PART 1

Representing Dynamic Systems Mathematically in time and s-domains

review (ish)
- Transfer functions & LaPlace
- State space

new (ish?) →
- Block diagrams

Transfer Functions & LaPlace (frequency domain/classical)

A **transfer function** of an LTI system of differential equations is the ratio of the LaPlace transform of the output to the LaPlace transform of the input, assuming all initial conditions are zero

We'll get to the details of the LaPlace transform later. For now, we can say it is similar to a Fourier Transform, but transforms from the time domain to the s domain, which is a complex variable that includes sinusoidal and exponential behaviors, and therefore useful for capturing the behavior of systems that exhibit both, like spring-mass-damper systems.

$$L(s) = \int_0^\infty f(t)e^{-st}dt \qquad s = \sigma + jw$$

So why is this useful?

- → The transfer function captures the behavior of the system to create a mathematical map of how inputs are translated to outputs
- → The transfer function can be analyzed to chacterize and understand the system
- → Transfer functions of induvidual components can be combined (easily) to understand and analyze more complex systems
- → These are key to controls - the controller is used to alter the overall system behavior to obtain the desired input-output relationship

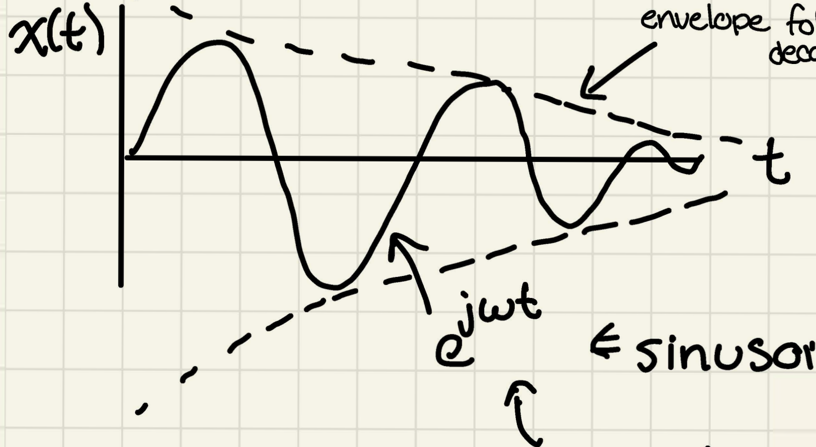
Now, I said these could easily be combined to characterize more complex systems - let's learn how to do that!

But what does the s-domain mean? Why do we care?

Let's consider some typical responses to odes:

1) Oscillating and decaying (damped, stable)

$e^{\sigma t}$ σ is real and <0 ⊖



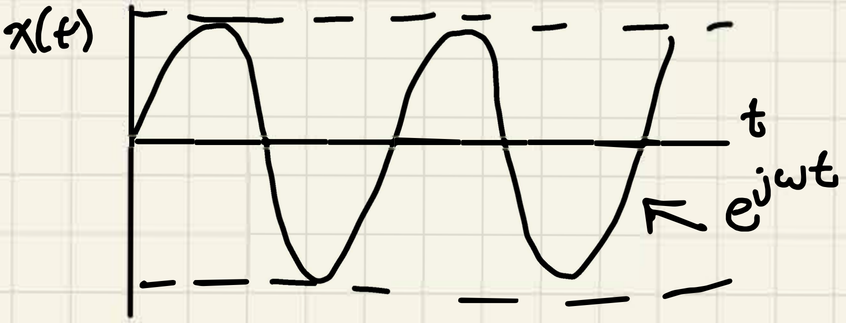
includes both a frequency (sinusoidal) and an exponetial

$e^{j\omega t}$ ← sinusoid (Euler's formula)

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

periodic w/ period $2\pi/\omega$

2) Oscillating only (neutral)



$\sigma = 0$ (no exponetial piece to the response)

← periodic with period $2\pi/\omega$

3) Oscillating and exponetially increasing (unstable)

