Precalculus Practice Problems: Midterm 1

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Contents

1 Trig (I): Right Triangle and Unit Circle

1.1 Review problems

1. *Unit conversions for angles.*

   (a) 360 degrees to radians

   (b) π radians to degrees

   (c) 60 degrees to radians

   (d) $3\pi/4$ radians to degrees

   (e) $\pi/5$ degrees to radians

2. *Trig functions as ratios of lengths.* Let ABC be a triangle with a right angle at B. Suppose $AB = 8$ and $BC = 15$.

   (a) Evaluate $\tan A$ and $\cot A$.

   (b) Find the length of AC.

   (c) Evaluate $\sin A$, $\cos A$, $\sec A$, and $\csc A$.

3. *Using one trig function to compute another.* Throughout, assume θ is acute.

   (a) If $\sin\theta = 1/3$, what is $\cos\theta$?

   (b) If $\sec\theta = \sqrt{10}$, what is $\tan\theta$?

   (c) If $\tan\theta = 2/5$, what is $\csc\theta$?

4. *Important acute angles.*

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
30							
45							
	$\frac{\pi}{3}$						

5. *Unit circle calculations.*

   (a) $\cos(0)$
   (b) $\sin(150°)$
   (c) $\cos(-3\pi/4)$
   (d) $\sin(7\pi/3)$
   (e) $\cos(330°)$
   (f) $\sin(-\pi/4)$

6. *Unit circle identities.* Express each of the following in terms of $\sin\theta$ and/or $\cos\theta$.

   (a) $\sin(\pi - \theta)$
   (b) $\cos(\pi - \theta)$
   (c) $\sin(\pi + \theta)$
   (d) $\cos(\pi + \theta)$
   (e) $\sin(-\theta)$
   (f) $\cos(-\theta)$
   (g) $\sin(\frac{\pi}{2} - \theta)$
   (h) $\cos(\frac{\pi}{2} - \theta)$
   (i) $\sin(\frac{\pi}{2} + \theta)$
   (j) $\cos(\frac{\pi}{2} + \theta)$

7. *Some triangle geometry.* In acute triangle ABC, it is given that $AB = 13$, that $BC = 14$, and that $\sin B = 12/13$.

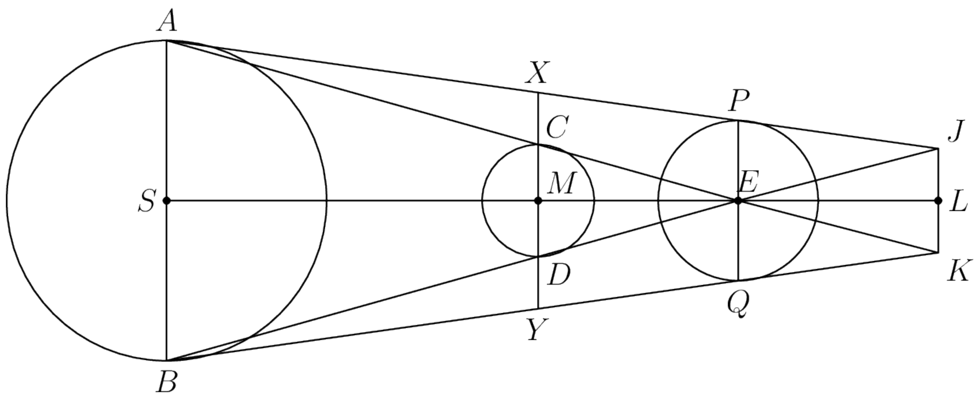
   (a) Find the area of triangle ABC.
   (b) Find the length of AC.
   (c) Find $\sin A$.

1.2 Challenge problems

8. Much of classical trigonometry was done in terms of the *chord function*, which is defined for angles $0 < \theta < 180°$ as follows. Let O be the center of a circle of radius 1, and let A and B be points on the circle so that $\angle AOB = \theta$. Then $\operatorname{chd}\theta = AB$.

   (a) Compute $\operatorname{chd} 90°$, $\operatorname{chd} 60°$, and $\operatorname{chd} 30°$.
   (b) Express $\operatorname{chd}\theta$ in terms of the sine function.
   (c) Prove that $\operatorname{chd}^2\theta + \operatorname{chd}^2(180° - \theta) = 4$.

9. A method for estimating the distances to the Sun and the Moon using measurements of solar and lunar eclipses dates back to the Greek astronomer Hipparchus (c.~190 BC to c.~120 BC). In this problem, we work through the relevant geometric argument.



In the diagram, points S, M, E, and L lie on a line, with S, M, and E representing the centers of the Sun, Moon, and Earth, respectively. Segments $\overline{AB}$, $\overline{CD}$, and $\overline{PQ}$ are diameters of their respective circles, all perpendicular to $\overline{SL}$. Segment $\overline{JK}$ is also perpendicular to $\overline{SL}$. Points A, X, P, J are collinear, points B, Y, Q, K are collinear, points X, C, D, Y are collinear, points A, C, E are collinear, and points B, D, E are collinear. Finally, E is the midpoint of $\overline{LM}$. Let $PE = 1$ (so lengths are in terms of Earth radii) and define

$$\ell = EM = EL, \quad s = ES, \quad \theta = \angle CEM, \quad \phi = \angle JEL.$$

(a) Show that

$$s = \frac{\ell}{(\tan\theta + \tan\phi)\ell - 1}$$

(b) (Calculator recommended) The measurements used by Hipparchus were

$$\ell \approx 67\tfrac{1}{3}, \quad \theta \approx 0.277^\circ, \quad \phi \approx 0.693^\circ.$$

Given these measurements, what value do we get for s?

(c) (Calculator recommended) Currently, our measurements for the same quantities are

$$\ell = 60.268, \quad \theta \approx 0.267^\circ, \quad \phi \approx 0.746^\circ.$$

Given these measurements, what value do we get for s?

