Precalculus Practice Problems: Midterm 2

Alan Zhou

2024-2025

The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. *SAS congruence.* Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Using the law of sines, or otherwise, find $\sin A$ and $\sin B$.
   (d) Show that $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$, and hence compute $\angle A$ and $\angle B$. (Hint: For which angles does $\arcsin(\sin\theta) = \theta$ hold?)
2. *SSS congruence.* Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Using the law of cosines, or otherwise, find $\cos A$, $\cos B$, and $\cos C$.
   (b) Compute $\angle A$, $\angle B$, and $\angle C$.
   (c) Find $[ABC]$.
3. *ASA/AAS congruence.* Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.
4. *SSA non-congruence.* Let ABC be a triangle with $\angle A = 30°$, $a = 6$, and $b = 9$.
   (a) Show that $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) Find all possible values of c.
5. *Extended law of sines.* If ABC is a triangle with circumradius R, then the **extended law of sines** states that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Express $\sin C$ in terms of a, b, and $[ABC]$.
   (b) Assuming the extended law of sines, show that $R = \dfrac{abc}{4[ABC]}$.
   (c) Given that $a = 13$, $b = 14$, and $c = 15$, compute R.
6. *Single-cevian computations.* Let ABC be a triangle and let D be a point on side $\overline{BC}$. (The line segment $\overline{AD}$ is a *cevian* from A.)
   (a) Express $\angle ADB$ in terms of $\angle CDA$.

(b) *Ratio lemma.* Using the law of sines, or otherwise, show that

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(c) *Angle bisector theorem.* Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

(d) *Stewart's theorem.* Let $\angle ADB = \theta$ and let $AD = d$, $BD = x$, and $DC = y$. Show that

$$c^2 = d^2 + x^2 - 2dx\cos\theta,$$
$$b^2 = d^2 + y^2 + 2dy\cos\theta,$$

and conclude that

$$b^2x + c^2y = a(d^2 + xy).$$

7. *Concurrent cevians.* Let ABC be a triangle and let D, E, F lie on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively.

(a) Show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}.$$

(b) *Ceva's theorem.* Show that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent if and only if the two sides of the above equation are equal to 1.

1.2 Challenge problems

8. Points O, A, B, and C are placed in the plane so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. Find all possible lengths of BC.

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Assuming that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. *Eisenstein triples.* An **Eisenstein triple** is an unordered triple of positive integers $\{a, b, c\}$ for which a triangle with side lengths a, b, and c has an angle of measure either $60°$ or $120°$. If the Eisenstein triple corresponds to a triangle with an angle of measure $60°$, we will call it an Eisenstein triple of **acute type**, and otherwise, we call it an Eisenstein triple of **obtuse type**. (The "acute type" and "obtuse type" names are non-standard.)

(a) Suppose $a, b < c$ are such that $\{a, b, c\}$ is an Eisenstein triple of obtuse type. Show that $\{a, a+b, c\}$ and $\{a+b, b, c\}$ are Eisenstein triples of acute type.

(b) Conversely, show that every Eisenstein triple of acute type either corresponds to an equilateral triangle or arises from an Eisenstein triple of obtuse type in the above manner.

(c) Show that if $\{a, b, c\}$ is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that

$$\{a, b, c\} = \{2mn + n^2, m^2 - n^2, m^2 + mn + n^2\}.$$

(Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104^\circ) \approx 2.426$
   (b) $c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{26 - 10\cos(104^\circ)} \approx 5.331$
   (c) $\sin A = \frac{a\sin C}{c} \approx \frac{\sin(104^\circ)}{5.331} \approx 0.182$
   $\sin B = \frac{b\sin C}{c} \approx \frac{5\sin(104^\circ)}{5.331} \approx 0.910$
   (d) Since $\angle C$ is obtuse, $\angle A$ and $\angle B$ are acute. Since acute angles are included in the range of arcsin, we have $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$.
   $\angle A = \arcsin(\sin A) \approx \arcsin(0.182) \approx 10.49^\circ$
   $\angle B = \arcsin(\sin B) \approx \arcsin(0.910) \approx 65.51^\circ$

2. (a) $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3}{5}$
   $\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{33}{65}$
   $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{5}{13}$
   (b) The range of arccos is $[0^\circ, 180^\circ]$, so we can always use it to extract triangle angles.
   $\angle A = \arccos(\cos A) = \arccos(\frac{3}{5}) \approx 53.13^\circ$
   $\angle B = \arccos(\cos B) = \arccos(\frac{33}{65}) \approx 59.49^\circ$
   $\angle C = \arccos(\cos C) = \arccos(\frac{5}{13}) \approx 67.38^\circ$
   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7 \cdot 15 \cdot \frac{4}{5} = 84$

3. (a) $\angle C = 180^\circ - \angle A - \angle B = 91^\circ$
   (b) $a = \frac{c}{\sin C} \cdot \sin A = \frac{2\sin 12^\circ}{\sin 91^\circ} \approx 0.416$
   $b = \frac{c}{\sin C} \cdot \sin B = \frac{2\sin 77^\circ}{\sin 91^\circ} \approx 1.949$
   (c) $[ABC] = \frac{1}{2}ac\sin B \approx 0.416\sin 77^\circ \approx 0.405$

4. (a) By the law of cosines,
   $$a^2 = b^2 + c^2 - 2bc\cos A \implies 36 = 81 + c^2 - 18\cos(30^\circ)c.$$
   Evaluating $\cos(30^\circ) = \sqrt{3}/2$ and rearranging gives us $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) By the quadratic formula,
   $$c = \frac{9\sqrt{3} \pm \sqrt{(9\sqrt{3})^2 - 4 \cdot 1 \cdot 45}}{2} = \frac{9\sqrt{3} \pm 3\sqrt{7}}{2}.$$

5. (a) $\sin C = \frac{2[ABC]}{ab}$.
   (b) $R = \dfrac{c}{2\sin C} = \dfrac{c}{\frac{4[ABC]}{ab}} = \dfrac{abc}{4[ABC]}$
   (c) $R = 65/8$

6. (a) $\angle ADB = 180^\circ - \angle CDA$

(b) By the law of sines,

$$\frac{AB}{\sin(\angle ADB)} = \frac{BD}{\sin(\angle BAD)} \quad \text{and} \quad \frac{AC}{\sin(\angle CDA)} = \frac{DC}{\sin(\angle DAC)}.$$

Dividing one equation by the other and using the fact that $\sin(\angle ADB) = \sin(\angle CDA)$ gets us the desired result after rearranging.

(c) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (b).

(d) The law of cosines in triangle ADB, using $\angle ADB$, gives us

$$(AB)^2 = (AD)^2 + (BD)^2 - 2(AD)(BD)\cos(\angle ADB) \implies c^2 = d^2 + x^2 - 2dx\cos\theta,$$

while the law of cosines in triangle ADC, using $\angle CDA$, gives us

$$b^2 = d^2 + y^2 - 2dy\cos(180^\circ - \theta) = d^2 + y^2 + 2dy\cos\theta.$$

Adding y times the first equation to x times the second equation, so as to eliminate $\cos\theta$,

$$\begin{aligned} b^2x + c^2y &= (d^2x + y^2 + 2dy\cos\theta)x + (d^2y + x^2 - 2dx\cos\theta)y \\ &= d^2(x+y) + xy(x+y) = a(d^2 + xy). \end{aligned}$$

7. (a) By the ratio lemma (problem 6b),

$$\begin{aligned} \frac{AF}{FB} &= \frac{CA}{CB} \cdot \frac{\sin(\angle ACF)}{\sin(\angle FCB)}, \\ \frac{BD}{DC} &= \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}, \\ \frac{CE}{EA} &= \frac{BC}{BA} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}. \end{aligned}$$

Multiplying these equations together gives us the desired result.

(b) First suppose $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur at point P. Then by the law of sines,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle DAC)} &= \frac{\sin(\angle ACP)}{\sin(\angle PAC)} = \frac{AP}{CP}, \\ \frac{\sin(\angle BAD)}{\sin(\angle EBA)} &= \frac{\sin(\angle BAP)}{\sin(\angle PBA)} = \frac{BP}{AP}, \\ \frac{\sin(\angle CBE)}{\sin(\angle FCB)} &= \frac{\sin(\angle CBP)}{\sin(\angle PCB)} = \frac{CP}{BP}. \end{aligned}$$

Multiplying these equations,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)} &= \frac{\sin(\angle ACF)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle EBA)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle FCB)} \\ &= \frac{AP}{CP} \cdot \frac{BP}{AP} \cdot \frac{CP}{BP} = 1. \end{aligned}$$

Conversely, suppose both sides of the equation from part (a) are 1, so in particular

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Let $\overline{AD}$ and $\overline{BE}$ intersect at point Q, and let line $\overline{CQ}$ intersect side $\overline{AB}$ at point F'. Then using what we just showed,

$$\frac{AF'}{F'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

This means $AF/FB = AF'/F'B$. With F and F' both interior to segment $\overline{AB}$, this can only happen if $F = F'$, which means that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur (at Q) as desired.

8. Fix points A and B on a circle of radius 4 centered at O so that $AB = 2$. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

Since $CO = 4$, we know C lies on this circle as well, and since $AC = 1$, there are two possible locations for C, one on either side of $\overline{OA}$. When C and B lie on the same side of $\overline{OA}$, we have $\angle BOC = \angle AOB - \angle AOC$, which gives us

$$\begin{aligned} BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8} \cdot \frac{31}{32} + \frac{\sqrt{15}}{8} \cdot \frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

When C and B lie on opposite sides of $\overline{OA}$, we instead have $\angle BOC = \angle BOA + \angle AOC$. A similar calculation to the first case yields

$$BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. By Stewart's theorem,

$$\begin{aligned} (AB)^2 \cdot (DC) + (AC)^2 \cdot (BD) &= (BC)[(AD)^2 + (BD)(DC)] \\ 8^2 \cdot 7x + (8x)^2 \cdot 7 &= (7 + 7x)(5^2 + 7 \cdot 7x) \\ (8^2 \cdot 7)(x + x^2) &= 7(1 + x)(25 + 49x). \end{aligned}$$

Since $x > 0$, we can safely divide by $7(1 + x)$ on both sides to get

$$64x = 25 + 49x \implies x = \frac{5}{3}.$$

Therefore, $CD = 7x = 35/3$.

