Precalculus Practice Problems: Final

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The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

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1 Matrices in 2D

1.1 Review Problems

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector. We also let $\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathsf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.

   (a) $\mathbf{u} + \mathbf{v}$

   (b) $2\mathbf{v}$

   (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

   (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$

   (e) The angle between $\mathbf{u}$ and $\mathbf{v}$ (in terms of an inverse trig function)

   (f) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{u}}(\mathbf{v})$

2. *Applying matrices to vectors.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

   (a) Compute $\mathsf{A}\mathbf{v}$

   (b) Find a vector $\mathbf{u}$ for which $\mathsf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.

3. *Matrix operations.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\mathsf{A} + \mathsf{B}$

   (b) $-3\mathsf{A}$

   (c) AB

   (d) BA

   (e) B^T (the transpose of B)

4. *Geometric transformations.* Write down matrices for each of the following.

   (a) Dilation about the origin by a factor of 4

   (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2

   (c) Rotation about the origin by $\pi/4$ counterclockwise

   (d) Projection onto the line $y = (3/2)x$

   (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\det \mathsf{A}$ and $\det \mathsf{B}$

   (b) $\det(\mathsf{AB})$

   (c) $\det(\mathsf{A}^T)$

   (d) $\det(\mathsf{A}+\mathsf{B})$

   (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) A^{-1} and B^{-1}

   (b) $\mathsf{A}^{-1}\mathsf{B}^{-1}$ and $\mathsf{B}^{-1}\mathsf{A}^{-1}$

   (c) $(\mathsf{AB})^{-1}$

   (d) $(\mathsf{A}^T)^{-1}$

   (e) $(\mathsf{A}+\mathsf{B})^{-1}$

   (f) $\det(\mathsf{A}^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

   (a) Describe the image of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.

   (b) By what factor does a horizontal shear multiply areas?

   (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

   (The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

   (a) For the matrices A, B in problems 3, 5, 6, compute $\operatorname{tr}\mathsf{A}$, $\operatorname{tr}\mathsf{B}$, $\operatorname{tr}(\mathsf{A}+\mathsf{B})$, and $\operatorname{tr}(\mathsf{AB})$.

   (b) Show that for any 2×2 matrices P and Q, we have $\operatorname{tr}(\mathsf{PQ}) = \operatorname{tr}(\mathsf{QP})$.

   (c) In general, must it be true that $\operatorname{tr}(\mathsf{ABC}) = \operatorname{tr}(\mathsf{ACB})$?

9. Two matrices A, B are **similar**, written $\mathsf{A} \sim \mathsf{B}$, if there is an invertible P with $\mathsf{B} = \mathsf{P}^{-1}\mathsf{AP}$.

   (a) Show that the only matrix similar to I is I.

   (b) Show that if $\mathsf{A} \sim \mathsf{B}$, then $\det\mathsf{A} = \det\mathsf{B}$ and $\operatorname{tr}\mathsf{A} = \operatorname{tr}\mathsf{B}$.

   (c) Let $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $\mathsf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $\mathsf{D} \sim \mathsf{A}$. Find D.

10. If A is a square matrix, the **characteristic polynomial** of A is defined by

$$f_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I}).$$

   (a) Compute the characteristic polynomial $f_{\mathsf{A}}(X)$ of the matrix $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_{\mathsf{A}}(X)$.

   (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $\mathsf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, 2$. (In general, if $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call $\mathbf{v}$ an **eigenvector** of A corresponding to the **eigenvalue** λ.)

   (d) Let P be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Compute $\mathsf{P}^{-1}\mathsf{AP}$.

   (e) Find A^{100}.

   (f) *Cayley-Hamilton theorem.* Suppose $f_{\mathsf{A}}(X) = a_0 + a_1 X + a_2 X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0\mathsf{I} + a_1\mathsf{A} + a_2\mathsf{A}^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

   (c) Both are 5. In general, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

   (d) $\|\mathbf{u}\| = \sqrt{13}$
   $\|\mathbf{v}\| = \sqrt{17}$
   $\|\mathbf{u} + \mathbf{v}\| = \sqrt{40} = 2\sqrt{10}$

   (e) $\arccos\left(\frac{5}{\sqrt{221}}\right)$

   (f) $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

   $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

2. (a) $\begin{pmatrix} 18 \\ 7 \end{pmatrix}$

   (b) Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

   $$\mathsf{A}\mathbf{u} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 4b \\ a + b \end{pmatrix},$$

   so we require $2a + 4b = 5$ and $a + b = 2$. The solution to this system is that $a = 3/2$ and $b = 1/2$, so then $\mathbf{u} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.

   Remark: We can also compute $\mathbf{u} = \mathsf{A}^{-1}\mathbf{v}$ once we have A^{-1} (see Problem 6).

3. (a) $\begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix}$

   (b) $\begin{pmatrix} -6 & -12 \\ -3 & -3 \end{pmatrix}$

   (c) $\begin{pmatrix} 14 & -20 \\ 2 & -3 \end{pmatrix}$

   (d) $\begin{pmatrix} -2 & -8 \\ 3 & 13 \end{pmatrix}$

   (e) $\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$

4. (a) $4\mathsf{I} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{pmatrix}$

(e) $2\mathsf{P} - \mathsf{I} = \begin{pmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{pmatrix}$

5. (a) $\det \mathsf{A} = -2$
$\det \mathsf{B} = 1$

(b) $\det(\mathsf{AB}) = \det(\mathsf{A}) \cdot \det(\mathsf{B}) = -2$

(c) $\det(\mathsf{A}^T) = \det \mathsf{A} = -2$

(d) $\det(\mathsf{A} + \mathsf{B}) = \det \begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix} = -42$

(e) $|\det \mathsf{A}| \cdot (\text{unit circle area}) = 2\pi$

6. (a) $\mathsf{A}^{-1} = \dfrac{1}{\det \mathsf{A}} \begin{pmatrix} 1 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 1/2 & -1 \end{pmatrix}$