Remark: The true value of s is $s \approx 23{,}455$, so some of the approximations made in order to set up the diagram turn out to be substantial sources of error.

10. A *Pythagorean triple* is a triple (X, Y, Z) of positive integers for which $X^2 + Y^2 = Z^2$. Note that if (X, Y, Z) is a Pythagorean triple, then $(x, y) = (X/Z, Y/Z)$ is a point on the unit circle whose coordinates are rational numbers.

    (a) Let $O = (-1, 0)$. If $P \neq O$ has rational coordinates and lies on the unit circle, then the slope of $\overline{OP}$ is rational. Conversely, show that if ℓ is a line passing through O which has rational slope, then the other point $P \neq O$ at which ℓ intersects the unit circle must have rational coordinates.

    (b) Use part (a) to show that every point on the unit circle with rational coordinates can be written in the form $\left(\dfrac{n^2 - m^2}{n^2 + m^2}, \dfrac{2mn}{n^2 + m^2} \right)$ for integers m and n.

    Remark: A result going back to Euclid states that every *primitive* Pythagorean triple, meaning a Pythagorean triple (X, Y, Z) where $\gcd(X, Y, Z) = 1$, can be written as either

$$X = n^2 - m^2, \quad Y = 2mn, \quad Z = n^2 + m^2$$

    with $\gcd(m, n) = 1$ and $m + n$ odd, or in the same form with X and Y swapped.

1.3 Answers

1. (a) 2π radians
   (b) $180°$
   (c) $\pi/3$ radians
   (d) $135°$
   (e) $\pi^2/900$ radians
2. (a) $\tan A = \frac{15}{8}$; $\cot A = \frac{8}{15}$
   (b) 17
   (c) $\sin A = \frac{15}{17}$; $\cos A = \frac{8}{17}$; $\sec A = \frac{17}{8}$; $\csc A = \frac{17}{15}$
3. Since θ is acute, all six basic trig functions of θ have positive values.
   (a) Since $\sin^2\theta + \cos^2\theta = 1$, we know that $\cos^2\theta = \frac{8}{9}$, so $\cos\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.
   (b) Since $\tan^2\theta + 1 = \sec^2\theta = 10$, we have $\tan\theta = 3$.
   (c) We build a right triangle ABC with a right angle at C and with leg lengths $AC = 5$ and $BC = 2$, so that $\tan\angle A = \frac{2}{5}$ and hence $\angle A = \theta$. Then, $AB = \sqrt{(AC)^2 + (BC)^2} = \sqrt{29}$ and $\csc\theta = \csc A = \frac{AB}{BC} = \frac{\sqrt{29}}{2}$.
4. The completed table is below.

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$

5. (a) 1
   (b) $1/2$
   (c) $-1/\sqrt{2}$
   (d) $\sqrt{3}/2$
   (e) $\sqrt{3}/2$
   (f) $-1/\sqrt{2}$
6. (a) $\sin\theta$
   (b) $-\cos\theta$

(c) $-\sin\theta$

(d) $-\cos\theta$

(e) $-\sin\theta$

(f) $\cos\theta$

(g) $\cos\theta$

(h) $\sin\theta$

(i) $\cos\theta$

(j) $-\sin\theta$

7. (a) We use the formula $[ABC] = \frac{1}{2} \cdot BA \cdot BC \cdot \sin B = 84$.

   (b) Let D be the point on $\overline{BC}$ for which $\overline{AD} \perp \overline{BC}$. In right triangle ABD, we have that $AD = AB \sin B = 12$, so $BD = \sqrt{(AB)^2 - (AD)^2} = 5$. Then, $CD = BC - BD = 9$. (This is where we use the fact that the triangle is acute: it implies that D is between B and C.) Finally, $AC = \sqrt{(CD)^2 + (DA)^2} = 15$.

   (c) From $[ABC] = \frac{1}{2} \cdot AB \cdot AC \cdot \sin A$, we have $\sin A = \frac{2 \cdot [ABC]}{AB \cdot AC} = \frac{56}{65}$.

8. (a) For $\operatorname{chd} 90^\circ$, we have $AB = \sqrt{(AO)^2 + (OB)^2} = \sqrt{2}$.
   For $\operatorname{chd} 60^\circ$, note that triangle AOB is equilateral, so $AB = 1$.
   For $\operatorname{chd} 30^\circ$, let C be the point on $\overline{OB}$ for which $\overline{AC} \perp \overline{OB}$. Then $AC = \sin 30^\circ = \frac{1}{2}$ and $OC = \cos 30^\circ = \frac{\sqrt{3}}{2}$, so $BC = \frac{2-\sqrt{3}}{2}$. The Pythagorean theorem gives

$$AB = \sqrt{(AC)^2 + (CB)^2} = \sqrt{\frac{1}{4} + \frac{7 - 4\sqrt{3}}{4}} = \frac{\sqrt{8 - 4\sqrt{3}}}{2} = \frac{\sqrt{6} - \sqrt{2}}{2}.$$

   (b) Let M be the foot of the perpendicular from O to $\overline{AB}$. Since $OA = OB$, we have $MA = MB$ and $\angle AOM = \theta/2$, so $AB = 2(MA) = 2\sin(\theta/2)$.

   (c) Let $\overline{AC}$ be a diameter of a unit circle with center O and let B lie on the circle so that $\angle AOB = \theta$. Then $\angle BOC = 180^\circ - \theta$, so $AB = \operatorname{chd}\theta$ and $BC = \operatorname{chd}(180^\circ - \theta)$. As $\overline{AC}$ is a diameter, $\angle ABC = 90^\circ$. By the Pythagorean theorem,

$$\operatorname{chd}^2\theta + \operatorname{chd}^2(180^\circ - \theta) = (AB)^2 + (BC)^2 = (AC)^2 = 4.$$

9. (a) Since $\overline{PE}$ is the midline of trapezoid $JLMX$, we have $JL + XM = 2(PE) = 2$. From right triangles JLE and CME, we have $JL = \ell \tan\phi$ and $CM = \ell \tan\theta$, so $XC = 2 - \ell(\tan\theta + \tan\phi)$. Now, using the similarity $\triangle AXC \sim \triangle APE$,

$$\frac{XC}{PE} = \frac{\text{height from } A \text{ to } \overline{XC}}{\text{height from } A \text{ to } \overline{PE}} = \frac{s - \ell}{s} = 1 - \frac{\ell}{s}.$$

   Solving for s gives the desired formula.