10. (a) Let ABC be the corresponding triangle, with $\angle C = 120°$ opposite side $AB = c$. Extend ray $\overrightarrow{AC}$ to point D so that $CD = CB$. Then triangle BCD is equilateral, so triangle ADB has side lengths $AD = a + b$, $DB = a$, and $AB = c$ with $\angle D = 60°$. This means $\{a, a+b, c\}$ is an Eisenstein triple of acute type. A similar construction where we extend $\overrightarrow{BC}$ tells us that $\{a + b, b, c\}$ is also an Eisenstein triple of acute type.

(b) Let a, b, c be such that $\{a, b, c\}$ is an Eisenstein triple of acute type with $AB = c$ opposite the $60°$ angle in the corresponding triangle ABC. If $a = b$, then the triangle is equilateral. Otherwise, suppose without loss of generality that $a < b$. Let point D on side $\overline{AC}$ be such that $CD = CB$ and let $a' = AD = b - a$. Then triangle ADB has $\angle ADB = 120°$, so $\{a', a, c\}$ is an Eisenstein triple of obtuse type. As $b = a' + a$, the original Eisenstein triple $\{a, b, c\}$ can be constructed from $\{a', a, c\}$ according to part (a).

(c) Suppose without loss of generality that c is the side opposite the $120°$ angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

This equation describes an ellipse in the plane passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$.[1] Moreover, every point with positive rational coordinates can be connected to the point $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$\begin{aligned} x^2 + x \cdot (tx - 1) + (tx - 1)^2 &= 1, \\ (t^2 + t + 1)x^2 - (2t + 1)x &= 0. \end{aligned}$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

[1] More specifically, we can rewrite the equation as $\frac{u^2}{2/3} + \frac{v^2}{2} = 1$ where $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$. This is an ellipse centered at $(0, 0)$ whose semimajor axis has length $\sqrt{2}$ lying along the v-axis, i.e. the line $u = 0$ or equivalently $y = -x$. The semiminor axis has length $\sqrt{2/3}$ lying along the u-axis, i.e. the line $v = 0$ or equivalently $y = x$.

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m,n) = 1$,

$$\begin{aligned}\gcd(2mn+n^2, m^2+mn+n^2) &= \gcd(n\cdot(2m+n), m^2+mn+n^2)\\ &= \gcd(2m+n, m^2+mn+n^2)\\ &= \gcd(2m+n, m^2+mn+n^2-n\cdot(2m+n))\\ &= \gcd(2m+n, m^2-mn) = \gcd(2m+n, m\cdot(m-n))\\ &= \gcd(2m+n, m-n) = \gcd(3n, m-n).\end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n+3k$. Then $\gcd(n,k) = 1$ and

$$\gcd(3n, m-n) = \gcd(3n, 3k) = 3\gcd(n,k) = 3.$$

Otherwise,

$$\gcd(3n, m-n) = \gcd(n, m-n) = \gcd(n,m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn+n^2}{3}, \quad b = \frac{m^2-n^2}{3}, \quad c = \frac{m^2+mn+n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r-s$ and $m = r+2s$. Then

$$\begin{aligned}a &= \frac{2(r+2s)(r-s)+(r-s)^2}{3} = r^2-s^2,\\ b &= \frac{(r+2s)^2-(r-s)^2}{3} = 2rs+s^2,\\ c &= \frac{(r+2s)^2+(r+2s)(r-s)+(r-s)^2}{3} = r^2+rs+s^2,\end{aligned}$$

so the result still holds with r and s in place of m and n.

2 Complex Numbers I: Algebra

Throughout, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

2.1 Review problems

1. *Arithmetic.* Let $z = -3 + 3i$ and $w = -4 - 2i$. Compute each of the following:
   (a) $\operatorname{Re} z$
   (b) $\operatorname{Im} w$
   (c) $z + w$
   (d) $z - w$
   (e) zw
   (f) z/w
2. *A quadratic with real coefficients.* Find all complex solutions to the equation $z^2 + 5 = 4z$.
3. *Real-valued products.* Find all complex numbers z for which $(-4 + 2i)z$ is real.
4. *Powers of i and periodic sequences.*
   (a) Show that $i^4 = 1$.
   (b) Find all complex solutions to the equation $z^4 = 1$ and write each one as a power of i.
   (c) Let $z_1, z_2, z_3, \ldots$ be a 4-periodic sequence of complex numbers, meaning that $z_{n+4} = z_n$ for all positive integers n. Show that there exist complex numbers a, b, c, d such that
   $$z_n = a + b \cdot i^n + c \cdot i^{2n} + d \cdot i^{3n}$$
   for all n.
5. *Complex conjugation.* Given a complex number $z = x + yi$, the **complex conjugate** of z is defined to be $\overline{z} = x - yi$.
   (a) Compute $\overline{943 - 319i}$.
   (b) Prove the following properties of complex conjugation:
      i. $\overline{(\overline{z})} = z$ for all complex numbers z.
      ii. $\overline{z + w} = \overline{z} + \overline{w}$ for all complex numbers z and w.
      iii. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ for all complex numbers z and w.
      iv. $\operatorname{Re} z = (z + \overline{z})/2$
      v. $\operatorname{Im} z = (z - \overline{z})/2i$

   Remark: From these, we can show that $\overline{z - w} = \overline{z} - \overline{w}$ for all complex numbers z and w, that $\overline{z/w} = \overline{z}/\overline{w}$ for all complex numbers z and $w \neq 0$, and that $\overline{z^n} = \overline{z}^n$ for all complex numbers z and for all integers n (with $z \neq 0$ when $n \leq 0$).
6. *Magnitude.* Given a complex number $z = x + yi$, the **magnitude** or **absolute value** of z is defined to be $|z| = \sqrt{x^2 + y^2}$.

(a) Compute $|21 + 20i|$.

(b) Prove the following properties of the magnitude:

i. $|z|^2 = z \cdot \overline{z}$ for all complex numbers z.

ii. $|zw| = |z| \cdot |w|$ for all complex numbers z and w.

Remark: From these, it follows that $|z/w| = |z|/|w|$ for all complex numbers z and $w \neq 0$, and that $|z^n| = |z|^n$ for all integers n (with $z \neq 0$ when $n \leq 0$).

7. *Square roots of complex numbers.*

(a) Find a complex number w for which $w^2 = -16 + 30i$.

(b) Find the two complex numbers z satisfying $2z^2 - (8 + 4i)z + (14 - 7i) = 0$.

(c) Prove that for every complex number z, there is a complex number w for which $w^2 = z$.
Remark: It follows from this that every quadratic polynomial with complex coefficients has complex roots (with roots given by the familiar quadratic formula).

2.2 Challenge problems

8. (a) Show that $\operatorname{Re} z \leq |z|$ for all complex numbers z. When does equality occur?

(b) *Triangle inequality.* Using part (a), or otherwise, show that

$$|z + w| \leq |z| + |w|$$

for all complex numbers z and w. When does equality occur?

9. In this problem, we work through one formal construction of the complex numbers.

Let $\mathcal{C}$ be the set of all ordered pairs of real numbers, and define operations $\oplus$ and $\otimes$ on $\mathcal{C}$ by

$$\begin{aligned}(a, b) \oplus (c, d) &= (a + c, b + d),\\ (a, b) \otimes (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

We call $\oplus$ and $\otimes$ the addition and multiplication on $\mathcal{C}$, respectively.

(a) Let $u = (-2, -4)$, $v = (-3, 1)$, and $w = (0, 4)$. Verify each of the following:

$$\begin{aligned}u \oplus (v \oplus w) &= (u \oplus v) \oplus w,\\ u \otimes (v \otimes w) &= (u \otimes v) \otimes w,\\ u \oplus v &= v \oplus u,\\ u \otimes v &= v \otimes u,\\ u \otimes (v \oplus w) &= (u \otimes v) \oplus (u \otimes w),\\ u \oplus (0, 0) &= u,\\ v \otimes (1, 0) &= v.\end{aligned}$$

Also find pairs a_v and m_u for which $v \oplus a_v = (0, 0)$ and $m_u \otimes u = (1, 0)$.

One can show that these equalities hold in general and that for any $z \in \mathcal{C}$, we can find a unique $a_z \in \mathcal{C}$ for which $z \oplus a_z = (0,0)$. This a_z is the **additive inverse** of z and is denoted $-z$. Similarly, for any non-zero $z \in \mathcal{C}$, we can find a unique $m_z \in \mathcal{C}$ for which $z \otimes m_z = (1,0)$. This m_z is the **multiplicative inverse** of z and is denoted z^{-1}.

These properties, collectively called the "field axioms," are enough to derive all of the usual algebraic facts that we are familiar with in the context of real number algebra. What remains is to check that $\mathcal{C}$ "does what we expect the complex numbers to do."

(b) Prove that for any two real numbers x and y,

$$(x,0) \oplus (y,0) = (x+y,0) \quad \text{and} \quad (x,0) \otimes (y,0) = (xy,0).$$

This shows that the elements $(r,0)$ for $r \in \mathbb{R}$, with operations $\oplus$ and $\otimes$, "act like" the real numbers with the usual addition and multiplication operations $+$ and $\times$. As such, we can regard $\mathbb{R}$ as being contained within $\mathcal{C}$ by identifying $r \in \mathbb{R}$ with $(r,0) \in \mathcal{C}$, and then $\otimes$ and $\otimes$ extend $+$ and $\times$ from $\mathbb{R}$ to all of $\mathcal{C}$. As such, when r is a real number we simply write r instead of $(r,0)$. Moreover, from now on, we write $+$ and $\times$ (or $\cdot$) instead of $\oplus$ and $\otimes$. We also introduce the subtraction and division operations as $z-w = z+(-w)$ and $z/w = z \cdot w^{-1}$.

(c) Show that $(0,1) \times (0,1) = -1$ and $(0,-1) \times (0,-1) = -1$.

