Rules to sketch a Root Locus By Hand (pg 522)

1) Number of Branches

→ Since branches start at open loop poles and end at open loop zeros, the number of branches (N) is either the number of open loop poles or zeros, whichever is greater.

2) Symmetry

→ The root locus is symmetric about the real axis

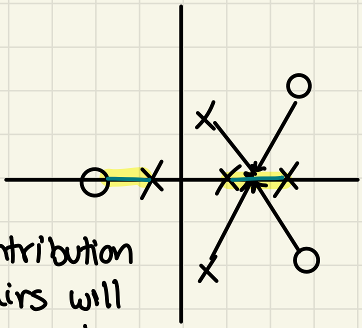
3) Real axis branches

⇒ This rule stems from applying the angle property

⇒ If the angle of the open loop transfer function at a point is an odd multiple of 180°, that point is on the root locus

⇒ The root locus exists to the left of an odd number of real axis finite poles or zeros

⇒ Here's why:



- For a test point on the real axis, the angle contribution of complex conjugate pairs will cancel, so only real axis poles and zeros contribute

- The angle is 0 for all poles and zeros to the left of the test point, and 180° to the right → an odd number to the right of the test point satisfies the criteria

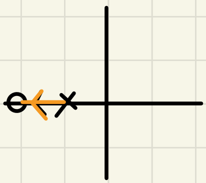
4) Starting and ending points
→ the root locus starts at OL poles and ends at OL zeros

→ what if we don't have the same number of poles and zeros?

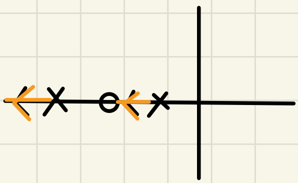
→ if $P > Z$, then the lines of extra poles go to ∞

→ if $Z < P$, then the lines to the extra zeros go to ∞

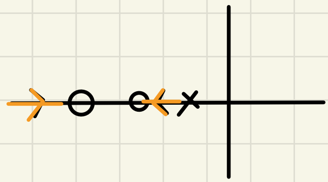
$P = Z$



$P > Z$



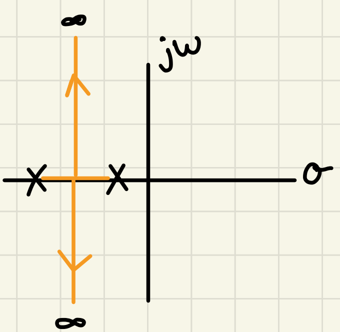
$P < Z$

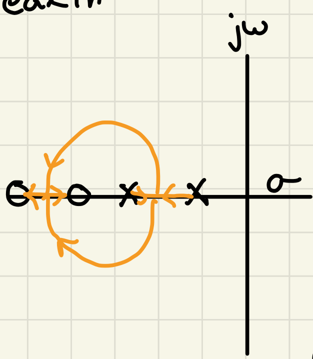


5) No path crossing - At no point will any one path cross over itself

6) Breakaway Points - Lines enter and leave the real axis at 90°. Whenever there is a real axis segment:
between two poles ⇒ Breakaway
two zeros ⇒ Breakin

ex:





to find location of breakaways:

$$\sum \frac{1}{\sigma + z_i} = \sum \frac{1}{\sigma + p_i}$$

solve for σ
z_i and p_i are negatives of poles and zeros

Breakaway & Breakin points represent locations where a value of K exists that creates a double root in the closed loop characteristic equation

$$1 + KG(s)H(s) = 0$$

Segments of the root locus on the real axis ($s = \sigma$) occur when

$$K = -\frac{1}{G(\sigma)H(\sigma)}$$ which we can write as $-\frac{D(\sigma)}{N(\sigma)}$

The breakaways occur when the real axis K is maximized, break ins when real axis K is minimized → so at critical points of K along the real axis

$$\frac{dK}{d\sigma} = 0 = -\frac{d}{d\sigma}\left(\frac{D(\sigma)}{N(\sigma)}\right)$$