   (b) $s \approx 481.038$

   (c) $s \approx 918.779$

10. (a) Let t be the slope of ℓ, so the point-slope form of ℓ using O as the reference point is $y = t(x+1)$. Substituting this into $x^2+y^2 = 1$, we get $x^2+t^2(x+1)^2 = 1$. Rearranging,

$$(t^2+1)x^2 + 2t^2x + (t^2-1) = 0.$$

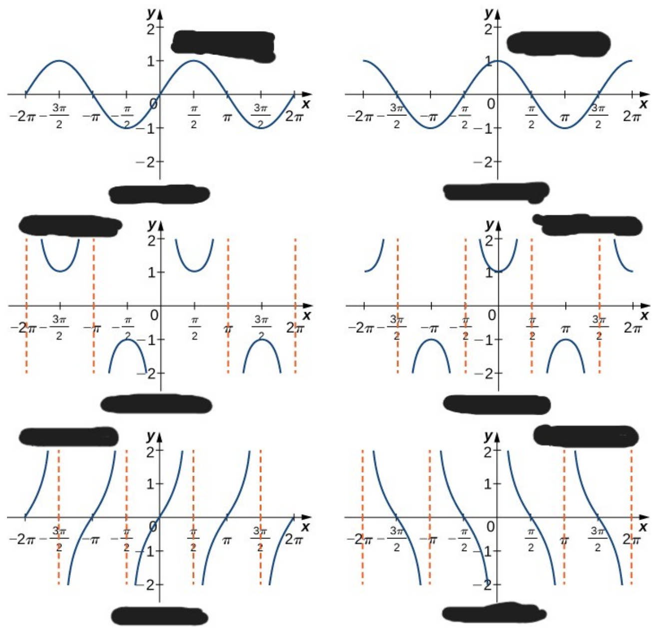
We know that $x = -1$ is one solution (corresponding to point O), so the other solution, by Vieta's formula for the product, is $x = \frac{1-t^2}{1+t^2}$. Substituting this into the equation for ℓ gives $y = \frac{2t}{1+t^2}$. When t is rational, these coordinates are both rational.

(b) In our result from the previous part, let $t = m/n$, where m and n are relatively prime, to get the desired form for any point $P \neq O$. For the point O itself, let $n = 0$ and $m = 1$. (This corresponds to the vertical line $x = -1$ tangent to the unit circle at O.)

2 Trig (II): Graphs and Inverses

2.1 Review problems

1. *The six basic graphs.* Match each of the graphs below to one of the six basic trig functions.



2. *Period and frequency.* A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* if there is a non-zero real number T such that $f(x + T) = f(x)$ for all real x. Any such T is a *period* of f, and the smallest such positive T, if it exists, is the *fundamental period* of f. If f is a periodic function with fundamental period T, the *natural frequency* of f is $\nu = 1/T$ while the *angular frequency* of f is $\omega = 2\pi/T = 2\pi\nu$. Usually, "the period" of a periodic function means its fundamental period, while "a period" can be any period (not just the fundamental period). For each of the six basic trig functions, what are the period, natural frequency, and angular frequency?

3. *Transformed sinusoidal waves.* For each of the following functions $\mathbb{R} \to \mathbb{R}$, find the period, amplitude, and phase shift relative to $\sin(\omega x)$, where ω is the angular frequency of the function.

   (a) $\sin x$

   (b) $\cos x$

   (c) $3\sin(5x - \frac{\pi}{7})$

   (d) $2\sin(2x) - 2\cos(2x)$

4. *Domain and range.* What are the standard domains and ranges of arcsin, arccos, and arctan?

5. *Calculating inverses.*

   (a) $\arcsin(1/2)$

   (b) $\arccos\left(-1/\sqrt{2}\right)$

   (c) $\arcsin\left(-\sqrt{3}/2\right)$

   (d) $\arctan(-1)$

   (e) $\arcsin(\sin(7\pi/6))$

   (f) $\cos(\arccos(-1/3))$

   (g) $\cos(\arctan(1/2))$

6. *Counting solutions.* Find the number of solutions for x in each of the following equations.

   (a) $\sin\theta = 0.5$ when $0 \le \theta < 2\pi$

   (b) $\cos\theta = -2$ when $0 \le \theta < 2\pi$

   (c) $\sec\theta = 1$ when $-\pi < \theta \le \pi$

   (d) $\tan\theta = 1$ when $-\pi < \theta \le \pi$

   (e) $\sin(3\theta) = 0.2024$ when $0 \le \theta < 10\pi$

   (f) $\cos(\frac{22}{7}\theta) = 0.5$ when $-20 < \theta < 20$

7. What are the standard domains and ranges of $\sec^{-1}$, $\csc^{-1}$, and $\cot^{-1}$?

2.2 Challenge problems

8. Karen has a calculator which only has seven buttons: sin, cos, tan, arcsin, arccos, arctan, and Reset. The first six apply these functions to the number in the display, while Reset changes the display back to its default state of showing 0. All calculations assume radian measure.

   (a) Starting from a positive real number x in the display, show that there is a sequence of buttons that changes the display to $1/x$.

   (b) Starting from a non-negative real number x in the display, show that there is a sequence of buttons that changes the display to $\sqrt{x^2+1}$.

   (c) Show that for every positive rational number q, there is a sequence of buttons that changes the display from 0 to $\sqrt{q}$.