σ is real and >0 ⊕

← periodic with period $2\pi/\omega$

4) Purely exponetial

now $s = \sigma + j\omega$ ← sinusoidal / frequency term

↑ exponetial behavior term

the real part of s can tell us about the stability

⇒ negative = stable, will converge

⇒ zero = nuetral, will niether converge nor diverge

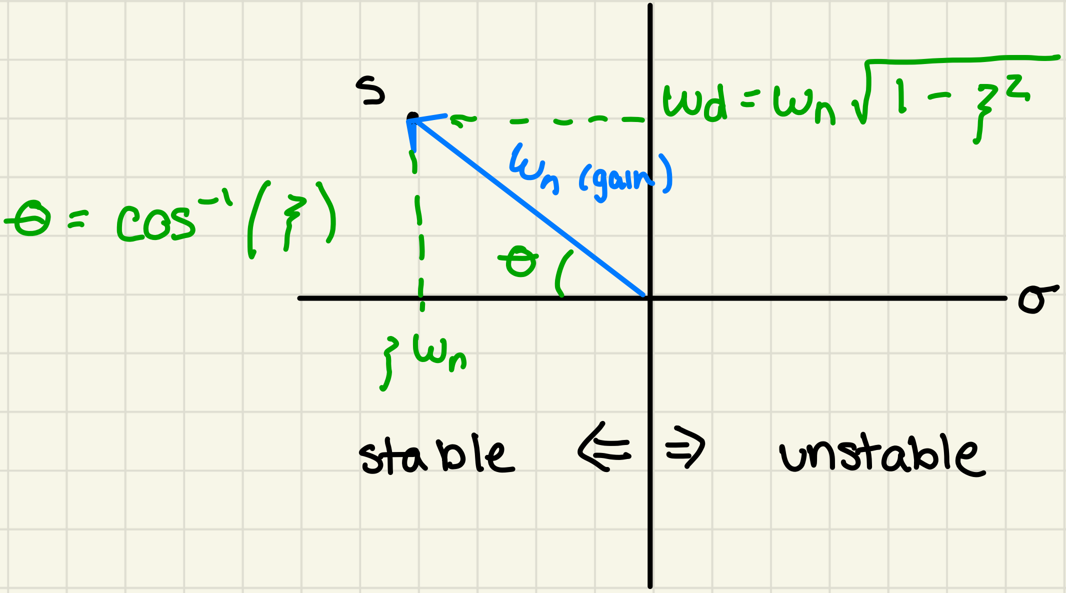
⇒ positive = unstable, will diverge

we can plot solutions (poles) in the s-space on a root-locus

ζ = damping ratio

ω_n = natural freq.

ω_d = damped freq



(note that the complex conjugate is omitted)

this will give us critical information about stability and insight on how to design ζ and ω_n

Later, we will revisit root locus plots and their associated variables in detail

For now, let's keep the general criteria for stability in mind as we consider why this is useful

LaPlace Transforms

We mentioned briefly the fact that the contents of the blocks are the LaPlace transforms of the dynamic equations for the systems.

$$\text{Time Domain} \overset{\mathcal{L}}{\Rightarrow} \text{Frequency Domain}$$

Let's dig into that more, what it means, and why it's useful

The LaPlace transform is defined as:

$$L(s) = \int_0^\infty f(t)e^{-st}dt \qquad s = \sigma + j\omega$$

These can help us solve complex differential equations

Let's look at some transforms for the f(t)s we may frequently encounter

Function	$f(t)$	$L(s)$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
unit step	$u(t) = 1(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
nth power (integer n)	t^n	$\frac{n!}{s^{n+1}}$

Table 2-1 in your textbook offers a more complete list of LaPlace transforms for common functions.

It's also worth understanding how transforms of derivatives work:

f(t)	L(t)	if IC=0
$x(t)$	$X(s)$	$X(s)$
$x'(t)$	$sX(s) - x(0) \swarrow$ inital position	$sX(s)$
$x''(t)$	$s^2X(s) - \underset{\substack{\uparrow\\ \text{initial}\\ \text{position}}}{sX(0)} - \underset{\substack{\nwarrow\\ \text{inital}\\ \text{velocity}}}{X'(0)}$	$s^2X(s)$

See table 2-2 in your book for a more extensive list
This is sometimes called the "classical technique"

You can also go the other way, from the s-domain to the t-domain. This is called the inverse LaPlace

