Homework 6

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22.12 Prove: For every positive integer n, the Tower of Hanoi puzzle with n disks can be solved in $2^n - 1$ moves.

Proof. $n = 1$: If there is only 1 disk, it can be moved to the correct position in $1 = 2^1 - 1$ moves, so the base case is satisfied. Suppose a puzzle with k disks is solved in $2^k - 1$ moves. Then for a puzzle with $k + 1$ discs, we can equivalently move the first k disks to the middle position in $2^k - 1$ moves. Then we move the bottom disk to the correct position, and then move the k middle discs to the correct position in $2^k - 1$ moves. The total number of moves is $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$, so the formula holds for $k + 1$, and the statement is proved by induction. □

22.16 (e) Let $e_0 = 1, e_1 = 4$, and for $n > 1$, let $e_n = 4(e_{n-1} - e_{n-2})$. What are the first five terms of the sequence $e_0, e_1, e_2, \cdots$? Prove $e_n = (n + 1)2^n$.

Proof. we have

$$\begin{aligned} e_0 &= 1 \\ e_2 &= 4 \\ e_2 &= 4(4 - 1) = 12 \\ e_3 &= 4(12 - 4) = 32 \\ e_4 &= 4(32 - 12) = 80 \end{aligned}$$

Now proceed by strong induction. For $n = 0$, we have $e_0 = 1 = (0 + 1)2^0$, so the base case is satisfied. Then suppose that the formula holds for all of 0 to k. That means $e_k = (k + 1)2^k$ and $e_{k-1} = k2^{k-1}$. Then

$$\begin{aligned} e_{k+1} &= 4(e_k - e_{k-1}) = 4\left[(k + 1)2^k - k2^{k-1}\right] \\ &= 4k2^k + 4 \cdot 2^k - 4k2^{k-1} = k2^{k+2} + 2^{k+2} - k2^{k+1} \\ &= 2^{k+1}(2k + 2 - k) = [(k + 1) + 1]\, 2^{k+1} \end{aligned}$$

so the formula holds for $k + 1$ and the statement is proved by strong induction. □

3. Let n be a positive integer. Use induction to prove that

$$\sum_{j=1}^{n} j^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

Proof. $n = 1$: The base case is satisfied because

$$1^4 = 1 = \frac{6 + 15 + 10 - 1}{30}$$

Now suppose the formula holds for arbitrary k. Then we have

$$\begin{aligned}\sum_{j=1}^{k+1} j^4 = \sum_{j=1}^{k} j^4 + (k+1)^4 &= \frac{6k^5+15k^4+10k^3-k}{30} + (k+1)^4 \\ &= \frac{(6k^5+15k^4+10k^3-k)+30(k^4+4k^3+6k^2+4k+1)}{30} \\ &= \frac{6k^5+45k^4+130k^3+180k^2+119k+30}{30} \\ &= \frac{6(k+1)^5+15(k+1)^4+10(k+1)^3-(k+1)}{30}\end{aligned}$$

so the formula holds for $k+1$ and the statement is proved by induction. □

4. Consider the following nonlinear recurrence relation defined for $n \in \mathbb{N}$:

$$a_0 = 1, \quad a_n = na_0 + (n-1)a_1 + (n-2)a_2 + \cdots + 2a_{n-2} + 1a_{n-1}$$

(a) Calculate a_1, a_2, a_3, a_4

Solution.

$$\begin{aligned} a_1 &= 1a_0 = 1 \\ a_2 &= 2a_0 + 1a_1 = 3 \\ a_3 &= 3a_0 + 2a_1 + 1a_2 = 8 \\ a_4 &= 4a_0 + 3a_1 + 2a_2 + 1a_3 = 21 \end{aligned}$$

□

(b) Use induction to prove for all positive integers n that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

Proof. $n = 1$: The base case is satisfied because

$$1 = a_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^1 - \left(\frac{3-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2}$$

Now suppose the formula holds for arbitrary k. Note that

$$\begin{aligned} a_k &= ka_0 + (k-1)a_1 + (k-2)a_2 + \cdots + 2a_{k-2} + 1a_{k-1} \\ a_{k+1} &= (k+1)a_0 + ka_1 + (k-1)a_2 + \cdots + 3a_{k-2} + 2a_{k-1} + 1a_k \\ \implies a_{k+1} - a_k &= a_0 + a_1 + a_2 + \cdots + a_{k-2} + a_{k-1} + a_k \end{aligned}$$

