Homework 1

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Section 1.1

8. Write the number in the form $a + bi$.

$$\frac{(8+2i)-(1-i)}{(2+i)^2}$$

Solution.

$$\begin{aligned}\frac{(8+2i)-(1-i)}{(2+i)^2} &= \frac{(7+3i)(2-i)^2}{(2+i)^2(2-i)^2} = \frac{(7+3i)(3-4i)}{(2^2+1^2)^2} \\ &= \frac{33-19i}{(4+1)^2} = \boxed{\frac{33}{25} - \frac{19}{25}i}\end{aligned}$$

□

10. Write the number in the form $a + bi$.

$$\left[\frac{2+i}{6i-(1-2i)}\right]^2$$

Solution.

$$\begin{aligned}\left[\frac{2+i}{6i-(1-2i)}\right]^2 &= \left(\frac{2+i}{-1+8i}\right)^2 = \frac{(2+i)^2(-1-8i)^2}{(-1+8i)^2(-1-8i)^2} \\ &= \frac{(3+4i)(-63+16i)}{(1^2+8^2)^2} = \frac{-253-204i}{65^2} = \boxed{-\frac{253}{4225} - \frac{204}{4225}i}\end{aligned}$$

□

Section 1.2

7. (e) Describe the set of points z in the complex plane that satisfies $|z| = \operatorname{Re} z + 2$.

Solution. Let $z = a + bi$, Then we have

$$\begin{aligned}|z| &= |a+bi| = \sqrt{a^2+b^2} \\ \operatorname{Re} z + 2 &= a + 2 \\ \implies \sqrt{a^2+b^2} &= a + 2 \implies a^2 + b^2 = a^2 + 4a + 4 \\ \implies b^2 &= 4a + 4 \implies a = \frac{1}{4}b^2 - 1\end{aligned}$$

This traces out a parabola in the complex plane. □

16. Prove that if $|z| = 1$ $(z \neq 1)$, then $\text{Re}[1/(1-z)] = \frac{1}{2}$.

Proof. Let $z = a + bi$. Then we have $|z| = |a + bi| = \sqrt{a^2 + b^2} = 1 \implies a^2 + b^2 = 1$. Then

$$\begin{aligned}
\text{Re}\left(\frac{1}{1-z}\right) &= \text{Re}\left(\frac{1}{1-a-bi}\right) = \text{Re}\left[\frac{(1-a)+bi}{(1-a-bi)(1-a+bi)}\right] \\
&= \text{Re}\left[\frac{1-a+bi}{(1-a)^2+b^2}\right] = \frac{1-a}{1-2a+a^2+b^2} \\
&= \frac{1-a}{1-2a+1} = \frac{1-a}{2-2a} = \frac{1}{2}
\end{aligned}$$

as desired. □

Section 1.3

5. (d) Find the value of

$$\left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right|$$

Solution. Let $z = \pi + i$. Then we have

$$\left|\frac{z^{100}}{(\overline{z})^{100}}\right| = \frac{|z|^{100}}{|\overline{z}|^{100}} = \left(\frac{|z|}{|\overline{z}|}\right)^{100} = \boxed{1}$$

□

7. (h) Find the argument of this complex number and write it in polar form.

$$\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i}$$

Solution. We have

$$\begin{aligned}
\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i} &= \frac{-\sqrt{7}(1+i)(\sqrt{3}-i)}{3+1^2} = \frac{-\sqrt{7}\left[(1+\sqrt{3})+(\sqrt{3}-1)i\right]}{4} \\
&= \frac{-\sqrt{7}-\sqrt{21}}{4} + \frac{\sqrt{7}-\sqrt{21}}{4}i
\end{aligned}$$

Now, we have

$$\theta = \tan^{-1}\left(\frac{\sqrt{7}-\sqrt{21}}{-\sqrt{7}-\sqrt{21}}\right) = \tan^{-1}(2-\sqrt{3}) = \frac{\pi}{12}$$

$$r = \sqrt{\left(\frac{-\sqrt{7}-\sqrt{21}}{4}\right)^2 + \left(\frac{\sqrt{7}-\sqrt{21}}{4}\right)^2} = \frac{\sqrt{14}}{2}$$

Since this lies in the third quadrant, we adjust to get $\theta_0 = -\frac{11\pi}{12}$, so the polar form is $\boxed{\frac{\sqrt{14}}{2}\text{ cis}\left(-\frac{11\pi}{12}\right)}$. □

28. Let the crankshaft pivot O lie at the right of the origin of the coordinate system, and let z be the complex number giving the location of the base of the piston rod, as depicted in Fig 1.14,

$$z = \ell + id$$

where ℓ gives the piston's linear excursion and d is a fixed offset. The crank arm is described by $A = a(\cos\theta_1 + i\sin\theta_1)$ the connecting arm by $B = b(\cos\theta_2 + i\sin\theta_2)$ (θ_2 is negative in Fig 1.14). Exploit the obvious identity $A + B = z = \ell + id$ to derive the expression relating the piston position to the crankshaft angle:

$$\ell = \cos\theta_1 + b\cos\left[\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right]$$