9. Find the period of the following functions, or show that no period exists.

   (a) $\sin(3x) + \sin(4x)$

   (b) $\sin(20x) + \sin(24x)$

   (c) $\sin(x) + \sin(\sqrt{2}x)$

10. A function $f : \mathbb{C} \to \mathbb{C}$ is *doubly-periodic* if there are non-zero constants $u, v \in \mathbb{C}$ such that u/v is non-real and $f(z) = f(z+u) = f(z+v)$ for all complex numbers z. Write down an example of a non-constant doubly-periodic function.

2.3 Answers

1. top left is sine; top right is cosine;
   middle left is cosecant; middle right is secant;
   bottom left is tangent; bottom right is cotangent

2. sine, cosine, secant, cosecant have period 2π, natural frequency $\frac{1}{2\pi}$, angular frequency 1;
   theirangent, cotangent have period π, natural frequency $\frac{1}{\pi}$, angular frequency 2

3. (a) period 2π; amplitude 1; phase shift 0
   (b) period 2π; amplitude 1; phase shift $-\pi/2$ since $\cos x = \sin\left(x + \frac{\pi}{2}\right)$
   (c) period $2\pi/5$; amplitude 3; phase shift $\pi/35$
   (d) period π; amplitude $2\sqrt{2}$ since $2\sin(2x) - 2\cos(2x) = 2\sqrt{2}\sin\left(2x - \frac{\pi}{4}\right)$; phase shift $\pi/8$

4. arcsin has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$;
   arccos has domain $[-1, 1]$ and range $[0, \pi]$;
   arctan has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$

5. (a) $\pi/6$
   (b) $3\pi/4$
   (c) $-\pi/3$
   (d) $-\pi/4$
   (e) $-\pi/6$
   (f) $-1/3$
   (g) $2/\sqrt{5}$

6. (a) 2
   (b) 0
   (c) 1
   (d) 2
   (e) 30
   (f) 40

7. $\sec^{-1}$ has domain $(-\infty, -1] \cup [1, +\infty)$ and range $[0, \pi/2) \cup (\pi/2, \pi]$;
   $\csc^{-1}$ has domain $(-\infty, -1] \cup [1, +\infty)$ and range $[-\pi/2, 0) \cup (0, \pi/2]$;
   $\cot^{-1}$ has domain $\mathbb{R}$ and range $(0, \pi)$

8. First note for any acute angle θ that $C(\theta) = \arccos(\sin\theta) = \pi/2 - \theta$.
   (a) Given $x > 0$, the angle $\arctan x$ is acute, so $A(x) = \tan(C(\arctan x)) = 1/x$.
   (b) Given $x \geq 0$, we have $\cos(\arctan x) = 1/\sqrt{x^2+1}$, so $B(x) = A(\cos(\arctan x)) = \sqrt{x^2+1}$.

(c) Let $q = m/n$ where $\gcd(m,n) = 1$. Note that $B^{-1}(\sqrt{q}) = \sqrt{q-1}$ and $A^{-1}(\sqrt{q}) = \sqrt{1/q}$ correspond to steps of the Euclidean algorithm on m and n. Since $\gcd(m,n) = 1$, we can work backwards until we reach $0/1 = 0$. Running our steps in reverse gives us a sequence of button presses that goes from 0 to $\sqrt{q}$.

9. For each part, denote the function by f and the period by T.

(a) From $f(T) = f(0)$, we get $\sin(3T) + \sin(4T) = 0$, while from $f(\pi) = f(\pi + T)$, we get $-\sin(3T) + \sin(4T) = 0$, which means $\sin(3T) = \sin(4T) = 0$. The only values which satisfy this are multiples of π. However, $f(-\pi/2) = 1$ while $f(\pi/2) = -1$, so $T \neq \pi$. Therefore, the smallest positive real number that T could be is 2π, and $f(x+2\pi) = f(x)$ for all x since each term individually has period 2π, so $T = 2\pi$ works.

(b) From $f(T) = f(0)$, we get $\sin(20T) + \sin(24T) = 0$, while from $f(\pi/4) = f(\pi/4 + T)$, we get $-\sin(20T) + \sin(24T) = 0$, so $\sin(20T) = \sin(24T) = 0$. This holds when T is a multiple of $\pi/4$, and by similar reasoning to part (a), $\pi/4$ fails while $T = \pi/2$ works.

(c) From $f(T) = f(0)$, we get $\sin T + \sin(\sqrt{2}T) = 0$. However, from $f(2\pi) = f(2\pi + T)$,

$$\sin(2\sqrt{2}\pi) = \sin T + \sin(2\sqrt{2}\pi + \sqrt{2}T).$$

Subtracting these two equations and rearranging,

$$\sin(2\sqrt{2}\pi + \sqrt{2}T) = \sin(2\sqrt{2}\pi) + \sin(\sqrt{2}T).$$

In general,

$$\sin a + \sin b = \sin(a+b) = \sin a \cos b + \sin b \cos a$$

only when $\sin b = 0$ and $\cos b = 1$ or when $\sin b = -\sin a$ and $\cos b = \cos a$; this can be proved by holding a fixed and solving for $\sin b$ and $\cos b$ using $\sin^2 b + \cos^2 b = 1$. With $a = 2\sqrt{2}\pi$ and $b = \sqrt{2}T$, the first case would give us $\sin T = \sin(\sqrt{2}T) = 0$. However, this would force both T and $\sqrt{2}T$ to be multiples of π, which is impossible when T is non-zero since $\sqrt{2}$ is irrational. For the second case, the conditions on $\sin b$ and $\cos b$ tell us that $a + b = 2m\pi$ for an integer m. We can run the exact same argument with $f(-2\pi) = f(-2\pi + T)$ to show that $a' + b = 2n\pi$ for an integer n, where $a' = -2\sqrt{2}\pi$. Subtracting gives us $4\sqrt{2}\pi = (2m - 2n)\pi$, which is impossible by irrationality of $\sqrt{2}$. Hence f has no period.

