Precalculus Practice Problems: Final

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2024-2025

The focus of these review problems is on the material covered in Weeks 25 through 36, but keep in mind that prior material can still appear on the exam.

Contents

1 Matrices in 2D

1.1 Review Problems

Review problems are meant to cover "standard" definitions and calculations as well as the use of important results.

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector.
We also let $\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathsf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.

   (a) $\mathbf{u} + \mathbf{v}$

   (b) $2\mathbf{v}$

   (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

   (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$

   (e) The angle between $\mathbf{u}$ and $\mathbf{v}$

   (f) $\operatorname{proj}_{\mathbf{v}}(\mathbf{u})$ and $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$

2. *Applying matrices to vectors.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

   (a) Compute $\mathsf{A}\mathbf{v}$

   (b) Find a vector $\mathbf{u}$ for which $\mathsf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.

3. *Matrix operations.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\mathsf{A} + \mathsf{B}$

   (b) $-3\mathsf{A}$

   (c) AB

   (d) BA

   (e) B^T (the transpose of B)

4. *Geometric transformations.* Write down matrices for each of the following.

   (a) Dilation about the origin by a factor of 4

   (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2

   (c) Rotation about the origin by $\pi/4$ counterclockwise

   (d) Projection onto the line $y = (3/2)x$

   (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

    (a) $\det A$ and $\det B$

    (b) $\det(AB)$

    (c) $\det(A^T)$

    (d) $\det(A + B)$

    (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

    (a) A^{-1} and B^{-1}

    (b) $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$

    (c) $(AB)^{-1}$

    (d) $(A^T)^{-1}$

    (e) $(A + B)^{-1}$

    (f) $\det(A^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

    (a) Describe the image of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.

    (b) By what factor does a horizontal shear multiply areas?

    (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

    (The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

Challenge problems are meant to provide optional extensions of the ideas from class.

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

   (a) For the matrices A and B in problems 3, 5, and 6, compute $\mathrm{tr}\,A$, $\mathrm{tr}\,B$, and $\mathrm{tr}(AB)$.

   (b) Show that for any 2×2 matrices P and Q, we have $\mathrm{tr}(PQ) = \mathrm{tr}(QP)$.

   (c) In general, must it be true that $\mathrm{tr}(ABC) = \mathrm{tr}(ACB)$?

9. Two matrices A, B are **similar**, written $A \sim B$, if there is an invertible P with $B = P^{-1}AP$.

   (a) Show that the only matrix similar to I is I.

   (b) Show that if $A \sim B$, then $\det A = \det B$ and $\mathrm{tr}\,A = \mathrm{tr}\,B$.

   (c) Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $D \sim A$. Find D.

10. If A is a square matrix, the **characteristic polynomial** of A is defined by

$$f_A(X) = \det(A - XI).$$

   (a) Compute the characteristic polynomial $f_A(X)$ of the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_A(X)$.

   (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, 2$. (In general, if $A\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call $\mathbf{v}$ an **eigenvector** of A corresponding to the **eigenvalue** λ.)

   (d) Let P be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Compute $P^{-1}AP$.

   (e) *Cayley-Hamilton theorem.* Suppose $f_A(X) = a_0 + a_1 X + a_2 X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0 I + a_1 A + a_2 A^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

2. (a) $\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -9 \end{pmatrix}$ (there are many other options)

   (b) Let p and q be the value of the parameters for $\mathbf{x}_1$ and $\mathbf{x}_2$ at the point of intersection, so $\mathbf{x}_1(p) = \mathbf{x}_2(q)$. Then
   $$\begin{pmatrix} 0 \\ 3 \end{pmatrix} + p \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + q \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
   which means $p = -4 + 2q$ and $3 - p = 2 + 3q$. Solving the system, $q = 1$ and $p = -2$, and the point of intersection is $(-2, 5)$.

3. (a) $\|\mathbf{u}\| = \sqrt{13}$

   $\mathbf{u} \cdot \mathbf{v} = 5$

   (b) $\hat{\mathbf{v}} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{pmatrix}$

   (c) $\cos\theta = \dfrac{5}{\sqrt{221}}$

   (d) $\pm \dfrac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

4. (a) If $a\mathbf{u} + b\mathbf{v} = 0$, then $2a + 4b = 0$ and $3a - b = 0$. Adding 4 times the second equation to the first, $14a = 0$, so $a = 0$ and hence $b = 0$ as well.

