Homework 6

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1. *Solution.* If X and Y are the results of the two dice, then $Z = XY$. The rolls are independent of each other, and we have

$$E[X] = E[Y] = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$
$$E[X^2] = E[Y^2] = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$

Now, since X and Y are independent, it also holds that X^2 and Y^2 are independent, so

$$\begin{aligned} E[Z] &= E[XY] = E[X]E[Y] = \frac{7}{2}\cdot\frac{7}{2} = \frac{49}{4} \\ \mathrm{Var}(Z) &= E[Z^2] - (E[Z])^2 = E[X^2Y^2] - (E[XY])^2 = E[X^2]E[Y^2] - (E[XY])^2 \\ &= \frac{91}{6}\cdot\frac{91}{6} - \left(\frac{49}{4}\right)^2 = \frac{11515}{144} \end{aligned}$$

□

2. (a) *Solution.* We wish to find α that minimizes

$$\begin{aligned} L &= \frac{1}{2}\mathrm{Var}[\alpha r_A + (1-\alpha)r_B] = \frac{1}{2}\left(\mathrm{Var}(\alpha r_A) + \mathrm{Var}[(1-\alpha)r_B] + 2\mathrm{Cov}(\alpha r_A, (1-\alpha)r_B)\right) \\ &= \frac{1}{2}\alpha^2\sigma_A^2 + \frac{1}{2}(1-\alpha)^2\sigma_B^2 + \alpha(1-\alpha)\rho\sigma_A\sigma_B \end{aligned}$$

Taking the derivative with respect to α, we have

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \alpha\sigma_A^2 - (1-\alpha)\sigma_B^2 + (1-2\alpha)\rho\sigma_A\sigma_B = 0 \\ \implies \alpha &= \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B} = 0.8261 \\ \implies 1-\alpha &= 0.1739 \end{aligned}$$

□

(b) *Solution.* Evaluating at α, we have

$$\sigma = \sqrt{2L(\alpha)} = 1.94\%$$

□

(c) *Solution.* The expected return is

$$E[r] = E[\alpha r_A + (1-\alpha)r_B] = \alpha\bar{r}_a + (1-\alpha)\bar{r}_B = 11.39\%$$

□

3. *Solution.* If α is the weight of asset 1 and $1-\alpha$ is the weight of asset 2, then from above, we have

$$\alpha = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

minimizes the variance of the portfolio. The expected return is

$$\begin{aligned} E[r] &= E[\alpha r_1 + (1-\alpha)r_2] = \alpha\bar{r}_1 + (1-\alpha)\bar{r}_2 \\ &= \bar{r}_1 + \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}(\bar{r}_1 - \bar{r}_2) \end{aligned}$$

□

4. (a) *Solution.* Each asset has the same return $\bar{r}_0$, so if the portfolio was entirely a single asset, all of these points would lie on the horizontal line $\bar{r} = \bar{r}_0$. Since they are uncorrelated, this line is exactly the minimum variance set, and the efficient frontier is the entire line. □

   (b) *Solution.* We have the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{n} w_i^2\sigma_i^2 - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

Taking the partial derivatives with each of the w_i, the weights with minimum variance satisfy

$$\begin{aligned} \frac{\partial L}{\partial w_i} &= w_i\sigma_i^2 - \mu = 0 \implies w_i = \frac{\mu}{\sigma_i^2} \\ \implies \sum_{i=1}^{n} w_i &= \sum_{i=1}^{n}\frac{\mu}{\sigma_i^2} = \frac{\mu}{\bar{\sigma}^2} = 1 \implies \mu = \bar{\sigma}^2 \\ \implies w_i &= \frac{\bar{\sigma}^2}{\sigma_i^2} \end{aligned}$$

□

5. (a) *Solution.* If w is the column vector of weights, we seek to minimize

$$\begin{aligned} \frac{1}{2}w^TVw &= \frac{1}{2}\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2w_1 + w_2 & w_1 + 2w_2 + w_3 & w_2 + 2w_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &= \frac{1}{2}\left[(2w_1^2 + w_1w_2) + (w_1w_2 + 2w_2^2 + w_2w_3) + (w_2w_3 + 2w_3^2)\right] \\ &= w_1^2 + w_2^2 + w_3^2 + w_1w_2 + w_2w_3 \end{aligned}$$

The Lagrangian and its derivatives are given by

$$\begin{aligned} L &= \frac{1}{2}w^TVw - \mu(w_1 + w_2 + w_3 - 1) \\ \frac{\partial L}{\partial w_1} &= 2w_1 + w_2 - \mu = 0 \\ \frac{\partial L}{\partial w_2} &= 2w_2 + w_1 + w_3 - \mu = 0 \\ \frac{\partial L}{\partial w_3} &= 2w_3 + w_2 - \mu = 0 \\ \frac{\partial L}{\partial \mu} &= -w_1 - w_2 - w_3 + 1 = 0 \end{aligned}$$

and solving this system gives $w_1 = 0.5, w_2 = 0, w_3 = 0.5$. □

(b) *Solution.* Setting $\lambda = 1, \mu = 0$, we are not as concerned with the weight restriction so we have

$$\begin{aligned} 2v_1 + v_2 &= 0.4 \\ v_1 + 2v_2 + v_3 &= 0.8 \\ v_2 + 2v_3 &= 0.8 \\ \implies \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \\ \implies \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \end{bmatrix} \end{aligned}$$

