# 3.3) Frequency Response Techniques (Bode & Nyqurst)

## 3.3.1) Bode!

Frequency Response methods add another perspective we can use to help us design our control systems.

But first - what is frequency response?

In the steady state, sinusoidal inputs to an LTI system generate sinusoidal responses of the same frequency, although they can differ in amplitude & phase

But wait! Our inputs don't look like sinusoids! Why are we suddenly talking about sinusoids?

Recall that any arbitary signal can be represented by a series of sinusoids (Fourier & LaPlace are based on this) and all our system outputs take the general form of sinusoids. If we understand how a system responds to sinusoids, we can predict now any input will affect it.

So back to sinusoids.

Let's consider the generic form $\underbrace{M_1}_{\text{amplitude}} \cos(\omega t + \underbrace{\phi_1}_{\text{phase}})$

Since frequency isn't changing, we can represent this with a phasor $M_1 \angle \phi_1$ (complex number in polar form)

Since a system can change both $M_1$ and $\phi_1$, we can think of the system itself as a function that when multiplied by the input phasor gives the output phasor

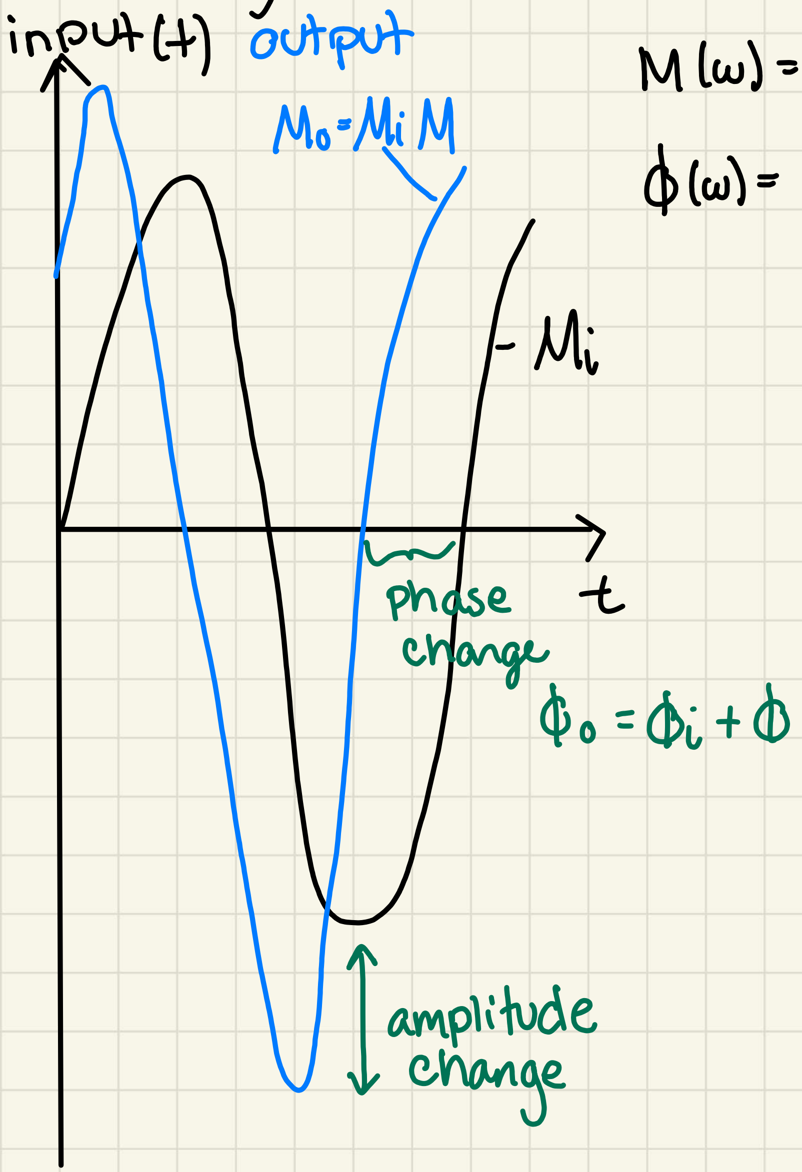
$$M_i(\omega)\angle\phi_i(\omega) \longrightarrow \boxed{M(\omega)\angle\phi(\omega)} \xrightarrow{M_o(\omega)\angle\phi_o(\omega)}$$

