Homework 4

Aleck Zhao

March 9, 2017

14.15 Prove: A relation R on a set is antisymmetric if and only if

$$R \cap R^{-1} \subseteq \{ (a,a) \mid a \in A \}$$

Proof. ($\implies$) : Suppose $(x,y) \in R \cap R^{-1}$. Then $(x,y) \in R$ and $(x,y) \in R^{-1} \implies (y,x) \in R$. Since R is antisymmetric, it follows that $x = y$, so

$$R \cap R^{-1} \subseteq \{ (a,a) \mid a \in A \}$$

($\impliedby$) : Thus, if $(x,y) \in R \cap R^{-1}$, we have $(x,y) \in R$ and $(x,y) \in R^{-1} \implies (y,x) \in R$. Since $R \cap R^{-1} \subseteq \{ (a,a) \mid a \in A \}$, that means all elements in $R \cap R^{-1}$ are of the form (z,z) for $z \in A$, so that means $x = y$, and thus R is antisymmetric. □

2. A relation R on a nonempty set A is said to be circular if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(c,a) \in R$.

   (a) Is circular just another name for transitive?

   Answer. No. Circular also implies symmetric. If $(a,b) \in R$ and $(a,b) \in R$, then $(b,a) \in R$ which is the requirement for symmetry.

   (b) Prove that a relation R on a nonempty set A is an equivalence relation if and only if R is reflexive and circular.

   Proof. ($\implies$) : If R is an equivalence relation, it is reflexive by definition. It is also transitive, so $(a,b),(b,c) \in R \implies (a,c) \in R$. Since equivalence relations are also symmetric, we have $(a,c) = (c,a) \in R$, so R is circular as well.

   ($\impliedby$) : If R is circular, it is symmetric by above. Then since $(a,b),(b,c) \in R \implies (c,a) = (a,c) \in R$, it follows that R is also transitive, so since R is also reflexive, it is an equivalence relation. □

3. Let $A = \{ x \in \mathbb{N} \mid 1 \leq x \leq 5 \}$ and let R be a relation on A defined as follows: $(x,y) \in R$ if $3 \mid (x-y)$.

   (a) Which of the five main properties of relations does R have?

   Solution. We have $(x,x) \in R$ since $x - x = 0$ and $3 \mid 0$, so R is reflexive. Obviously then R is not irreflexive. If $(x,y) \in R$ then $3 \mid (x-y)$, so let $x - y = 3k$ for some $k \in \mathbb{Z}$. Then $y - x = 3(-k)$, so $3 \mid (y-x)$, and thus $(y,x) \in R$, so R is symmetric. If $x = 2$ and $y = 5$, then $(2,5) \in R$ and $(5,2) \in R$ since $3 \mid (2-5)$ and $3 \mid (5-2)$, but $2 \neq 5$, so R is not antisymmetric. If $(x,y),(y,z) \in R$, then $3 \mid (x-y)$ and $3 \mid (y-z)$. Suppose $x - y = 3k$ and $y - z = 3m$ for some $k, m \in \mathbb{Z}$. Then adding the two equations, we have $(x-y)+(y-z) = x - z = 3k + 3m = 3(k+m)$, so $3 \mid (x-z)$ and thus $(x,z) \in R$, so R is transitive. □

   (b) Is R an equivalence relation?

   Answer. Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

   (c) Is R a partial order relation?

   Answer. Since R is not antisymmetric, it is not a partial order relation.

4. Let R be a relation on a set A. We say that R is complete if for every x and y in A either $(x,y) \in R$ or $(y,x) \in R$ (or both).

   Let $S = \{a,b,c\}$ and let $A = 2^S$. Let R be a relation on A defined as follows: $(x,y) \in R$ if $x \subseteq y$.

   (a) Is R a partial order relation?

   Solution. Reflexive: clearly $x \subseteq x$ so $(x,x) \in R$.
   Antisymmetric: If $(x,y) \in R$ and $(y,x) \in R$, then $x \subseteq y$ and $y \subseteq x$, so $x = y$ as sets.
   Transitive: If $(x,y),(y,z) \in R$, then $x \subseteq y$ and $y \subseteq z$, so clearly $x \subseteq z$, and thus $(x,z) \in R$.
   R is indeed a partial order relation. □

   (b) Is R an equivalence relation?

   Solution. No, since R is not symmetric. Let $x = \{a\}$ and $y = \{a,b\}$, then $(x,y) \in R$ since $\{a\} \subseteq \{a,b\}$, but $(y,x) \notin R$ since $\{a,b\} \nsubseteq \{a\}$. □

   (c) Is R complete?

   Solution. No. Let $x = \{a\}$ and $y = \{b\}$. Then $x \nsubseteq y$ and $y \nsubseteq z$, so $(x,y) \notin R$ and $(y,x) \notin R$. □

5. Let $S = \{1,2,3,4,5\}$ and let $A = 2^S$. Let R be a relation on A defined as follows: $(x,y) \in R$ if $x \cup \{3,4\} = y \cup \{3,4\}$.

   (a) Prove R is an equivalence relation.

