Homework 11

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Chapter 15: Fourier Series

7. Define $f(x) = (\pi - x)^2$ for $0 \le x \le 2\pi$, and extend f to a 2π-periodic continuous function on $\mathbb{R}$ in the obvious way. Show that the Fourier series for f is $\pi^2/3 + 4\sum_{n=1}^\infty \cos nx/n^2$. Since the series is uniformly convergent, it actually converges to f. In particular, note that setting $x = 0$ yields the familiar formula $\sum_{n=1}^\infty 1/n^2 = \pi^2/6$.

Proof. We have

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)^2\, dt = \frac{1}{2\pi}\left[-\frac{1}{3}(\pi - x)^3\right]\Big|_0^{2\pi} = \frac{\pi^2}{3}$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx = \frac{1}{\pi}\left(\int_0^{2\pi} \pi^2 \cos nx\, dx - 2\pi\int_0^{2\pi} x\cos nx\, dx + \int_0^{2\pi} x^2 \cos nx\, dx\right)$$

Integrating by parts, we have

$$\int_0^{2\pi} x\cos nx\, dx = \left[\frac{x}{n}\sin nx\right]\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nx\, dx = 0$$
$$\int_0^{2\pi} x^2\cos nx\, dx = \left[\frac{x^2}{n}\sin nx\right]\Big|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} x\sin nx\, dx = -\frac{2}{n}\left(\left[-\frac{x}{n}\cos nx\right]\Big|_0^{2\pi} - \int_0^{2\pi} -\frac{1}{n}\cos nx\right)$$
$$= \frac{4\pi}{n^2}$$

Thus, since $\int_0^{2\pi} \pi^2 \cos nx\, dx = 0$, we have

$$a_n = \frac{1}{\pi}\cdot\frac{4\pi}{n^2} = \frac{4}{n^2}$$

Similarly, we have

$$b_n = \frac{1}{\pi}\left(\int_0^{2\pi} \pi^2 \sin nx\, dx - 2\pi\int_0^{2\pi} x\sin nx\, dx + \int_0^{2\pi} x^2\sin nx\, dx\right)$$

Integrating by parts, we have

$$\int_0^{2\pi} x\sin nx\, dx = \left[-\frac{x}{n}\cos nx\right]\Big|_0^{2\pi} = -\frac{2\pi}{n}$$
$$\int_0^{2\pi} x^2\sin nx\, dx = \left[-\frac{x^2}{n}\cos nx\right]\Big|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} -x\cos nx\, dx = -\frac{4\pi^2}{n}$$

Thus, since $\int_0^{2\pi} \pi^2 \sin nx\, dx = 0$, we have

$$b_n = \frac{1}{\pi}\left(-2\pi\cdot\left(-\frac{2\pi}{n}\right) + \left(-\frac{4\pi^2}{n}\right)\right) = 0$$

so the Fourier series of $(\pi - x)^2$ is given by

$$(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

as desired. Substituting $x = 0$, we have

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

8. Fix $n \geq 1$ and $\varepsilon > 0$.
   (a) Show that there is a continuous function $f \in C^{2\pi}$ satisfying $\|f\|_\infty = 1$ and $(1/\pi)\int_{-\pi}^{\pi} |f(t) - \operatorname{sign} D_n(t)|\, dt < \varepsilon/(n+1)$.
   (b) Show that $s_n(f)(0) \geq \lambda_n - \varepsilon$ and hence, that $\|s_n(f)\|_\infty \geq \lambda - \varepsilon$.

