Homework 10

Aleck Zhao

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Chapter 10: Sequences of Functions

7. Let (f_n) and (g_n) be real-valued functions on a set X, and suppose that (f_n) and (g_n) converge uniformly on X. Show that $(f_n + g_n)$ converges uniformly on X. Give an example showing that $(f_n g_n)$ need not converge uniformly on X.

   Proof. Suppose $f_n \to f$ and $g_n \to g$ uniformly. Then we claim that $(f_n + g_n) \to f + g$ uniformly. Take $\varepsilon > 0$, so there exists N and M such that $|f_n(x) - f(x)| < \varepsilon/2$ and $|g_n(x) - g(x)| < \varepsilon/2$ for $n \geq N$ and $n \geq M$, respectively. Take $K = \max\{N, M\}$, so we have

   $$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

   for all $n \geq K$, so $(f_n + g_n)$ is uniformly convergent.

   Take $f_n(x) = 1/x$ and $g_n(x) = 1/n$ over $(0, 1)$. Then $f_n \to 1/x$ and $g_n \to 0$ uniformly. Suppose $f_n g_n \to fg = 0$ uniformly. Then let $\varepsilon > 0$ and there exists N such that

   $$\left|\frac{1}{nx} - 0\right| < \varepsilon$$

   for all $n \geq N$. However, this is clearly impossible since given any n, if $0 < x < 1/n$, then $\left|\frac{1}{nx}\right| > 1$ so $(f_n g_n)$ does not converge uniformly. □

12. Prove that a sequence of functions $f_n : X \to \mathbb{R}$, where X is any set, is uniformly convergent if and only if it is uniformly Cauchy. That is, prove that there exists some $f : X \to \mathbb{R}$ such that $f_n \rightrightarrows f$ on X if and only if, for each $\varepsilon > 0$, there exists an $N \geq 1$ such that $\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon$ whenever $m, n \geq N$. (Hint: Notice that if (f_n) is uniformly Cauchy, then it is also pointwise Cauchy. That is, if $\sup_{x \in X} |f_n(x) - f_m(x)| \to 0$ as $m, n \to \infty$, then $(f_n(x))$ is Cauchy in $\mathbb{R}$ for each $x \in X$.)

    Proof. ($\Longrightarrow$) : If there is an f such that $f_n \rightrightarrows f$, then for any $\varepsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \varepsilon/2$ and $|f(x) - f_m(x)| < \varepsilon/2$ for all $n, m \geq N$ and $x \in X$. Then

    $$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ \Longrightarrow \sup_{x \in X} |f_n(x) - f_m(x)| &< \varepsilon \end{aligned}$$

    so (f_n) is uniformly Cauchy.

    ($\Longleftarrow$) : If (f_n) is uniformly Cauchy, then it is also pointwise Cauchy, so $(f_n(x))$ is Cauchy in $\mathbb{R}$ for any x, and therefore convergent since $\mathbb{R}$ is complete. Thus, set $f(x) := \lim_{n\to\infty} f_n(x)$ to be this limit. Then $|f_n(x) - f(x)| \to 0$ as $n \to \infty$ so $f_n \rightrightarrows r$. □

18. Here is a partial converse to Theorem 10.4, called Dini's theorem. Let X be a compact metric space, and suppose that the sequence (f_n) in $C(X)$ increases pointwise to a continuous function $f \in C(X)$; that is, $f_n(x) \le f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is actually uniform. The same is true if (f_n) decreases pointwise to f. (Hint: First reduce to the case where (f_n) decreases pointwise to 0. Now, given $\varepsilon > 0$, consider the (open) sets $U_n = \{x \in X : f_n(x) < \varepsilon\}$.) Give an example showing that $f \in C(X)$ is necessary.

Proof. Since f_n converges pointwise to f, let $g_n(x) := f(x) - f_n(x)$. Since f_n is increasing pointwise, g_n is decreasing pointwise, and converges pointwise to 0. Now given $\varepsilon > 0$, consider the open set $U_n = \{x \in X : g_n(x) < \varepsilon\}$. We have $U_n \subset U_{n+1}$ since $g_n(x) \ge g_{n+1}(x)$, and $g_n(x) \to 0$, so $\bigcup_{n=1}^{\infty} U_n = X$. Since this is an open cover and X is compact, there exists a finite subcover, and since $U_n \subset U_{n+1}$, there exists some N such that $U_N = X$. Thus, $g_N(x) < \varepsilon$ for all x, and therefore $g_n(x) < \varepsilon$ for all x and $n \ge N$, so g_n is uniformly convergent to 0. Thus, f_n is uniformly convergent to f.

If $X = [0,1]$ and $f_n(x) = x^{1/n}$, then $f_n \to g$ where $g(0) = 0$ and $g(x) = 1$ for $x \in (0,1]$. Here, the convergence is not uniform, but g is also not continuous. □

19. Suppose that (f_n) is a sequence of functions in $C[0,1]$ and that $f_n \rightrightarrows f$ on $[0,1]$. True or false? $\int_0^{1-(1/n)} f_n \to \int_0^1 f$.

