Using Asymptotic Approximations to Sketch Bode Plots:

Genralizing what we observed,

if $$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

then $$|G(j\omega)| = \left.\frac{K|s+z_1||s+z_2|\cdots|s+z_3|}{|s^m||s+p_1||s+p_2|\cdots|s+p_n|}\right|_{s\to j\omega}$$

or in log space (dB)

$$20\log|G(j\omega)| = 20\log K + 20\log|s+z_1| + 20\log|s+z_2| + \cdots$$
$$\left.\cdots -20\log|s^m| - 20\log|s+p_1| - \cdots\right|_{s\to j\omega}$$

and similarly the phase response is the sum of the phase terms

So if we can sketch each term, we can sketch the total frequency response by adding them up

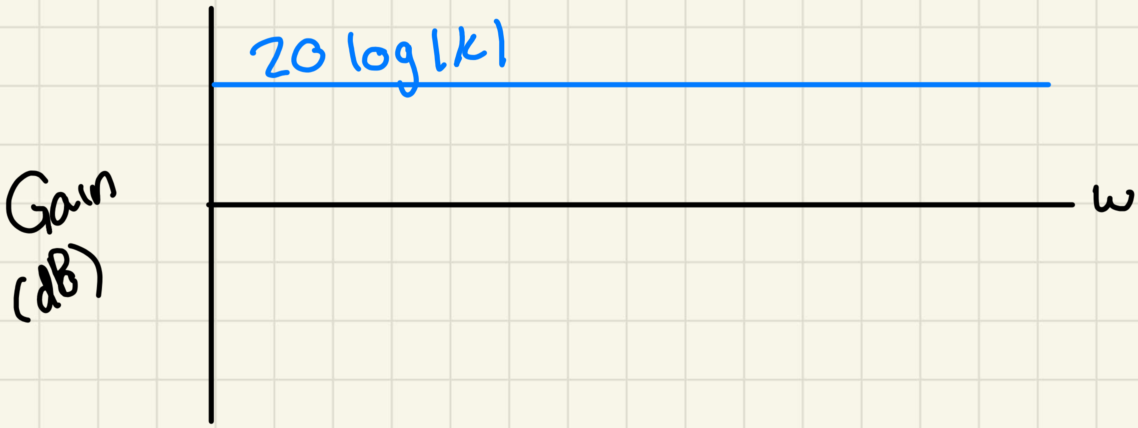
Ok, so let's look at each type of term and how to sketch it

- Constant Terms

$G(s) = K$

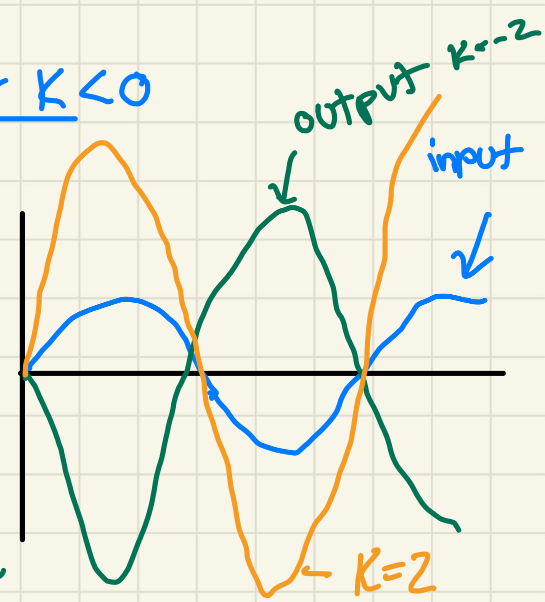
gain $= |G(j\omega)| = |K| =$ positive K
phase $= \text{atan2}(0, K)$ ← calculates unique atan angle using sign of both terms, allows us to differentiate angles in all 4 quadrants
$= 0^\circ$ if $K > 0$
$= -180^\circ$ if $K < 0$

on the bode plot, gain $= 20 \log |K|$



$\sin(\omega t) \rightarrow \boxed{K} \rightarrow K\sin(\omega t)$

if $K = 2$
double amplitude, no phase
if $K = -2$
double amplitude, -180° phase



- Real Poles $\omega_0/(s+\omega_0)$

Let's assume a single real pole

$$G(s) = \frac{\omega_0}{\omega_0 + s} = \frac{1}{1 + s/\omega_0}$$

← we will generally use this form for Bode Plots

ω_0 is called the "break frequency"

now, let's set $s = j\omega$

$$G(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} = \frac{1 - j\,\omega/\omega_0}{1 + \omega^2/\omega_0^2}$$

