Homework 11

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Section 6.1

3. Evaluate each of the following integrals by means of the Cauchy residue theorem.

(a) $\oint_{|z|=5} \frac{\sin z}{z^2-4}\,dz$

Solution. The integrand has simple poles at $z=2, z=-2$, which are both contained inside $|z|=5$. Then we calculate the residues as

$$\begin{aligned}
&\frac{\sin z}{(z-2)(z+2)} = a_{-1}(z-2)^{-1} + a_0 + \cdots \implies \frac{\sin z}{z+2} = a_{-1} + a_0(z-2) + \cdots \\
&\implies \text{Res}(2) = a_{-1} = \left.\frac{\sin z}{z+2}\right|_2 = \frac{\sin 2}{4} \\
&\frac{\sin z}{(z-2)(z+2)} = b_{-1}(z+2)^{-1} + b_0 + \cdots \implies \frac{\sin z}{z-2} = b_{-1} + b_0(z+2) + \cdots \\
&\implies \text{Res}(-2) = b_{-1} = \left.\frac{\sin z}{z-2}\right|_{-2} = \frac{\sin(-2)}{-4} = \frac{\sin 2}{4}
\end{aligned}$$

so by the Residue theorem, the integral is equal to

$$2\pi i\left(\text{Res}(2) + \text{Res}(-2)\right) = 2\pi i\left(\frac{\sin 2}{4} + \frac{\sin 2}{4}\right) = \pi i \sin 2$$

□

(d) $\oint_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz$

Solution. The integrand has simple poles at 2 and $-5i$, and a pole of order 2 at 0. Only the poles at 2 and 0 are contained inside $|z|=3$, so we calculate the residues as

$$\begin{aligned}
&\frac{e^{iz}}{z^2(z-2)(z+5i)} = a_{-1}(z-2)^{-1} + a_0 + \cdots \implies \frac{e^{iz}}{z^2(z+5i)} = a_{-1} + a_0(z-2) + \cdots \\
&\implies \text{Res}(2) = a_{-1} = \left.\frac{e^{iz}}{z^2(z+5i)}\right|_2 = \frac{e^{2i}}{2^2(2+5i)} = \frac{e^{2i}(2-5i)}{116} \\
&\frac{e^{iz}}{z^2(z-2)(z+5i)} = b_{-2}z^{-2} + b_{-1}z^{-1} + b_0 + \cdots \implies \frac{e^{iz}}{(z-2)(z+5i)} = b_{-2} + b_{-1}z + b_0 + \cdots \\
&\implies \text{Res}(0) = b_{-1} = \left.\frac{d}{dz}\left[\frac{e^{iz}}{(z-2)(z+5i)}\right]\right|_0 = \frac{12-5i}{-100}
\end{aligned}$$

so by the Residue theorem, the integral is equal to

$$2\pi i\left(\text{Res}(2) + \text{Res}(0)\right) = 2\pi i\left(\frac{e^{2i}(2-5i)}{116} - \frac{12-5i}{100}\right)$$

□

(e) $\oint_{|z|=1} \frac{1}{z^2 \sin z}\, dz$

Solution. The integrand has a pole of order 2 at 0, and simple poles at $k\pi$ for $k \in \mathbb{Z}$, so the only one inside $|z| = 1$ is the pole $z = 0$ of order 3. Then we calculate the residue as

$$\frac{1}{z^2 \sin z} = a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \cdots \implies \frac{z}{\sin z} = a_{-3} + a_{-2}z + a_{-1}z^2 + a_0 z^3 + \cdots$$
$$\implies \operatorname{Res}(0) = a_{-1} = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z}{\sin z}\right] = \frac{1}{6}$$

so by the Residue theorem, the integral is equal to

$$2\pi i \operatorname{Res}(0) = \frac{\pi i}{3}$$

□

(f) $\oint_{|z|=3} \frac{3z+1}{z^4+1}\, dz$

Solution. The integrand has simple poles at the solutions to

$$z^4 + 1 = 0 \implies z^4 = -1 = e^{\pi i} \implies z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4} = r_1, r_2, r_3, r_4$$

