Question 1

(a) the primal problem

$$\begin{aligned}\max \quad & 4x_1 + 3x_2 \\ \text{st} \quad & x_1 + 4x_2 \le 140 \\ & 6x_1 + 3x_2 \le 180 \\ & x_2 \le 30 \\ & x_1, x_2 \ge 0\end{aligned}$$

$\Rightarrow$

the dual problem

$$\begin{aligned}\min \quad & 140y_1 + 180y_2 + 30y_3 \\ \text{st} \quad & y_1 + 6y_2 \ge 4 \\ & 4y_1 + 3y_2 + y_3 \ge 3 \\ & y_1, y_2, y_3 \ge 0\end{aligned}$$

b)

$$\begin{array}{l} 4x_1 + 3x_2 = z \\ \hline x_1 + 4x_2 + w_1 = 140 \\ 6x_1 + 3x_2 + w_2 = 180 \\ x_2 + w_3 = 30 \end{array}$$

$w_1 = 140$, $w_2 = 180$, $w_3 = 30$

$x_1 = x_2 = 0$, $z = 0$

increasing x_1 up to $\min\{140, 180/6\} = 30$

$$\begin{array}{l} x_2 - \frac{2}{3}w_2 = z - 120 \\ \hline \frac{7}{2}x_2 + w_1 - \frac{1}{6}w_2 = 110 \\ x_1 + \frac{1}{2}x_2 + \frac{1}{6}w_2 = 30 \\ x_2 + w_3 = 30 \end{array}$$

$w_1 = 110$ $x_1 = 30$

$w_3 = 30$

$x_2 = w_2 = 0$, $z = 120$

increasing x_2 up to $\min\{110/\frac{7}{2}, 30/\frac{1}{2}, 30\} = 30$

$$\begin{array}{l} -\frac{2}{3}w_2 - w_3 = z - 150 \\ \hline w_1 - \frac{1}{6}w_2 - \frac{7}{2}w_3 = 5 \\ x_1 + \frac{1}{6}w_2 - \frac{1}{2}w_3 = 15 \\ x_2 + w_3 = 30 \end{array}$$

$w_1 = 5$, $x_1 = 15$, $x_2 = 30$, $w_2 = w_3 = 0$, $z = 150$

the optimal solution $(x_1, x_2, w_1, w_2, w_3) = (15, 30, 5, 0, 0)$

the maximum $4x_1 + 3x_2 = 150$

(c) Using the last system of equation in Part b.

the optimal solution to the dual problem

is $(0, \frac{2}{3}, 1, 0, 0)$

(d) let $(\hat{x}_1, \hat{x}_2)$ be the feasible solution for primal problem

let (x_1^*, x_2^*) be the optimal solution for primal problem

let $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ be the feasible solution for dual problem

let (y_1^*, y_2^*, y_3^*) be the optimal solution for dual problem

Using the Implication of Weak Duality

we can see

$$140\hat{y}_1 + 180\hat{y}_2 + 30\hat{y}_3 \geq 140y_1^* + 180y_2^* + 30y_3^* \geq 4x_1^* + 3x_2^* \geq 4\hat{x}_1 + 3\hat{x}_2$$

from part b & c.

$(x_1^*, x_2^*) = (15, 30)$

the optimal objective value is $4x_1 + 3x_2 = 150$

$(y_1^*, y_2^*, y_3^*) = (0, \frac{2}{3}, 1)$

the optimal objective value is $140y_1 + 180y_2 + 30y_3 = 150$

they have the same objective value.

Therefore, the optimal solution to the dual problem in part b

is indeed the optimal solution to the dual problem.

(e) Increase the right side of the second constraint by a small amount ϵ,

Consider replacing all of the appearaces of w_2 in the system equations with $w_2 - \epsilon$

$$\begin{array}{rcl} 4x_1 + 3x_2 & = & Z \\ \hline x_1 + 4x_2 + w_1 & = & 140 \\ 6x_1 + 3x_2 \quad + (w_2 - \epsilon) & = & 180 \\ x_2 \quad + w_3 & = & 30 \end{array}$$

$$\begin{array}{rcl} -\frac{2}{3}(w_2 - \epsilon) - w_3 & = & Z - 150 \\ \hline w_1 \quad - \frac{1}{6}(w_2 - \epsilon) - \frac{7}{2}w_3 & = & 5 \\ x_1 \quad + \frac{1}{6}(w_2 - \epsilon) - \frac{1}{2}w_3 & = & 15 \\ x_2 \quad + w_3 & = & 30 \end{array}$$

moving all the terms that involve an ϵ to the right side. we can obtain

$$\begin{array}{rcl} -\frac{2}{3}w_2 \quad - w_3 & = & Z - 150 - \frac{2}{3}\epsilon \\ \hline w_1 \quad - \frac{1}{6}w_2 \quad - \frac{7}{2}w_3 & = & 5 - \frac{1}{6}\epsilon \\ x_1 \quad + \frac{1}{6}w_2 \quad - \frac{1}{2}w_3 & = & 15 + \frac{1}{6}\epsilon \\ x_2 \quad + w_3 & = & 30 \end{array}$$

$w_1 = 5 - \frac{1}{6}\epsilon$, $x_1 = 15 + \frac{1}{6}\epsilon$, $x_2 = 30$

$w_2 = w_3 = 0$, $Z = 150 + \frac{2}{3}\epsilon$

$\therefore$ the optimal objective value change is $\frac{2}{3}\epsilon$

(f) $\begin{cases} w_1 = 5 - \frac{1}{6}\epsilon \geq 0 \\ x_1 + 4x_2 = 15 + \frac{1}{6}\epsilon + 30 \cdot 4 \leq 140 \\ 6x_1 + 3x_2 = 6(15 + \frac{1}{6}\epsilon) + 3 \cdot 30 \leq 180 + \epsilon \end{cases}$

$\Rightarrow \epsilon \leq 30$

$\therefore$ the largest value of ϵ is 30

(g) when $\epsilon = 6$

Variables:	x1	x2
	16	30
Max	=4*B2+3*C2	
St		
=B2+4*C2	<=	140
=6*B2+3*C2	<=	186
=C2	<=	30

$x_1 = 16$, $x_2 = 30$. the optimal objective value is $4x_1 + 3x_2 = 154$

the optimal objective value change is $154 - 150 = 4$

from part e, the change is $\frac{2}{3}\epsilon = \frac{2}{3} \times 6 = 4$

$\therefore$ the answer is matched.

(h) when $\epsilon = 36$.

Variables:	x1	x2
	21.1428571421678	29.7142857156643
Max	=4*B2+3*C2	
St		
=B2+4*C2	<=	140
=6*B2+3*C2	<=	216
=C2	<=	30

the optimal objective value is 173.714

the change should be $173.714 - 150 = 23.714$

because the largest ϵ is 30. when $\epsilon = 36$, it will not follow the rule of the optimal objective value change in part e.