$\mathsf{B}^{-1} = \dfrac{1}{\det \mathsf{B}} \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix}$

(b) $\mathsf{A}^{-1}\mathsf{B}^{-1} = \begin{pmatrix} -13/2 & -4 \\ 3/2 & 1 \end{pmatrix}$

$\mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

(c) $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

(d) $(\mathsf{A}^T)^{-1} = (\mathsf{A}^{-1})^T = \begin{pmatrix} -1/2 & 1/2 \\ 2 & -1 \end{pmatrix}$

(e) $(\mathsf{A} + \mathsf{B})^{-1} = \dfrac{1}{\det(\mathsf{A} + \mathsf{B})} \begin{pmatrix} -6 & -8 \\ -6 & -1 \end{pmatrix} = \begin{pmatrix} 1/7 & 4/21 \\ 1/7 & 1/42 \end{pmatrix}$

(f) $\det(\mathsf{A}^{-1}) = 1/\det \mathsf{A} = -1/2$

7. (a) A parallelogram with vertices $(0,0), (1,0), (3,1), (2,1)$

(b) $\det \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = 1$

(c) Multiplying the right two matrices, $\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & ak \\ 0 & b \end{pmatrix}$. Looking at the image of vector $\hat{\mathbf{i}}$, we need $\begin{pmatrix} a \\ 0 \end{pmatrix}$ to rotate to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. This can be achieved with a rotation by $\theta = \arccos(4/5)$ and $a = 5$. To find b, taking the determinant on both sides and noting that rotations have determinant 1, we require $ab = 25$, so $b = 5$. Finally, to get k, we need $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ to rotate to $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$. Comparing lengths and noting that $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ must be in the first quadrant, $k = 1$.

8. (a) $\operatorname{tr} \mathsf{A} = 3$
   $\operatorname{tr} \mathsf{B} = -10$
   $\operatorname{tr}(\mathsf{A}+\mathsf{B}) = \operatorname{tr} \mathsf{A} + \operatorname{tr} \mathsf{B} = -7$
   $\operatorname{tr}(\mathsf{AB}) = 11$

   (b) Let $\mathsf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathsf{Q} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$\mathsf{PQ} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \quad \text{and} \quad \mathsf{QP} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix},$$

   so $\operatorname{tr}(\mathsf{PQ}) = \operatorname{tr}(\mathsf{QP}) = ae+bg+cf+dh$.

   (c) In general, the answer is **no**. For example, let

$$\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Then

$$\mathsf{ABC} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix},$$
$$\mathsf{ACB} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 3 \end{pmatrix},$$

   so $\operatorname{tr}(\mathsf{ABC}) = 6$ while $\operatorname{tr}(\mathsf{ACB}) = 7$.

9. (a) Suppose $\mathsf{I} \sim \mathsf{B}$. Then there is an invertible matrix P such that $\mathsf{B} = \mathsf{P}^{-1}\mathsf{IP}$, but the right hand side simplifies to $\mathsf{P}^{-1}\mathsf{P} = \mathsf{I}$.

   (b) If $\mathsf{A} \sim \mathsf{B}$ with $\mathsf{B} = \mathsf{P}^{-1}\mathsf{AP}$, then

$$\det \mathsf{B} = \det(\mathsf{P}^{-1}\mathsf{AP}) = \det(\mathsf{P})^{-1} \cdot \det \mathsf{A} \cdot \det \mathsf{P} = \det \mathsf{A}.$$

   For the trace, Problem 8b gives us

$$\operatorname{tr} \mathsf{B} = \operatorname{tr}(\mathsf{P}^{-1}(\mathsf{AP})) = \operatorname{tr}((\mathsf{AP})\mathsf{P}^{-1}) = \operatorname{tr} \mathsf{A}.$$

   (c) We have $\det \mathsf{A} = 4$ and $\operatorname{tr} \mathsf{A} = 5$, so

$$\det \mathsf{D} = d_1 d_2 = 4 \quad \text{and} \quad \operatorname{tr} \mathsf{D} = d_1 + d_2 = 5.$$

   This is satisfied by $d_1 = 4$ and $d_2 = 1$, so $\mathsf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

10. (a) We compute

$$f_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I}) = \det \begin{pmatrix} 3-X & 1 \\ 2 & 2-X \end{pmatrix} = (3-X)(2-X) - 2 = X^2 - 5X + 4.$$

   (b) The roots are $\lambda_1 = 4$ and $\lambda_2 = 1$.

(c) Note that the equation $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, which has a non-zero solution if and only if $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$. Moreover, we can use this version of the equation to find solutions more easily.

For $\lambda_1 = 4$, we have $\mathsf{A} - \lambda_1\mathsf{I} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$, so we can take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathsf{A} - \lambda_1\mathsf{I})\mathbf{v} = \mathbf{0}$.

For $\lambda_2 = 1$, we have $\mathsf{A} - \lambda_2\mathsf{I} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, so we can take $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathsf{A} - \lambda_2\mathsf{I})\mathbf{v} = \mathbf{0}$.

(d) Here $\mathsf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, so then $\mathsf{P}^{-1} = -\frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. We compute

$$\begin{aligned} \mathsf{P}^{-1}\mathsf{A}\mathsf{P} &= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 12 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Remark 1: If we produced different valid choices of $\mathbf{v}_1$ and $\mathbf{v}_2$ from part (c), P and P^{-1} would change, but the end result would be the same. If we swapped the order of the columns of P, then we would swap the order of the diagonal entries correspondingly.

Remark 2: The fact that we got a diagonal matrix with entries λ_1, λ_2, the same one as in Problem 9c, is not a coincidence. The process we went through in this problem is called **diagonalisation**. (Not all $n \times n$ matrices are diagonalisable, but one sufficient condition for diagonalisability is that the characteristic polynomial has n distinct roots.)