where we have

$$\lim_{z\to r_i} \frac{z^4+1}{z - r_i} = \frac{d}{dz}\left[z^4 + 1\right]\bigg|_{r_i} = 4r_i^3$$

for all i, so

$$\begin{aligned}
\operatorname{Res}\left(e^{\pi i/4}\right) &= \frac{3z+2}{(z-r_2)(z-r_3)(z-r_4)}\bigg|_{r_1} = \frac{3r_1+2}{4r_1^3} = \frac{3e^{\pi i/4}+2}{4e^{3\pi i/4}} \\
\operatorname{Res}\left(e^{3\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_3)(z-r_4)}\bigg|_{r_2} = \frac{3r_2+2}{4r_2^3} = \frac{3e^{3\pi i/4}+2}{4e^{9\pi i/4}} = \frac{3e^{3\pi i/4}+2}{4e^{\pi i/4}} \\
\operatorname{Res}\left(e^{5\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_2)(z-r_4)}\bigg|_{r_3} = \frac{3r_3+2}{4r_3^3} = \frac{3e^{5\pi i/4}+2}{4e^{15\pi i/4}} = \frac{3e^{5\pi i/4}+2}{4e^{7\pi i/4}} \\
\operatorname{Res}\left(e^{7\pi i/4}\right) &= \frac{3z+2}{(z-r_1)(z-r_2)(z-r_3)}\bigg|_{r_4} = \frac{3r_4+2}{4r_4^3} = \frac{3e^{7\pi i/4}+2}{4e^{21\pi i/4}} = \frac{3e^{7\pi i/4}+2}{4e^{5\pi i/4}}
\end{aligned}$$

where the sum of the residues is

$$\begin{aligned}
&\frac{3e^{\pi i/4}+2}{4e^{3\pi i/4}} + \frac{3e^{3\pi i/4}+2}{4e^{\pi i/4}} + \frac{3e^{5\pi i/4}+2}{4e^{7\pi i/4}} + \frac{3e^{7\pi i/4}+2}{4e^{5\pi i/4}} \\
&= \frac{1}{4e^{\pi i/4}}\left(\frac{3e^{\pi i/4}+2}{e^{\pi i/2}} + \frac{3e^{3\pi i/4}+2}{1} + \frac{3e^{5\pi i/4}+2}{e^{3\pi i/2}} + \frac{3e^{7\pi i/4}+2}{e^{\pi i}}\right) \\
&= \frac{1}{4e^{\pi i/4}}\left(-i\left(3e^{\pi i/4}+2\right) + \left(3e^{3\pi i/4}+2\right) + i\left(3e^{5\pi i/4}+2\right) - \left(3e^{7\pi i/4}+2\right)\right) \\
&= \frac{3}{4e^{\pi i/4}}\left[-i\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + i\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right),\right] \\
&= \frac{3}{4e^{\pi i/4}}\left[-i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] \\
&= 0
\end{aligned}$$

so by the Residue theorem, the integral is equal to 0. □

5. Is there a function f having a simple pole at z_0 with $\operatorname{Res}(f; z_0) = 0$? How about a function with a pole of order 2 at z_0 and $\operatorname{Res}(f; z_0) = 0$?

   Answer. The first scenario is impossible, because since poles are isolated, there exists a sufficiently small ε such that z_0 is the only pole contained in $B_\varepsilon(z_0)$, and the integral around this circle would be 0 because $\operatorname{Res}(f; z_0) = 0$, but it should be $2\pi i$.

   For the second scenario, we can take $f(z) = 1/z^2$, which has a pole of order 2 at 0 but $\operatorname{Res}(f; 0) = 0$.

