Precalculus Practice Problems: Midterm 1

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Contents

1 Trig (I): Right Triangle and Unit Circle

1.1 Review problems

1. *Unit conversions for angles.*
   (a) 360 degrees to radians
   (b) π radians to degrees
   (c) 60 degrees to radians
   (d) $3\pi/4$ radians to degrees
   (e) $\pi/5$ degrees to radians

2. *Trig functions as ratios of lengths.* Let ABC be a triangle with a right angle at B. Suppose $AB = 8$ and $BC = 15$.
   (a) Evaluate $\tan A$ and $\cot A$.
   (b) Find the length of AC.
   (c) Evaluate $\sin A$, $\cos A$, $\sec A$, and $\csc A$.

3. *Using one trig function to compute another.* Throughout, assume θ is acute.
   (a) If $\sin\theta = 1/3$, what is $\cos\theta$?
   (b) If $\sec\theta = \sqrt{10}$, what is $\tan\theta$?
   (c) If $\tan\theta = 2/5$, what is $\csc\theta$?

4. *Important acute angles.*

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
30							
45							
	$\pi/3$						

5. *Unit circle calculations.*

   (a) $\cos(0)$
   (b) $\sin(150°)$
   (c) $\cos(-3\pi/4)$
   (d) $\sin(7\pi/3)$
   (e) $\cos(330°)$
   (f) $\sin(-\pi/4)$

6. *Unit circle identities.* Express each of the following in terms of $\sin\theta$ and/or $\cos\theta$.

   (a) $\sin(\pi - \theta)$
   (b) $\cos(\pi - \theta)$
   (c) $\sin(\pi + \theta)$
   (d) $\cos(\pi + \theta)$
   (e) $\sin(-\theta)$
   (f) $\cos(-\theta)$
   (g) $\sin(\frac{\pi}{2} - \theta)$
   (h) $\cos(\frac{\pi}{2} - \theta)$
   (i) $\sin(\frac{\pi}{2} + \theta)$
   (j) $\cos(\frac{\pi}{2} + \theta)$

7. *Some triangle geometry.* In acute triangle ABC, it is given that $AB = 13$, that $BC = 14$, and that $\sin B = 12/13$.

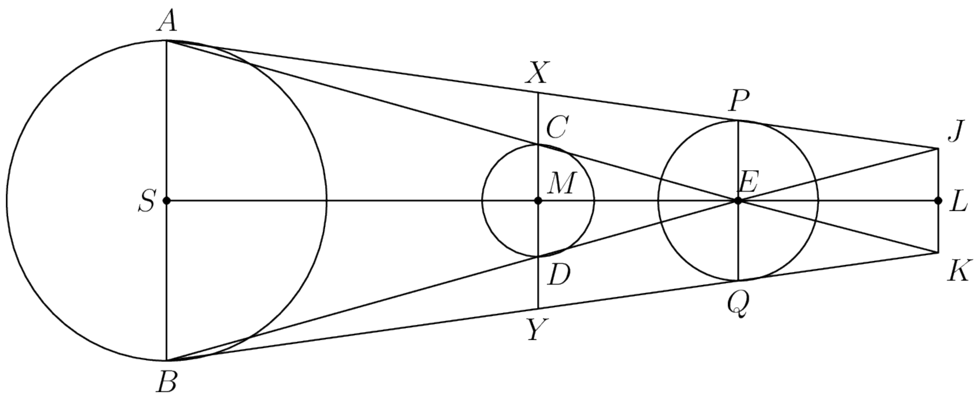
   (a) Find the area of triangle ABC.
   (b) Find the length of AC.
   (c) Find $\sin A$.

1.2 Challenge problems

8. Much of classical trigonometry was done in terms of the *chord function*, which is defined for angles $0 < \theta < 180°$ as follows. Let O be the center of a circle of radius 1, and let A and B be points on the circle so that $\angle AOB = \theta$. Then $\operatorname{chd}\theta$ is defined to be the length AB.

