

Homework 4 Solutions

ALECK ZHAO

March 11, 2018

1. MATLAB exercise. When trying to use Cramer's rule on the selected exercise, the answer is unreasonably large because the matrix is singular.
2. MATLAB exercise.
3. (a) Let $\mathbf{u} = (-4, 4, -1)$, $\mathbf{v} = (-1, -2, 2)$. Compute $\mathbf{u} \times \mathbf{v}$.

Solution. We have

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 4 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \mathbf{i} \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -4 & -1 \\ -1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} \\ &= \mathbf{i}(4 \cdot 2 - (-1) \cdot (-2)) - \mathbf{j}(-4 \cdot 2 - (-1) \cdot (-1)) + \mathbf{k}(-4 \cdot (-2) - 4 \cdot (-1)) \\ &= 6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k} = (6, 9, 12)\end{aligned}$$

□

- (b) Prove for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Let $\mathbf{u} = (a, b, c)$ and let $\mathbf{v} = (x, y, z)$. Then

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{bmatrix} = \mathbf{i} \begin{vmatrix} b & c \\ y & z \end{vmatrix} - \mathbf{j} \begin{vmatrix} a & c \\ x & z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a & b \\ x & y \end{vmatrix} \\ &= \mathbf{i}(bz - cy) + \mathbf{j}(cx - az) + \mathbf{k}(ay - bx) \\ \implies \|\mathbf{u} \times \mathbf{v}\|^2 &= (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2 \\ &= b^2z^2 - 2bcyz + c^2y^2 + c^2x^2 - 2acxz + a^2z^2 + a^2y^2 - 2abxy + b^2x^2\end{aligned}\quad (1)$$

We also have

$$\begin{aligned}\|\mathbf{u}\|^2 &= a^2 + b^2 + c^2 \\ \|\mathbf{v}\|^2 &= x^2 + y^2 + z^2 \\ \mathbf{u} \cdot \mathbf{v} &= ax + by + cz \\ \implies \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 &= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ &= a^2x^2 + a^2y^2 + a^2z^2 + b^2x^2 + b^2y^2 + b^2z^2 + c^2x^2 + c^2y^2 + c^2z^2 \\ &\quad - (a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz) \\ &= a^2y^2 + a^2z^2 + b^2x^2 + b^2z^2 + c^2x^2 + c^2y^2 - 2abxy - 2acxz - 2bcyz\end{aligned}$$

and we can see that this expression is equivalent to the one in equation (1), so the two quantities are equal, as desired. □

4. (a) Express $\mathbf{u} = 1\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ as a sum of vectors parallel and perpendicular to $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

Solution. We have

$$\mathbf{u}_{\parallel} = \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{1 \cdot -2 + 4 \cdot 5 + 3 \cdot 4}{(-2)^2 + 5^2 + 4^2} \mathbf{v} = \frac{30}{45}(-2, 5, 4) = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right)$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u} = (1, 4, 3) - \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) = \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

so now

$$\mathbf{u} = (1, 4, 3) = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) + \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

□

- (b) Show that the vectors \mathbf{u}_{\parallel} and \mathbf{u}_{\perp} you obtained in part (a) are orthogonal.

Proof. The dot product between orthogonal vector is 0, and here

$$\mathbf{u}_{\parallel} \cdot \mathbf{u}_{\perp} = \left(-\frac{4}{3}, \frac{10}{3}, \frac{8}{3}\right) \cdot \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right) = -\frac{4}{3} \cdot \frac{7}{3} + \frac{10}{3} \cdot \frac{2}{3} + \frac{8}{3} \cdot \frac{1}{3} = \frac{-28}{3} + \frac{20}{3} + \frac{8}{3} = 0$$

so these two vectors are orthogonal, as desired.

□

5. MATLAB exercise. For the vectors $\mathbf{u} = (2, 8, -3, -1, 2)$ and $\mathbf{v} = (-5, 3, 1, 1, 6)$, we have

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{2^2 + 8^2 + (-3)^2 + (-1)^2 + 2^2} = \sqrt{82} \approx 9.055 \\ \|\mathbf{v}\| &= \sqrt{(-5)^2 + 3^2 + 1^2 + 1^2 + 6^2} = 6\sqrt{2} \approx 8.485 \\ \mathbf{u} \cdot \mathbf{v} &= 2 \cdot -5 + 8 \cdot 3 + -3 \cdot 1 + -1 \cdot 1 + 2 \cdot 6 = 22 \\ \|\mathbf{u} - \mathbf{v}\| &= \|(7, 5, -4, -2, -4)\| = \sqrt{7^2 + 5^2 + (-4)^2 + (-2)^2 + (-4)^2} = \sqrt{110} \approx 10.488 \\ \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{22}{\sqrt{82} \cdot 6\sqrt{2}} \approx 0.286\end{aligned}$$

6. (E&P 4.1.30) V is the set of all (x, y, z) such that $x + y + z = 0$.

Proof. Let $\mathbf{u}, \mathbf{v} \in V$, where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, and let $k \in \mathbb{R}$. Then since \mathbf{u} and \mathbf{v} are in V , they have the property that $u_1 + u_2 + u_3 = 0$ and $v_1 + v_2 + v_3 = 0$. Now,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

where

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0$$

so V is closed under addition. Then we have

$$k\mathbf{u} = (ku_1, ku_2, ku_3)$$

where

$$ku_1 + ku_2 + ku_3 = k(u_1 + u_2 + u_3) = k \cdot 0 = 0$$

so V is closed under scalar multiplication.

□

7. (E&P 4.1.36) V is the set of all (x, y, z) such that $xyz = 1$.

Solution. This is not a subspace. Consider the vector $\mathbf{u} = (1, 1, 1) \in V$ since $1 \cdot 1 \cdot 1 = 1$, and let $k = 2$. Then $k\mathbf{u} = (2, 2, 2)$, but $2 \cdot 2 \cdot 2 = 8 \neq 1$, so V is not closed under scalar multiplication. \square

8. (E&P 4.2.8) W is the set of all vectors in \mathbb{R}^2 such that $(x_1)^2 + (x_2)^2 = 0$.

Proof. Let $\mathbf{u}, \mathbf{v} \in V$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, and let $k \in \mathbb{R}$. Then it follows that $u_1^2 + u_2^2 = 0 \implies u_1 = u_2 = 0$, and similarly $v_1 = v_2 = 0$. Thus,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (0 + 0, 0 + 0) = (0, 0) \in V$$

so V is closed under addition, and

$$k\mathbf{u} = k(u_1, u_2) = k(0, 0) = (0, 0) \in V$$

so V is closed under scalar multiplication. In a sense, V is vacuously a subspace because it contains only the 0 vector. \square