Algebra 1 Practice Problems II

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The focus of these review problems and the second midterm is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Graphing

1.1 Review problems

1. Quadrilateral $ABCD$ is positioned in the coordinate plane so that its vertices have coordinates

$$A = (5, 7); \quad B = (5, 6); \quad C = (3, 1); \quad D = (-4, -5).$$

   Points E, F, G, H are the midpoints of segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively.

   (a) Find the coordinates of E, F, G, and H.
   (b) Compute the midpoints of segments $\overline{EG}$ and $\overline{FH}$.

   To check your work, the two midpoints computed in part (b) should be the same.

2. Maurine needs to get from $(2, 3)$ to $(17, 11)$.

   (a) If they take the shortest path possible, how much distance would they cover?
   (b) Suppose Maurine gets distracted while pondering the meaning of life and goes from $(2, 3)$ to $(6, 6)$, then to $(11, 18)$, then to $(17, 10)$, and finally to $(17, 11)$. What is the minimum distance Maurine can cover which is consistent with this information?

3. A line passes through the point $(-5, 2)$ and has slope $1/2$.

   (a) Write down an equation for this line in point-slope form.
   (b) Find the slope-intercept form and the standard form of the line.

4. A line passes through the points $(-3, 4)$ and $(-3, -4)$. Find an equation for this line.

5. A line passes through the points $(-3, 3)$ and $(0, -4)$.

   (a) Find the slope of the line.
   (b) For each of the two given points, find the point-slope form of the line using that point.
   (c) Find the slope-intercept form and the standard form of the line.

   As a check of your answers, rearranging either of the point-slope equations from part (b) should give you the same slope-intercept form and standard form.

6. Let ℓ be the line with equation $y = -4x - 2$ and let $P = (-4, 2)$. The point Q on line ℓ which is closest to P is the point for which $\overline{PQ} \perp \ell$. As such, if ℓ' is the line through P perpendicular to line ℓ, then Q is the intersection of ℓ and ℓ'.

   (a) Find the slope of ℓ'.
   (b) Find an equation for ℓ'.
   (c) Find the coordinates of Q.

7. (a) Of the equations

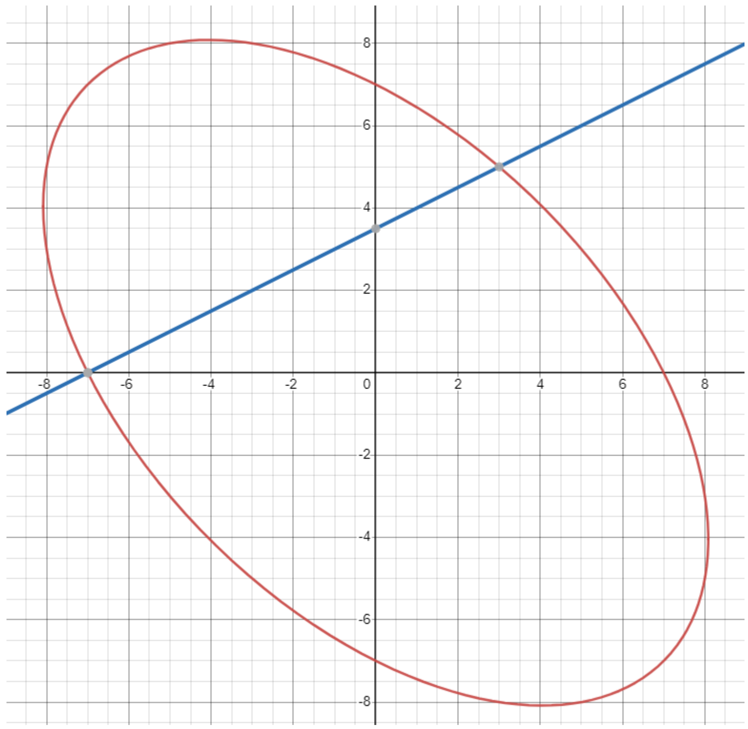
$$5x + 4y = 35; \quad (x + 4)^2 + (y - 1)^2 = 10; \quad x^2 + xy + y^2 = 49; \quad x - 2y = -7,$$

   which one is an equation for the blue line below?

(b) Of the equations

$$5x + 4y = 35; \quad (x+4)^2 + (y-1)^2 = 10; \quad x^2 + xy + y^2 = 49; \quad x - 2y = -7,$$

which one is an equation for the red curve below?



1.2 Challenge problems

8. (Median and centroid) Given any triangle ABC, the *medians* are the lines that pass through one vertex and the midpoint of the side opposite that vertex. For example, the A-median is the line that passes through A and the midpoint of side $\overline{BC}$. The three medians of a triangle all pass through a single point G, called the *centroid* of triangle ABC.

   (a) Let $A = (0,0)$, $B = (14,0)$, and $C = (5,12)$. Let D, E, and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Find the coordinates of D, E, and F.

   (b) Find equations for the medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

   (c) Find the coordinates of the centroid G of ABC. How do they relate to the coordinates of A, B, and C?

   (d) Compute the ratios AG/GD, BG/GE, and CG/GF.

9. (Altitude and orthocenter) Given any triangle ABC, the *altitudes* are the lines that pass through one vertex which are perpendicular to the opposite side. For example, the A-altitude is the line that passes through A and is perpendicular to line $\overline{BC}$. The three altitudes of a triangle all pass through a single point H, called the *orthocenter* of triangle ABC.

(a) Let $A = (0, 0)$, $B = (14, 0)$, and $C = (5, 12)$. Find equations for the altitudes of ABC.

(b) Find the coordinates of the orthocenter H of ABC.

10. Let $A = (1, 1)$, $B = (5, 2)$, and $C = (-4, 3)$. There are three parallelograms in the plane for which A, B, and C are three of the four vertices. What are the possible coordinates for the fourth vertex?

1.3 Answers

1. (a) $E = (5, 13/2)$; $F = (4, 7/2)$; $G = (-1/2, -2)$; $H = (1/2, 1)$

   (b) The midpoint of $\overline{EG}$ is $(9/4, 9/4)$, which is also the midpoint of $\overline{FH}$.

2. (a) $\boxed{17}$

   (b) $5 + 13 + 10 + 1 = \boxed{29}$

3. (a) $y - 2 = \frac{1}{2}(x + 5)$

   (b) Slope-intercept form: $y = \frac{1}{2}x + \frac{9}{2}$

   Standard form: $x - 2y = -9$

4. The line through the two points is the vertical line $\boxed{x = -3}$.

5. (a) $\dfrac{3 - (-4)}{(-3) - 0} = \boxed{-\dfrac{7}{3}}$

   (b) Using $(-3, 3)$, we get $\boxed{y - 3 = -\dfrac{7}{3}(x + 3)}$.

   Using $(0, -4)$, we get $\boxed{y + 4 = -\dfrac{7}{3}(x - 0)}$.

   (c) Slope-intercept form: $\boxed{y = -\dfrac{7}{3}x - 4}$

   Standard form: $\boxed{7x + 3y = -12}$

6. (a) Given a line with slope $m \neq 0$, the slope of any line perpendicular to it is given by $-1/m$. Applying this formula with $m = -4$, the slope of ℓ, tells us the slope of ℓ' must be $\boxed{1/4}$.

