Homework 10

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Chapter 15: The Black-Scholes-Merton Model

4. Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

   Solution. We have

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(50/50) + (0.10 + 0.30^2/2)\sqrt{\frac{1}{4}}}{0.30\sqrt{\frac{1}{4}}} = 0.483$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.333$$

   so the price of the put option is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1) = 50e^{-0.10\cdot\frac{1}{4}}N(-0.333) - 50N(-0.483) = \boxed{2.295}$$

   □

5. What difference does it make to your calculation in Problem 15.4 if a dividend of \$1.50 is expected in 2 months?

   Solution. The present value of the dividend is $1.50e^{-0.10\cdot\frac{1}{6}} = 1.475$, so using $S_0 = 50 - 1.475 = 48.525$,

$$d_1 = \frac{\ln(48.525/50) + (0.10 + 0.30^2/2)\sqrt{\frac{1}{4}}}{0.30\sqrt{\frac{1}{4}}} = 0.234$$

$$d_2 = d_1 - 0.30\sqrt{\frac{1}{4}} = 0.084$$

   so the price of the put option is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1) = 50e^{-0.10\cdot\frac{1}{4}}N(-0.084) - 48.525N(-0.234) = \boxed{2.977}$$

   □

11. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.

    (a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time T.

    Solution. The risk-free rate is μ, so the price of the security at time T is is $e^{-\mu(T-t)}\ln S_T$. □

(b) Confirm that your price satisfies the differential equation (15.16).

Solution. If $f = e^{-\mu(T-t)} \ln S_T$, then the price of the derivative does not depend on S, so

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = \mu e^{-\mu(T-t)} \ln S_T = rf$$

so the differential equation is satisfied, as desired. □

13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months?

Solution. We have

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(52/50) + (0.12^2 + 0.30^2/2)\sqrt{\frac{1}{4}}}{0.30\sqrt{\frac{1}{4}}} = 0.811$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.811 - 0.30\sqrt{\frac{1}{4}} = 0.661$$

so the price of the call option is

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2) = 52N(0.811) - 50e^{-0.12\cdot\frac{1}{4}}N(0.661) = \boxed{4.966}$$

□

15. Consider an American call option on a stock. The stock price is \$70, the time to maturity is 8 months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after 3 months and again after 6 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Proof. We have $D_1 = 1$ and $D_2 = 1$, with $t_1 = 1/4$ and $t_2 = 1/2$. Then we have

$$1 \leq 1.605 = 65\left[1 - e^{-0.10\left(\frac{1}{2}-\frac{1}{4}\right)}\right]$$

so it is not optimal to exercise immediately prior to time $t = 1/4$. We also have

$$1 \leq 1.074 = 65\left[1 - e^{-0.10\left(\frac{2}{3}-\frac{1}{2}\right)}\right]$$

so it is not optimal exercise immediately prior to time $t = 1/2$ either.

According to DerivaGem, the price of the option is \$12.363. □

17. With the notation used in this chapter:

(a) What is $N'(x)$?

Solution. We have

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$
$$\implies N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

□

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$ where S is the stock price at time t and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Proof. We have

$$\begin{aligned}
\frac{SN'(d_1)}{Ke^{-r(T-t)}N'(d_2)} &= \frac{S \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\ln^2(S/K)+(r+\sigma^2/2)^2(T-t)^2+2\ln(S/K)(r+\sigma^2/2)(T-t)}{2\sigma^2(T-t)}\right\}}{Ke^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\ln^2(S/K)+(r-\sigma^2/2)^2(T-t)^2+2\ln(S/K)(r-\sigma^2/2)(T-t)}{2\sigma^2(T-t)}\right\}} \\
&= \frac{S\exp\left\{-\frac{(r^2+r\sigma^2+\sigma^4/4)(T-t)+2\ln(S/K)(r+\sigma^2/2)}{2\sigma^2}\right\}}{Ke^{-r(T-t)}\exp\left\{-\frac{(r^2-r\sigma^2+\sigma^4/4)(T-t)+2\ln(S/K)(r-\sigma^2/2)}{2\sigma^2}\right\}} \\
&= \frac{S\exp\left\{-\frac{r\sigma^2(T-t)+2\ln(S/K)\sigma^2/2}{2\sigma^2}\right\}}{Ke^{-r(T-t)}\exp\left\{-\frac{-r\sigma^2(T-t)-2\ln(S/K)\sigma^2/2}{2\sigma^2}\right\}} \\
&= \frac{S}{K}e^{r(T-t)}e^{-r(T-t)-\ln(S/K)} \\
&= \frac{S}{K}e^{r(T-t)}e^{-r(T-t)}\frac{K}{S} = 1
\end{aligned}$$

so the two are equal, as desired. □

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

Solution. We have

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}} = \frac{\partial d_2}{\partial S}$$

□

(d) Show that when $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, it follows that