More generally:

$$a_n \frac{d^ny}{dt^n} + \dots + a_1\frac{dy}{dt} + a_0y(t) = b_m\frac{d^mu}{dt^m} + \dots + b_1\frac{du}{dt} + b_0u(t)$$

↙ does this look familiar? (3530?)

has a LaPlace Transform with zero I.C of

$$a_n s^n Y(s) + \dots + a_1 s Y(s) + a_0 Y(s) = b_m s^m U(s) + \dots + b_1 s U(s) + b_0 U(s)$$

so the transfer function of such a system is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s_1 + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s_1 + a_0}$$

Let's go a bit further with this xfer fxn:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s_1 + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s_1 + a_0}$$

It will be useful to us in genral to factor our transfer functions to get zero-pole-gain form

$$G(s) = K \frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)}$$

$K = \frac{b_m}{a_n}$ and is called the system gain

z_i are called the zeros (the roots of the numerator polynomial that make it 0

p_i are called the poles and are the roots of the denominator polynomial that make it zero

both the poles and zeros may be complex values

The values of the poles and zeros give us insight into both the transient & steady state response and can help us determine if the system is stable

↳ we'll come back to this point soon.

First, let's look at some examples

Example: Find the factored form of the transfer function for the linear system defined by

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt}$$

remember that a transfer function assumes zero I.C.

Laplace-ing

$$\underbrace{s^2Y(s) - s\cancelto{0}{Y(0)} - \cancelto{0}{y'(0)}}_{\frac{d^2y}{dt}} + \underbrace{5\left(sY(s) + \cancelto{0}{Y(0)}\right)}_{5\frac{dy}{dt}} + \underbrace{6Y(s)}_{6y}$$

$$= 2\left(sU(s) + \cancelto{0}{y(0)}\right)$$

Rearranging

$$Y(s)\left[s^2+5s+6\right] = 2sU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s}{s^2+5s+6}$$

Factoring

$$G(s) = 2\left(\frac{s}{(s+3)(s+2)}\right)$$

The system has a real zero at $s=0$ and two real poles at $s=-3$ and $s=-2$

Let's play with this in Matlab and look at how it responds to a impulse function

```
t = 0:.01:10

num = [1  0]

den = [1  5  6]

sys = tf(num, den)

p = pole(sys)

z = zero(sys)

y = impulse(sys, t)

plot (t,y)

grid
```

let's change the form of the transfer function and see how the poles, zeros, and response changes

⇒ try it, put some observations in chat

Now lets go the other way. Given the poles, zeros, and gain, can we find the transfer function?

$p_1, p_2 = -1 \pm j2$

$z_1 = -4$

$K = 3$

let's put this into a transfer function

$$G(s) = K\frac{(s - z_1)}{(s-p_1)(s-p_2)}$$

$$= \frac{3(s+4)}{(s-(-1+2j))(s-(-1-2j)}$$

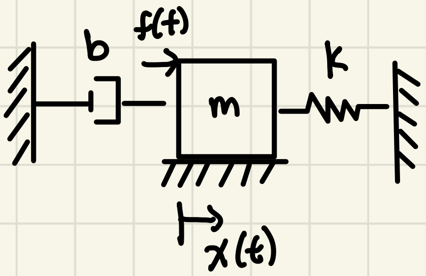
$$= \frac{3(s+4)}{s^2+2s+5}$$

$$= \frac{3s+12}{s^2+2s+5}$$

Inverse LaPlace

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u$$

To understand all this, let's look at a familar system



in response to forcing function f(t), the block is displaced x(t)

$$\Sigma F = ma \Rightarrow f(t) - kx(t) - b\dot{x}(t) = m\ddot{x}(t)$$

or $$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

Let's assume f(t) is a constant force applied at t=0 and $x(0)=0$ and $\dot{x}(0)=0$

$$\underset{\text{unit step}}{f(t) = F \cdot u(t)} \overset{L}{\Rightarrow} F(s) = \frac{F}{s}$$

note: recall from prior class that
mass + spring ⇒ oscillatory
damper ⇒ exponetial decay

