Homework 6 Solutions

Aleck Zhao

April 1, 2018

1. (E&P 3.6.62) Prove this as follows. Given x_1, x_2, and x_3, define the cubic polynomial $P(y)$ to be

$$P(y) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & y & y^2 & y^3 \end{bmatrix} \tag{26}$$

Because $P(x_1) = P(x_2) = P(x_3) = 0$ (why?), the roots of $P(y)$ are x_1, x_2, and x_3. It follows that

$$P(y) = k(y - x_1)(y - x_2)(y - x_3)$$

where k is the coefficient of y^3 in $P(y)$. Finally, observe that expansion of the 4×4 determinant in (26) along its last row gives $k = V(x_1, x_2, x_3)$ and that $V(x_1, x_2, x_3, x_4) = P(x_4)$.

Proof. First, note that

$$P(x_1) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_1 & x_1^2 & x_1^3 \end{bmatrix}$$

Here, this matrix has two identical rows, so its determinant is 0, and thus $P(x_1) = 0$. Similarly, $P(x_2) = P(x_3) = 0$, so x_1, x_2, and x_3 are the roots of P, so it factors in the form given. Now, expanding the determinant along the 4th row, the coefficient of y^3 is

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = V(x_1, x_2, x_3)$$

as desired, and if we substitute $y = x_4$, then $P(x_4)$ takes on the form of $V(x_1, x_2, x_3, x_4)$, as desired. $\square$

2. (E&P 4.7.26) Use the method of Example 8 to find a basis for the 2-dimensional solution space of the given differential equation. $y'' + 10y' = 0$.

Solution. Let $v(x) = y'(x)$. Then this equation says $v' + 10v = 0 \implies v' = -10v$, which has solution $v(x) = Ce^{-10x}$. Thus,

$$y(x) = \int y'(x)\,dx = \int v(x)\,dx = \int Ce^{-10x}\,dx = -\frac{1}{10}Ce^{-10x} + A$$

so the general solution is given by $y(x) = A + Be^{-10x}$, so a basis of this solution space is $\left\{1, e^{-10x}\right\}$. $\square$

3. (E&P 4.7.32) Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 (real) matrices partitioned into 2×2 submatrices or "block":

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{B_{21}} & \mathbf{B_{22}} \end{bmatrix}$$

Then verify that $\mathbf{AB}$ can be calculated in "blockwise" fashion:

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A_{11}B_{11}} + \mathbf{A_{12}B_{21}} & \mathbf{A_{11}B_{12}} + \mathbf{A_{12}B_{22}} \\ \mathbf{A_{21}B_{11}} + \mathbf{A_{22}B_{21}} & \mathbf{A_{21}B_{12}} + \mathbf{A_{22}B_{22}} \end{bmatrix}$$

Solution. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Consider only the top left block of $\mathbf{AB}$, since every other quadrant is computed similarly. We have

$$\begin{aligned}
\mathbf{A_{11}} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B_{11}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
\mathbf{A_{12}} &= \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}, \quad \mathbf{B_{21}} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \\
\implies \mathbf{A_{11}B_{11}} + \mathbf{A_{12}B_{21}} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} + a_{14}b_{41} & a_{13}b_{32} + a_{14}b_{42} \\ a_{23}b_{31} + a_{24}b_{41} & a_{23}b_{32} + a_{24}b_{42} \end{bmatrix}
\end{aligned}$$

which is the same as the upper left 2×2 block we obtain if we multiply $\mathbf{AB}$ directly. □

4. Determine if $y_1 = x^3 - 1, y_2 = x + 1, y_3 = x^3 + x^2$, and $y_4 = x^2 + x$ form a basis for $\mathcal{P}_4$ (the space of all polynomials of degree less than 4.)

Solution. We claim this is a basis for $\mathcal{P}_4$. We must show these span $\mathcal{P}_4$ and are linearly independent. Since $\mathcal{P}_4$ is a vector space of dimension 4, these are equivalent, so we only need to show one of them.

