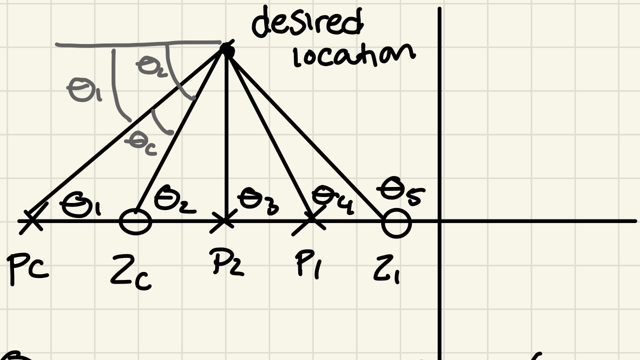
Lead Compensator and the Root Locus

In a lead compensator, the pole is further from the imaginary axis than the zero, and not necessarily right by the origin

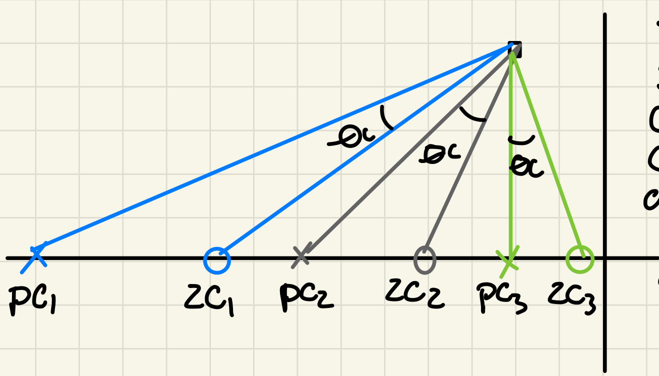


$$\underbrace{\theta_2 - \theta_1}_{\theta_c} - \theta_3 - \theta_4 + \theta_5 = (2k+1)180^\circ$$ ← by angle condition

$\theta_c = \theta_2 - \theta_1$ and is determined such that the desired pole location falls on the RL

Thus, the lead compensator is able to affect the transient behavior

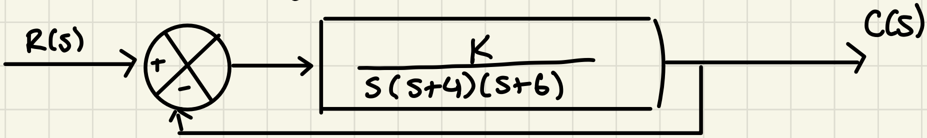
However, there are lots of pole-zero comensator combinations that might satisfy the angle condition



θ_c is the same for all 3 compensator designs! (and there are ∞ more)

so how to pick?

Let's use an example to help us understand lead compensator design

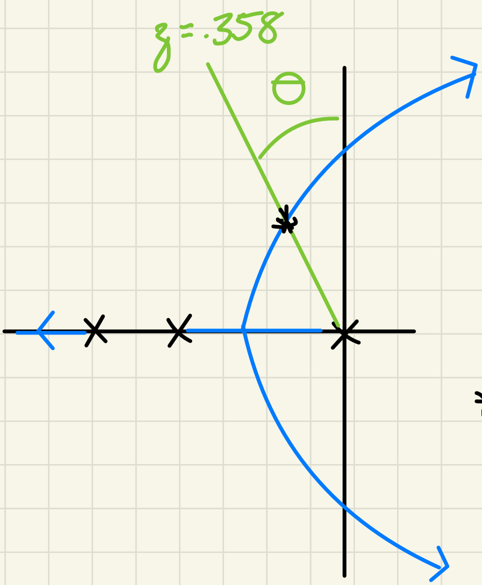


Goal: Design 3 different lead compensators that will reduce the settling time by a factor of 2 and maintain a 30% OS. We already have 0 $SSE_{(step)}$

First, let's use the design criteria to determine where we want our dominant poles

$$30\% \Rightarrow \zeta = .358 \qquad \text{from } \%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

for the uncompensated system, we can search on the RL for the intersection with $\zeta = .358$



$\theta = \sin^{-1}(.358) = 21°$

Using matlab, or by finding the s at which the ζ line intesects the root locus, we find

