Homework 5

Aleck Zhao

October 12, 2016

Section 2.4: Cyclic Groups and the Order of an Element

4. In each case determine whether G is cyclic.

(a) $G = \mathbb{Z}_7^*$

Solution. Here, $G = \{1, 2, 3, 4, 5, 6\}$, where these are understood to be the equivalence classes, and the operation is multiplication. Then we have

$$\begin{aligned} 1 &\equiv 1 \\ 2 &\equiv 3^2 \\ 3 &\equiv 3^1 \\ 4 &\equiv 3^4 \\ 5 &\equiv 3^5 \\ 6 &\equiv 3^3 \end{aligned}$$

so $G = \langle 3 \rangle$, and $\boxed{G \text{ is cyclic.}}$

$\square$

(b) $G = \mathbb{Z}_{12}^*$

Solution. Here, $G = \{1, 5, 7, 11\}$, where these are understood to be equivalence classes, so the order of G is 4. However, $\langle 5 \rangle = \{1, 5\}$ and $\langle 7 \rangle = \{1, 7\}$, and these subgroups both have order 2, so $\boxed{G \text{ is not cyclic.}}$

$\square$

(c) $G = \mathbb{Z}_{16}^*$

Solution. Here, $G = \{1, 3, 5, 7, 9, 11, 13, 15\}$ so the order of G is 8. Now, we have

$$\begin{aligned} \langle 3 \rangle &= \{1, 3, 9, 11\} \\ \langle 5 \rangle &= \{1, 5, 9, 13\} \end{aligned}$$

so G has two distinct subgroups of order 4, so $\boxed{G \text{ is not cyclic.}}$

$\square$

(d) $G = \mathbb{Z}_{11}^*$

Solution. Here, $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and we have

$$\begin{aligned} 1 &\equiv 1 \\ 2 &\equiv 2^1 \\ 3 &\equiv 2^8 \\ 4 &\equiv 2^2 \\ 5 &\equiv 2^4 \\ 6 &\equiv 2^9 \\ 7 &\equiv 2^7 \\ 8 &\equiv 2^3 \\ 9 &\equiv 2^6 \\ 10 &\equiv 2^5 \end{aligned}$$

so $G = \langle 2 \rangle$ so $\boxed{G \text{ is cyclic.}}$

□

20. (a) Find three elements of $C_6 \times C_{15}$ of maximum order.

Solution. Let $C_6 = \left\{1, g, \cdots, g^5\right\}$ and $C_{15} = \left\{1, f, \cdots, f^{14}\right\}$. Then the element of max order in C_6 is g^5, where $o(g^5) = 6$ since 5 and 6 are relatively prime. Similarly, the elements of max order in C_{15} are the elements f^k where k is relatively prime to 15, which are $k = 7, 11, 13$. Then the elements in $C_6 \times C_{15}$ of max order are

$$\boxed{(g^5, f^7), \quad (g^5, f^{11}), \quad (g^5, f^{13})}$$

which all have order $\operatorname{lcm}(6, 15) = 30$.

□

(b) Find one element of maximum order in $C_m \times C_n$.

Solution. If $C_m = \langle g \rangle$ and $C_n = \langle f \rangle$ then we are guaranteed $o(g^{m-1}) = m$ and $o(f^{n-1}) = n$ since $m - 1$ and $n - 1$ are relatively prime to m and n, respectively. Thus, the element $\boxed{(g^{m-1}, f^{n-1})}$ is of maximum order $\operatorname{lcm}(m, n)$.

□

28. Let H be a subgroup of a group G and let $a \in G, o(a) = n$. If m is the smallest positive integer such that $a^m \in H$, show that $m | n$.

Section 2.5: Homomorphisms and Isomorphisms

3. If G is any group, define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that G is abelian if and only if α is a homomorphism.

Proof. If G is abelian, then $gf = fg$ for any $f, g \in G$. Then $\alpha(gf) = \alpha(fg) = (fg)^{-1} = g^{-1}f^{-1} = \alpha(g)\alpha(f)$ so $\alpha(gf) = \alpha(g)\alpha(f)$, so α is a homomorphism, as desired.

If α is a homomorphism, then $\alpha(fg) = \alpha(f)\alpha(g)$ for all $f, g \in G$. Then $(fg)^{-1} = f^{-1}g^{-1} = (gf)^{-1}$ so in fact $fg = gf$ since inverses are unique, and G is abelian, as desired.

□

6. Show that there are exactly two homomorphisms $\alpha : C_6 \to C_4$.

13. Show that $G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a subgroup of $GL_2(\mathbb{Z})$ isomorphic to $\{1, -1, i, -i\}$.

25. Are the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ isomorphic?

33. If $Z(G) = \{1\}$, show that $G \cong \text{inn}G$.

Section 2.6: Cosets and Lagrange's Theorem

1. In each case find the right and left cosets in G of the subgroups H and K of G.
   (e) $G = D_4 = \left\{1, a, a^2, a^3, b, ba, ba^2, ba^3\right\}$, $o(a) = 4$, $o(b) = 2$, and $aba = b$; $H = \left\langle a^2 \right\rangle$, $K = \langle b \rangle$.
   (f) $G =$ any group; H is any subgroup of index 2.

17. Let $|G| = p^2$, where p is a prime. Show that every proper subgroup of G is cyclic.