Homework 4 Solutions

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1. MATLAB exercise. When trying to use Cramer's rule on the selected exercise, the answer is unreasonably large because the matrix is singular.
2. MATLAB exercise.
3. (a) Let $\mathbf{u} = (-4, 4, -1), \mathbf{v} = (-1, -2, 2)$. Compute $\mathbf{u} \times \mathbf{v}$.

Solution. We have

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 4 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \mathbf{i} \left| \begin{bmatrix} 4 & -1 \\ -2 & 2 \end{bmatrix} \right| - \mathbf{j} \left| \begin{bmatrix} -4 & -1 \\ -1 & 2 \end{bmatrix} \right| + \mathbf{k} \left| \begin{bmatrix} -4 & 4 \\ -1 & -2 \end{bmatrix} \right| \\ &= \mathbf{i}(4 \cdot 2 - (-1) \cdot (-2)) - vj(-4 \cdot 2 - (-1) \cdot (-1)) + \mathbf{k}(-4 \cdot (-2) - 4 \cdot (-1)) \\ &= 6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k} = (6, 9, 12)\end{aligned}$$

□

(b) Prove for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Let $\mathbf{u} = (a, b, c)$ and let $\mathbf{v} = (x, y, z)$. Then

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{bmatrix} = \mathbf{i} \left| \begin{bmatrix} b & c \\ y & z \end{bmatrix} \right| - \mathbf{j} \left| \begin{bmatrix} a & c \\ x & z \end{bmatrix} \right| + \mathbf{k} \left| \begin{bmatrix} a & b \\ x & y \end{bmatrix} \right| \\ &= \mathbf{i}(bz - cy) + \mathbf{j}(cx - az) + \mathbf{k}(ay - bx) \\ \implies \|\mathbf{u} \times \mathbf{v}\|^2 &= (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2 \\ &= b^2z^2 - 2bcyz + c^2y^2 + c^2x^2 - 2acxz + a^2z^2 + a^2y^2 - 2abxy + b^2x^2 \qquad (1)\end{aligned}$$

We also have

$$\begin{aligned}\|\mathbf{u}\|^2 &= a^2 + b^2 + c^2 \\ \|\mathbf{v}\|^2 &= x^2 + y^2 + z^2 \\ \mathbf{u} \cdot \mathbf{v} &= ax + by + cz \\ \implies \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 &= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ &= a^2x^2 + a^2y^2 + a^2z^2 + b^2x^2 + b^2y^2 + b^2z^2 + c^2x^2 + c^2y^2 + c^2z^2 \\ &- \left(a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz\right) \\ &= a^2y^2 + a^2z^2 + b^2x^2 + b^2z^2 + c^2x^2 + c^2y^2 - 2abxy - 2acxz - 2bcyz\end{aligned}$$

and we can see that this expression is equivalent to the one in equation (1), so the two quantities are equal, as desired.

Alternatively, we can use the identities

$$\begin{aligned}
\|\mathbf{u}\times\mathbf{v}\| &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta \\
\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\
\implies \|\mathbf{u}\times\mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2 &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\sin^2\theta + \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\cos^2\theta \\
&= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\left(\sin^2\theta+\cos^2\theta\right) = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 \\
\implies \|\mathbf{u}\times\mathbf{v}\|^2 &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2
\end{aligned}$$

This also proves that this equality generalizes to any dimension, not just $\mathbb{R}^3$! □

4. (a) Express $\mathbf{u} = 1\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ as a sum of vectors parallel and perpendicular to $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

   Solution. We have

$$\begin{aligned}
\mathbf{u}_\parallel &= \operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{1\cdot -2 + 4\cdot 5 + 3\cdot 4}{(-2)^2+5^2+4^2}\mathbf{v} = \frac{30}{45}(-2,5,4) = \left(-\frac{4}{3},\frac{10}{3},\frac{8}{3}\right) \\
\mathbf{u}_\perp &= \mathbf{u} - \operatorname{proj}_{\mathbf{v}}\mathbf{u} = (1,4,3) - \left(-\frac{4}{3},\frac{10}{3},\frac{8}{3}\right) = \left(\frac{7}{3},\frac{2}{3},\frac{1}{3}\right)
\end{aligned}$$

   so now

$$\mathbf{u} = (1,4,3) = \left(-\frac{4}{3},\frac{10}{3},\frac{8}{3}\right) + \left(\frac{7}{3},\frac{2}{3},\frac{1}{3}\right)$$

   □

   (b) Show that the vectors $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$ you obtained in part (a) are orthogonal.

