orbit equation

$$r(f) = \frac{p}{1 + e\cos f}$$

$r(f)$ forms a conic section as the true anomaly f is varied

(a) $p=2,\ e=0$
(b) $p=6,\ e=0.2$
(c) $p=6,\ e=0.6$
(d) $p=3,\ e=1$
(e) $p=2,\ e=2$

See MATLAB plots for (a-e)

v_s = circular orbit speed at surface of planet of radius r_s & r_p (periapse radius of hyperbolic orbit abt the planet

(a) $\mu = r_s v_s^2$

specific nrg $\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$ for circular orbit, $r = r_s = a$

vis-viva $v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

$$v_s = \sqrt{\mu\left(\frac{2}{r_s} - \frac{1}{r_s}\right)}$$

$$v_s^2 = \mu\left(\frac{1}{r_s}\right)$$

$$r_s v_s^2 = \mu \rightarrow v_s = \sqrt{\frac{\mu}{r_s}}$$

(b) $e = 1 + \psi$ $\qquad \psi = e - 1$

$$\psi = \left[\frac{v_\infty}{v_s}\right]^2\left[\frac{r_p}{r_s}\right]$$

for hyperbolic orbit, $a = \infty$, $e = 1$

at $r = \infty$,

↳ using vis viva, $v_\infty = \sqrt{\mu\left(\cancel{\frac{2}{r}} - \frac{1}{a}\right)} = \sqrt{-\frac{\mu}{a}}$

$$v_\infty^2 = -\frac{\mu}{a} \quad (a<0)$$

$$\frac{v_\infty}{v_s} = \frac{\sqrt{-\mu/a}}{\sqrt{\mu/r_s}}$$

$$\left[\frac{v_\infty}{v_s}\right]^2 = \frac{-\mu/a}{\mu/r_s} = -\frac{r_s}{a}$$

$$\left[\frac{v_\infty}{v_s}\right]^2\left[\frac{r_p}{r_s}\right] = -\frac{r_s}{a}\frac{r_p}{r_s} = -\frac{r_p}{a} = -\frac{a(1-e)}{a} = -(1-e) = e-1$$

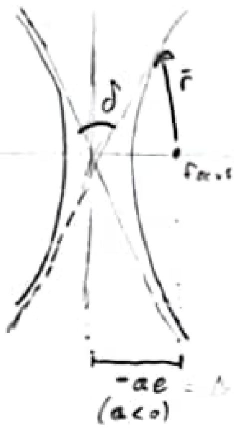
↳ $r_p = a(1-e)$

$$\left[\frac{v_\infty}{v_s}\right]^2\left[\frac{r_p}{r_s}\right] = e - 1$$

$$\psi = e - 1$$

$$\boxed{e = 1 + \psi}$$

(c) $\sin\left(\frac{\delta}{2}\right) = \frac{1}{1+\psi}$ $\qquad \psi = \left(\frac{v_\infty}{v_s}\right)^2 \left(\frac{r_p}{r_s}\right)$



~~sin ... 1/e~~

$\frac{\delta}{2} = \theta_\infty - \frac{\pi}{2}$

$\sin\left(\frac{\delta}{2}\right) = \frac{1}{e} = \frac{1}{1+\psi}$

$e = 1+\psi$

$\boxed{\sin\frac{\delta}{2} = \frac{1}{1+\psi}}$

(d) $\frac{\Delta}{r_s} = \frac{r_p}{r_s}\sqrt{1+2/\psi}$

$e = 1+\psi$

$\psi = e-1$

$\sqrt{1+\frac{2}{\psi}} = \sqrt{\frac{e-1}{e-1}+\frac{2}{e-1}} = \sqrt{\frac{e+1}{e-1}}$

$\frac{r_p}{r_s} = \psi \Big/ \left(\frac{v_\infty}{v_s}\right)^2 = \frac{e-1}{(v_\infty/v_s)^2}$

