Precalculus Practice Problems: Midterm 2

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The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. *SAS congruence.* Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104^\circ$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Using the law of sines, or otherwise, find $\sin A$ and $\sin B$.
   (d) Show that $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$, and hence compute $\angle A$ and $\angle B$. (Hint: For which angles does $\arcsin(\sin\theta) = \theta$ hold?)

2. *SSS congruence.* Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Using the law of cosines, or otherwise, find $\cos A$, $\cos B$, and $\cos C$.
   (b) Compute $\angle A$, $\angle B$, and $\angle C$.
   (c) Find $[ABC]$.

3. *ASA/AAS congruence.* Let ABC be a triangle with $c = 2$, $\angle A = 12^\circ$, and $\angle B = 77^\circ$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.

4. *SSA non-congruence.* Let ABC be a triangle with $\angle A = 30^\circ$, $a = 6$, and $b = 9$.
   (a) Show that $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) Find all possible values of c.

5. *Extended law of sines.* If ABC is a triangle with circumradius R, then the **extended law of sines** states that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Express $\sin C$ in terms of a, b, and $[ABC]$.
   (b) Assuming the extended law of sines, show that $R = \dfrac{abc}{4[ABC]}$.
   (c) Given that $a = 13$, $b = 14$, and $c = 15$, compute R.

6. *Single-cevian computations.* Let ABC be a triangle and let D be a point on side $\overline{BC}$. (The line segment $\overline{AD}$ is a *cevian* from A.)
   (a) Express $\angle ADB$ in terms of $\angle CDA$.

(b) *Ratio lemma.* Using the law of sines, or otherwise, show that

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(c) *Angle bisector theorem.* Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

(d) *Stewart's theorem.* Let $\angle ADB = \theta$ and let $AD = d$, $BD = x$, and $DC = y$. Show that

$$c^2 = d^2 + x^2 - 2dx\cos\theta,$$
$$b^2 = d^2 + y^2 + 2dy\cos\theta,$$

and conclude that

$$b^2x + c^2y = a(d^2 + xy).$$

7. *Concurrent cevians.* Let ABC be a triangle and let D, E, F lie on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively.

(a) Show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}.$$

(b) *Ceva's theorem.* Show that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent if and only if the two sides of the above equation are equal to 1.

1.2 Challenge problems

8. Points O, A, B, and C are placed in the plane so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. Find all possible lengths of BC.

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Assuming that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. *Eisenstein triples.* An **Eisenstein triple** is an unordered triple of positive integers $\{a, b, c\}$ for which a triangle with side lengths a, b, and c has an angle of measure either $60°$ or $120°$. If the Eisenstein triple corresponds to a triangle with an angle of measure $60°$, we will call it an Eisenstein triple of **acute type**, and otherwise, we call it an Eisenstein triple of **obtuse type**. (The "acute type" and "obtuse type" names are non-standard.)

(a) Suppose $a, b < c$ are such that $\{a, b, c\}$ is an Eisenstein triple of obtuse type. Show that $\{a, a + b, c\}$ and $\{a + b, b, c\}$ are Eisenstein triples of acute type.

(b) Conversely, show that every Eisenstein triple of acute type either corresponds to an equilateral triangle or arises from an Eisenstein triple of obtuse type in the above manner.

(c) Show that if $\{a, b, c\}$ is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that

$$\{a, b, c\} = \{2mn + n^2, m^2 - n^2, m^2 + mn + n^2\}.$$

(Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104^\circ) \approx 2.426$

   (b) $c = \sqrt{a^2+b^2-2ab\cos C} = \sqrt{26-10\cos(104^\circ)} \approx 5.331$

   (c) $\sin A = \frac{a\sin C}{c} \approx \frac{\sin(104^\circ)}{5.331} \approx 0.182$
   $\sin B = \frac{b\sin C}{c} \approx \frac{5\sin(104^\circ)}{5.331} \approx 0.910$

   (d) Since $\angle C$ is obtuse, $\angle A$ and $\angle B$ are acute. Since acute angles are included in the range of arcsin, we have $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$.
   $\angle A = \arcsin(\sin A) \approx \arcsin(0.182) \approx 10.49^\circ$
   $\angle B = \arcsin(\sin B) \approx \arcsin(0.910) \approx 65.51^\circ$