$$m\ddot{x}(t) \quad + \quad b\dot{x}(t) \quad + \quad kx(t)$$

$$\Downarrow L \qquad \Downarrow L \qquad \Downarrow L$$

$$m[s^2X(s) - s\underset{0}{\cancel{x(0)}} - \underset{0}{\cancel{\dot{x}(0)}}] + b[sX(s) - \underset{0}{\cancel{x(0)}}] + kX(s)$$

$$\boxed{F/s = ms^2X(s) + bsX(s) + kX(s)}$$

now, we want to know X(t), which we will get from finding X(s)

$$\boxed{X(s) = \frac{F}{s(ms^2 + bs + k)}}$$

⇒ This tells us displacement as a result of our forcing in s.

note: This is not the transfer fxn. This is

That's nice! How does it help?

$$\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}}$$

Let's make this more concrete:

assume: $m = 1\,kg$, $F = 5N$, $b = 4\,Ns/m$, $K = 5\,N/m$

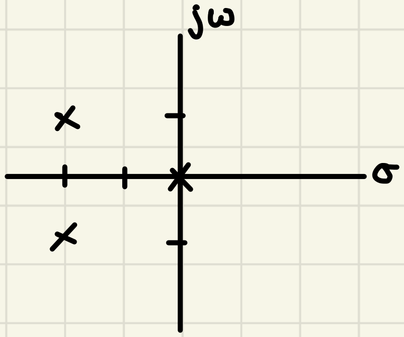
so $$X(s) = \frac{5}{s(s^2+4s+5)} \overset{\uparrow}{=} \frac{5}{s(s+2-j)(s+2+j)}$$

factor the quadratic
(it has complex roots!)

we have 3 poles in the system:

$s_1 = 0$
$s_2 = -2 + j$
$s_3 = -2 - j$

let's plot them!



Again, we will get into the meat of how to intepert this later, but at a glance, we see the system is stable $\Rightarrow$ easy to see in s-plane

Now we can do the inverse LaPlace to get back to the time domain (let's use partial fraction expansion)

$$X(s) = \frac{A}{s} + \frac{B}{s+2-j} + \frac{C}{s+2+j}$$

to get A, multiply both sides by s and set $s = 0$

B and C terms cancel

$$A = sX(s)\Big|_{s=0} = \frac{5\cancel{s}}{\cancel{s}(s^2+4s+5)}\Big|_{s=0} = \frac{5}{5} = 1$$

to get B, multiply both sides by $(s+2+j)$ and set $s=-2-j$

$$B=(s+2+j)X(s)\Big|_{s=-2-j} = \frac{5(\cancel{s+2+j})}{s(s+2-j)(\cancel{s+2+j})}\Big|_{s=-2-j}$$

$$B=-j-0.5$$

repeat for C, it will be

$$C=(s+2-j)X(s)\Big|_{s=-2+j} = j-0.5$$

so $X(s)=\frac{1}{s}+\frac{-j-0.5}{s+2+j}+\frac{j-0.5}{s+2-j}$ $\swarrow \mathcal{L}^{-1}\{X(s)\}$

