Homework 7

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Chapter 7: Completeness

35. Prove that a normed vector space X is complete if and only if its closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is complete.

Proof. $(\implies)$: Since B is closed in X and X is complete, B is also complete.

$(\impliedby)$: Suppose (x_n) is a sequence in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then there exists $K > 0$ such that $\|x_n\| \leq K$ for all n. Now consider the sequence $y_n = x_n/K$. Then $\|y_n\| = \left\|\frac{x_n}{K}\right\| \leq 1$, so y_n is a sequence in B. Since B is a complete normed vector space, it follows that $\sum_{n=1}^{\infty} y_n$ converges to $y \in B$. We have

$$\begin{aligned} y &= \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{x_n}{K} \\ \implies \sum_{n=1}^{\infty} x_n &= Ky \end{aligned}$$

so $\sum_{n=1}^{\infty} x_n$ converges to $Ky \in X$, so X is complete, as desired. □

40. Extend the result in Example 7.15 as follows: Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable in (a, b), and satisfies $F(a) < 0, F(b) > 0$, and $0 < K_1 \leq F'(x) \leq K_2$. Show that there is a unique solution to the equation $F(x) = 0$. (Hint: Consider the equation $f(x) = x$, where $f(x) = x - \lambda F(x)$ for some suitably chosen λ.)

Proof. Let $f : [a, b] \to \mathbb{R}$ by $x \mapsto x - \lambda F(x)$. Then $f'(x) = 1 - \lambda F'(x)$, and

$$0 < K_1 \leq F'(x) \leq K_2 \implies 1 - \lambda K_2 \leq 1 - \lambda F'(x) \leq 1 - \lambda K_1 < 1$$

Thus, as long as $1 - \lambda K_2 > -1 \implies \lambda < 2/K_2$, we will have

$$|f'(x)| = |1 - \lambda F'(x)| \leq \alpha < 1$$

and since F is continuous on $[a, b]$ and differentiable on (a, b), it follows that f is as well. We have

$$\begin{aligned} f(a) &= a - \lambda F(a) > a \\ f(b) &= b - \lambda F(b) < b \end{aligned}$$

If we take $\lambda < 1/K_2$, then $f'(x) = 1-\lambda F'(x) > 0$, so f is monotone increasing, and thus $f([a, b]) \subset [a, b]$. Thus, letting $f : [a, b] \to [a, b]$, by the mean value theorem we have $|f(x) - f(y)| \leq \alpha\, |x - y|$, so f is a strict contraction. Since every strict contraction has a unique fixed point, there exists a unique $x_0 \in [a, b]$ such that

$$f(x_0) = x_0 - \lambda F(x_0) = x_0 \implies F(x_0) = 0$$

□

47. A function $f : (M, d) \to (n, \rho)$ is said to be uniformly continuous if f is continuous and if, given $\varepsilon > 0$, there is always a single $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ for any $x, y \in M$ with $d(x, y) < \delta$. That is, δ is allowed to depend on f and ε but not on x or y. Prove that any Lipschitz map is uniformly continuous.

Proof. If f is Lipschitz, then there exists $K > 0$ such that $\rho(f(x), f(y)) \le Kd(x, y)$ for all $x, y \in M$. Given $\varepsilon > 0$, take $\delta = \varepsilon/K$. Then for all $x, y \in M$ where $d(x, y) < \delta = \varepsilon/K$, we have

$$\rho(f(x), f(y)) \le Kd(x, y) < K \cdot \frac{\varepsilon}{K} = \varepsilon$$

Since all Lipschitz maps are continuous, it follows that f is uniformly continuous. □

Chapter 8: Compactness

2. Let $E = \left\{x \in \mathbb{Q} : 2 < x^2 < 3\right\}$, considered as a subset of $\mathbb{Q}$ (with its usual metric). Show that E is closed and bounded but not compact.

Proof. Consider the complement

$$\begin{aligned} E^c &= \left\{x \in \mathbb{Q} : 0 \le x^2 \le 2\right\} \cup \left\{x \in \mathbb{Q} : 3 \le x^2\right\} \\ &= \left\{x \in \mathbb{Q} : x^2 < 2\right\} \cup \left\{0 < x \in \mathbb{Q} : x^2 > 3\right\} \cup \left\{0 > x \in \mathbb{Q} : x^2 > 3\right\} \end{aligned}$$

Since $\mathbb{Q}$ is a subspace of $\mathbb{R}$, if $U \subset \mathbb{R}$ is open in $\mathbb{R}$, then $U \cap \mathbb{Q}$ is open in $\mathbb{Q}$. Then we have

$$\begin{aligned} \left\{x \in \mathbb{Q} : x^2 < 2\right\} &= \left(-\sqrt{2}, \sqrt{2}\right) \cap \mathbb{Q} \\ \left\{0 < x \in \mathbb{Q} : x^2 > 3\right\} &= \left(\sqrt{3}, \infty\right) \cap \mathbb{Q} \\ \left\{0 > x \in \mathbb{Q} : x^2 > 3\right\} &= \left(-\infty, -\sqrt{3}\right) \cap \mathbb{Q} \end{aligned}$$

and each of their corresponding sets is open in $\mathbb{R}$, so each set is open in $\mathbb{Q}$. Thus, since unions of open sets are open, E^c is open so E is closed. It is obviously bounded above and below by 2 and -2, respectively.