$$* \frac{1 - j\omega/\omega_0}{1 - j\omega/\omega_0}$$

the real part of $G(j\omega)$ is $\frac{1}{1+\omega^2/\omega_0^2}$

the imag part of $G(j\omega)$ is $\frac{-\omega/\omega_0}{1 + \frac{\omega^2}{\omega_0^2}}$

$$\text{gain} = 20 \log \left[\left(\frac{1}{1+\frac{\omega^2}{\omega_0^2}} \right)^2 + \left(\frac{-\omega/\omega_0}{1+\frac{\omega^2}{\omega_0^2}} \right)^2 \right]^{1/2}$$

skipping algebra

$$= -20 \log \left(\sqrt{1 + \frac{\omega^2}{\omega_0^2}} \right)$$

⇐ gain equation

$$\text{gain} = -20\log\left(\sqrt{1+\frac{\omega^2}{\omega_0^2}}\right)$$

a decade is 10 time the inital freq.

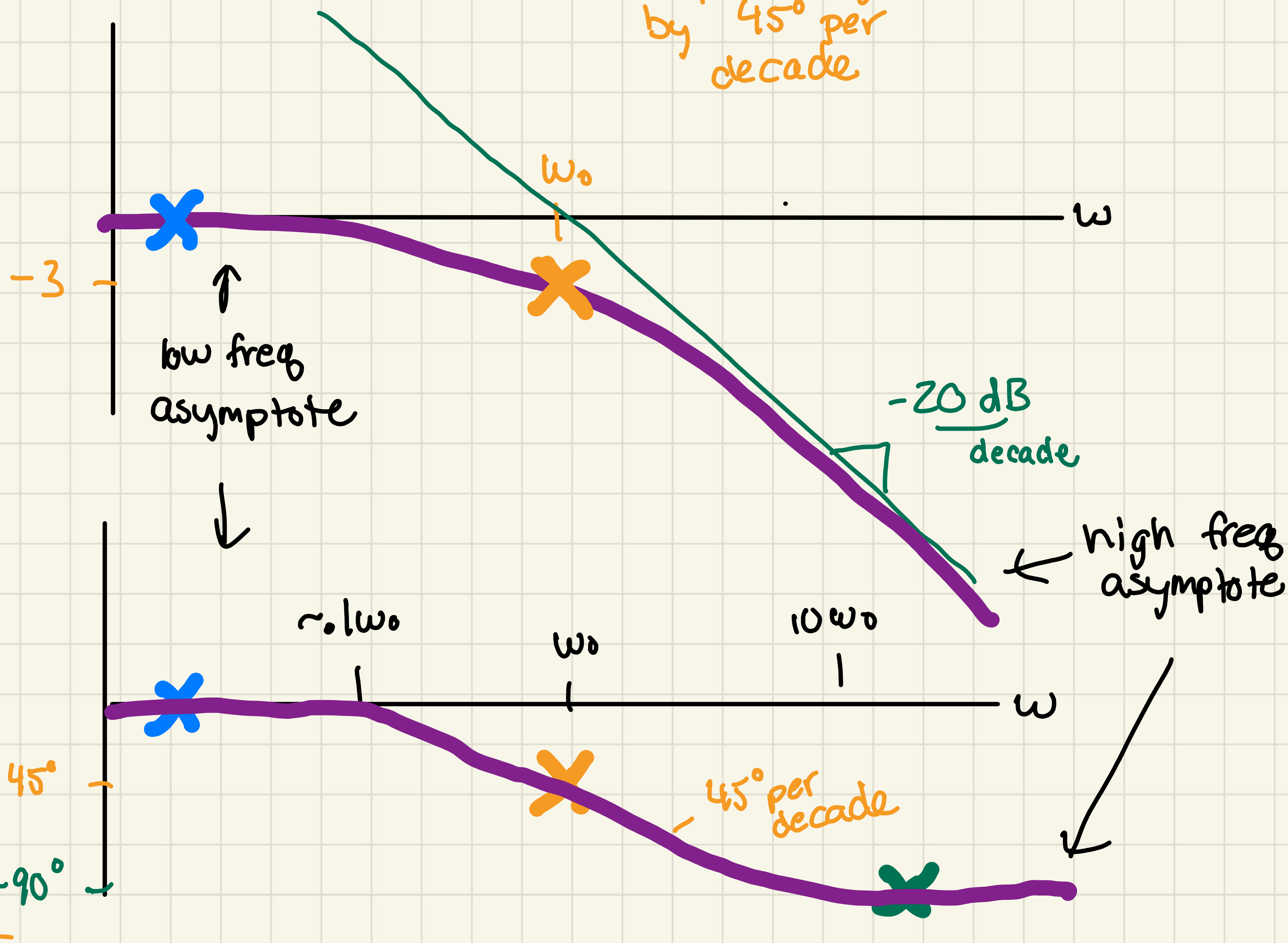
$$\text{phase} = \operatorname{atan}\left(\frac{\text{imag}}{\text{real}}\right) = \operatorname{atan}\left(-\frac{\omega}{\omega_0}\right)$$