and $x(t)=u(t)+u(t)\left[(-j-0.5)e^{-(2+j)t}+(j-0.5)e^{-(2-j)t}\right]$

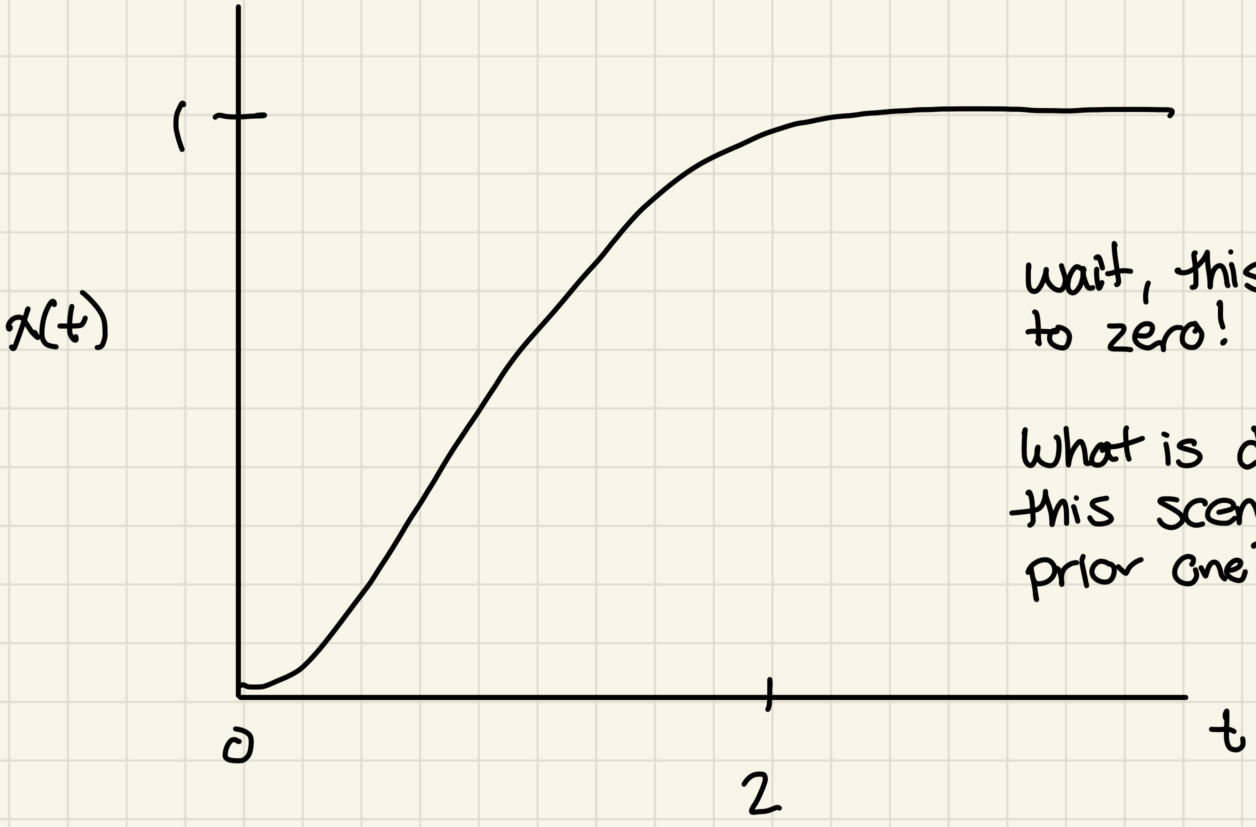
Euler $\Rightarrow$ $e^{jt}=\cos t + j\sin t$

this is the unit step

so $\boxed{x(t)=\left[1-e^{-2t}(\cos t+2\sin t)\right]u(t)}$

this is our response in the time domain

- note that Matlab's symbolic math toolbox can also be used (laplace, ilaplace, zpk, residue...)
  → check it out!



wait, this didn't return to zero! Why?

What is different about this scenario than the prior one?

to plot in matlab

```
t= 0:.01:10
num=[ 1]
den =[1 4  5]        or [m  b  k]
sys = tf(num, den)
opt = stepDataOptions ('StepAmplitude', 5 );
 y = step (sys, t, opt)
plot (t, y)
grid
```

