Homework 4

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1. *Solution.* The coupons are 4% semi-annually, and the semi-annual yield is 5%, so the price is

$$P = \sum_{k=1}^{20} \frac{4\%}{(1+5\%)^k} + \frac{1}{(1+5\%)^{20}} = \frac{4\%}{5\%}\left(1 - \frac{1}{(1+5\%)^{20}}\right) + \frac{1}{(1+5\%)^{20}}$$
$$= 0.875378 = 87.5378\%$$

of the par value. The duration in number of periods is given by

$$\begin{aligned}
D' &= \frac{1}{PV} \cdot \left(\sum_{k=1}^{20} \frac{4\%}{(1+5\%)^k} \cdot k + \frac{1}{(1+5\%)^{20}} \cdot 20\right) \\
&= \frac{1}{PV} \cdot \left[4\%\left(\frac{1}{1.05} + \frac{2}{1.05^2} + \frac{3}{1.05^3} + \cdots + \frac{20}{1.05^{20}}\right) + \frac{20}{1.05^{20}}\right] \\
D' \cdot \frac{1}{1.05} &= \frac{1}{PV} \cdot \left[4\%\left(\frac{0}{1.05} + \frac{1}{1.05^2} + \frac{2}{1.05^3} + \cdots + \frac{19}{1.05^{20}} + \frac{20}{1.05^{21}}\right) + \frac{20}{1.05^{21}}\right] \\
\implies D'\left(1 - \frac{1}{1.05}\right) &= \frac{1}{PV} \cdot \left[4\%\left(\frac{1}{1.05} + \frac{1}{1.05^2} + \frac{1}{1.05^3} + \cdots + \frac{1}{1.05^{20}} - \frac{20}{1.05^{21}}\right) + \frac{20}{1.05^{20}}\left(1 - \frac{1}{1.05}\right)\right] \\
&= \frac{1}{PV} \cdot 4\% \cdot \left[\frac{1}{5\%}\left(1 - \frac{1}{1.05^{20}}\right) - \frac{20}{1.05^{21}}\right] + \frac{1}{PV} \cdot \frac{20}{1.05^{20}}\left(1 - \frac{1}{1.05}\right) \\
\implies D' &= 13.6807 \\
\implies D &= 6.8404
\end{aligned}$$

□

2. (a) *Solution.* The bond prices are

$$\begin{aligned}
P_A &= \frac{100}{1+15\%} + \frac{100}{(1+15\%)^2} + \frac{1100}{(1+15\%)^3} = 885.84 \\
P_B &= \frac{50}{1+15\%} + \frac{50}{(1+15\%)^2} + \frac{1050}{(1+15\%)^3} = 771.68 \\
P_C &= \frac{1000}{(1+15\%)^3} = 657.52 \\
P_D &= \frac{1000}{1+15\%} = 869.57
\end{aligned}$$

□

(b) *Solution.* The bond durations are

$$\begin{aligned}
D_A &= \frac{1}{P_A} \cdot \left(\frac{100 \cdot 1}{1+15\%} + \frac{100 \cdot 2}{(1+15\%)^2} + \frac{1100 \cdot 3}{(1+15\%)^3}\right) = 2.718 \\
D_B &= \frac{1}{P_B} \cdot \left(\frac{50 \cdot 1}{1+15\%} + \frac{50 \cdot 2}{(1+15\%)^2} + \frac{1050 \cdot 3}{(1+15\%)^3}\right) = 2.838 \\
D_C &= \frac{1}{P_C} \cdot \frac{1000 \cdot 3}{(1+15\%)^3} = 3 \\
D_D &= \frac{1}{P_D} \cdot \frac{1000}{1+15\%} = 1
\end{aligned}$$

□

(c) *Solution.* Bond C is the most sensitive to a change in yield because it has the greatest duration. □

(d) *Solution.* The present value and duration of the obligation are

$$P_O = \frac{2000}{(1+15\%)^2} = 1512.29$$
$$D_O = \frac{1}{P_O} \cdot \frac{2000 \cdot 2}{(1+15\%)^2} = 2$$

To immunize the obligation, we must have

$$V_A + V_B + V_C + V_D = 1512.29$$
$$2.718V_A + 2.838V_B + 3V_C + V_D = 1512.29 \cdot 2$$

□

(e) *Solution.* We should choose Bond D because it is the only one with duration less than 2, which is the duration of the obligation. We have

$$V_C + V_D = 1512.29$$
$$3V_C + V_D = 3024.58$$
$$\Longrightarrow V_C = 756.15$$
$$\Longrightarrow V_D = 756.15$$

□

(f) *Solution.* No, other combinations would not lead to lower costs. In order to immunize, the present value of our portfolio must always be \$1512.29. □

3. *Solution.* If λ is the continuously compounded annual rate, then we have

$$P = \frac{1}{e^{\lambda T}} = e^{-\lambda T}$$
$$C = \frac{1}{P} \cdot \frac{\partial^2 P}{\partial \lambda^2} = e^{\lambda T} \cdot T^2 e^{-\lambda T} = T^2$$

□

4. Suppose that an obligation occurring at a single time period is immunized against interest rate changes with bonds that have only non-negative cash flows. Let $P(\lambda)$ be the value of the resulting portfolio, including the obligation, when the interest rate is $r+\lambda$ and r is the current interest rate. By construction $P(0) = 0$ and $P'(0) = 0$. In this exercise we show that $P(0)$ is a local minimum; that is, $P''(0) \geq 0$.