(e) Let $\mathsf{D} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, so then $\mathsf{A} = \mathsf{P}\mathsf{D}\mathsf{P}^{-1}$. Then

$$\begin{aligned} \mathsf{A}^{100} &= \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \cdot \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \cdot \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \cdot \ldots \cdot \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \cdot \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \\ &= \mathsf{P}\mathsf{D} \cdot \mathsf{D} \cdot \mathsf{D} \cdot \ldots \cdot \mathsf{D} \cdot \mathsf{D}\mathsf{P}^{-1} = \mathsf{P}\mathsf{D}^{100}\mathsf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4^{100} & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{-1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 4^{100} & 1 \\ 4^{100} & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 \cdot 4^{100} + 1 & 4^{100} - 1 \\ 2 \cdot 4^{100} - 2 & 4^{100} + 2 \end{pmatrix}. \end{aligned}$$

(f) Here $(a_0, a_1, a_2) = (4, -5, 1)$, so

$$\begin{aligned} a_0\mathsf{I} + a_1\mathsf{A} + a_2\mathsf{A}^2 &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -15 & -5 \\ -10 & -10 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -5 \\ -10 & -6 \end{pmatrix} + \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix} = \mathsf{0}. \end{aligned}$$

2 Vectors in 3D

2.1 Review Problems

1. *Operations.* Let
$$\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}.$$
Compute each of the following. (Write “Err” or similar for any undefined expressions.)

   (a) $2\mathbf{a} + \mathbf{b} - \mathbf{c}$

   (b) $\|\mathbf{a}\| + \|\mathbf{b}\| - \|\mathbf{a} + \mathbf{b}\|$

   (c) $\mathbf{b} \cdot \mathbf{c}$

   (d) $\mathbf{a} \times \mathbf{b}$

   (e) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

   (f) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

2. *Distances and spheres.*

   (a) Find the distance between the points $(2, -5, -2)$ and $(1, -5, 0)$.

   (b) Write down an equation for the sphere with center $(5, -1, 0)$ and radius 5.

   (c) Find the center and radius of the sphere with equation
$$x^2 + y^2 + z^2 - 2x + 8y + 8z + 17 = 0.$$

3. *Angles.* Let $A = (-20, -2, 1)$, $B = (-15, 3, 21)$, and $C = (-16, 14, 5)$. Compute $\angle BAC$.

4. *Cross products.* Let $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$.

   (a) Find all vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ with norm 1.

   (b) Find the area of the parallelogram with vertices at $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$.

   (c) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Compute $\sin\theta$.

5. *Planes.* Let $A = (4, -5, 5)$, $B = (-2, 5, -5)$, and $C = (3, -3, -3)$. Find an equation for the plane passing through A, B, and C

   (a) in parametric form;

   (b) in cartesian form $ax + by + cz = d$.

   Then find a parametric form for the intersection of this plane and the plane $x + 2y + 3z = 4$.

6. *Projections and reflections.*

   (a) What point on the line through $(-5, 0, -2)$ and $(2, 5, 2)$ is closest to $(3, 1, -4)$?

   (b) Find the reflection of the point $P = (4, -4, 5)$ across the plane $-4x + 4y + 3z = 3$.

   (c) $(*)$ Let $\mathcal{C}$ be the circle centered at $(0, 0, 1)$ of radius 1 lying in the plane $z = 1$ and let $\mathcal{P}$ be the plane passing through the origin as well as the points $(5, 1, 1)$ and $(1, 3, 1)$. What is the shortest possible distance between a point on $\mathcal{C}$ and a point on $\mathcal{P}$?

7. *Using cross products in 2D problems.*

   (a) Let ABC be a triangle in the xy-plane with area 14. If the points A, B, C are listed in clockwise order going around the triangle, what is $\overrightarrow{AB} \times \overrightarrow{AC}$?

   (b) $(*)$ Let $ABCD$ be a convex quadrilateral and let points P and Q lie on segments $\overline{AB}$ and $\overline{CD}$ respectively so that $AP/AB = CQ/CD$. Let R be the intersection of $\overline{AQ}$ and $\overline{PD}$ and let S be the intersection of $\overline{BQ}$ and $\overline{PC}$. Show that

$$[PSQR] = [ARD] + [BCS].$$

2.2 Challenge Problems

Suppose we sample n members of a population and measure quantities X and Y for each of the n observations. (For example, perhaps X and Y denote height and wingspan that we measure for several people.) The observed values of X and Y are stored in vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

8. In statistics, often more useful than the ordinary dot product is a rescaled version,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{n}(\mathbf{x} \cdot \mathbf{y}) = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n}.$$

   (a) Let $\mathbf{1}$ (or $\mathbf{1}_n$) denote the vector with components that are all equal to 1. Express the sample mean $\overline{x}$ of the observed values of X in terms of $\mathbf{x}$, $\mathbf{1}$, and $\langle\ ,\ \rangle$.

   (b) Let $\mathcal{H}$ be the "hyperplane" of all points in n-dimensional space with the property that the sum of the coordinates is 0. Show that the projection of $\mathbf{x}$ onto $\mathcal{H}$ is $\mathbf{x} - \overline{x}\mathbf{1}$.

   (c) The *(uncorrected) sample variance* of the observed values of X is

$$s_x^2 = \langle \mathbf{x} - \overline{x}\mathbf{1}, \mathbf{x} - \overline{x}\mathbf{1} \rangle,$$

   while the *sample standard deviation* s_x is the square root of the sample variance. Show that $s_x^2 = \langle \mathbf{x}, \mathbf{x} \rangle - (\overline{x})^2$.

9. The *sample covariance* of the observed values of X and Y is

$$s_{xy} = \langle \mathbf{x} - \overline{x}\mathbf{1}, \mathbf{y} - \overline{y}\mathbf{1} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle - \overline{x} \cdot \overline{y},$$

   and the *sample correlation* is $r_{xy} = \frac{s_{xy}}{s_x \cdot s_y}$ (when $s_x, s_y \neq 0$).

   (a) Show that $-1 \leq r \leq 1$.

   (b) When does $r = 1$? When does $r = -1$?

