Homework 4

Aleck Zhao

March 1, 2018

Chapter 16: Lebesgue Measure

2. Prove statements (i) and (ii) of Proposition 16.2.

   (i) $0 \leq m^*(E) \leq \infty$

   Proof. Let $\varepsilon > 0$. Then there exists a sequence of intervals (I_n) covering E such that

   $$\sum_{n=1}^{\infty} \ell(I_n) < m^*(E) + \varepsilon$$

   Since $\ell(I_n) \geq 0$, it follows that $m^*(E) + \varepsilon > 0 \implies m^*(E) \geq 0$. Then if $E = \mathbb{R}$, any covering must include an unbounded interval, so $m^*(E) = \infty$, so the upper bound can be achieved. □

   (ii) If $E \subset F$, then $m^*(E) \leq m^*(F)$.

   Proof. Let $\varepsilon > 0$. Then there exists a sequence of intervals (I_n) covering F such that

   $$\sum_{n=1}^{\infty} \ell(I_n) < m^*(F) + \varepsilon$$

   Then since $E \subset F$, this sequence also covers E, so

   $$\begin{aligned} m^*(E) &\leq \sum_{n=1}^{\infty} \ell(I_n) < m^*(F) + \varepsilon \\ \implies m^*(E) &\leq m^*(F) \end{aligned}$$

   □

3. Earlier attempts at defining the measure of a (bounded) set were similar to Lebesgue's, except that the infimum was typically taken over finite unions of intervals covering the set. Show that if $\mathbb{Q} \cap [0, 1]$ is contained in a finite union of open intervals $\bigcup_{i=1}^{n}(a_i, b_i)$, then $\sum_{i=1}^{n}(b_i - a_i) \geq 1$. Thus, $\mathbb{Q} \cap [0, 1]$ would have "measure" 1 by this definition.

   Proof. Suppose $\sum_{i=1}^{n}(b_i - a_i) < 1$. Then these intervals would not cover $[0, 1]$, so there must exist some open interval. Since rationals are dense in $\mathbb{R}$, there must exist a rational q in this interval, and thus these intervals would not cover $\mathbb{Q} \cap [0, 1]$. Contradiction, so $\sum_{i=1}^{n}(b_i - a_i) \geq 1$. □

5. If we define $rE = \{rx : x \in E\}$, what is $m^*(rE)$ in terms of $m^*(E)$?

Solution. We claim that $m^*(rE) = |r|\, m^*(E)$. If E has measure ∞, then it is clear that rE also has measure ∞. Otherwise, they are both bounded. Let $\varepsilon > 0$. Then there exists a sequence of intervals (a_n, b_n) covering E such that

$$\sum_{n=1}^{\infty}(b_n - a_n) < m^*(E) + \frac{\varepsilon}{|r|}$$

Then if $r \geq 0$, it follows that (ra_n, rb_n) covers rE, and likewise if $r < 0$, the intervals (rb_n, ra_n) covers rE. In either case, we have

$$\begin{aligned} m^*(rE) &\leq \sum_{n=1}^{\infty} |r|\,(b_n - a_n) < |r|\, m^*(E) + \varepsilon \\ \implies m^*(rE) &\leq |r|\, m^*(E) \end{aligned}$$

By a similar argument, there exists a sequence of intervals (c_k, d_k) covering rE such that

$$\sum_{k=1}^{\infty}(d_k - c_k) < m^*(rE) + \varepsilon$$

Then if $r \geq 0$, the intervals $\left(\frac{c_k}{r}, \frac{d_k}{r}\right)$ covers E and if $r < 0$, the intervals $\left(\frac{d_k}{r}, \frac{c_k}{r}\right)$ cover E. Thus

$$\begin{aligned} m^*(E) &\leq \sum_{k=1}^{\infty} \frac{1}{|r|}(d_k - c_k) < \frac{1}{|r|} m^*(rE) + \frac{1}{|r|}\varepsilon \\ \implies |r|\, m^*(E) &\leq m^*(rE) \end{aligned}$$

so in fact $m^*(rE) = rm^*(E)$. □

8. Given $\delta > 0$, show that $m^*(E) = \inf \sum_{n=1}^{\infty} \ell(I_n)$ where the infimum is taken over all coverings of E by sequences of intervals (I_n), where each I_n has diameter less than δ.

