3.2) Root Locus Method

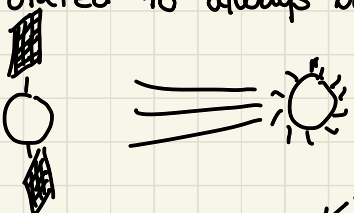
A root locus is a graphical presentation of how the closed loop poles move as a system parameter is varied

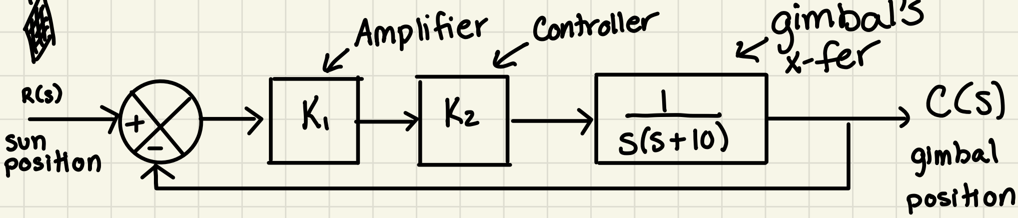
Can be used to determine how the system's performance parameters change as something (controller gain, for example) is varied

You can use this method to visually impose design criteria to derimine regions of acceptable root placement

Let's consider a simple motivating example:

A set of solar panels are gimbaled on a satellite and a control system is put in place so that the solar panel can be rotated to always be facing the sun. The system takes in the position of the sun and outputs the position of the solar panel. The control system is shown:

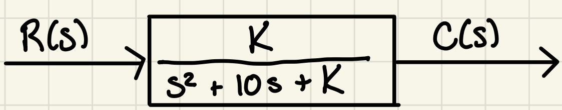




Note that I've oversimplified a bit here....

