Homework 3

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1. Consider the linear program (LP) min $c^T x$ such that $Ax = b, x \geq 0$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad c = \begin{bmatrix} 3 & -4 & 5 \end{bmatrix}$$

a) Find all basic feasible solutions. (There are three possibilities, two of which are basic feasible solutions and one isn't. Be clear why the third possibility fails to be a BFS.)

Solution. Suppose $x_3 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \implies B^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$
$$x_B = B^{-1}b = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$
$$x = \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \end{bmatrix}$$

is a BFS.

Now suppose $x_2 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \implies B^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$
$$x_B = B^{-1}b = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is a BFS.

Now suppose $x_1 = 0$, then

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$x_B = B^{-1}b = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$$

and this is not a BFS because $x_B \not\geq 0$.

$\square$

b) Evaluate r_N for each of the two basic feasible solutions.

c) Identify the optimal solution to the (LP) and explain why it is the optimal solution.

2. Consider the linear program (LP) min $c^T x$ such that $AX = b, x \geq 0$ where

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

   a) Compute the BFS that comes from using the second, third, and fourth columns of A as the basis, and compute the associated vector r_N.

   Solution. If we use the second, third, and fourth columns as a basis, we have

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{4} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   where the inverse B^{-1} was calculated with MATLAB. $\square$

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   b) By examining the vector r_N from part a, explain how changing the different nonbasic variables will affect the objective function.

   c) Now compute the BFS that comes from using the first three columns of A as the basis, and compute the associated vector r_N.

   Solution. Now, we have

$$B = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix} \implies B^{-1} = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix}$$

   where the inverse B^{-1} was calculated with MATLAB. $\square$

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   d) Find the optimal solution for (LP).

3. Suppose you are handed a linear program which is given to you in the form min $c^T x$ such that $Ax \leq b, x \geq 0$, and suppose that the vector b happens to be non-negative. This linear program is currently not in standard form; explain how to then immediately identify a BFS once it is converted to standard form. Then explain what r_N will be for this BFS.

   Solution. We add a dummy variable z, so that

$$[A \,|\, I] \begin{bmatrix} x \\ z \end{bmatrix} = b$$

   where I is the identity matrix. Then if we simply let I be the basis, we have $Iz = b$ so the matrix $\begin{bmatrix} \vec{0} \\ z \end{bmatrix}$ will be a BFS. $\square$

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