Homework 4

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Section 2.2: Groups

13. If G is any group, define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that α is injective and surjective.

Section 2.3: Subgroups

2. If H is a subset of a group G, show that H is a subgroup if and only if H is nonempty and $ab^{-1} \in H$ whenever $a \in H$ and $b \in H$.

5. (a) If G is an abelian group, show that $H = \left\{a \in G | a^2 = 1\right\}$ is a subgroup of G.
   (b) Give an example where H is not a subgroup.

8. If X is a nonempty subset of a group G, let $\langle X \rangle$ be the set of all products of powers of elements of X; more formally

$$\langle X \rangle = \left\{x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \mid m \geq 1, x_i \in X\right\}$$

   (a) Show that $\langle X \rangle$ is a subgroup of G that contains X.
   (b) Show that $\langle X \rangle \subseteq H$ for every subgroup H such that $X \subseteq H$. Thus, $\langle X \rangle$ is the *smallest* subgroup of G that contains X, and is called the **subgroup generated** by X.

13. (a) If G is a group, show that $\{(g, g) | g \in G\}$ is a subgroup of $G \times G$.
    (b) Determine the groups G such that $\left\{(g, g^{-1}) | g \in G\right\}$ is a subgroup of $G \times G$.

22. Find $Z[GL_2(\mathbb{R})]$.

Section 2.4: Cyclic Groups and the Order of an Element

6. If G is a group and $g \in G$, show that $\langle g \rangle = \left\langle g^{-1} \right\rangle$.

7. Let $o(g) = 20$ in a group G. Compute
   (a) $o(g^2)$
   (b) $o(g^8)$
   (c) $o(g^5)$
   (d) $o(g^3)$

10. (a) If $gh = hg$ in a group and $o(g)$ and $o(h)$ are finite, show that $o(gh)$ is finite.
    (b) Show that (a) fails if $gh \neq hg$ by considering $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.

18. If $G = \langle g \rangle$ and $H = \langle h \rangle$, show that $G \times H = \langle (g, 1), (1, h) \rangle$