Homework 5

Aleck Zhao

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1. *Solution.* We have

$$\begin{aligned}(1+s_2)^2 &= (1+s_1)(1+f_{1,2})\\ \implies f_{1,2} &= \frac{(1+6.9\%)^2}{1+6.3\%} - 1 = \boxed{7.5\%}\end{aligned}$$

□

2. *Solution.* Since the spot curve follows expectations dynamics, the curve for the next year satisfies

$$s'_{j-1} = f_{1,j} = \left[\frac{(1+s_j)^j}{1+s_1}\right]^{1/(j-1)} - 1$$

We compute the spot curve for the next year as

$$\begin{aligned}s'_1 = f_{1,2} &= \left[\frac{(1+s_2)^2}{1+s_1}\right]^{1/1} - 1 = 5.6\%\\ s'_2 = f_{1,3} &= \left[\frac{(1+s_3)^3}{1+s_1}\right]^{1/2} - 1 = 5.9\%\\ s'_3 = f_{1,4} &= \left[\frac{(1+s_4)^4}{1+s_1}\right]^{1/3} - 1 = 6.07\%\\ s'_4 = f_{1,5} &= \left[\frac{(1+s_5)^5}{1+s_1}\right]^{1/4} - 1 = 6.25\%\\ s'_5 = f_{1,6} &= \left[\frac{(1+s_6)^5}{1+s_1}\right]^{1/5} - 1 = 6.32\%\end{aligned}$$

□

3. *Solution.* Consider a portfolio with a long position in 4.5 bonds of the 7% coupon and a short position in 3.5 bonds of the 9% coupon. The coupon payments cancel out, and the net face value is 100, and the net price is $4.5 \cdot 93.20 - 3.5 \cdot 101.00 = 65.90$. This portfolio is equivalent to a 5-year ZCB, so the price of a 5-year ZCB is 65.90. □

4. (a) *Solution.* We have

$$\begin{aligned}e^{s(t_1)t_1} \cdot e^{f(t_1,t_2)(t_2-t_1)} &= e^{s(t_2)t_2}\\ \implies f(t_1,t_2) &= \boxed{\frac{s(t_2)t_2 - s(t_1)t_1}{t_2 - t_1}}\end{aligned}$$

□

(b) *Proof.* By definition, we have

$$s'(t) = \lim_{t_2 \to t} \frac{s(t_2) - s(t)}{t_2 - t}$$

By simple algebra, we have

$$\begin{aligned}
sjs(t) + s'(t)t &= s(t) + \lim_{t_2 \to t} \frac{s(t_2) - s(t)}{t_2 - t} \cdot t \\
&= \lim_{t_2 \to t} \left(s(t) + \frac{s(t_2)t - s(t)t}{t_2 - t} \right) = \lim_{t_2 \to t} \left(s(t_2) + \frac{s(t_2)t - s(t)t}{t_2 - t} \right) \\
&= \lim_{t_2 \to t} \left(\frac{s(t_2)t_2 - s(t_2)t}{t_2 - t} + \frac{s(t_2)t - s(t)t}{t_2 - t} \right) \\
&= \lim_{t_2 \to t} \frac{s(t_2)t_2 - s(t)t}{t_2 - t} \\
&= r(t)
\end{aligned}$$

□

(c) *Solution.* Rearranging, we have

$$\frac{1}{x(t)}\, dx = r(t)\, dt$$

and integrating both sides, we get

$$\begin{aligned}
&\int \frac{1}{x(t)}\, dx = \int r(t)\, dt \\
\implies &\ln(x(t)) = \int \left(s(t) + s'(t)t\right)\, dt = \int s(t)\, dt + \int s'(t)t\, dt
\end{aligned}$$

Now, let $u = t$ and $dv = s'(t)\, dt$, so $du = dt$ and $v = s(t)$. Integrating by parts, we have

$$\begin{aligned}
&\int s'(t)t\, dt = s(t)t - \int s(t)\, dt \\
\implies &\ln(x(t)) = \int s(t)\, dt + s(t)t - \int s(t)\, dt = s(t)t + C \\
\implies &x(t) = e^{s(t)t + C} = Ce^{s(t)t}
\end{aligned}$$