To show they are spanning, take $ax^3 + bx^2 + cx + d \in \mathcal{P}_4$ to be an arbitrary element of $\mathcal{P}_4$. Now, let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$\begin{aligned}
ax^3 + bx^2 + cx + d &= k_1(x^3 - 1) + k_2(x + 1) + k_3(x^3 + x^2) + k_4(x^2 + x) \\
&= (k_1 + k_3)x^3 + (k_3 + k_4)x^2 + (k_2 + k_4)x + (k_2 - k_1)
\end{aligned}$$

so by matching coefficients, we have the 4 equations

$$\begin{aligned}
a &= k_1 + k_3 \\
b &= k_3 + k_4 \\
c &= k_2 + k_4 \\
d &= k_2 - k_1
\end{aligned}$$

$$\implies \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Now, by performing Gaussian elimination on this matrix, we find that it is non-singular, and thus a solution exists for k_1, k_2, k_3, k_4, so any arbitrary element of $\mathcal{P}_4$ can be represented as a linear combination of the 4 given polynomials, so y_1, y_2, y_3, y_4 span $\mathcal{P}_4$, and thus form a basis.

Alternatively, we can show that the polynomials are linearly independent. Suppose

$$0 = k_1y_1 + k_2y_2 + k_3y_3 + k_4y_4$$

$$\implies \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

By performing Gaussian elimination, we find that $k_1 = k_2 = k_3 = k_4$, so it follows that y_1, y_2, y_3, y_4 are linearly independent, and thus they form a basis for $\mathcal{P}_4$. □

5. MATLAB Practice Lesson 10
6. MATLAB Practice Lesson 11
7. Let $\mathbf{A}$ be an $m \times n$ matrix with $m > n$. Consider the linear system $\mathbf{Ax} = \mathbf{b}$. This system is usually inconsistent so instead, we attempt to find a vector $\hat{\mathbf{x}}$ such that $\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2$ is as small as possible. The vector $\hat{\mathbf{x}}$ is called the least squares solution for the system.

   - We may find $\hat{\mathbf{x}}$ by solving
   $$\mathbf{A^T A}\hat{\mathbf{x}} = \mathbf{A^T b}$$
   - The orthogonal projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$ is given by
   $$\hat{\mathbf{b}} = \mathbf{A}\hat{\mathbf{x}}$$

   Let
   $$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

   (a) Find the least squares solution for the system $\mathbf{Ax} = \mathbf{b}$.

   Solution. We have
   $$\mathbf{A^T A}\hat{\mathbf{x}} = \mathbf{A^T b} \implies \hat{\mathbf{x}} = (\mathbf{A^T A})^{-1}\mathbf{A^T b}$$
   so the least squares solution can be solved as
   $$\begin{aligned} \mathbf{A^T A} &= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \\ \implies (\mathbf{A^T A})^{-1} &= \frac{1}{3 \cdot 6 - (-2) \cdot (-2)} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \\ \implies \hat{\mathbf{x}} &= \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 14 \\ 49 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \end{bmatrix} \end{aligned}$$
   □

   (b) Find the orthogonal projection of $\hat{b}$ onto the column space of $\mathbf{A}$.

   Solution. We have
   $$\hat{\mathbf{b}} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 5/2 \\ 6 \end{bmatrix}$$
   □

8. (E&P 5.1.23) Determine whether the pair of functions are linearly independent or linearly dependent on the real line. $f(x) = xe^x, y(x) = |x|\, e^x$.

   Solution. These functions are linearly dependent if they are constant multiples on the real line. Suppose
   $$\frac{f(x)}{g(x)} = k \implies \frac{xe^x}{|x|\, e^x} = \frac{x}{|x|} = k$$
   Now, if $x \geq 0$, then $|x| = x$, but if $x < 0$ then $|x| = -x$, so it is clear that there is no value of k that satisfies this equation, and thus f and g are linearly independent. □