10. In *simple linear regression*, we seek values β_0, β_1 so that the linear model $Y = \beta_0 + \beta_1 X$ is "best possible." This is usually taken to mean that the *mean squared error*

$$MSE = \langle \mathbf{y} - \hat{\mathbf{y}}, \mathbf{y} - \hat{\mathbf{y}} \rangle$$

   should be as small as possible, where $\hat{\mathbf{y}} = \beta_0\mathbf{1} + \beta_1\mathbf{x}$. Show, using projection or otherwise, that this is achieved when

$$\beta_1 = r_{xy} \cdot \frac{s_y}{s_x} \quad \text{and} \quad \beta_0 = \overline{y} - \beta_1\overline{x}.$$

2.3 Answers

1. (a) $2\mathbf{a}+\mathbf{b}-\mathbf{c} = \begin{pmatrix} 2(-2)+3-(-1) \\ 2(-1)+0-1 \\ 2(2)+(-4)-5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -5 \end{pmatrix}$

   (b) $\|\mathbf{a}\|+\|\mathbf{b}\|-\|\mathbf{a}+\mathbf{b}\| = 3+5-\left\| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\| = 8-\sqrt{6}$

   (c) $\mathbf{b}\cdot\mathbf{c} = 3\cdot(-1)+0\cdot 1+(-4)\cdot 5 = -23$

   (d) $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} (-1)\cdot(-4)-2\cdot 0 \\ 2\cdot 3-(-2)\cdot(-4) \\ (-2)\cdot 0-(-1)\cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

   (e) Err ($\mathbf{b}\cdot\mathbf{c}$ produces a real number, which cannot be dotted with $\mathbf{a}$)

   (f) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} (-2)\cdot 5-3\cdot 1 \\ 3\cdot(-1)-4\cdot 5 \\ 4\cdot 1-(-2)\cdot(-1) \end{pmatrix} = \begin{pmatrix} -13 \\ -23 \\ 2 \end{pmatrix}$

2. (a) $\sqrt{(2-1)^2+((-5)-(-5))^2-((-2)-0)^2} = \sqrt{5}$

   (b) $(x-5)^2+(y+1)^2+z^2 = 25$

   (c) We complete the square:

$$\begin{aligned} x^2+y^2+z^2-2x+8y+8z+17 &= 0, \\ (x^2-2x)+(y^2+8y)+(z^2+8z) &= -17, \\ (x^2-2x+1)+(y^2+8y+16)+(z^2+8z+16) &= -17+1+16+16 \\ (x-1)^2+(y+4)^2+(z+4)^2 &= 16. \end{aligned}$$

   This is a sphere with center $(1,-4,-4)$ and radius $\sqrt{16}=4$.

3. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$, so that $\theta = \angle BAC$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We compute

$$\mathbf{u} = \begin{pmatrix} (-15)-(-20) \\ 3-(-2) \\ 21-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 20 \end{pmatrix},$$
$$\mathbf{v} = \begin{pmatrix} (-16)-(-20) \\ 14-(-2) \\ 5-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ 4 \end{pmatrix},$$

   so then

$$\begin{aligned} \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} &= \frac{5\cdot 4+5\cdot 16+20\cdot 4}{\sqrt{5^2+5^2+20^2}\cdot\sqrt{4^2+16^2+4^2}} \\ &= \frac{180}{5\sqrt{1^2+1^2+4^2}\cdot 4\sqrt{1^2+4^2+1^2}} = \frac{9}{18} = \frac{1}{2}. \end{aligned}$$

   This means that $\theta = \pi/3 = 60°$.

4. (a) Any vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ must be parallel to

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \cdot (-5) - (-5) \cdot 2 \\ (-5) \cdot (-3) - 3 \cdot (-5) \\ 3 \cdot 2 - 3 \cdot (-3) \end{pmatrix} = \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix}.$$

The two vectors parallel to $\mathbf{n}$ of length 1 are

$$\frac{\pm 1}{\|\mathbf{n}\|}\mathbf{n} = \frac{\pm 1}{\sqrt{(-5)^2 + 30^2 + 15^2}} \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix} = \frac{\pm 1}{5\sqrt{46}} \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix} = \frac{\pm 1}{\sqrt{46}} \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

(b) Since $\mathbf{u} = \mathbf{v} + (\mathbf{u} - \mathbf{v})$, this parallelogram is the one defined by $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. Its area is

$$\|\mathbf{v} \times (\mathbf{u} - \mathbf{v})\| = \|\mathbf{v} \times \mathbf{u} - \mathbf{v} \times \mathbf{v}\| = \|\mathbf{v} \times \mathbf{u}\| = \| - \mathbf{n}\| = 5\sqrt{46}.$$

(c) We compute

$$\sin\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5\sqrt{46}}{\sqrt{3^2 + 3^2 + (-5)^2} \cdot \sqrt{(-3)^2 + 2^2 + (-5)^2}} = \frac{5\sqrt{23}}{\sqrt{817}}.$$

5. (a) If $P = (x, y, z)$ is an arbitrary point in the plane, then there exist s and t for which

$$\overrightarrow{P} = \overrightarrow{A} + s(\overrightarrow{AB}) + t(\overrightarrow{AC}) = \begin{pmatrix} 4 \\ -5 \\ 5 \end{pmatrix} + s \begin{pmatrix} -6 \\ 10 \\ -10 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}.$$

(b) A normal vector to the plane is given by

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 10 \cdot (-8) - (-10) \cdot 2 \\ (-10) \cdot (-1) - (-6) \cdot (-8) \\ (-6) \cdot 2 - 10 \cdot (-1) \end{pmatrix} = \begin{pmatrix} -60 \\ -38 \\ -2 \end{pmatrix}.$$

Therefore, an equation for the plane is

$$0 = \mathbf{n} \cdot (\overrightarrow{P} - \overrightarrow{A}) = -60(x - 4) - 38(y + 5) - 2(z - 5),$$

which can be rearranged to $30x + 19y + z = 30$.