   (b) Using point-slope form, since ℓ' passes through P, one equation for ℓ' is

   $$\boxed{y - 2 = \frac{1}{4}(x + 4)}.$$

   (c) Solving the system of equations

   $$y = -4x - 2 \qquad \text{and} \qquad y - 2 = \frac{1}{4}(x + 4),$$

   we find that $Q = \boxed{(-20/17, 46/17)}$.

7. (a) The blue line passes through $(0, 3.5)$, which only satisfies the equation $\boxed{x - 2y = -7}$.

   (b) The first and last equation describe lines, so the answer cannot be those. Of the remaining two equations, the point $(7, 0)$ on the red curve only satisfies $\boxed{x^2 + xy + y^2 = 49}$.

8. (a) $D = (19/2, 6)$; $E = (5/2, 6)$; $F = (7, 0)$

(b) AD: $y = \frac{12}{19}x$
BE: $y = -\frac{12}{23}(x - 14)$
CF: $y = -6(x - 7)$

(c) $G = (19/3, 4)$; the x-coordinate of G is the average of the x-coordinates of A, B, and C, and a similar statement holds for the y-coordinates.

(d) $AG/GD = BG/GE = CG/GF = 2$

9. (a) A-altitude: $y = \frac{3}{4}x$
B-altitude: $y = -\frac{5}{12}(x - 14)$
C-altitude: $x = 5$

(b) $(5, 15/4)$

10. Each possible fourth vertex D comes from picking two of the three points A, B, and C to be opposite each other, and then D will be opposite the third point. We will show how to find D in the case that A and C are opposite each other, i.e. the parallelogram is $ABCD$. We will then provide answers for the case that A and B are opposite each other (so the parallelogram is $ACBD$) and the case that B and C are opposite each other (so the parallelogram is $BACD$).

Solution 1: If $ABCD$ is a parallelogram, then $\overline{AB} \parallel \overline{CD}$, so $\overline{AB}$ and $\overline{CD}$ have the same slope. The slope of $\overline{AB}$ is $\frac{1-2}{1-5} = \frac{1}{4}$, so letting $D = (x, y)$,

$$\frac{y-3}{x+4} = \frac{1}{4}.$$

Multiplying both sides by $4(x + 4)$ (i.e. cross multiplying) turns this into the linear equation $4y - 12 = x + 4$. To get another equation, $\overline{BC} \parallel \overline{AD}$, so $\overline{BC}$ and $\overline{AD}$ have the same slope. The slope of $\overline{BC}$ is $\frac{2-3}{5-(-4)} = -\frac{1}{9}$, so

$$\frac{y-1}{x-1} = -\frac{1}{9}.$$

Cross multiplying gives us $9y - 9 = 1 - x$. We now have two linear equations for the variables x and y, and solving the resulting system of equations gives us $(x, y) = \boxed{(-8, 2)}$. By similar reasoning, the point which makes $ACBD$ a parallelogram is $D = \boxed{(10, 0)}$, and the point which makes $BACD$ a parallelogram is $D = \boxed{(0, 4)}$.

Solution 2: A geometry theorem states that a quadrilateral $WXYZ$ is a parallelogram if and only if the midpoints of $\overline{WY}$ and $\overline{XZ}$ coincide. Thus for $ABCD$ to be a parallelogram with $D = (x, y)$, we need

$$\text{midpoint of } \overline{AC} = \text{midpoint of } \overline{BD},$$
$$\left(\frac{1 + (-4)}{2}, \frac{1+3}{2}\right) = \left(\frac{5+x}{2}, \frac{2+y}{2}\right).$$

Matching the first coordinate gives $x = -8$, and matching the second coordinate gives $y = 2$, so $D = \boxed{(-8, 2)}$. Similarly, $ACBD$ is a parallelogram when $D = \boxed{(10, 0)}$, and $BACD$ is a parallelogram when $D = \boxed{(0, 4)}$.

2 Linear Inequalities

2.1 Review problems

1. Solve each of the following inequalities, expressing the solution set in interval notation and graphing the solution set on a number line.

   (a) $x > -1$

   (b) $2x + 5 \leq 7$

   (c) $7 - 3x < 5x - 9$

   (d) $(x + 4)^2 + x^2 \geq (x + 2)^2 + (x + 1)^2$

2. For each of the following combinations of inequalities, write the solution set in interval notation and graph the solution set on a number line.

   (a) $2 \leq 3 + x < 6$

   (b) $x \geq 2 - x$ or $5 - 3x \geq x + 9$

   (c) $-9x + 4 < 8x + 2 < 2x + 7$

   (d) $-6x - 8 \leq 4$ and $-x - 10 \leq -7$

   (e) $-6x - 8 \leq 4$ or $-x - 10 \leq -7$

3. (a) Write down a pair of linear inequalities, each involving a single variable x, for which no real number satisfies both of the inequalities simultaneously. In this case, the solution set for "[first inequality] and [second inequality]" is the *empty set*, denoted $\emptyset$.

   (b) Write down a pair of linear inequalities, each involving a single variable x, for which every real number satisfies at least one of the inequalities. In this case, the solution set for "[first inequality] or [second inequality]" is the entire set of real numbers, denoted $\mathbb{R}$. We could also write $(-\infty, \infty)$.

   (c) Write down a pair of linear inequalities, each involving a single variable x, for which the only real number which does not satisfy at least one of the inequalities is 6.626. Express this solution set in interval notation as a union ($\cup$) of two intervals.

4. Solve each of the following inequalities, expressing the solution set in interval notation and graphing the solution set on a number line.

   (a) $|x| < 4$

   (b) $|x - 4| \geq 2$

5. Graph the solution set of each of the following inequalities.

   (a) $x + y < -2$

   (b) $3x - y \geq 7$

   (c) $(x - 1)^2 + y^2 \leq (x - 5)^2 + (y - 2)^2$

   (d) $|x| + |y| < 1$

6. Graph the solution set of each of the following combinations of inequalities.

   (a) $x + y \leq -2$ and $3x - y > 7$
   (b) $x + y \leq -2$ or $3x - y > 7$
   (c) $2x + 3y \geq 4$ and $6y \leq 12 - 4x$
   (d) $3x + 3y < -5x < 5y$

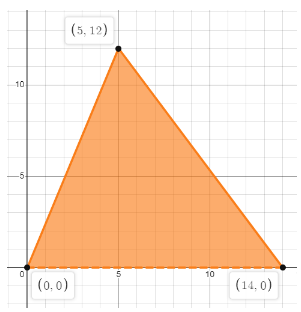
7. Supposing $a \leq b$ and $c \leq d$, which of the following must also be true?

   (I) $a + c \leq b + d$
   (II) $a - c \leq b - d$
   (III) $ac \leq bd$
   (IV) $a/c \leq b/d$

   For the statements that were not guaranteed to be true, which ones are guaranteed to be true when we also assume $a \geq 0$ and $c \geq 0$?

2.2 Challenge problems

8. Write down three inequalities for which the set of all points (x, y) satisfying all of the three inequalities is the set shown below.



9. Miko and Yuu plan to sell alms bowls and cowrie shell necklaces at their school's culture festival. Each bowl sold earns them \$30 while each necklace sold earns them \$23. Their stand will allow them to stock up to 120 items in total, and the amounts of these two items cannot differ by more than 40. (Otherwise, Miko and Yuu will start arguing so much that they drive away all potential customers.) To make the items, they need to borrow materials from the art club. Each bowl uses up 2 units of a 170-unit material allowance that the art club gives them, while each necklace uses up 1 unit of that allowance. Given all of these restrictions, what is the maximum possible amount of revenue that Miko and Yuu could bring in?