now we plot these equations

Case 1: $\omega \ll \omega_0$
gain $= -20\log(1) = 0$
phase $= \operatorname{atan}(0) = 0°$

case 2: $\omega = \omega_0$
gain $= -20\log(\sqrt{2}) = -3\text{dB}$
phase $= \operatorname{atan}(-1) = -45°$
↳ slope changes by 45° per decade

case 3: $\omega \gg \omega_0$
gain $= -20\log\left(\frac{\omega}{\omega_0}\right) =$ line w/ slope -20dB/dec through 0 at ω_0
phase $= \operatorname{atan}(\infty) = -90$



we approx w/ straight lines called asymptotes

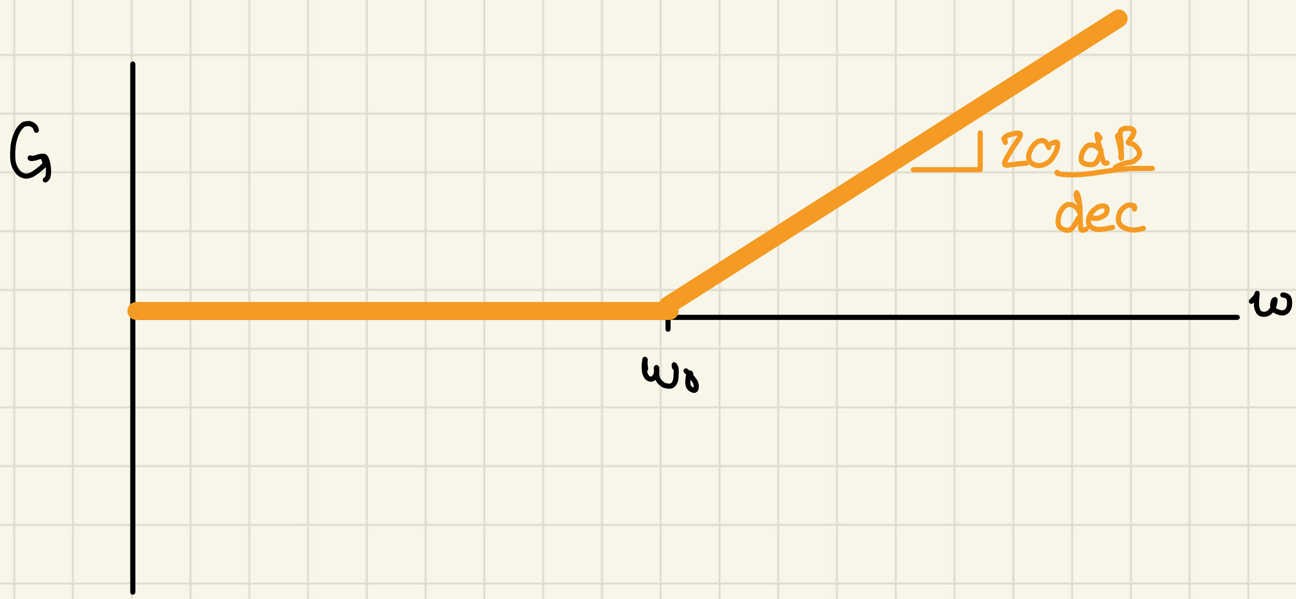
• Real Zeros $(1 + s/\omega_0)$

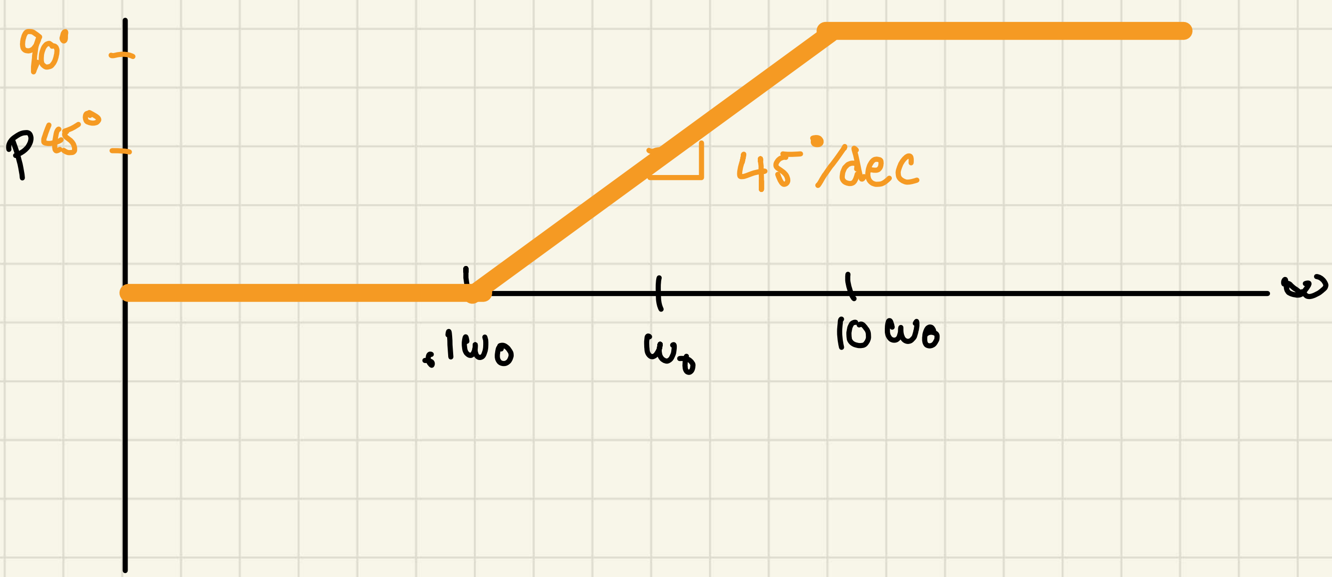
Your book has the derivation for real zeros, but we are not going to go through it in class

For now, we'll simply recognize that a zero is the opposite of a pole

$$\text{gain} = 20 \log\left(\sqrt{1 + \frac{\omega^2}{\omega_0^2}}\right)$$

$$\text{phase} = \text{atan}\left(\frac{\text{imag}}{\text{real}}\right) = \text{atan}\left(\frac{\omega}{\omega_0}\right)$$





• Poles & Zeros at the origin

$$G(s) = \frac{1}{s} \quad \text{or} \quad s$$

real $= 0$

imag $= -\frac{1}{\omega}$

Let's do the pole first

$$G(j\omega) = 20\log\left(\frac{1}{j\omega}\right) = -20\log\left(\frac{1}{\omega}j\right)$$

$$\text{gain} = |G(j\omega)| = \left[\left(-\frac{1}{\omega}j\right)^2\right]^{1/2} = \frac{1}{\omega}$$

$$\text{phase} = \operatorname{atan}\left(-\frac{1}{\omega}, 0\right) = -90^\circ$$

in a block diagram

phase

$$\sin(\omega t) \rightarrow \boxed{1/s} \rightarrow \int \sin \omega t = -\frac{1}{\omega}\cos(\omega t)$$