Proof. We trivially have $m^*(E) \leq \inf \sum_{n=1}^{\infty} \ell(I_n)$. Let $\varepsilon > 0$. Then there exists a sequence of intervals (J_k) covering E such that

$$\sum_{k=1}^{\infty} \ell(J_k) < m^*(E) + \varepsilon$$

Now, for each interval J_k, we can write $J_k = \bigcup_{i=1}^{\infty} I_{k,i}$ where $I_{k,i}$ are pairwise disjoint and $\ell(I_{k,i}) < \delta$. Thus,

$$\begin{aligned} \inf \sum_{n=1}^{\infty} \ell(I_n) &\leq \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty} \ell(J_k) < m^*(E) + \varepsilon \\ \implies \inf \sum_{n=1}^{\infty} \ell(I_n) &\leq m^*(E) \end{aligned}$$

so we have $m^*(E) = \inf \sum_{n=1}^{\infty} \ell(I_n)$ as desired. □

13. Show that $m^*(E \cup F) \le m^*(E) + m^*(F)$ for any sets E, F.

Proof. If E or F has measure ∞, the inequality trivially holds. Otherwise, they both have bounded measure. Let $\varepsilon > 0$. Then there exist sequences of intervals (I_n) and (J_k) covering E and F, respectively, such that

$$\sum_{n=1}^{\infty} \ell(I_n) < m^*(E) + \frac{\varepsilon}{2}$$
$$\sum_{k=1}^{\infty} \ell(J_k) < m^*(F) + \frac{\varepsilon}{2}$$

Since $E \cup F \subset \left(\bigcup_{n=1}^{\infty} I_n\right) \cup \left(\bigcup_{k=1}^{\infty} J_k\right)$, we have

$$\begin{aligned} & m^*(E \cup F) \le \sum_{n=1}^{\infty} \ell(I_n) + \sum_{k=1}^{\infty} \ell(J_k) < \left(m^*(E) + \frac{\varepsilon}{2}\right) + \left(m^*(F) + \frac{\varepsilon}{2}\right) = m^*(E) + m^*(F) + \varepsilon \\ \implies & m^*(E \cup F) \le m^*(E) + m^*(F) \end{aligned}$$

□

15. Prove that a subset of a set of outer measure zero is again a set of outer measure zero. Prove that a finite union of sets of outer measure zero has outer measure zero.

Proof. Let $F \subset E$ where E has measure 0. Then by property (ii), we have $m^*(F) \le m^*(E) = 0$, but since measure is at least, 0, it follows that $m^*(F) = 0$.

If E_1 and E_2 are sets of outer measure zero, then by the result of 13, we have

$$m^*(E_1 \cup E_2) \le m^*(E_1) + m^*(E_2) = 0$$

and since measure is at least 0, it follows that $m^*(E_1 \cup E_2) = 0$. By induction, it follows that any finite union of measure zero sets has measure 0. □

16. If $m^*(E) = 0$, show that $m^*(E \cup A) = m^*(A) = m^*(A \setminus E)$ for any A.

Proof. By countable subadditivity, we have $m^*(E \cup A) \le m^*(E) + m^*(A) = m^*(A)$. Since $A \subset E \cup A$, we also have $m^*(A) \le m^*(E \cup A)$, so it follows that $m^*(E \cup A) = m^*(A)$.

Similarly, we have

$$\begin{aligned} & m^*(A) \le m^*(A \setminus E) + m^*(E) = m^*(A \setminus E) \\ & A \setminus E \subset A \implies m^*(A \setminus E) \le m^*(A) \\ \implies & m^*(A) = m^*(A \setminus E) \end{aligned}$$

□

21. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le K\,|x - y|$ for all x and y, show that $m^*(f(E)) \le K m^*(E)$ for any $E \subset \mathbb{R}$.

Proof. If E has measure ∞, the inequality trivially holds. Otherwise, $m^*(E) < \infty$. Let $\varepsilon > 0$. Then there exists a sequence of intervals (a_n, b_n) such that $E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then we have $\sum_{n=1}^{\infty}(b_n - a_n) < m^*(E) + \frac{\varepsilon}{K}$. We also have that $f(E) \subset \bigcup_{n=1}^{\infty} f\left[(a_n, b_n)\right]$, where

$$f[(a_n, b_n)] = (f(c_n), f(d_n)), \quad c_n, d_n \in (a_n, b_n)$$

since Lipschitz functions are continuous. Then since $f[(a_n, b_n)]$ is a covering of $f(E)$, we have

$$\begin{aligned} m^*(f(E)) &\leq \sum_{n=1}^{\infty} \ell\bigg(f[(a_n, b_n)]\bigg) = \sum_{n=1}^{\infty}(f(d_n) - f(c_n)) \leq \sum_{n=1}^{\infty} K(d_n - c_n) \leq \sum_{n=1}^{\infty} K(b_n - a_n) \\ &< Km^*(E) + \varepsilon \\ \implies m^*(f(E)) &\leq Km^*(E) \end{aligned}$$

$\square$