This shows that $\mathcal{C}$ has square roots of -1, as expected. We can now recover the usual notation by defining $i = (0,1)$ and then observing that $(x,y) = x + y \cdot i$. Henceforth, we can forget about the underlying ordered pairs and replace $\mathcal{C}$ with the usual $\mathbb{C}$.

10. A function $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-*automorphism of* $\mathbb{C}$ if

$$f(z+w) = f(z) + f(w) \quad \text{and} \quad f(zw) = f(z) \cdot f(w)$$

for all $z, w \in \mathbb{C}$ and $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$, then $f(i) = i$ or $f(i) = -i$.

(b) Show that the only two $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity function $f(z) = z$ and the conjugation function $f(z) = \overline{z}$.

2.3 Answers

1. (a) -3
   (b) -2
   (c) $-7+i$
   (d) $1+5i$
   (e) $18-6i$
   (f) $\frac{3}{10}-\frac{9}{10}i$

2. $2+i$ and $2-i$

3. Let $z=a+bi$. Then $(-4+2i)(a+bi)=(-4a-2b)+(2a-4b)i$. For this product to be real, we need $2a-4b=0$, so $a=2b$. Hence $z=2b+bi=b(2+i)$, so $\boxed{\text{any real multiple of } 2+i}$ would do the job.

4. (a) We have $i^2=-1$, so then $i^3=-i$, so then $i^4=-i^2=-(-1)=1$.
   (b) $1=i^0$, $i=i^1$, $-1=i^2$, and $-i=i^3$.
   (c) The right hand side is also 4-periodic, so it suffices to show that there exist a,b,c,d with

   $$z_n=a+b\cdot i^n+c\cdot i^{2n}+d\cdot i^{3n}$$

   for $n=1,2,3,4$. This gives us the system of linear equations

   $$\begin{aligned} a+ib-c-id&=z_1,\\ a-b+c-d&=z_2,\\ a-ib-c+id&=z_3,\\ a+b+c+d&=z_4. \end{aligned}$$

   This system does indeed have a solution, namely

   $$\begin{aligned} a&=\frac{z_1+z_2+z_3+z_4}{4},\\ b&=\frac{-iz_1-z_2+iz_3+z_4}{4},\\ c&=\frac{-z_1+z_2-z_3+z_4}{4},\\ d&=\frac{iz_1-z_2-iz_3+z_4}{4}. \end{aligned}$$

5. (a) $943+319i$
   (b) Let $z=x+yi$ and $w=a+bi$ throughout.
      i. $\overline{(\overline{z})}=\overline{x-yi}=x+yi=z$
      ii. $\overline{z+w}=\overline{(x+a)+(y+b)i}=(x+a)-(y+b)i$
      $\overline{z}+\overline{w}=(x-yi)+(a-bi)=(x+a)-(y+b)i$

iii. $\overline{z \cdot w} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$
$\overline{z} \cdot \overline{w} = (x - yi)(a - bi) = (xa - yb) - (xb + ya)i$

iv. $\frac{z+\overline{z}}{2} = \frac{(x+yi)+(x-yi)}{2} = x = \operatorname{Re} z$

v. $\frac{z-\overline{z}}{2i} = \frac{(x+yi)-(x-yi)}{2i} = y = \operatorname{Im} z$

6. (a) 29

(b) Let $z = x + yi$ and $w = a + bi$ throughout.

i. $z \cdot \overline{z} = (x + yi)(x - yi) = x^2 - (yi)^2 = x^2 + y^2 = |z|^2$

ii. $|zw| = \sqrt{|zw|^2} = \sqrt{zw \cdot \overline{zw}} = \sqrt{z\overline{z} \cdot w\overline{w}} = \sqrt{|z|^2 \cdot |w|^2} = |z| \cdot |w|$

7. (a) Let $w = a + bi$, so then $w^2 = (a^2 - b^2) + 2abi$. Therefore, we need $a^2 - b^2 = -16$ and $ab = 15$. Substituting $b = 15/a$,

$$a^2 - \frac{225}{a^2} = -16 \iff a^4 + 16a^2 - 225 = 0.$$

Since a is real, $a^2 > 0$, so the only viable solution to the quadratic in a^2 is

$$a^2 = \frac{-16 + \sqrt{16^2 - 4(1)(-225)}}{2} = \frac{-16 + \sqrt{1156}}{2} = \frac{-16 + 34}{2} = 9.$$

Therefore, $a = 3$, in which case $b = 5$, or $a = -3$, in which case $b = -5$. Hence the two square roots of $-16 + 30i$ are $\pm(3 + 5i)$.

(b) By the quadratic formula,

$$\begin{aligned} z &= \frac{(8+4i) \pm \sqrt{(8+4i)^2 - 4(2)(14-7i)}}{2(2)} \\ &= \frac{(8+4i) \pm \sqrt{(64+64i-16) - (112-56i)}}{4} \\ &= \frac{(8+4i) \pm \sqrt{-64+120i}}{4} \\ &= \frac{(8+4i) \pm 2\sqrt{-16+30i}}{4} \\ &= \frac{(4+2i) \pm (3+5i)}{2} \\ &= \frac{7+7i}{2} \text{ or } \frac{1-3i}{2}. \end{aligned}$$

(c) If $z = x + yi$ and $w = a + bi$ satisfies $w^2 = z$, then following the same procedure as in part (a) yields, when $y \neq 0$,

$$a^4 - xa^2 - \frac{y^2}{4} = 0.$$

The product of the roots of the quadratic $T^2 - xT - y^2/4$ is $-y^2/4 < 0$, so there is a positive root α and a negative root β. Taking $a^2 = \alpha$, so then $a = \sqrt{\alpha}$ and $b = y/2a$, gives us a square root of z.

In the case $y = 0$, either $a = 0$ or $b = 0$. If $x \geq 0$, then take $a = \sqrt{x}$ and $b = 0$, and if $x < 0$, then take $a = 0$ and $b = \sqrt{-x}$.

8. (a) For the inequality,

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geq \sqrt{(\operatorname{Re} z)^2} = |\operatorname{Re} z| \geq \operatorname{Re} z.$$

For the first inequality step, we have equality if and only if $\operatorname{Im} z = 0$. For the second inequality step, we have equality if and only if $\operatorname{Re} z \geq 0$. Putting these together, equality holds in the overall inequality if and only if z is a non-negative real number.

(b) As both sides are non-negative, it suffices to prove the squared inequality. We have

$$\begin{aligned} |z+w|^2 &= (z+w) \cdot \overline{(z+w)} \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z| \cdot |w| + |w|^2 \\ &= (|z| + |w|)^2, \end{aligned}$$

as required. Equality holds when we have equality in $\operatorname{Re}(z\overline{w}) = |z\overline{w}|$, and by part (a), we know that this requires $z\overline{w} \geq 0$. If $w = 0$, this condition holds. Otherwise, $w\overline{w} > 0$ and $z\overline{w} \geq 0$, so $z/w \geq 0$. That is, $z = \lambda w$ for a non-negative real number λ.

9. (a) $u \oplus (v \oplus w) = (u \oplus v) \oplus w = (-5, 1)$
$u \otimes (v \otimes w) = (u \otimes v) \otimes w = (-40, 40)$
$u \oplus v = v \oplus u = (-5, -3)$
$u \otimes v = v \otimes u = (10, 10)$
$u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w) = (26, 2)$
$a_v = (3, -1)$
$m_u = (-1/10, /5)$

(b) $(x, 0) \oplus (y, 0) = (x + y, 0 + 0) = (x + y, 0)$
$(x, 0) \otimes (y, 0) = (x \cdot y - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0)$

(c) $(0, 1) \times (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1$
$(0, -1) \times (0, -1) = (0 \cdot 0 - (-1) \cdot (-1), 0 \cdot (-1) + (-1) \cdot 0) = (-1, 0) = -1$

10. (a) We have

$$f(i)^2 = f(i) \cdot f(i) = f(i \cdot i) = f(-1) = -1,$$

so either $f(i) = i$ or $f(i) = -i$.

(b) Let $z = x + yi$. Then

$$f(z) = f(x + yi) = f(x) + f(yi) = x + f(y) \cdot f(i) = x + y \cdot f(i).$$

If $f(i) = i$, then $f(z) = x + yi = z$. If $f(i) = -i$, then $f(z) = x - yi = \overline{z}$.

3 Complex Numbers II: Exponentials

For real numbers θ, we define $e^{i\theta} = \cos\theta + i\sin\theta$, with this expression also sometimes denoted $\operatorname{cis}\theta$. For complex numbers $z = x + yi$, we define $e^z = e^x \cdot e^{iy}$, where e^x is the real exponential function with base $e \approx 2.718$ evaluated at x.

3.1 Review problems

1. *Magnitude and argument.* For each of the following complex numbers z, find the magnitude $|z|$ and all possible values of the argument $\arg z$ (if defined).

   (a) 1

   (b) $2 + 2i$

   (c) $-3i$

   (d) $-2 + 2\sqrt{3}i$

   (e) 0

2. *More on arguments.* For any non-zero real number z, we define the **principal value** of the argument to be the value of $\arg z$ in the interval[2] $(-\pi, \pi]$ and denote this by $\operatorname{Arg} z$.

   (a) Let z and w be complex numbers with arguments $2\pi/3$ and $3\pi/4$, respectively. What is (a possible value of) the argument of z^2/w?

   (b) In general, $\arg(zw) = \arg z + \arg w$ is true when interpreted to mean that if θ is a possible value for the argument of z and ϕ is a possible value for the argument of w, then $\theta + \phi$ is a possible value for the argument of zw.
   Does $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$ hold?

3. *Exponential form.* Fill in the table below.

Standard form	Exponential form
$7i$	$7e^{i\pi/2}$
	$e^{\pi i}$
$1 + i$	
$3 - 3\sqrt{3}i$	
	$2e^{-2i\pi/3}$
	$4e^{5i\pi/12}$

4. *Exact values of roots of unity.* For any positive integer n, an ***n*-th root of unity** is a solution to the equation $z^n = 1$. Compute the n-th roots of unity for $n = 1, 2, 3, 4, 6, 8$.

[2] Some authors may opt for another interval like $[0, 2\pi)$, but $(-\pi, \pi]$ is the most common.