10. Find all values of x for which

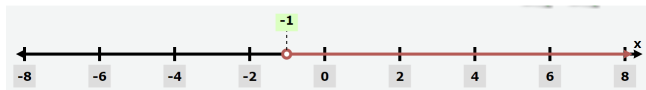
$$||x - 1| - |2x - 8|| \leq 3/2.$$

    Express your answer in interval notation.

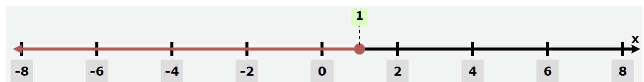
2.3 Answers

In all instances of interval notation, writing ∞ in place of $+\infty$ would be valid (but writing ∞ in place of $-\infty$ would not).

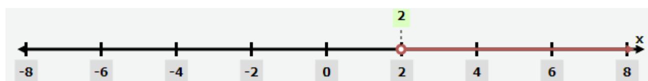
1. (a) $(-1, +\infty)$



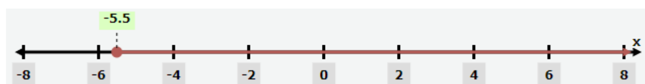
   (b) $(-\infty, 1]$



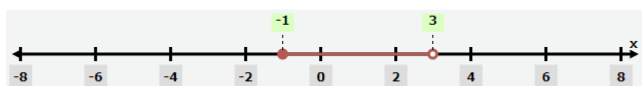
   (c) $(2, +\infty)$



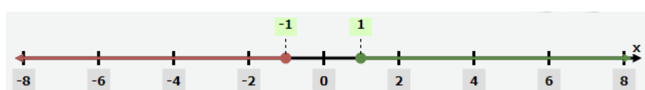
   (d) $[-11/2, +\infty)$



2. (a) $[-1, 3)$

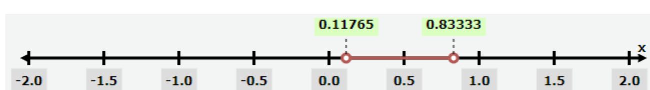


   (b) $(-\infty, -1] \cup [1, +\infty)$



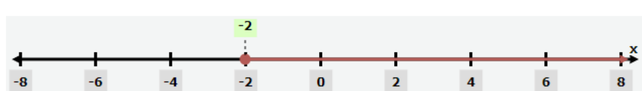
   (The colors on the graph are irrelevant.)

   (c) $(2/17, 5/6)$

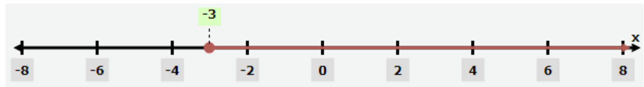


   (The decimals labeling the endpoints are approximations.)

   (d) $[-2, +\infty)$



(e) $[-3, +\infty)$



3. There are several possible options for each of these questions.

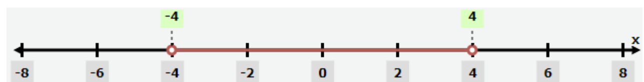
   (a) $x > 0$ and $x < 0$

   (b) $x > 0$ or $x \leq 0$

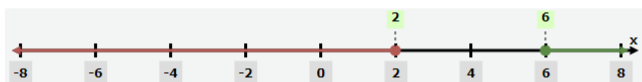
   (c) $x > 6.626$ or $x < 6.626$

4. The *absolute value* of a real number x, denoted $|x|$, is the distance between x and 0 on the number line. A useful corollary of this is that $|a - b|$ is the distance between a and b.

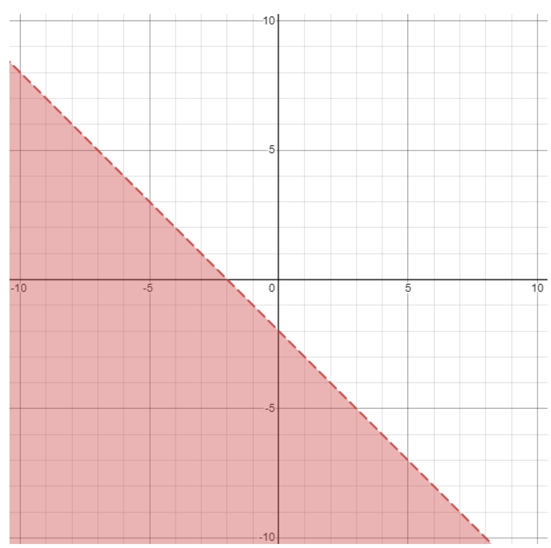
   (a) We want the numbers whose distance from 0 is less than 4, so the answer is $\boxed{(-4, 4)}$.



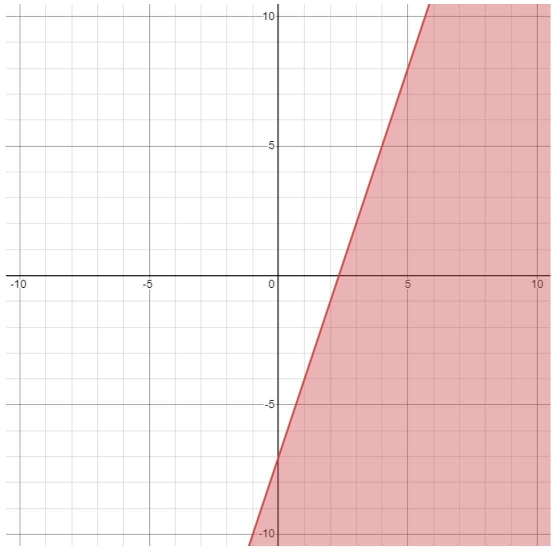
   (b) We want the numbers whose distance from 4 is greater than or equal to 2, so the answer is $\boxed{(-\infty, 2] \cup [6, +\infty)}$.



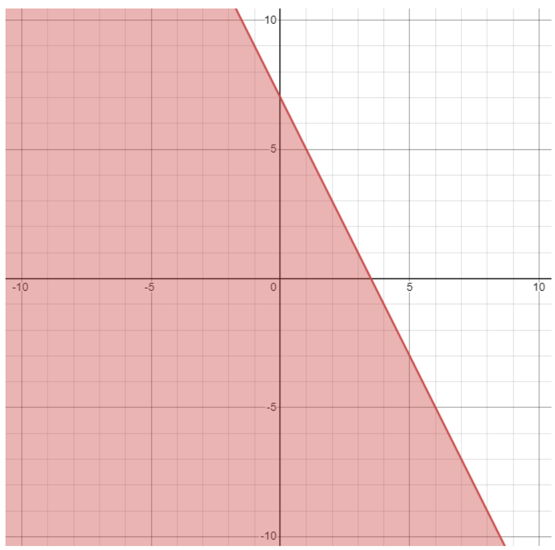
5. (a)



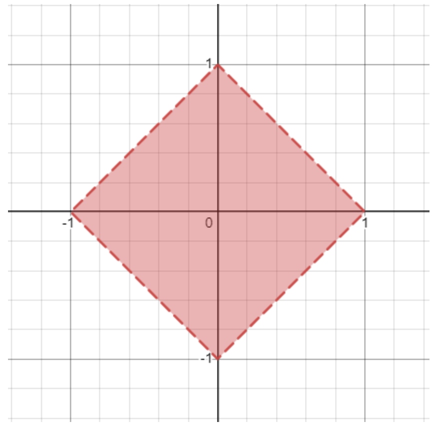
(b)



