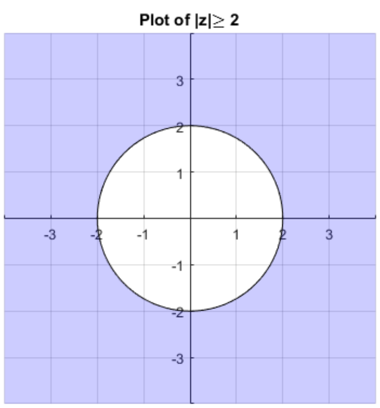
(b) Plot of $|\arg z| < \pi/4$



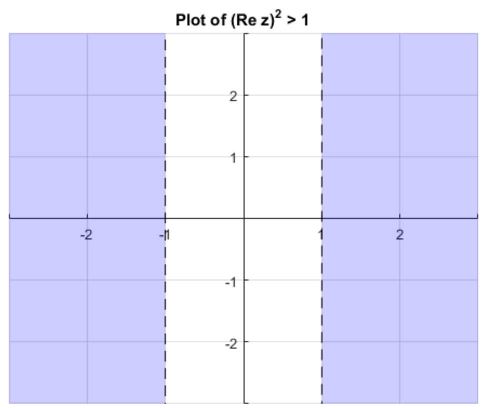
(c) Plot of $0 < |z-2| < 3$



(d) Plot of $-1 < \operatorname{Im} z \leq 1$



(e) Plot of $|z| \geq 2$



(f) Plot of $(\operatorname{Re} z)^2 > 1$