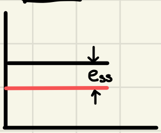
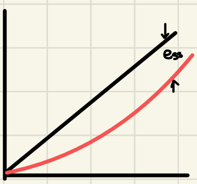
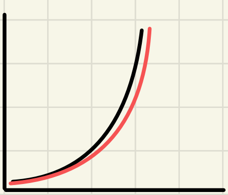
A TIME OUT TO REVISIT SSE IN MORE DETAIL

Steady State Error

At this point, we are going to take a time out from Root Locus, and revisit the concept of steady state error in a bit more detail, thinking about the "integrator"

We previously explored the concept of SS error in response to step inputs → we will now explore this concept in regard to a wider range of inputs.

Let's look at 3 inputs that might be given to a position control system

Name	Form	Physical Meaning	Time Function	$\mathcal{L}$
Step		constant position	1	$\frac{1}{s}$
Ramp		constant velocity	t	$\frac{1}{s^2}$
Parabola		constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

In real systems, SSE can occur as a result of the system configuration and the applied input

Real examples for each input type:

1) STEP: A system with a position control system trying to hold a steady, commanded position

→ a satellite in geostationary orbit
→ position control of an antenna or camera

2) Ramp: A system with a position control system is commanded to hold a constant velocity

→ A system to track a satellite across the sky
→ A ground radar tracking an aircraft moving at a constant velocity

3) Parabola: A system with position control that is holding constant acceleration

→ tracking an accelerating missile or rocket

To know the SSE, we need to know the behavior of the function as $t \to \infty$, or it's final value

So we need to know what $f(\infty)$ is.

The final value theorem tells us that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Only if students want to see it

$$\mathcal{L}\{\dot{f}(t)\} = \int_0^{\infty} \dot{f}(t) e^{-st} dt = sF(s) - f(0)$$

as $s \to 0$

$$\int_0^{\infty} \dot{f}(t) dt = f(\infty) - \cancel{f(0)} = \lim_{s \to 0} sF(s) - \cancel{f(0)}$$

$$f(\infty) = \lim_{s \to 0} sF(s)$$

$\Leftarrow$ only applies to stable systems with no more than 1 pole at origin

What we actually want to know is the error at ∞

if $T(s) = \frac{C(s)}{R(s)} \Rightarrow$ CLTF

$$E(s) = R(s) - C(s) = R(s)[1 - T(s)]$$

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

So let's do a quick example

$$T(s) = \frac{5}{(s^2 + 7s + 10)} \qquad R(s) = \frac{1}{s} \text{ step}$$

find the SSE:

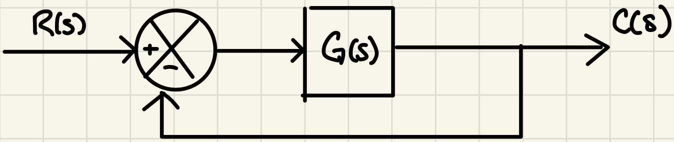
$$E(s) = R(s)[1 - T(s)] = \frac{1}{s}\left[1 - \frac{5}{s^2 + 7s + 10}\right] = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

$$e(\infty) = \lim_{s \to 0} sE(s) = \frac{s}{s}\left[\frac{s^2 + 7s + 5}{s^2 + 7s + 10}\right] = \frac{5}{10} = \frac{1}{2}$$

what is $R(s) = 1$? (impulse)

now $e(\infty) = \lim_{s \to 0} sE(s) = \frac{s(s^2 + 7s + 5)}{s^2 + 7s + 10} = 0$

But often, we are dealing with systems that look like this:



So let's figure out $e(\infty)$ in terms of $G(s)$

$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

so $e(\infty) = \lim_{s \to 0} sE(s) = \boxed{\lim_{s \to \infty} s\left[\frac{R(s)}{1 + G(s)}\right]}$

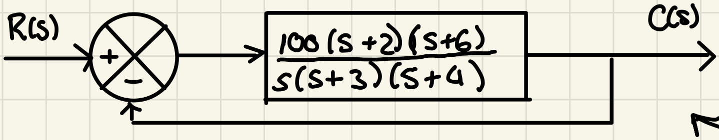
Now, we can look at our input forms

Input [R(s)]	substitute	e(∞) equation
$\frac{1}{s}$ (step)	$\lim_{s \to 0} \frac{s(1/s)}{1 + G(s)}$	$e(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)}$
$\frac{1}{s^2}$ (ramp)	$\lim_{s \to 0} \frac{s(1/s^2)}{1 + G(s)}$	$e(\infty) = \frac{1}{\lim_{s \to 0} sG(s)}$
$\frac{1}{s^3}$ (parabolic)	$\lim_{s \to 0} \frac{s(1/s^3)}{1 + G(s)}$	$e(\infty) = \frac{1}{\lim_{s \to 0} s^2 G(s)}$

Example:

Find the steady state error for the following inputs:

a) $5u(t)$ b) $5tu(t)$ c) $5t^2u(t)$ $u(t)$ = unit step



first we should verify closed loop system is stable - it is!

a) step: $\mathcal{L}(R(s)) = 5/s$

$$e(\infty) = 5e_{step}(\infty) = \frac{5}{1 + \underbrace{\lim_{s\to 0} G(s)}_{=\infty}} = \frac{5}{\infty} = 0$$

b) ramp: $\mathcal{L}(R(s)) = 5/s^2$

$$e(\infty) = 5e_{ramp}(\infty) = \frac{5}{\underbrace{\lim_{s\to 0} sG(s)}_{100}} = \frac{5}{100} \text{ or } \frac{1}{20}$$

c) parabola: $\mathcal{L}(R(s)) = 10/s^3$

$$e_\infty = 10e_{parabola}(\infty) = \frac{10}{\underbrace{\lim_{s\to 0} s^2G(s)}_{0}} = \infty$$

Static Error Constants and System Type

e(∞) equation

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

$$e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)}$$

$$e_{parabola}(\infty) = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

the limit terms of the denomintor in these equations define the character of the SSE.