To find the intersection of this plane with $x + 2y + 3z = 4$, we eliminate x to get

$$\begin{aligned} 30(x + 2y + 3z) - (30x + 19y + z) &= 30 \cdot 4 - 30, \\ 41y + 89z &= 90. \end{aligned}$$

If $z = t$, then $y = \dfrac{90}{41} - \dfrac{89}{41}t$ and

$$x = 4 - 2y - 3z = 4 - \left(\frac{90}{41} - \frac{89}{41}t\right) - 3t = -\frac{16}{41} + \frac{55}{41}t.$$

In vector parametric form,

$$\overrightarrow{P} = \begin{pmatrix} -16/41 \\ 90/41 \\ 0 \end{pmatrix} + t \begin{pmatrix} 55/41 \\ -89/41 \\ 1 \end{pmatrix}.$$

6. (a) Translating $(-5, 0, -2)$ to the origin, the problem is equivalent to finding the point on the line generated by $\mathbf{u} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}$ closest to $\mathbf{v} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$. This is

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = \frac{8 \cdot 7 + 1 \cdot 5 + (-2) \cdot 4}{7 \cdot 7 + 5 \cdot 5 + 4 \cdot 4} \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} = \frac{53}{90} \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}.$$

Translating back, the desired point is $\left(\frac{53}{90} \cdot 7 - 5, \frac{53}{90} \cdot 5, \frac{53}{90} \cdot 4 - 2\right) = \left(\frac{-79}{90}, \frac{53}{18}, \frac{16}{45}\right)$.

(b) Let Q be the desired reflection. Since $\overrightarrow{PQ}$ is normal to the plane, we can write

$$\vec{Q} = \vec{P} + t \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 - 4t \\ -4 + 4t \\ 5 + 3t \end{pmatrix}$$

for some value of t. The midpoint of $\overline{PQ}$ lies on the plane, so

$$(-4) \cdot \frac{4 + (4 - 4t)}{2} + 4 \cdot \frac{-4 + (-4 + 4t)}{2} + 3 \cdot \frac{5 + (5 + 3t)}{2} = 3.$$

Solving this equation yields $t = 40/41$ and $Q = \left(\frac{4}{41}, \frac{-4}{41}, \frac{325}{41}\right)$.

(c) A parameterization for $\mathcal{C}$ is given by $\mathbf{v}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}$. The shortest possible distance between an arbitrary point on $\mathcal{C}$ and a point on $\mathcal{P}$ can be found by projecting $\mathbf{v}(\theta)$ onto a normal vector for $\mathcal{P}$. One such normal vector is

$$\mathbf{n} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 14 \end{pmatrix},$$

so then the distance between $\mathbf{v}(\theta)$ and the plane $\mathcal{P}$ is

$$\|\operatorname{proj}_{\mathbf{n}}(\mathbf{v}(\theta))\| = \frac{|\mathbf{v}(\theta) \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-2\cos\theta - 4\sin\theta + 14|}{6\sqrt{6}}.$$

We can write

$$2\cos\theta + 4\sin\theta = 2\sqrt{5}\left(\frac{1}{\sqrt{5}}\cos\theta + \frac{2}{\sqrt{5}}\sin\theta\right) = 2\sqrt{5}\sin(\phi + \theta),$$

where $\sin\phi = 1/\sqrt{5}$ and $\cos\phi = 2/\sqrt{5}$. Therefore,

$$\|\operatorname{proj}_{\mathbf{n}}(\mathbf{v}(\theta))\| = \frac{|14 - 2\sqrt{5}\sin(\phi + \theta)|}{6\sqrt{6}} \geq \boxed{\frac{14 - 2\sqrt{5}}{6\sqrt{6}}},$$

with equality when $\sin(\phi + \theta) = 1$.

7. (a) Since ABC lies in the xy-plane, the cross product is parallel to $\hat{\mathbf{k}}$. By the right-hand rule, $\overrightarrow{AB} \times \overrightarrow{AC}$ points downward since A, B, C go around the triangle in clockwise order. The norm of $\overrightarrow{AB} \times \overrightarrow{AC}$ is twice the area of triangle ABC. Therefore, $\overrightarrow{AB} \times \overrightarrow{AC} = (0, 0, -28)$.

   (b) Without loss of generality, suppose that the vertices of $ABCD$ are in counterclockwise order and that the quadrilateral lies in the xy-plane. Then the z-coordinate of the vector

$$\overrightarrow{AP} \times \overrightarrow{AD} + \overrightarrow{QB} \times \overrightarrow{QA} + \overrightarrow{BC} \times \overrightarrow{BP} \tag{$*$}$$

is precisely $2([ARD] - [PSQR] + [BSC])$, so it suffices to show that $(*)$ is $\mathbf{0}$. Let $\mathbf{a} = \overrightarrow{A}$, $\mathbf{b} = \overrightarrow{B}$, etc., so then $(*)$ becomes

$$(\mathbf{p} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) + (\mathbf{b} - \mathbf{q}) \times (\mathbf{a} - \mathbf{q}) + (\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}).$$

Let $r = AP/AB = CQ/CD$. Then

$$\overrightarrow{P} = (1 - r)\mathbf{a} + r\mathbf{b} \quad \text{and} \quad \overrightarrow{Q} = (1 - r)\mathbf{c} + r\mathbf{d},$$

so

$$\begin{aligned}
(\mathbf{p} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) &= r(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) \\
&= r(\mathbf{b} - \mathbf{a}) \times \mathbf{d} - r\mathbf{b} \times \mathbf{a} \\
(\mathbf{b} - \mathbf{q}) \times (\mathbf{a} - \mathbf{q}) &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{q} - \mathbf{q} \times \mathbf{a} \\
&= \mathbf{b} \times \mathbf{a} + \mathbf{q} \times (\mathbf{b} - \mathbf{a}) \\
&= \mathbf{b} \times \mathbf{a} + ((1 - r)\mathbf{c} + r\mathbf{d}) \times (\mathbf{b} - \mathbf{a}) \\
&= \mathbf{b} \times \mathbf{a} + (1 - r)\mathbf{c} \times (\mathbf{b} - \mathbf{a}) - r(\mathbf{b} - \mathbf{a}) \times \mathbf{d} \\
(\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}) &= (\mathbf{c} - \mathbf{b}) \times (1 - r)(\mathbf{a} - \mathbf{b}) \\
&= -(1 - r)\mathbf{c} \times (\mathbf{b} - \mathbf{a}) - (1 - r)\mathbf{b} \times \mathbf{a}.
\end{aligned}$$

Adding these up gives $\mathbf{0}$, as required.