5. *Roots of other complex numbers.* Let n be a positive integer and let z be any complex number. Writing z in exponential form, let real numbers $r \geq 0$ and θ be such that $z = re^{i\theta}$.

   (a) In terms of r, θ, and n, write down an n-th root of z.

   (b) Show that if w is an n-th root of z and ζ is an n-th root of unity, then ζw is also an n-th root of z.

   (c) Conversely, show that if w_1 and w_2 are two n-th roots of z, then there is an n-th root of unity ζ for which $w_2 = \zeta w_1$.

   (d) Let $\zeta = e^{2\pi i/7}$. Express all solutions to $z^7 = 128$ in terms of ζ.

6. *Primitive n-th roots of unity.* Let n be a positive integer. A **primitive n-th root of unity** is an n-th root of unity which is not a k-th root of unity for any positive integer $k < n$.

   (a) For $n = 1, 2, 3, 4, 6, 8$, what are the primitive n-th roots of unity?

   (b) Any n-th root of unity can be written in the form $e^{2\pi ik/n}$ for an integer k. Show that this is a primitive n-th root of unity if and only if $\gcd(k, n) = 1$.

   (c) Let ζ be a primitive n-th root of unity. Show that for any integer k,

$$1 + \zeta^k + \zeta^{2k} + \cdots + \zeta^{(n-1)k} = \begin{cases} n & n \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

   (d) *Roots of unity filter.* Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{mn} z^{mn}$$

   be a polynomial and let $\zeta = e^{2\pi i/n}$. Show that

$$a_0 + a_n + a_{2n} + \cdots + a_{mn} = \frac{f(1) + f(\zeta) + f(\zeta^2) + \cdots + f(\zeta^{n-1})}{n}.$$

   (e) Compute $\binom{2025}{0} + \binom{2025}{3} + \binom{2025}{6} + \cdots + \binom{2025}{2025}$.

7. *Sine and cosine as exponentials.*

   (a) Let θ be a real number. Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

   (b) For any complex number z, we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

   Other trig functions of z are defined in terms of sine and cosine as usual, e.g. $\tan z = \frac{\sin z}{\cos z}$. Calculate $\cos(i)$ and $\sin(i)$.

   (c) Prove that $\cos^2 z + \sin^2 z = 1$ for all complex numbers z.

   (d) Prove that $2 \sin z \cos z = \sin(2z)$ for all complex numbers z.

3.2 Challenge problems

8. Let $\ln : (0, \infty) \to \mathbb{R}$ (temporarily) denote the natural logarithm function, i.e. the inverse of the real exponential function $x \mapsto e^x$. Define $L : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by

$$L(z) = \ln|z| + i \arg z.$$

Since arg is multi-valued, L is also multi-valued, with its possible output values differing by integer multiples of $2\pi i$.

(a) Show that $e^{L(z)} = z$ (no matter what value of $\arg z$ we take). As such, L acts as an "inverse" to the complex exponential, so we call it the **complex logarithm** and denote it by $\log z$ instead of $L(z)$. We define the **principal value of the logarithm** to be

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

(b) Compute the following:

i. $\operatorname{Log} 1$

ii. $\operatorname{Log} i$

iii. $\operatorname{Log}(-3 - 3i)$

(c) Prove that $\log(zw) = \log z + \log w$ for all $z, w \in \mathbb{C}\backslash\{0\}$.
Is it true that $\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w$ for all $z, w \in \mathbb{C}\backslash\{0\}$?

(d) Show that the solutions to $\tan w = z$ are

$$w = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right).$$

This makes the expression on the right hand side the (multi-valued) **inverse tangent function** for complex numbers.

9. The **fundamental theorem of algebra** states that for every non-constant one-variable polynomial function $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and the a_i are complex numbers with $a_n \neq 0$, there exists $r \in \mathbb{C}$ with $p(r) = 0$. In this problem, we provide one proof of this theorem. Note that without loss of generality, we can suppose $a_n = 1$, and since $p(0) = 0$ when $a_0 = 0$, we only need to address the case that $a_0 \neq 0$.

(a) Let $R = 2(|a_0| + |a_1| + \cdots + |a_{n-1}| + 1)$. Show that if $|z| \geq R$, then

$$\left|\frac{p(z)}{z^n}\right| \geq \frac{1}{2}.$$

(b) A theorem of mathematical analysis tells us that since $|p(z)|$ is a continuous function of z, there is a point z_0 with $|z_0| \leq R$ for which $m = |p(z_0)|$ is the minimum value of $|p(z)|$ as z ranges over all complex numbers with magnitude at most R. Show that in fact, m is the minimum value of $|p(z)|$ as z ranges over all complex numbers.

(c) Suppose for the sake of contradiction that $m > 0$. By translating, we can suppose that $z_0 = 0$, so that $m = |a_0|$. Let $0 < k \leq n$ be the smallest positive integer with $a_k \neq 0$, let ω be any complex number satisfying $\omega^k = -a_0/a_k$, and let

$$\epsilon = \frac{1}{2} \min \left(1, |a_0| \cdot \left[|a_{k+1}\omega^{k+1}| + |a_{k+2}\omega^{k+2}| + \cdots + |a_n\omega^n|\right]^{-1}\right).$$

Show that

$$|p(\epsilon \cdot \omega) - (a_0 - a_0\epsilon^k)| \leq \frac{|a_0|}{2}\epsilon^k,$$

and hence deduce that $|p(\epsilon \cdot \omega)| < m$, contradicting minimality of m.

10. Let $\zeta_1, \ldots, \zeta_{\varphi(n)}$ be all the primitive n-th roots of unity (where $\varphi(n)$ is the number of positive integers $k \leq n$ satisfying $\gcd(k, n) = 1$). We define the ***n*-th cyclotomic polynomial** to be

$$\Phi_n(X) = (X - \zeta_1)(X - \zeta_2) \cdots (X - \zeta_{\varphi(n)}).$$

(a) Compute $\Phi_n(X)$ for $n = 1, 2, 3, 4, 6, 8$.

(b) Show that $X^n - 1 = \prod_{d|n} \Phi_d(X)$.

(c) Show that $\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$ whenever p is prime.

(d) An important fact for number theory is that the cyclotomic polynomials are **irreducible** (over the integers), meaning that they cannot be written as products of polynomials of lower degree with integer coefficients. We will not go through a general proof here, but some special cases are easier to tackle.

Let p be a prime. By considering a shifted cyclotomic polynomial $f(X) = \Phi_p(X + 1)$, or otherwise, show that $\Phi_p(X)$ is irreducible.

3.3 Answers

1. In all of the below, k can be any integer.
   (a) $|1| = 1$ and $\arg 1 = 2\pi ik$
   (b) $|2+2i| = 2\sqrt{2}$ and $\arg(2+2i) = \pi/4 + 2\pi ik$
   (c) $|-3i| = 3$ and $\arg(-3i) = -\pi/2 + 2\pi ik$
   (d) $|-2+2\sqrt{3}i| = 4$ and $\arg(-2+2\sqrt{3}i) = 2\pi/3 + 2\pi ik$
   (e) $|0| = 0$ and $\arg 0$ is undefined

2. (a) $2 \cdot \frac{2\pi}{3} - \frac{3\pi}{4} = \frac{7\pi}{12}$ (any integer multiple of 2π can be added to this)
   (b) Let $z = w = -i$. Then $\operatorname{Arg} z = \operatorname{Arg} w = -\pi/2$, but

$$\operatorname{Arg}(zw) = \operatorname{Arg}(-1) = \pi \neq \operatorname{Arg} z + \operatorname{Arg} w.$$

3. When converting from standard form to exponential form, we can add any integer multiple of $2\pi i$ to the exponent to get another valid exponential form expression.

Standard form	Exponential form
$7i$	$7e^{i\pi/2}$
-1	$e^{\pi i}$
$1+i$	$\sqrt{2}e^{i\pi/4}$
$3-3\sqrt{3}i$	$6e^{-i\pi/3}$
$-1-\sqrt{3}i$	$2e^{-2i\pi/3}$
$(\sqrt{6}-\sqrt{2})+(\sqrt{6}+\sqrt{2})i$	$4e^{5i\pi/12}$

4. See the table below.

n	n-th roots of unity
1	1
2	$1,\ -1$
3	$1,\ \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}$
4	$1,\ i,\ -1,\ -i$
6	$1,\ \frac{1+\sqrt{3}i}{2},\ \frac{-1+\sqrt{3}i}{2},\ -1,\ \frac{-1-\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2}$
8	$1,\ \frac{1+i}{\sqrt{2}},\ i,\ \frac{1-i}{\sqrt{2}},\ -1,\ \frac{-1-i}{\sqrt{2}},\ -i,\ \frac{1-i}{\sqrt{2}}$

5. (a) $r^{1/n}e^{i(\theta/n)}$
   (b) If $w^n = z$ and $\zeta^n = 1$, then $(\zeta w)^n = \zeta^n w^n = 1 \cdot z = z$.

(c) If $w_1^n = w_2^n = z$, then $(w_1/w_2)^n = 1$, so $w_1/w_2 = \zeta$ for some n-th root of unity ζ.

(d) $z = 2\zeta^k$ for $k = 0, 1, \ldots, 6$.

6. (a) See the table below.

n	primitive n-th roots of unity
1	1
2	-1
3	$\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$
4	i, $-i$
6	$\frac{1+\sqrt{3}i}{2}$, $\frac{1-\sqrt{3}i}{2}$
8	$\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, $\frac{-1-i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$

(b) Suppose that $\gcd(k,n) = 1$ and $(e^{2\pi ik/n})^m = e^{2\pi ikm/n} = 1$ for some positive integer m. Then km/n must be an integer, and with $\gcd(k,n) = 1$, this forces m to be divisible by n. Hence $e^{2\pi ik/n}$ is not an m-th root of unity for any $m < n$.

Inversely, suppose $\gcd(k,n) = d > 1$. Then $(e^{2\pi ik/n})^{n/d} = e^{2\pi i(k/d)} = 1$, so $e^{2\pi ik/n}$ is an (n/d)-th root of unity and hence not a primitive n-th root of unity.

