Great, we can now make these plots!

But what do we do with them?

Let's see how we might use them to design a controller

But first, we need to understand what our design requirements look like in frequency space

So let's learn some terms sometimes called "safety margins"

Gain → how much a signal is scaled

Margin → how much extra you have - safety net

→ how much you can increase gain before your system goes unstable

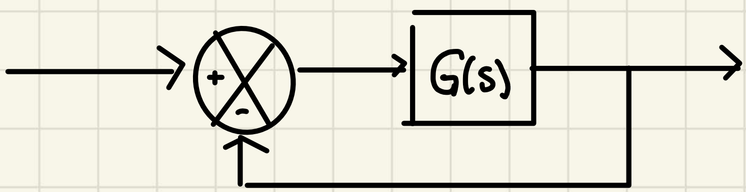
In real life, we need margins to ensure stable behavior when real world conditions vary

→ helps us account for uncertainty

So what makes a system unstable?

Or, more specifically:
what properties of an open loop system will make a closed loop system unstable?

We have



The CLTF is $\dfrac{G(s)}{1+G(s)}$

pole in RHP

but, this corresponds to a situation in which the response grows without bound meaning

$$\frac{G(s)}{1+G(s)} = \infty \quad \text{which happens when} \quad 1+G(s)=0$$

remember this?

$$\begin{cases} \text{gain: } |G(s)| = 1 \quad \text{or } 0\,dB \\ \text{phase: } \angle G(s) = -180 \end{cases}$$

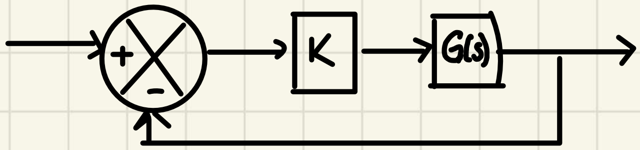
or $G(s) = -1$

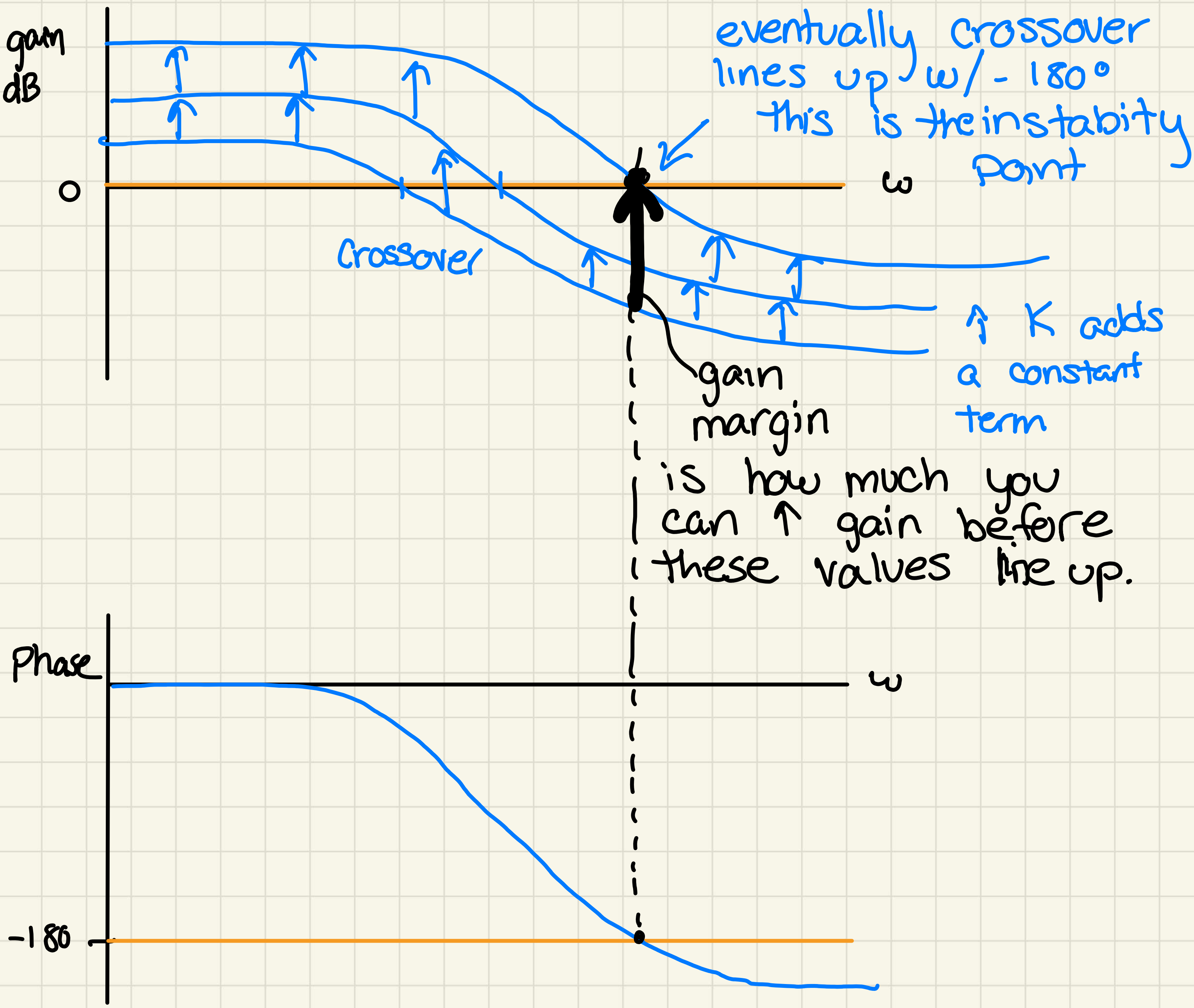
The gain margin and phase margin tell us how far we are from this point