7. Evaluate

$$\oint_{|z|=1} e^{1/z} \sin(1/z)\, dz$$

   Solution. The integrand has a pole at 0. We have the Taylor series

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \cdots \\ \sin(1/z) &= \frac{1}{z} - \frac{(1/z)^3}{3!} + \cdots \\ \implies e^{1/z}\sin(1/z) &= \left(1 + \frac{1}{z} + \cdots\right)\left(\frac{1}{z} - \cdots\right) = \frac{1}{z} + \cdots \\ \implies \operatorname{Res}(0) &= 1 \end{aligned}$$

   so the integral is $2\pi i \operatorname{Res}(0) = 2\pi i$. □

Section 6.2

1. $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$

   Solution. Using the substitution $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ along the parametrization $e^{i\theta}$ of the unit circle,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_{|z|=1} \frac{1}{2 + \frac{1}{2i}\left(z - \frac{1}{z}\right)} \cdot \frac{1}{iz}\, dz = \oint_{|z|=1} \frac{2}{(z^2 + 4iz - 1)}$$

   The poles are at the roots

$$r_1, r_2 = \frac{-4i \pm \sqrt{(4i)^2 + 4}}{2} = \left(-2 \pm \sqrt{3}\right) i$$

   where only the root $r_1 = \left(-2 + \sqrt{3}\right) i$ lies inside the unit circle, so

$$\operatorname{Res}(r_1) = \left.\frac{2}{z - r_2}\right|_{r_1} = \frac{2}{\left(-2+\sqrt{3}\right) i - \left(-2 - \sqrt{3}\right) i} = \frac{1}{i\sqrt{3}}$$

   so the integral is $2\pi i \operatorname{Res}(r_1) = 2\pi i \cdot \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$. □

5. $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \quad a^2 < 1$

   Solution. Using the substitution $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ along the parametrization $e^{i\theta}$ of the unit circle,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \oint_{|z|=1} \frac{1}{1 + a \cdot \frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{1}{iz}\, dz = \oint_{|z|=1} \frac{2}{ai\left(z^2 + \frac{2}{a}z + 1\right)}\, dz$$

The poles are at the roots

$$r_1, r_2 = \frac{-\frac{2}{a} \pm \sqrt{\left(\frac{2}{a}\right)^2 - 4}}{2} = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

where only the root $r_1 = \frac{-1+\sqrt{1-a^2}}{a}$ lies inside the unit circle, so

$$\operatorname{Res}(r_1) = \left.\frac{2}{ai(z-r_2)}\right|_{r_1} = \frac{2}{ai\left(\frac{-1+\sqrt{1-a^2}}{a} - \frac{-1-\sqrt{1-a^2}}{a}\right)} = \frac{1}{i\sqrt{1-a^2}}$$

so the integral is $2\pi i \operatorname{Res}(r_1) = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$. □

8. $\int_0^{2\pi} \frac{d\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta} = \frac{2\pi}{ab}, \quad a, b > 0$

Solution. Using the substitutions on the unit circle, we have

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a^2 \sin^2\theta + b^2\cos^2\theta} &= \oint \frac{1}{a^2 \cdot \left(\frac{z^2-1}{2iz}\right)^2 + b^2 \cdot \left(\frac{z^2+1}{2z}\right)} \cdot \frac{1}{iz}\, dz \\ &= \frac{1}{i} \oint \frac{4z}{(b^2-a^2)z^4 + (2b^2+2a^2)z^2 + (b^2-a^2)}\, dz\end{aligned}$$

We have

$$z^2 = \frac{-(2b^2+2a^2) \pm \sqrt{(2b^2+2a^2)^2 - 4(b^2-a^2)^2}}{2(b^2-a^2)} = \frac{b^2 \pm 2ab + a^2}{a^2-b^2}$$

WLOG $a \geq b$, so the roots are

$$r_1 = \frac{a-b}{\sqrt{a^2-b^2}}, \quad r_2 = \frac{b-a}{\sqrt{a^2-b^2}}, \quad r_3 = \frac{b+a}{\sqrt{a^2-b^2}}, \quad r_4 = \frac{-b-a}{\sqrt{a^2-b^2}}$$