Therefore, we define these as static error constants

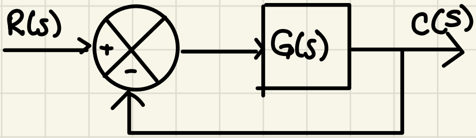
position constant: $K_p = \lim_{s\to 0} G(s)$

velocity constant: $K_v = \lim_{s\to 0} sG(s)$

acceleration constant: $K_a = \lim_{s\to 0} s^2 G(s)$

Don't forget these all were derived for unity feedback with G(s) as the feedforward function

Example: Let's look at 3 unity feedback systems and find their K_p, K_v, and K_a and the associated SSE for a step, ramp, or parabolic input.



$$G_1 = \frac{500(s+2)(s+5)}{(s+8)(s+10)(s+12)}$$

no integrator

$$G_2 = \frac{500(s+2)(s+5)(s+6)}{s(s+8)(s+10)(s+12)}$$

1 integrator

$$G_3 = \frac{500(s+2)(s+5)(s+6)(s+7)}{s^2(s+8)(s+10)(s+12)}$$

2 integrators

For G_1: $K_p = \lim_{s \to 0} G_1(s) = \frac{500(2)(5)}{8(10)(12)} = 5.208$

$K_v = \lim_{s \to 0} sG_1(s) = 0$

$K_a = \lim_{s \to 0} s^2 G_1(s) = 0$

stable response to step but SSE present

step: $e_{step}(\infty) = \frac{1}{1+K_p} = .161$

ramp: $e_{ramp}(\infty) = \frac{1}{K_v} = \infty$

parabola: $e_{parabola}(\infty) = \frac{1}{K_a} = \infty$

∞ SSE response to ramp or parabola

error grows w/ time!

for G_2: $K_p = \lim_{s \to 0} G_2(s) = \infty$

$$K_v = \lim_{s \to 0} s\,G_2(s) = \frac{500(2)(5)(6)}{(8)(10)(12)} = 31.25$$

$$K_a = \lim_{s \to 0} s^2 G_2(s) = 0$$

stable response w/ no error

step: $e(\infty) = \frac{1}{1+K_p} = 0$

ramp: $e(\infty) = \frac{1}{K_v} = .032$ ← stable w/ SSE

parabola: $e(\infty) = \frac{1}{K_a} = \infty$ ← ∞ SSE

for G_3

$K_p = \infty$

$K_v = \infty$

$$K_a = \frac{500(2)(4)(5)(6)(7)}{(8)(10)(12)} = 875$$

step: $e(\infty) = 0$

ramp: $e(\infty) = 0$

$\}$ Stable w/ no SSE

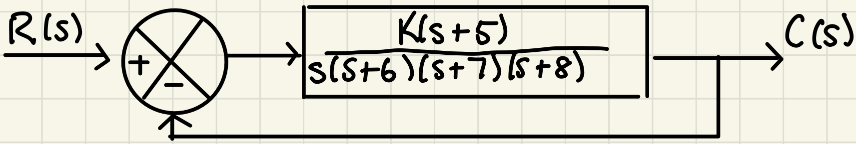
parabola: $e(\infty) = \frac{1}{K_a} = 1.14 \times 10^{-3}$ ← stable w/ SSE

Example takeaway ⇒ the number of integrators in the forward path has a direct impact on the values of K_p, K_v, K_a

The number of integrators in the forward path is assigned as an attribute of a system, called system type

	Step	Ramp	Parabola
Error fxn	$\frac{1}{1+K_p}$	$\frac{1}{K_v}$	$\frac{1}{K_a}$
Type 0 (no integrator)	$K_p = \text{const}$ $e(\infty) = \frac{1}{1+K_p}$	$K_v = 0$ $e(\infty) = \infty$	$K_a = 0$ $e(\infty) = \infty$
Type 1 (1 integrator)	$K_p = \infty$ $e(\infty) = 0$	$K_v = \text{const}$ $e(\infty) = \frac{1}{K_v}$	$K_a = 0$ $e(\infty) = \infty$
Type 2 (2 integrators)	$K_p = \infty$ $e(\infty) = 0$	$K_v = \infty$ $e(\infty) = 0$	$K_a = \text{const}$ $e(\infty) = \frac{1}{K_a}$

Quick Example: Find K so that there is 10% error in the steady state



1) What type of input will produce a SSE?

A: Ramp (Type 1 system)

2) What is the K_v corresponding to 10% SSE?

$$e(\infty) = .1 = \frac{1}{K_v} \Rightarrow K_v = 10$$

3) What should the gain K be?

$$K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K(5)}{(6)(7)(8)}$$

$$\Rightarrow K = 672$$

Why do we care?

Up until now, we have focused on transient response improvements as our main design criteria

But, as we saw in our PID root locus example, SSE is also a design criteria of concen

A PI or PID adds 1 integrator to the system, which eliminates SSE in response to a step, but whether it can do more depends on the dynamics of the plant

$\Rightarrow$ Increases system type by 1

In addition to our transient criteria, we might also get requirements on SSE

Ex: $K_v = 1000$ is the requirement, which means:

- $\Rightarrow$ the system is stable
- $\Rightarrow$ the system is type 1
- $\Rightarrow$ the test input is a ramp
- $\Rightarrow$ the $SSE = 1/K_v$ per unit of input slope

<u>But</u> this tells us nothing about the transient response

Let's learn how to put together everything we know so far to design controllers that meet all our design criteria thus far

- stability
- transient response
- SSE