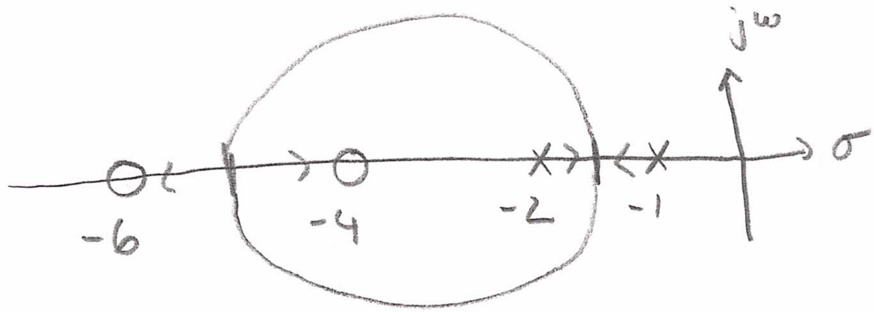
which gives

$$N(\sigma)D'(\sigma) - N'(\sigma)D(\sigma) = 0$$

Ex: $G(s)H(s) = \dfrac{s^2 + 10s + 24}{s^2 + 3s + 2}$

Find breakaway & break-in

Z: $-6, -4$
P: $-1, -2$



we find the roots of

$$N(\sigma)D'(\sigma) - N'(\sigma)D(\sigma) = 0$$

$$(\sigma^2 + 10\sigma + 24)(2\sigma + 3) - (\sigma^2 + 3\sigma + 2)(2\sigma + 10) = 0$$

$$7\sigma^2 + 44\sigma + 52 = 0$$

$$\sigma_{1,2} = -4.708, \ -1.578$$

↑ break in ↑ breakaway

Notice these are not nescisarily directly in the middle!

7) Behavior at Infinity

→ Lines go to infinity along asymptotes

→ If there are n poles and m zeros then:

- The angle of the asymptotes is

$$\phi_A = \frac{(2k+1)}{n-m} \cdot 180^\circ$$

where $K = 0, 1, 2 \ldots (n-m-1)$
or # lines to ∞ -1

- The asymptotes all come together at the centroid, or "CG of the poles"

$$\sigma_A = \frac{\sum p_j - \sum z_i}{n-m}$$

So the asymptotes that the infinity lines approach leave the real axis at σ_A and go to infinity at ϕ_A

If 1 line to ∞

$$\phi_A = \frac{2 \cdot 0 + 1}{1} \cdot 180^\circ = 180^\circ$$

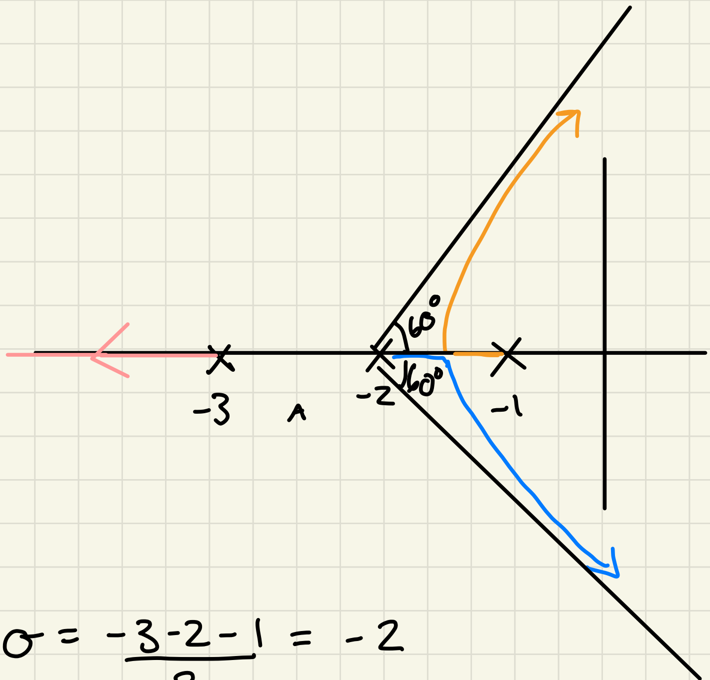
If 2 lines to ∞

$$\phi_{A_1} = \frac{2 \cdot 0 + 1}{2} \cdot 180^\circ = 90^\circ$$

$$\phi_{A_2} = \frac{2 \cdot 1 + 1}{2} \cdot 180^\circ = 270^\circ = -90^\circ$$

and the breakaway will be at the midpoint

Let's look at a quick example with 3 → ∞



$$\sigma = \frac{-3-2-1}{3} = -2$$

$$\phi_{A_1} = \frac{2\cdot 0 + 1}{3} \cdot 180^\circ = 60^\circ$$

$$\phi_{A_2} = \frac{2\cdot 1 + 1}{3} \cdot 180^\circ = 180^\circ$$

$$\phi_{A_3} = \frac{2\cdot 2 + 1}{3} \cdot 180^\circ = 300^\circ = -60^\circ$$

There are some other rules that could help you further refine your hand sketch, for ex:

- → find imaginary axis crossing
- → angles of departure and arrival

You can check these out in your book, but honestly, if you need that level of detail in your plot, just use Matlab! (pg 552 in Ogata 5D)

The value in being able to sketch by hand has to do with 1) understanding why they look as they do
2) intuitively learning to assess stability changes in response to changing the transfer function → adding a controller