HW9

Question 1

(a) the primal problem

max $4x_1 + 3x_2$

s.t. $x_1 + 4x_2 \le 140 \quad (y_1)$

$6x_1 + 3x_2 \le 180 \quad (y_2)$

$x_2 \le 30 \quad (y_3)$

$x_1, x_2 \ge 0$

the dual problem is:

min $140y_1 + 180y_2 + 30y_3$

s.t. $y_1 + 6y_2 \ge 4$

$4y_1 + 3y_2 + y_3 \ge 3$

$y_1, y_2, y_3 \ge 0$

(b)

$$
\begin{array}{rl}
4x_1 + 3x_2 & = z \\
\hline
x_1 + 4x_2 + w_1 & = 140 \\
\frac{1}{6} \; 6x_1 + 3x_2 \qquad + w_2 & = 180 \\
x_2 \qquad\qquad + w_3 & = 30
\end{array}
$$

$-\frac{2}{3}$, $-\frac{1}{6}$

$x_1 = x_2 = 0 \quad z = 0$

$w_1 = 140 \quad w_2 = 180 \quad w_3 = 30$

increase x_1 to $\min\{30, 140\} = 30$

$$
\begin{array}{rl}
x_2 \quad -\frac{2}{3}w_2 & = z - 120 \\
\hline
\frac{7}{2}x_2 + w_1 - \frac{1}{6}w_2 & = 110 \\
x_1 + \frac{1}{2}x_2 \quad + \frac{1}{6}w_2 & = 30 \\
x_2 \qquad + w_3 & = 30
\end{array}
$$

$-\frac{7}{2}$, -1, $-\frac{1}{2}$

$x_1 = 30 \quad w_1 = 110 \quad w_3 = 30$

$x_2 = w_2 = 0 \quad z = 120$

increase x_2 to $\min\{\frac{220}{7}, 30, 60\} = 30$

$$
\begin{array}{rl}
-\frac{2}{3}w_2 - w_3 & = z - 150 \\
\hline
w_1 - \frac{1}{6}w_2 - \frac{7}{2}w_3 & = 5 \\
x_1 \qquad + \frac{1}{6}w_2 - \frac{1}{2}w_3 & = 15 \\
x_2 \qquad + w_3 & = 30
\end{array}
$$

$x_1 = 15 \quad x_2 = 30 \quad z = 150$

$w_1 = 5 \quad w_2 = w_3 = 0$

The optimal solution is $(15, 30, 5, 0, 0)$

c) By using the last system of equations, the optimal of the dual problem is $(0, \frac{2}{3}, 1, 0, 0)$

(d) Check to feasible solution:

for primal problem $\quad x_1+4x_2+w_1 = 15+4\times30+5 = 140 \quad (x_1+4x_2 \le 140)$

$6x_1+3x_2+w_2 = 6\times15+3\times30+0 = 180 \quad (6x_1+3x_2 \le 180)$

$\Rightarrow 4x_1+3x_2 = 4\times15+3\times30 = 150$

for dual problem $\quad y_1+6y_2 = 0+6\times\frac{2}{3} \ge 4$

$4y_1+3y_2+y_3 = 3\times\frac{2}{3}+1 \ge 3$

$\Rightarrow 140y_1+180y_2+30y_3 = 0+180\times\frac{2}{3}+30\times1 = 150$

equal

Because of weak duality, the optimal solution to the dual problem that start in Part c is indeed the optimal solution to the dual problem.

(e) Replacing all of the appearance of w_2 with $w_2-\varepsilon$. Thus, we get

$$\begin{array}{lllll} 4x_1 + 3x_2 & & & & = z \\ \hline x_1 + 4x_2 + w_1 & & & & = 140 \\ 6x_1 + 3x_2 & & +(w_2-\varepsilon) & & = 180 \\ \quad x_2 & & & +w_3 & = 30 \end{array}$$

Apply the same sequence of row operation, we can obtain

$$\begin{aligned} -\tfrac{2}{3}(w_2-\varepsilon)-w_3 &= z-150 \\ w_1-\tfrac{1}{6}(w_2-\varepsilon)-\tfrac{7}{2}w_3 &= 5 \\ x_1+\tfrac{1}{6}(w_2-\varepsilon)-\tfrac{1}{2}w_3 &= 15 \\ x_2+w_3 &= 30 \end{aligned} \quad\Rightarrow\quad \begin{aligned} -\tfrac{2}{3}w_2-w_3 &= z-150-\tfrac{2}{3}\varepsilon \\ w_1-\tfrac{1}{6}w_2-\tfrac{7}{2}w_3 &= 5-\tfrac{1}{6}\varepsilon \\ x_1+\tfrac{1}{6}w_2-\tfrac{1}{2}w_3 &= 15+\tfrac{1}{6}\varepsilon \\ x_2+w_3 &= 30 \end{aligned}$$

Thus, the optimal solution is $(15+\frac{1}{6}\varepsilon, 30, 5-\frac{1}{6}\varepsilon, 0, 0)$ and the objective value is $150+\frac{2}{3}\varepsilon$, thus the change between objective values is $\frac{2}{3}\varepsilon$

(f) if we cannot generate the optimal solution using the system above, we cannot make the objective value increase when we add ε to the right of the constrains.

When $\varepsilon > 30$, $5-\frac{1}{6}\varepsilon$ is smaller than 0, which means we can't find the optimal value for w_1 since $w_1 \ge 0$. Thus, the largest value of ε is 30

(g) ε=6, using excel to solve the problem. We set the second constrain to 180+6=186, the figure is shown as follows:

```
$C$6 <= 140
$C$7 <= 186
$C$8 <= 30
```

We get the optimal solution:

x1	16	
x2	30	
4x1+3x2	154	
st	x1 + 4x2	136
	6 x1 + 3 x2	186
	x2	30

We find that the optimal solution goes to (16, 30) and the objective value goes to 154. As we mentioned in Part e, x1 = 15+ε/6 = 16 and objective value = 150+2ε/3 = 150+4 = 154.

The optimal objective value increases 4 and matches my solution in Part e.

(h) ε=36, using excel to solve the problem. We set the second constrain to 180+36=216, the figure is shown as follows:

```
$C$6 <= 140
$C$7 <= 216
$C$8 <= 30
```

We get the optimal solution:

x1	21.1428571	
x2	29.7142857	
4x1+3x2	173.714286	
st	x1 + 4x2	140
	6 x1 + 3 x2	216
	x2	29.7142857

We find that the optimal solution goes to (21.142857, 29.714285) and the objective value goes to 173.714286. The objective value increases 23.714286.

If we use the solution mentioned in Part e, $x1$ should be $15+\varepsilon/6 = 21$, $x2$ should be 30 and objective value should be $150+2\varepsilon/3 = 150+24 = 174$. Which does not match our solution in Part e. But using the solution in Part f, our solution in Part e can only be usable when $\varepsilon <= 30$, so the result of Part h matches the solution of Part f.