So now, we have a new design criteria we can consider.

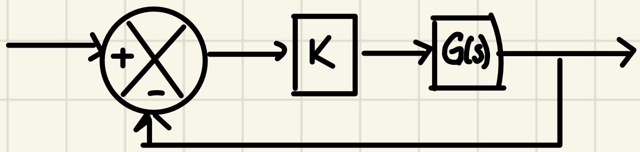
Instead of just saying "the system must be stable" we can set targets or evaluate the gain and phase margin to help us determine how robust we are to uncertainty

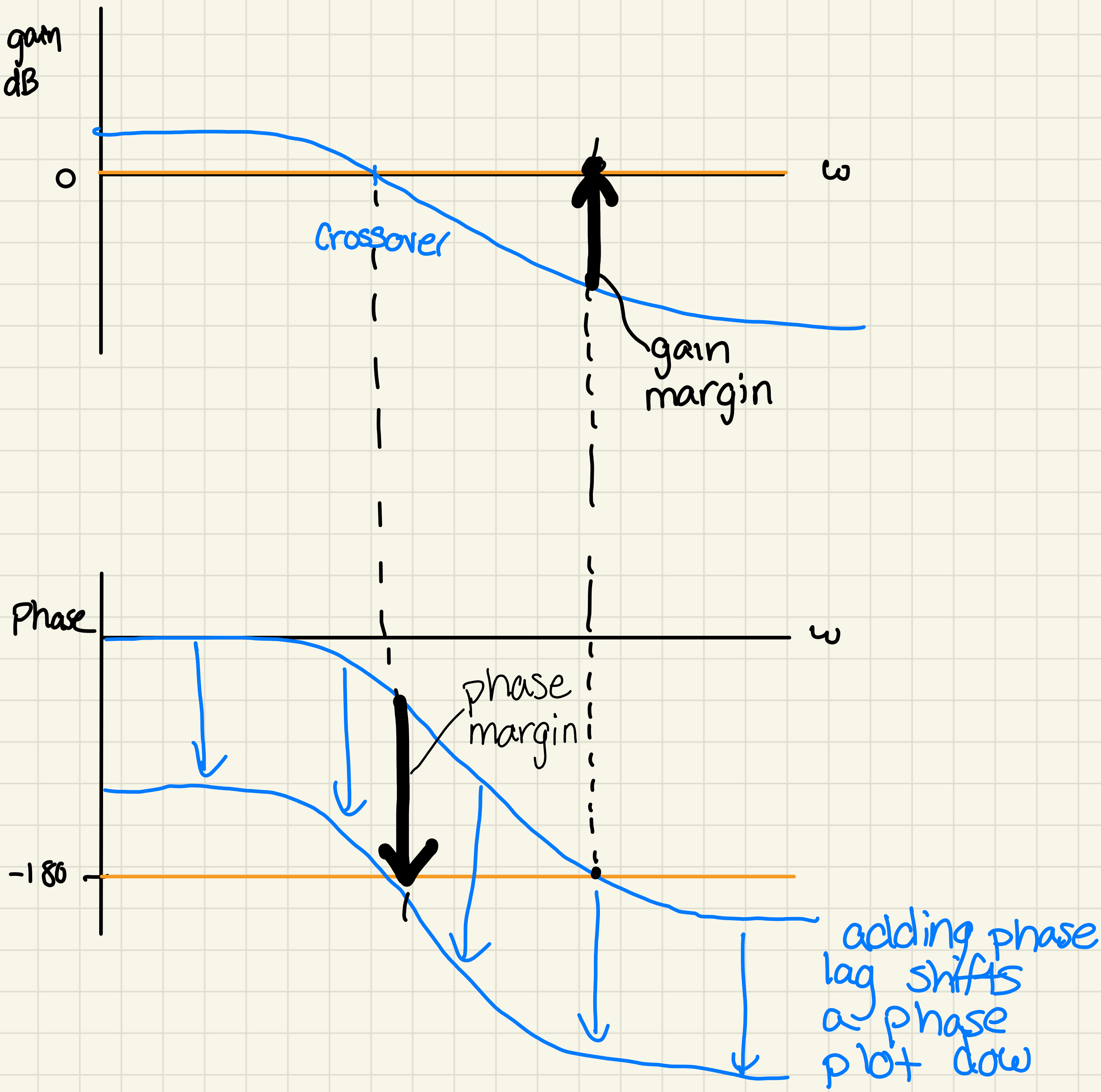
Now let's see how we determine the gain and phase margin from our bode plot





