Homework 2

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Chapter 14: The Riemann-Stieltjes Integral

29. Show that $|S_\alpha(f,P,T)| \leq \|f\|_\infty V(\alpha,P)$.

Proof. We have $V(\alpha,P) \geq |\alpha(b)-\alpha(a)| \geq \alpha(b)-\alpha(a)$ for any partition P. Thus,

$$\begin{aligned} S_\alpha(f,P,T) &= \sum_{i=1}^n f(t_i)\left[\alpha(x_i)-\alpha(x_{i-1}\right] \leq \sum_{i=1}^n \|f\|_\infty \left[\alpha(x_i)-\alpha(x_{i-1})\right] \\ &= \|f\|_\infty \sum_{i=1}^n \left[\alpha(x_i)-\alpha(x_{i-1})\right] = \|f\|_\infty \left[\alpha(b)-\alpha(a)\right] \\ &\leq \|f\|_\infty V(\alpha,P) \end{aligned}$$

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31. Let $a < c < b$, and suppose that $f \in \mathcal{R}_\alpha[a,c] \cap \mathcal{R}_\alpha[c,b]$. Show that $f \in \mathcal{R}_\alpha[a,b]$ and that $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. In fact, if any two of these integrals exist, then so does the third and the equation above still holds.

Proof. Since $f \in \mathcal{R}_\alpha[a,c]$ and $f \in \mathcal{R}_\alpha[c,b]$, let I_1 and I_2 be $\int_a^c f\,d\alpha$ and $\int_c^b f\,d\alpha$, respectively. Let $\varepsilon > 0$. There exists partitions P^* and Q^* of $[a,c]$ and $[c,b]$ such that

$$\begin{aligned} |S_\alpha(f,P,T_1) - I_1| &< \frac{\varepsilon}{2} \\ |S_\alpha(f,Q,T_2) - I_2| &< \frac{\varepsilon}{2} \end{aligned}$$

for all $P \supset P^*$ and $Q \supset Q^*$ and all choices T_1 and T_2. Then let $R^* = P^* \cup Q^*$ be a partition of $[a,b]$. Then for any $R \supset R^*$,

$$|S_\alpha(f,R,T_3) - (I_1+I_2)| \left| \Big[S_\alpha(f,P,T_1) + S_\alpha(f,P,T_2)\Big] - (I_1+I_2) \right| \quad \leq |S_\alpha(f,P,T_1) - I_1| + |S_\alpha(f,Q,T_2) - I_2|$$

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36. If $\alpha \in BV[a,b]$ and $f \in \mathcal{R}_\alpha[a,b]$, show that $f \in \mathcal{R}_\alpha[c,d]$ for every subinterval $[c,d] \subset [a,b]$.

39. Given $\alpha \in BV[a,b]$, let p and n be the positive and negative variations of α. Show that $\mathcal{R}_\alpha = \mathcal{R}_p \cap \mathcal{R}_n$ and that $\int_a^b f\,d\alpha = \int_a^b f\,dp - \int_a^b f\,dn$ for any $f \in \mathcal{R}_\alpha$.

Proof. Since $\alpha = p + n$, it follows that $R_p \cap R_n \subset R_{p+n} = R_\alpha$. □

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41. Suppose that (α_n) is a sequence in $BV[a,b]$ and that $V_a^b(\alpha_n - \alpha) \to 0$. Show that $\int_a^b f\,d\alpha_n \to \int_a^b f\,d\alpha$ for all $f \in C[a,b]$.

    Proof. Since $f \in C[a,b]$, it is integrable, $f \in \mathcal{R}_\alpha \cap \mathcal{R}_{\alpha_n}$, so

    $$\left| \int_a^b f\,d\alpha_n - \int_a^b f\,d\alpha \right| = \left| \int_a^b f\,d(\alpha_n - \alpha) \right|$$

    From the result of Problem 29, we have

    $$|S_{\alpha_n - \alpha}(f, P, T)| \leq \|f\|_\infty V(\alpha_n - \alpha, P) \leq \|f\|_\infty V_a^b(\alpha_n - \alpha) \to 0$$

    Thus, $\left| \int_a^b f\,d(\alpha_n - \alpha) \right| \to 0$, so $\left| \int_a^b f\,d\alpha_n - \int_a^b f\,d\alpha \right| \to 0$. □

42. Suppose that φ is a strictly increasing continuous function from $[c,d]$ onto $[a,b]$. Given $\alpha \in BV[a,b]$ and $f \in \mathcal{R}_\alpha[a,b]$, show that $\beta = \alpha \circ \varphi \in BV[c,d]$ and that $g = f \circ \varphi \in \mathcal{R}_\beta[c,d]$. Moreover, $\int_c^d g\,d\beta = \int_a^b f\,d\alpha$.

50. If f is continuous on $[a,b]$, and if $\int_a^b |f(x)|\,dx = 0$, show that $f = 0$.