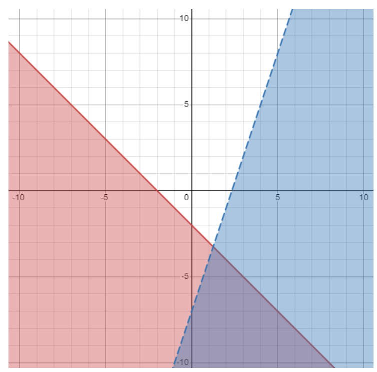
(c)



(d) In the first quadrant, $|x| + |y| = x + y$, so the first-quadrant points in the solution set are the ones satisfying $x + y < 1$. In the second quadrant, $|x| + |y| = -x + y$, so the second-quadrant points in the solution set are the ones satisfying $-x+y < 1$. In the third quadrant, $|x|+|y| = -x-y$, so the third-quadrant points in the solution set are the ones satisfying $-x - y < 1$. In the fourth quadrant, $|x| + |y| = x - y$, so the fourth-quadrant points in the solution set are the ones satisfying $x - y < 1$. Putting these together gives us a square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$, with the boundaries not included.

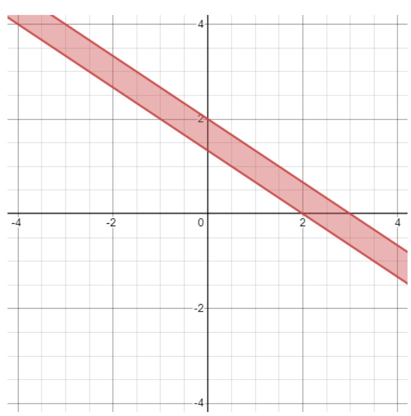


6. (a) In the image below, the solution set to this problem is the region covered by both shadings. The intersection point of the lines is not included since it does not satisfy the strict inequality.

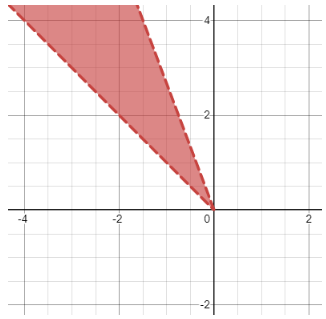


(b) In the image above, the solution set to this problem is the region covered by at least one shading. The intersection point is included since it satisfies the non-strict inequality.

(c)



(d)



7. Statement (I) is the only one that is guaranteed to hold. If we introduce the additional condition that $a, c \geq 0$, so that all variables are non-negative, then statement (III) is also guaranteed to hold.

8. The bottom boundary is a dashed line segment along the x-axis, so to get the region above it, we can use the inequality $\boxed{y > 0}$.

   The left boundary is a full line segment along $y = \frac{12}{5}x$, so to get a region below it, we can use the inequality $\boxed{y \leq \dfrac{12}{5}x}$.

   The right boundary is a full line segment along $y = -\frac{4}{3}x + \frac{56}{3}$, so to get a region below it, we can use the inequality $\boxed{y \leq -\dfrac{4}{3}x + \dfrac{56}{3}}$.

9. Let x be the number of alms bowls and y be the number of cowrie shell necklaces. The problem statement imposes the following constraints on x and y:

$$\begin{aligned} x &\geq 0, \\ y &\geq 0, \\ x + y &\leq 120, && \text{(maximum stock)} \\ x - y &\leq 40, && \text{(cannot have too many more bowls)} \\ y - x &\leq 40, && \text{(cannot have too many more necklaces)} \\ 2x + y &\leq 170. && \text{(materials allowance)} \end{aligned}$$

   The revenue we wish to maximise is $30x + 23y$. Our key observation for solving this type of problem (a linear program) is that a revenue R can be achieved if and only if the line $30x + 23y = R$ intersects the region defined by the constraints above, so we want to find the maximum such R. As we increase R, our line moves up and to the right while retaining its slope, and when R is at the desired maximum, the line will pass through a vertex of the region. (You can experiment with this here.) By drawing such lines, we can identify that the vertex of interest is the intersection of the lines $x + y = 120$ and $2x + y = 170$. This intersection point is $(50, 70)$, and the corresponding revenue is

$$30x + 23y = 30 \cdot 50 + 23 \cdot 70 = \boxed{3110}.$$

Remark: In general, the maximum or minimum for a linear program is achieved at a vertex of the feasible region, so even without drawing lines to figure out the right vertex, we can test all of the vertices and see which one results in the largest revenue.

10. A useful tool for handling complicated expressions involving absolute values is to consider graphs. We can interpret the given inequality as saying that we want the values of x for which the vertical distance between the graphs $y = |x-1|$ and $y = |2x-8|$ should be at most $3/2$. (For reference, the graphs are here.)

    From the graphs, we can see two intersection points whose x coordinates will be part of the solution set, as the vertical distance at those values is just 0. For the left intersection point, the lines intersecting are $y = x - 1$ and $y = -(2x - 8)$, and the intersection point is $(3, 2)$. For the right intersection point, the lines intersecting are $y = x - 1$ and $y = 2x - 8$, and the intersection point is $(7, 6)$.

    As we move leftward from $(3, 2)$, the blue line $y = -(2x - 8)$ moves up 2 units for every 1 unit left while the red line $y = x - 1$ moves down 1 unit for every 1 unit left, so the vertical distance between them grows by 3 units for every 1 unit left we move. Therefore, the graphs stay within $3/2$ of each other until we have moved $1/2$ units to the left and reached $x = 5/2$. Similarly, they stay within $3/2$ of each other as we move rightward from $(3, 2)$ until we have moved $1/2$ units to the right and reached $x = 7/2$.

    As we move from $(7, 6)$, we can use similar reasoning to see that the vertical distance between the graphs grows by 1 unit for every 1 unit left or right we move. Thus we can move $3/2$ units left to reach $x = 11/2$ or $3/2$ units right to reach $x = 17/2$.

    The final answer is $\boxed{[5/2, 7/2] \cup [11/2, 17/2]}$.

3 Quadratics (I)

3.1 Review problems

1. Solve each of the following equations for all real solutions.

   (a) $(x+1)(x+3)=0$

   (b) $4(x+3)(x^2-3)=0$

   (c) $(x-1)(\sqrt{x}-3)(x-4)=0$

2. Factor each of the following expressions as much as possible. All coefficients and constants in the factorisations should be integers.

   (a) x^2+5x

   (b) $6x^2+2x$

   (c) x^2-16

   (d) x^2+3x+2

   (e) $x^2+8x+16$

   (f) $4x^2-18x+8$

3. For each of the following quadratics, find the sum and product of all real roots.

   (a) x^2-25

   (b) x^2+x

   (c) x^2+3x+2

   (d) $4x^2+14x-8$

   (e) $16x^2+8x+1$

4. For each of the following descriptions, find the quadratic ax^2+bx+c fitting the description. The coefficients a, b, c should be integers with $a>0$ and $\gcd(a, b, c)=1$.

   (a) The roots are -1 and 5.

   (b) The roots are 2 and -2.

   (c) The roots are $-2/3$ and 0.

   (d) The only root is $-1/5$.

   (e) The sum of the roots is 3 and the product of the roots is 2.

   (f) The sum of the roots is $-9/2$ and the product of the roots is $-5/2$.