Let's combine our two gains into a single gain K where $K = K_1 K_2$

Now we can find our closed loop transfer function

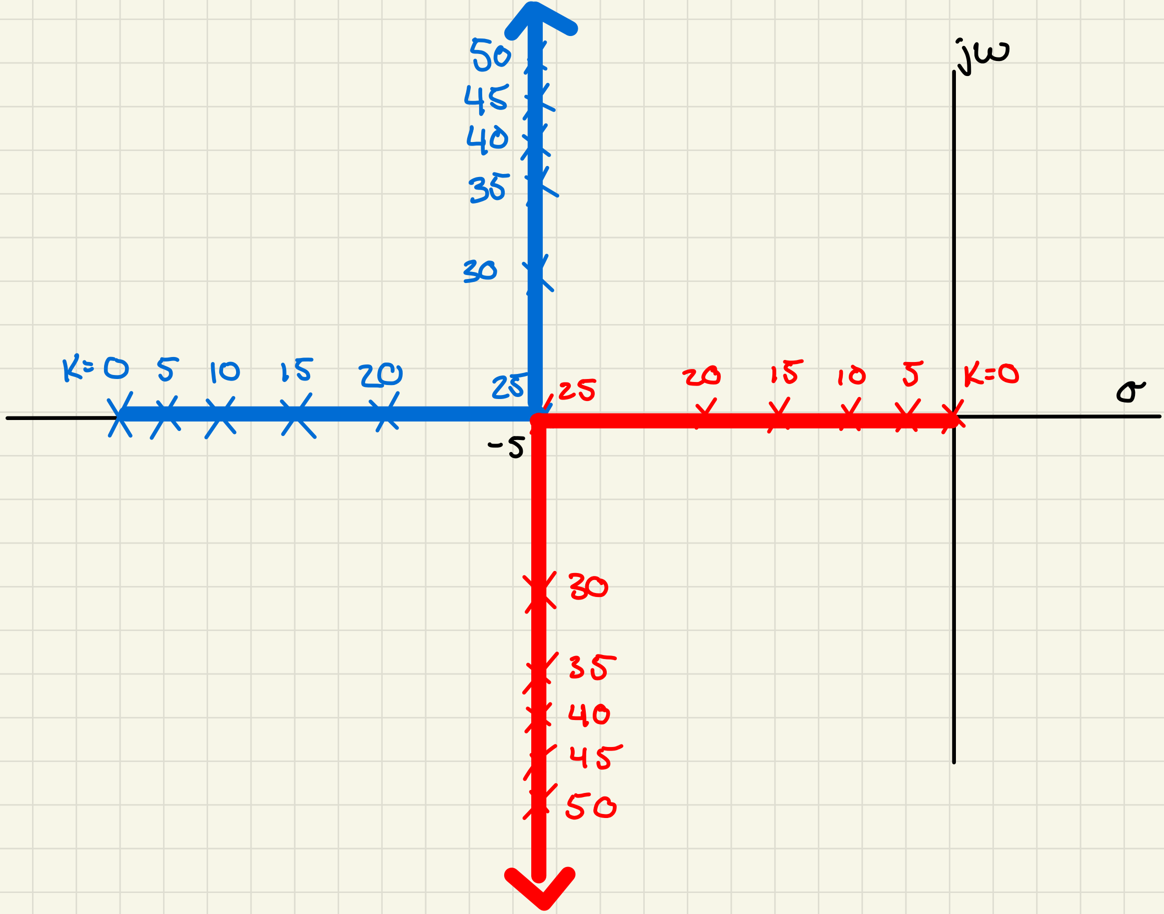


$$R(s) \longrightarrow \boxed{\dfrac{K}{s^2 + 10s + K}} \longrightarrow C(s)$$

Starting with $K=0$, let's examine how the pole location varies as K is increased

K	pole 1	pole 2
0	-10	0
5	-9.47	$-.53$
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5+2.24j$	$-5-2.24j$
35	$-5+3.16j$	$-5-3.16j$
40	$-5+3.87j$	$-5-3.87j$
45	$-5+4.47j$	$-5-4.47j$
50	$-5+5j$	$-5-5j$

Now let's plot them



What does this tell us?

- for all values of $K>0$, the system will be stable
- the system is critically damped at $K=25$
- for all $K \geq 25$, the settling time will remain constant
- As gain increases, damping ratio decreases and percent overshoot increases
- The ω_d increases w/ gain, thus reducing peak time

These are all valuable insight to us as designers.

Just by looking at this, we can quickly narrow our acceptable range of K

Now, you are probably silently protesting again:

1) But that was a pain to create!
2) But Matlab can do it for me!
3) But I could have found K to meet requirements from my handy second order system equations!
4) Any others?

The root locus is the locus of the roots of the characteristic equation when K is varied from $0 \to \infty$

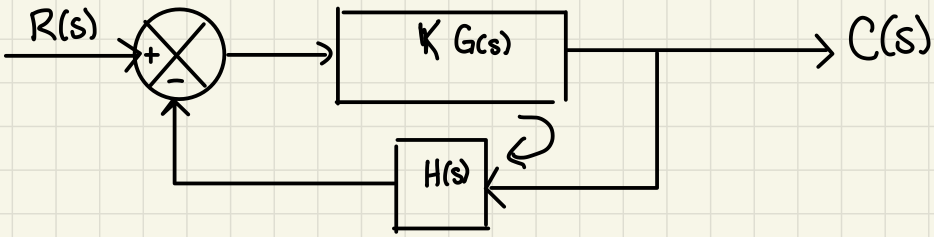
We are now going to learn how to create these much more quickly (and in Matlab) and how they can be useful for higher order systems.

Although you will do this in Matlab after this class, understanding how they are created and why they behave as they do will help build understanding and intuition - important qualities in an engineer

Now you might also say, "but what about the zeros?" Well, they are going to play a role, too! Let's get to it!