2. (a) $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3}{5}$
   $\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{33}{65}$
   $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{5}{13}$

   (b) The range of arccos is $[0^\circ, 180^\circ]$, so we can always use it to extract triangle angles.
   $\angle A = \arccos(\cos A) = \arccos(\frac{3}{5}) \approx 53.13^\circ$
   $\angle B = \arccos(\cos B) = \arccos(\frac{33}{65}) \approx 59.49^\circ$
   $\angle C = \arccos(\cos C) = \arccos(\frac{5}{13}) \approx 67.38^\circ$

   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7\cdot 15\cdot \frac{4}{5} = 84$

3. (a) $\angle C = 180^\circ - \angle A - \angle B = 91^\circ$

   (b) $a = \frac{c}{\sin C}\cdot \sin A = \frac{2\sin 12^\circ}{\sin 91^\circ} \approx 0.416$
   $b = \frac{c}{\sin C}\cdot \sin B = \frac{2\sin 77^\circ}{\sin 91^\circ} \approx 1.949$

   (c) $[ABC] = \frac{1}{2}ac\sin B \approx 0.416\sin 77^\circ \approx 0.405$

4. (a) By the law of cosines,

   $$a^2 = b^2 + c^2 - 2bc\cos A \implies 36 = 81 + c^2 - 18\cos(30^\circ)c.$$

   Evaluating $\cos(30^\circ) = \sqrt{3}/2$ and rearranging gives us $c^2 - (9\sqrt{3})c + 45 = 0$.

   (b) By the quadratic formula,

   $$c = \frac{9\sqrt{3} \pm \sqrt{(9\sqrt{3})^2 - 4\cdot 1\cdot 45}}{2} = \frac{9\sqrt{3} \pm 3\sqrt{7}}{2}.$$

5. (a) $\sin C = \frac{2[ABC]}{ab}$.

   (b) $R = \dfrac{c}{2\sin C} = \dfrac{c}{\frac{4[ABC]}{ab}} = \dfrac{abc}{4[ABC]}$

   (c) $R = 65/8$

6. (a) $\angle ADB = 180^\circ - \angle CDA$

(b) By the law of sines,

$$\frac{AB}{\sin(\angle ADB)} = \frac{BD}{\sin(\angle BAD)} \quad \text{and} \quad \frac{AC}{\sin(\angle CDA)} = \frac{DC}{\sin(\angle DAC)}.$$

Dividing one equation by the other and using the fact that $\sin(\angle ADB) = \sin(\angle CDA)$ gets us the desired result after rearranging.

(c) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (b).

(d) The law of cosines in triangle ADB, using $\angle ADB$, gives us

$$(AB)^2 = (AD)^2 + (BD)^2 - 2(AD)(BD)\cos(\angle ADB) \implies c^2 = d^2 + x^2 - 2dx\cos\theta,$$

while the law of cosines in triangle ADC, using $\angle CDA$, gives us

$$b^2 = d^2 + y^2 - 2dy\cos(180^\circ - \theta) = d^2 + y^2 + 2dy\cos\theta.$$

Adding y times the first equation to x times the second equation, so as to eliminate $\cos\theta$,

$$\begin{aligned} b^2x + c^2y &= (d^2x + y^2 + 2dy\cos\theta)x + (d^2y + x^2 - 2dx\cos\theta)y \\ &= d^2(x+y) + xy(x+y) = a(d^2 + xy). \end{aligned}$$

7. (a) By the ratio lemma (problem 6b),

$$\begin{aligned} \frac{AF}{FB} &= \frac{CA}{CB} \cdot \frac{\sin(\angle ACF)}{\sin(\angle FCB)}, \\ \frac{BD}{DC} &= \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}, \\ \frac{CE}{EA} &= \frac{BC}{BA} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}. \end{aligned}$$

Multiplying these equations together gives us the desired result.