10. Given two periodic functions $g, h : \mathbb{R} \to \mathbb{R}$, we can define $f : \mathbb{C} \to \mathbb{C}$ by $f(x + yi) = g(x)h(y)$ for x, y real, and this will be doubly-periodic. In complex notation, this is

$$f(z) = g\left(\frac{z + \overline{z}}{2}\right) h\left(\frac{z - \overline{z}}{2i}\right).$$

Writing down an example that only uses z (without $\overline{z}$ or $|z|$) turns out to be substantially more difficult. A classic example central to the theory of "nice" doubly-periodic functions is the *Weierstrass elliptic function*: if we want the periods to take the form $m\omega_1 + n\omega_2$, where ω_1/ω_2 is non-real and m, n are integers, then we define

$$\wp(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{m,n}(z), \quad f_{m,n}(z) = \begin{cases} \frac{1}{[z-(m\omega_1+n\omega_2)]^2} - \frac{1}{(m\omega_1+n\omega_2)^2} & (m,n) \neq (0,0); \\ \frac{1}{z^2} & (m,n) = (0,0). \end{cases}$$

3 Trig (III): Identities

3.1 Review problems

1. *Angle sum and difference identities.*
   (a) $\sin(\alpha + \beta) =$
   (b) $\sin(\alpha - \beta) =$
   (c) $\cos(\alpha + \beta) =$
   (d) $\cos(\alpha - \beta) =$
   (e) $\tan(\alpha + \beta) =$
   (f) $\tan(\alpha - \beta) =$
2. *Calculation.*
   (a) Compute $\cos(75°)$.
   (b) Supposing $\sin(\alpha) = 5/13$ and $\sin(\beta) = 3/5$, compute $\sin(\alpha + \beta)$.
3. *Double angle identities.* (There are three useful expressions for $\cos(2\theta)$.)
   (a) $\sin(2\theta) =$
   (b) $\cos(2\theta) =$
   (c) $\cos(2\theta) =$
   (d) $\cos(2\theta) =$
   (e) $\tan(2\theta) =$
4. *Half angle calculations.*
   (a) If $\cos\theta = 3/5$, what are the possible values of $\cos(\theta/2)$?
   (b) Calculate $\tan(\pi/8)$.
5. *The tangent half-angle substitution.* Let $t = \tan(\theta/2)$. Show that

$$\cos\theta = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin\theta = \frac{2t}{1+t^2}.$$

   These are sometimes used in calculus to change expressions involving trig functions into expressions involving rational functions.
6. *Product-to-sum and sum-to-product identities.*
   (a) $\cos\alpha\cos\beta =$
   (b) $\sin\alpha\sin\beta =$
   (c) $\sin\alpha\cos\beta =$
   (d) $\sin\theta + \sin\phi =$
   (e) $\cos\theta + \cos\phi =$

7. *Some equations.* Find all real solutions to the following equations. (These will generally involve an integer parameter and may involve inverse trig functions.)

   (a) $\sin\alpha = 1$

   (b) $2\cos(2t) + 5 = 8\cos t$

   (c) $\sin\theta = \cos^2(2\pi/9) - \sin^2(2\pi/9)$

   (d) $3\sin x + 5\cos x = 23/4$

3.2 Challenge problems

8. For each non-negative integer n, the *degree-n Chebyshev polynomial of the first kind*, denoted $T_n(X)$, is defined by the property that $T_n(\cos\theta) = \cos(n\theta)$ for all real θ. Thus $T_0(X) = 1$ and $T_1(X) = X$.

   (a) Compute $T_2(X)$, $T_3(X)$, and $T_4(X)$.

   (b) Find the roots of $T_n(X)$ for $n = 0, 1, 2, 3, 4$.

   (c) Prove that $T_{n+1}(X) = 2X \cdot T_n(X) - T_{n-1}(X)$ for all positive integers n.

9. Show that for all positive integers n,

$$1 + 2\cos\theta + 2\cos(2\theta) + \cdots + 2\cos(n\theta) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}.$$

   Remark: If we denote either side of this equation by $D_n(\theta)$, then the sequence of functions $D_0, D_1, D_2, \ldots$ is known as the *Dirichlet kernel*.

10. Prove the following for a triangle ABC:

   (a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

   (b) $\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2}) = \cot(\frac{A}{2})\cot(\frac{B}{2})\cot(\frac{C}{2})$

   (c) $\sin(2A) + \sin(2B) + \sin(2C) = 4\sin A \sin B \sin C$

3.3 Answers

1. (a) $\sin\alpha\cos\beta + \cos\alpha\sin\beta$
   (b) $\sin\alpha\cos\beta - \cos\alpha\sin\beta$
   (c) $\cos\alpha\cos\beta - \sin\alpha\sin\beta$
   (d) $\cos\alpha\cos\beta + \sin\alpha\sin\beta$
   (e) $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
   (f) $\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
2. (a) $\frac{\sqrt{6}-\sqrt{2}}{4}$
   (b) $56/65$
3. The answers for (b), (c), and (d) can be rearranged.
   (a) $2\sin\theta\cos\theta$
   (b) $\cos^2\theta - \sin^2\theta$
   (c) $2\cos^2\theta - 1$
   (d) $1 - 2\sin^2\theta$
   (e) $\frac{2\tan\theta}{1-\tan^2\theta}$
4. (a) $\pm 2/\sqrt{5}$
   (b) $\sqrt{2} - 1$
5. We have $\sec^2(\theta/2) = 1 + \tan^2(\theta/2) = 1 + t^2$, so $\cos^2(\theta/2) = \frac{1}{1+t^2}$. Then,
   $$\cos\theta = 2\cos^2(\theta/2) - 1 = \frac{1-t^2}{1+t^2}.$$
   To find $\sin\theta$, we compute
   $$\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{2t}{1-t^2}$$
   and hence $\sin\theta = \cos\theta\tan\theta = \frac{2t}{1+t^2}$.
   Remark: Another method to solve this is to set up the situation in Section 1 Problem 10(a).
6. (a) $\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$
   (b) $\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$
   (c) $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$
   (d) $2\sin(\frac{\theta+\phi}{2})\sin(\frac{\theta-\phi}{2})$
   (e) $2\cos(\frac{\theta+\phi}{2})\cos(\frac{\theta-\phi}{2})$
7. For all of the below, n ranges over all integers.
   (a) $\alpha = \frac{\pi}{2} + 2\pi n$