(c) If n divides k, then $\zeta^k = 1$ so the sum evaluates to n. If n does not divide k, then since ζ is a primitive n-th root of unity, $\zeta^k \neq 1$. Therefore,

$$1 + \zeta^k + \cdots + \zeta^{(n-1)k} = \frac{1 - \zeta^{nk}}{1 - \zeta^k} = \frac{1 - 1}{1 - \zeta^k} = 0.$$

(d) We start by writing the right hand side as

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\zeta^k) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{mn} a_j(\zeta^k)^j = \frac{1}{n}\sum_{j=0}^{mn}\left(a_j \sum_{k=0}^{n-1}(\zeta^k)^j\right).$$

The inner sum, by part (c), is n whenever $n \mid j$ and 0 otherwise, so the sum becomes

$$\frac{1}{n}(na_0 + na_n + na_{2n} + \cdots + na_{mn}) = a_0 + a_n + \cdots + a_{mn}.$$

(e) Let $\zeta = e^{2\pi i/3}$ and let $f(z) = (1+z)^{2025}$, so that $a_j = \binom{2025}{j}$. Then

$$\binom{2025}{0} + \binom{2025}{3} + \cdots + \binom{2025}{2025} = \frac{f(1) + f(\zeta) + f(\zeta^2)}{3} = \frac{2^{2025} - 2}{3}.$$

7. (a) We have $e^{i\theta} = \cos\theta + i\sin\theta$ and

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

Adding the equations and dividing by 2 gives us the cosine formula, while subtracting the equations and dividing by $2i$ gives us the sine formula.

(b) $\cos i = \frac{e+e^{-1}}{2}$ and $\sin i = \frac{e^{-1}-e}{2i} = \frac{(e-e^{-1})i}{2}$

(c) $\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \frac{e^{2iz}+2+e^{-2iz}}{4} - \frac{e^{2iz}-2+e^{-2iz}}{4} = 1$

(d) $2\sin z \cos z = 2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{e^{2iz}-e^{-2iz}}{2i} = \sin(2z)$

8. (a) $e^{L(z)} = e^{\ln|z|}e^{i\arg z} = |z|e^{i\arg z} = z$

(b) i. 0

ii. $\pi i/2$

iii. $\ln(3\sqrt{2}) - 3\pi i/4$

(c) $\log(zw) = \ln|zw| + i\arg(zw) = (\ln|z| + \ln|w|) + i(\arg z + \arg w) = \log z + \log w$

The statement $\operatorname{Log}(zw) \overset{?!}{=} \operatorname{Log} z + \operatorname{Log} w$ fails for the same reason that $\operatorname{Arg}(zw) \overset{?!}{=} \operatorname{Arg} z + \operatorname{Arg} w$ fails. In particular, when $|z| = 1$, we have $\operatorname{Log} z = i\operatorname{Arg} z$, so we can just take $z = w = -i$ as in our counterexample for problem 2(b).

(d) We have

$$\tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1}.$$

If $\tan w = z$, then

$$\begin{aligned}
\frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1} &= z, \\
e^{2iw} - 1 &= iz(e^{2iw} + 1), \\
(1 - iz)e^{2iw} &= 1 + iz, \\
e^{2iw} &= \frac{1 + iz}{1 - iz}, \\
w &= \frac{1}{2i}\log\left(\frac{1 + iz}{1 - iz}\right).
\end{aligned}$$

9. (a) By the triangle inequality,

$$\left|\frac{p(z)}{z^n}\right| = \left|1 + \sum_{j=0}^{n-1}\frac{a_j}{z^{n-j}}\right| \geq 1 - \left|\sum_{j=0}^{n-1}\frac{a_j}{z^{n-j}}\right|,$$

so it suffices to show the last term is at most $1/2$. Since $|z| \geq 1$,

$$\left|\sum_{j=0}^{n-1}\frac{a_j}{z^{n-j}}\right| \leq \sum_{j=0}^{n-1}\frac{|a_j|}{|z|^{n-j}} \leq \frac{1}{|z|}\sum_{j=0}^{n-1}|a_j|.$$

That this is at most $1/2$ then follows from $|z| \geq R > 2\sum_{j=0}^{n-1}|a_j|$.

(b) We have $m \leq |p(z)|$ whenever $|z| \leq R$, so in particular $m \leq |p(0)| = |a_0|$. When $|z| \geq R$, part (a) gives us $|p(z)| \geq R^n/2$. If $m \leq 1$, then $R^n/2 \geq 2^{n-1} \geq m$, and if $m > 1$, then $R^n/2 \geq |a_0|^n > |a_0| \geq m$ as well. Hence m is indeed the minimum of $|p(z)|$ over all z.

(c) We start by writing

$$p(\epsilon \cdot \omega) = a_0 + a_k \epsilon^k \omega^k + \sum_{j=k+1}^{n} a_j \epsilon^j \omega^j = a_0 - a_0 \epsilon^k + \epsilon^k \sum_{j=k+1}^{n} a_j \epsilon^{j-k} \omega^j.$$

Since $\epsilon \leq 1/2 < 1$, the sum can be bounded by

$$\left| \sum_{j=k+1}^{n} a_j \epsilon^{j-k} \omega^j \right| \leq \sum_{j=k+1}^{n} |a_j \omega^j| \cdot \epsilon^{j-k} \leq \epsilon \sum_{j=k+1}^{n} |a_j \omega^j|.$$

This last expression is bounded above by $|a_0|/2$, as required. To finish,

$$|p(\epsilon \cdot \omega)| \leq |a_0| - |a_0|\epsilon^k + \frac{|a_0|}{2}\epsilon^k = m\left(1 - \frac{\epsilon^k}{2}\right) < m.$$

10. (a) See the table below.

n	$\Phi_n(X)$
1	$X - 1$
2	$X + 1$
3	$X^2 + X + 1$
4	$X^2 + 1$
6	$X^2 - X + 1$
8	$X^4 + 1$

(b) First, note that when $d \neq d'$, the polynomials $\Phi_d(X)$ and $\Phi_{d'}(X)$ share no roots because a primitive d-th root of unity cannot also be a (d')-th root of unity or vice versa. Therefore, all roots of the right hand side occur with multiplicity 1, as is the case on the left hand side. As such, it suffices to show that the left and right hand sides have the same roots.

If ζ is a root of the right hand side, then ζ is a root of $\Phi_d(X)$ for some $d \mid n$. Since $\zeta^d = 1$ and $d \mid n$, we also have $\zeta^n = 1$, so ζ is a root of the left hand side.

Conversely, let ζ be a root of $X^n - 1$ and suppose ζ is a primitive d-th root of unity. Let $n = qd + r$, where $0 \leq r < d$ is the remainder when n is divided by d. Then $\zeta^(n - qd) = \zeta^r = 1$. Since ζ is a primitive d-th root of unity, we must have $r = 0$, so $d \mid n$. This means ζ is a root of $\Phi_d(X)$ and hence a root of the right hand side.

(c) When p is prime, $X^p - 1 = \Phi_1(X)\Phi_p(X)$, so

$$\Phi_p(X) = \frac{X^p - 1}{X - 1} = X^{p-1} + \cdots + X + 1.$$

(d) For $p = 2$, we saw that $\Phi_2(X) = X + 1$, which cannot be factored, so for the rest of the argument, we can suppose $p > 2$. Any factorisation of $f(X) = \Phi_p(X + 1)$ yields a factorisation of $\Phi_p(X)$, so it suffices to show that $f(X)$ is irreducible. We calculate

$$f(X) = \frac{(X+1)^p - 1}{(X+1) - 1} = X^{p-1} + pX^{p-2} + \binom{p}{2}X^{p-3} + \cdots + \binom{p}{p-2}X + p.$$

Suppose for the sake of contradiction that we have a (non-trivial) factorisation

$$f(X) = (X^k + a_{k-1}X^{k-1} + \cdots + a_0)(X^{p-1-k} + b_{p-2-k}X^{p-2-k} + \cdots + b_0),$$

where $0 < k < p-1$. Expanding, $a_0b_0 = p$, so we can suppose without loss of generality that $a_0 = \pm p$ and $b_0 = \pm 1$; whatever the signs are, $p \mid a_0$ but $p \nmid b_0$. Looking at the linear term, $a_0b_1 + a_1b_0 = \binom{p}{p-2}$, which is divisible by p. Since $p \mid a_0$ but $p \nmid b_0$, we must have $p \mid a_1$. Looking at the quadratic term, $a_0b_2 + a_1b_1 + a_2b_0 = \binom{p}{p-3}$, which is divisible by p. Since $p \mid a_0, a_1$ but $p \nmid b_0$, we must have $p \mid a_2$. Continuing in this way, $p \mid a_j$ for all $0 \le j \le k-1$. (If $k \ge p-1-k$, then we take $b_{p-1-k} = 1$ and $b_j = 0$ for all $j \ge p-k$.) The coefficient of X^k is then $a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + b_0$, which is not divisible by p. However, the coefficient of X^k should also be $\binom{p}{p-1-k}$, which is divisible by p since $0 < k < p-1$. This is a contradiction, so $f(X)$ cannot factor non-trivially over the integers.

4 Complex Numbers III: Geometry

4.1 Review Problems

1. *Geometric interpretations.* Let points Z and W in the plane be represented by complex numbers z and w, respectively. Let O be the origin and let A be the point represented by 1. For each of the following points or quantities described, write down an expression producing that point or quantity.

   (a) The distance from Z to O

   (b) The reflection of Z across the real axis

   (c) The distance between Z and W

   (d) The angle $\angle AOW$, measured counterclockwise from ray $\overrightarrow{OA}$

   (e) The point P which makes $OZPW$ a parallelogram

2. *Distances.*

   (a) Compute the distance between $2+4i$ and $-5-i$.

   (b) Show that if z lies on the circle centered at $3+i$ with radius 2, then

   $$z\overline{z} - (3-i)z - (3+i)\overline{z} = -6.$$

   (c) Find all complex numbers z for which $|z| = 10$ and $|z-21| = 17$.

3. *Angles.* Let A, B, C be distinct points in the plane represented by a, b, c, respectively.

   (a) Explain why (up to sign) $\angle BAC = \arg(\frac{c-a}{b-a})$.

