Homework 7 Solutions

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1. (E&P 5.2.36) Suppose that one solution $y_1(x)$ of the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{18}$$

is known. The method of reduction of order consists of substituting $y_2(x) = v(x)y_1(x)$ in (18) and attempting to determine the function $v(x)$ so that $y_2(x)$ is a second linearly independent solution of (18). After substituting $y = v(x)y_1(x)$ in Eq. (18), use the fact that $y_1(x)$ is a solution to deduce that

$$y_1 v'' + (2y_1' + py_1)v' = 0 \tag{19}$$

Proof. Let $y_2(x) = v(x)y_1(x)$. Then if y_2 is a solution to (18), we have

$$\begin{aligned} y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1'' \\ \implies y_2'' + py_2' + qy_2 &= (v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + qvy_1 = 0 \\ &= v''y_1 + (2y_1' + py')v' + (y_1'' + py_1' + qy_1)v \end{aligned}$$

and since y_1 is a solution to (18), the coefficient of v in this equation is 0, so our transformed equation is

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

as desired. □

2. (E&P 5.2.43) First note that $y_1(x) = x$ is one solution of Legendre's equation of order 1,

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

Then use the method of reduction of order to derive the second solution

$$y_2(x) = 1 - \frac{x}{2}\ln\frac{1+x}{1-x} \quad (\text{for } -1 < x < 1)$$

Solution. Suppose $y_2 = vy_1 = vx$ is another solution. Then

$$\begin{aligned} y_2' &= v + v'x \\ y_2'' &= v' + v' + v''x = 2v' + v''x \\ \implies (1-x^2)(2v' + v''x) - 2x(v + v'x) + 2vx &= x(1-x^2)v'' + 2(1-x^2)v' - 2xv - 2x^2v' + 2vx \\ &= x(1-x)(1+x)v'' + (2 - 4x^2)v' = 0 \end{aligned}$$

Let $w = v'$, so this equation is

$$0 = x(1-x)(1+x)w' + (2-4x^2)w \implies x(1-x)(1+x)w' = (4x^2-2)w \implies \frac{w'}{w} = \frac{4x^2-2}{x(1-x)(1+x)}$$

Now, we find the partial fraction decomposition as

$$\frac{4x^2-2}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$
$$\implies 4x^2 - 2 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$$

Substituting $x = 0, 1, -1$, we have the equations

$$4 \cdot 0^2 - 2 = A(1-0)(1+0) \implies A = -2$$
$$4 \cdot 1^2 - 2 = B(1)(1+1) \implies B = 1$$
$$4(-1)^2 - 2 = C(-1)(1-(-1)) \implies C = -1$$

Thus, we can integrate both sides of the equation

$$\int \frac{w'}{w}\,dx = \int \frac{4x^2-2}{x(1-x)(1+x)}\,dx = \int \left(-\frac{2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right) dx$$
$$\implies \ln w = -2\ln|x| - \ln|1-x| - \ln|1+x| = \ln \frac{1}{x^2(1-x)(1+x)}$$
$$\implies w = v' = \frac{1}{x^2(1-x)(1+x)}$$

Here, the partial fraction decomposition is given by

$$\frac{1}{x^2(1-x)(1+x)} = \frac{D}{x^2} + \frac{E}{x} + \frac{F}{1-x} + \frac{G}{1+x}$$
$$\implies 1 = D(1-x)(1+x) + Ex(1-x)(1+x) + Fx^2(1+x) + Gx^2(1-x)$$

Substituting $x = 0, 1, -1$, we have the equations

$$1 = D(1-0)(1+0) \implies D = 1$$
$$1 = F(1+1) \implies F = \frac{1}{2}$$
$$1 = G(1-(-1)) \implies G = \frac{1}{2}$$

and finally looking at the coefficient of x^3, we have $\left(-E + \frac{1}{2} - \frac{1}{2}\right)x^3 = 0 \implies E = 0$. Thus, we can integrate to solve for v as

$$\int v' = \int \frac{1}{x^2(1-x)(1+x)} = \int \left(\frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{1-x} + \frac{1}{2}\cdot\frac{1}{1+x}\right) dx$$
$$\implies v = -\frac{1}{x} - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|$$

and finally, our second solution $y_2 = vx$ can be written as

$$y_2 = xv = x\left(-\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\right) = -1 + \frac{x}{2}\ln\frac{1+x}{1-x}$$