$$\Rightarrow \quad M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$

$$M_o(\omega)\,\phi_o(\omega) = M_i(\omega)\,M(\omega)\angle[\phi_i(\omega) + \phi(\omega)]$$

graphically



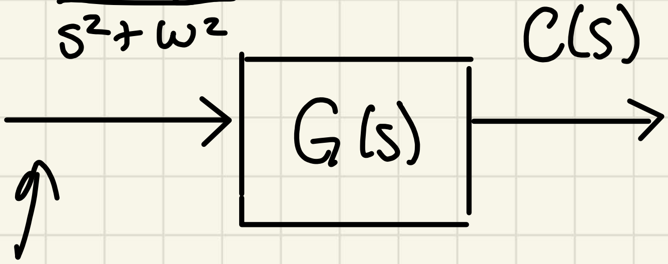

$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$ = magnitude response

$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$ = phase response

The magnitude and phase frequency response tells you how the system will transform an input to an output

So let's look at this analytically

$R(s) = \frac{As + B\omega}{s^2 + \omega^2}$




sinusoidal
Input in s
domain

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cos[\omega t - \tan^{-1}(B/A)]$$

$$\mathcal{L}\{A\cos(\omega t) + B\sin(\omega t) = \frac{As + B\omega}{s^2 + \omega^2} = \frac{As + B\omega}{(s - j\omega)(s + j\omega)}$$

We can represent the input phasor as:

1) polar form $M_i \angle \phi_i$ where $M_i = \sqrt{A^2 + B^2}$
$\phi_i = \tan^{-1}(B/A)$

2) rectangular form as a complex number $A - jB$

3) Euler's formula $M_i e^{j\phi_i}$

This will be important later. But let's write the form of the response

$$C(s) = \frac{As + B\omega}{(s - j\omega)(s + j\omega)} G(s)$$

$$= \frac{K_1}{s - j\omega} + \frac{K_2}{s + j\omega} + \text{partial fractions from } G(s)$$

since we can represent $G(s)$ as it's phasor $M_G \angle \phi_G$

$$K_1 = \frac{As + B\omega}{s - j\omega} G(s) \Big|_{s \to -j\omega} = \frac{1}{2}(A + jB) G(-j\omega) = \frac{1}{2} M_i e^{-j\phi_i} M_G e^{-j\phi_G}$$

$$= \frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \frac{As + B\omega}{s + j\omega} G(s) \Big|_{s \to j\omega} = \frac{1}{2}(A - jB) G(j\omega) = \frac{1}{2} M_i e^{j\phi_i} M_G e^{j\phi_G}$$

$$= \frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)} \Leftarrow \text{complex conj of } K_1$$

note that $M_G = |G(j\omega)|$ and $\phi_G$ = angle of $G(j\omega)$

$$C(s) = \underbrace{\frac{K_1}{s - j\omega} + \frac{K_2}{s + j\omega}}_{\text{these terms are the steady state response}} + \text{partial fractions from } G(s)$$

these terms are the steady state response

$$\text{so} \quad C_{ss}(s) = \frac{\frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}}{s + j\omega} + \frac{\frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)}}{s - j\omega}$$

which, skipping the details, inverse Laplaces to

$$c(t) = M_i M_G \cos(\omega t + \phi_i + \phi_G)$$

$$M_o \angle \phi_o = (M_i \angle \phi_i) \underbrace{(M_G \angle \phi_G)}_{= G(j\omega) \text{ from above}}$$

so $G(j\omega) = G(s)\big|_{s \to j\omega}$

meaning we can evaluate $G(s)\big|_{s \to j\omega}$ to find out the frequency response of a system as a function of frequency.