Consider the sequence $1, 1.7, 1.73, 1.732, \cdots$ of rationals converging to $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$. This is a sequence in E, but any subsequence also converges to $\sqrt{3}$ and thus fails to converge in E. Thus, E is not compact. □

8. Prove that the set $\{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $\mathbb{R}^n$ under the Euclidean norm.

Proof. Consider $B = \{x \in \mathbb{R}^n : \|x\|_1 \le 1\}$. Then B is a closed, bounded subset of $\mathbb{R}^n$, and therefore compact. Then $S = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} \subset B$ is closed in B because $S^c = \{x \in \mathbb{R}^n : \|x\|_1 < 1\}$ is open. Thus, since B is compact and S is closed in B, it follows that S is compact. □

10. Show that the Heine-Borel theorem (closed, bounded sets in $\mathbb{R}$ are compact) implies the Bolzano-Weierstrass theorem. Conclude that the Heine-Borel theorem is equivalent to the completeness of $\mathbb{R}$.

Proof. Consider a bounded, infinite set A, so $A \subset [a, b]$ for some $a < b$. Suppose A has no limit points in $[a, b]$, so for each $x \in [a, b]$, there exists $\varepsilon_x > 0$ such that $(B_{\varepsilon_x}(x) \setminus x) \cap A = \varnothing$. Then take the open cover $\{B_{\varepsilon_x}(x) : x \in [a, b]\}$. Since $[a, b]$ is compact by the Heine-Borel theorem, we can take a finite subcover, which necessarily covers A, but since each of the $B_{\varepsilon_x}(x)$ only contains a single point x, it follows that this subcover, and therefore A, is finite. Contradiction, so A must have a limit point in $[a, b]$, which is the Bolzano-Weierstrass theorem. □

37. A real-valued function f on a metric space M is called lower semi-continuous if, for each real α, the set $\{x \in M : f(x) > \alpha\}$ is open in M. prove that f is lower semi-continuous if and only if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x$ in M.

Proof. ($\implies$) : Suppose $x_n \to x$ in M but $f(x) > \liminf_{n\to\infty} f(x_n) = L$. Then there exists α with $f(x) > \alpha > L$. Then since $f^{-1}((\alpha,\infty)) = \{x \in M : f(x) > \alpha\} = S$ is open in M and $x \in S$, the sequence x_n must eventually be in S. Thus the sequence $f(x_n)$ must eventually be in (α, ∞), so $\liminf_{n\to\infty} f(x_n) \geq \alpha$. Contradiction, since $\liminf_{n\to\infty} f(x_n) = L < \alpha$. Thus $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

($\impliedby$) : Suppose $f(x) \leq \liminf_{n\to\infty} f(x_n)$ for $x_n \to x$ in M. Take $\alpha \in \mathbb{R}$, and let $T = f^{-1}((\alpha,\infty))$. Then for any $x \in S$, we have $f(x) > \alpha$, so

$$\liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} \inf \{f(x_k) : k \geq n\} \geq f(x) > \alpha$$

Thus, since the sequence $\inf \{f(x_k) : k \geq n\}$ is increasing, there exists N such that $f(x_n) > \alpha$ for all $n \geq N$, which means that x_n is eventually in S, so S is open in M. □

40. Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto. (Hint: If $B_\varepsilon(x) \cap f(M) = \varnothing$, consider the sequence $f^n(x)$.)

Proof. Since f is an isometry, it is continuous, so $f(M) \subset M$ is compact, and therefore closed. Suppose f is not onto. Then there exists $x \in M \setminus f(M)$, so $B_\varepsilon(x) \cap f(M) = \varnothing$ for some $\varepsilon > 0$. Now, consider the sequence $(f(x), f(f(x)), f(f(f(x))), \cdots) = (f^n(x))$ in $f(M)$. Such a sequence cannot have a Cauchy subsequence because for any $n > m$, we have

$$d(f^n(x), f^m(x)) = d(f^{n-m}(x), x) \geq \varepsilon$$

since $B_\varepsilon(x) \cap f(M) = \varnothing$. Thus, $f(M)$ is not totally bounded, and therefore not compact. Contradiction, so f must be onto. □