Letting $t = 0$, we have

$$x(0) = x_0 = Ce^0 \implies C = x_0$$

so finally the expression for $x(t)$ is

$$x(t) = \boxed{x_0 e^{s(t)t}}$$

□

5. *Solution.* The discount factors satisfy $d_{i,k} = d_{i,j}d_{j,k}$. Thus, we have

$$\begin{aligned}
d_{0,1} &= 0.950 \\
d_{0,2} &= d_{0,1}d_{1,2} = 0.950 \cdot 0.940 = 0.893 \\
d_{0,3} &= d_{0,2}d_{2,3} = 0.893 \cdot 0.932 = 0.832 \\
d_{0,4} &= d_{0,3}d_{3,4} = 0.832 \cdot 0.925 = 0.770 \\
d_{0,5} &= d_{0,4}d_{4,5} = 0.770 \cdot 0.919 = 0.707 \\
d_{0,6} &= d_{0,5}d_{5,6} = 0.707 \cdot 0.913 = 0.646
\end{aligned}$$

□

6. (a) *Solution.* The present value V of the principal payment stream is

$$\begin{aligned}
V &= \sum_{k=1}^{n} \frac{P(k)}{(1+r)^k} = \sum_{k=1}^{n} \frac{B - rM(k-1)}{(1+r)^k} \\
&= \sum_{k=1}^{n} \frac{B}{(1+r)^k} - r\sum_{k=1}^{n} \frac{(1+r)^{k-1}M - \frac{(1+r)^{k-1}-1}{r}\cdot B}{(1+r)^k} \\
&= B\sum_{k=1}^{n} \frac{1}{(1+r)^k} - r\sum_{k=1}^{n} \frac{M}{1+r} + \sum_{k=1}^{n} \frac{(1+r)^{k-1}-1}{(1+r)^k}\cdot B \\
&= B\sum_{k=1}^{n} \frac{1}{(1+r)^k} - \frac{nrM}{1+r} + \sum_{k=1}^{n} \frac{B}{1+r} - B\sum_{k=1}^{n} \frac{1}{(1+r)^k} \\
&= \boxed{\frac{n}{1+r}(B - rM)}
\end{aligned}$$

□

(b) *Solution.* Substituting the expression for B, we have

$$\begin{aligned}
V &= \frac{n}{1+r}(B - rM) = \frac{n}{1+r}\left(\frac{r(1+r)^n M}{(1+r)^n - 1} - rM\right) \\
&= \frac{nrM}{1+r}\left(\frac{(1+r)^n}{(1+r)^n - 1} - 1\right) \\
&= \boxed{\frac{nrM}{1+r}\cdot\frac{1}{(1+r)^n - 1}}
\end{aligned}$$

□

(c) *Solution.* The present value W of the interest payment stream is

$$\begin{aligned}
W &= \sum_{k=1}^{n} \frac{I(k)}{(1+r)^k} = \sum_{k=1}^{n} \frac{B - P(k)}{(1+r)^k} = \sum_{k=1}^{n} \frac{B}{(1+r)^k} - V \\
&= B\cdot\frac{1}{r}\cdot\frac{(1+r)^n - 1}{(1+r)^n} - \frac{nrM}{1+r}\cdot\frac{1}{(1+r)^n - 1} \\
&= \frac{r(1+r)^n M}{(1+r)^n - 1}\cdot\frac{1}{r}\cdot\frac{(1+r)^n - 1}{(1+r)^n} - \frac{nrM}{1+r}\cdot\frac{1}{(1+r)^n - 1} \\
&= \boxed{M - \frac{nrM}{1+r}\cdot\frac{1}{(1+r)^n - 1}}
\end{aligned}$$