   (b) Calculate $\angle BAC$ when $a = 3-2i$, $b = -1+i$, and $c = 2+5i$.

4. *Perpendicular bisectors.* Let A and B be represented by $a = 2i$ and $b = 6-2i$, respectively.

   (a) Let M be the midpoint of $\overline{AB}$, represented by m. Compute m.

   (b) Find the point Z on the imaginary axis for which $ZA = ZB$.

5. *Altitudes.* Let A, B, C be distinct points on the unit circle represented by a, b, c, respectively.

   (a) Show that for any point z on the unit circle, $\overline{z} = 1/z$.

   (b) Show that the altitudes of triangle ABC all pass through the point H represented by $a+b+c$. This point H is the **orthocenter** of ABC.

6. *Transformation calculations.* For the next few problems,

   - $\tau_a : \mathbb{C} \to \mathbb{C}$ denotes translation by a;
   - $\delta_{c,r} : \mathbb{C} \to \mathbb{C}$ denotes dilation by factor r centered at c;
   - $\rho_{c,\theta} : \mathbb{C} \to \mathbb{C}$ denotes (CCW) rotation by angle θ centered at c;
   - $\sigma_\ell : \mathbb{C} \to \mathbb{C}$ denotes reflection across line ℓ.

Compute each of the following.

(a) $\tau_{7+9i}(12-24i)$
(b) $\delta_{1+i,3}(-4+3i)$
(c) $\rho_{0,\pi/7}(e^{i\pi/11})$
(d) $\rho_{2-i,\pi/4}(6+i)$
(e) $\sigma_\ell(2+5i)$ when ℓ passes through 1 and has slope $1/\sqrt{3}$

7. *Isometries.* An **isometry** of $\mathbb{C}$ is a function $f : \mathbb{C} \to \mathbb{C}$ that preserves distances, i.e.

$$|f(z) - f(w)| = |z - w|$$

for all $z, w \in \mathbb{C}$.

(a) Verify that translations, rotations centered at 0, and reflection across the real axis are all isometries of $\mathbb{C}$.
(b) Show that the composition of two isometries is an isometry.
(c) By SSS congruence, isometries must also preserve (unsigned) angles. If f is an isometry and $\arg(\frac{f(c)-f(a)}{f(b)-f(a)}) = \arg(\frac{c-a}{b-a})$, then we say that f is **orientation-preserving**, but if we have $\arg(\frac{f(c)-f(a)}{f(b)-f(a)}) = -\arg(\frac{c-a}{b-a})$ instead, then we say that f is **orientation-reversing**. Of the isometries mentioned in part (a), which are orientation-preserving and which are orientation-reversing?

4.2 Challenge Problems

These problems feature the **extended complex plane** $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Definitions involving ∞ will be provided as needed; they can be justified using limits for those who are familiar.

8. A **Möbius transformation** is a function of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. If $c = 0$, then f is initially defined as a function $\mathbb{C} \to \mathbb{C}$, and we extend it to $\mathbb{C}_\infty$ by setting $f(\infty) = \infty$. If $c \neq 0$, then f is initially defined as a function $\mathbb{C}\backslash\{-d/c\} \to \mathbb{C}$, and we extend it to $\mathbb{C}_\infty$ by setting $f(-d/c) = \infty$ and $f(\infty) = a/c$.

(a) Show that every Möbius transformation can be written as a composition of translations $z \mapsto z + \alpha$, spiral similarities $z \mapsto \beta z$, and (conformal) inversions $z \mapsto 1/z$.
(b) Show that every Möbius transformation is a bijection $\mathbb{C}_\infty \to \mathbb{C}_\infty$ and the inverse of a Möbius transformation is also a Möbius transformation.
(c) Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Show that either $f(z) = z$ for all $z \in \mathbb{C}_\infty$ or $f(z) = z$ for at most two $z \in \mathbb{C}_\infty$.
(d) A **circline** is any curve which is either a circle or a line, and the lines are precisely the circlines that contain the point ∞.

Show that Möbius transformations map circlines to circlines.

9. For distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the **cross ratio** is defined as

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

(The variables may be permuted in other sources, but the theory is the same as long as one is consistent.) When one of the variables is ∞, we "cancel out" the two factors in which the variable appears and evaluate with the rest. For example, $[z_1, z_2; z_3, \infty] = \frac{z_1 - z_3}{z_2 - z_3}$.

(a) Show that if f is a Möbius transformation, then f preserves cross ratios.

(b) Show that for any distinct $z_1, z_2, z_3 \in \mathbb{C}_\infty$, there is a Möbius transformation f for which $f(z_1) = 1$, $f(z_2) = 0$, and $f(z_3) = \infty$. (In fact, it is unique.)

(c) Show that if $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct, then they lie on a single circline if and only if their cross ratio $[z_1, z_2; z_3, z_4]$ is real.

(d) *Ptolemy's theorem.* Using the computation

$$[z_1, z_2; z_3, z_4] + [z_1, z_3; z_2, z_4] = 1,$$

or otherwise, prove that if A, B, C, D are distinct points in the plane, then

$$(AB)(CD) + (AD)(BC) \geq (AC)(BD),$$

with equality if and only if the points lie on a circline in (cyclic) order.

10. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc, i.e. the points within the unit circle, and let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the open upper half plane, i.e. the points above the real axis.

(a) Show that if $|a| < 1$, the **Blaschke factor**

$$f(z) = \frac{z - a}{1 - \overline{a}z}$$

is a bijection from the unit circle to the unit circle and from D to D.

(b) Show that if $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$,

$$f(z) = \frac{az + b}{cz + d}$$

is a bijection from $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R} \cup \{\infty\}$. When is f a bijection from $\mathcal{H}$ to $\mathcal{H}$?

(c) Write down a Möbius transformation which is a bijection $\mathcal{H} \to D$.

4.3 Answers

1. (a) $|z|$
   (b) $\overline{z}$
   (c) $|z-w|$
   (d) $\arg w$
   (e) $z+w$
2. (a) $\sqrt{74}$
   (b) From $|z-c|=2$, where $c=3+i$, we square both sides to get
   $$\begin{aligned} |z-c|^2 &= 4, \\ (z-c)(\overline{z}-\overline{c}) &= 4, \\ z\overline{z}-\overline{c}z-c\overline{z}+c\overline{c} &= 4. \end{aligned}$$
   The result follows by substituting $c=3+i$ back in and computing $c\overline{c}=10$.
   (c) Squaring the given equations and using $|a|^2=a\overline{a}$ gives us
   $$z\overline{z}=100 \quad \text{and} \quad z\overline{z}-21z-21\overline{z}+441=289.$$
   Substituting the first equation into the second and simplifying,
   $$z+\overline{z}=12.$$
   This means $\operatorname{Re} z=6$, so from $|z|=10$, we must have $\operatorname{Im} z=\pm 8$. Both solutions $z=6\pm 8i$ work.
3. (a) Let $\arg(c-a)=\theta$ and $\arg(b-a)=\phi$. Then $\angle BAC$ is $\theta-\phi=\arg(\frac{c-a}{b-a})$.
   (b) $-\pi/4$ (As phrased, $\pi/4$ would also be acceptable.)
4. (a) $m=3$
   (b) If $ZA=ZB$, then $\overline{ZM}\perp\overline{AB}$, so $\frac{z-3}{6-4i}$ should be purely imaginary, i.e.
   $$\frac{z-3}{6-4i}=-\frac{\overline{z}-3}{6+4i}.$$
   If Z is on the imaginary axis, then $\overline{z}=-z$, so
   $$\frac{z-3}{6-4i}=\frac{z+3}{6+4i}.$$
   The solution is $z=-\frac{9}{2}i$.
5. (a) If z is on the unit circle, $|z|=1$, so $z\overline{z}=|z|^2=1$.
   (b) We show that $\overline{AH}\perp\overline{BC}$; the other altitudes proceed similarly. For this, we must show that $\frac{(a+b+c)-a}{b-c}=\frac{b+c}{b-c}$ is purely imaginary. Indeed,
   $$\overline{\left(\frac{b+c}{b-c}\right)}=\frac{\overline{b}+\overline{c}}{\overline{b}-\overline{c}}=\frac{1/b+1/c}{1/b-1/c}=\frac{c+b}{c-b}=-\frac{b+c}{b-c}.$$

6. (a) $19 - 15i$
   (b) $-14 + 7i$
   (c) $e^{8i\pi/77}$
   (d) $(2 + \sqrt{2}) + (3\sqrt{2} - 1)i$
   (e) $\frac{3+5\sqrt{3}}{2} + \frac{-5+\sqrt{3}}{2}i$

7. (a) $|\tau_a(z) - \tau_a(w)| = |(z + a) - (w + a)| = |z - w|$
   $|\rho_{0,\theta}(z) - \rho_{0,\theta}(w)| = |e^{i\theta}z - e^{i\theta}w| = |e^{i\theta}||z - w| = |z - w|$
   $|\overline{z} - \overline{w}| = |\overline{z - w}| = |z - w|$
   (b) If $f, g : \mathbb{C} \to \mathbb{C}$ are isometries, then
   $$|(f \circ g)(z) - (f \circ g)(w)| = |f(g(z)) - f(g(w))| = |g(z) - g(w)| = |z - w|.$$
   (c) Translations and rotations are orientation-preserving
   Reflections are orientation-reversing

8. (a) If $c = 0$, then we take $z \mapsto \frac{a}{d}z \mapsto \frac{a}{d}z + \frac{b}{d} = \frac{az+b}{d}$.
   If $c \neq 0$, then we take $z \mapsto cz \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b-ad/c}{cz+d} \mapsto \frac{b-ad/c}{cz+d} + \frac{a}{c} = \frac{az+b}{cz+d}$.
   (b) The inverse of $z \mapsto \frac{az+b}{cz+d}$ is $w \mapsto \frac{dw-b}{-cw+a}$.
   (c) If $c = 0$, then $(a/d)z + (b/d) = z$ has exactly one solution in $\mathbb{C}$ unless $a/d = 1$, in which case either there is no solution ($b \neq 0$) or every z is a solution. For the $c = 0$ case, $z = \infty$ is also a solution, giving us at most two solutions in $\mathbb{C}_\infty$.
   If $c \neq 0$, then $\frac{az+b}{cz+d} = z$ rearranges to a quadratic in z, so there are at most two solutions in $\mathbb{C}\backslash\{-d/c\}$. As $-d/c$ is mapped to ∞ and ∞ is mapped to a/c, there are at most two solutions in $\mathbb{C}_\infty$.
   (d) It suffices to show that translations $z \mapsto z + \alpha$, spiral similarities $z \mapsto \beta z$, and inversion $z \mapsto 1/z$ send circlines to circlines. The first two are straightforward, so it remains to check inversion.
   Circlines are precisely the curves specified by equations of the form $az\overline{z} + \overline{b}z + b\overline{z} + c = 0$ with $a, c \in \mathbb{R}$. When we substitute $1/z$ for z and clear denominators, we get another equation of this form, so circlines go to circlines.