(b) First suppose $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur at point P. Then by the law of sines,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle DAC)} &= \frac{\sin(\angle ACP)}{\sin(\angle PAC)} = \frac{AP}{CP}, \\ \frac{\sin(\angle BAD)}{\sin(\angle EBA)} &= \frac{\sin(\angle BAP)}{\sin(\angle PBA)} = \frac{BP}{AP}, \\ \frac{\sin(\angle CBE)}{\sin(\angle FCB)} &= \frac{\sin(\angle CBP)}{\sin(\angle PCB)} = \frac{CP}{BP}. \end{aligned}$$

Multiplying these equations,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)} &= \frac{\sin(\angle ACF)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle EBA)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle FCB)} \\ &= \frac{AP}{CP} \cdot \frac{BP}{AP} \cdot \frac{CP}{BP} = 1. \end{aligned}$$

Conversely, suppose both sides of the equation from part (a) are 1, so in particular

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Let $\overline{AD}$ and $\overline{BE}$ intersect at point Q, and let line $\overline{CQ}$ intersect side $\overline{AB}$ at point F'. Then using what we just showed,

$$\frac{AF'}{F'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

This means $AF/FB = AF'/F'B$. With F and F' both interior to segment $\overline{AB}$, this can only happen if $F = F'$, which means that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur (at Q) as desired.

8. Fix points A and B on a circle of radius 4 centered at O so that $AB = 2$. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

Since $CO = 4$, we know C lies on this circle as well, and since $AC = 1$, there are two possible locations for C, one on either side of $\overline{OA}$. When C and B lie on the same side of $\overline{OA}$, we have $\angle BOC = \angle AOB - \angle AOC$, which gives us

$$\begin{aligned} BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8} \cdot \frac{31}{32} + \frac{\sqrt{15}}{8} \cdot \frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

When C and B lie on opposite sides of $\overline{OA}$, we instead have $\angle BOC = \angle BOA + \angle AOC$. A similar calculation to the first case yields

$$BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. By Stewart's theorem,

$$\begin{aligned} (AB)^2 \cdot (DC) + (AC)^2 \cdot (BD) &= (BC)[(AD)^2 + (BD)(DC)] \\ 8^2 \cdot 7x + (8x)^2 \cdot 7 &= (7 + 7x)(5^2 + 7 \cdot 7x) \\ (8^2 \cdot 7)(x + x^2) &= 7(1 + x)(25 + 49x). \end{aligned}$$

Since $x > 0$, we can safely divide by $7(1 + x)$ on both sides to get

$$64x = 25 + 49x \implies x = \frac{5}{3}.$$

Therefore, $CD = 7x = 35/3$.

10. (a) Let ABC be the corresponding triangle, with $\angle C = 120^\circ$ opposite side $AB = c$. Extend ray $\overrightarrow{AC}$ to point D so that $CD = CB$. Then triangle BCD is equilateral, so triangle ADB has side lengths $AD = a + b$, $DB = a$, and $AB = c$ with $\angle D = 60^\circ$. This means $\{a, a+b, c\}$ is an Eisenstein triple of acute type. A similar construction where we extend $\overrightarrow{BC}$ tells us that $\{a + b, b, c\}$ is also an Eisenstein triple of acute type.

(b) Let a, b, c be such that $\{a, b, c\}$ is an Eisenstein triple of acute type with $AB = c$ opposite the 60° angle in the corresponding triangle ABC. If $a = b$, then the triangle is equilateral. Otherwise, suppose without loss of generality that $a < b$. Let point D on side $\overline{AC}$ be such that $CD = CB$ and let $a' = AD = b - a$. Then triangle ADB has $\angle ADB = 120^\circ$, so $\{a', a, c\}$ is an Eisenstein triple of obtuse type. As $b = a' + a$, the original Eisenstein triple $\{a, b, c\}$ can be constructed from $\{a', a, c\}$ according to part (a).

(c) Suppose without loss of generality that c is the side opposite the 120° angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

This equation describes an ellipse in the plane passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$.[1] Moreover, every point with positive rational coordinates can be connected to the point $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$\begin{aligned} x^2 + x \cdot (tx - 1) + (tx - 1)^2 &= 1, \\ (t^2 + t + 1)x^2 - (2t + 1)x &= 0. \end{aligned}$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

[1] More specifically, we can rewrite the equation as $\frac{u^2}{2/3} + \frac{v^2}{2} = 1$ where $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$. This is an ellipse centered at $(0, 0)$ whose semimajor axis has length $\sqrt{2}$ lying along the v-axis, i.e. the line $u = 0$ or equivalently $y = -x$. The semiminor axis has length $\sqrt{2/3}$ lying along the u-axis, i.e. the line $v = 0$ or equivalently $y = x$.