5. Two real numbers have a sum of 18.

   (a) What is their largest possible product?

   (b) If their product is -63, what are the two numbers?

6. The quadratic x^2-x-1 has roots r and s. Compute r^2+s^2.

3.2 Challenge problems

7. Suppose $a + b = 7$ and $ab = 3$.
   (a) Find a quadratic whose roots are a and b.
   (b) Find a quadratic whose roots are a^2 and b^2.
   (c) Find a quadratic whose roots are $\frac{1}{a}$ and $\frac{1}{b}$.
8. (a) Factor $x^4 + x^2 + 1$.
   (b) (Sophie-Germain) Factor $x^4 + 4$.
9. Suppose x is a real number satisfying $x + \frac{1}{x} = 3$.
   (a) Compute $x^2 + \frac{1}{x^2}$.
   (b) Compute $x^3 + \frac{1}{x^3}$.
10. Find all real solutions of

$$2x^4 - x^3 - 6x^2 - x + 2 = 0.$$

3.3 Answers

1. (a) -3, -1
   (b) -3, $-\sqrt{3}$, $\sqrt{3}$
   (c) 1, 4, 9

2. (a) $x(x+5)$
   (b) $2x(3x+1)$
   (c) $(x-4)(x+4)$
   (d) $(x+1)(x+2)$
   (e) $(x+4)^2$
   (f) $2(2x-1)(x-4)$

3. (a) The sum is 0 and the product is -25.
   (b) The sum is -1 and the product is 0.
   (c) The sum is -3 and the product is 2.
   (d) The sum is $-14/4 = -7/2$ and the product is $-8/4 = -2$.
   (e) The sum is $-1/4$ and the product is $-1/4$.
   Vieta's formulas would give us a sum of $-8/16 = -1/2$ *and a product of* $1/16$, *but this quadratic only has one root* $-1/4$ *(which is a double root).*

4. (a) $(x+1)(x-5) = \boxed{x^2 - 4x - 5}$
   (b) $(x-2)(x+2) = \boxed{x^2 - 4}$
   (c) $(3x+2) \cdot x = \boxed{3x^2 + 2x}$
   (d) $(5x+1)(5x+1) = \boxed{25x^2 + 10x + 1}$
   (e) Using Vieta's formulas, $\boxed{x^2 - 3x + 2}$ would work.
   (f) Using Vieta's formulas, $x^2 + \frac{9}{2}x - \frac{5}{2}$ has the desired properties, but we need integer coefficients. Multiplying by 2 does not change the roots and gives $\boxed{2x^2 + 9x - 5}$.

5. (a) Let the two numbers be $9+d$ and $9-d$, where $d \geq 0$. Their product is

$$(9+d)(9-d) = 81 - d^2 \leq 81.$$

   Therefore, the maximum possible product is $\boxed{81}$, which is attained when $d = 0$ and the two numbers are 9 and 9.
   (b) If the product is -63, then we have

$$81 - d^2 = -63 \implies d = 12.$$

   The numbers are $9 + 12 = \boxed{21}$ and $9 - 12 = \boxed{-3}$.

6. By Vieta's formulas, $r + s = 1$ and $rs = -1$. Then,

$$r^2 + s^2 = (r+s)^2 - 2rs = 1^2 - 2 \cdot (-1) = \boxed{3}.$$

7. (a) By Vieta's formulas, $\boxed{x^2 - 7x + 3}$ would work.

   (b) The sum of the roots would be

   $$a^2 + b^2 = (a+b)^2 - 2ab = 7^2 - 2 \cdot 3 = 43$$

   and the product of the roots would be

   $$a^2b^2 = (ab)^2 = 3^2 = 9.$$

   Hence we could take $\boxed{x^2 - 43x + 9}$.

   (c) The sum of the roots would be

   $$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{7}{3}$$

   and the product of the roots would be

   $$\frac{1}{ab} = \frac{1}{3}.$$

   Hence we could take $x^2 - \frac{7}{3}x + \frac{1}{3}$, or after rescaling, $\boxed{3x^2 - 7x + 1}$.
   Alternatively, if $x = 1/y$, then the solutions for y in the equation

   $$\left(\frac{1}{y}\right)^2 - 7 \cdot \frac{1}{y} + 3 = 0$$

   would be $y = 1/a$ and $y = 1/b$, by part (a). Multiplying through by y^2 gives us $1 - 7y + 3y^2 = 0$. This does not have $y = 0$ as a solution, so we did not introduce any new solutions after multiplying by y^2. Hence this quadratic has the required roots (in agreement with the first method).

8. (a) $x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2 = (x^2+1)^2 - x^2 = \boxed{(x^2 + 1 - x)(x^2 + 1 + x)}$

   (b) $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2+2)^2 - (2x)^2 = \boxed{(x^2 + 2 - 2x)(x^2 + 2 + 2x)}$

9. (a) We have

   $$9 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2},$$

   so $x^2 + \dfrac{1}{x^2} = \boxed{7}$.

   (b) We have

   $$27 = \left(x + \frac{1}{x}\right)^3 = \left(x^2 + 2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3},$$

   so

   $$x^3 + \frac{1}{x^3} = 27 - 3\left(x + \frac{1}{x}\right) = 27 - 3 \cdot 3 = \boxed{18}.$$

10. First, $x = 0$ is not a root. Thus we can safely divide both sides by x^2 to get

$$2\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) - 6 = 0.$$

Letting $y = x + \frac{1}{x}$, we get $x^2 + \frac{1}{x^2} = y^2 - 2$, so our equation becomes

$$2(y^2 - 2) - y - 6 = 0,$$

or $2y^2 - y - 10 = 0$. This factors as

$$(2y - 5)(y + 2) = 0,$$

so $y = 5/2$ and $y = -2$ are the two solutions for y.

When $y = -2$, we have

$$x + \frac{1}{x} = -2 \implies x^2 + 2x + 1 = 0.$$

This factors as $(x + 1)^2 = 0$, so the lone solution is $x = -1$.

When $y = 5/2$, we have

$$x + \frac{1}{x} = \frac{5}{2} \implies 2x^2 - 5x + 2 = 0.$$

This factors as $(2x - 1)(x - 2) = 0$, so the solutions are $x = 1/2$ and $x = 2$.

In conclusion, the solutions are $\boxed{-1, 1/2, 2}$.

Remark: This sort of trick can be used whenever the list of coefficients is palindromic, i.e. reads the same forwards or backwards.

4 Complex Numbers

Given a complex number $z = x + yi$, the *magnitude* of z, denoted $|z|$, is $\sqrt{x^2 + y^2}$. For example,

$$|-1 + 3i| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}.$$

4.1 Review problems

1. Let $z = 2 + i$. Compute each of the following:
   (a) $\operatorname{Re} z$, the real part of z
   (b) $\operatorname{Im} z$, the imaginary part of z
   (c) $\overline{z}$, the complex conjugate of z
   (d) $|z|$, the magnitude of z

2. Let $z = 5 + 3i$ and $w = 7 - 9i$. Compute each of the following:
   (a) $z + w$
   (b) $z - w$
   (c) zw
   (d) z/w

3. (a) Compute $(1 + i)^2$.
   (b) Compute $(1 + i)^{2022}$.

4. (a) Find the two complex numbers whose square is -180.
   (b) Find the four complex numbers whose fourth power is 16.