9. Prove that $\|\sigma_n(f)\|_2 \leq \|f\|_2$ and $\|\sigma_n(f)\|_\infty \leq \|f\|_\infty$.

   Proof. We have

$$\|\sigma_n(f)\|_2 = \left\| \frac{1}{n}\sum_{k=0}^{n-1} s_k(f) \right\|_2 = \frac{1}{n}\left\| \sum_{k=0}^{n-1} s_k(f) \right\|_2 \leq \frac{1}{n}\sum_{k=0}^{n-1} \|s_k(f)\|_2$$

   and by Bessel's inequality, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \|s_k(f)\|_2 \leq \frac{1}{n}\sum_{k=0}^{n-1} \|f\|_2 = \|f\|_2$$

□

Chapter 19: Additional Topics

1. Find a sequence of integrable functions (f_n) such that $\int |f_n| \to 0$ for $f_n \not\to 0$ pointwise a.e.

   Solution. Let $f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, f_4 = \chi_{[0,1/3]}, f_5 = \chi_{[1/3,2/3]}$, etc. Then the integrals approach 0 but f_n does not converge pointwise to anything. □

2. Find a sequence of integrable functions (f_n) such that $f_n \to 0$ uniformly but $\int |f_n| = 1$ for all n.

   Solution. Let $f_n = \frac{1}{n} \cdot \chi_{[0,n]}$. Then $\int |f_n| = \frac{1}{n} \cdot n = 1$ for all n, but $f_n \to 0$ uniformly. □

26. If $m(E) < \infty$ and $f \in L_p(E)$, show that $\|f\|_p \leq (m(E))^{1/p - 1/q}\|f\|_q$ for $1 \leq p < q < \infty$. Thus, as sets, $L_q(E) \subset L_p(E)$ whenever $m(E) < \infty$. (Hint: Holder's inequality). In particular, if $E = [0,1]$, notice that the L_p-norms increase with p; that is, $\|f\|_p \leq \|f\|_q$ for $1 \leq p < q < \infty$.

Proof. Let $1/p' = \frac{p}{q}$ and $1/q' = 1 - \frac{p}{q}$, where $p', q' > 0$ since $p < q$ and $1/p' + 1/q' = 1$. Then if we take $g \equiv 1$, by Holder's inequality, we have

$$\begin{aligned} \int_E |f^p| \le \|f^p\|_{p'} \|g\|_{q'} &= \left(\int_E |f^p|^{q/p}\right)^{p/q} \left(\int_E 1^{1/\left(1-\frac{p}{q}\right)}\right)^{1-\frac{p}{q}} \\ \implies \left(\int_E |f|^p\right)^{1/p} &= \left(\int_E |f|^q\right)^{1/q} (m(E))^{\frac{1}{p}\left(1-\frac{p}{q}\right)} \\ \implies \|f\|_p &\le \|f\|_q (m(E))^{1/p-1/q} \end{aligned}$$

□

33. If f and g are disjointly supported elements of L_p, that is, if $fg = 0$ a.e., show that $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$.

Proof. We can partition $\mathbb{R}$ as

$$\mathbb{R} = \{f = 0, g \neq 0\} \cup \{f \neq 0, g = 0\} \cup \{f = 0, g = 0\} \cup \{f \neq 0, g \neq 0\}$$

So we can write

$$\begin{aligned} \|f+g\|_p^p &= \int |f+g|^p \\ &= \int_{\{f=0,g\neq 0\}} |f+g|^p + \int_{\{f\neq 0,g=0\}} |f+g|^p + \int_{\{f=0,g=0\}} |f+g|^p + \int_{\{f\neq 0,g\neq 0\}} |f+g|^p \end{aligned}$$

Since $fg = 0$ a.e., we have $m(\{f \neq 0, g \neq 0\}) = 0$, so the last integral equals 0, and the third integral trivially equals 0, so this is equal to

$$\begin{aligned} \int_{\{f=0,g\neq 0\}} |f+g|^p + \int_{\{f\neq 0,g=0\}} |f+g|^p &= \int_{\{g\neq 0\}} |g|^p + \int_{\{f\neq 0\}} |f|^p \\ &= \left[\|g\|_p^p - \int_{\{g=0\}} |g|^p\right] + \left[\|f\|_p^p - \int_{\{f=0\}} |f|^p\right] \\ &= \|g\|_p^p + \|f\|_p^p \end{aligned}$$

since integrating $|g|^p$ and $|f|^p$ over $\{f = 0\}$ and $\{g = 0\}$ trivially result in 0. □