Solution. Because of the identity

$$\begin{aligned} A + B &= a(\cos\theta_1 + i\sin\theta_1) + b(\cos\theta_2 + i\sin\theta_2) \\ &= (a\cos\theta_1 + b\cos\theta_2) + i(a\sin\theta_1 + b\sin\theta_2) \\ &= \ell + id \end{aligned}$$

we must have

$$\begin{aligned} a\cos\theta_1 + b\cos\theta_2 &= \ell \\ a\sin\theta_1 + b\sin\theta_2 &= d \implies \theta_2 = \sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right) \\ &\implies \ell = a\cos\theta_1 + b\cos\left[\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right] \end{aligned}$$

as desired. □

Section 1.4

11. Determine which of the following properties of the real exponential function remain true for the complex exponential function

    (a) e^x is never zero.

    Answer. This is true.

    (b) e^x is a one-to-one function.

    Answer. This is false. We have $1 = e^0 = e^{2\pi i}$.

    (c) e^x is defined for all x.

    Answer. This is true.

    (d) $e^{-x} = 1/e^x$.

    Answer. This is true.

18. Sketch the curves that are given for $0 \le t \le 2\pi$ by

(a) $z(t) = e^{(1+i)t}$

Answer. Plots all created in MATLAB.
This is $e^{(1+i)t} = e^t e^{it} = e^t \cos t + ie^t \sin t$, so we may model this parametrically in $\mathbb{R}^2$.

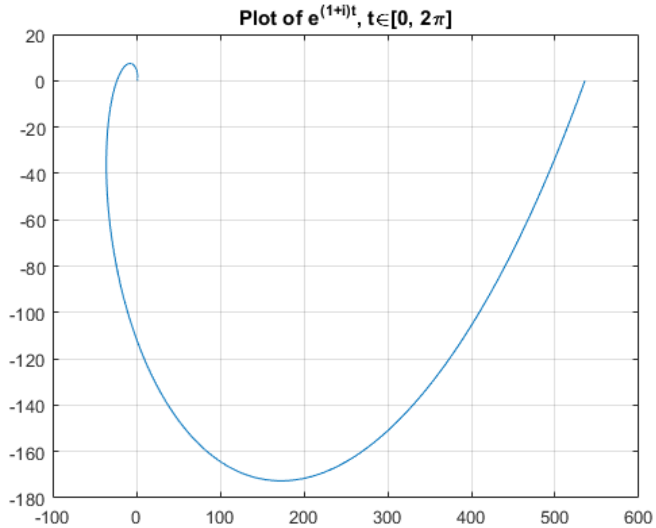

Figure 1: Plot of $e^{(1+i)t}, t \in [0, 2\pi]$

(b) $z(t) = e^{(1-i)t}$

Answer. Similar to above, $e^{(1-i)t} = e^t e^{-it} = e^t \cos(-t) + ie^t \sin(-t) = e^t \cos t - ie^t \sin t$.

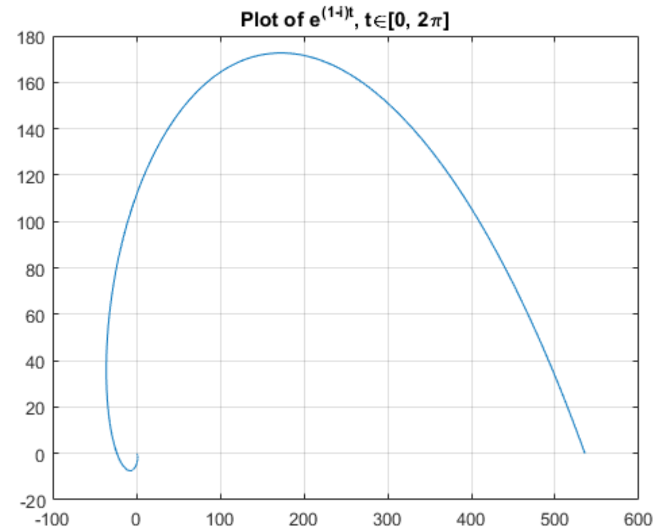

Figure 2: Plot of $e^{(1-i)t}, t \in [0, 2\pi]$

(c) $z(t) = e^{(-1+i)t}$

Answer. Similar to above, $e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}\cos t + ie^{-t}\sin t$.

