# Homework 3

Aleck Zhao

September 27, 2017

## Chapter 2: Countable and Uncountable Sets

22. Show that $\Delta$ contains no nonempty open intervals. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $z \in [0,1] \setminus \Delta$ with $x < z < y$.

*Proof.* Suppose $(a,b) \subset [0,1]$, and let $\Delta = \bigcap_{j=0}^{\infty} I_j$. We will show that some $I_n$ does not contain $(a,b)$.

$$0 < b - a < 1 \implies -\infty < \log_3(b-a) < 0$$

Thus, there exists some $n > 0$ such that $-n < \log_3(b-a)$. Then we have

$$-n < \log_3(b-a) \implies 3^{-n} < 3^{\log_3(b-a)} = b - a$$

Since the measure of $I_n$ is $3^{-n}$, it cannot contain an interval of strictly greater measure, so $(a,b) \not\subset \Delta$.

Suppose that if $x < z < y$, then $z \in \Delta$. Then $\Delta$ would contain the interval $(x,y)$, which is a contradiction. Thus, some $z \in (x,y)$ must lie outside of $\Delta$. □

23. The endpoints of $\Delta$ are those points in $\Delta$ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in $\Delta$ of the form $a/3^n$ for some integers $n$ and $0 \le a \le 3^n$. Show that the endpoints of $\Delta$ other than 0 and 1 can be written as $0.a_1a_2\cdots a_{n+1}$ (base 3), where each $a_k$ is 0 or 2, except $a_{n+1}$, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2\cdots a_n1, 0.a_1a_2\cdots a_n2)$, where both entries are points of $\Delta$ written in base 3.

*Proof.* By Theorem 2.15, all elements of $\Delta$ can be written as the infinite sum $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for $a_i \in \{0,2\}$. Then if $x$ is an endpoint, it can be expressed as a finite sum. It obviously can't end in 0, since otherwise it would end at the farthest non-zero digit. Now,

Case 1: $a_i = 0$ for all $i \ge k$ for some $k$, in which case $x$ ends with 2.

Case 2: $a_i = 2$ for all $i \ge k$ for some $k$, in which case $x$ ends with 1.

Thus, $x$ must take on the desired form. □

26. Let $f : \Delta \to [0,1]$ be the Cantor function and let $x, y \in \Delta$ with $x < y$. Show that $f(x) \le f(y)$. If $f(x) = f(y)$, show that $x$ has two distinct binary expansions. Finally show that $f(x) = f(y)$ if and only if $x$ and $y$ are "consecutive" endpoints of the form $x = 0.a_1a_2\cdots a_n1$ and $y = 0.a_1a_2\cdots a_n2$ (base 3).

*Proof.* Since $x, y \in \Delta$, we may write them as

$$x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \qquad y = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \qquad f(y) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

Now suppose that the first $k$ terms of both sums are equal, but that $b_{k+1} - a_{k+1} = 1$. Then

$$y - x = \frac{2}{3^{k+1}} + \sum_{n=k+2}^{\infty} \frac{2(b_n - a_n)}{3^n} \geq \frac{2}{3^{k+1}} + \sum_{n=k+2}^{\infty} \frac{2(-1)}{3^n} = \frac{1}{3^{k+1}} > 0$$

so this is sufficient for $x < y$. Now,

$$\begin{aligned} f(y) - f(x) &= \frac{1}{2^{k+1}} + \sum_{n=k+2}^{\infty} \frac{b_n - a_n}{2^n} \geq \frac{1}{2^{k+1}} + \sum_{n=k+2}^{\infty} \frac{-1}{2^n} = 0 \\ \implies f(y) \geq f(x) \end{aligned}$$

as desired. Equality occurs if and only if $b_i - a_i = -1$, that is, if $b_i = 0$ and $a_i = 1$ for all $i \geq k+2$.

$$\begin{aligned} y &= 0.c_1c_2\cdots c_k2 \\ x &= 0.c_1c_2\cdots c_k0222\cdots \\ &= 0.c_1c_2\cdots c_k1 \\ c_i &\in \{0, 2\} \end{aligned}$$

Thus, $x$ has two ternary (question says binary, but assuming this is a typo) expansions, and $x$ and $y$ are consecutive endpoints of the desired form. □

29. Prove that the extended Cantor function $f : [0,1] \to [0,1]$ is increasing.

*Proof.* Suppose $p, q \in [0,1]$ and WLOG $q \geq p$. We wish to show $f(q) \geq f(p)$. Consider 4 cases:

Case 1: $p, q \in \Delta$. Then $f(q) \geq f(p)$ by the first part of 26.

