Homework 9

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Section 3.1: Examples and Basic Properties

1. In each case explain why R is not a ring.

   (a) $R = \{0, 1, 2, 3, \cdots\}$, operations of $\mathbb{Z}$.

   Answer. R does not contain the additive inverses.

   (b) $R = 2\mathbb{Z}$.

   Answer. R does not contain a multiplicative identity.

   (c) $R =$ the set of all mappings $f : \mathbb{R} \to \mathbb{R}$; addition is point-wise but using composition as the multiplication.

   Answer. If $f, g, h \in R$, the condition $f(g+h) = fg+fh$ does not hold. For example, if $f(x) = \sqrt{x}$ and $g(x) = h(x) = x$, we have $f(g+h) = \sqrt{2x}$ but $fg + gh = 2\sqrt{x}$, and the two are not equal.

3. (c) Show that $S = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & a \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ is a subring of $R = M_3(\mathbb{R})$.

   Proof. We have

$$0_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$$

   and

$$1_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S$$

   Now, let

$$S = \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & a \end{bmatrix}, \quad T = \begin{bmatrix} w & 0 & x \\ 0 & y & z \\ 0 & 0 & w \end{bmatrix}$$

   so then

$$S - T = \begin{bmatrix} a-w & 0 & b-x \\ 0 & c-y & d-z \\ 0 & 0 & a-w \end{bmatrix} \in S$$
$$ST = \begin{bmatrix} aw & 0 & ax+bw \\ 0 & cy & cz+dw \\ 0 & 0 & aw \end{bmatrix} \in S$$

   Thus, S is a subring of R, as desired.

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Section 3.2: Integral Domains and Fields

1. Find all the roots of $x^2 + 3x - 4$ in

   (a) $\mathbb{Z}$

   Solution. This quadratic factors as

   $$x^2 + 3x - 4 = (x + 4)(x - 1)$$

   so the roots are $1, -4 \in \mathbb{Z}$. $\square$

   (b) $\mathbb{Z}_6$

   Solution. Similarly to part (a), this quadratic factors as

   $$x^2 + 3x - 4 = (x + 4)(x - 1)$$

   Thus $x + 4 = \bar{0}$ so $x = \bar{2}$ and $x - 1 = 0$ so $x = \bar{1}$ are solutions. $\square$

   (c) $\mathbb{Z}_4$

   Solution. Similarly to part (a), this quadratic factors as

   $$x^2 + 3x - 4 = (x + 4)(x - 1)$$

   Thus $x + 4 = \bar{0}$ so $x = \bar{0}$ and $x - 1 = 0$ so $x = \bar{1}$ are solutions. $\square$

5. Show that $M_n(R)$ is never a domain if $n \geq 2$.

   Proof. Consider the element $A = \begin{bmatrix} 0 & \cdots & r \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in M_n(R)$ where $0 \neq r \in R$. Then A^2 is a matrix of all 0's, but A itself is not a matrix of all 0's. Thus, $M_n(R)$ is never a domain if $n \geq 2$. $\square$

10. If $F = \{0, 1, a, b\}$ is a field, fill in the addition and multiplication tables for F.

    Solution. Since F is a field, it must contain the multiplicative inverses of a and b. Thus, $a^{-1} = b$ and vice versa. Similarly, it must contain the additive inverses of a and b, which are again each other. $\square$

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Section 3.3: Ideals and Factor Rings

1. (a) Decide whether $\mathbb{Z}$ is an ideal of $\mathbb{C}$. Support your answer.

   Solution. $\mathbb{Z}$ is not an ideal of $\mathbb{C}$. Consider $z = 1 + i \in \mathbb{C}$. Then if $a = 2 \in \mathbb{Z}$ we have $az = 2 + i \notin \mathbb{Z}$. $\square$

4. (a) If m is an integer, show that $mR = \{\, mr \mid r \in R \,\}$ and $A_m = \{\, r \in R \mid mr = 0 \,\}$ are ideals of R.

6. If A is an ideal of R, show that $M_2(A)$ is an ideal of $M_2(R)$.

Section 3.4: Homomorphisms

1. In each case determine whether the map θ is a ring homomorphism. Support your answer.
   (a) $\theta : \mathbb{Z}_3 \to \mathbb{Z}_{12}$, where $\theta(r) = 4r$.
   (b) $\theta : \mathbb{Z}_4 \to \mathbb{Z}_{12}$, where $\theta(r) = 3r$.
   (c) $\theta : R \times R \to R$, where $\theta(r, s) = r + s$.
   (d) $\theta : R \times R \to R$, where $\theta(r, s) = rs$.
   (e) $\theta : F(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, where $\theta(f) = f(1)$.

15. If $\sigma : R \to S$ is a ring isomorphism, show that the same is true of the inverse map $\sigma^{-1} : S \to R$.