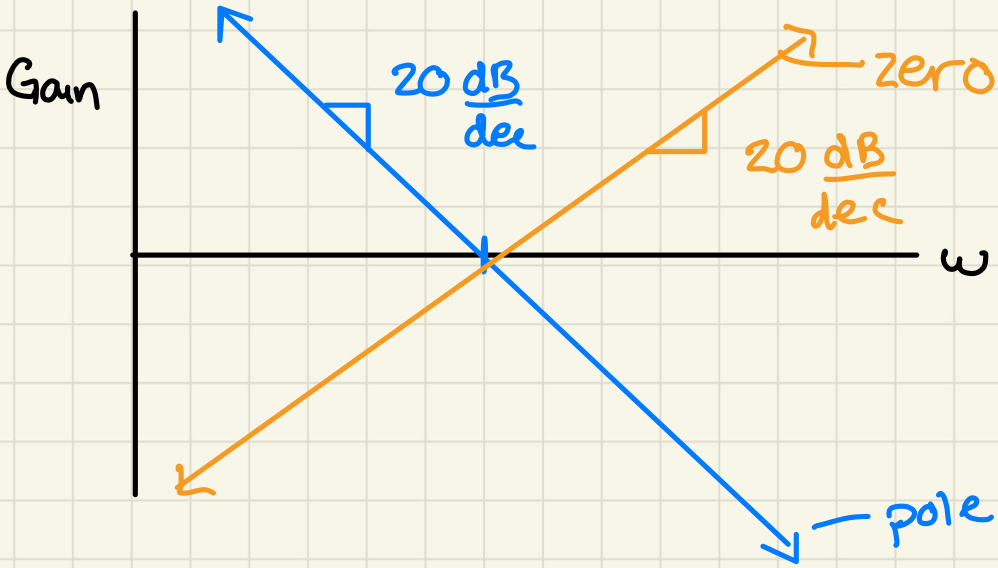
gain

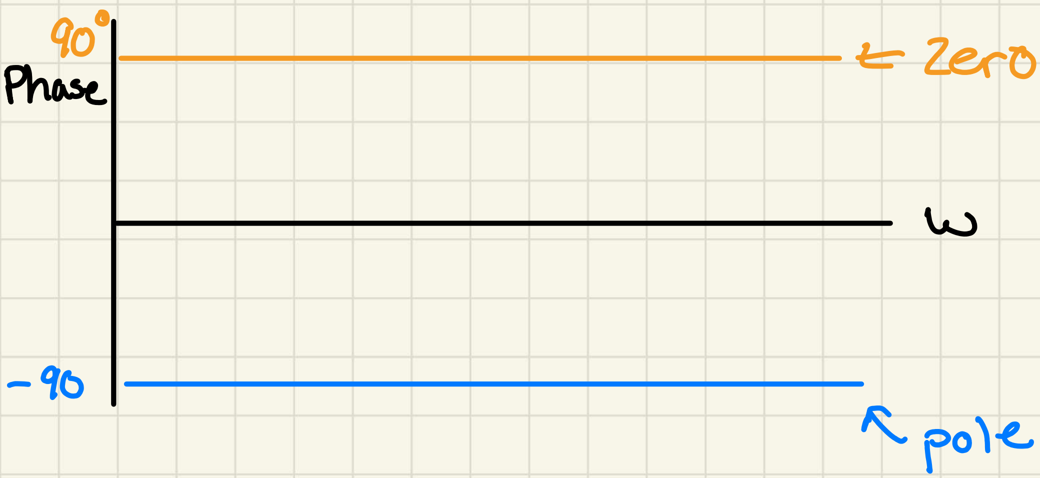
zero: $G(j\omega) = j\omega$

so gain $= 20\log(\omega)$

phase $= \operatorname{atan}\left(\frac{\omega}{0}\right) = 90^\circ$

we can plot these :





Note that we can also do this for complex poles and zeros but we aren't going to cover that right now

Let's use these approximations to assemble a Bode plot for a more complex transfer function

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

① Rewrite the transfer function in appropriate form:

$$G(s) = \frac{\frac{3}{2}K\left(\frac{s}{3}+1\right)}{s\,(s+1)\left(\frac{s}{2}+1\right)}$$

② Determine the break frequencies

→ these will be at 1, 2, and 3

→ note that frequency is typically expressed in rad/s, even though phase is typically expressed in degrees

③ Determine the range of the plot.

Rule of thumb: begin a decade below the lowest break frequency and extend at least a decade above the highest break frequency, so we will choose .1 to 100 as our range

4 Choose a value for K

K is simply going to shift the mag curve by $20\log(K)$ and has no effect on phase (our Ks are typically ⊕)

so let's just choose $K=1$, and then we can adjust K later to scale our response

5 Find the mag contribution and start point for the slope from each element

	frequency (rad/s) .1	1	2	3
pole at 0	-20	-20	-20	-20
pole at -1	0	-20	-20	-20
pole at -2	0	0	-20	-20
zero at -3	0	0	0	20
3/2 constant	0	0	0	0
total slope	-20	-40	-60	-40