Solution. This is false. We have

$$\left| \int_0^1 f - \int_0^{1-(1/n)} f_n \right| = \left| \int_0^1 f - \int_0^1 f_n + \int_{1-(1/n)}^1 f_n \right| \le \left| \int_0^1 f - \int_0^1 f_n \right| + \left| \int_{1-(1/n)}^1 f_n \right|$$

The left hand absolute value tends to 0 because $f_n \rightrightarrows f$, but we can construct f_n such that the right hand absolute value does not tend to 0. □

21. Use Dini's theorem to conclude that the sequence $(1 + (x/n))^n$ converges uniformly to e^x on every compact interval in $\mathbb{R}$. How does this explain the findings in Example 10.1 (a)?

Proof. Fix some x. Consider the numbers $x_1 = 1, x_2 = x_3 = \cdots = x_{n+1} = 1 + \frac{x}{n}$. Since these are all non-negative, by the AM-GM inequality, we have

$$\begin{aligned}
\sqrt[n+1]{x_1 x_2 \cdots x_{n+1}} &\le \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1} \\
\implies \left(1 + \frac{x}{n}\right)^{\frac{n}{n+1}} &\le \frac{1 + n\left(1 + \frac{x}{n}\right)}{n+1} = \frac{n+1+x}{n+1} \\
\implies \left(1 + \frac{x}{n}\right)^n &\le \left(1 + \frac{x}{n+1}\right)^{n+1}
\end{aligned}$$

Thus, the sequence $(1 + (x/n))^n$ is increasing in n. It is well known that it converges to e^x, so by Dini's theorem, it is uniformly convergent to e^x. Since the sequence is uniformly convergent to e^x, the sequences of derivatives and integrals also converge to the derivatives and integrals of e^x. □

23. Show that $B(X)$ is an algebra of functions; that is, if $f, g \in B(X)$, then so is fg and $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$. Moreover, if $f_n \to f$ and $g_n \to g$ in $B(X)$, show that $f_n g_n \to fg$ in $B(X)$.

Proof. We have $|f(x)| \le M$ and $|g(x)| \le N$ for all $x \in X$ and some M, N. Then $|f(x)g(x)| \le MN$ for all x, so $fg \in B(X)$. We have $\sup_{x \in X} |f(x)| \ge f(x)$ and $\sup_{x \in X} |g(x)| \ge g(x)$ for all x, so

$$\sup_{x \in X} |f(x)| \cdot \sup_{x \in X} |g(x)| \ge f(x)g(x), \quad \forall x \in X \implies \|f\|_\infty \|g\|_\infty \ge \|fg\|_\infty$$

Now, f_n and g_n are bounded, so suppose $|f_n(x)| \leq M_f$ and $|g_n(x)| \leq M_g$ for all n and x and some M_f, M_g. Let $M = \max\{M_f, M_g\}$. Then since f_n and g_n are also uniformly convergent to f and g, respectively, for $\varepsilon > 0$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{2M}$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$. Next, $|f_n(x)| \leq M$ and $|g(x)| \leq M$, so

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)|\,|g_n(x) - g(x)| + |g(x)|\,|f_n(x) - f(x)| \\
&< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon
\end{aligned}$$

so $f_n g_n \to fg$. □

29. (a) For which values of x does $\sum_{n=1}^{\infty} ne^{-nx}$ converge? On which intervals is the convergence uniform?

Solution. For any interval $[a, \infty)$ with $a > 0$, we have $ne^{-nx} \leq ne^{-na}$ for all $x \in [a, b]$, so by the M-test, we have

$$\sum_{n=1}^{\infty} ne^{-nx} \leq \sum_{n=1}^{\infty} ne^{-na} = \frac{e^{-a}}{(1 - e^{-a})^2}$$

It is also uniformly convergent on $[a, \infty)$ since $\sup_{x \in [a,\infty)} ne^{-nx} = ne^{-na} \to 0$ as $n \to 0$. □

(b) Conclude that $\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\, dx = e/(e^2 - 1)$

Solution. Since it is uniformly convergent, we may switch the order of summation and integration:

$$\begin{aligned}
\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\, dx &= \sum_{n=1}^{\infty} \int_1^2 ne^{-nx}\, dx = \sum_{n=1}^{\infty} (-e^{-nx})\Big|_1^2 \\
&= \sum_{n=1}^{\infty} \left(e^{-n} - e^{-2n}\right) = \sum_{n=1}^{\infty} e^{-n} - \sum_{n=1}^{\infty} e^{-2n} \\
&= \frac{\frac{1}{e}}{1 - \frac{1}{e}} - \frac{\frac{1}{e^2}}{1 - \frac{1}{e^2}} = \frac{1}{e - 1} - \frac{1}{e^2 - 1} \\
&= \frac{e}{e^2 - 1}
\end{aligned}$$

□