Gain Margin is how much gain there is at the frequency where phase = -180

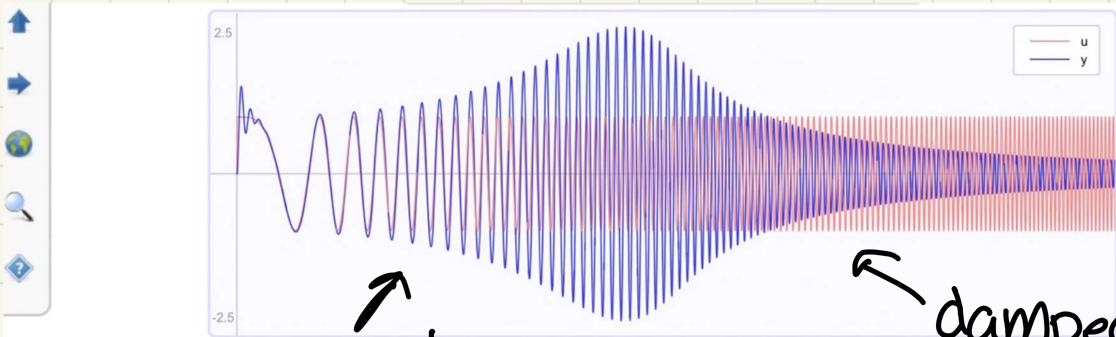




Phase Margin tells us, for the crossover freq, how much phase lag will make the system unstable

Now, these new gain and phase margins are related to all that other stuff we care about too

Remember this?