(b) By the double-angle identity $\cos(2t) = 2\cos^2 t - 1$,

$$4\cos^2 t + 3 = 8\cos t \quad \iff \quad (2\cos t - 1)(2\cos t - 3) = 0.$$

Only $\cos t = 1/2$ is possible, and we get the solutions $t = \pi/3 + 2\pi n$ and $t = -\pi/3 + 2\pi n$.

(c) The right hand side is $\cos(4\pi/9) = \sin(\pi/18)$, so $\sin\theta = \sin(\pi/18)$. Hence $\theta = \pi/18 + 2\pi n$ or $\theta = 17\pi/18 + 2\pi n$.

(d) Dividing both sides by $\sqrt{3^2+5^2} = \sqrt{34}$ and setting $\varphi = \arccos(\frac{3}{\sqrt{34}})$ and $C = \frac{23}{4\sqrt{34}}$, we have $\sin x \cos\alpha + \cos x \sin\alpha = C$, so $\sin(x+\alpha) = C$. Hence

$$x = \arcsin C - \alpha + 2\pi n \quad \text{or} \quad x = \pi - \arcsin C - \alpha + 2\pi n.$$

8. (a) $T_2(X) = 2X^2 - 1$
$T_3(X) = 4X^3 - 3X$
$T_4(X) = 8X^4 - 8X^2 + 1$

(b) $T_0(X)$ has no roots
$T_1(X)$ has one root, 0
$T_2(X)$ has two roots, $\pm 1/\sqrt{2}$
$T_3(X)$ has three roots, 0 and $\pm\sqrt{3}/2$
$T_4(X)$ has four roots: letting $Y = X^2$, then $8Y^2 - 8Y + 1 = 0$ when $Y = 4 \pm 2\sqrt{2}$, which in turn gives us $X = \pm\sqrt{4 \pm 2\sqrt{2}}$

(c) It suffices to show that $\cos((n+1)\theta) = 2\cos\theta\cos(n\theta) - \cos((n-1)\theta)$, or equivalently,

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta).$$

This follows by expanding $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ with $\alpha = n\theta$ and $\beta = \theta$, or (since the work was already done before) using by a sum-to-product or product-to-sum identity.

9. Multiplying through by $\sin(\theta/2)$ and expanding the left hand side using product-to-sum,

$$\begin{aligned}
&\sin\left(\frac{1}{2}\theta\right)[1 + 2\cos\theta + \cdots + 2\cos(n\theta)] \\
&= \sin\left(\frac{1}{2}\theta\right) + \left[\sin\left(\frac{3}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)\right] + \cdots + \left[\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\left(n-\frac{1}{2}\right)\theta\right)\right] \\
&= \sin\left(\left(n+\frac{1}{2}\right)\theta\right).
\end{aligned}$$

10. (a) We compute

$$\tan C = \tan(\pi - (A+B)) = -\tan(A+B) = \frac{\tan A + \tan B}{\tan A \tan B - 1}.$$

With this, both sides simplify to $\dfrac{\tan A \tan B(\tan A + \tan B)}{\tan A \tan B - 1}$.

(b) Let $x = A/2$ and $y = B/2$. Then

$$\cot\left(\frac{C}{2}\right) = \cot\left(\frac{\pi}{2} - (x+y)\right) = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\cot x + \cot y}{\cot x \cot y - 1}.$$

With this, both sides simplify to $\dfrac{\cot x \cot y(\cot x + \cot y)}{\cot x \cot y - 1}$.

(c) First, we use the sine double-angle formula and $C = \pi - (A+B)$ to reduce the problem to showing that

$$\sin A \cos A + \sin B \cos B - \sin(A+B)\cos(A+B) \stackrel{?}{=} 2 \sin A \sin B \sin(A+B).$$

We compute

$$\begin{aligned}
&\sin A \cos A + \sin B \cos B - \sin(A+B)\cos(A+B) \\
&= \sin A \cos A + \sin B \cos B - (\sin A \cos B + \cos A \sin B)(\cos A \cos B - \sin A \sin B) \\
&= \sin A \cos A + \sin B \cos B - \sin A \cos A \cos^2 B \\
&\qquad + \sin^2 A \sin B \cos B - \cos^2 A \sin B \cos B + \sin A \cos A \sin^2 B \\
&= \sin A \cos A(1 - \cos^2 B + \sin^2 B) + \sin B \cos B(1 + \sin^2 A - \cos^2 A) \\
&= 2 \sin A \cos A \sin^2 B + 2 \sin^2 A \sin B \cos B \\
&= 2 \sin A \sin B(\cos A \sin B + \sin A \cos B) \\
&= 2 \sin A \sin B \sin(A+B).
\end{aligned}$$

Remark: Both sides of the original equation are equal to twice the area of a triangle with circumradius 1.