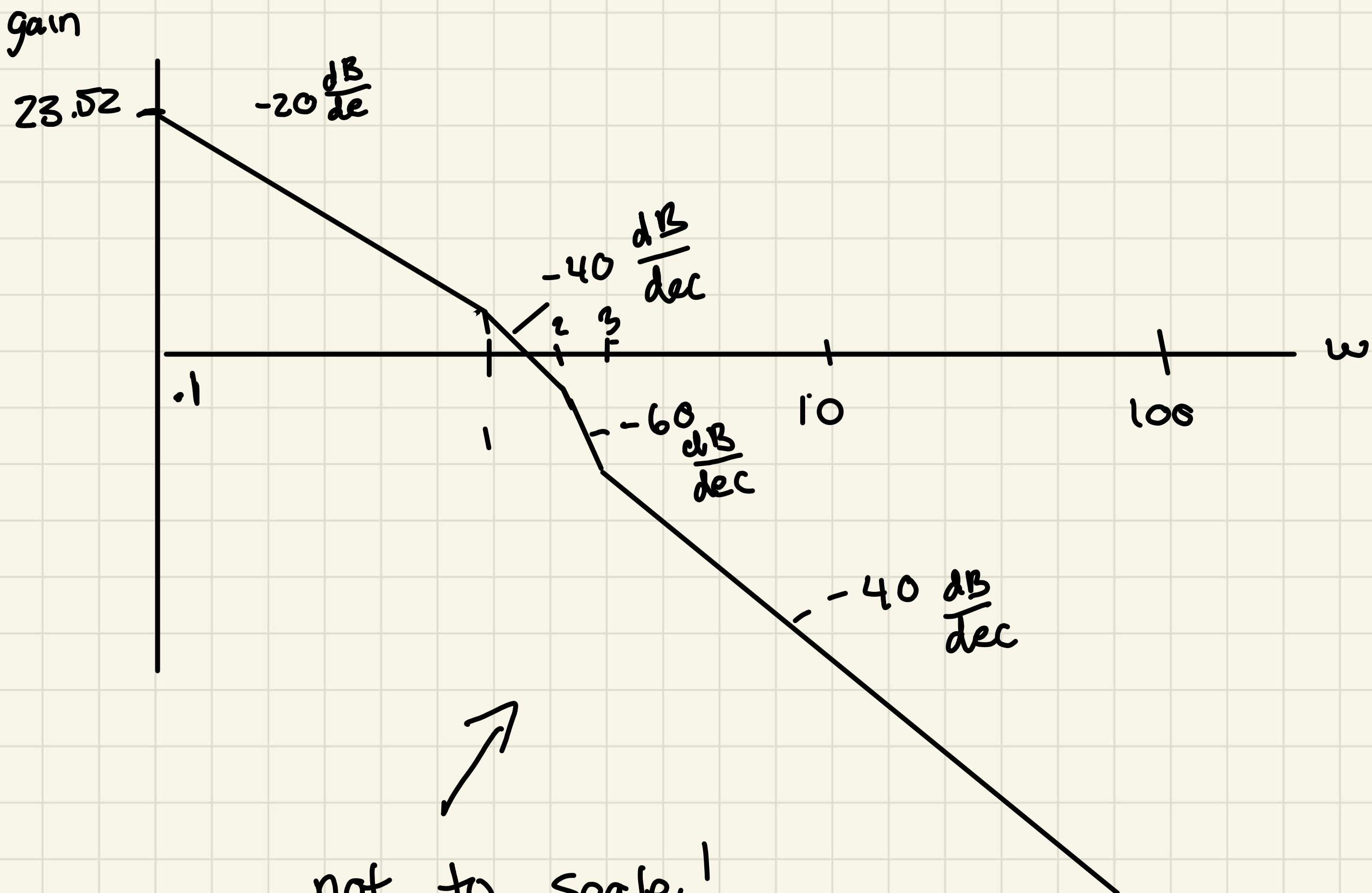
⑥ Find the starting value for the magnitude plot

At $\omega = .1$, the low frequency value of the function is found by assuming $s=0$ for all $(\frac{s}{a}+1)$ terms and $s=.1$ for the s term

$$G(j(.1)) = \frac{3}{2} / .1 = 15$$

and in dB scale $20\log(15) = 23.52$

⑦ Sketch the magnitude plot



not to scale!

⑧ Determine slope contributions to phase plot

	.1 (start pole @ -1)	.2 (start pole at -2)	.3 (start zero @ -3)	10 (end pole @ -1)	20 (end pole @ -2)	30 (end zero @ -3)
pole @ -1	-45	-45	-45	0	0	0
pole @ -2	0	-45	-45	-45	0	0
zero @ -3	0	0	45	45	45	0
total	-45	-90	-45	0	45	0