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m,n)=1$,

$$\begin{aligned}
\gcd(2mn+n^2, m^2+mn+n^2) &= \gcd(n\cdot(2m+n), m^2+mn+n^2)\\
&= \gcd(2m+n, m^2+mn+n^2)\\
&= \gcd(2m+n, m^2+mn+n^2-n\cdot(2m+n))\\
&= \gcd(2m+n, m^2-mn) = \gcd(2m+n, m\cdot(m-n))\\
&= \gcd(2m+n, m-n) = \gcd(3n, m-n).
\end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n+3k$. Then $\gcd(n,k)=1$ and

$$\gcd(3n, m-n) = \gcd(3n, 3k) = 3\gcd(n,k) = 3.$$

Otherwise,

$$\gcd(3n, m-n) = \gcd(n, m-n) = \gcd(n,m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn+n^2}{3}, \quad b = \frac{m^2-n^2}{3}, \quad c = \frac{m^2+mn+n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r-s$ and $m = r+2s$. Then

$$\begin{aligned}
a &= \frac{2(r+2s)(r-s)+(r-s)^2}{3} = r^2-s^2,\\
b &= \frac{(r+2s)^2-(r-s)^2}{3} = 2rs+s^2,\\
c &= \frac{(r+2s)^2+(r+2s)(r-s)+(r-s)^2}{3} = r^2+rs+s^2,
\end{aligned}$$

so the result still holds with r and s in place of m and n.

2 Complex Numbers I: Algebra

Throughout, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

2.1 Review problems

1. *Arithmetic.* Let $z = -3 + 3i$ and $w = -4 - 2i$. Compute each of the following:
   (a) $\operatorname{Re} z$
   (b) $\operatorname{Im} w$
   (c) $z + w$
   (d) $z - w$
   (e) zw
   (f) z/w
2. *A quadratic with real coefficients.* Find all complex solutions to the equation $z^2 + 5 = 4z$.
3. *Real-valued products.* Find all complex numbers z for which $(-4 + 2i)z$ is real.
4. *Powers of i and periodic sequences.*
   (a) Show that $i^4 = 1$.
   (b) Find all complex solutions to the equation $z^4 = 1$ and write each one as a power of i.
   (c) Let $z_1, z_2, z_3, \ldots$ be a 4-periodic sequence of complex numbers, meaning that $z_{n+4} = z_n$ for all positive integers n. Show that there exist complex numbers a, b, c, d such that
   $$z_n = a + b \cdot i^n + c \cdot i^{2n} + d \cdot i^{3n}$$
   for all n.
5. *Complex conjugation.* Given a complex number $z = x + yi$, the **complex conjugate** of z is defined to be $\overline{z} = x - yi$.
   (a) Compute $\overline{943 - 319i}$.
   (b) Prove the following properties of complex conjugation:
      i. $\overline{(\overline{z})} = z$ for all complex numbers z.
      ii. $\overline{z + w} = \overline{z} + \overline{w}$ for all complex numbers z and w.
      iii. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ for all complex numbers z and w.
      iv. $\operatorname{Re} z = (z + \overline{z})/2$
      v. $\operatorname{Im} z = (z - \overline{z})/2i$

   Remark: From these, we can show that $\overline{z - w} = \overline{z} - \overline{w}$ for all complex numbers z and w, that $\overline{z/w} = \overline{z}/\overline{w}$ for all complex numbers z and $w \neq 0$, and that $\overline{z^n} = \overline{z}^n$ for all complex numbers z and for all integers n (with $z \neq 0$ when $n \leq 0$).
6. *Magnitude.* Given a complex number $z = x + yi$, the **magnitude** or **absolute value** of z is defined to be $|z| = \sqrt{x^2 + y^2}$.

(a) Compute $|21 + 20i|$.

(b) Prove the following properties of the magnitude:

i. $|z|^2 = z \cdot \overline{z}$ for all complex numbers z.

ii. $|zw| = |z| \cdot |w|$ for all complex numbers z and w.