This is a solution up to constant multiple, so we can negate it to get the other solution

$$y_2 = 1 - \frac{x}{2}\ln\frac{1+x}{1-x}$$

□

3. Solve the following IVP

$$y''' - 3y'' + 4y' - 2y = 0 \quad y(0) = 1, y'(0) = 0, y''(0) = 0$$

Solution. This IVP has characteristic equation $r^3 - 3r^2 + 4r - 2 = (r-1)(r^2 - 2r + 2)$, which has roots

$$\begin{aligned} r_1 &= 1 \\ r_2, r_3 &= 1 \pm i \end{aligned}$$

so the general solution is of the form

$$\begin{aligned} y(t) &= c_1 e^t + e^t\left(c_2 \cos t + c_3 \sin t\right) = e^t(c_1 + c_2 \cos t + c_3 \sin t) \\ \implies y'(t) &= e^t(-c_2 \sin t + c_3 \cos t) + e^t(c_1 + c_2 \cos t + c_3 \sin t) = e^t(c_1 + (c_2 + c_3)\cos t + (c_3 - c_2)\sin t) \\ \implies y''(t) &= e^t(-(c_2 + c_3)\sin t + (c_3 - c_2)\cos t) + e^t(c_1 + (c_2 + c_3)\cos t + (c_3 - c_2)\sin t) \\ &= e^t(c_1 + 2c_3 \cos t - 2c_2 \sin t) \end{aligned}$$

Using the initial conditions, we have the equations

$$\begin{aligned} y(0) &= c_1 + c_2 \cos 0 + c_3 \sin 0 = c_1 + c_2 = 1 \\ y'(0) &= c_1 + (c_2 + c_3)\cos 0 + (c_3 - c_2)\sin 0 = c_1 + c_2 + c_3 = 0 \\ y''(0) &= c_1 + 2c_3 \cos 0 - 2c_2 \sin 0 = c_1 + 2c_3 = 0 \\ \implies \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \end{aligned}$$

so the particular solution is

$$y(t) = e^t(2 - \cos t - \sin t)$$

□

4. (E&P 5.3.58) Make the substitution $v = \ln x$ of Problem 51 to find general solutions (for $x > 0$) of the Euler equation

$$x^3 y''' + 6x^2 y'' + 7xy' + y = 0$$

Solution. Here, $a = 1, b = 6, c = 7, d = 1$, so the substitution transforms the equation into

$$1 \cdot \frac{d^3y}{dv^3} + (6 - 3 \cdot 1) \cdot \frac{d^2y}{dv^2} + (7 - 6 + 2 \cdot 1)\frac{dy}{dv} + dy = \frac{d^3y}{dv^3} + 3\frac{d^2y}{dv^2} + 3\frac{dy}{dv} + dy = 0$$

which has characteristic equation $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, which has a repeated root $r = -1$ of order 3, so the general solution is

$$\begin{aligned} y(v) &= e^{-v}\left(c_1 + c_2 v + c_3 v^2\right) \\ \implies y(x) &= e^{-\ln x}\left(c_1 + c_2 \ln x + c_3 \ln^2 x\right) = \frac{1}{x}\left(c_1 + c_2 \ln x + c_3 \ln^2 x\right) \end{aligned}$$

□

5. Use undetermined coefficients to find a general solution for

$$y'' - 4y = \sinh x$$

Solution. For the complementary part, we have the characteristic equation $r^2 - 4 = (r-2)(r+2)$, so the two roots are -2 and 2, so

$$y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$$

Now, since $\sinh x = \frac{e^x - e^{-x}}{2}$, we expect the particular form to be

$$\begin{aligned}
y_p(x) &= Ae^x + Be^{-x} \\
\implies y_p''(x) &= Ae^x + Be^{-x} \\
\implies y'' - 4y &= (Ae^x + Be^{-x}) - 4(Ae^x + Be^{-x}) = -3Ae^x - 3Be^{-x} = \frac{e^x}{2} - \frac{e^{-x}}{2} \\
\implies A &= -\frac{1}{6}, B = \frac{1}{6}
\end{aligned}$$

so the general solution is given by

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{6}e^x + \frac{1}{6}e^{-x}$$

