# Homework 10

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## Chapter 15: Fourier Series

6. Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic and Riemann integrable on $[-\pi, \pi]$. Prove that $\lim_{x\to 0} \int_{-\pi}^{\pi} |f(x+t) - f(t)|^2 \, dt = 0$.

*Proof.* Since $f$ is $2\pi$-periodic, we have

$$\begin{aligned}\lim_{x\to 0}\int_{-\pi}^{\pi}|f(x+t)-f(t)|^2\,dt &= \lim_{x\to 0}\int_{-\pi}^{\pi} f^2(x+t)\,dt - 2\lim_{x\to 0}\int_{-\pi}^{\pi} f(x+t)f(t)\,dt + \lim_{x\to 0}\int_{-\pi}^{\pi} f^2(t)\,dt \\ &= \lim_{x\to 0}\int_{-\pi+x}^{\pi+x} f^2(t)\,dt - 2\lim_{x\to 0}\int_{-\pi}^{\pi} f(x+t)f(t)\,dt + \int_{-\pi}^{\pi} f^2(t)\,dt\end{aligned}$$

By Parseval's equation, since $f$ is Riemann integrable, we have

$$\int_{-\pi}^{\pi} f^2(t)\,dt = \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right] = \lim_{x\to 0}\int_{-\pi+x}^{\pi+x} f^2(t)\,dt$$

Now, since $f(x+t)f(t)$ uniformly converges to $f^2(t)$, we can switch the order of limit and integration, so

$$2\lim_{x\to 0}\int_{-\pi}^{\pi} f(x+t)f(t)\,dt = 2\int_{-\pi}^{\pi}\lim_{x\to 0} f(x+t)f(t)\,dt = 2\int_{-\pi}^{\pi} f^2(t)\,dt$$

so combining this with the first result, we get that the resulting integral evaluates to 0. □

## Chapter 18: The Lebesgue Integral

38. If $f \in L_1[0,1]$, show that $x^n f(x) \in L_1[0,1]$ for $n = 1, 2, \cdots$ and compute $\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx$.

*Proof.* Since $f \in L_1[0,1]$, it follows that $|f| \in L_1[0,1]$. Because $x \in [0,1]$, we have

$$|x^n f(x)| \le |f(x)| \implies |x^n f(x)| \in L_1[0,1] \implies x^n f(x) \in L_1[0,1]$$

Then the sequence $(x^n f(x))$ converges to the function

$$g(x) = \begin{cases} f(1) & x = 1 \\ 0 & x \ne 1 \end{cases}$$

so $g(x) \equiv 0$ a.e., and since $|x^n f(x)| \le |f(x)| \in L_1[0,1]$, by the DCT we have

$$\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx = \int_0^1 g(x)\,dx = 0$$

□

40. Let $(f_n), (g_n)$, and $g$ be integrable, and suppose that $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \le g_n$ a.e., for all $n$, and that $\int g_n \to \int g$. Prove that $f \in L_1$ and that $\int f_n \to \int f$. (Hint: Revise the proof of the DCT)

*Proof.* Since $|f_n| \le g_n$, the sequences $(g_n + f_n)$ and $(g_n - f_n)$ are non-negative, so by Fatou's lemma, we have

$$\int \liminf_{n\to\infty}(g_n + f_n) \le \liminf_{n\to\infty} \int (g_n + f_n)$$
$$\implies \int \liminf_{n\to\infty} g_n + \int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int g_n + \liminf_{n\to\infty} \int f_n$$
$$\implies \int g + \int f \le \int g + \liminf_{n\to\infty} \int f_n \implies \int f \le \liminf_{n\to\infty} \int f_n$$
$$\int \liminf_{n\to\infty}(g_n - f_n) \le \liminf_{n\to\infty} \int (g_n - f_n)$$
$$\implies \int \liminf_{n\to\infty} g_n - \int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int g_n - \limsup_{n\to\infty} \int f_n$$
$$\implies \int g - \int f \le \int g - \limsup_{n\to\infty} \int f_n \implies \int f \ge \limsup_{n\to\infty} \int f_n$$

so we have the inequality

$$\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n$$

and thus $f \in L_1$ and $\int f = \lim_{n\to\infty} \int f_n$. □

41. Let $(f_n), f$ be integrable, and suppose that $f_n \to f$ a.e. Prove that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

*Proof.* $(\implies)$ : We have

$$|f_n - f| + |f| \ge |f_n| \implies |f_n - f| \ge |f_n| - |f|$$
$$\implies \int (|f_n| - |f|) \le \int |f_n - f| \to 0 \implies \int (|f_n| - |f|) \to 0$$

$(\impliedby)$ : We have $|f_n - f| + |f| - |f_n| \to 0$ and $|f_n - f| + |f| - |f_n| \le 2|f| \in L_1$, so by the DCT,

$$\int \lim_{n\to\infty} (|f_n - f| + |f| - |f_n|) = \lim_{n\to\infty} \int (|f_n - f| + |f| - |f_n|)$$
$$\implies 0 = \lim_{n\to\infty} \int |f_n - f| + \left[\int |f| - \lim_{n\to\infty} \int |f_n|\right] = \lim_{n\to\infty} \int |f_n - f|$$

□

42. Let $(f_n)$ be a sequence of integrable functions and suppose that $|f_n| \le g$ a.e., for all $n$, for some integrable function $g$. Prove that

$$\int \left(\liminf_{n\to\infty} f_n\right) \le \liminf_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f_n \le \int \left(\limsup_{n\to\infty} f_n\right)$$

*Proof.* The second inequality is trivial as a property between $\liminf$ and $\limsup$.

For the first inequality, we have the sequence $(g + f_n)$ is non-negative, so by Fatou's lemma, we have

$$\int \liminf(g + f_n) \leq \liminf \int (g + f_n)$$
$$\implies \int g + \int \liminf f_n \leq \int g + \liminf \int f_n \implies \int \liminf f_n \leq \liminf \int f_n$$

and for the third inequality, we have the sequence $(g - f_n)$ is non-negative, so again by Fatou's lemma, we have

$$\int \liminf(g - f_n) \leq \liminf \int (g - f_n)$$
$$\implies \int g + \int \liminf(-f_n) \leq \int g - \limsup \int f_n \implies \limsup \int f_n \leq -\int \liminf(-f_n) = \int \limsup f_n$$

□

43. Let $f$ be measurable and finite a.e. on $[0, 1]$.

(a) If $\int_E f = 0$ for all measurable $E \subset [0, 1]$ with $m(E) = 1/2$, prove that $f = 0$ a.e. on $[0, 1]$.

*Proof.* Suppose $m(\{f \neq 0\}) > 0$. Then we have

$$\{f \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

so WLOG $\{f > 0\}$ has positive measure. Then

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\}$$

so one of the sets $\left\{f > \frac{1}{k}\right\}$ has positive measure, and there exists some $E \subset [0, 1]$ with $m(E) = 0$ such that $E \cap \left\{f > \frac{1}{k}\right\}$ has positive measure. Thus, we have

$$0 = \int_{E \cap \left\{f > \frac{1}{k}\right\}} f \geq \int_{E \cap \left\{f > \frac{1}{k}\right\}} \frac{1}{k} > 0$$

which is a contradiction. Thus, $\{f \neq 0\}$ has measure 0, and thus $f = 0$ a.e. □