Homework 2

Aleck Zhao

February 9, 2018

Chapter 14: The Riemann-Stieltjes Integral

1. If $f, g \in \mathcal{R}_\alpha[a,b]$ with $f \leq g$, show that $\int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$.

3. If $f \in \mathcal{R}_\alpha[a,b]$, show that $|f| \in \mathcal{R}_\alpha[a,b]$ and that $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$. (Hint: $U(|f|, P) - L(|f|, P) \leq U(f,P) - L(f,P)$. Why?)

6. Define increasing functions α, β, and γ on $[-1,1]$ by $\alpha = \chi_{(0,1]}, \beta = \chi_{[0,1]}$, and $\gamma = \frac{1}{2}(\alpha + \beta)$. Given $f \in B[-1,1]$, show that:
   (a) $f \in \mathcal{R}_\alpha[-1,1]$ if and only if $f(0+) = f(0)$.
   (b) $f \in \mathcal{R}_\beta[-1,1]$ if and only if $f(0-) = f(0)$.
   (c) $f \in \mathcal{R}_\gamma[-1,1]$ if and only if f is continuous at 0.
   (d) If $f \in \mathcal{R}_\gamma[-1,1]$, then $\int_{-1}^1 f\,d\alpha = \int_{-1}^1 f\,d\beta = \int_{-1}^1 f\,d\gamma = f(0)$.

7. Let $P = \{x_0, \cdots, x_n\}$ be a (fixed) partition of $[a,b]$, and let α be an increasing step function on $[a,b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the x_i, where $\alpha_0 = \alpha(a+) - \alpha(a)$ and $\alpha_n = \alpha(b) - \alpha(b-)$. If $f \in B[a,b]$ is continuous at each of the x_i, show that $f \in \mathcal{R}_\alpha$ and $\int_a^b f\,d\alpha = \sum_{i=1}^n f(x_i)\alpha_i$.

9. If f is monotone and α is continuous (and still increasing), show that $f \in \mathcal{R}_\alpha[a,b]$.

10. If $f \in \mathcal{R}_\alpha[a,b]$, show that $f \in \mathcal{R}_\alpha[c,d]$ for every subinterval $[c,d]$ of $[a,b]$. Moreover, $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ for every $a < c < b$. In fact, if any two of these integrals exist, then so does the third and the equation above still holds.

23. Suppose that φ is a strictly increasing continuous function from $[c,d]$ onto $[a,b]$. Given $f \in \mathcal{R}_\alpha[a,b]$, show that $g = f \circ \varphi \in \mathcal{R}_\beta[c,d]$, where $\beta = \alpha \circ \varphi$. Moreover, $\int_c^d g\,d\beta = \int_a^b f\,d\alpha$.

27. Give an example of a sequence of Riemann integrable functions on $[0,1]$ that converges pointwise to a non-integrable function.