   (a) Compute $\operatorname{chd} 90°$, $\operatorname{chd} 60°$, and $\operatorname{chd} 30°$.
   (b) Express $\operatorname{chd}\theta$ in terms of the sine function.
   (c) Prove that $\operatorname{chd}^2\theta + \operatorname{chd}^2(180° - \theta) = 4$.

9. A method for estimating the distances to the Sun and the Moon using measurements of solar and lunar eclipses dates back to the Greek astronomer Hipparchus (c.1̃90 BC to c.1̃20 BC). In this problem, we work through the relevant geometric argument.



In the diagram, points S, M, E, and L lie on a line, with S, M, and E representing the centers of the Sun, Moon, and Earth, respectively. Segments $\overline{AB}$, $\overline{CD}$, and $\overline{PQ}$ are diameters of their respective circles, all perpendicular to $\overline{SL}$. Segment $\overline{JK}$ is also perpendicular to $\overline{SL}$. Points A, X, P, J are collinear, points B, Y, Q, K are collinear, points X, C, D, Y are collinear, points A, C, E are collinear, and points B, D, E are collinear. Finally, E is the midpoint of $\overline{LM}$. Let $PE = 1$ (so lengths are in terms of Earth radii) and define

$$\ell = EM = EL, \quad s = ES, \quad \theta = \angle CEM, \quad \phi = \angle JEL.$$

(a) Show that

$$s = \frac{\ell}{(\tan\theta + \tan\phi)\ell - 1}$$

(b) (Calculator recommended) The measurements used by Hipparchus were

$$\ell \approx 67\tfrac{1}{3}, \quad \theta \approx 0.277^\circ, \quad \phi \approx 0.693^\circ.$$

Given these measurements, what value do we get for s?

(c) (Calculator recommended) Currently, our measurements for the same quantities are

$$\ell = 60.268, \quad \theta \approx 0.267^\circ, \quad \phi \approx 0.746^\circ.$$

Given these measurements, what value do we get for s?

For reference, the true value of s is $s \approx 23{,}455$, so some of the approximations made in order to set up the diagram turn out to be substantial sources of error.

10. A *Pythagorean triple* is a triple (X, Y, Z) of positive integers for which $X^2 + Y^2 = Z^2$. Note that if (X, Y, Z) is a Pythagorean triple, then $(x, y) = (X/Z, Y/Z)$ is a point on the unit circle whose coordinates are rational numbers.

    (a) Let $O = (-1, 0)$. If $P \neq O$ has rational coordinates and lies on the unit circle, then the slope of $\overline{OP}$ is rational. Conversely, show that if ℓ is a line passing through O which has rational slope, then the other point $P \neq O$ at which ℓ intersects the unit circle must have rational coordinates.

    (b) Use part (a) to show that aside from O, every point on the unit circle with rational coordinates can be written in the form $\left(\frac{n^2 - m^2}{n^2 + m^2}, \frac{2mn}{n^2 + m^2}\right)$ for integers m and n.

    A result going back to Euclid states that every *primitive* Pythagorean triple, meaning a Pythagorean triple (X, Y, Z) where $\gcd(X, Y, Z) = 1$, can be written as either

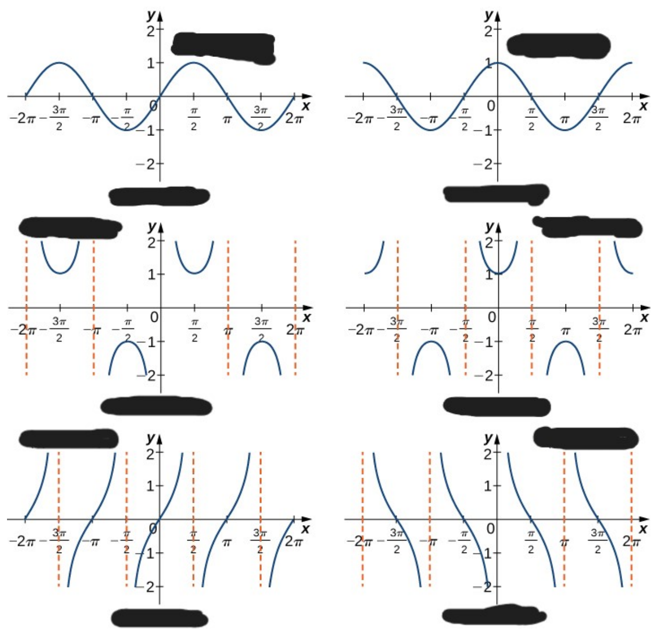
    $$X = n^2 - m^2, \quad Y = 2mn, \quad Z = n^2 + m^2$$