The RHS is given by

$$\sum_{i=0}^{k} \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^i - \left(\frac{3-\sqrt{5}}{2}\right)^i\right] = \frac{1}{\sqrt{5}}\left[\sum_{i=0}^{k}\left(\frac{3+\sqrt{5}}{2}\right)^i - \sum_{i=0}^{k}\left(\frac{3-\sqrt{5}}{2}\right)^i\right]$$

These are the sums of two geometric series, and the closed form is

$$\frac{1}{\sqrt{5}}\left[\frac{\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-1}{\frac{3+\sqrt{5}}{2}-1}-\frac{\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}-1}{\frac{3-\sqrt{5}}{2}-1}\right]=\frac{1}{\sqrt{5}}\left[\frac{\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-1}{\frac{1+\sqrt{5}}{2}}-\frac{\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}-1}{\frac{1-\sqrt{5}}{2}}\right]$$

$$=\frac{1}{\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)}\left(\left(\frac{1-\sqrt{5}}{2}\right)\left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-1\right]-\left(\frac{1+\sqrt{5}}{2}\right)\left[\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}-1\right]\right)$$

$$=-\frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}+\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-\sqrt{5}\right]$$

Then a_{k+1} is obtained by adding a_k to the result above, which is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-\sqrt{5}\right]+\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\left(\frac{3-\sqrt{5}}{2}\right)^{k}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\cdot\frac{3-\sqrt{5}}{2}-1\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\cdot\frac{3+\sqrt{5}}{2}-1\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}-5}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k}-\left(\frac{-\sqrt{5}-5}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{5+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\left(\frac{5-\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k}-\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(1+\frac{3+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\left(1+\frac{3-\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{k}-\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}+\frac{3+\sqrt{5}}{2}-\frac{3-\sqrt{5}}{2}-\sqrt{5}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}\right]$$

Thus, the formula holds for $k+1$, so the statement is proved by induction. □

24.1 For each of the following relations, please answer these questions:

(1) Is it a function? If not, explain why and stop.

(2) What are its domain and image?

(3) Is the function one-to-one? If not, explain why and stop.

(4) What is its inverse function?

(a) $\{(1,2),(3,4)\}$

Answer. This is a function. Its domain is $\{1, 3\}$ and its range is $\{2, 4\}$. The function is one-to-one. The inverse function is $\{(2, 1), (4, 3)\}$.

(b) $\{ (x, y) \mid x, y \in \mathbb{Z}, y = 2x \}$

Answer. This is a function. Its domain is $\mathbb{Z}$ and its image is $2\mathbb{Z}$. The function is one-to-one. The inverse function is $\{ (x, y) \mid x, y \in \mathbb{Z}, x = 2y \}$.

(c) $\{ (x, y) \mid x, y \in \mathbb{Z}, x + y = 0 \}$

Answer. This is a function. Its domain is $\mathbb{Z}$ and its image is $\mathbb{Z}$. The function is one-to-one. The inverse function is $\{ (x, y) \mid x, y \in \mathbb{Z}, x + y = 0 \}$.

(d) $\{ (x, y) \mid x, y \in \mathbb{Z}, xy = 0 \}$

Answer. This is not a function. When $x = 0$, then $(0, y)$ satisfies the relation for all $y \in \mathbb{Z}$, so x is mapped to more than a single value.

(e) $\{ (x, y) \mid x, y \in \mathbb{Z}, y = x^2 \}$

Answer. This is a function. Its domain is $\mathbb{Z}$ and its image is $\mathbb{Z}_{\geq 0}$. The function is not one-to-one because $(2, 4)$ and $(-2, 4)$ are both in the relation, but $2 \neq -2$.

(f) $\varnothing$

Answer. This is a function. Its domain is $\varnothing$ and its image is $\varnothing$. The function is one-to-one. The inverse function is $\varnothing$.

(g) $\{ (x, y) \mid x, y \in \mathbb{Q}, x^2 + y^2 = 1 \}$

Answer. This is not a function. The pairs $(0.6, 0.8)$ and $(0.6, -0.8)$ are both in the relation, so 0.6 is mapped to more than a single value.

(h) $\{ (x, y) \mid x, y \in \mathbb{Z}, x \mid y \}$

Answer. This is not a function. The pairs $(1, 2)$ and $(1, 3)$ are both in the relation, so 1 is mapped to more than a single value.

(i) $\{ (x, y) \mid x, y \in \mathbb{N}, x \mid y, y \mid x \}$

Answer. This is a function since the condition is equivalent to $x = y$. The domain is $\mathbb{N}$ and the image is $\mathbb{N}$. The function is one-to-one. The inverse function is $\{ (x, y) \mid x, y \in \mathbb{N}, x = y \}$.