   (b) $(a, b, c, d) = (1/14, 3/14, 2/7, -1/7)$

   (c) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. If $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$, then

   $$x_1 a + y_1 b = 0, \tag{1}$$
   $$x_2 a + y_2 b = 0. \tag{2}$$

   Taking the combination $y_2 \cdot (1) - y_1 \cdot (2)$,

   $$(x_1 y_2 - x_2 y_1)a = 0.$$

   If $x_1 y_2 - x_2 y_1 = 0$, then we consider further subcases.

   - If $\mathbf{x} = \mathbf{0}$, then taking $(a, b) = (1, 0)$ gives a non-zero solution, so $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Moreover, linear combinations $c\mathbf{x} + d\mathbf{y} = d\mathbf{y}$ can only produce multiples of $\mathbf{y}$, hence will not span all of $\mathbb{R}^2$.
   - If $\mathbf{x} \neq \mathbf{0}$, either $x_1 \neq 0$ or $x_2 \neq 0$. If $x_1 \neq 0$, then let $\lambda = y_1/x_1$, so $y_1 = \lambda x_1$. Substituting, we find $y_2 = \lambda x_2$, so $\mathbf{y} = \lambda \mathbf{x}$. This means that $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Also, any linear combination $c\mathbf{x} + d\mathbf{y} = (c + d\lambda)\mathbf{x}$ will be a multiple of $\mathbf{x}$, hence will not span all of $\mathbb{R}^2$.

If $x_1y_2 - x_2y_1 \neq 0$, then $a = 0$. At least one of y_1 and y_2 is non-zero in this case, and substituting into the relevant equation gives $b = 0$ as well. Therefore, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. To see that they span $\mathbb{R}^2$, let $\begin{pmatrix} p \\ q \end{pmatrix}$ be arbitrary. We look for coefficients c, d such that $c\mathbf{x} + d\mathbf{y} = \begin{pmatrix} p \\ q \end{pmatrix}$, or

$$x_1 c + y_1 d = p,$$
$$x_2 c + y_2 d = q.$$

This has a solution, namely $(c, d) = \left(\dfrac{py_2 - qy_1}{x_1 y_2 - x_2 y_1}, \dfrac{qx_1 - px_2}{x_1 y_2 - x_2 y_1} \right).$

(d) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be given. If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are as well. Otherwise, by part (c), there exist a, b for which $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. Then $a\mathbf{x} + b\mathbf{y} - \mathbf{z}$ is a non-trivial linear combination of the three which equals $\mathbf{0}$, so they are linearly dependent.

5. (a) $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

(b) It suffices to show $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Suppose $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$. Then

$$0 = \mathbf{x} \cdot (a\mathbf{x} + b\mathbf{y}) = a(\mathbf{x} \cdot \mathbf{x}) + b(\mathbf{x} \cdot \mathbf{y}) = a(1) + b(0) = a.$$

By a similar argument, $b = 0$.

(c) Translating up by 1 unit, it suffices to find the distance between the head of $\mathbf{u} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and the line $y = 2x$, which is spanned by the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We compute

$$\| \text{proj}_{\mathbf{v}}(\mathbf{u}) \| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{v}\|} = \frac{16}{\sqrt{5}}.$$

6. (a) $y = (-2/3)x + (4/3)$

(b) $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $d = 5$ (there are many choices that work)

(c) If $\mathbf{x}_1, \mathbf{x}_2$ are two position vectors for points on the line, then $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$ points along the line. Since

$$\hat{\mathbf{n}} \cdot \mathbf{v} = \hat{\mathbf{n}} \cdot (\mathbf{x}_2 - \mathbf{x} - 1) = \hat{\mathbf{n}} \cdot \mathbf{x}_2 - \hat{\mathbf{n}} \cdot \mathbf{x}_1 = d - d = 0,$$

$\hat{\mathbf{n}}$ is perpendicular to the line. Then, for any position vector $\mathbf{x}$ on the line, we compute the distance from the origin to the line as

$$\| \text{proj}_{\hat{\mathbf{n}}}(\mathbf{x}) \| = \frac{|\hat{\mathbf{n}} \cdot \mathbf{x}|}{\|\hat{\mathbf{n}}\|} = d.$$

6

7. (a) By the angle bisector theorem, $BD/DC = AB/AC = 1/2$. Therefore, $\mathbf{D} = \frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}$.

   (b) The points on line $\overline{AD}$ are those with position vector of the form

   $$t\mathbf{A} + (1-t)\mathbf{D} = t\mathbf{A} + (1-t)\left(\frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}\right)$$

   for a real number t. Setting $t = 5/14$ gives us $\mathbf{I}$.