undamped

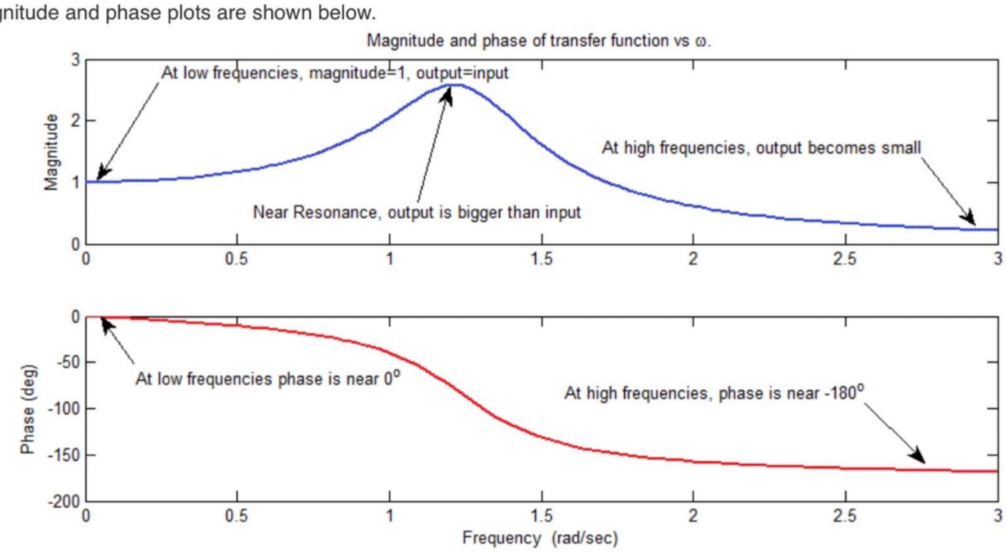
damped

Animation by Ames Bielenberg

The transfer function of the system is given by (with m=1, b=0.5, k=1.6, u=input to system, y=output (the position of the mass):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{ms^2 + bs + k} = \frac{1.6}{s^2 + 0.5s + 1.6}$$

The magnitude and phase plots are shown below.



As the phase changes, so does the damping behavior

For a second order system, the phase margin and damping ratio (%OS) are related

Recall that the phase margin is evaluated at the frequency where

$$|G(j\omega)| = 1 = \frac{\omega_n}{\|-\omega^2 + j2\zeta\omega_n\omega\|} = 1$$

$$\omega_1 = \omega_n\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}$$

the phase at ω_1 is

$$\angle G(j\omega) = -90 - \tan^{-1}\left(\frac{\omega_1}{2\zeta\omega_n}\right) = -90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}{2\zeta}\right)$$

the phase margin at this point is the difference between that angle and $-180°$, so

$$\phi_M = 90 - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}{2\zeta}\right)$$

$$\phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right)$$

In other words → we can find the phase margin needed to achieve a particular ζ / %OS and vice versa

Note that we can also relate the peak of the magnitude response to the damping ratio

$$M = |G(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

If we square this, differentiate wrt ω^2, and set the derivative equal to 0, we can find the peak value (M_p)

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

occuring at $\omega_p = \omega_n\sqrt{1-2\zeta^2}$

and we can relate damping ratio to peak values

The speed of the time response can also be related to the frequency response

Recall that for a real pole, the gain at $\omega = \omega_0$ $= -3dB$

case 2: $\omega = \omega_0$

gain $= -20\log(\sqrt{2}) = -3dB$

phase $= \operatorname{atan}(-1) = -45°$

↳ slope changes by 45° per decade

This value is used as a reference point we call bandwidth

Bandwidth is the frequency ω_{BW} at which the freq. response curve is down 3dB from its value at 0 frequency

Skipping the derivations

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

and $\omega_n = \dfrac{4}{T_s \zeta} = \dfrac{\pi}{T_p(\sqrt{1-\zeta^2})}$

so

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$= \frac{\pi}{T_p\sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Graphically

Quick Example:

Find the closed loop bandwith required for 20% overshoot and a settling time of 2 s

$$20 = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$$

$$\zeta = \frac{-\ln(20/100)}{\sqrt{\pi^2 + \ln^2(20/100)}} = .4559$$

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

plug in $T_s = 2$, $\zeta = .4559$

$\omega_{BW} = 5.79$ rad/s