after normalizing. □

(c) *Solution.* If the risk free rate $r_f = 0.2$, then the equations become

$$\begin{aligned} 2v_1 + v_@ &= 0.4 - 0.2 = 0.2 \\ v_1 + 2v_2 + v_3 &= 0.8 - 0.2 = 0.6 \\ v_2 + 2v_3 &= 0.8 - 0.2 = 0.6 \\ \implies \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \end{bmatrix} \\ \implies \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

after normalizing. □

6. (a) *Solution.* If σ_{iM} is the covariance between r_i and r_M and σ_M^2 is the variance of r_M, we have

$$\begin{aligned} \mathrm{Var}(r - r_M) &= \mathrm{Var}(r) + \mathrm{Var}(r_M) - 2\mathrm{Cov}(r, r_M) \\ &= \mathrm{Var}\left(\sum_{i=1}^n \alpha_i r_i\right) + \sigma_M^2 - 2\mathrm{Cov}\left(\sum_{i=1}^n \alpha_i r_i, r_M\right) \\ &= \sum_{i,j} \alpha_i \alpha_j \sigma_{ij} + \sigma_M^2 - 2\sum_{i=1}^n \alpha_i \sigma_{iM} \end{aligned}$$

Now, we have the Lagrangian

$$L = \frac{1}{2}\sum_{i,j} \alpha_i \alpha_j \sigma_{ij} + \frac{1}{2}\sigma_M^2 - \sum_{i=1}^n \alpha_i \sigma_{iM} - \lambda\left(\sum_{i=1}^n \alpha_i - 1\right)$$

and its partial derivatives set equal to 0:

$$\begin{aligned} \frac{\partial L}{\partial \alpha_i} &= \sum_{j=1}^n \alpha_j \sigma_{ij} - \sigma_{iM} - \lambda = 0, \quad \forall i = 1, 2, \cdots, n \\ \frac{\partial L}{\partial \lambda} &= 1 - \sum_{i=1}^n \alpha_i = 0 \end{aligned}$$

are $n + 1$ equations in $n + 1$ variables $\alpha_1, \cdots, \alpha_n, \lambda$. □

(b) *Solution.* If we also wish to achieve a mean return of $\bar{r}_M$, we have the additional constraint $\sum_{i=1}^n \alpha_i \bar{r}_i = \bar{r}_M$, so the Lagrangian and its derivatives are

$$\begin{aligned}
L &= \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j\sigma_{ij} + \frac{1}{2}\sigma_M^2 - \sum_{i=1}^n \alpha_i\sigma_{iM} - \lambda\left(\sum_{i=1}^n \alpha_i\bar{r}_i - \bar{r}_M\right) - \mu\left(\sum_{i=1}^n \alpha_i - 1\right) \\
\frac{\partial L}{\partial \alpha_i} &= \sum_{j=1}^n \alpha_j\sigma_{ij} - \sigma_{iM} - \lambda\bar{r}_i - \mu = 0, \quad \forall i = 1, 2, \cdots, n \\
\frac{\partial L}{\partial \lambda} &= \bar{r}_M - \sum_{i=1}^n \alpha_i\bar{r}_i = 0 \\
\frac{\partial L}{\partial \mu} &= 1 - \sum_{i=1}^n \alpha_i = 0
\end{aligned}$$

which are $n+2$ equations in $n+2$ variables. □

7. *Solution.* We have

$$\frac{\partial}{\partial w_k}\left(\sum_{i,j}^n \sigma_{ij}w_iw_j\right)^{1/2} = \left(\sum_{i,j}^n \sigma_{ij}w_iw_j\right)^{-1/2}\sum_{i=1}^n \sigma_{ik}w_i$$

Taking the derivative of $\tan\theta$ with respect to w_k, we have

$$\begin{aligned}
\frac{\partial}{\partial w_k}\tan\theta &= \frac{\partial}{\partial w_k}\left[\frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{1/2}}\right] \\
&= \frac{\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{1/2}\cdot\frac{\partial}{\partial w_k}\left[\sum_{i=1}^n w_i(\bar{r}_i - r_f)\right] - \sum_{i=1}^n w_i(\bar{r}_i - r_f)\cdot\frac{\partial}{\partial w_k}\left[\sum_{i,j}\sigma_{ij}w_iw_j\right]^{1/2}}{\sum_{i,j}\sigma_{ij}w_iw_j} \\
&= \frac{\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{1/2}(\bar{r}_k - r_f) - \sum_{i=1}^n w_i(\bar{r}_i - r_f)\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{-1/2}\sum_{i=1}^n \sigma_{ik}w_i}{\sum_{i,j}\sigma_{ij}w_iw_j} = 0 \\
\implies \bar{r}_k - r_f &= \sum_{i=1}^n \sigma_{ik}w_i\sum_{i=1}^n w_i(\bar{r}_i - r_f)\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{-1} = \sum_{i=1}^n \sigma_{ik}\lambda w_i
\end{aligned}$$

where

$$\lambda = \sum_{i=1}^n w_i(\bar{r}_i - r_f)\left(\sum_{i,j}\sigma_{ij}w_iw_j\right)^{-1}$$

as desired. □