Homework 4

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1. Let R be a ring, assumed commutative for simplicity, and let $f \in R[x]$ be a polynomial of degree $n \geq 1$ whose leading coefficient is a unit in R. show that $R[x]/\langle f \rangle$, regarded as an R-module via the composite ring homomorphism $R \to R[x] \twoheadrightarrow R[x]/\langle f \rangle$, is a free R-module containing a basis with n elements.

Section 7.1: Modules

8. Let R be an integral domain. Given ${}_RM$ let $T(M) = \{\, t \in M \mid t \text{ is torsion} \,\}$.

   (a) Show that $T(M)$ is a submodule of M - called the torsion submodule.

   Proof. Clearly, 0 is torsion since it annihilates with every ring element, so $0 \in T(M)$. Now, if $s, t \in T(M)$, then they are both torsion, so suppose $sx = ty = 0$ for $x, y \in R$ nonzero. Now, we have
   $$(s+t)(xy) = sxy + txy = (sx)y + (ty)x = 0$$
   and since R is an integral domain, $xy \neq 0$, so $s+t$ is also torsion. Then $-s$ is also torsion because $(-s)x = -(sx) = 0$. Thus, $T(M)$ is a subgroup of M.
   Now, for any $r \in R$, we have $(rs)x = r(sx) = 0$, so $rs \in T(M)$, and thus $T(M)$ is a submodule of M, as desired. $\square$

   (b) Show that $T[M/T(M)] = 0$. We say $M/T(M)$ is torsion-free.

11. Let $M = \mathbb{Z} \oplus \mathbb{Z}$, and $K = \{\, (k,k) \mid k \in \mathbb{Z} \,\}$. Determine if $M = K \oplus X$ in case:

   (a) $X = \{\, (k, 0) \mid k \in \mathbb{Z} \,\}$

   Solution. If $y \in K \cap X$, then $y = (k,k) = (i,0)$ for some k, i so $k = 0 \implies y = (0,0)$. Now consider $(m,n) \in M$, which has a unique decomposition $(m,n) = (n,n) + (m-n, 0)$, where $(n,n) \in K$ and $(m-n, 0) \in X$. Thus, $M = K \oplus X$. $\square$

   (b) $X = \{\, (0, k) \mid k \in \mathbb{Z} \,\}$

   Solution. If $y \in K \cap X$, then $y = (k,k) = (0,i)$ for some k, i so $k = 0 \implies y = (0,0)$. Now consider $(m,n) \in M$, which has a unique decomposition $(m,n) = (m,m) + (0, n-m)$, where $(m,m) \in K$ and $(0, n-m) \in X$. Thus, $M = K \oplus X$. $\square$

   (c) $X = \{\, (2k, 3k) \mid k \in \mathbb{Z} \,\}$

   Solution. If $y \in K \cap X$, then $y = (k,k) = (2i, 3i)$ for some k, i, so $2i = 3i \implies i = 0 \implies y = (0,0)$. Now consider $(m,n) \in M$, which has a unique decomposition
   $$(m,n) = (3m-2n, 3m-2n) + (2(n-m), 3(n-m))$$
   where $(3m-2n, 3m-2n) \in K$ and $(2(n-m), 3(n-m)) \in X$. Thus, $M = K \oplus X$. $\square$

   (d) $X = \{\, (k, -k) \mid k \in \mathbb{Z} \,\}$

Solution. Suppose $(m, n) \in M$ had a decomposition $(m, n) = (k, k) + (i, -i) = (k + i, k - i)$ for some k, i. Now, $m + n = (k + i) + (k - i) = 2k$, so the only elements that have this decomposition are the ones where $m + n$ is even. Thus, $M \neq K \oplus X$. $\square$

16. Given ${}_RM$, an R-linear map $\pi : M \to M$ is called a projection if $\pi^2 = \pi$.
    (a) If π is a projection, show that $M = \pi(M) \oplus \ker \pi$.
    (b) If $M = N \oplus K$, find a projection π such that $N = \pi(M)$ and $K = \ker \pi$.

23. If ${}_RM$ and ${}_RN$ are simple, prove Schur's Lemma: If $\alpha : M \to N$ is R-linear, then either $\alpha \equiv 0$ or α is an isomorphism.

24. Show that the following conditions on a finitely generated module P are equivalent:
    (1) P is projective
    (2) P is isomorphic to a direct summand of a free module.
    (3) If α, β are R-linear and α is onto in the diagram, then γ exists such that $\alpha\gamma = \beta$.
    (4) If $\alpha : M \to P$ is onto and R-linear, there exists $\gamma : P \to M$ such that $a\gamma = 1_P$.

25. Show that $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module that is not free.