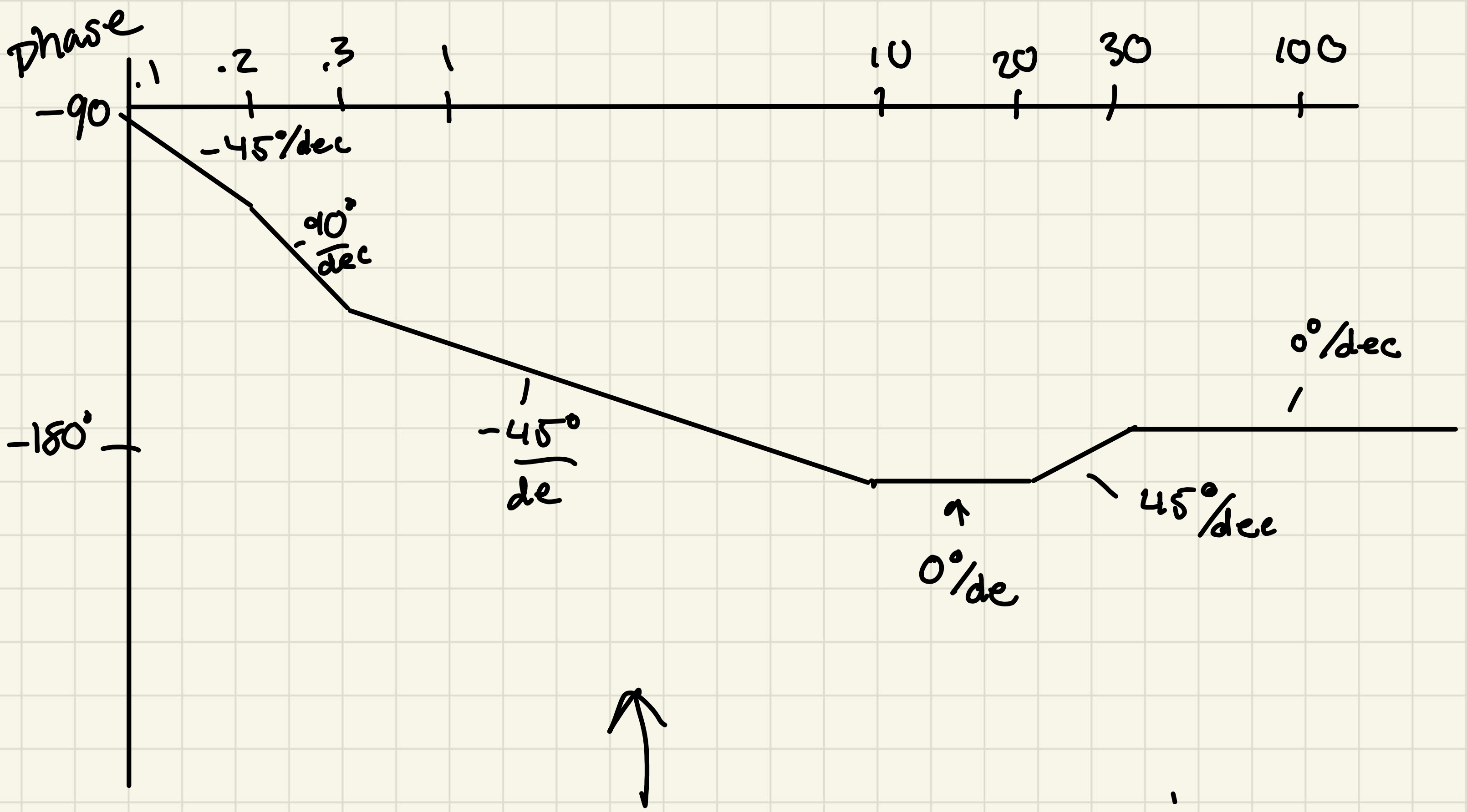
when the influence starts, it contributes that slope until it ends, then becomes 0

⑨ Determine the starting point for the phase plot

The constant term contributes 0°

The pole at the origin contributes a constant -90°, so this is the start

10 Make the phase plot



not to scale!

• What if you have complex conjugate poles? (zeros are just the inverse, so we'll focus on the poles)

→ Using the same derivation method used for the others:

If we have $G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$

and solve for $G(j\omega)$, we will find:

$$|G(j\omega)| = -20 \log\left(\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}\right)$$

$$\angle G(j\omega) = -\arctan\left(\frac{2\zeta\, \omega/\omega_0}{1 - (\omega/\omega_0)^2}\right)$$

If we look at our 3 cases, we'll see a constant 0 at low frequencies and a linear plot at high frequencies, but the break region is slightly more complicated

Case 1: $\omega \ll \omega_0$ $\quad |G(j\omega)| = -20\log(1) = 0$
$\angle G(j\omega) = -\arctan(0) = 0°$

Case 2: $\omega = \omega_0$
$|G(j\omega)| = -20\log(2\zeta)$

- for $\zeta < .5$, this yields a positive mag
- for $\zeta = .5$, this yields 0
- for $\zeta > .5$, the magnitude is slighty negative, and can be approximated by the high frequency asymptote

→ when $\zeta < .5$, the plot will go up to a peak before going into the high frequency asymptote