(j) $\{ (x, y) \mid x, y \in \mathbb{N}, \binom{x}{y} = 1 \}$

Answer. This is not a function. Since $\binom{2}{0} = \binom{2}{2} = 1$, the pairs $(2, 0)$ and $(2, 2)$ are in the relation, so 2 is mapped to more than a single value.

24.23 (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = |x|$. If $X = \{-1, 0, 1, 2\}$, find $f(X)$.

Answer. We have $f(X) = \{f(-1), f(0), f(1), f(2)\} = \{0, 1, 2\}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$. If $X = [0, \pi]$, find $f(X)$.

Answer. The sin function takes on values from 0 to 1 inclusive over $[0, \pi]$, so $f(X) = [0, 1]$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^x$. If $X = [-1, 1]$, find $f(X)$.

Answer. Since 2^x is an increasing function, its minimum value over X is $2^{-1} = 1/2$ and its maximum value is $2^1 = 2$, so $f(X) = \left[\frac{1}{2}, 2\right]$.

(d) Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 3x - 1$. What is $f(\{1\})$? Is it the same as $f(1)$?

Answer. We have $f(\{1\}) = \{f(1)\} = \{2\}$. It is not the same as $f(1) = 2$ because the former is a set, while the latter is a number.

(e) Let $f : A \to B$ be a function. What is $f(A)$?

Answer. Here, $f(A)$ is the image of f as a function.

24.24 (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = |x|$. If $Y = \{1, 2, 3\}$ find $f^{-1}(Y)$.

Answer. Under absolute value, both 1 and -1 are mapped to 1, and similarly for 2 and 3. So $f^{-1}(Y) = \{-3, -2, -1, 1, 2, 3\}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. If $Y = [1, 2]$, find $f^{-1}(Y)$.

Answer. Under f, the interval $[1, \sqrt{2}]$ maps to Y since $f(1) = 1$ and $f(\sqrt{2}) = 2$ and f is increasing over $[1, \sqrt{2}]$. The interval $[-\sqrt{2}, -1]$ also maps to Y since $f(-\sqrt{2}) = 2$ and $f(-1) = 1$ and f is decreasing over $[-\sqrt{2}, -1]$. Thus $f^{-1}(Y) = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Find $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$.

Answer. We have

$$\frac{1}{2} = \frac{1}{1 + 1^2} = \frac{1}{1 + (-1)^2}$$

so $f^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \{-1, 1\}$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Find $f^{-1}\left(\left\{-\frac{1}{2}\right\}\right)$.

Answer. Since f is strictly positive over $\mathbb{R}$, there are no values of x such that $f(x) = -1/2$, so $f^{-1}\left(\left\{-\frac{1}{2}\right\}\right) = \varnothing$.

26.1 For each pair of functions f and g please do the following:

- Determine which of $g \circ f$ and $f \circ g$ is defined.
- If one or both are defined, find the resulting function(s).
- If both are defined, determine whether $g \circ f = f \circ g$.

(e) $f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$ and $g = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}$

Solution. We have $\operatorname{im} f = \operatorname{im} g = \operatorname{dom} f = \operatorname{dom} g = \{1, 2, 3, 4, 5\}$, so both $g \circ f$ and $f \circ g$ are defined. Then

$$g \circ f = \{(1, 4), (2, 5), (3, 1), (4, 2), (5, 3)\}$$
$$f \circ g = \{(1, 4), (2, 5), (3, 1), (4, 2), (5, 3)\}$$

and thus $g \circ f = f \circ g$. □

(g) $f(x) = x + 3$ and $g(x) = x - 7$ (both for all $x \in \mathbb{Z}$)

Solution. We have $\operatorname{im} f = \operatorname{im} g = \operatorname{dom} f = \operatorname{dom} g = \mathbb{Z}$ so both $g \circ f$ and $f \circ g$ are defined. Then

$$(g \circ f)(x) = g(x + 3) = x - 4$$
$$(f \circ g)(x) = f(x - 7) = x - 4$$

and thus $g \circ f = f \circ g$. □

(i) $f(x) = \frac{1}{x}$ for $x \in \mathbb{Q}$ except $x = 0$ and $g(x) = x + 1$ for all $x \in \mathbb{Q}$

Solution. $g \circ f$ is defined because

$$\operatorname{im} f = \mathbb{Q} \setminus \{0\} \subseteq \mathbb{Q} = \operatorname{dom} g$$

However, $f \circ g$ is not defined because

$$\operatorname{im} g = \mathbb{Q} \not\subseteq \mathbb{Q} \setminus \{0\}$$

Then we have

$$(g \circ f)(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1$$

□