Homework 1

Aleck Zhao

February 8, 2017

1. Provide a definition for each of the following terms.

   (a) convex set

   Definition. A convex set A is a subset of $\mathbb{R}^n$ such that for any two points $p_1, p_2 \in A$, the line joining the two points is also contained in A.

   (b) convex function

   Definition. A mapping $f : \mathbb{R} \to \mathbb{R}$ is a convex function if for any two values $a, b \in \mathbb{R}$, the line joining the points $(a, f(a))$ and $(b, f(b))$ lies on or above the graph of f.

   (c) linear combination

   Definition. Given $a_1, \cdots, a_n \in \mathbb{R}$, a linear combination of the a_i is given by

   $$a_1c_1 + a_2c_2 + \cdots + a_nc_n$$

   where $c_i \in \mathbb{R}, \forall i$.

   (d) convex combination

   Definition. Given $a_1, \cdots, a_n \in \mathbb{R}$, a convex combination of the a_i is a linear combination

   $$a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$$

   where $\lambda_i \in [0, 1], \forall i$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

2. Based on your definitions in the previous problem,

   (a) which of the following six sets are convex and which at not? Justify.

   Solution. Set 1 is not convex, consider a segment joining the top-most point and the top-left corner; this segment is not contained in the set.

   Set 2 is not convex, consider a segment joining the top-most point and the left-most point; this segment is not contained in the set.

   Set 3 is convex. For any two points in the set, the line joining them will always be in the set.

   Set 4 is not convex, consider a point above the hole and a point beneath the hole within the set; the segment joining these two points is not contained in the set.

   Set 5 is convex. For any two points on the line, the segment between them will still be a subset of the line.

   Set 6 is convex. For any two points in the set, the line joining them will always be in the set.

   □

(b) which of the following functions are convex and which are not? Justify.

Solution. Function 1 is convex. If the two points are on different branches, the segment joining them will lie above the function. If they are on the same branch, the segment joining them will be a subset of the graph.

Function 2 is not convex, consider a segment joining the left and right endpoints; this segment intersects the graph so it does not lie on or above it.

Function 3 is not convex, consider a segment joining the left and right endpoints; this segment lies entirely below the graph. □

3. How would you define the

(a) binary representation for a natural number n?

Definition. For $n \in \mathbb{N}$, the binary representation of n is

$$n = d_k 2^k + d_{k-1} 2^{k-1} + \cdots + d_1(2) + d_0$$

where $d_i \in \{0, 1\}, 0 \leq i \leq k$ and $d_k \neq 0$.

(b) ternary representation for a natural number n?

Definition. For $n \in \mathbb{N}$, the ternary representation of n is

$$n = d_k 3^k + d_{k-1} 3^{k-1} + \cdots + d_1(3) + d_0$$

where $d_i \in \{0, 1, 2\}, 0 \leq i \leq k$ and $d_k \neq 0$.

(c) decimal representation for a rational number q such that $0 < q < 1$?

Definition. For $q \in \mathbb{Q}$ where $0 < q < 1$, the decimal representation of q is

$$\sum_{i=1}^{\infty} d_i 10^{-i}$$

where $d_i \in \{0, 1, \cdots, 9\}$. Note that this is an infinite sum because some rational numbers do not terminate. For the ones that do, $d_k = 0$ for $k > N$ sufficiently large.

4. Let w be a positive integer, Then the sum of any w consecutive integers is always divisible by w.

(a) How would you define two consecutive integers?

Answer. Two integers a and b are consecutive if $|a - b| = 1$.

(b) Prove that the sum of any two consecutive integers is always odd.

Proof. Let a, b be consecutive integers. WLOG, $a < b$, so $b - a = 1 \implies b = a + 1$. Then

$$a + b = a + (a + 1) = 2a + 1 \equiv 1 \pmod 2$$

so the sum of any two consecutive integers is odd, as desired. □

(c) Prove that the sum of any three consecutive integers is always divisible by 3.

Proof. Let a, b, c be consecutive integers. WLOG, $a < b < c$, so $b = a + 1$ and $c = b + 1 = a + 2$. Then

$$a + b + c = a + (a + 1) + (a + 2) = 3a + 3 \equiv 0 \pmod 3$$

so the sum of any three consecutive integers is divisible by 3, as desired. □

(d) Prove that the sum of any five consecutive integers is always divisible by 5.

Proof. Let a, b, c, d, e be consecutive integers. WLOG, $a < b < c < d < e$, so similarly to parts (b) and (c), the sum is given by

$$a + b + c + d + e = a + (a + 1) + (a + 2) + (a + 3) + (a + 4) = 5a + 10 \equiv 0 \pmod 5$$

so the sum of any five consecutive integers is divisible by 5, as desired. □

(e) Disprove the statement given at the start of this exercise.

Solution. If $w = 2$, the statement is false. The sum of two consecutive integers is odd, which is never divisible by w which is 2. □

(f) Conjecture: for what values of w do you think the statement is true?

Answer. The statement is true for odd values of w.

5. Prove for all $x \in \mathbb{R}$ and for all $m \in \mathbb{Z}$,

$$\lfloor x + m \rfloor = \lfloor x \rfloor + m$$

Proof. If x is an integer, then

$$\lfloor x + m \rfloor = x + m = \lfloor x \rfloor + m$$

Otherwise, $n < x < n + 1$ for some integer n, and $n + m < x + m < n + m + 1$. Then

$$\lfloor x + m \rfloor = n + m = \lfloor x \rfloor + m$$

as desired. □

6. Let $x \in \mathbb{R}$. Prove that if $\lfloor x \rfloor = \lceil x \rceil$ then x is an integer.

Proof. We prove the contrapositive. If x is not an integer, it is between two integers $m < x < n + 1$. Then

$$\lfloor x \rfloor = m \neq m + 1 = \lceil x \rceil$$

Thus, it must be that x is an integer, as desired. □

7. Prove/Disprove: For all primes $p, 2p + 1$ is also prime.

Proof. This is false. A counterexample is $p = 7$, which is prime, but $2 \cdot 7 + 1 = 15$ is not prime. □

8. Prove using direct proof by cases: Every integer that is a perfect cube is either a multiple of 9, or 1 more than a multiple of 9, or 1 less than a multiple of 9.

Proof. Any integer can be represented as one of $3k - 1, 3k$, or $3k + 1$ for $k \in \mathbb{Z}$. It follows that any perfect cube can be represented as one of $(3k - 1)^3, (3k)^3$, or $(3k + 1)^3$.

Case 1:

$$(3k - 1)^3 = 27k^3 - 27k^2 + 9k - 1 \equiv -1 \pmod 9$$

Case 2:

$$(3k)^3 = 27k^3 \equiv 0 \pmod 9$$

Case 3:

$$(3k + 1)^3 = 27k^3 + 27k^2 + 9k + 1 \equiv 1 \pmod 9$$

Thus, the statement is proven. □