□

6. (E&P 5.5.31) Use undetermined coefficients.

$$y'' + 4y = 2x, \quad y(0) = 1, y'(0) = 2$$

Solution. For the complementary part, we have the characteristic equation $r^2 + 4 = 0 \implies r = \pm 2i$, so

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

Now, we expect the particular form to be

$$\begin{aligned}
y_p(x) &= Ax + B \\
\implies y_p''(x) &= 0 \\
\implies y'' + 4y &= 0 + 4(Ax + B) = 2x \\
\implies A &= \frac{1}{2}, B = 0
\end{aligned}$$

so the general solution is given by

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x$$

□

7. (E&P 5.5.47) Use the method of variation of parameters to find a particular solution

$$y'' + 3y' + 2y = 4e^x$$

Solution. For the complementary part, the characteristic equation is $r^2 + 3r + 2 = (r+1)(r+2)$, so the roots are -1 and -2, so

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x} = c_1 y_1 + c_2 y_2$$

Now, suppose $y = \mu_1 y_1 + \mu_2 y_2$ is also a solution. We have

$$W(y_1, y_2) = \det \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} = e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x}) = -e^{-3x}$$

Then by the formulas in the textbook, we have

$$\begin{aligned}
\mu_1 &= \int \frac{-y_2(x)f(x)}{W(x)}\,dx = \int \frac{-e^{-2x}\cdot 4e^{x}}{-e^{-3x}}\,dx = \int 4e^{2x}\,dx = 2e^{2x} \\
\mu_2 &= \int \frac{y_1(x)f(x)}{W(x)}\,dx = \int \frac{e^{-x}\cdot 4e^{x}}{-e^{-3x}}\,dx = \int -4e^{3x}\,dx = -\frac{4}{3}e^{3x} \\
\implies y_p(x) &= \mu_1 y_1 + \mu_2 y_2 = 2e^{2x}\cdot e^{-x} - \frac{4}{3}e^{3x}\cdot e^{-2x} = \frac{2}{3}e^{x}
\end{aligned}$$

□

8. Use variation of parameters to find a general solution for

$$4x^2y'' - 4xy' + 3y = 8x^{4/3}$$

Solution. This is a Cauchy equation, with characteristic equation

$$\begin{aligned}
4m(m-1) - 4m + 3 = 4(m^2 - m) - 4m + 3 = 4m^2 - 8m + 3 = (2m-1)(2m-3) \\
\implies m = \frac{1}{2}, \frac{3}{2}
\end{aligned}$$

so the complementary solution is given by

$$y_c(x) = c_1x^{1/2} + c_2x^{3/2}$$

Now, suppose $y = \mu_1 y_1 + \mu_2 y_2$ is also a solution. We have

$$W(y_1, y_2) = \det\begin{bmatrix} x^{1/2} & x^{3/2} \\ \frac{1}{2}x^{-1/2} & \frac{3}{2}x^{1/2}\end{bmatrix} = x^{1/2}\left(\frac{3}{2}x^{1/2}\right) - x^{3/2}\left(\frac{1}{2}x^{-1/2}\right) = x$$

Converting the original equation to standard form, $y'' - \frac{1}{x}y' + \frac{3}{x^2}y = 2x^{-2/3}$, we have

$$\begin{aligned}
\mu_1 &= \int \frac{-y_2(x)f(x)}{W(x)}\,dx = \int \frac{-x^{3/2}\cdot 2x^{-2/3}}{x}\,dx = \int 2x^{-1/6}\,dx = \frac{12}{5}x^{5/6} \\
\mu_2 &= \int \frac{y_1(x)f(x)}{W(x)}\,dx = \int \frac{x^{1/2}\cdot 2x^{-2/3}}{x}\,dx = \int 2x^{-7/6}\,dx = -12x^{-1/6} \\
\implies y_p(x) &= \mu_1 y_1 + \mu_2 y_2 = \frac{12}{5}x^{5/6}\cdot x^{1/2} - 12x^{-1/6}\cdot x^{3/2} = -\frac{72}{5}x^{4/3}
\end{aligned}$$

and thus a general solution is given by

$$y(x) = c_1x^{1/2} + c_2x^{3/2} - \frac{72}{5}x^{4/3}$$

□