4 Parametric and polar equations

4.1 Review problems

1. *Graphing parametric curves.* Graph each of the following and write down equivalent equations using x and y only.
   (a) $x(t) = 2 + 3t$ and $y(t) = 4 - t$
   (b) $x(t) = \cos t$ and $y(t) = \sin t$
   (c) $x(t) = t$ and $y(t) = t^3$
   (d) $x(t) = 1 + 2\sin t$ and $y(t) = 3 - 2\cos t$

2. *Parameterizing standard curves.* Write down parametric equations for each of the following curves in the plane.
   (a) The line passing through $(3, 2)$ and $(6, 7)$
   (b) The circle with center $(-2, 3)$ and radius 4
   (c) The ray emanating from $(0, 1)$ passing through $(-4, -4)$

3. *Plane polar coordinates.* Make each of the following conversions.
   (a) Polar $(3, 0)$ to cartesian (rectangular, (x, y))
   (b) Polar $(2, 2\pi/3)$ to cartesian
   (c) Cartesian $(-4, 0)$ to polar
   (d) Cartesian $(1, 1)$ to polar

4. *Lines and circles.* Graph each of the following and write down equivalent equations using cartesian coordinates (x and y).
   (a) $r = 5$
   (b) $\theta = \arctan(1/4)$, where r is allowed to be any real number
   (c) $r = -2\sin\theta$
   (d) $r = 2\cos\theta + 4\sin\theta$
   (e) $r = \frac{7}{3\cos\theta - 2\sin\theta}$

5. *Limaçons and roses.* Match each equation with its corresponding graph. (Not every graph has a corresponding equation.)

Equation	Graph
$r = 1 + 3\cos\theta$	Limaçon with indentation
$r = 3 + 3\cos\theta$	Rose with 3 petals
$r = 5 + 3\cos\theta$	Limaçon with inner loop
$r = 7 + 3\cos\theta$	Convex limaçon
$r = \cos\theta$	Rose with 4 petals
$r = \cos(2\theta)$	Circle
$r = \cos(3\theta)$	Rose with 2 petals
	Rose with 6 petals
	Cardioid (limaçon with cusp)

6. *Arc length parameterization.* An *arc length parameterization* of a curve in the plane is a parameterization $(x(s), y(s))$ with the property that whenever a particle moves along the curve from $s = s_1$ to $s = s_2$, where $s_1 < s_2$, the distance it travels is $s_2 - s_1$.

   (a) Prove that $x(s) = 1 + \frac{3}{5}s$ and $y(s) = 1 + \frac{4}{5}s$ is an arc length parameterization for the line passing through $(1, 1)$ and $(4, 5)$.

   (b) Find an arc length parameterization for the circle centered at $(1, -1)$ with radius 2 that traverses the circle clockwise starting at $(3, -1)$ when $s = 0$.

7. *Roulettes.* Let P be a point on a circle of radius 1. Parameterize the path traced by P as the circle rolls along each of the following, assuming that P is at the point of contact at time $t = 0$. (You are free to choose exactly where the initial point of contact is.)

   (a) *Cycloid.* Rolling on top of the x-axis

   (b) *Cardioid.* Rolling on the outside of a circle of radius 1

   (c) *Nephroid.* Rolling on the outside of a circle of radius 2

   (d) *Deltoid.* Rolling on the inside of a circle of radius 3

   (e) *Astroid.* Rolling on the inside of a circle of radius 4

   Remark: The cardioid and nephroid are examples of *epicycloids* while the deltoid and astroid are examples of *hypocycloids*.

4.2 Challenge problems

8. Let $e > 0$ and $\ell > 0$ be given. The conic section with eccentricity e whose focus is the point $F = (0, 0)$ and whose directrix is the line $x = -\ell$ is the set of all points satisfying

$$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to the directrix}} = e.$$

   Note that when $e = 1$, this matches the usual focus-directrix definition of the parabola.

   (a) Show that the above curve has polar equation

$$r = \frac{e\ell}{1 - e\cos\theta}.$$

   (b) Convert the polar equation to cartesian form.

   (c) Show that when $0 < e < 1$, the conic is an ellipse. What is the other focus of the ellipse?

   (d) Show that when $e > 1$, the conic is a hyperbola. In terms of e and/or ℓ, what are the slopes of the asymptotes?

   (e) Suppose the conic passes through $(-p, 0)$, where $p > 0$. Express ℓ in terms of e and p.

   (f) Holding p fixed, what happens to the conic and the directrix as e approaches 0?

9. Parameterization works for curves in three dimensions as well.

   (a) Graph the line $x(t) = t$; $y(t) = 1 + 2t$; $z(t) = 2 + t$.

   (b) Graph the *helix* $x(t) = \cos t$; $y(t) = \sin t$; $z(t) = t$.

   (c) Find the intersection point (if it exists) of the lines

   $$x_1(t) = 2 - t; \quad y_1(t) = 2 + t; \quad z_1(t) = 3t$$

   and

   $$x_2(t) = -1 + t; \quad y_2(t) = 7 - 2t; \quad z_2(t) = 2 + t.$$

10. By using two parameters, we can describe surfaces in three dimensions. For each pair of values (u, v), we get a corresponding point on the surface.

    (a) Graph the plane $x(u, v) = u + v$, $y(u, v) = u - v$, and $z(u, v) = 1 + 4u + 6v$. Write down an equation for this plane using only x, y, and z.

    (b) Graph the (infinite) cylinder $x(u, v) = 2\cos u$, $y(u, v) = 2\sin u$, and $z(u, v) = v$.

    (c) Assuming the Earth is a perfect sphere of radius R centered at the origin with the prime meridian and equator intersecting at $(R, 0, 0)$, let ϕ denote latitude, ranging from $-90°$ at the south pole to $90°$ at the north pole, and let θ denote longitude, increasing from $-180°$ to $180°$ moving eastward with the prime meridian at $\theta = 0°$. Parameterize points on the surface of the Earth using θ and ϕ.

4.3 Answers

1. Checking parametric graphs with Desmos

   (a) $x + 3y = 14$

   (b) $x^2 + y^2 = 1$

   (c) $y = x^3$

   (d) $(x-1)^2 + (y-3)^2 = 4$

2. (a) $x(t) = 3 + 4t$ and $y(t) = 2 + 5t$

   (b) $x(t) = -2 + 4\cos t$ and $y(t) = 3 + 4\sin t$

   (c) $x(t) = -4t$ and $y(t) = 1 - 4t$ with $t \geq 0$

3. (a) $(3, 0)$

   (b) $(-1, \sqrt{3})$

   (c) $(4, \pi)$

   (d) $(\sqrt{2}, \pi/4)$

4. Checking polar graphs with Desmos

   (a) $x^2 + y^2 = 25$

   (b) $y = x/4$

   (c) $x^2 + (y+1)^2 = 1$

   (d) $(x-1)^2 + (y-2)^2 = 5$

   (e) $3x - 2y = 7$

5. “Rose with 2 petals” and “Rose with 6 petals” are unused.