$* \ s = -1.007 + 2.627j$
$K = 63.21$
at the intersection

using the real part of $*$

$$T_s = \frac{4}{1.007} = 3.972$$

Our goal is reduce T_s by twofold while maintaining 30% OS (in other words we want to stay on the $\zeta = .358$ line, but shift the RL up along the line)

$$T_{s\,goal} = \frac{3.972}{2} = 1.986 \text{ s}$$

so, the real part of the desired pole location

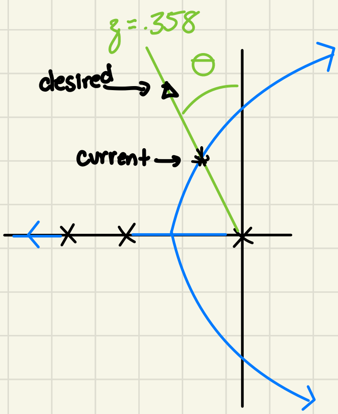
is $-\zeta\omega_n = \frac{-4}{T_s} = -2.014$

because we wish to stay on the $\zeta = .358$ line, which is 111° from the x-axis, the imaginary part of the desired pole location is:

$$\omega_d = -2.014\tan(111°) = 5.252$$

So we desire to design a compensator such that the root locus includes

$$s_{1,2} = -2.014 \pm 5.252j$$



So let's design a lead compensator

The first step is to arbitrarily choose a zero for the compensator

$\theta_{p0} = 111°$

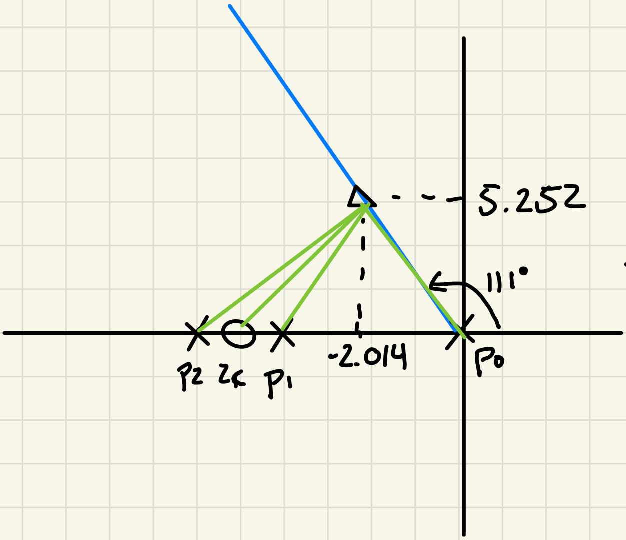
let's start with a zero at -5

$\theta_{p1} = \tan^{-1}\left(\frac{5.252}{4-2.014}\right)$

$= 69.2$

$\theta_{zc} = 60.38$

$\theta_{p2} = 52.8$

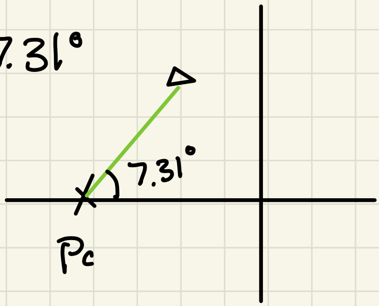


Next, we find the angles from all the existing poles and this zero to the test point

$$\theta_{zc} - \theta_{p0} - \theta_{p1} - \theta_{p2} = -172.62$$

So now we can find how much angle our new pole needs to add to achieve an odd multiple of 180°