   Assume a yearly compounding convention. The discount factor for time t is $d_t(\lambda) = (1 + r + \lambda)^{-t}$. Let $d_t = d_t(0)$. For convenience assume that the obligation has magnitude 1 and is due at time $\bar{t}$. The conditions for immunization are then

$$P(0) = \sum_t c_t d_t - d_{\bar{t}} = 0$$

$$P'(0)(1+r) = \sum_t t c_t d_t - \bar{t} d_{\bar{t}} = 0$$

   (a) Show that for all values of α and β there holds

$$P''(0)(1+r)^2 = \sum_t (t^2 + \alpha t + \beta) c_t d_t - (\bar{t}^2 + \alpha \bar{t} + \beta) d_{\bar{t}}$$

Proof. We have

$$\begin{aligned}
P(\lambda) &= \sum_t c_t(1+r+\lambda)^{-t} - (1+r+\lambda)^{-\bar{t}} \\
P'(\lambda) &= \sum_t -tc_t(1+r+\lambda)^{-t-1} + \bar{t}(1+r+\lambda)^{-\bar{t}-1} \\
P''(\lambda) &= \sum_t t(t+1)c_t(1+r+\lambda)^{-t-2} - \bar{t}(\bar{t}+1)(1+r+\lambda)^{-\bar{t}-2} \\
\implies P''(0) &= \sum_t t(t+1)c_t(1+r)^{-t-2} - \bar{t}(\bar{t}+1)(1+r)^{-\bar{t}-2} \\
\implies P''(0)(1+r)^2 &= \sum_t (t^2+t)c_t d_t - (\bar{t}^2+\bar{t})d_{\bar{t}} \qquad (1)
\end{aligned}$$

Now, from the immunization conditions, we have

$$\begin{aligned}
P(0) = 0 &= \sum_t c_t d_t - d_{\bar{t}} \\
\implies 0 &= \beta\left(\sum_t c_t d_t - d_{\bar{t}}\right) = \sum_t \beta c_t d_t - \beta d_{\bar{t}} \qquad (2) \\
P'(0)(1+r) = 0 &= \sum_t t c_t d_t - \bar{t} d_{\bar{t}} \\
\implies 0 &= (\alpha-1)\left(\sum_t t c_t d_t - \bar{t} d_{\bar{t}}\right) = \sum_t (\alpha-1)t c_t d_t - (\alpha-1)\bar{t} d_{\bar{t}} \qquad (3)
\end{aligned}$$

Adding (2) and (3) to (1), we get our desired

$$P''(0)(1+r)^2 = \sum_t (t^2+\alpha t+\beta)c_t d_t - (\bar{t}^2+\alpha\bar{t}+\beta)d_{\bar{t}}$$

□

(b) Show that α and β can be selected so that the function $t^2+\alpha t+\beta$ has a minimum at $\bar{t}$ and has a value of 1 there. Use these values to conclude that $P''(0) \geq 0$.

Proof. If $f(t) = t^2+\alpha t+\beta$, then

$$f'(t) = 2t+\alpha = 0 \implies t = -\frac{\alpha}{2}$$

so we can choose α to be $-2\bar{t}$, so the derivative is 0, and therefore the minimum of the function is at $\bar{t}$. Then we can choose β such that

$$f(\bar{t}) = \bar{t}^2 - 2\bar{t}\cdot\bar{t} + \beta = 1 \implies \beta = \bar{t}^2+1$$

Thus, the function $f(t) = t^2 - 2\bar{t}\cdot t + (\bar{t}^2+1)$ has a minimum value of 1 at $t=\bar{t}$. Now

$$\begin{aligned}
P''(0)(1+r)^2 &= \sum_t (t^2 - 2\bar{t}\cdot t + (\bar{t}^2+1))c_t d_t - (\bar{t}^2 - 2\bar{t}\cdot\bar{t} + (\bar{t}^2+1))d_{\bar{t}} \\
&\geq \sum_t 1\cdot c_t d_t - 1\cdot d_{\bar{t}} = P(0) = 0 \\
\implies P''(0) &\geq 0
\end{aligned}$$

as desired. □

5. (a) *Solution.* The settlement date is 10-Oct-2016, and the time to maturity is 339 days, so sold at

$$P = \left(1 - Y \cdot \frac{d}{360}\right) \cdot F = \left(1 - 0.65\% \cdot \frac{339}{360}\right) \cdot \$10M = \$9,938,791.67$$

□

(b) *Solution.* The time to maturity is 338 days, so we bought at

$$P = \left(1 - Y \cdot \frac{d}{360}\right) \cdot F = \left(1 - 0.75\% \cdot \frac{338}{360}\right) \cdot \$10M = \$9,929,583.33$$

□

(c) *Solution.* For the reverse repo, we borrow \$10M of T-bills on 10-Oct-2016 and lend out \$10M of cash, then sell the T-bills for the price from part (a). On 11-Oct-2016, we buy back the T-bills for the price from part (b), and return them to the reverse repo counterparty for

$$\$10M\left(1 + 0.4\% \cdot \frac{1}{360}\right) = \$10,000,111.11$$

□

(d) *Solution.* The profit from the reverse repo is \$111.11, and the profit from the buying and selling of the T-bills is \$9208.34, so the total profit is \$9319.45. □