5. Which of the following statements is always true (whenever both sides are defined)?
   (I) $\overline{z + w} = \overline{z} + \overline{w}$
   (II) $\overline{z - w} = \overline{z} - \overline{w}$
   (III) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
   (IV) $\overline{z/w} = \overline{z}/\overline{w}$

6. Which of the following statements is always true (whenever both sides are defined)?
   (I) $|z + w| = |z| + |w|$
   (II) $|z - w| = |z| - |w|$
   (III) $|z \cdot w| = |z| \cdot |w|$
   (IV) $|z/w| = |z|/|w|$

4.2 Challenge problems

7. Let $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Compute ω^{2024}.

8. Every complex number can be represented as a point in the plane by associating $a + bi$ with the point (a, b). (When we do this, we get the *complex plane* or *Argand diagram.*)

   (a) Complex numbers with negative real part and positive imaginary part are represented by points in which quadrant?

   (b) What geometric transformation is represented by complex conjugation?

   (c) What geometric transformation is represented by multiplication by i?

9. Let $ax^2 + bx + c$ be a quadratic with <u>real</u> coefficients. Show that if r is a root, then so is $\overline{r}$.

10. Find a complex number z such that $z^2 = -15 - 8i$.

    One can show that in fact, every complex number has a complex number square root: we do not need to add any new numbers besides i!

4.3 Answers

1. (a) 2
   (b) 1
   (c) $2 - i$
   (d) $\sqrt{5}$

2. (a) $12 - 6i$
   (b) $-2 + 12i$
   (c) $62 - 24i$
   (d) $\frac{4}{65} + \frac{33}{65}i$

3. (a) $(1+i)^2 = 1 + 2i + i^2 = \boxed{2i}$
   (b) $(1+i)^{2022} = (2i)^{1011} = 2^{1011}i^{1011} = 2^{1011}i^3 = \boxed{-2^{1011}i}$

4. (a) The two square roots of -180 are
$$\pm\sqrt{-180} = \pm\sqrt{180}i = \boxed{\pm 6\sqrt{5}i}.$$
   (b) Every fourth root is a square root of a square root, so we can start by finding the square roots of 16, which are 4 and -4. The square roots of 4 are ± 2, while the square roots of -4 are $\pm 2i$, so the final list is $\boxed{\pm 2, \pm 2i}$.

5. All four statements (I), (II), (III), (IV) are always true.

6. Statements (III) and (IV) are always true, while a counterexample for (I) and (II) can be found by taking $z = 1$ and $w = i$.

7. Computing several powers of ω, we can eventually find that $\omega^6 = 1$. Therefore,
$$\omega^{2024} = \omega^{2022} \cdot \omega^2 = \omega^2 = \boxed{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}.$$

8. (a) Second quadrant (II)
   (b) Reflection across the x-axis
   (c) Rotation by 90° counterclockwise about the origin

9. If r is a root, then $ar^2 + br + c = 0$. Taking the complex conjugate on both sides of the equation, and using the facts from question 5, the right hand side stays 0 while the left hand side becomes
$$\begin{aligned}\overline{ar^2 + br + c} &= \overline{ar^2} + \overline{br} + \overline{c} \\ &= \overline{a} \cdot \overline{rr} + \overline{b} \cdot \overline{r} + \overline{c} \\ &= a \cdot \overline{r} \cdot \overline{r} + b \cdot \overline{r} + c && (a, b, c \text{ real}) \\ &= a\overline{r}^2 + b\overline{r} + c.\end{aligned}$$
   Thus when we plug $x = \overline{r}$ into the quadratic, we get 0, so $\overline{r}$ is a root.

10. Suppose $a + bi$ is a square root of $-15 - 8i$, so that $(a + bi)^2 = -15 - 8i$. (Here a and b are real.) Expanding the left hand side, we get

$$(a^2 - b^2) + (2ab) \cdot i = -15 - 8i,$$

so we have the system of equations

$$a^2 - b^2 = -15 \quad \text{and} \quad 2ab = -8.$$

From the second equation, neither a nor b are zero, so we can safely divide by $2a$ to get $b = -4/a$. Substituting into the first equation yields

$$a^2 - \left(-\frac{4}{a}\right)^2 = -15 \implies a^2 - \frac{16}{a^2} = -15.$$

Letting $x = a^2$ and multiplying both sides by x, we get $x^2 - 16 = -15x$, so

$$x^2 + 15x - 16 = 0 \implies (x + 16)(x - 1) = 0.$$

The two solutions to this quadratic are $x = -16$ and $x = 1$, but since $x = a^2$ where a is a real number, the only valid solution is $x = 1$. This means that $a = 1$ or $a = -1$. When $a = 1$, we get $b = -4$, and when $a = -1$, we get $b = 4$, so the two square roots of $-15 - 8i$ are $\boxed{1 - 4i}$ and $\boxed{-1 + 4i}$.

5 Quadratics (II)

5.1 Review problems

1. Find all solutions to the following equations. You may use any method.

   (a) $x^2 = 2x + 80$

   (b) $8x = 4x^2 + 1$

   (c) $x^2 + 25 = 10x$

   (d) $-6x^2 - 1 - 5x = 0$

   (e) $x^2 + x + 10 = 0$

2. Complete the square in each of the following expressions. That is, write each of the following in the form $a(x - h)^2 + k$ for constants a, h, k.

   (a) $x^2 - 18x + 81$

   (b) $x^2 + 16x + 59$

   (c) $-6x^2 + 24x + 5$

   (d) $9x^2 - 7x + 5$

   (e) $-\frac{x^2}{2s} - 2\pi tx$, where s and t are constants.

3. For what real values of c does the quadratic $x^2 + 8x + c$ have

   (a) two non-real roots?

   (b) one real root?

   (c) two real roots?

   (d) For what positive integers c does the quadratic have two rational roots?

4. (a) A parabola is given by $y = x^2 + 18x + 81$. Find the intercepts and the vertex.

   (b) A parabola has x-intercepts $(-8, 0)$ and $(10, 0)$ and y-intercept $(0, 560)$. Find an equation for the parabola and find the vertex.

   (c) A parabola has vertex $(-8, -4)$ and passes through the point $(6, 192)$. Find an equation for the parabola and find the intercepts.

5. When Linda sells boxes of cookies, she finds that when she prices each box at $\$x$, she manages to sell $56 - 4x$ boxes.

   (a) What price should she set in order to maximize revenue?

   (b) It costs Linda \$2 to make each box. What price should she set to maximize profit?

6. (a) Write down an equation for a circle centered at $(-5, 1)$ with radius 7.

   (b) Find the center and radius of the circle with equation $x^2 + y^2 + 6x + 10y + 3 = 0$.

   (c) Find the value of E for which the graph of the equation $x^2 + y^2 - 4x - 4y = E$ is a single point. What is that point?

7. Find the two points where the circles $x^2 + y^2 = 125$ and $x^2 - 16x + y^2 - 12y = 25$ intersect.

5.2 Challenge problems

8. (a) Write down a quadratic with real coefficients whose discriminant is a perfect square integer but whose roots are irrational.

   (b) Write down a quadratic with complex coefficients whose discriminant is a negative real number but whose roots are real.

9. Every parabola has a point F, called the *focus*, and a line ℓ, called the *directrix*, with the property that the points P on the parabola are precisely those for which the distance from P to F is the same as the distance from P to ℓ. The line through F perpendicular to ℓ is the axis of symmetry of the parabola.

   (a) Let F be the point $(1, 3)$ and let ℓ be the line $y = -1$. For the parabola with focus F and directrix ℓ, what point is the vertex?