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

Proof. We have

$$\begin{aligned}
\frac{\partial d_1}{\partial t} &= -\frac{r+\sigma^2/2}{2\sigma\sqrt{T-t}} \\
\frac{\partial d_2}{\partial t} &= -\frac{r-\sigma^2/2}{2\sigma\sqrt{T-t}} \\
\implies \frac{\partial c}{\partial t} &= S\frac{\partial d_1}{\partial t}N'(d_1) + \frac{\partial S}{\partial t}N(d_1) - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{\partial d_2}{\partial t}N'(d_2) \\
&= S\frac{\partial d_1}{\partial t}N'(d_1) - rKe^{-r(T-t)}N(d_2) - S\frac{\partial d_2}{\partial t}N'(d_1) \\
&= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\left(-\frac{r-\sigma^2/2}{2\sigma\sqrt{T-t}} + \frac{r+\sigma^2/2}{2\sigma\sqrt{T-t}}\right) \\
&= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}
\end{aligned}$$

as desired. □

(e) Show that $\partial c/\partial S = N(d_1)$.

Proof. We have

$$\frac{\partial c}{\partial S} = N(d_1) + S\frac{\partial d_1}{\partial S}N'(d_1) - Ke^{-r(T-t)}\frac{\partial d_2}{\partial S}N'(d_2) = N(d_1)$$

since $\partial d_1/\partial S = \partial d_2/\partial S$ and by the result of part (b). □

(f) Show that c satisfies the Black-Scholes-Merton differential equation.

Proof. We have

$$\begin{aligned}
\frac{\partial^2 c}{\partial S^2} &= \frac{\partial d_1}{\partial S}N'(d_1) = \frac{1}{S\sigma\sqrt{T-t}}N'(d_1) \\
&\implies \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} \\
&= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rSN(d_1) + \frac{1}{2}\sigma^2 S^2\frac{1}{S\sigma\sqrt{T-t}}N'(d_1) \\
&= -rKe^{-r(T-t)}N(d_2) + rSN(d_1) \\
&= rc
\end{aligned}$$

as desired. □

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S_T - K, 0)$ as $t \to T$.

Proof. If $S > K$, then

$$\begin{aligned}
\lim_{t\to T} d_1 &= \lim_{t\to T}\frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \to \infty \\
\lim_{t\to T} d_2 &\to \infty \\
\implies \lim_{t\to T} N(d_1) &= \lim_{t\to T} N(d_2) = 1
\end{aligned}$$

so then $c = S_T - K$. If $S < K$, then $d_1 \to -\infty$ and $d_2 \to -\infty$, so $N(d_1) \to 0$ and $N(d_2) \to 0$, so $c \to 0$. Thus, $c \to \max\{S_T - K, 0\}$. □

28. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

$$30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2$$

Estimate the stock price volatility. What is the standard error of your estimate?

Solution. The log-returns are given by $u_i = \ln(S_i/S_{i-1})$. We have

$$\begin{aligned}
u_1 &= \ln\left(\frac{S_1}{S_0}\right) = \ln\left(\frac{32.0}{30.2}\right) = 0.058 \\
u_2 &= \ln\left(\frac{S_2}{S_1}\right) = \ln\left(\frac{31.1}{32.0}\right) = -0.029 \\
u_3 &= \ln\left(\frac{S_3}{S_2}\right) = \ln\left(\frac{30.1}{31.1}\right) = -0.033 \\
u_4 &= \ln\left(\frac{S_4}{S_3}\right) = \ln\left(\frac{30.2}{30.1}\right) = 0.003 \\
u_5 &= \ln\left(\frac{S_5}{S_4}\right) = \ln\left(\frac{30.3}{30.2}\right) = 0.003 \\
u_6 &= \ln\left(\frac{S_6}{S_5}\right) = \ln\left(\frac{30.6}{30.3}\right) = 0.010 \\
u_7 &= \ln\left(\frac{S_7}{S_6}\right) = \ln\left(\frac{33.0}{30.6}\right) = 0.076 \\
u_8 &= \ln\left(\frac{S_8}{S_7}\right) = \ln\left(\frac{32.9}{33.0}\right) = -0.003 \\
u_9 &= \ln\left(\frac{S_9}{S_8}\right) = \ln\left(\frac{33.0}{32.9}\right) = 0.003 \\
u_{10} &= \ln\left(\frac{S_{10}}{S_9}\right) = \ln\left(\frac{33.5}{33.0}\right) = 0.015 \\
u_{11} &= \ln\left(\frac{S_{11}}{S_{10}}\right) = \ln\left(\frac{33.7}{33.5}\right) = 0.006 \\
u_{12} &= \ln\left(\frac{S_{12}}{S_{11}}\right) = \ln\left(\frac{33.5}{33.5}\right) = 0 \\
u_{13} &= \ln\left(\frac{S_{13}}{S_{12}}\right) = \ln\left(\frac{33.5}{33.7}\right) = -0.006 \\
u_{14} &= \ln\left(\frac{S_{14}}{S_{13}}\right) = \ln\left(\frac{33.2}{33.5}\right) = -0.009
\end{aligned}$$