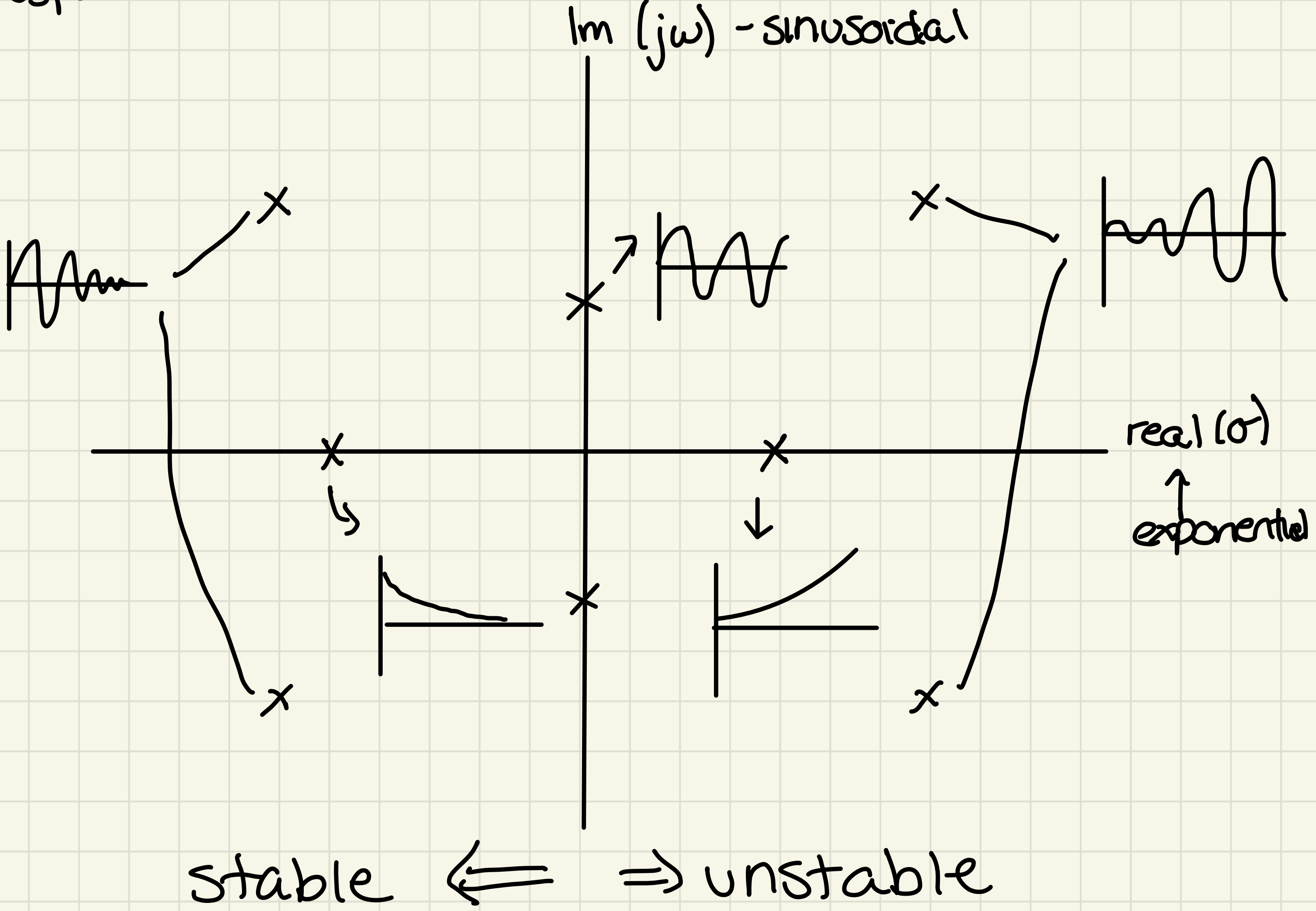
now lets play with changing the damping & spring

change to impulse - Now what happens?

With our observations in mind, let's revisit our plot of the s-space

natural

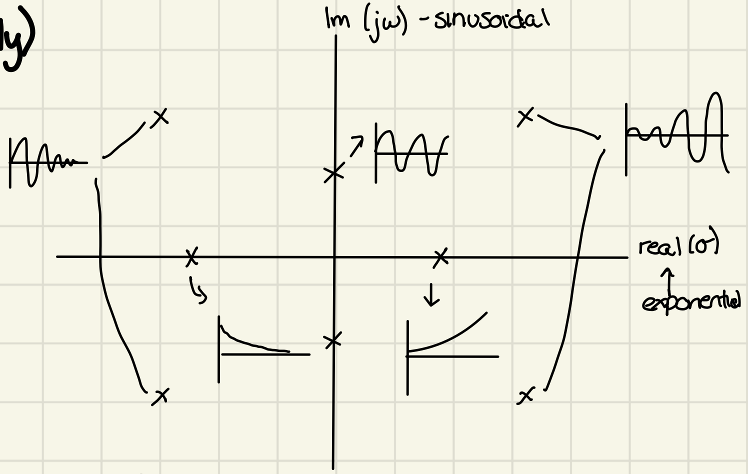
The system poles define components in the homogeneous response



so from the poles, we can know a lot about the form of the response

Consider this a spoiler alert!