$\frac{\Delta^2}{r_s^2} = \frac{(e-1)^2}{(v_\infty/v_s)^4}\,\frac{e+1}{e-1}$

~~$\sin(x+\frac{\pi}{2}) = \cos(x)$ $\quad \sin(x) = -\sin(-x)$~~

~~$\cos(x-\frac{\pi}{2}) = \sin(x)$ $\quad \cos x = \cos(-x)$~~

★ $\boxed{\sin(\frac{\pi}{2}-\theta) = \cos\theta}$

$\Delta = -ae\sin\left(\frac{\pi}{2}-\frac{\delta}{2}\right)$

$\frac{\Delta}{-ae} = \sin\left(\frac{\pi}{2}-\frac{\delta}{2}\right)$

$\frac{\Delta}{-ae} = \cos\left(\frac{\delta}{2}\right)$

$\Delta = -ae\cos\left(\frac{\delta}{2}\right)$

$\Delta^2 = a^2e^2\cos^2\left(\frac{\delta}{2}\right)$ $\qquad \sin\frac{\delta}{2} = \frac{1}{e}$

$= a^2e^2\left(1-\sin^2\left(\frac{\delta}{2}\right)\right)$

$= a^2e^2\left(1-\frac{1}{e^2}\right)$

$= a^2e^2\left(\frac{e^2}{e^2}-\frac{1}{e^2}\right)$

$= a^2e^2\left(\frac{e^2-1}{e^2}\right)$

$\Delta^2 = a^2(e^2-1) = -a^2(1-e)(1+e)$ $\qquad r_p^2 = a^2(1-e)(1-e)$

~~$= a^2(1+e)(e-1)$~~ $\qquad r_p = a(1-e)$, $\quad 1-e = \frac{r_p}{a}$ $\qquad \frac{\psi}{\psi}+\frac{2}{\psi} = 1+\frac{2}{\psi}$

$= -a^2\left(\frac{r_p}{a}\right)(1+e)$

$= -a r_p(1+e) = -a r_p^2(1+e)\,\frac{1}{a(1-e)}$

$= -r_p^2\left(\frac{1+e}{1-e}\right) = r_p^2\left(\frac{e+1}{e-1}\right) = r_p^2\left(\frac{\psi+2}{\psi}\right) = \boxed{r_p^2\left(1+\frac{2}{\psi}\right)}$

(3) eqn of motion - 2 body → see Question 3 of ae6353_hw1.mlx

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r}$$

t Span = 30000 s

r_{earth} = 6378 km

r_0 = [3858.213; -5798.143; 14.693] km

v_0 = [-0.563; -0.542; 7.497] km/s

convert to 1st order:

$$\ddot{\bar{r}} = \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} = -\frac{\mu}{r^3}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$\begin{aligned}
x_1 &= r_x & \dot{x}_1 &= x_2 \\
x_1 &= \dot{r}_x & \dot{x}_2 &= -\mu/r^3\, x_1 \\
x_3 &= r_y & \dot{x}_3 &= x_4 \\
x_4 &= \dot{r}_y & \dot{x}_4 &= -\mu/r^3\, x_3 \\
x_5 &= r_z & \dot{x}_5 &= x_6 \\
x_6 &= \dot{r}_z & \dot{x}_6 &= -\frac{\mu}{r^3} x_5
\end{aligned}$$

$\overset{aka}{\Longrightarrow}$

$$\begin{aligned}
\dot{x} &= f(t,x) \\
\bar{r} &= (r_x\ r_y\ r_z)^T = (x_1\ x_2\ x_3)^T \\
\bar{v} &= (v_x\ v_y\ v_z)^T = (x_4\ x_5\ x_6)^T \\
\bar{a} &= -\frac{\mu}{r^3}\bar{r} \\
\dot{x} &= (\bar{v}\ \ \bar{a})^T
\end{aligned}$$

(4) n-body

$$m_i\ddot{\bar{r}}_i = G\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}^3}\bar{r}_{ij}$$

$\bar{r}_{ij} = -\bar{r}_{ji}$

$\bar{r}_{ij} = \bar{r}_j - \bar{r}_i$

$\bar{r}_i + \bar{r}_{ij} = \bar{r}_j$

$\bar{r}_{ij} = \bar{r}_j - \bar{r}_i$

if n = 4 , $m = m_1 = m_2 = m_3 = m_4$

$$m_1\ddot{\bar{r}}_1 = G\left[\frac{m_1 m_2}{r_{12}^3}\bar{r}_{12} + \frac{m_1 m_3}{r_{13}^3}\bar{r}_{13} + \frac{m_1 m_4}{r_{14}^3}\bar{r}_{14}\right]$$