8. (a) $\overline{x} = \langle \mathbf{x}, \mathbf{1} \rangle$

   (b) We note that $\mathcal{H}$ consists of all vectors orthogonal to $\mathbf{x}$, and since

$$\langle \mathbf{x} - \overline{x}\mathbf{1}, \mathbf{1} \rangle = \langle \mathbf{x}, \mathbf{1} \rangle - \overline{x}\langle \mathbf{1}, \mathbf{1} \rangle = \overline{x} - \overline{x} = 0,$$

we see that $\mathbf{x} - \overline{x}\mathbf{1} \in \mathcal{H}$. The decomposition $\mathbf{x} = (\mathbf{x} - \overline{x}\mathbf{1}) + \overline{x}\mathbf{1}$ writes $\mathbf{x}$ as the sum of a vector in $\mathcal{H}$ and a vector orthogonal to $\mathcal{H}$, so $\mathbf{x} - \overline{x}\mathbf{1}$ is the projection of $\mathbf{x}$ onto $\mathcal{H}$.

   (c) We compute

$$\begin{aligned}
s_x^2 &= \langle \mathbf{x} - \overline{x}\mathbf{1}, \mathbf{x} - \overline{x}\mathbf{1} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - \overline{x}\langle \mathbf{x}, \mathbf{1} \rangle - \overline{x}\langle \mathbf{1}, \mathbf{x} \rangle + (\overline{x})^2\langle \mathbf{1}, \mathbf{1} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - \overline{x} \cdot \overline{x} - \overline{x} \cdot \overline{x} + (\overline{x})^2 \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - (\overline{x})^2.
\end{aligned}$$

9. (a) Note that

$$s_x = \sqrt{\langle \mathbf{x} - \overline{x}\mathbf{1}, \mathbf{x} - \overline{x}\mathbf{1} \rangle} = \frac{1}{\sqrt{n}} \|\mathbf{x} - \overline{x}\mathbf{1}\|,$$

where $\| \ \|$ is the usual (euclidean) norm. Therefore,

$$r_{xy} = \frac{s_{xy}}{s_x \cdot s_y} = \frac{\frac{(\mathbf{x}-\overline{x}\mathbf{1})\cdot(\mathbf{y}-\overline{y}\mathbf{1})}{n}}{\frac{1}{\sqrt{n}}\|\mathbf{x} - \overline{x}\mathbf{1}\| \cdot \frac{1}{\sqrt{n}}\|\mathbf{y} - \overline{y}\mathbf{1}\|} = \frac{(\mathbf{x} - \overline{x}\mathbf{1}) \cdot (\mathbf{y} - \overline{y}\mathbf{1})}{\|\mathbf{x} - \overline{x}\mathbf{1}\|\|\mathbf{y} - \overline{y}\mathbf{1}\|} = \cos\theta,$$

where θ is the angle between $\mathbf{x} - \overline{x}\mathbf{1}$ and $\mathbf{y} - \overline{y}\mathbf{1}$. The result follows.

(b) We have $r_{xy} = 1$ when $\mathbf{y} - \overline{y}\mathbf{1}$ is a positive multiple of $\mathbf{x} - \overline{x}\mathbf{1}$, so that the angle between them satisfies $\cos\theta = 1$. This means $\mathbf{y} - \overline{y}\mathbf{1} = m(\mathbf{x} - \overline{x}\mathbf{1})$ for some $m > 0$. Separating into components, $y_i - \overline{y} = m(x_i - \overline{x})$ for all i, so the observed data lies on a single line of slope $m > 0$ when the points (x_i, y_i) are plotted in the plane.

Similarly, $r_{xy} = -1$ when the data points (x_i, y_i) lie on a single line of negative slope.

10. Picking β_0 and β_1 as specified corresponds to projecting $\mathbf{y}$ onto the plane $\mathcal{X}$ generated by $\mathbf{1}$ and $\mathbf{x}$. Suppose $\mathbf{y} = \beta_0\mathbf{1} + \beta_1\mathbf{x} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is normal to $\mathcal{X}$. Then

$$\overline{y} = \langle \mathbf{y}, \mathbf{1} \rangle = \langle \beta_0\mathbf{1} + \beta_1\mathbf{x} + \boldsymbol{\varepsilon}, \mathbf{1} \rangle = \beta_0 + \beta_1\overline{x},$$

which establishes the formula for β_0 in terms of β_1. For β_1,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{x}, \beta_0\mathbf{1} + \beta_1\mathbf{x} + \boldsymbol{\varepsilon} \rangle = \beta_0\overline{x} + \beta_1\langle \mathbf{x}, \mathbf{x} \rangle \\ &= (\overline{y} - \beta_1\overline{x})\overline{x} + \beta_1\langle \mathbf{x}, \mathbf{x} \rangle. \end{aligned}$$

Solving for β_1 gives us

$$\beta_1 = \frac{\langle \mathbf{x}, \mathbf{y} \rangle - \overline{xy}}{\langle \mathbf{x}, \mathbf{x} \rangle - (\overline{x})^2} = \frac{s_{xy}}{s_x^2} = r_{xy} \cdot \frac{s_y}{s_x}.$$

3 Matrices in 3D

Problems and solutions can be found at https://azhou5849.github.io/teaching/

3.1 Review Problems

Throughout this section, let

$$\mathsf{A} = \begin{pmatrix} 3 & 3 & 0 \\ 2 & -1 & 2 \\ 3 & 2 & 3 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 2 & 2 & -1 \\ -3 & 3 & -3 \\ -3 & -1 & 1 \end{pmatrix}, \qquad \mathsf{U} = \begin{pmatrix} -3 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -7 \\ -3 \\ 8 \end{pmatrix}.$$