$$180° - 172.62° = 7.31°$$



Now we can find the location of our pole

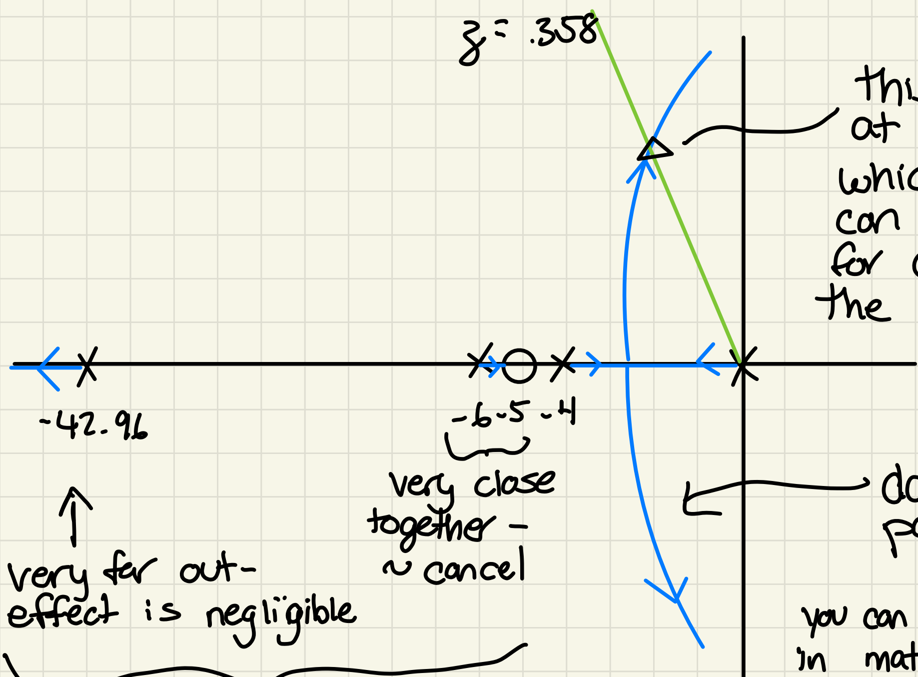
$$\frac{5.252}{p_c - 2.014} = \tan(7.31^\circ) \Rightarrow p_c = 42.96$$

The lead lag compensator is $\dfrac{K(s+5)}{(s+42.96)}$

Our OLTF for our compensated system is now

$$G_c G_p = \frac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$$

and all that's left is to find the value of K such that the closed loop poles lie on $\zeta = .358$ line



this occurs at K = 1423, which we can either solve for or get the help of matlab

dominant poles

$\underbrace{-6\,-5\,-4}$ very close together - ~ cancel

very far out - effect is negligible

$\underbrace{\text{very far out - effect is negligible} \quad \text{very close together - \~{} cancel}}$

second order approx is ok

you can try all this in matlab

leadcompensator-design example1.m

Now, the question you are probably asking is:

"but you picked that zero randomly! Was it a good zero? what if you picked a different zero? What if that zero was between (or on top of) the dominant poles? What if it completely changed the shape of the RL?!?"

Let's explore this by choosing some different zero locations and repeating the process.

lead compensator.m

We we will try

$z_c = 5$, $z_c = 4$, $z_c = 2$
$P_c = 42.96$, $P_c = 20$, $P_c = 8.97$

Note that all 3 systems have the same dominant poles (though the gain at those poles is different.

Dominant Poles: $-2.014 \pm 5.252j$ (by design)

for $z_c = 5$, $K = 1423$	2^{nd} order approx will be
$z_c = 4$, $K = 698.1$	$\zeta = .358$, $\omega_n = 5.625$, %OS = 30
$z_c = 2$, $K = 345.6$	$T_s = 1.986$, $T_p = 1.196$ for all

However, looking at the actual response we can see that this approx is valid for $z_c = 5, 4$ but not valid when $z_c = 2$. We can also see this on RL, b/c we have no pole-zero cancellation

none have steady state error in response to a step.

However, the three systems will have very different SSE in response to a ramp

$$K_v = \lim_{s \to 0} sG(s) = \frac{K(z_c)}{(p_c)(4)(6)} \quad , \quad e_\infty = \frac{1}{K_v}$$

	$z_c, p_c = 0$	$z_c = 5, p_c = 42.9$	$z_c = 4, p_c = 20.9$	$z_c = 2, p_c = 8.9$
K	63.21	1423	698	345.6
K_v	2.63	6.9	5.791	3.21
e_∞	.380	.145	.173	.321

In this example we saw how to design a lead compensator to improve transient response

Now let's put these together and learn about using lead - lag compensation to improve both transient response and SSE