OLG Write-up

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This note describes two OLG models and their equilibrium to be used for computational exercises.

Model 1: Based on Huggett, 1996

This is a heterogeneous agent OLG model based on Hugget, 1996.

In this economy, there is a continuum of agents born each period. The population has the following features:

- agents live for N periods max
- for each age t, agents have a probability of s_t of surviving up to s_t (conditional on being alive at $t-1$)
- population grows at rate n

Agents have CRRA utility with coefficient σ: $u(c) = \frac{u^(1-\sigma)}{1-\sigma}$ and lifetime utility:

$$E[\sum_{t=1}^{N} \beta^t (\Pi_{j=1}^{t} s_j) u(c_t)] \tag{1}$$

The labour productivity of an agent, $e(z,t)$ depends on their age, t and an idiosyncratic shock z that is Markov. Every period, agents choose consumption c and assets a', while supplying labour inelastically in exchange for a wage w.

The dynamic programming problem for an agent is:

$$
\begin{aligned}
&V(a,z,t) = \max_{c,a'} u(c) + \beta s_{t+1} E[V(a',z',t+1)|(a,z)] \\
&\text{s.t. } c + a' \geq a(1 + r(1-\tau)) + (1-\theta-\tau)e(z,t)w + T + b_t \\
&\quad c \geq 0, a' \geq \bar{a}
\end{aligned}
$$

T is a lump-sum transfer, b_t is a social security benefit, r is the risk-free interest rate, τ is a tax rate on capital and labour income. τ is an additional social security tax on labour income. Furthermore, b_t has the following form: $b_t = 0$ if $t < R$ and $b_t = b$ otherwise, where R is the retirement age. $\bar{a}$ is a borrowing limit and agents in their last period of life cannot borrow: $a' \geq 0$ if $t = N$. If agents die before age N their assets are collected and given back to agents as transfers T.

There is a representative firm with production function $Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$ that hires labour and capital each period.

In addition, there is a government who spends G in expenditures and collects taxes on capital, and labour. It must satisfy: $G = \tau(rK + wL)$ We need to define some more notation before going into the definition of equilibrium.

- $X = [\bar{a}, \infty)xZ$ is the state space over assets and idiosyncratic productivity shocks
- $B(X)$ is the Borel $\sigma-$algebra on X
- $\psi_t(B)$ is the fraction of age t agents with states in B
- μ_t is the fraction of age t agents in the economy.
- $P(x, t-1, B)$ is a transition function that gives the probability an agent at t agent transitions to set B given he started in state x

Equilibrium

A stationary eq. consists of $c(a, z, t)$, $a'(a, z, t), r, w, K, L, T, G, \tau, \theta, b$ and distributions $\psi_1, ...\psi_N$ s.t.:

1. $c(a, z, t), a'(a, z, t)$ solve the household's problem.
2. K, L solve the problem of the firm:

$$w = F_L(K, L) \ , \ r = F_K(K, L) - \delta$$

3. Markets clear:
   (a) Goods: $\sum_t \mu_t \int_X (c(a, z, t) + a'(a, z, t))d\psi_t + G = F(K, L) + (1 - \delta)K$
   (b) Assets: $\sum_t \mu_t \int_X a'(a, z, t)d\psi_t = (1 + n)K$
   (c) Labour: $\sum_t \mu_t \int_X e(z, t)d\psi_t = L$
   (d) The transition function is consistent with agent's behaviour: $\psi_{t+1}(B) = \int_X P(x, t, B)d\psi_t$, $\forall t, B$
   (e) Government budget constraint: $G = \tau(rK + wL)$
   (f) Social security benefits equal taxes: $\theta wL = b(\sum_{t=R}^{N} \mu_t)$
   (g) Transfers equal accidental bequests:

$$T = \{\sum_t \mu_t(1 - s_{t+1}) \int_X a(x, t)(1 + r(1 - \tau)d\psi_t\}/(1 - n) \qquad (2)$$

Model 2: Based on Krueger and Kubler, 2003

This is a discrete-time OLG model with aggregate uncertainty.The authors solve the model using a projection algorithm (Smolyak's method). The aggregate shocks follow a Markov Chain with transition matrix Π. Let z^t denote the history of aggregate states. There are overlapping generations of agents that live for a finite number periods, N. One agent is born at each date z^t, i.e. there is no within generation heterogeneity.

Problem of an Agent

The agent gets utility from consumption and supplies one unit of labour inelastically. The problem for an agent who was born at time s and is currently alive in period t is:

$$\max E_s[\sum_{t=s}^{s+N-1} \beta^{t-s} u(c_t^s)]$$
$$\text{s.t. } c_t^s + a_t^s = r_t a_{t-1}^s + w_t l^{t-s}$$

$u(\cdot)$ is the utility function of the agent and satisfies Inada conditions. a_t^s is holdings of capital that the agent can rent out to a firm at price r_t. l^{t-s} is the labour productivity of the agent and is dependent on the agent's age and the aggregate state of the economy, z_t.

The Euler equation for the household is:

$$u'(c_t^s(z^t)) = \beta E_{z_t} u'(c_{t+1}^s(z^t, z_{t+1})) r(z^t, z_{t+1}) \tag{3}$$

The agents take expectations over next period's realization of z, z_{t+1}. Other assumptions about the households are:

- $a_{s+N-1}^s \geq 0$ i.e. at period $t = s + N - 1$ (the last period of the agent's life), the agent cannot borrow
- agents are born with no assets : $a_{s-1}^s = 0$
- at $t = 0$ there is one agent of each age $1, ..., N$ with respective assets $a_{-1}^0, ...a_{-1}^{-N+1}$ where $a_{-1}^0 = 0$ by the previous point
- denote $\mathcal{I}_t$ to be the set of agents alive at time t

Firms

There is a single firm who hires capital K_t, L_t each period t, after seeing the shock z_t. Therefore firms don't have uncertainty and solve the following problem to maximize current period profits:

$$\max_{K_t, L_t} f(K_t, L_t, z_t) - w_t L_t - r_t K_t$$

The optimal choice for labour satisfies:

$$w(z^t) = \eta(z^t) F_L(K(z^t), L(z^t)) \tag{4}$$

The optimal choice for capital satisfies:

$$w(z^t) = \eta(z^t) F_K(K(z^t), L(z^t)) + (1 - \delta(z^t)) \tag{5}$$

The authors take: $f(K, L, z) = \eta(z)F(K, L) + K(1 - \delta(z))$ where $\eta(z)$ is a stochastic productivity shock, and δ is depreciation (that is also a function of the aggregate state).

Equilibrium

Given an initial aggregate state z_0, an initial level of assets $\{a^s_{-1}\}^0_{s=-N+1}$, a competitive equilibrium, is a sequence of choices $\{c^i_t, a^i_t\}^{i \in \mathcal{I}_t}$, and choices $\{K_t, L_t\}$ such that $\forall\ t = 0, ...\infty$:

1. $\forall\ s = 0, ...,$, given $\{r_t, w_t\}^{\infty}_{t=0}, \{c^i_t, a^i_t\}^{s+N-1}_{t=s}$ solve the household's problem, i.e. they satisfy equation (1).
2. For r_t, w_t, K_t and L_t satisfy (2) and (3).
3. Market Clearing: $\forall t$,:
   - Labour: $L_t = \sum_{i \in \mathcal{I}_t} l^i_t$
   - Capital: $K_t = \sum_{i \in \mathcal{I}_t} a^i_{t-1}$