The sample standard deviation of these values is 2.9%, so using $\tau = 1/52$, the stock price volatility is $2.9/\sqrt{1/52} = 40.2\%$. □

Chapter 17: Options on Stock Indices and Currencies

4. A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value

   (a) a European four-month call option with a strike price of 0.79

   Solution. We have

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}} = e^{0.12\sqrt{1/6}} = 1.05 \\
d &= e^{-\sigma\sqrt{\Delta t}} = e^{-0.12\sqrt{1/6}} = 0.952 \\
p &= \frac{e^{(r_d - r_f)\Delta t} - d}{u - d} = \frac{e^{(0.06-0.08)\cdot 1/6} - 0.952}{1.05 - 0.952} = 0.454
\end{aligned}
$$

   Then we have

$$
\begin{aligned}
f_{uu} &= 0.80u^2 - 0.79 = 0.092 \\
f_{ud} &= f_{du} = 0.80ud - 0.79 = 0.01 \\
f_{dd} &= 0 \\
\implies f_u &= e^{-r_d T}\left[pf_{uu} + (1-p)f_{ud}\right] = 0.0468 \\
f_d &= e^{-r_d T}\left[pf_{du} + (1-p)f_{dd}\right] = 0.0045 \\
\implies f &= e^{-r_d T}\left[pf_u + (1-p)f_d\right] = 0.0235
\end{aligned}
$$

   □

   (b) an American four-month call option with the same strike price.

   Solution. After 2 months, the price of the currency can be either $0.80u = 0.84$ or $0.80d = 0.7616$. If it went up, the payoff from early exercise is $0.84 - 0.79 = 0.05$, whereas the price of the option at that point was 0.0468, so it would be optimal to exercise early. Thus, $f_u = 0.05$ in this case, and $f_d = 0.0045$ still, so

$$
f = e^{-r_d T}\left[pf_u + (1-p)f_d\right] = 0.0249
$$

   □

22. Can an option on the yen/euro exchange rate be created from two options, one on the dollar/euro exchange rate, and the other on the dollar/yen exchange rate? Explain your answer.

    Solution. This is not possible. There are scenarios in which one option will be exercised and not the other, so they cannot accurately model a single option. □

23. The Dow Jones Industrial Average on January 12, 2007 was 12,556 and the price of the March 126 call was \$2.25. Use the DerivaGem software to calculate the implied volatility of this option. Assume the risk-free rate was 5.3% and the dividend yield was 3%. The option expires on March 20, 2007. Estimate the price of a March 126 put. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.)

    Solution. According to DerivaGem, the implied volatility is 10.34%. By put-call parity, we have

$$
\begin{aligned}
c + Ke^{-rT} &= p + S_0 e^{-qT} \\
\implies p &= 2.25 + 126e^{-0.053\cdot\frac{67}{365}} - 125.56e^{-0.03\cdot\frac{67}{365}} = 2.16
\end{aligned}
$$

    According to DerivaGem, the implied volatility is also 10.34%. □

Chapter 18: Futures Options

7. Calculate the value of a five-month European put futures option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

   Solution. We have

$$\begin{aligned} d_1 &= \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(19/20) + 0.20^2 \cdot \frac{5}{12} \cdot \frac{1}{2}}{0.20\sqrt{\frac{5}{12}}} = -0.333 \\ d_2 &= d_1 - \sigma\sqrt{T} = -0.462 \\ \implies p &= e^{-rT}\left[KN(-d_2) - F_0 N(-d_1)\right] = 1.504 \end{aligned}$$

   □

8. Suppose you buy a put option contract on October gold futures with a strike price of \$1400 per ounce. Each contract is for the delivery of 100 ounces. What happens if you exercise when the October futures price is \$1380?

   Solution. I receive a cash amount of $100(1400 - 1380) = \$2000$ and a short position in the contract. □

15. A futures price is currently 70, its volatility is 20% per annum, and the risk-free interest rate is 6% per annum. What is the value of a five-month European put on the futures with a strike price of 65?

   Solution. We have

$$\begin{aligned} d_1 &= \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(70/65) + \sigma^2 T/2}{\sigma\sqrt{T}} = 0.639 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.509 \\ \implies p &= e^{-rT}\left[KN(-d_2) - F_0 N(-d_1)\right] = 1.512 \end{aligned}$$

   □

22. A futures price is currently 40. It is known that at the end of three months the price will be either 35 or 45. What is the value of a three-month European call option on the futures with a strike price of 42 if the risk-free interest rate is 7% per annum?

   Solution. We have $u = 45/40 = 1.125$ and $d = 35/40 = 0.875$. Then $p - \frac{1-d}{u-d} = 0.5$. Then $f_u = 3$ and $f_d = 0$, so we have

$$f = e^{-rT}\left[pf_u + (1-p)f_d\right] - = e^{-0.07 \cdot \frac{1}{4}}\left[0.5 \cdot 3\right] = 1.474$$

   □