Homework 9

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Section 3.1: Examples and Basic Properties

1. In each case explain why R is not a ring.

   (a) $R = \{0, 1, 2, 3, \cdots\}$, operations of $\mathbb{Z}$.

   Answer. R does not contain the additive inverses.

   (b) $R = 2\mathbb{Z}$.

   Answer. R does not contain a multiplicative identity.

   (c) $R =$ the set of all mappings $f : \mathbb{R} \to \mathbb{R}$; addition is point-wise but using composition as the multiplication.

   Answer. If $f, g, h \in R$, the condition $f(g+h) = fg+fh$ does not hold. For example, if $f(x) = \sqrt{x}$ and $g(x) = h(x) = x$, we have $f(g+h) = \sqrt{2x}$ but $fg + gh = 2\sqrt{x}$, and the two are not equal.

3. (c) Show that $S = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & a \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ is a subring of $R = M_3(\mathbb{R})$.

   Proof. We have

$$0_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$$

   and

$$1_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S$$

   Now, let

$$S = \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & a \end{bmatrix}, \quad T = \begin{bmatrix} w & 0 & x \\ 0 & y & z \\ 0 & 0 & w \end{bmatrix}$$

   so then

$$S - T = \begin{bmatrix} a-w & 0 & b-x \\ 0 & c-y & d-z \\ 0 & 0 & a-w \end{bmatrix} \in S$$

$$ST = \begin{bmatrix} aw & 0 & ax+bw \\ 0 & cy & cz+dw \\ 0 & 0 & aw \end{bmatrix} \in S$$

   Thus, S is a subring of R, as desired.

   □

Section 3.2: Integral Domains and Fields

1. Find all the roots of $x^2 + 3x - 4$ in

   (a) $\mathbb{Z}$

   Solution. This quadratic factors as

   $$x^2 + 3x - 4 = (x + 4)(x - 1)$$

   so the roots are $1, -4 \in \mathbb{Z}$. □

   (b) $\mathbb{Z}_6$

   Solution. We have $x + 4 = \bar{0}$ so $x = \bar{2}$ and $x - 1 = \bar{0}$ so $x = \bar{1}$ are solutions. □

   (c) $\mathbb{Z}_4$

   Solution. We have $x + 4 = \bar{0}$ so $x = \bar{0}$ and $x - 1 = \bar{0}$ so $x = \bar{1}$ are solutions. □

5. Show that $M_n(R)$ is never a domain if $n \geq 2$.

   Proof. Consider the element $A = \begin{bmatrix} 0 & \cdots & r \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in M_n(R)$ where $0 \neq r \in R$. Then A^2 is a matrix of all 0's, but A itself is not a matrix of all 0's. Thus, $M_n(R)$ is never a domain if $n \geq 2$. □

10. If $F = \{0, 1, a, b\}$ is a field, fill in the addition and multiplication tables for F.

   Solution. We have $(1 + 1) \cdot (1 + 1) = 1 + 1 + 1 + 1 = 0$ since $(F, +)$ has a group structure, and since we are in a field, we must have $1 + 1 = 0$. Thus, $0 = a \cdot (1 + 1) = a + a$ and $b + b = 0$ as well. Now consider $a + 1$. This is not equal to a or 1, and cannot be 0 because then $a = 1$, but a is distinct from 1. Thus $a + 1 = b$, and similarly $b + 1 = a$. Then $a + b = a + (a + 1) = (a + a) + 1 = 1 = b + a$. Thus, the addition table is summarized as

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

   For multiplication, we obviously can't have $a \cdot a = 0$ because then $a = 0$, but a is distinct from 0, and we can't have $a \cdot a = a$ because otherwise $a(a-1) = 0$, so $a = 0$ or $a-1 = 0$ since we're in a field, but we know that a is distinct from 0 and 1. If $a \cdot a = 1$, then $(a-1) \cdot (a+1) = (a+1) \cdot (a+1) = b \cdot b = 0$, but we assumed that b was distinct from 0. Thus, $a \cdot a = b$, and similarly $b \cdot b = a$. Then $a \cdot b = a \cdot (a+1) = a \cdot a + a = b + a = 1$. Similarly, $b \cdot a = 1$. Thus, the multiplication table is summarized as

$\times$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

□

Section 3.3: Ideals and Factor Rings

1. (a) Decide whether $\mathbb{Z}$ is an ideal of $\mathbb{C}$. Support your answer.