Remark: From these, it follows that $|z/w| = |z|/|w|$ for all complex numbers z and $w \neq 0$, and that $|z^n| = |z|^n$ for all integers n (with $z \neq 0$ when $n \leq 0$).

7. *Square roots of complex numbers.*

(a) Find a complex number w for which $w^2 = -16 + 30i$.

(b) Find the two complex numbers z satisfying $2z^2 - (8 + 4i)z + (14 - 7i) = 0$.

(c) Prove that for every complex number z, there is a complex number w for which $w^2 = z$.
Remark: It follows from this that every quadratic polynomial with complex coefficients has complex roots (with roots given by the familiar quadratic formula).

2.2 Challenge problems

8. (a) Show that $\operatorname{Re} z \leq |z|$ for all complex numbers z. When does equality occur?

(b) *Triangle inequality.* Using part (a), or otherwise, show that

$$|z + w| \leq |z| + |w|$$

for all complex numbers z and w. When does equality occur?

9. In this problem, we work through one formal construction of the complex numbers.
Let $\mathcal{C}$ be the set of all ordered pairs of real numbers, and define operations $\oplus$ and $\otimes$ on $\mathcal{C}$ by

$$\begin{aligned}(a, b) \oplus (c, d) &= (a + c, b + d),\\ (a, b) \otimes (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

We call $\oplus$ and $\otimes$ the addition and multiplication on $\mathcal{C}$, respectively.

(a) Let $u = (-2, -4)$, $v = (-3, 1)$, and $w = (0, 4)$. Verify each of the following:

$$\begin{aligned}u \oplus (v \oplus w) &= (u \oplus v) \oplus w,\\ u \otimes (v \otimes w) &= (u \otimes v) \otimes w,\\ u \oplus v &= v \oplus u,\\ u \otimes v &= v \otimes u,\\ u \otimes (v \oplus w) &= (u \otimes v) \oplus (u \otimes w),\\ u \oplus (0, 0) &= u,\\ v \otimes (1, 0) &= v.\end{aligned}$$

Also find pairs a_v and m_u for which $v \oplus a_v = (0, 0)$ and $m_u \otimes u = (1, 0)$.

One can show that these equalities hold in general and that for any $z \in \mathcal{C}$, we can find a unique $a_z \in \mathcal{C}$ for which $z \oplus a_z = (0,0)$. This a_z is the **additive inverse** of z and is denoted $-z$. Similarly, for any non-zero $z \in \mathcal{C}$, we can find a unique $m_z \in \mathcal{C}$ for which $z \otimes m_z = (1,0)$. This m_z is the **multiplicative inverse** of z and is denoted z^{-1}.

These properties, collectively called the "field axioms," are enough to derive all of the usual algebraic facts that we are familiar with in the context of real number algebra. What remains is to check that $\mathcal{C}$ "does what we expect the complex numbers to do."

(b) Prove that for any two real numbers x and y,

$$(x,0) \oplus (y,0) = (x+y,0) \quad \text{and} \quad (x,0) \otimes (y,0) = (xy,0).$$

This shows that the elements $(r,0)$ for $r \in \mathbb{R}$, with operations $\oplus$ and $\otimes$, "act like" the real numbers with the usual addition and multiplication operations $+$ and $\times$. As such, we can regard $\mathbb{R}$ as being contained within $\mathcal{C}$ by identifying $r \in \mathbb{R}$ with $(r,0) \in \mathcal{C}$, and then $\otimes$ and $\otimes$ extend $+$ and $\times$ from $\mathbb{R}$ to all of $\mathcal{C}$. As such, when r is a real number we simply write r instead of $(r,0)$. Moreover, from now on, we write $+$ and $\times$ (or $\cdot$) instead of $\oplus$ and $\otimes$. We also introduce the subtraction and division operations as $z-w = z+(-w)$ and $z/w = z \cdot w^{-1}$.

(c) Show that $(0,1) \times (0,1) = -1$ and $(0,-1) \times (0,-1) = -1$.