Case 2: $p, q \in [1,0] \setminus \Delta$. Since $p \leq q$, it follows that

$$\begin{aligned} &\{f(y) : y \in \Delta, y \leq p\} \subseteq \{f(y) : y \in \Delta, y \leq q\} \\ \implies &f(q) = \sup\{f(y) : y \in \Delta, y \leq p\} \leq \sup\{f(y) : y \in \Delta, y \leq q\} = f(q) \end{aligned}$$

Case 3: $p \in \Delta, q \in [1,0] \setminus \Delta$. Then since $p \leq q$, we have

$$\begin{aligned} &f(p) \in \{f(y) : y \in \Delta, y \leq q\} \\ \implies &f(p) \leq \sup\{f(y) : y \in \Delta, y \leq q\} = f(q) \end{aligned}$$

Case 4: $p \in [1,0] \setminus \Delta, q \in \Delta$. From case 1, we have shown that $q \geq y \implies f(q) \geq f(y)$ for $q, y \in \Delta$. Thus, $f(q)$ is an upper bound for $\{f(y) : y \in \Delta, y \leq p\}$ since $y \leq p \leq q$. Thus,

$$f(p) = \sup\{f(y) : y \in \Delta, y \leq p\} \leq f(q)$$

Thus, $f$ is increasing, as desired. □

30. Check that the construction of the generalized Cantor set with parameter $\alpha$, as described above, leads to a set of measure $1 - \alpha$; that is, check that the discarded intervals now have total length $\alpha$.

*Proof.* Going from $I_n$ to $I_{n+1}$, we will be removing a total of $2^n$ middle segments, each of length $\alpha 3^{-n-1}$. The total measure of the removed intervals is thus

$$\sum_{n=0}^{\infty} 2^n \cdot \alpha 3^{-n-1} = \frac{\alpha}{3} \sum_{n=0}^{\infty} \cdot \left(\frac{2}{3}\right)^n = \frac{\alpha}{3} \cdot \frac{1}{1 - \frac{2}{3}} = \alpha$$

□

32. Deduce from Theorem 2.17 that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has points of continuity in every open interval.

    *Solution.* Consider an open interval $(a, b) \subset \mathbb{R}$. Then $f$ restricted to $(a, b)$ must also be monotone, so it has at most countably many points of discontinuity in $(a, b)$. Since $(a, b)$ is uncountable, there must exist a point of continuity in $(a, b)$, as desired. $\square$

33. Let $f : [a, b] \to \mathbb{R}$ be monotone. Given $n$ distinct points $a < x_1 < x_2 < \cdots < x_n < b$, show that $\sum_{i=1}^{n} |f(x_i+) - f(x_i-)| \le |f(b) - f(a)|$. Use this to give another proof that $f$ has at most countably many (jump) discontinuities.

    *Proof.* WLOG $f$ is is monotone increasing. Then it holds that $f(x_i+) \ge f(x_i-)$. Thus,

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i+) - f(x_i-)| &= \sum_{i=1}^{n} [f(x_i+) - f(x_i-)] \\ &= -f(x_1-) + \sum_{i=1}^{n-1} [f(x_i+) - f(x_{i+1}-)] + f(x_n+)\end{aligned}$$

    Now, since $f$ is monotone increasing and $x_{i+1} > x_i$ for all $i$, each term of the summation is non-positive. Additionally, $a < x_1 \implies -f(x_1-) \le -f(a)$ and $b > x_n \implies f(x_n+) \le f(b)$. Thus, we have

$$\sum_{i=1}^{n} |f(x_i+) - f(x_i-)| \le -f(a) + f(b) = |f(b) - f(a)|$$

    as desired. $\square$