Now, this probably doesn't seem to useful yet. But hidden in this is important information about transient response and steady state error characteristics.

→ We'll get there! (after we learn to make them!)

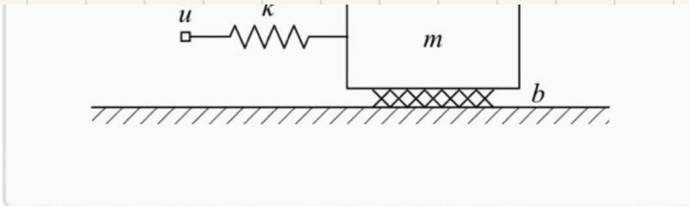
But first, let's learn how to represent this visually so we can interpret it better - Bode & Nyquist

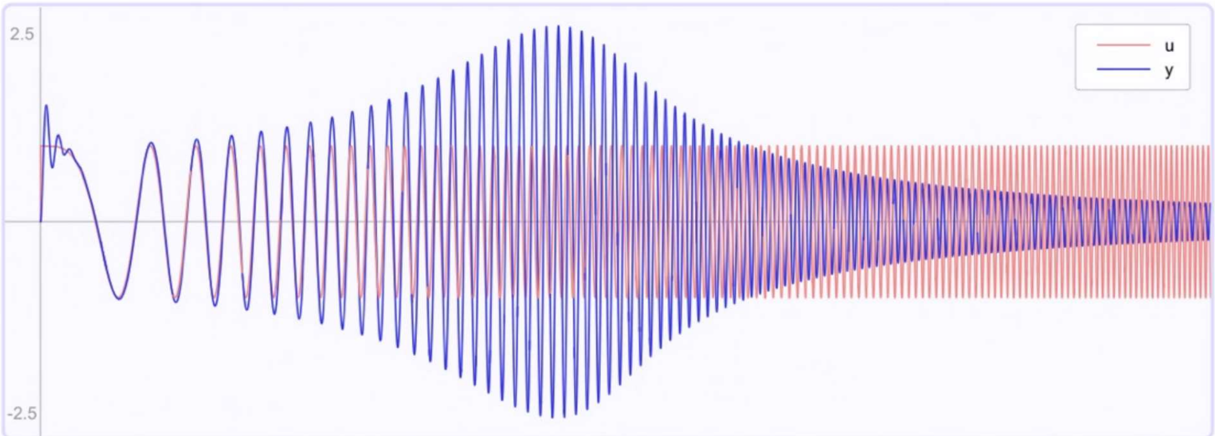
Let's look at an example that shows what a Bode plot is telling us

