Homework 1

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Section 2.3

4. Using Definition 4, show that each of the following functions is nowhere differentiable.

   (a) $\operatorname{Re} z$

   Proof. Suppose $\operatorname{Re} z$ was differentiable at $z_0 = a_0 + b_0 i$. Then

$$\begin{aligned}\frac{d(\operatorname{Re} z)}{dz}(z_0) &= \lim_{h\to 0}\frac{\operatorname{Re}(z_0+h) - \operatorname{Re} z}{h} = \lim_{a+bi\to 0}\frac{\operatorname{Re}\left[(a_0+b_0i)+(a+bi)\right] - \operatorname{Re}(a_0+b_0i)}{a+bi}\\ &= \lim_{a+bi\to 0}\frac{(a_0+a)-a_0}{a+bi} = \lim_{a+bi\to 0}\frac{a}{a+bi}\end{aligned}$$

   This limit does not exist because if we go along the real axis, the limit is 1, but if we go along the imaginary axis, the limit is 0. Thus, $\operatorname{Re} z$ is not differentiable at any point. □

   (c) $|z|$

   Proof. Suppose $|z|$ was differentiable at $z_0 = a_0 + b_0 i$. Then

$$\begin{aligned}\frac{d(|z|)}{dz}(z_0) &= \lim_{h\to 0}\frac{|z_0+h| - |z_0|}{z} = \lim_{a+bi\to 0}\frac{|(a_0+b_0i)+(a+bi)| - |a_0+b_0i|}{a+bi}\\ &= \lim_{a+bi\to 0}\frac{\sqrt{(a_0+a)^2+(b_0+b)^2} - \sqrt{a_0^2+b_0^2}}{a+bi}\end{aligned}$$

   If we approach along the real axis, $b = 0$, so the limit is

$$\lim_{a\to 0}\frac{\sqrt{(a_0+a)^2+b_0^2} - \sqrt{a_0^2}}{a} \to \infty$$

   so the limit does not exist. □

8. Suppose that f is analytic at z_0 and $f'(z_0) \neq 0$. Show that

$$\lim_{z\to z_0}\frac{|f(z)-f(z_0)|}{|z-z_0|} = |f'(z_0)|$$

   and

$$\lim_{z\to z_0}\left\{\arg\left[f(z)-f(z_0)\right] - \arg(z-z_0)\right\} = \arg f'(z_0)$$

11. Discuss the analyticity of each of the following functions.

   (b) $\frac{x}{\bar{z}+2}$

   (f) $\left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$

   (g) $|z|^2 + 2z$

Section 2.4

3. Use Theorem 5 to show that $g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is entire. Write this function in terms of z.

   Proof. Here, $u = 3x^2 + 2x - 3y^2 - 1$ and $v = 6xy + 2y$. We have
   $$\frac{\partial u}{\partial x} = 6x + 2$$
   $$\frac{\partial v}{\partial y} = 6x + 2$$
   $$\frac{\partial u}{\partial y} = -6y$$
   $$\frac{\partial v}{\partial x} = 6y$$
   so the Cauchy-Riemann equations are satisfied, and they are satisfied at all points in $\mathbb{C}$. The first partials are also all continuous, so g is entire. □

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5. Show that the function $f(z) = e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right]$ is entire, and find its derivative.

   Proof. Here, $u = e^{x^2-y^2}\cos(2xy)$ and $v = e^{x^2-y^2}\sin(2xy)$. We have
   $$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$$
   $$\frac{\partial v}{\partial y} = -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy)$$
   $$\frac{\partial u}{\partial y} = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy)$$
   $$\frac{\partial v}{\partial x} = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy)$$
   so the Cauchy-Riemann equations are satisfied, the first partials are continuous, and thus f is analytic at every point in $\mathbb{C}$, so f is also entire. By De Moivre's theorem, the derivative is
   $$\begin{aligned} f(z) &= e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right] = e^{x^2-y^2}e^{2xyi} \\ &= e^{x^2+2xyi-y^2} = e^{(x+yi)^2} = e^{z^2} \\ \implies f'(z) &= 2ze^{z^2} \end{aligned}$$
   □

8. Show that if f is analytic in a domain D and either $\operatorname{Re} f(x)$ or $\operatorname{Im} f(x)$ is constant in D, then $f(z)$ must be constant in D.

15. The Jacobian of a mapping
   $$u = u(x, y), \quad v = v(x, y)$$
   from the xy-plane to the uv-plane is defined to be the determinant
   $$J(x_0, y_0) := \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$
   where the partial derivatives are all evaluated at (x_0, y_0). Show that if $f = u + iv$ is analytic at $z_0 = x_0 + iy_0$, then $J(x_0, y_0) = |f'(z_0)|^2$.

Section 2.5

8. Suppose that the functions u and v are harmonic in a domain D.

   (a) Is the sum $u + v$ necessarily harmonic in D?

   (b) Is the product uv necessarily harmonic in D?

   (c) Is $\partial u/\partial x$ harmonic in D?

12. Prove that if r and θ are polar coordinates, then the functions $r^n \cos n\theta$ and $r^n \sin n\theta$, where n is an integer, are harmonic as functions of x and y.

13. Find a function harmonic inside the wedge bounded by the non-negative x-axis and the half-line $y = x \quad (x \geq 0)$ that goes to 0 on these sides but is not identically zero.