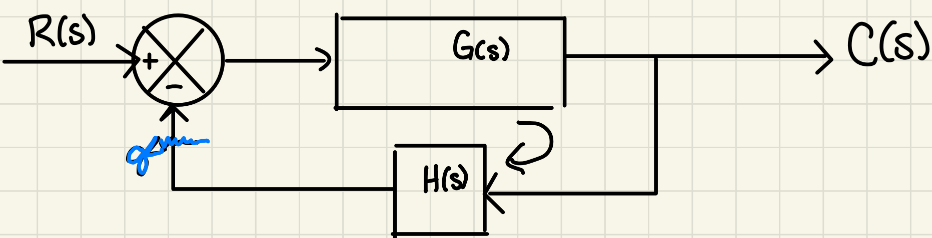
How to make a sketch of the root locus:

Let's consider a general feedback control system of the form:



with the closed loop transfer function $T(s) = \dfrac{KG(s)}{\underbrace{1 + KG(s)H(s)}_{\text{characteristic eqn}}}$

We covered this briefly during our unit on block diagrams, but let's revisit an important definition



The open loop transfer function (OLTF) is the product of everything in the loop

Basically, what you would get if you "opened" the loop

So the OLTF $= G(s)H(s)$

we have closed loop transfer function $T(s) = \frac{KG(s)}{\underbrace{1 + KG(s)H(s)}_{\text{characteristic eqn}}}$

Let's check out the characterisitic eqn

$$1 + KG(s)H(s) = 0$$

which we will represent as $1 + K\frac{N_{OLTF}(s)}{D_{OLTF}(s)} = 0$

or $1 + \frac{KN(s)}{D(s)} = 0$

Then, we will rearrange:

$$D(s) + KN(s) = 0$$

Now recall that root locus involves a sweep of K from $0 \to \infty$, so let's look at our boundary cases

Case 1: $K = 0$ $\therefore$ $D(s) = 0 \Rightarrow$ the closed loop poles are equal to the open loop poles

Case 2: $K = \infty$ rewrite $\underbrace{\frac{1}{K}}_{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$

the closed loop poles are equal to the open loop zeros

In other words, the root locus will <u>START</u> at the open loop poles and <u>END</u> at the open loop zeros

Up until now, our methods have primarily focused on the CLTF. But Root Locus, along with a few that are coming, work from the OLTF

The next question is how these points are connected. For this we will again consider the characteristic equation

For it to be 0, $KG(s)H(s) = 1$ or in polar form
$1\angle(2n+1)180°$
for integer values of n

Lets break this into 2 parts

$$|KG(s)H(s)| = 1 \Leftarrow \text{magnitude}$$

$$\angle KG(s)H(s)| = (2k+1)180° \Leftarrow \text{angle}$$

what this says is that if a value s is substituted into $KG(s)H(s)$, a complex number results.

If the angle of that complex number is an odd multiple of 180°, there exists a value of K for which s is a system pole.

AND That value of K is $K = \dfrac{1}{|G(s)||H(s)|}$

These are called the angle and magnitude criteria, and every point on the root locus must satisfy it

BUT - What on earth is the angle of $KG(s)H(s)$?
(and how do we find it?)

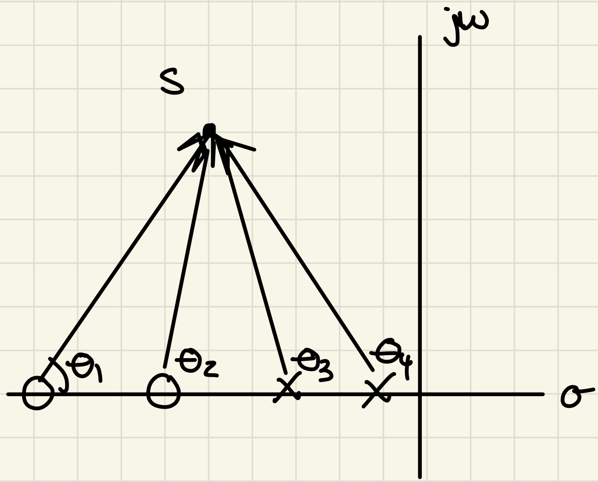
Skipping the derivation (I'll post it if you really want to see it)

the angle of $F(s) = \frac{\prod (s+z_i)}{\prod (s+p_j)} = \frac{\text{product of zeros}}{\text{product of poles}}$

can be calculated as

$$\theta = \Sigma \text{ zero angles} - \Sigma \text{ pole angles}$$

or, perhaps more easily understoood visually



$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = \theta$$

this is the angle that must be an odd multiple of 180°

We'll come back to this but the key point is that this will drive the shape of the path between the poles and zeros, as we will see as we move forward