Solution. $\mathbb{Z}$ is not an ideal of $\mathbb{C}$. Consider $z = 1 + i \in \mathbb{C}$. Then if $a = 2 \in \mathbb{Z}$ we have $az = 2 + i \notin \mathbb{Z}$. □

4. (a) If m is an integer, show that $mR = \{\, mr \mid r \in R \,\}$ and $A_m = \{\, r \in R \mid mr = 0 \,\}$ are ideals of R.

Proof. We first show that mR is a subgroup. It clearly contains 0_R because $m \cdot 0_R = 0_R$. Now, for two elements $mp, mq \in mR$, we have $mp + mq = m(p + q)$ by the distributive law, and since $p, q \in R$, it follows that $p + q \in R$, so $m(p + q) \in mR$. Next, if $mp \in mR$, its additive inverse $-mp = m(-p)$ is also in mR. Thus mR is an additive subgroup of R.

Now, consider some element $a \in R$, so then $a(mR) = \{\, amr \mid r \in R \,\}$. Consider $amr_0 \in a(mR)$. Since m is an integer, we have $amr_0 = m(ar_0)$ and $ar_0 \in R$, thus $amr_0 \in mR$ so $a(mR) \subset mR$. Then consider some element $b \in R$, so then $(mR)b = \{\, rmb \mid r \in R \,\}$. Consider the element $mr_1b \in (mR)b$. Clearly $r_1b \in R$, so it follows that $m(r_1b) \in mR$, and thus $(mR)b \subset mR$. Thus, mR is an ideal of R, as desired.

Next, we first show that A_m is a subgroup. Clearly $m \cdot 0_R = 0_R$ so $0_R \in A_m$. Then if two elements $r_0, r_1 \in A_m$ such that $mr_0 = mr_1 = 0_R$, then $mr_0 + mr_1 = m(r_0 + r_1) = 0_R$, so $r_0 + r_1 \in A_m$. Finally, if $mr_0 = 0_R$, then $-mr_0 = m(-r_0) = 0_R$, so $-r_0 \in A_m$ as well. Thus, A_m is a subgroup of R.

Now, consider some element $s \in R$, so then $sA_m = \{\, sr \mid r \in R, mr = 0 \,\}$. Clearly if $mr = 0_R$, then $s(mr) = m(sr) = 0_R$, so it follows that $sr \in A_m$ and thus $sA_m \subset A_m$. Now, consider some element $t \in R$, so then $A_mt = \{\, rt \mid r \in R, mr = 0 \,\}$. Similarly, if $mr = 0_R$, then $(mr)t = 0_R = m(rt)$, so $rt \in A_m$, and thus $A_mt \in A_m$. Thus, A_m is an ideal of R, as desired. □

6. If A is an ideal of R, show that $M_2(A)$ is an ideal of $M_2(R)$.

Proof. Let $a, b, c, d \in A$ and $p, q, r, s \in R$. Then we have

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(A), \qquad S = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M_2(R)$$

Then

$$SB = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{bmatrix}$$

Since A is an ideal of R, it follows that every term in this matrix is also an element of A, so then since A is also a subgroup of R, the sum in each entry is also in A. Thus, $SB \in M_2(A)$ so $SM_2(A) \subset M_2(A)$ for any $S \in M_2(R)$. Similarly,

$$BS = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

so $M_2(A)S \subset M_2(A)$ for any $S \in M_2(R)$, so $M_2(A)$ is an ideal of $M_2(R)$, as desired. □

Section 3.4: Homomorphisms

1. In each case determine whether the map θ is a ring homomorphism. Support your answer.

   (a) $\theta : \mathbb{Z}_3 \to \mathbb{Z}_{12}$, where $\theta(r) = 4r$.