Using pole location in s-space to understand the natural response (intuitively)



1) A real pole in the left half of the s-plane defines on an exponentially decaying pole in the response. The rate of decay is proportional to the distance from the origin

$p_i = -\sigma \Rightarrow Ce^{-\sigma t}$

2) A pole at the origin ($p_i = 0$) defines a constant amplitude component defined by initial conditions

3) A real pole in the right half $p_i = \sigma$ corresponds to an exponentially increasing component $Ce^{\sigma t}$

4) A complex conjugate pole pair $\sigma \pm j\omega$ in the left half combine to generate a decaying sinusoid $Ae^{-\sigma t}\sin(\omega t + \phi)$ where A and ϕ are functions of the initial condition

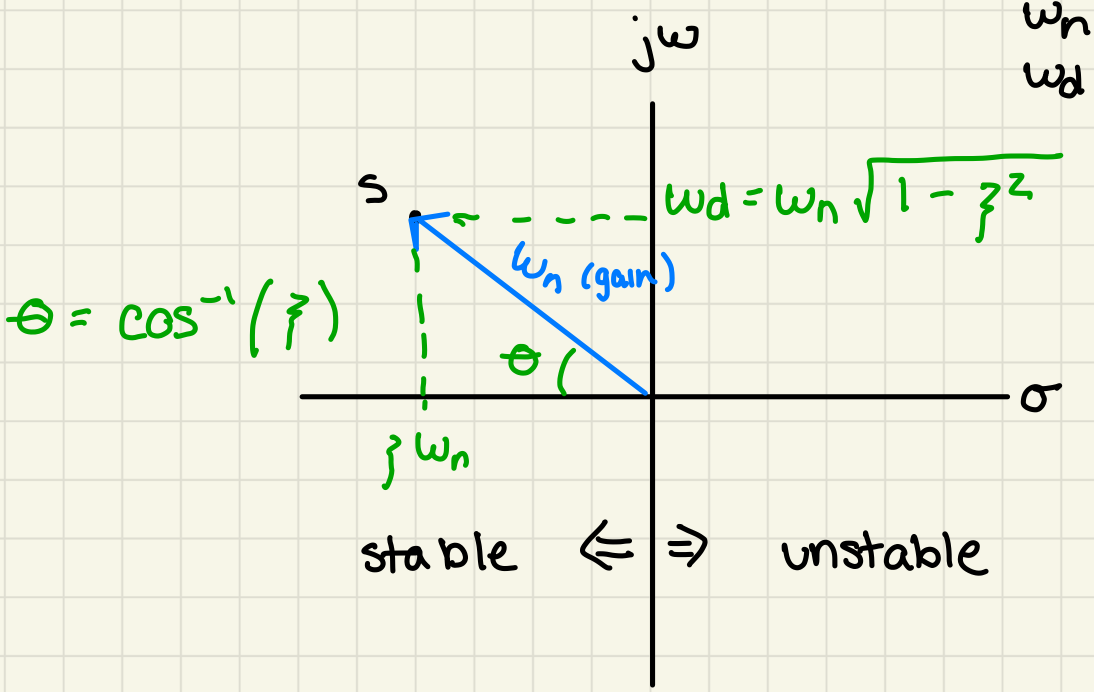
5) An imaginary pole pair $\pm j\omega$ generates an oscillatory component of constant amplitude determined by initial conditions

6) A complex pole pair in the right half generates an exponentially increasing sinusoidal component

By looking at the poles in the s-space, we can directly understand the natural response of the system

Further, we can use this plot to evaluate some key values

ζ = damping ratio
ω_n = natural freq.
ω_d = damped freq



(note that the complex conjugate is omitted)

We'll be using these plots a good bit going forward

This is just a first introduction

Let's look at one more example, this time not a spring-mass-damper, of how we can use a transfer function to help us represent a real system.