   Proof. Reflexive: Since

$$x \cup \{3,4\} = x \cup \{3,4\}$$

   trivially, we have $(x,x) \in R$ and R is reflexive.
   Symmetric: If $(x,y) \in R$, then

$$x \cup \{3,4\} = y \cup \{3,4\} = x \cup \{3,4\}$$

   so $(y,x) \in R$ and R is symmetric.
   Transitive: If $(x,y),(y,z) \in R$, then

$$\begin{aligned} & x \cup \{3,4\} = y \cup \{3,4\} \\ & y \cup \{3,4\} = z \cup \{3,4\} \\ \implies & x \cup \{3,4\} = z \cup \{3,4\} \end{aligned}$$

   so $(x,z) \in R$ and R is transitive.
   Thus, R is indeed an equivalence relation. □

   (b) What is the equivalence class of $\{1,3\}$?

   Solution. We have $\{1,3\} \cup \{3,4\} = \{1,3,4\}$ so if $x \in [\{1,3\}]$ we must have $x \cup \{3,4\} = \{1,3,4\}$. Thus, the class is explicitly

$$[\{1,3\}] = \{\{1,3,4\}, \{1,3\}, \{1,4\}, \{1\}\}$$

   □

(c) List all the distinct equivalence classes resulting from this relation.

Solution. We have

$$\begin{aligned}
\varnothing \cup \{3,4\} = \{3,4\} &\implies [\varnothing] = \{\varnothing, \{3\}, \{4\}, \{3,4\}\} \\
\{1\} \cup \{3,4\} = \{1,3,4\} &\implies [\{1\}] = \{\{1\}, \{1,3\}, \{1,4\}, \{1,3,4\}\} \\
\{2\} \cup \{3,4\} = \{2,3,4\} &\implies [\{2\}] = \{\{2\}, \{2,3\}, \{2,4\}, \{2,3,4\}\} \\
\{5\} \cup \{3,4\} = \{3,4,5\} &\implies [\{5\}] = \{\{5\}, \{3,5\}, \{4,5\}, \{3,4,5\}\} \\
\{1,2\} \cup \{3,4\} = \{1,2,3,4\} &\implies [\{1,2\}] = \{\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\} \\
\{1,5\} \cup \{3,4\} = \{1,3,4,5\} &\implies [\{1,5\}] = \{\{1,5\}, \{1,3,5\}, \{1,4,5\}, \{1,3,4,5\}\} \\
\{2,5\} \cup \{3,4\} = \{2,3,4,5\} &\implies [\{2,5\}] = \{\{2,5\}, \{2,3,5\}, \{2,4,5\}, \{2,3,4,5\}\} \\
\{1,2,5\} \cup \{3,4\} = \{1,2,3,4,5\} &\implies [\{1,2,5\}] = \{\{1,2,5\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,2,3,4,5\}\}
\end{aligned}$$

$\square$

16.2 How many anagrams can be made from each of the following?

(a) staple

Solution. There are 6 distinct letters, so there are $6! = \boxed{720}$ anagrams. $\square$

(b) discrete

Solution. There are 8 total letters, but E appears twice, so their relative ordering doesn't matter. Thus, there are $8!/2! = \boxed{20160}$ anagrams. $\square$

(c) mathematics

Solution. There are 11 total letters, but M, A, and T each appear twice, and within a letter, the relative ordering doesn't matter. Thus, there are $\boxed{\dfrac{11!}{2!2!2!}}$ anagrams. $\square$

(d) success

Solution. There are 7 total letters, but C appears twice, and S appears 3 times, and for each letter, the relative ordering doesn't matter. Thus, there are $\frac{7!}{2!3!} = \boxed{420}$ anagrams. $\square$

(e) Mississippi

Solution. There are 11 total letters, but I appears 4 times, S appears 4 times, and P appears 2 times, and for each letter, the relative ordering doesn't matter. Thus, there are $\boxed{\dfrac{11!}{4!4!2!}}$ anagrams $\square$

16.4 How many different anagrams can be made from FACETIOUSLY if we require that all six vowels must remain in alphabetical order?

Solution. There are 11 distinct letters, so without restriction, there are 11! ways to order them. However, for any given ordering, only 1/6! are valid and have the six vowels in alphabetical order, so the number of anagrams is $\boxed{11!/6!.}$ $\square$

8. List all the partitions of $\{ x \in \mathbb{N} \mid 1 \leq x \leq 4 \}$ that have 3 parts. Now generalize: let A be a set with $|A| > 1$. Explain why the number of partitions of A into $|A| - 1$ parts must equal the number of 2-element subsets of A.

   Solution. We have $A = \{1, 2, 3, 4\}$, so the partitions with 3 parts are:

   $$\begin{array}{ll} \{1,2\} \cup \{3\} \cup \{4\} & \{1,3\} \cup \{2\} \cup \{4\} \\ \{1,4\} \cup \{2\} \cup \{3\} & \{2,3\} \cup \{1\} \cup \{4\} \\ \{2,4\} \cup \{1\} \cup \{3\} & \{3,4\} \cup \{1\} \cup \{2\} \end{array}$$

   If we want to partition A into $|A| - 1$ parts, then there must be $|A| - 2$ parts with 1 element each, and 1 part with 2 elements. Thus, the partition is uniquely determined by what this 2-element part is, so the number of partitions is equal to the number of 2-element subsets of A. $\square$