9. (a) It suffices to show that translations $z \mapsto z + \alpha$, spiral similarities $z \mapsto \beta z$, and inversion $z \mapsto 1/z$ preserve cross ratios. These are all straightforward calculation.
   (b) The required Möbius transformation is $f(z) = \frac{z_1-z_3}{z_1-z_2}\frac{z-z_2}{z-z_3} = [z, z_1; z_2, z_3]$.
   (c) By applying a Möbius transformation, which preserves cross ratios and circlines, we can suppose without loss of generality that $z_2 = 1$, $z_3 = 0$, and $z_4 = \infty$. The cross ratio is then just z_1, which is real if and only if z_1 also lies on the real axis.
   (d) Let A, B, C, D be represented by a, b, c, d. By the triangle inequality,
   $$1 = |[a, b; d, c] + [a, d; b, c]| \leq |[a, b; d, c]| + |[a, d; b, c]|,$$
   and clearing denominators from the cross ratios gives us the desired inequality. Equality holds if and only if $[a, b; d, c]$ and $[a, d; b, c]$ are positive real numbers, so the four points

must lie on a circline. Mapping them to the real axis with a Möbius transformation, which preserves (cyclic) order, we can suppose without loss of generality that b, c, d are $1, 0, \infty$, respectively. Then $[a, b; d, c] = 1/a$ and $[a, d; b, c] = (a-1)/a$, so for these to be positive, we need $a > 1$. This means that A, B, C, D are in the correct order along the real axis, hence on the original circline as well.

10. (a) To see it is a bijection from the unit circle to the unit circle,

$$\begin{aligned} |f(z)| = 1 &\iff \frac{z-a}{1-\overline{a}z}\frac{\overline{z}-\overline{a}}{1-a\overline{z}} = 1 \\ &\iff z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} = 1 - \overline{a}z - a\overline{z} + a\overline{a}z\overline{z} \\ &\iff |z|^2(1-|a|^2) = 1 - |a|^2. \end{aligned}$$

Since $|a| < 1$, this occurs if and only if $|z| = 1$.

Since f is a Möbius transformation, it is a bijection $\mathbb{C}_\infty \to \mathbb{C}_\infty$, so we only need to show that $f(z) \in D$ if and only if $z \in D$. If $z \in D$, then let ℓ be the line segment connecting z with a. Since ℓ does not intersect the unit circle, neither does $f(\ell)$. Since $f(a) = 0 \in D$, this means $f(z) \in D$ as well, as otherwise $f(\ell)$ would have to cross the unit circle to go from $f(a)$ to $f(z)$. Conversely, if $f(z) \in D$, then let ℓ be the line segment connecting $f(z)$ with 0. Then $f^{-1}(\ell)$ does not intersect the unit circle, and since $f^{-1}(0) = a \in D$, this means $z = f^{-1}(f(z)) \in D$ as well.

(b) As in part (a), we just need to check that $f(z) \in \mathbb{R} \cup \{\infty\}$ if and only if $z \in \mathbb{R} \cup \{\infty\}$. Since $a, b, c, d \in \mathbb{R}$, we have $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}_\infty$, where $\overline{\infty} = \infty$. Then $f(z)$ is real or ∞ if and only if $f(z) = \overline{f(z)} = f(\overline{z})$. Since f is bijective, this happens if and only if $z = \overline{z}$, i.e. z is real or ∞.

By a similar "path crossing" argument to part (a), f bijectively maps $\mathcal{H}$ to either $\mathcal{H}$ or $-\mathcal{H}$, the open lower half plane. The former occurs when $\operatorname{Im} f(i) > 0$, so we compute

$$f(i) = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2}.$$

The imaginary part is $\frac{ad-bc}{c^2+d^2}$, so f is a bijection $\mathcal{H} \to \mathcal{H}$ precisely when $ad - bc > 0$.

(c) We can take $z \mapsto \frac{z-i}{z+i}$, as $|\frac{z-i}{z+i}| < 1$ precisely when z is closer to i than $-i$, i.e. $z \in \mathcal{H}$.

5 Vectors in 2D

Throughout, $\mathbb{R}^2$ denotes the set of (column) vectors with two components. The standard basis vectors in $\mathbb{R}^2$ are $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

5.1 Review Problems

1. *Vector operations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.

   (a) $\mathbf{u} + \mathbf{v}$

   (b) $2\mathbf{v}$

2. *Parametric equations.*

   (a) Find a vector parametric equation for the line through $(2, 1)$ and $(4, -8)$.

   (b) Lines ℓ_1 and ℓ_2 are described by the parametric equations

   $$\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

   Find the point of intersection of ℓ_1 and ℓ_2.

3. *Lengths and angles.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

   (a) Find $\|\mathbf{u}\|$ and $\mathbf{u} \cdot \mathbf{v}$.

   (b) Find $\hat{\mathbf{v}}$, the unit vector in the direction of $\mathbf{v}$.

   (c) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$.

   (d) Find the two unit vectors orthogonal to $\mathbf{u}$.

4. *Linear dependence.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

   (a) Show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

   (b) Find the (unique) real numbers a, b, c, d for which $a\mathbf{u} + b\mathbf{v} = \hat{\mathbf{i}}$ and $c\mathbf{u} + d\mathbf{v} = \hat{\mathbf{j}}$.

   (c) Show that a list of two vectors in $\mathbb{R}^2$ is linearly independent if and only if it spans $\mathbb{R}^2$.

   (d) Show that any list of three vectors in $\mathbb{R}^2$ is linearly dependent.

5. *Projections.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

   (a) Compute $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ and $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$.

   (b) Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are orthogonal unit vectors. Show that $\mathbf{x}$ and $\mathbf{y}$ span $\mathbb{R}^2$.

   (c) Find the distance between the point $(4, 5)$ and the line $y = 2x - 1$.

6. *Lines via dot products.*

   (a) Find the slope-intercept form of the line $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \mathbf{x} = 4$.

   (b) Find a vector $\mathbf{n}$ and constant d for which the cartesian equation $y = -3x + 5$ is equivalent to the vector equation $\mathbf{n} \cdot \mathbf{x} = d$.

   (c) Suppose $\hat{\mathbf{n}}$ is a unit vector and $d \geq 0$. Show that $\hat{\mathbf{n}}$ is perpendicular to the line $\hat{\mathbf{n}} \cdot \mathbf{x} = d$ and d is the distance from the origin to the line.

7. *Angle bisectors.* Let ABC be a triangle with $AB = 3$, $BC = 5$, and $CA = 6$, and let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be the position vectors of the respective vertices.

   (a) Let D be the point on $\overline{BC}$ for which $\overline{AD}$ bisects $\angle BAC$, and let $\mathbf{D}$ be its position vector. Using the angle bisector theorem, or otherwise, express $\mathbf{D}$ in terms of $\mathbf{B}$ and $\mathbf{C}$.

   (b) Show that $\overline{AD}$ passes through the point I with position vector

   $$\mathbf{I} = \frac{5}{14}\mathbf{A} + \frac{6}{14}\mathbf{B} + \frac{3}{14}\mathbf{C}.$$

   (Similar calculations show that I lies on all three angle bisectors.)

5.2 Challenge Problems

8. Given a non-zero vector $\mathbf{v}$, the **reflection across v**, denoted $\mathrm{refl}_{\mathbf{v}} : \mathbb{R}^2 \to \mathbb{R}^2$, is the reflection across the line in the direction of $\mathbf{v}$ passing through the origin.

   (a) Show that $\mathrm{refl}_{\mathbf{v}}(\mathbf{x}) = 2\,\mathrm{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}$.

   (b) A **linear operator** on $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. Using the fact that projection is a linear operator, or otherwise, show that reflection is also a linear operator.

9. In general, a **projection** $\mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator $P : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $P \circ P = P$. A projection is an **orthogonal projection** if $\mathbf{x} - P(\mathbf{x})$ is orthogonal to $P(\mathbf{x})$ for all $\mathbf{x}$.

   (a) Show for any non-zero vector $\mathbf{v}$ that $\mathrm{proj}_{\mathbf{v}}$ is an orthogonal projection.

   (b) Let P be a projection other than the zero and identity operators. (That is, $P(\mathbf{x})$ is not always $\mathbf{0}$ and not always $\mathbf{x}$.) Show that there exist non-zero linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ for which $P(\mathbf{u}) = \mathbf{u}$ and $P(\mathbf{v}) = \mathbf{0}$.

   (c) With notation as in part (b), what is the range of P?

10. Given any vector $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$, there is a corresponding sequence $(x_n) = x_0, x_1, x_2, \ldots$ defined by $x_0 = a$, $x_1 = b$, and $x_{n+2} = x_{n+1} + 2x_n$ for all $n \geq 0$.

    (a) Let (x_n) and (y_n) be the sequences corresponding to vectors $\mathbf{x}$ and $\mathbf{y}$. If (s_n) is the sequence corresponding to $a\mathbf{x} + b\mathbf{y}$, show that $s_n = ax_n + by_n$ for all $n \geq 0$.

    (b) Find all $\lambda \in \mathbb{C}$ for which the sequence $e_n = \lambda^n$ satisfies $e_{n+2} = e_{n+1} + 2e_n$ for all $n \geq 0$.