An aircraft is coming into it's final approach with its turbofan engines set to idle. As it breaks through the clouds, the pilot notices another aircraft stopped on the runway. The pilot hits the throttle to abort landing. When applying throttle to this large of an engine, the rate of increase of the engine's RPM is porportional to the difference between the actual RPM (Ω) and the RPM corresponding to the new throttle setting ($\Omega_{desired}$). We will assume a porportionality constant K to quantify the porportional relationship.

a) Write the differential equation for this system and convert to a transfer function. (focus on highlight)

$$\underset{\substack{\uparrow \\ \text{rate of increase} \\ \text{of RPM}}}{\dot{\Omega}} = \underset{\substack{\uparrow \\ \text{porportionality} \\ \text{constant}}}{K} \underset{\substack{\nwarrow \\ \text{the difference between} \\ \text{desired \& actual}}}{(\Omega_{desired} - \Omega)}$$

→ What is the input? What is the output?

$\Omega_{desired}$ = input ; Ω = output

→ The transfer function should be output/input (Ω / Ω_{des})

→ Let's LaPlace (with 0 IC)!

$$s\Omega = K\Omega_{desired} - K\Omega \xRightarrow{\text{rearrange}} G(s) = \frac{\Omega}{\Omega_{desired}} = \frac{K}{s+K}$$

b) What will be the form of the natural response to this system? How do you know?

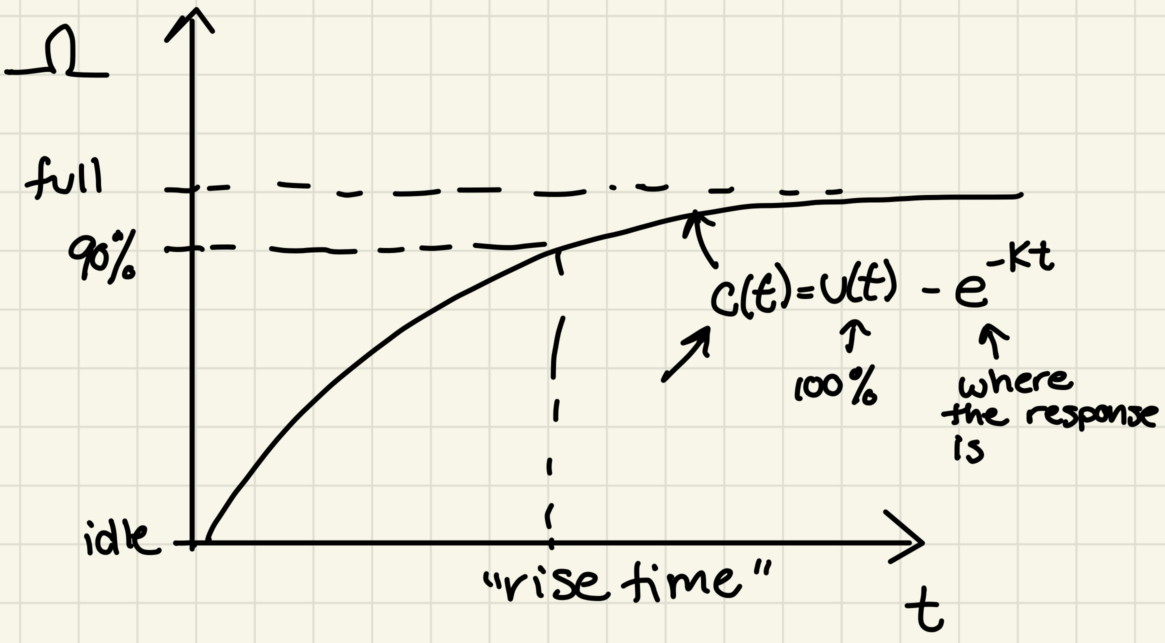
We have one pole (at $s=-K$)

$X(s) = \frac{A}{s+K}$ if we inverse LaPlace this, $\mathcal{L}^{-1}\left\{\frac{A}{s+K}\right\} = Ae^{-Kt}$

so the response with a time constant K

i.e. immediate application of full throttle

c) Sketch the expected response to a unit step. How long will it take to get to 90% of the desired value, as a function of K?



so when $e^{-Kt} = 10\%$, you will be at 90% of the value

solve for $t \Rightarrow t = \frac{\ln(10\%)}{K}$