   (b) Find the two points on the line $y = 3$ that lie on the parabola.

   (c) Write down an equation for the parabola.

10. (Even more challenging than usual) Find all real numbers x for which

$$\sqrt{5 - x} = 5 - x^2.$$

5.3 Answers

1. (a) -8 and 10
   (b) $1 \pm \frac{\sqrt{3}}{2}$
   (c) 5
   (d) $-1/2$ and $-1/3$
   (e) $\frac{-1 \pm \sqrt{39} i}{2}$
2. (a) $(x-9)^2$
   (b) $(x+8)^2 - 5$
   (c) $-6(x-2)^2 + 29$
   (d) $9\left(x - \frac{7}{18}\right)^2 + \frac{131}{36}$
   (e) $-\frac{1}{2s}(x + 2\pi st)^2 + 4\pi^2 st^2$
3. In this problem, we are interested in the discriminant $\Delta = 8^2 - 4 \cdot 1 \cdot c = 64 - 4c$. Note that all coefficients are real.
   (a) We have two non-real roots if and only if $\Delta < 0$, so
   $$64 - 4c < 0 \implies \boxed{c > 16}.$$
   (b) We have one real root if and only if $\Delta = 0$, so
   $$64 - 4c = 0 \implies \boxed{c = 16}.$$
   (c) We have two real roots if and only if $\Delta > 0$, so
   $$64 - 4c > 0 \implies \boxed{c < 16}.$$
   (d) All coefficients are integers, so we get two rational roots if and only if Δ is a positive perfect square. Here we have that Δ is even and $\Delta < 64$ since c must be a positive integer, so the solutions come from setting
   $$\begin{aligned} 64 - 4c = 4 &\implies c = \boxed{15}, \\ 64 - 4c = 16 &\implies c = \boxed{12}, \\ 64 - 4c = 36 &\implies c = \boxed{7}. \end{aligned}$$
4. (a) x-intercept $(-9, 0)$; y-intercept $(0, 81)$; vertex $(-9, 0)$
   (b) equation $y = -7(x+8)(x-10)$; vertex $(1, 567)$
   (c) equation $y = (x+8)^2 - 4$; x-intercepts $(-10, 0)$ and $(-6, 0)$; y-intercept $(0, 60)$

5. (a) Linda's revenue, in dollars, is $x(56-4x)$. This is maximised when the value of x is the vertex of the parabola $y = x(56-4x)$. The quadratic here has roots $x = 0$ and $x = 14$, so the vertex has x-coordinate $(0+14)/2 = \boxed{7}$.

   (b) Linda's profit, in dollars, is $(x-2)(56-4x)$, as we profit $x-2$ from each box. This time, the vertex has x-coordinate $(2+14)/2 = \boxed{8}$.

6. (a) $(x+5)^2 + (y-1)^2 = 49$

   (b) center $(-3, -5)$; radius $\sqrt{31}$

   (c) Completing the square, the equation becomes

$$(x-2)^2 + (y-2)^2 = E + 8.$$

   This has exactly one solution precisely when $E + 8 = 0$, in which case $\boxed{E = -8}$. The lone solution in that case is the point $\boxed{(2, 2)}$.

7. Subtracting the two given equations from each other yields

$$16x + 12y = 100, \tag{$\dagger$}$$

or $y = -\frac{4}{3}x + \frac{25}{3}$. Substituting this into the first given equation,

$$\begin{aligned}
x^2 + \left(-\frac{4}{3}x + \frac{25}{3}\right)^2 &= 125, \\
9x^2 + (-4x + 25)^2 &= 1125, \\
9x^2 + 16x^2 - 200x + 625 &= 1125, \\
25x^2 - 200x &= 500, \\
x^2 - 8x - 20 &= 0, \\
(x-10)(x+2) &= 0.
\end{aligned}$$

Thus the two possible x-coordinates for a point of intersection are $x = 10$ and $x = -2$. Substituting $x = 10$ into ($\dagger$) yields $y = -5$, while substituting $x = -2$ yields $y = 11$, so the two points of intersection are $\boxed{(10, -5)}$ and $\boxed{(-2, 11)}$.

8. (a) $x^2 - 2\sqrt{2}x + 1$

   (b) $ix^2 - i$

9. (a) The line perpendicular to ℓ through F is $x = 1$, so it hits ℓ at the point $A = (1, -1)$. The vertex is then the midpoint of F and A, which is $\boxed{(1, 1)}$.

   (b) Let $P = (x, 3)$ be a point on the parabola. Then the distance from P to ℓ is $3 - (-1) = 4$, while $PF = |x-1|$, so we have

$$|x-1| = 4.$$

   The two solutions are $x = -3$ and $x = 5$, so the points are $\boxed{(-3, 3)}$ and $\boxed{(5, 3)}$.

(c) Since the vertex is $(1,1)$, we can write down an equation

$$y = a(x-1)^2 - 1$$

for some constant a. Plugging in the point $(5,3)$ gives us

$$3 = a(5-1)^2 - 1,$$

from which it follows that $a = 1/4$. Hence our equation is $\boxed{y = \frac{1}{4}(x-1)^2 - 1}$.

10. Squaring both sides,

$$5 - x = 5^2 - 2 \cdot 5 \cdot x^2 + x^4,$$

which rearranges to

$$5^2 - (2x^2+1) \cdot 5 + (x^4 + x) = 0.$$

Using the quadratic formula to solve for 5 (?!?!),

$$\begin{aligned} 5 &= \frac{(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4 \cdot 1 \cdot (x^4+x)}}{2} \\ &= \frac{(2x^2+1) \pm \sqrt{(4x^4+4x^2+1) - (4x^4+4x)}}{2} \\ &= \frac{(2x^2+1) \pm \sqrt{4x^2-4x+1}}{2} \\ &= \frac{(2x^2+1) \pm (2x-1)}{2}. \end{aligned}$$

We now consider two cases based on the $\pm$ sign. For the "plus" case,

$$\begin{aligned} \frac{(2x^2+1) + (2x-1)}{2} &= 5, \\ x^2 + x &= 5, \\ x^2 + x - 5 &= 0, \\ x &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-5)}}{2} \\ &= \frac{-1 \pm \sqrt{21}}{2}. \end{aligned}$$

The conditions for x to be a valid solution to the initial equation are that $x \leq 5$, so that $\sqrt{5-x}$ is defined, and that $x^2 \leq 5$, so that $5 - x^2$ is non-negative. Of the two solutions found in this "plus" case, only $x = \boxed{\frac{-1+\sqrt{21}}{2}}$ satisfies this.

For the “minus” case,

$$\begin{aligned}
\frac{(2x^2+1)-(2x-1)}{2} &= 5, \\
x^2 - x + 1 &= 5, \\
x^2 - x - 4 &= 0, \\
x &= \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-4)}}{2} \\
&= \frac{1 \pm \sqrt{17}}{2}.
\end{aligned}$$

This time, the solution that satisfies the initial equation is $x = \boxed{\dfrac{1-\sqrt{17}}{2}}$.