$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1$

$\bar{r}_{21} = -\bar{r}_{12}$

⋮

$$m_2\ddot{\bar{r}}_2 = G\left[\frac{m_2 m_1}{r_{21}^3}\bar{r}_{21} + \frac{m_2 m_3}{r_{23}^3}\bar{r}_{23} + \frac{m_2 m_4}{r_{24}^3}\bar{r}_{24}\right]$$

$$m_3\ddot{\bar{r}}_3 = G\left[\frac{m_3 m_1}{r_{31}^3}\bar{r}_{31} + \frac{m_3 m_2}{r_{32}^3}\bar{r}_{32} + \frac{m_3 m_4}{r_{34}^3}\bar{r}_{34}\right] \rightarrow$$

$$m_4\ddot{\bar{r}}_4 = G\left[\frac{m_4 m_1}{r_{41}^3}\bar{r}_{41} + \frac{m_4 m_2}{r_{42}^3}\bar{r}_{42} + \frac{m_4 m_3}{r_{43}^3}\bar{r}_{43}\right]$$

Writing a general Nbody solver in MATLAB rather than simplifying (aka factoring out mass) for future more general problems.

See Question 4 of mlx file.

nBody Model (configure for ode45)

$$\bar{x} = \begin{bmatrix} \bar{r}_1 \\ \bar{v}_1 \\ \vdots \\ \bar{r}_n \\ \bar{v}_n \end{bmatrix}_{6n\times 1} \qquad \dot{\bar{x}} = \begin{bmatrix} \bar{v}_1 \\ \bar{a}_1 \\ \vdots \\ \bar{v}_n \\ \bar{a}_n \end{bmatrix}_{6n\times 1}$$

$$\bar{r}_1 = [x_1 \quad x_2 \quad x_3]^T$$
$$\bar{r}_2 = [x_7 \quad x_8 \quad x_9]^T$$
$$\bar{v}_1 = [x_4 \quad x_5 \quad x_6]^T$$
$$v_2 = [x_{10} \quad x_{11} \quad x_{12}]^T$$
$$\vdots$$

center of mass: $\bar{r}_{cm} = \dfrac{\sum_i m_i \bar{R}_i}{\sum_i m_i}$

$$\cancel{\begin{bmatrix} r_{cmx1} & r_{cmy1} & r_{cmz1} \\ r_{cmx2} & r_{cmy2} & r_{cmz2} \\ & \vdots & \end{bmatrix} \sum_i m_i}$$

$$\bar{r}_{cm} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + m_3\bar{r}_3 + m_4\bar{r}_4}{m_1 + m_2 + m_3 + m_4}$$

$$h = r \times v = \frac{H}{m}$$
$$H = m(r \times v)$$

total angular momentum: $\sum_{i=1}^{n} m_i \bar{r}_i \times \dot{\bar{r}}_i = \bar{c}_3$ $\qquad h_{tot} = \sum_{i=1}^{n} (\bar{r}_i \times \bar{v}_i)\, m_i$

$$m_1\bar{r}_1 \times \dot{\bar{r}}_1 + m_2\bar{r}_2 \times \dot{\bar{r}}_2 + m_3\bar{r}_3 \times \dot{\bar{r}}_3 + m_4\bar{r}_4 \times \dot{\bar{r}}_4 = \bar{c}_3$$

total energy: $\cancel{T + V = \bar{c}_4}$ $\qquad \varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{E}{m}$

$$\cancel{\frac{1}{2}\sum_{i=1}^{n} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}}} \qquad E_{tot} = \sum_{i=1}^{n} m_i \left(\frac{1}{2}v_i^2 - \frac{\mu}{r_i}\right)$$

vectorization trick: $A \times B$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \times \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} A_2B_3 - A_3B_2 \\ -A_1B_3 + A_3B_1 \\ A_1B_2 - A_2B_1 \end{bmatrix}$$

$$\begin{bmatrix} A_{x1} & A_{y1} & A_{z1} \\ A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3} \\ & \vdots & \end{bmatrix} \times \begin{bmatrix} B_{x1} & B_{y1} & B_{z1} \\ B_{x2} & B_{y2} & B_{z2} \\ & \vdots & \end{bmatrix} = \begin{bmatrix} A_1 \times B_1 \\ A_2 \times B_2 \\ \vdots \end{bmatrix}$$

nvm can just use cross()

(4b) Total angular momentum rose from the (4a) configuration b/c there is more net initial velocity being injected into the system. Center of mass is also moving now, compared to (4a) b/c the initial conditions are not symmetric anymore about the center of mass.

(4c) Propagating the system for longer allows for "oscillations" due to numerical round off errors / instabilities to be visualized for total energy. The total angular momentum also decreases.