Homework 1

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Chapter 1: Calculus Review

3. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $s = \sup A$ if and only if

   (i) s is an upper bound for A

   (ii) for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$.

   State and prove the corresponding result for the infimum of a nonempty subset of $\mathbb{R}$ that is bounded below.

7. If $a < b$, then there is also an irrational $x \in \mathbb{R} \setminus \mathbb{Q}$ with $a < x < b$. (Hint: Find an irrational of the form $p\sqrt{2}/q$.)

15. Show that a Cauchy sequence with a convergent subsequence actually converges.

17. Given real numbers a and b, establish the following formulas:

   - $|a + b| \leq |a| + |b|$
   - $||a| - |b|| \leq |a - b|$
   - $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$
   - $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

37. If (E_n) is a sequence of subsets of a fixed set S, we define

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$$
$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right)$$

   Show that

   - $\liminf\limits_{n\to\infty} E_n \subset \limsup\limits_{n\to\infty} E_n$
   - $\liminf\limits_{n\to\infty}(E_n^c) = \left(\limsup\limits_{n\to\infty} E_n\right)^c$

45. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(x) = 0$ whenever x is rational. Show that $f(x) = 0$ for every x in $[a, b]$.

46. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

   (a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some open interval $(-a, a)$.

   (b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with ≥ 0 replaced by > 0? Explain.