https://lpsa.swarthmore.edu/Bode/BodeWhat.html

scroll to animation of SMD system

We'll come back to Nyquist. For now, let's look a Bode plots



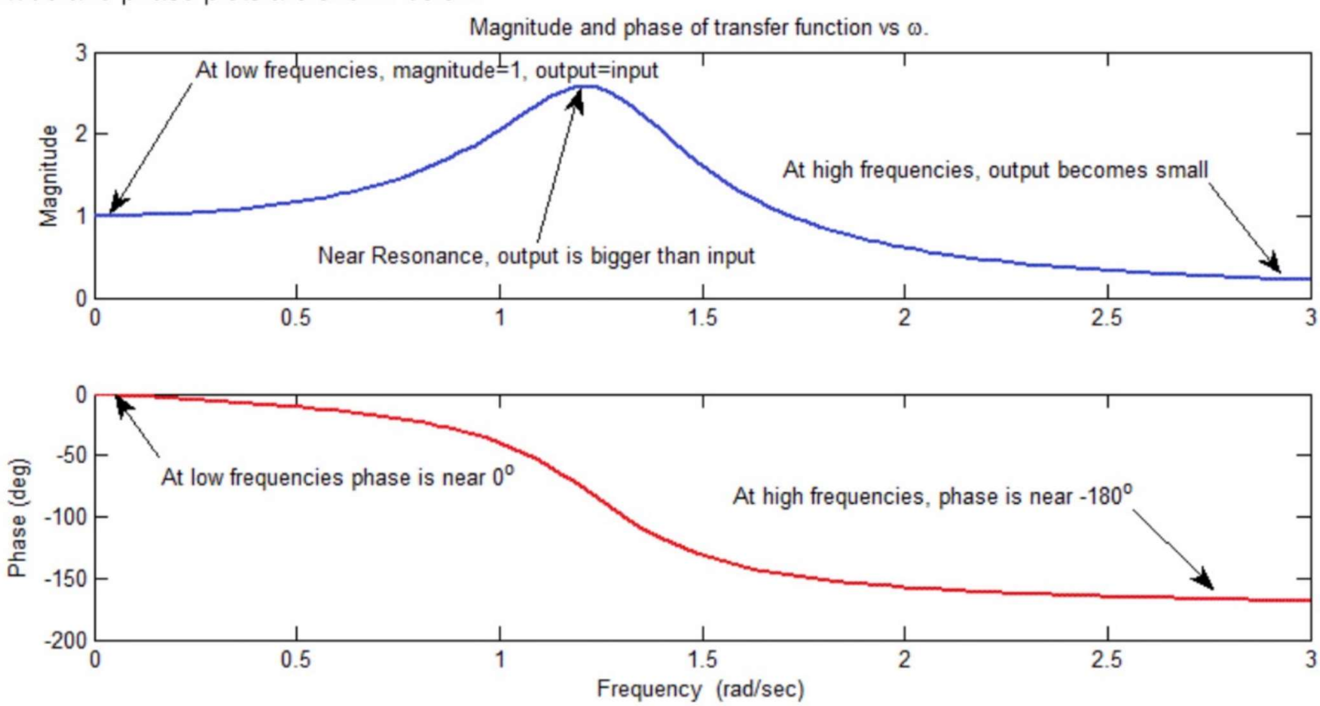




*Animation by Ames Bielenberg*

The transfer function of the system is given by (with m=1, b=0.5, k=1.6, u=input to system, y=output (the position of the mass):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{ms^2 + bs + k} = \frac{1.6}{s^2 + 0.5s + 1.6}$$

The magnitude and phase plots are shown below.




The input is a sinusoidal function whose frequency increases with time. You can see by the animation that at low frequencies (and low times) the input and output are equal in magnitude, and in phase (after the initial startup transient dies out). This is shown by a magnitude of one and a phase of zero on the plots of magnitude and phase of H(*j*ω). At intermediate frequencies (and times) the system is somewhat resonant, and the output actually gets larger than the input (but there is a growing phase lag, i.e., negative phase). As frequency increases further, the output decreases; again, you can see this both in the animation and in the magnitude plot. The outline of the peaks of the output plot is similar to the magnitude plot above. The phase is not as obvious, but it obviously starts at 0° and then decreases to -180° (you may need to zoom in to see the phase shift). At high frequencies (phase near -180°) the two waveforms are completely out of phase; when one is at a maximum, the other is at a minimum.

Now, let's understand the relationship between the shape of these plots and the transfer function.

Consider that we usually look at transfer functions as being made up of 3 things:
1. gain, K
2. zeros
3. poles

We'll start by looking at the impact of each of these induvidual terms on the Bode plot.

bode_introduction.m

vary K, z, p and look at how the Bode plot changes

→ ± K, ↑ mag
→ ± z, ↑ mag
→ ± p, ↑ mag

incrament value and run section for each

then look at the combined bode $G(s) = \frac{K(s+z)}{(s+p)}$.

Note the use of the logarithmic scale for magnitude
→ this allows us to add and subtract mag contributions like we do phase contributions

In general we know we can use tools like Matlab to draw these plots, but it still useful to know how to sketch an approximation

Just like w/ RL, this allows us to build intuition about the impact of adding a controller, <u>and</u> it lets us estimate the form of a transfer function for a system from a frequency sweep