8. (a) We show that $\mathbf{y} = 2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}$ has the defining properties of reflection, namely that the line ℓ spanned by $\mathbf{v}$ is orthogonal to $\mathbf{x} - \mathbf{y}$ and passes through the midpoint of the segment connecting $\mathbf{x}$ and $\mathbf{y}$. First, $(\mathbf{x} + \mathbf{y})/2 = \operatorname{proj}_{\mathbf{v}}(\mathbf{x})$, so ℓ passes through the midpoint. For orthogonality,

   $$\mathbf{v} \cdot (\mathbf{x} - \mathbf{y}) = 2\mathbf{v} \cdot (\mathbf{x} - \operatorname{proj}_{\mathbf{v}}(\mathbf{x})) = 0.$$

   (b) We compute

   $$\begin{aligned}
   \operatorname{refl}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) &= 2\operatorname{proj}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) - (a\mathbf{x} + b\mathbf{y}) \\
   &= 2(a\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{proj}_{\mathbf{v}}(\mathbf{y})) - (a\mathbf{x} + b\mathbf{y}) \\
   &= a(2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}) + b(2\operatorname{proj}_{\mathbf{v}}(\mathbf{y}) - \mathbf{y}) \\
   &= a\operatorname{refl}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{refl}_{\mathbf{v}}(\mathbf{y}).
   \end{aligned}$$

9. (a) Linearity of $\operatorname{proj}_{\mathbf{v}}$ follows from (bi)linearity of the dot product.
   To see that $\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}} = \operatorname{proj}_{\mathbf{v}}$, we compute

   $$\begin{aligned}
   (\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}})(\mathbf{x}) &= \operatorname{proj}_{\mathbf{v}}\left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v}\right) \\
   &= \left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
   &= \left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} = \operatorname{proj}_{\mathbf{v}}(\mathbf{x}).
   \end{aligned}$$

   Finally, orthogonality is part of how $\operatorname{proj}_{\mathbf{v}}$ was defined.

   (b) Let $\mathbf{x}$ be a vector for which $P(\mathbf{x}) \neq \mathbf{0}$ and let $\mathbf{y}$ be a vector for which $P(\mathbf{y}) \neq \mathbf{y}$. (These conditions imply $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.) Let $\mathbf{u} = P(\mathbf{x})$ and $\mathbf{v} = \mathbf{y} - P(\mathbf{y})$. These are non-zero, and

   $$\begin{aligned}
   P(\mathbf{u}) &= P(P(\mathbf{x})) = P(\mathbf{x}) = \mathbf{u}, \\
   P(\mathbf{v}) &= P(\mathbf{y} - P(\mathbf{y})) = P(\mathbf{y}) - P(P(\mathbf{y})) = P(\mathbf{y}) - P(\mathbf{y}) = \mathbf{0}.
   \end{aligned}$$

   For linear independence, suppose $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$. Applying P,

   $$\mathbf{0} = P(\mathbf{0}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u},$$

   so $a = 0$. Then, $b\mathbf{v} = \mathbf{0}$, so $b = 0$ as well.

(c) Since $\mathbf{u}, \mathbf{v}$ are linearly independent in $\mathbb{R}^2$, they also span $\mathbb{R}^2$. Given any vector $\mathbf{x} \in \mathbb{R}^2$, let $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$. Then

$$P(\mathbf{x}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u}.$$

As a and b range over all real numbers, we find the range of P is the span of $\mathbf{u}$.

10. (a) We proceed by induction. By definition, $s_0 = ax_0 + by_0$ and $s_1 = ax_1 + by_1$. Then, if $s_n = ax_n + by_n$ and $s_{n+1} = ax_{n+1} + by_{n+1}$,

$$\begin{aligned}
s_{n+2} &= s_{n+1} + 2s_n \\
&= (ax_{n+1} + by_{n+1}) + 2(ax_n + by_n) \\
&= a(x_{n+1} + 2x_n) + b(y_{n+1} + 2y_n) \\
&= ax_{n+2} + by_{n+2}.
\end{aligned}$$

(b) We need $\lambda^n(\lambda^2 - \lambda - 2) = 0$ for all $n \geq 0$. This holds when $\lambda = -1$ and when $\lambda = 2$.

(c) The sequence $e_n = (-1)^n$ corresponds to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, while the sequence $f_n = 2^n$ corresponds to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We can find

$$\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{11}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so

$$x_n = \frac{1}{3}e_n + \frac{11}{3}f_n = \frac{1}{3}\cdot(-1)^n + \frac{11}{3}\cdot 2^n.$$