1. *Matrix-vector calculations.*
   (a) Compute $\mathsf{A}\mathbf{u}$, $\mathsf{B}\mathbf{u}$, $\mathsf{U}\mathbf{u}$, $\mathsf{A}\mathbf{v}$, and $\mathsf{B}\mathbf{w}$.
   (b) Compute $2(\mathsf{A}\mathbf{u}) + 3(\mathsf{A}\mathbf{v})$ and $\mathsf{A}(2\mathbf{u} + 3\mathbf{v})$.
   (c) How does $\mathsf{B}\mathbf{w}$ relate to $\mathbf{w}$? Use this to compute $\mathsf{B}^5\mathbf{w}$.
2. *Matrix-matrix calculations.*
   (a) Compute $3\mathsf{A}$ and $\mathsf{B} - \mathsf{U}$.
   (b) Compute $(3\mathsf{A} + \mathsf{B} - \mathsf{U})\mathbf{v}$ and $3(\mathsf{A}\mathbf{v}) + \mathsf{B}\mathbf{v} - \mathsf{U}\mathbf{v}$.
   (c) Find all triples (r, s, t) of real numbers such that $r\mathsf{A} + s\mathsf{B} + t\mathsf{U} = \mathsf{0}$.
3. *Matrix products.*
   (a) Compute AB, BA, and AU.
   (b) Compute $\mathsf{AB} + \mathsf{AU}$ and $\mathsf{A}(\mathsf{B} + \mathsf{U})$.
   (c) Compute $(\mathsf{AB})\mathsf{U}$ and $\mathsf{A}(\mathsf{BU})$.
4. *Geometric transformations.*
   (a) Compute the matrix for scaling the x and y coordinates by $3/2$.
   (b) Compute the matrix for projecting onto $\mathbf{w}$.
   (c) Compute the matrix for reflecting across the plane $2x + 2y + 3z = 0$.
   (d) Compute the matrix for rotating around the z-axis with the property that $(-3, 4, -12)$ is rotated to $(0, 5, -12)$.
   (e) $(*)$ Compute the matrix for rotating by an angle of π around the axis passing through the origin and the point $(-3, 4, -12)$.

5. *Determinants.*

   (a) Compute $\det \mathsf{A}$, $\det \mathsf{B}$, and $\det \mathsf{U}$.

   (b) Compute $\det(\mathsf{AB})$ and $(\det \mathsf{A})(\det \mathsf{B})$.

6. *Inverses.*

   (a) Compute the inverses of A and U.

   (b) Show that $\mathsf{B} - 4\mathsf{I}$ is not invertible.

   (c) Identify a non-zero vector $\mathbf{x}$ with the property that $(\mathsf{B} - 4\mathsf{I})\mathbf{x} = \mathbf{0}$.

7. *Cross products and matrices.* Given a matrix M, let $[\mathsf{M}]_{ij}$ denote the entry in the i-th row and j-th column. Recall that the **transpose** of M is the matrix M^T with the property that $[\mathsf{M}^T]_{ij} = [\mathsf{M}]_{ji}$, i.e. the rows of M^T are the columns of M and vice-versa. A square matrix M is called **symmetric** if $\mathsf{M}^T = \mathsf{M}$ and **skew-symmetric** if $\mathsf{M}^T = -\mathsf{M}$.

   (a) Show that for every vector $\mathbf{a}$, there is a corresponding skew-symmetric matrix $\mathsf{R}_{\mathbf{a}}$ such that $\mathsf{R}_{\mathbf{a}}\mathbf{x} = \mathbf{a} \times \mathbf{x}$ for all vectors $\mathbf{x}$.

   (b) Conversely, show that for every skew-symmetric matrix M, there is a corresponding vector $\mathbf{a}_{\mathsf{M}}$ for which $\mathsf{M}\mathbf{x} = \mathbf{a}_{\mathsf{M}} \times \mathbf{x}$ for all vectors $\mathbf{x}$.

   (c) Show that $\mathsf{R}_{\mathbf{a}\times\mathbf{b}} = \mathsf{R}_{\mathbf{a}}\mathsf{R}_{\mathbf{b}} - \mathsf{R}_{\mathbf{b}}\mathsf{R}_{\mathbf{a}}$.

   (d) Independently of the previous parts, show that every square matrix M can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

3.2 Challenge Problems

The results in this section are stated for $\mathbb{R}^3$ and for 3×3 matrices, but they generalise directly to any (finite) dimension.

8. In general, if A is an $\ell \times m$ matrix and B is an $m \times n$ matrix, then AB is an $\ell \times n$ matrix. For $1 \leq i \leq \ell$ and $1 \leq j \leq n$, the entry of AB in row i and column j is found by taking the "dot product" of the i-th row of A with the j-th column of B. This also applies to matrix-vector multiplication if we regard an m-component vector as an $m \times 1$ matrix.

   (a) Describe the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ geometrically.

   (b) Let $\mathbf{u} \in \mathbb{R}^3$ be a vector of norm 1. Show that projection onto the line generated by $\mathbf{u}$ is given by the matrix $\mathbf{u}\mathbf{u}^T$.

   (c) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be vectors of norm 1 which are orthogonal to each other, and let Q be the matrix whose columns are $\mathbf{u}$ and $\mathbf{v}$. Show that projection onto the plane generated by $\mathbf{u}$ and $\mathbf{v}$ is given by the matrix $\mathsf{Q}\mathsf{Q}^T$.

   (d) More generally, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be any two linearly independent vectors and let A be the matrix whose columns are $\mathbf{u}$ and $\mathbf{v}$. Derive a formula in terms of A for the matrix that represents projection onto the plane generated by $\mathbf{u}$ and $\mathbf{v}$.

