Homework 1

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1. Let R be a commutative ring. Show that
$$p \in R \text{ is prime } \iff \langle p \rangle \text{ is a nonzero prime ideal.}$$

2. Let R be a UFD. Fill in the gap in the proof of §5.1 Theorem 10 by showing that if $p \in R[x]$ is irreducible of degree 0, then p is prime in $R[x]$.

3. Let R be an integral domain, and let $\delta : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ be a function satisfying condition DA. For nonzero $a \in R$, define
$$\tilde{\delta}(a) := \min_{x \in R \setminus \{0\}} \delta(xa).$$
Show that $\tilde{\delta}$ satisfies conditions DA and E in the text.

Section 5.1: Irreducibles and Unique Factorization

35. Let R be a UFD and let $g \mid f$ in $R[x]$, where $f \neq 0$. If f is primitive, show that g is also primitive.

38. Let R be a UFD with field of quotients F. if $p \in R[x]$ is primitive, and p is irreducible in $F[x]$, show that p is irreducible in $R[x]$.

Section 5.2: Principal Ideal Domains

5. If R is a PID and $A \neq 0$ is an ideal of R, show that R/A has a finite number of ideals, all of which are principal.

10. Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$. Show that R is a PID.

31. Show that every unit of $\mathbb{Z}[\sqrt{2}]$ has the form $\pm u^k$, where $k \in \mathbb{Z}$ and $u = 1 + \sqrt{2}$.