   Answer. We have $\theta(\bar{1}_3) = \bar{4}_{12} \neq \bar{1}_{12}$, so θ does not preserve the unity, so it is not a homomorphism

(b) $\theta : \mathbb{Z}_4 \to \mathbb{Z}_{12}$, where $\theta(r) = 3r$.

Answer. Similarly to part (a), $\theta(\bar{1}_4) = \bar{3}_{12} \neq \bar{1}_{12}$, so θ does not preserve the unity, so it is not a homomorphism.

(c) $\theta : R \times R \to R$, where $\theta(r, s) = r + s$.

Answer. We have

$$\theta\left[(a, b) \cdot (r, s)\right] = \theta(ar, bs) = ar + bs \neq \theta(a, b) \cdot \theta(r, s) = (a + b)(r + s)$$

so θ is not a homomorphism.

(d) $\theta : R \times R \to R$, where $\theta(r, s) = rs$.

Answer. We have

$$\theta\left[(a, b) + (r, s)\right] = \theta(a + b, r + s) = (a + b)(r + s) \neq \theta(a, b) + \theta(r, s) = ar + bs$$

so θ is not a homomorphism.

(e) $\theta : F(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, where $\theta(f) = f(1)$.

Solution. Here, f_0 where $f_0(x) \equiv 1$ is the unity in $F(\mathbb{R}, \mathbb{R})$ and $1 \in \mathbb{R}$ is the unity in $\mathbb{R}$. We have $\theta(f_0) = f_0(1) = 1$, so θ preserves the unity. For $f, g \in F(\mathbb{R}, \mathbb{R})$, we have

$$\theta(f + g) = (f + g)(1) = f(1) + g(1) = \theta(f) + \theta(g)$$

and

$$\theta(f \cdot g) = (f \cdot g)(1) = f(1) \cdot g(1) = \theta(f) \cdot \theta(g)$$

so θ preserves addition and multiplication. Thus, θ is a homomorphism. □

15. If $\sigma : R \to S$ is a ring isomorphism, show that the same is true of the inverse map $\sigma^{-1} : S \to R$.

Proof. Since σ is a bijective map, its inverse is also obviously bijective, since there is a bijection between each element $r \in R$ and $s \in S$.

Let $p, q \in S$. Since σ is bijective map it is surjective, so we may find $a, b \in R$ such that $\sigma(a) = p$ and $\sigma(b) = q$. Then we have

$$\sigma^{-1}(p + q) = \sigma^{-1}(\sigma(a) + \sigma(b))$$

and since σ is a ring homomorphism, we have $\sigma(a) + \sigma(b) = \sigma(a + b)$, so the above becomes

$$\sigma^{-1}(p + q) = \sigma^{-1}(\sigma(a + b)) = a + b = \sigma^{-1}(p) + \sigma^{-1}(q)$$

so σ^{-1} preserves addition.

Similarly, we have

$$\sigma^{-1}(p \cdot q) = \sigma^{-1}(\sigma(a) \cdot \sigma(b)) = \sigma^{-1}(\sigma(a \cdot b)) = a \cdot b = \sigma^{-1}(p) \cdot \sigma^{-1}(q)$$

so σ^{-1} preserves multiplication.

Finally, since σ is a ring homomorphism, we have $\sigma(1_R) = 1_S$, so $\sigma^{-1}(1_S) = \sigma^{-1}(\sigma(1_R)) = 1_R$, so σ^{-1} also preserves the unity. Thus, σ^{-1} is a ring isomorphism, as desired. □