□

(d) *Solution.* As $n \to \infty$, by l'Hopital's rule, we have

$$\begin{aligned}
\lim_{n\to\infty} V &= \lim_{n\to\infty} \frac{nrM}{1+r}\cdot\frac{1}{(1+r)^n - 1} \\
&= \frac{rM}{1+r}\cdot\lim_{n\to\infty}\frac{n}{(1+r)^n - 1} \\
&= \frac{rM}{1+r}\cdot\lim_{n\to\infty}\frac{1}{(1+r)^n \ln(1+r)} \\
&= 0
\end{aligned}$$

□

(e) *Solution.* As n grows, V decreases. Thus since duration is the weighted average with V in the denominator, the duration of the principal stream goes to infinity for arbitrarily large n. □

7. (a) *Solution.* The price, as a percentage of face value, is

$$P = \frac{0.04}{1+s_1} + \frac{1.04}{(1+s_2)^2} = \frac{0.04}{1.05} + \frac{1.04}{(1.06)^2} = \boxed{96.369\%}$$

□

(b) *Solution.* One year from now, the 1-year spot rate will be 6.5%, so the price will be

$$P = \frac{1.04}{1.065} = \boxed{97.653\%}$$

□

(c) *Solution.* After 1 year, we receive a coupon of 4%, so the return will be $\frac{97.653+4}{96.369} = \boxed{5.483\%.}$ □

(d) *Solution.* According to expectations dynamics, the 1-year spot rate one year from now should be

$$s_1' = \left[\frac{(1+s_2)^2}{1+s_1}\right]^{1/1} - 1 = 7\%$$

Using this spot rate, the price in one year would have been $\frac{1.04}{1.07} = 97.196\%$, and the return would be $\frac{97.196+4}{96.369} - 1 = \boxed{5.009\%.}$ □

8. (a) *Solution.* The spot rates are

$$\begin{aligned}
99.67 &= \frac{100.3125}{1+s_{0.5}/2} &&\Longrightarrow s_{0.5} = 1.289\% \\
100.30 &= \frac{0.875}{1+s_{0.5}/2} + \frac{100.875}{(1+s_1/2)^2} &&\Longrightarrow s_1 = 1.447\% \\
100.17 &= \frac{0.8125}{1+s_{0.5}/2} + \frac{0.8125}{(1+s_1/2)^2} + \frac{100.8125}{(1+s_{1.5}/2)^3} &&\Longrightarrow s_{1.5} = 1.511\% \\
99.36 &= \frac{0.625}{1+s_{0.5}/2} + \frac{0.625}{(1+s_1/2)^2} + \frac{0.625}{(1+s_{1.5}/2)^3} + \frac{100.625}{(1+s_2/2)^4} &&\Longrightarrow s_2 = 1.578\%
\end{aligned}$$

□

(b) *Solution.* The forward rates are

$$\begin{aligned}
f_{0.5,1} &= 2\left(\frac{(1+1.447\%/2)^2}{1+1.289/2\%} - 1\right) = 1.605\% \\
f_{1,1.5} &= 2\left(\frac{(1+1.511\%/2)^3}{(1+1.447\%/2)^2} - 1\right) = 1.639\% \\
f_{1.5,2} &= 2\left(\frac{(1+1.578\%/2)^4}{(1+1.511\%/2)^3} - 1\right) = 1.779\%
\end{aligned}$$

□

9. (a) *Solution.* The forward discount factor is given by

$$d_{10,t} = e^{-f_{10,t}\cdot(t-10)} = \boxed{e^{-0.05(t-10)}}$$

□

(b) *Solution.* The discount factor is given by

$$\begin{aligned} d_t = d_{0,t} = d_{0,10} d_{10,t} &= e^{-0.03\cdot 10} \cdot e^{-0.05(t-10)} \\ &= \boxed{e^{-0.05t+0.2}} \end{aligned}$$