    (c) Let (x_n) be the sequence corresponding to $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Find a non-recursive formula for x_n.

5.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

2. (a) $\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -9 \end{pmatrix}$ (there are many other options)

   (b) Let p and q be the value of the parameters for $\mathbf{x}_1$ and $\mathbf{x}_2$ at the point of intersection, so $\mathbf{x}_1(p) = \mathbf{x}_2(q)$. Then
   $$\begin{pmatrix} 0 \\ 3 \end{pmatrix} + p \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + q \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
   which means $p = -4 + 2q$ and $3 - p = 2 + 3q$. Solving the system, $q = 1$ and $p = -2$, and the point of intersection is $(-2, 5)$.

3. (a) $\|\mathbf{u}\| = \sqrt{13}$
   $\mathbf{u} \cdot \mathbf{v} = 5$

   (b) $\hat{\mathbf{v}} = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{pmatrix}$

   (c) $\cos\theta = \frac{5}{\sqrt{221}}$

   (d) $\pm \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

4. (a) If $a\mathbf{u} + b\mathbf{v} = 0$, then $2a + 4b = 0$ and $3a - b = 0$. Adding 4 times the second equation to the first, $14a = 0$, so $a = 0$ and hence $b = 0$ as well.

   (b) $(a, b, c, d) = (1/14, 3/14, 2/7, -1/7)$

   (c) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. If $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$, then
   $$x_1 a + y_1 b = 0, \tag{1}$$
   $$x_2 a + y_2 b = 0. \tag{2}$$
   Taking the combination $y_2 \cdot (1) - y_1 \cdot (2)$,
   $$(x_1 y_2 - x_2 y_1)a = 0.$$
   If $x_1 y_2 - x_2 y_1 = 0$, then we consider further subcases.
   - If $\mathbf{x} = \mathbf{0}$, then taking $(a, b) = (1, 0)$ gives a non-zero solution, so $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Moreover, linear combinations $c\mathbf{x} + d\mathbf{y} = d\mathbf{y}$ can only produce multiples of $\mathbf{y}$, hence will not span all of $\mathbb{R}^2$.
   - If $\mathbf{x} \neq \mathbf{0}$, either $x_1 \neq 0$ or $x_2 \neq 0$. If $x_1 \neq 0$, then let $\lambda = y_1/x_1$, so $y_1 = \lambda x_1$. Substituting, we find $y_2 = \lambda x_2$, so $\mathbf{y} = \lambda\mathbf{x}$. This means that $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Also, any linear combination $c\mathbf{x} + d\mathbf{y} = (c + d\lambda)\mathbf{x}$ will be a multiple of $\mathbf{x}$, hence will not span all of $\mathbb{R}^2$.

If $x_1y_2 - x_2y_1 \neq 0$, then $a = 0$. At least one of y_1 and y_2 is non-zero in this case, and substituting into the relevant equation gives $b = 0$ as well. Therefore, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. To see that they span $\mathbb{R}^2$, let $\begin{pmatrix} p \\ q \end{pmatrix}$ be arbitrary. We look for coefficients c, d such that $c\mathbf{x} + d\mathbf{y} = \begin{pmatrix} p \\ q \end{pmatrix}$, or

$$\begin{aligned} x_1c + y_1d &= p, \\ x_2c + y_2d &= q. \end{aligned}$$

This has a solution, namely $(c, d) = \left(\dfrac{py_2 - qy_1}{x_1y_2 - x_2y_1}, \dfrac{qx_1 - px_2}{x_1y_2 - x_2y_1} \right)$.

(d) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be given. If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are as well. Otherwise, by part (c), there exist a, b for which $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. Then $a\mathbf{x} + b\mathbf{y} - \mathbf{z}$ is a non-trivial linear combination of the three which equals $\mathbf{0}$, so they are linearly dependent.

5. (a) $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

(b) It suffices to show $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Suppose $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$. Then

$$0 = \mathbf{x} \cdot (a\mathbf{x} + b\mathbf{y}) = a(\mathbf{x} \cdot \mathbf{x}) + b(\mathbf{x} \cdot \mathbf{y}) = a(1) + b(0) = a.$$

By a similar argument, $b = 0$.

(c) Translating up by 1 unit, it suffices to find the distance between the head of $\mathbf{u} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and the line $y = 2x$, which is spanned by the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We compute

$$\|\operatorname{proj}_{\mathbf{v}}(\mathbf{u})\| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{v}\|} = \frac{16}{\sqrt{5}}.$$

6. (a) $y = (-2/3)x + (4/3)$

(b) $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $d = 5$ (there are many choices that work)

(c) If $\mathbf{x}_1, \mathbf{x}_2$ are two position vectors for points on the line, then $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$ points along the line. Since

$$\hat{\mathbf{n}} \cdot \mathbf{v} = \hat{\mathbf{n}} \cdot (\mathbf{x}_2 - \mathbf{x} - 1) = \hat{\mathbf{n}} \cdot \mathbf{x}_2 - \hat{\mathbf{n}} \cdot \mathbf{x}_1 = d - d = 0,$$

$\hat{\mathbf{n}}$ is perpendicular to the line. Then, for any position vector $\mathbf{x}$ on the line, we compute the distance from the origin to the line as

$$\|\operatorname{proj}_{\hat{\mathbf{n}}}(\mathbf{x})\| = \frac{|\hat{\mathbf{n}} \cdot \mathbf{x}|}{\|\hat{\mathbf{n}}\|} = d.$$

7. (a) By the angle bisector theorem, $BD/DC = AB/AC = 1/2$. Therefore, $\mathbf{D} = \frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}$.

   (b) The points on line $\overline{AD}$ are those with position vector of the form

$$t\mathbf{A} + (1-t)\mathbf{D} = t\mathbf{A} + (1-t)\left(\frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}\right)$$

   for a real number t. Setting $t = 5/14$ gives us $\mathbf{I}$.

8. (a) We show that $\mathbf{y} = 2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}$ has the defining properties of reflection, namely that the line ℓ spanned by $\mathbf{v}$ is orthogonal to $\mathbf{x} - \mathbf{y}$ and passes through the midpoint of the segment connecting $\mathbf{x}$ and $\mathbf{y}$. First, $(\mathbf{x} + \mathbf{y})/2 = \operatorname{proj}_{\mathbf{v}}(\mathbf{x})$, so ℓ passes through the midpoint. For orthogonality,

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{y}) = 2\mathbf{v} \cdot (\mathbf{x} - \operatorname{proj}_{\mathbf{v}}(\mathbf{x})) = 0.$$

   (b) We compute

$$\begin{aligned}
\operatorname{refl}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) &= 2\operatorname{proj}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) - (a\mathbf{x} + b\mathbf{y}) \\
&= 2(a\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{proj}_{\mathbf{v}}(\mathbf{y})) - (a\mathbf{x} + b\mathbf{y}) \\
&= a(2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}) + b(2\operatorname{proj}_{\mathbf{v}}(\mathbf{y}) - \mathbf{y}) \\
&= a\operatorname{refl}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{refl}_{\mathbf{v}}(\mathbf{y}).
\end{aligned}$$

9. (a) Linearity of $\operatorname{proj}_{\mathbf{v}}$ follows from (bi)linearity of the dot product.
   To see that $\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}} = \operatorname{proj}_{\mathbf{v}}$, we compute

$$\begin{aligned}
(\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}})(\mathbf{x}) &= \operatorname{proj}_{\mathbf{v}}\left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v}\right) \\
&= \left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
&= \left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} = \operatorname{proj}_{\mathbf{v}}(\mathbf{x}).
\end{aligned}$$

   Finally, orthogonality is part of how $\operatorname{proj}_{\mathbf{v}}$ was defined.

   (b) Let $\mathbf{x}$ be a vector for which $P(\mathbf{x}) \neq \mathbf{0}$ and let $\mathbf{y}$ be a vector for which $P(\mathbf{y}) \neq \mathbf{y}$. (These conditions imply $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.) Let $\mathbf{u} = P(\mathbf{x})$ and $\mathbf{v} = \mathbf{y} - P(\mathbf{y})$. These are non-zero, and

$$\begin{aligned}
&P(\mathbf{u}) = P(P(\mathbf{x})) = P(\mathbf{x}) = \mathbf{u}, \\
&P(\mathbf{v}) = P(\mathbf{y} - P(\mathbf{y})) = P(\mathbf{y}) - P(P(\mathbf{y})) = P(\mathbf{y}) - P(\mathbf{y}) = \mathbf{0}.
\end{aligned}$$

   For linear independence, suppose $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$. Applying P,

$$\mathbf{0} = P(\mathbf{0}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u},$$

   so $a = 0$. Then, $b\mathbf{v} = \mathbf{0}$, so $b = 0$ as well.

(c) Since $\mathbf{u}, \mathbf{v}$ are linearly independent in $\mathbb{R}^2$, they also span $\mathbb{R}^2$. Given any vector $\mathbf{x} \in \mathbb{R}^2$, let $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$. Then

$$P(\mathbf{x}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u}.$$

As a and b range over all real numbers, we find the range of P is the span of $\mathbf{u}$.

10. (a) We proceed by induction. By definition, $s_0 = ax_0 + by_0$ and $s_1 = ax_1 + by_1$. Then, if $s_n = ax_n + by_n$ and $s_{n+1} = ax_{n+1} + by_{n+1}$,

$$\begin{aligned} s_{n+2} &= s_{n+1} + 2s_n \\ &= (ax_{n+1} + by_{n+1}) + 2(ax_n + by_n) \\ &= a(x_{n+1} + 2x_n) + b(y_{n+1} + 2y_n) \\ &= ax_{n+2} + by_{n+2}. \end{aligned}$$

(b) We need $\lambda^n(\lambda^2 - \lambda - 2) = 0$ for all $n \geq 0$. This holds when $\lambda = -1$ and when $\lambda = 2$.

(c) The sequence $e_n = (-1)^n$ corresponds to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, while the sequence $f_n = 2^n$ corresponds to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We can find

$$\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{11}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so

$$x_n = \frac{1}{3}e_n + \frac{11}{3}f_n = \frac{1}{3} \cdot (-1)^n + \frac{11}{3} \cdot 2^n.$$