This shows that $\mathcal{C}$ has square roots of -1, as expected. We can now recover the usual notation by defining $i = (0,1)$ and then observing that $(x,y) = x + y \cdot i$. Henceforth, we can forget about the underlying ordered pairs and replace $\mathcal{C}$ with the usual $\mathbb{C}$.

10. A function $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-*automorphism of* $\mathbb{C}$ if

$$f(z+w) = f(z) + f(w) \quad \text{and} \quad f(zw) = f(z) \cdot f(w)$$

for all $z, w \in \mathbb{C}$ and $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$, then $f(i) = i$ or $f(i) = -i$.

(b) Show that the only two $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity function $f(z) = z$ and the conjugation function $f(z) = \overline{z}$.

2.3 Answers

1. (a) -3
   (b) -2
   (c) $-7+i$
   (d) $1+5i$
   (e) $18-6i$
   (f) $\frac{3}{10}-\frac{9}{10}i$
2. $2+i$ and $2-i$
3. Let $z=a+bi$. Then $(-4+2i)(a+bi)=(-4a-2b)+(2a-4b)i$. For this product to be real, we need $2a-4b=0$, so $a=2b$. Hence $z=2b+bi=b(2+i)$, so $\boxed{\text{any real multiple of } 2+i}$ would do the job.
4. (a) We have $i^2=-1$, so then $i^3=-i$, so then $i^4=-i^2=-(-1)=1$.
   (b) $1=i^0$, $i=i^1$, $-1=i^2$, and $-i=i^3$.
   (c) The right hand side is also 4-periodic, so it suffices to show that there exist a,b,c,d with

   $$z_n = a+b\cdot i^n + c\cdot i^{2n} + d\cdot i^{3n}$$

   for $n=1,2,3,4$. This gives us the system of linear equations

   $$\begin{aligned} a+ib-c-id &= z_1,\\ a-b+c-d &= z_2,\\ a-ib-c+id &= z_3,\\ a+b+c+d &= z_4. \end{aligned}$$

   This system does indeed have a solution, namely

   $$\begin{aligned} a &= \frac{z_1+z_2+z_3+z_4}{4},\\ b &= \frac{-iz_1-z_2+iz_3+z_4}{4},\\ c &= \frac{-z_1+z_2-z_3+z_4}{4},\\ d &= \frac{iz_1-z_2-iz_3+z_4}{4}. \end{aligned}$$

5. (a) $943+319i$
   (b) Let $z=x+yi$ and $w=a+bi$ throughout.
      i. $\overline{(\overline{z})}=\overline{x-yi}=x+yi=z$
      ii. $\overline{z+w}=\overline{(x+a)+(y+b)i}=(x+a)-(y+b)i$
      $\overline{z}+\overline{w}=(x-yi)+(a-bi)=(x+a)-(y+b)i$

iii. $\overline{z \cdot w} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$
$\overline{z} \cdot \overline{w} = (x - yi)(a - bi) = (xa - yb) - (xb + ya)i$

iv. $\frac{z + \overline{z}}{2} = \frac{(x+yi)+(x-yi)}{2} = x = \operatorname{Re} z$

v. $\frac{z - \overline{z}}{2i} = \frac{(x+yi)-(x-yi)}{2i} = y = \operatorname{Im} z$

6. (a) 29

(b) Let $z = x + yi$ and $w = a + bi$ throughout.

i. $z \cdot \overline{z} = (x + yi)(x - yi) = x^2 - (yi)^2 = x^2 + y^2 = |z|^2$

ii. $|zw| = \sqrt{|zw|^2} = \sqrt{zw \cdot \overline{zw}} = \sqrt{z\overline{z} \cdot w\overline{w}} = \sqrt{|z|^2 \cdot |w|^2} = |z| \cdot |w|$

7. (a) Let $w = a + bi$, so then $w^2 = (a^2 - b^2) + 2abi$. Therefore, we need $a^2 - b^2 = -16$ and $ab = 15$. Substituting $b = 15/a$,

$$a^2 - \frac{225}{a^2} = -16 \iff a^4 + 16a^2 - 225 = 0.$$

Since a is real, $a^2 > 0$, so the only viable solution to the quadratic in a^2 is

$$a^2 = \frac{-16 + \sqrt{16^2 - 4(1)(-225)}}{2} = \frac{-16 + \sqrt{1156}}{2} = \frac{-16 + 34}{2} = 9.$$

Therefore, $a = 3$, in which case $b = 5$, or $a = -3$, in which case $b = -5$. Hence the two square roots of $-16 + 30i$ are $\pm(3 + 5i)$.

