

## Homework 5 Solutions

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1. MATLAB exercise. When trying to use Cramer's rule on the selected exercise, the answer is unreasonably large because the matrix is singular.
2. MATLAB exercise.
3. (a) Let  $\mathbf{u} = (-4, 4, -1)$ ,  $\mathbf{v} = (-1, -2, 2)$ . Compute  $\mathbf{u} \times \mathbf{v}$ .

*Solution.* We have

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 4 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \mathbf{i} \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -4 & -1 \\ -1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} \\ &= \mathbf{i}(4 \cdot 2 - (-1) \cdot (-2)) - \mathbf{j}(-4 \cdot 2 - (-1) \cdot (-1)) + \mathbf{k}(-4 \cdot (-2) - 4 \cdot (-1)) \\ &= 6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k} = (6, 9, 12)\end{aligned}$$

□

- (b) Prove for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ ,

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

*Proof.* Let  $\mathbf{u} = (a, b, c)$  and let  $\mathbf{v} = (x, y, z)$ . Then

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{bmatrix} = \mathbf{i} \begin{vmatrix} b & c \\ y & z \end{vmatrix} - \mathbf{j} \begin{vmatrix} a & c \\ x & z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a & b \\ x & y \end{vmatrix} \\ &= \mathbf{i}(bz - cy) + \mathbf{j}(cx - az) + \mathbf{k}(ay - bx) \\ \implies \|\mathbf{u} \times \mathbf{v}\|^2 &= (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2 \\ &= b^2z^2 - 2bcyz + c^2y^2 + c^2x^2 - 2acxz + a^2z^2 + a^2y^2 - 2abxy + b^2x^2\end{aligned}\quad (1)$$

We also have

$$\begin{aligned}\|\mathbf{u}\|^2 &= a^2 + b^2 + c^2 \\ \|\mathbf{v}\|^2 &= x^2 + y^2 + z^2 \\ \mathbf{u} \cdot \mathbf{v} &= ax + by + cz \\ \implies \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 &= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ &= a^2x^2 + a^2y^2 + a^2z^2 + b^2x^2 + b^2y^2 + b^2z^2 + c^2x^2 + c^2y^2 + c^2z^2 \\ &\quad - (a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz) \\ &= a^2y^2 + a^2z^2 + b^2x^2 + b^2z^2 + c^2x^2 + c^2y^2 - 2abxy - 2acxz - 2bcyz\end{aligned}$$

and we can see that this expression is equivalent to the one in equation (1), so the two quantities are equal, as desired. □

4. (a) Express  $\mathbf{u} = 1\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$  as a sum of vectors parallel and perpendicular to  $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ .

*Solution.* We have

$$\mathbf{u}_{\parallel} = \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{1 \cdot -2 + 4 \cdot 5 + 3 \cdot 4}{(-2)^2 + 5^2 + 4^2} \mathbf{v} = \frac{30}{45}(-2, 5, 4) = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right)$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u} = (1, 4, 3) - \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) = \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

so now

$$\mathbf{u} = (1, 4, 3) = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) + \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

□

- (b) Show that the vectors  $\mathbf{u}_{\parallel}$  and  $\mathbf{u}_{\perp}$  you obtained in part (a) are orthogonal.

*Proof.* The dot product between orthogonal vector is 0, and here

$$\mathbf{u}_{\parallel} \cdot \mathbf{u}_{\perp} = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) \cdot \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right) = -\frac{4}{3} \cdot \frac{7}{3} + \frac{10}{3} \cdot \frac{2}{3} + \frac{8}{3} \cdot \frac{1}{3} = \frac{-28}{3} + \frac{20}{3} + \frac{8}{3} = 0$$

so these two vectors are orthogonal, as desired.

□

5. MATLAB exercise. For the vectors  $\mathbf{u} = (2, 8, -3, -1, 2)$  and  $\mathbf{v} = (-5, 3, 1, 1, 6)$ , we have

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{2^2 + 8^2 + (-3)^2 + (-1)^2 + 2^2} = \sqrt{82} \approx 9.055 \\ \|\mathbf{v}\| &= \sqrt{(-5)^2 + 3^2 + 1^2 + 1^2 + 6^2} = 6\sqrt{2} \approx 8.485 \\ \mathbf{u} \cdot \mathbf{v} &= 2 \cdot -5 + 8 \cdot 3 + -3 \cdot 1 + -1 \cdot 1 + 2 \cdot 6 = 22 \\ \|\mathbf{u} - \mathbf{v}\| &= \|(7, 5, -4, -2, -4)\| = \sqrt{7^2 + 5^2 + (-4)^2 + (-2)^2 + (-4)^2} = \sqrt{110} \approx 10.488 \\ \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{22}{\sqrt{82} \cdot 6\sqrt{2}} \approx 0.286\end{aligned}$$

6. (E&P 4.1.30)  $V$  is the set of all  $(x, y, z)$  such that  $x + y + z = 0$ .

*Proof.* Let  $\mathbf{u}, \mathbf{v} \in V$ , where  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ , and let  $k \in \mathbb{R}$ . Then since  $\mathbf{u}$  and  $\mathbf{v}$  are in  $V$ , they have the property that  $u_1 + u_2 + u_3 = 0$  and  $v_1 + v_2 + v_3 = 0$ . Now,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

where

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0$$

so  $V$  is closed under addition. Then we have

$$k\mathbf{u} = (ku_1, ku_2, ku_3)$$

where

$$ku_1 + ku_2 + ku_3 = k(u_1 + u_2 + u_3) = k \cdot 0 = 0$$

so  $V$  is closed under scalar multiplication.

□

7. (E&P 4.1.36)  $V$  is the set of all  $(x, y, z)$  such that  $xyz = 1$ .

*Solution.* This is not a subspace. Consider the vector  $\mathbf{u} = (1, 1, 1) \in V$  since  $1 \cdot 1 \cdot 1 = 1$ , and let  $k = 2$ . Then  $k\mathbf{u} = (2, 2, 2)$ , but  $2 \cdot 2 \cdot 2 = 8 \neq 1$ , so  $V$  is not closed under scalar multiplication.  $\square$

8. (E&P 4.2.8)  $W$  is the set of all vectors in  $\mathbb{R}^2$  such that  $(x_1)^2 + (x_2)^2 = 0$ .

*Proof.* Let  $\mathbf{u}, \mathbf{v} \in V$ , where  $\mathbf{u} = (u_1, u_2)$  and  $\mathbf{v} = (v_1, v_2)$ , and let  $k \in \mathbb{R}$ . Then it follows that  $u_1^2 + u_2^2 = 0 \implies u_1 = u_2 = 0$ , and similarly  $v_1 = v_2 = 0$ . Thus,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (0 + 0, 0 + 0) = (0, 0) \in V$$

so  $V$  is closed under addition, and

$$k\mathbf{u} = k(u_1, u_2) = k(0, 0) = (0, 0) \in V$$

so  $V$  is closed under scalar multiplication. In a sense,  $V$  is vacuously a subspace because it contains only the 0 vector.  $\square$