□

(c) *Solution.* Using the forward rate, we have

$$\begin{aligned} f_{10,t} &= \frac{s_t \cdot t - s_{10} \cdot 10}{t-10} = \frac{s_t \cdot t - 0.03 \cdot 10}{t-10} = 0.05 \\ \implies s_t &= \boxed{0.05 - \frac{0.2}{t}} \end{aligned}$$

□

(d) *Solution.* The present value of this perpetuity is

$$\begin{aligned} PV &= \sum_{t=11}^{\infty} \$1M \cdot d_t = \sum_{t=11}^{\infty} \$1M \cdot e^{-0.05t+0.2} \\ &= \$1M \cdot e^{0.2} \sum_{t=11}^{\infty} (e^{-0.05})^t \\ &= \$1M \cdot e^{0.2} \cdot \frac{e^{-0.55}}{1-e^{-0.05}} \\ &= \boxed{\$14.449M} \end{aligned}$$

□

10. (a) *Solution.* Each coupon payment on this inverse floater is $\frac{10\%-2L_3}{4} = 2.5\% - \frac{L_3}{2}$.
Consider a portfolio with a long position in a 10-year coupon bond paying 2.5% each coupon, a short position in $\frac{1}{2}$ of a 10-year floating bond paying L_3 each coupon, and a long position in $\frac{1}{2}$ of a 10-year ZCB. Then each net coupon payment is exactly $2.5\% - \frac{L_3}{2}$ and the net face value is 100, so this portfolio is equivalent to the inverse floater.
The quarterly rate is 1%, so the price of the 10-year coupon bond is

$$P_1 = \sum_{i=1}^{40} \frac{2.5}{(1+1\%)^i} + \frac{100}{(1+1\%)^{40}} = \frac{2.5}{0.01}\left(1 - \frac{1}{1.01^{40}}\right) + \frac{100}{1.01^{40}} = 149.252$$

The price of the ZCB is

$$P_2 = \frac{100}{1.01^{40}} = 67.165$$

and the price of the floating bond is 100. Thus, the price of the portfolio is

$$P_1 + \frac{1}{2}P_2 - \frac{1}{2}\cdot 100 = \boxed{132.835}$$

which is the price of the inverse floater. □

(b) *Solution.* Let $P(\lambda)$ be the price as a function of interest rate. Since the floating rate bond is always 100, we have

$$P(\lambda) = P_1(\lambda) + \frac{1}{2}P_2(\lambda) - \frac{1}{2}\cdot 100$$

where

$$\begin{aligned}
P_1(\lambda) &= \sum_{i=1}^{40} \frac{2.5}{(1+\lambda)^i} + \frac{100}{(1+\lambda)^{40}} = \frac{2.5}{\lambda}\left(1 - \frac{1}{(1+\lambda)^{40}}\right) + \frac{100}{(1+\lambda)^{40}} \\
P_2(\lambda) &= \frac{100}{(1+\lambda)^{40}} \\
\implies P(\lambda) &= \frac{2.5}{\lambda}\left(1 - \frac{1}{(1+\lambda)^{40}}\right) + \frac{100}{(1+\lambda)^{40}} + \frac{50}{(1+\lambda)^{40}} - 50 \\
&= 2.5\lambda^{-1} - 2.5\lambda^{-1}(1+\lambda)^{-40} + 150(1+\lambda)^{-40} - 50
\end{aligned}$$

Now, the modified duration is given by

$$\begin{aligned}
D'_M &= -\frac{1}{P} \cdot \frac{\partial P}{\partial \lambda} \\
&= -\frac{1}{P}\left(-2.5\lambda^{-2} + 2.5\lambda^{-2}(1+\lambda)^{-40} + 100\lambda^{-1}(1+\lambda)^{-41} - 6000(1+\lambda)^{-41}\right)
\end{aligned}$$

and substituting $\lambda = 1\%$, we get

$$D'_M = 41.771$$

which is in quarters, so the modified duration in terms of years is $\boxed{10.443.}$ □

(c) *Solution.* The duration is longer than the maturity, which is not surprising. This bond has added sensitivity to interest rates due to the LIBOR component of its coupon. □