Equation	Graph
$r = 1 + 3\cos\theta$	Limaçon with inner loop
$r = 3 + 3\cos\theta$	Cardioid (limaçon with cusp)
$r = 5 + 3\cos\theta$	Limaçon with indentation
$r = 7 + 3\cos\theta$	Convex limaçon
$r = \cos\theta$	Circle
$r = \cos(2\theta)$	Rose with 4 petals
$r = \cos(3\theta)$	Rose with 3 petals

6. (a) Let $s_1 < s_2$. The parametric equations given for $x(s)$ and $y(s)$ describe a line, so we can compute distance according to the distance formula (Pythagorean theorem). The

distance traveled from $(x(s_1), y(s_1))$ to $(x(s_2), y(s_2))$ along the line is

$$\begin{aligned}
&\sqrt{(x(s_2)-x(s_1))^2+(y(s_2)-y(s_1))^2}\\
&=\sqrt{\left(1+\frac{3}{5}s_2-1-\frac{3}{5}s_1\right)^2+\left(1+\frac{4}{5}s_2-1-\frac{4}{5}s_1\right)^2}\\
&=\sqrt{(s_2-s_1)^2\left(\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2\right)}\\
&=s_2-s_1.
\end{aligned}$$

(b) The "usual" parameterization $x(t) = 1 + 2\cos t$ and $y(t) = -1 + 2\sin t$ starts at $(3, -1)$ and travels counterclockwise. We travel 1 radian per unit time and the radius is 2, so the arc length traveled per unit time is $1 \cdot 2 = 2$. Therefore, to get a clockwise traversal at unit speed, we need to go backwards and take twice as long. Hence we set $s = -2t$, or $t = -s/2$, to get a new parameterization $x(s) = 1+2\cos(s/2)$ and $y(s) = -1-2\sin(s/2)$.

7. (a) $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$
   (b) $x(t) = 2\cos t - \cos(2t)$ and $y(t) = 2\sin t - \sin(2t)$
   (c) $x(t) = 3\cos t - \cos(3t)$ and $y(t) = 3\sin t - \sin(3t)$
   (d) $x(t) = 2\cos t + \cos(2t)$ and $y(t) = 2\sin t - \sin(2t)$
   (e) $x(t) = 3\cos t + \cos(3t)$ and $y(t) = 3\sin t - \sin(3t)$

8. (a) The distance from P to $F = (0, 0)$ is just r, while the distance from P to the directrix is $x + \ell = r\cos\theta + \ell$, so

$$\frac{r}{r\cos\theta+\ell}=e.$$

   Solving for r in terms of e, ℓ, θ gives the desired result.

   (b) Writing the above equation in the form $r = e(x + \ell)$, we can square both sides and rearrange to get

$$(1-e^2)x^2-2e^2\ell x+y^2=e^2\ell^2.$$

   (c) When $0 < e < 1$, the coefficients of x^2 and y^2 have the same sign and there is no xy term, so we get an ellipse. The center of the ellipse has x-coordinate $\frac{e^2\ell}{1-e^2}$ and y-coordinate 0, while one focus is $F = (0, 0)$, so the other focus must be $F' = (\frac{2e^2\ell}{1-e^2}, 0)$.

   (d) When $e > 1$, the coefficients of x^2 and y^2 have opposite sign, so we get a hyperbola. The asymptotes have slope $\pm\sqrt{e^2-1}$: when we divide both sides by x^2 and consider very large values of x, we have $(y/x)^2 + (1 - e^2) \approx 0$. Alternatively, the asymptotes correspond to when the denominator is 0 in the polar equation, and when $1 - e\cos\theta = 0$ we have $\tan\theta = \pm\sqrt{e^2-1}$.

   (e) We set $\theta = \pi$ and $r = p$ in the polar equation to get $p = \frac{e\ell}{1+e}$, or $\ell = p(1 + e)/e$.

   (f) Holding p fixed, as e approaches 0 we see that ℓ grows without bound, so the directrix slides to infinity to the left. In terms of p and e we have $r = \frac{p(1+e)}{1-e\cos\theta}$, and as e approaches 0 with p fixed, the equation approaches $r = p$, the circle of radius p.

9. (a) Desmos graph
   (b) Desmos graph
   (c) If there is an intersection point, suppose that for the first line it corresponds to parameter value $t = u$ and that in the second line it corresponds to parameter value $t = v$ (where u and v may be but are not necessarily equal). Matching coordinates, we have the system

$$\begin{aligned} 2 - u &= -1 + v, \\ 2 + u &= 7 - 2v, \\ 3u &= 2 + v. \end{aligned}$$

   This system does not have a solution, so there is no intersection point.

   Remark: If the second line had $z_2(t) = 1 + t$ instead, then we would get the solution $u = 1$ and $v = 2$, corresponding to the point $(1, 3, 3)$. Note that we would not find this point just by setting $x_1(t) = x_2(t)$, etc., so we do need to use a new variable for each line when setting up the system. If we imagine particles traveling according along their respective lines according to the given parametric equations, then they both pass through the same point but at different times (so the particles would not meet, but their trajectories do).

10. (a) Desmos graph, $z = 1 + 5x - y$
    (b) Desmos graph
    (c) $x(\phi, \theta) = R\cos\phi\cos\theta$
    $y(\phi, \theta) = R\cos\phi\sin\theta$
    $z(\phi, \theta) = R\sin\phi$