(b) By the quadratic formula,

$$\begin{aligned} z &= \frac{(8 + 4i) \pm \sqrt{(8 + 4i)^2 - 4(2)(14 - 7i)}}{2(2)} \\ &= \frac{(8 + 4i) \pm \sqrt{(64 + 64i - 16) - (112 - 56i)}}{4} \\ &= \frac{(8 + 4i) \pm \sqrt{-64 + 120i}}{4} \\ &= \frac{(8 + 4i) \pm 2\sqrt{-16 + 30i}}{4} \\ &= \frac{(4 + 2i) \pm (3 + 5i)}{2} \\ &= \frac{7 + 7i}{2} \text{ or } \frac{1 - 3i}{2}. \end{aligned}$$

(c) If $z = x + yi$ and $w = a + bi$ satisfies $w^2 = z$, then following the same procedure as in part (a) yields, when $y \neq 0$,

$$a^4 - xa^2 - \frac{y^2}{4} = 0.$$

The product of the roots of the quadratic $T^2 - xT - y^2/4$ is $-y^2/4 < 0$, so there is a positive root α and a negative root β. Taking $a^2 = \alpha$, so then $a = \sqrt{\alpha}$ and $b = y/2a$, gives us a square root of z.

In the case $y = 0$, either $a = 0$ or $b = 0$. If $x \geq 0$, then take $a = \sqrt{x}$ and $b = 0$, and if $x < 0$, then take $a = 0$ and $b = \sqrt{-x}$.

8. (a) For the inequality,

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geq \sqrt{(\operatorname{Re} z)^2} = |\operatorname{Re} z| \geq \operatorname{Re} z.$$

For the first inequality step, we have equality if and only if $\operatorname{Im} z = 0$. For the second inequality step, we have equality if and only if $\operatorname{Re} z \geq 0$. Putting these together, equality holds in the overall inequality if and only if z is a non-negative real number.

(b) As both sides are non-negative, it suffices to prove the squared inequality. We have

$$\begin{aligned} |z+w|^2 &= (z+w) \cdot \overline{(z+w)} \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z| \cdot |w| + |w|^2 \\ &= (|z| + |w|)^2, \end{aligned}$$

as required. Equality holds when we have equality in $\operatorname{Re}(z\overline{w}) = |z\overline{w}|$, and by part (a), we know that this requires $z\overline{w} \geq 0$. If $w = 0$, this condition holds. Otherwise, $w\overline{w} > 0$ and $z\overline{w} \geq 0$, so $z/w \geq 0$. That is, $z = \lambda w$ for a non-negative real number λ.

9. (a) $u \oplus (v \oplus w) = (u \oplus v) \oplus w = (-5, 1)$
$u \otimes (v \otimes w) = (u \otimes v) \otimes w = (-40, 40)$
$u \oplus v = v \oplus u = (-5, -3)$
$u \otimes v = v \otimes u = (10, 10)$
$u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w) = (26, 2)$
$a_v = (3, -1)$
$m_u = (-1/10, /5)$

(b) $(x, 0) \oplus (y, 0) = (x + y, 0 + 0) = (x + y, 0)$
$(x, 0) \otimes (y, 0) = (x \cdot y - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0)$

(c) $(0, 1) \times (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1$
$(0, -1) \times (0, -1) = (0 \cdot 0 - (-1) \cdot (-1), 0 \cdot (-1) + (-1) \cdot 0) = (-1, 0) = -1$

10. (a) We have

$$f(i)^2 = f(i) \cdot f(i) = f(i \cdot i) = f(-1) = -1,$$

so either $f(i) = i$ or $f(i) = -i$.

(b) Let $z = x + yi$. Then

$$f(z) = f(x + yi) = f(x) + f(yi) = x + f(y) \cdot f(i) = x + y \cdot f(i).$$

If $f(i) = i$, then $f(z) = x + yi = z$. If $f(i) = -i$, then $f(z) = x - yi = \overline{z}$.