Case 2 (con't): $\omega = \omega_0$

$$\angle G(j\omega) = -90°$$

Case 3: $\omega \gg \omega_0$

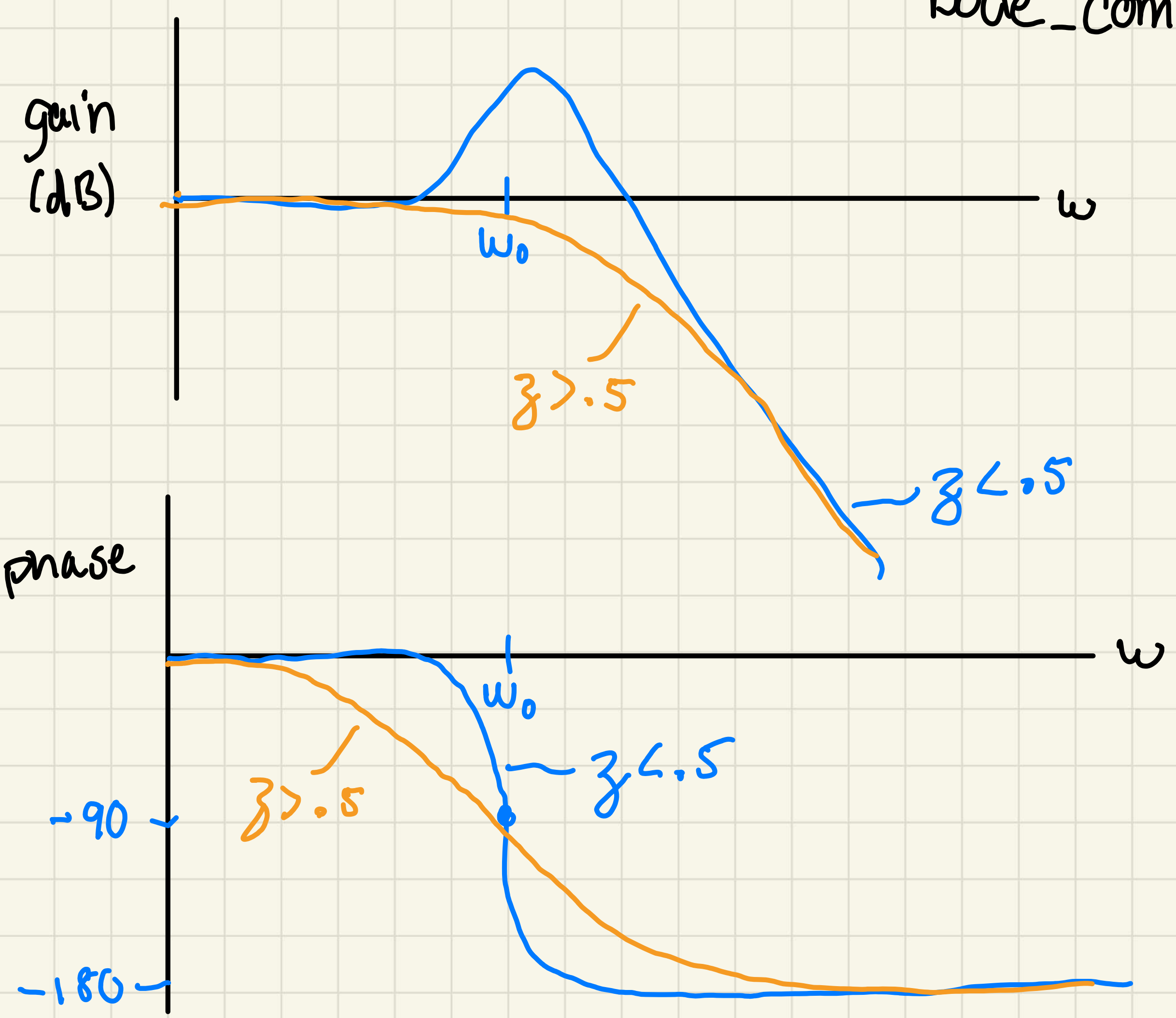
$$|G(j\omega)| = -20 \log\left(\left(\frac{\omega}{\omega_0}\right)^2\right) = -40 \log\left(\frac{\omega}{\omega_0}\right)$$

→ linear w/ slope of -40 dB/dec

$$\angle G(j\omega) = -180°$$

Let's look at matlab to understand how varying ζ affects these plots

bode_complexpoles.m



Great, we can now make these plots!

But what do we do with them?

Let's see how we might use them to design a controller

But first, we need to understand what our design requirements look like in frequency space

So let's learn some terms sometimes called "safety margins"

Gain → how much a signal is scaled

Margin → how much extra you have - safety net

→ how much you can increase gain before your system goes unstable

In real life, we need margins to ensure stable behavior when real world conditions vary

→ helps us account for uncertainty

So what makes a system unstable?

Or, more specifically:
what properties of an open loop system will make a closed loop system unstable?