9. Let A be a 3×3 matrix. For each pair of indices $1 \leq i, j \leq 3$, the (i, j)-th **minor** M_{ij} of A is the determinant of the 2×2 matrix formed by deleting the i-th row and j-th column of A. The **adjugate matrix** of A is the 3×3 matrix $\operatorname{adj} \mathsf{A}$ whose (i, j)-th entry is $(-1)^{i+j} M_{ji}$.

   (a) Show that $\mathsf{A}(\operatorname{adj} \mathsf{A}) = (\operatorname{adj} \mathsf{A})\mathsf{A} = (\det \mathsf{A})\mathsf{I}$.

   (b) Supposing $\det \mathsf{A} \neq 0$, express A^{-1} in terms of $\operatorname{adj} \mathsf{A}$ and $\det \mathsf{A}$.

10. Fix a matrix $\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Given any polynomial $f(X) = b_0 + b_1 X + \cdots + b_n X^n$, we can define scalar multiplication of a vector by a polynomial according to the formula

$$f(X)\mathbf{v} = (b_0 \mathsf{I} + b_1 \mathsf{A} + b_2 \mathsf{A}^2 + \cdots + b_n \mathsf{A}^n)\mathbf{v}.$$

   (a) Show that

$$\begin{pmatrix} a_{11} - X & a_{12} & a_{13} \\ a_{21} & a_{22} - X & a_{23} \\ a_{31} & a_{32} & a_{33} - X \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

   (b) Show that $p_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I})$, the **characteristic polynomial** of A, satisfies the relation $p_{\mathsf{A}}(X)\mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$.

   (c) Supposing $p_{\mathsf{A}}(X) = c_0 + c_1 X + c_2 X^2 + c_3 X^3$, show that $c_0 \mathsf{I} + c_1 \mathsf{A} + c_2 \mathsf{A}^2 + c_3 \mathsf{A}^3 = \mathsf{0}$. This is the **Cayley-Hamilton theorem** (3×3 case), sometimes written as $p_{\mathsf{A}}(\mathsf{A}) = \mathsf{0}$.

3.3 Answers

1. (a) $\mathsf{A}\mathbf{u} = \begin{pmatrix} 27 \\ 6 \\ 23 \end{pmatrix}$ $\quad$ $\mathsf{A}\mathbf{v} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$

   $\mathsf{B}\mathbf{u} = \begin{pmatrix} 18 \\ -3 \\ -19 \end{pmatrix}$ $\quad$ $\mathsf{B}\mathbf{w} = \begin{pmatrix} -28 \\ -12 \\ 32 \end{pmatrix}$

   $\mathsf{U}\mathbf{u} = \begin{pmatrix} -23 \\ -4 \\ 0 \end{pmatrix}$

   (b) The two computations should give the same result, as

   $$\mathsf{A}(2\mathbf{u} + 3\mathbf{v}) = \mathsf{A}(2\mathbf{u}) + \mathsf{A}(3\mathbf{v}) = 2(\mathsf{A}\mathbf{u}) + 3(\mathsf{A}\mathbf{v}).$$

   From what we computed in part (a),

   $$2(\mathsf{A}\mathbf{u}) + 3(\mathsf{A}\mathbf{v}) = 2\begin{pmatrix} 27 \\ 6 \\ 23 \end{pmatrix} + 3\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 45 \\ 21 \\ 49 \end{pmatrix}.$$

   (c) We observe that $\mathsf{B}\mathbf{w} = 4\mathbf{w}$. Therefore,

   $$\mathsf{B}^5\mathbf{w} = 4^5\mathbf{w} = 1024\begin{pmatrix} -7 \\ -3 \\ 8 \end{pmatrix} = \begin{pmatrix} -7168 \\ -3072 \\ 8192 \end{pmatrix}.$$

2. (a) $3\mathsf{A} = \begin{pmatrix} 9 & 9 & 0 \\ 6 & -3 & 6 \\ 9 & 6 & 9 \end{pmatrix}$ $\quad$ $\mathsf{B} - \mathsf{U} = \begin{pmatrix} 5 & 4 & -1 \\ -3 & 4 & -2 \\ -3 & -1 & -1 \end{pmatrix}$

   (b) The two computations should give the same result, as

   $$(3\mathsf{A} + \mathsf{B} - \mathsf{U})\mathbf{v} = (3\mathsf{A})\mathbf{v} + \mathsf{B}\mathbf{v} - \mathsf{U}\mathbf{v} = 3(\mathsf{A}\mathbf{v}) + \mathsf{B}\mathbf{v} - \mathsf{U}\mathbf{v}.$$

   From what we computed in part (a),

   $$3\mathsf{A} + \mathsf{B} - \mathsf{U} = \begin{pmatrix} 14 & 13 & -1 \\ 3 & 1 & 4 \\ 6 & 5 & 8 \end{pmatrix},$$

   so then

   $$(3\mathsf{A} + \mathsf{B} - \mathsf{U})\mathbf{v} = \begin{pmatrix} 14 & 13 & -1 \\ 3 & 1 & 4 \\ 6 & 5 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -14 \\ 3 \\ 3 \end{pmatrix}.$$

   (c) The upper right entry of $\mathsf{M} = r\mathsf{A} + s\mathsf{B} + t\mathsf{U} = \mathsf{0}$ is $-s$, so we need $s = 0$. Then, the lower left entry of $\mathsf{M} = r\mathsf{A} + t\mathsf{U} = \mathsf{0}$ is $3r$, so we need $r = 0$. Finally, $\mathsf{M} = t\mathsf{U} = \mathsf{0}$ forces $t = 0$, so the only solution is $(r, s, t) = (0, 0, 0)$.

3. (a) $\mathsf{AB} = \begin{pmatrix} -3 & 15 & -12 \\ 1 & -1 & 3 \\ -9 & 9 & -6 \end{pmatrix}$

$\mathsf{BA} = \begin{pmatrix} 7 & 2 & 1 \\ -12 & -18 & -3 \\ -8 & -6 & 1 \end{pmatrix}$

$\mathsf{AU} = \begin{pmatrix} -9 & -9 & -3 \\ -6 & -3 & 5 \\ -9 & -8 & 4 \end{pmatrix}$

(b)