where r_1, r_2 lie within the unit circle. We have

$$\begin{aligned}\operatorname{Res}(r_1) &= \left.\frac{4z}{(b^2-a^2)(z-r_2)\left(z^2 - \frac{(a+b)^2}{a^2-b^2}\right)}\right|_{r_1} = \frac{4 \cdot \frac{a-b}{\sqrt{a^2-b^2}}}{(b^2-a^2) \cdot 2 \cdot \frac{a-b}{\sqrt{a^2-b^2}}\left(\frac{(a-b)^2}{a^2-b^2} - \frac{(a+b)^2}{a^2-b^2}\right)} \\ &= \frac{2}{(a+b)^2 - (a-b)^2} = \frac{1}{2ab} \\ \operatorname{Res}(r_2) &= \left.\frac{4z}{(b^2-a^2)(z-r_1)\left(z^2 - \frac{(a+b)^2}{a^2-b^2}\right)}\right|_{r_2} = \frac{4 \cdot \frac{b-a}{\sqrt{a^2-b^2}}}{(b^2-a^2) \cdot 2 \cdot \frac{b-a}{\sqrt{a^2-b^2}}\left(\frac{(b-a)^2}{a^2-b^2} - \frac{(a+b)^2}{a^2-b^2}\right)} \\ &= \frac{2}{(a+b)^2 - (b-a)^2} = \frac{1}{2ab}\end{aligned}$$

so by the Residue theorem, the integral is

$$2\pi i \cdot \frac{1}{i}\left(\operatorname{Res}(r_1) + \operatorname{Res}(r_2)\right) = 2\pi\left(\frac{1}{2ab} + \frac{1}{2ab}\right) = \frac{2\pi}{ab}$$

□

9. $\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{\pi \cdot (2n)!}{2^{2n-1}(n!)^2}, \quad n = 1, 2, \cdots$

Solution. Using the substitutions on the unit circle, we have

$$\begin{aligned}
\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta &= \oint \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \cdot \frac{1}{iz}\, dz \\
&= \frac{1}{2^{2n} i} \oint \frac{1}{z}\left[z^{2n} + \binom{2n}{1} z^{2n-2} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n-1}\frac{1}{z^{2n-2}} + \frac{1}{z^{2n}}\right] dz \\
&= \frac{1}{2^{2n} i} \oint \left[z^{2n-1} + \binom{2n}{1} z^{2n-3} + \cdots + \binom{2n}{n}\frac{1}{z} + \cdots + \binom{2n}{2n-1}\frac{1}{z^{2n-1}} + \frac{1}{z^{2n+1}}\right] dz \\
&= \frac{1}{2^{2n} i} \cdot 2\pi i \binom{2n}{n} = \frac{\pi (2n)!}{2^{2n-1}(n!)^2}
\end{aligned}$$

□

10. $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}, \quad n = 1, 2, \cdots$

Solution. We have

$$\int_0^{2\pi} e^{\cos\theta} \cdot \frac{1}{2}\left[e^{i(n\theta - \sin\theta)} + e^{i(-n\theta + \sin\theta)}\right] d\theta = \frac{1}{2}\int_0^{2\pi} \left(e^{\cos\theta - i\sin\theta + in\theta} + e^{\cos\theta + i\sin\theta - in\theta}\right) d\theta$$

and using the substitution $z = e^{i\theta} = \cos\theta + i\sin\theta$ along the unit circle, this is

$$\begin{aligned}
&\frac{1}{2}\oint \left(e^{1/z} z^n + e^z z^{-n}\right) \cdot \frac{1}{iz}\, dz = \frac{1}{2i}\left(\oint e^{1/z} z^n\, dz + \oint e^z z^{-n}\, dz\right) \\
&= \frac{1}{2i}\left[\oint z^{n-1}\left(1 + \frac{1}{z} + \frac{(1/z)^2}{2} + \cdots + \frac{(1/z)^n}{n!}\right) dz + \oint z^{-n-1}\left(1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots\right) dz\right] \\
&= \frac{1}{2i}\left[\oint \left(\cdots + \frac{1}{n!}\frac{1}{z} + \cdots\right) dz + \oint \left(\cdots + \frac{1}{n!}\frac{1}{z} + \cdots\right) dz\right] = \frac{1}{2i} \cdot 2\pi i \cdot \frac{2}{n!} = \frac{2\pi}{n!}
\end{aligned}$$

□