    with $\gcd(m, n) = 1$ and $m + n$ odd, or in the same form with X and Y swapped. The result of part (b) gets most of the way to proving this, with a couple of details to fill in.

2 Trig (II): Graphs and Inverses

2.1 Review problems

1. *The six basic graphs.* Match each of the graphs below to one of the six basic trig functions.



2. *Period and frequency.* A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* if there is a positive real number T such that $f(x+T) = f(x)$ for all real x. The smallest such T, if it exists, is the *period* of f. If f is a periodic function with period T, the *natural frequency* of f is $\nu = 1/T$ while the *angular frequency* of f is $\omega = 2\pi/T = 2\pi\nu$. For each of the six basic trig functions, what are the period, natural frequency, and angular frequency?

3. *Transformed sinusoidal waves.* For each of the following functions $\mathbb{R} \to \mathbb{R}$, find the period, amplitude, and phase shift relative to $\sin(\omega x)$, where ω is the angular frequency of the function.

   (a) $\sin x$

   (b) $\cos x$

   (c) $3\sin(5x - \frac{\pi}{7})$

   (d) $2\sin(2x) - 2\cos(2x)$

4. *Domain and range.* What are the standard domains and ranges of arcsin, arccos, and arctan?

5. *Calculating inverses.*

   (a) $\arcsin(1/2)$

   (b) $\arccos\left(-1/\sqrt{2}\right)$

   (c) $\arcsin\left(-\sqrt{3}/2\right)$

   (d) $\arctan(-1)$

   (e) $\arcsin(\sin(7\pi/6))$

   (f) $\cos(\arccos(-1/3))$

   (g) $\cos(\arctan(1/2))$

6. *Counting solutions.* Find the number of solutions for x in each of the following equations.

   (a) $\sin\theta = 0.5$ when $0 \leq \theta < 2\pi$

   (b) $\cos\theta = -2$ when $0 \leq \theta < 2\pi$

   (c) $\sec\theta = 1$ when $-\pi < \theta \leq \pi$

   (d) $\tan\theta = 1$ when $-\pi < \theta \leq \pi$

   (e) $\sin(3\theta) = 0.2024$ when $0 \leq \theta < 10\pi$

   (f) $\cos(\frac{22}{7}\theta) = 0.5$ when $-20 < \theta < 20$

7. What are the standard domains and ranges of $\sec^{-1}$, $\csc^{-1}$, and $\cot^{-1}$?

2.2 Challenge problems

8. Karen has a calculator which only has seven buttons: sin, cos, tan, arcsin, arccos, arctan, and Reset. The first six apply these functions to the number in the display, while Reset changes the display back to its default state of showing 0. All calculations assume radian measure.

   (a) Starting from a positive real number x in the display, show that there is a sequence of buttons that changes the display to $1/x$.

   (b) Starting from a non-negative real number x in the display, show that there is a sequence of buttons that changes the display to $\sqrt{x^2+1}$.

   (c) Show that for every positive rational number q, there is a sequence of buttons that changes the display from 0 to $\sqrt{q}$.

9. Find the period of the following functions, or show that no period exists.

   (a) $\sin(3x) + \sin(4x)$

   (b) $\sin(20x) + \sin(24x)$

   (c) $\sin(x) + \sin(\sqrt{2}x)$

10. A function $f : \mathbb{C} \to \mathbb{C}$ is *doubly-periodic* if there are non-zero constants $u, v \in \mathbb{C}$ such that u/v is non-real and $f(z) = f(z+u) = f(z+v)$ for all complex numbers z. Write down an example of a non-constant doubly-periodic function.

3 Trig (III): Identities