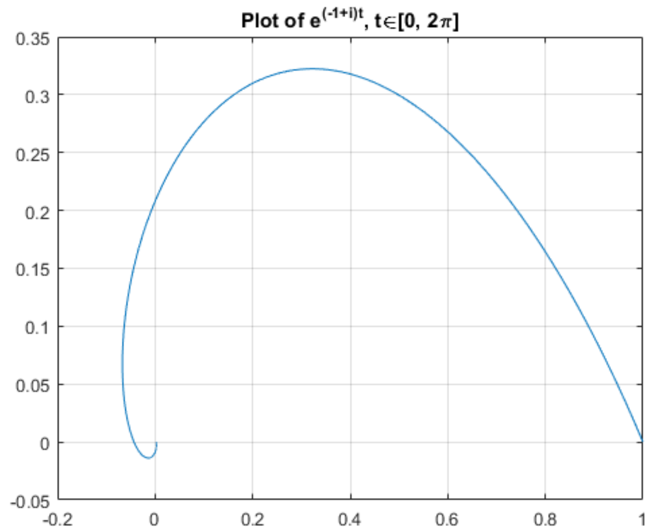

Figure 3: Plot of $e^{(-1+i)t}, t \in [0, 2\pi]$

(d) $z(t) = e^{(-1-i)t}$

Answer. Similar to above, $e^{(-1-i)t} = e^{-t}e^{-it} = e^{-t}\cos t - ie^{-t}\sin t$.

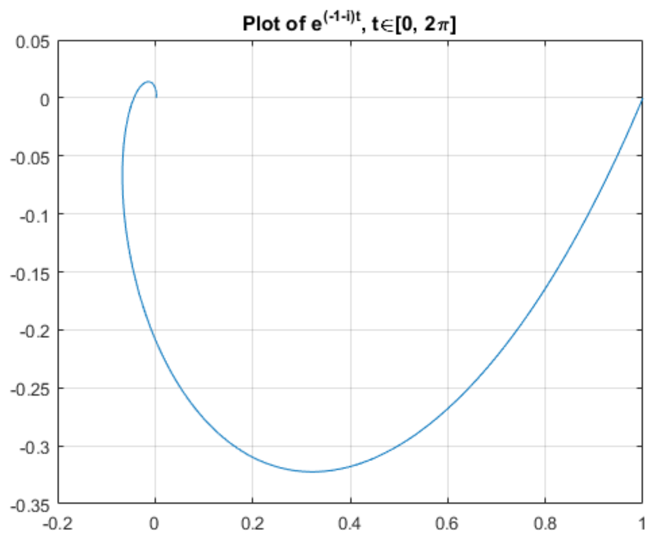

Figure 4: Plot of $e^{(-1-i)t}, t \in [0, 2\pi]$

22. Show that if n is an integer then

$$\int_0^{2\pi} e^{in\theta}\,d\theta = \int_0^{2\pi} \cos(n\theta)\,d\theta + i\int_0^{2\pi} \sin(n\theta)\,d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Proof. We have $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$, so

$$\int_0^{2\pi} e^{in\theta}\,d\theta = \int_0^{2\pi} \cos(n\theta)\,d\theta + i\int_0^{2\pi} \sin(n\theta)\,d\theta$$

If $n = 0$, then this is

$$\int_0^{2\pi} \cos 0\,d\theta + i\int_0^{2\pi} \sin 0\,d\theta = \int_0^{2\pi} 1\,d\theta + i\int_0^{2\pi} 0\,d\theta = 2\pi$$

Otherwise, this is

$$\frac{1}{n}\sin(n\theta)\Big|_0^{2\pi} - i\cdot\frac{1}{n}\cos(n\theta)\Big|_0^{2\pi} = 0$$

□

Section 1.5

4. Use the identity (1) to show that

(a) $(\sqrt{3} - i)^7 = -64\sqrt{3} + 64i$

Solution.

$$\begin{aligned} \sqrt{3} - i &= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] \\ \implies \left(\sqrt{3} - i\right)^7 &= 2^7\left[\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right] = 128\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= -64\sqrt{3} + 64i \end{aligned}$$

□

(b) $(1 + i)^{95} = 2^{47}(1 - i)$

Solution.

$$\begin{aligned} 1 + i &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ \implies (1 + i)^{95} &= \sqrt{2}^{95}\left(\cos\frac{95\pi}{4} + i\sin\frac{95\pi}{4}\right) = 2^{47}\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= 2^{47}(1 - i) \end{aligned}$$

□

5. (f) Find the value of $\left(\frac{2i}{1+i}\right)^{1/6}$

Solution.

$$\frac{2i}{1+i} = \frac{2i(1-i)}{1^2+1^2} = 1+i = \sqrt{2}e^{i\pi/4}$$

$$\implies \left(\frac{2\pi}{1+i}\right)^{1/6} = 2^{1/12}\exp\left\{i\left(\frac{\pi/4+2k\pi}{6}\right)\right\} = \boxed{2^{1/12}\exp\left\{i\left(\frac{\pi}{24}+\frac{k\pi}{3}\right)\right\}, k \in \mathbb{Z}}$$

$\square$