   Proof. The dot product between orthogonal vector is 0, and here

$$\mathbf{u}_\parallel\cdot\mathbf{u}_\perp = \left(-\frac{4}{3},\frac{10}{3},\frac{8}{3}\right)\cdot\left(\frac{7}{3},\frac{2}{3},\frac{1}{3}\right) = -\frac{4}{3}\cdot\frac{7}{3} + \frac{10}{3}\cdot\frac{2}{3} + \frac{8}{3}\cdot\frac{1}{3} = \frac{-28}{3} + \frac{20}{3} + \frac{8}{3} = 0$$

   so these two vectors are orthogonal, as desired. □

5. MATLAB exercise. For the vectors $\mathbf{u} = (2,8,-3,-1,2)$ and $\mathbf{v} = (-5,3,1,1,6)$, we have

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{2^2+8^2+(-3)^2+(-1)^2+2^2} = \sqrt{82} \approx 9.055 \\
\|\mathbf{v}\| &= \sqrt{(-5)^2+3^2+1^2+1^2+6^2} = 6\sqrt{2} \approx 8.485 \\
\mathbf{u}\cdot\mathbf{v} &= 2\cdot -5 + 8\cdot 3 + -3\cdot 1 + -1\cdot 1 + 2\cdot 6 = 22 \\
\|\mathbf{u}-\mathbf{v}\| &= \|(7,5,-4,-2,-4)\| = \sqrt{7^2+5^2+(-4)^2+(-2)^2+(-4)^2} = \sqrt{110} \approx 10.488 \\
\cos\theta &= \frac{\mathbf{u}\dot{\mathbf{v}}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{22}{\sqrt{82}\cdot 6\sqrt{2}} \approx 0.286
\end{aligned}$$

6. (E&P 4.1.30) V is the set of all (x,y,z) such that $x+y+z=0$.

   Proof. Let $\mathbf{u},\mathbf{v}\in V$, where $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{v} = (v_1,v_2,v_3)$, and let $k\in\mathbb{R}$. Then since $\mathbf{u}$ and $\mathbf{v}$ are in V, they have the property that $u_1+u_2+u_3 = 0$ and $v_1+v_2+v_3 = 0$. Now,

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$$

   where

$$(u_1+v_1)+(u_2+v_2)+(u_3+v_3) = (u_1+u_2+u_3)+(v_1+v_2+v_3) = 0+0 = 0$$

so V is closed under addition. Then we have

$$k\mathbf{u} = (ku_1, ku_2, ku_3)$$

where

$$ku_1 + ku_2 + ku_3 = k(u_1 + u_2 + u_3) = k \cdot 0 = 0$$

so V is closed under scalar multiplication. □

7. (E&P 4.1.36) V is the set of all (x, y, z) such that $xyz = 1$.

   Solution. This is not a subspace. Consider the vector $\mathbf{u} = (1, 1, 1) \in V$ since $1 \cdot 1 \cdot 1 = 1$, and let $k = 2$. Then $k\mathbf{u} = (2, 2, 2)$, but $2 \cdot 2 \cdot 2 = 8 \neq 1$, so V is not closed under scalar multiplication. □

8. (E&P 4.2.8) W is the set of all vectors in $\mathbb{R}^2$ such that $(x_1)^2 + (x_2)^2 = 0$.

   Proof. Let $\mathbf{u}, \mathbf{v} \in V$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, and let $k \in \mathbb{R}$. Then it follows that $u_1^2 + u_2^2 = 0 \implies u_1 = u_2 = 0$, and similarly $v_1 = v_2 = 0$. Thus,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (0 + 0, 0 + 0) = (0, 0) \in V$$

so V is closed under addition, and

$$k\mathbf{u} = k(u_1, u_2) = k(0, 0) = (0, 0) \in V$$

so V is closed under scalar multiplication. In a sense, V is vacuously a subspace because it contains only the 0 vector. □