6 Nonlinear Inequalities

6.1 Review problems

1. Express the solutions to each of the following in interval notation.
   (a) $x^2 - 9 > 0$
   (b) $x^2 + 6x - 27 \leq 0$
   (c) $2x^2 + 9x - 3 \geq 2$
   (d) $2x^2 + 5 < 6x$
2. Express the solutions to each of the following in interval notation.
   (a) $\sqrt{x} < \sqrt[4]{25}$
   (b) $(x-1)(x-2)(x-3) > 0$
   (c) $\frac{x^2+x+1}{2x^2+x-15} \leq 0$
   (d) $(x+1)(x+2)(x+3)(x+4) \geq (x-1)(x-2)(x-3)(x-4)$
3. Find the area of the region in the cartesian plane consisting of all points (x, y) satisfying
$$(x-2)^2 + (y-3)^2 \geq 1 \quad \text{and} \quad x^2 + y^2 - 4x - 6y \leq 3.$$
4. Two positive real numbers have a sum of 22. What is their maximum possible product?
5. Carl has 20 feet of fencing to use to build a rectangular pen for his sheep. One side of the pen will be along his barn and thus does not require fencing, but the other three sides need to be entirely fenced off. What is the maximum possible area that Carl can fence off?
6. Find the minimum possible value of the expression
$$4x^2 + 3y^2 - 32x + 30y - 8$$
when x and y are real numbers.

6.2 Challenge problems

7. For any finite list of positive real numbers, the *harmonic mean* is computed by taking the reciprocal of the arithmetic mean of the reciprocals, i.e. for two numbers,
$$HM(x, y) = \left(\frac{x^{-1} + y^{-1}}{2}\right)^{-1}.$$
   (a) Compute the harmonic mean of 40 and 60.
   (b) Eggsy drives to work at a speed of 60 miles per hour, then drives back along the same route at a speed of 40 miles per hour. What was Eggsy's average speed for the entire round trip?

8. In class, we stated the AM-GM inequality for two positive real numbers,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Throughout this problem, assume all variables represent positive real numbers.

(a) By manipulating $\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$, prove the two-variable AM-GM inequality.

(b) Use the two-variable AM-GM inequality to prove the four-variable AM-GM inequality

$$\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}.$$

(c) Use the four-variable AM-GM inequality to prove the three-variable AM-GM inequality

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}.$$

9. A rectangular box with an open top is made with a surface area of 18 square feet. What is the maximum possible volume within the box?

10. Are there any positive real numbers a, b, c satisfying the system of equations

$$a^2+ab+b^2=9, \quad a^2+ac+c^2=16, \quad b^2+bc+c^2=64?$$

6.3 Answers

1. (a) $(-\infty, -3) \cup (3, +\infty)$
   (b) $[-9, 3]$
   (c) $(-\infty, -5] \cup [1/2, +\infty)$
   (d) no solutions ($\emptyset$)

2. (a) $[0, 5)$
   (b) $(1, 2) \cup (3, +\infty)$
   (c) The discriminant of x^2+x+1 is $1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$, so x^2+x+1 is always positive or always negative. At $x = 0$, the numerator evaluates to 1, so it must always be positive. Therefore, we are looking for when the denominator is negative,

   $$2x^2 + x - 15 < 0.$$

   Factoring, we have $(2x-5)(x+3) < 0$, and the solution set is $\boxed{(-3, 5/2)}$.
   (d) Expanding both sides,

   $$\begin{aligned} x^4 + 10x^3 + 35x^2 + 40x + 24 &\geq x^4 - 10x^3 + 35x^2 - 40x + 24, \\ 20x^3 + 80x &\geq 0, \\ x^3 + 4x &\geq 0, \\ x(x^2+4) &\geq 0. \end{aligned}$$

   Since $x^2 + 4 > 0$, this is satisfied precisely when $x \geq 0$, so the solution set is $\boxed{[0, +\infty)}$.

3. Completing the square in the second inequality yields

   $$(x-2)^2 + (y-3)^2 \leq 16.$$

   Therefore, taking the two inequalities together, the region satisfying both is the region between two concentric circles centered at $(2, 3)$, one with radius 1 and one with radius 4. The area of this region (an annulus) is $\pi \cdot 4^2 - \pi \cdot 1^2 = \boxed{15\pi}$.

4. If the two numbers are x and y, then by AM-GM,

   $$\sqrt{xy} \leq \frac{x+y}{2} = 11.$$

   Hence $xy \leq \boxed{121}$, and equality is attained when $x = y = 11$.

5. Let x be the length of the pen parallel to the barn and let y be the width perpendicular to the barn. From the given conditions, $x + 2y = 20$. Then, by AM-GM,

   $$\sqrt{x \cdot 2y} \leq \frac{x+2y}{2} = 10.$$

   This means $2xy \leq 100$, so $xy \leq \boxed{50}$. Equality is attained when $x = 2y$, in which case we have $x = 10$ and $y = 5$.

6. Completing the square,

$$\begin{aligned}
4x^2 + 3y^2 - 32x + 30y - 8 &= 4[x^2 - 8x + \boxed{}] + 3[y^2 + 10y + \boxed{}] - 4\boxed{} - 3\boxed{} - 8 \\
&= 4(x-4)^2 + 3(y+5)^2 - 4 \cdot 16 - 3 \cdot 25 - 8 \\
&= 4(x-4)^2 + 3(y+5)^2 - 147.
\end{aligned}$$

Since the perfect squares are always non-negative, the minimum possible value of the given expression is $\boxed{-147}$, attained when $x = 4$ and $y = -5$.

7. (a) 48

   (b) 48 mph

8. (a) Expanding the left hand side,
$$x - 2\sqrt{xy} + y \geq 0,$$
so $x + y \geq 2\sqrt{xy}$. Dividing by 2 on both sides yields AM-GM for two variables.

   (b) Let $a = (w + x)/2$ and $b = (y + z)/2$. Then using AM-GM for two variables,
$$\begin{aligned}
\frac{w+x+y+z}{4} &= \frac{a+b}{2} \\
&\geq \sqrt{ab} \\
&= \sqrt{\frac{w+x}{2} \cdot \frac{y+z}{2}} \\
&\geq \sqrt{\sqrt{wx}\sqrt{yz}} \\
&= \sqrt[4]{wxyz}.
\end{aligned}$$

   (c) Using AM-GM for four variables,
$$\begin{aligned}
\frac{x+y+z}{3} &= \frac{x+y+z+\frac{x+y+z}{3}}{4} \\
&\geq \sqrt[4]{xyz \cdot \frac{x+y+z}{3}} \\
&= (xyz)^{1/4}\left(\frac{x+y+z}{3}\right)^{1/4}.
\end{aligned}$$
This gives us
$$\left(\frac{x+y+z}{3}\right)^{3/4} \geq (xyz)^{1/4},$$
and raising both sides to the 4/3 power proves three-variable AM-GM.

9. Let ℓ be the length, w be the width, and h be the height of this box, and suppose the open face is one of the $\ell \times w$ faces. Then
$$\ell w + 2\ell h + 2wh = 18.$$

By three-variable AM-GM,

$$\sqrt[3]{(\ell w)(2\ell h)(2wh)} \le \frac{\ell w + 2\ell h + 2wh}{3} = 6.$$

Cubing both sides gives us $4\ell^2 w^2 h^2 \le 216$, so $\ell^2 w^2 h^2 \le 54$ and $\ell wh \le \sqrt{54} = \boxed{3\sqrt{6}}$.

10. From the third equation,

$$b + c = \sqrt{b^2 + 2bc + c^2} \ge \sqrt{b^2 + bc + c^2} = 8.$$

However, from the first and second equations,

$$b \le \sqrt{a^2 + ab + b^2} = 3 \quad \text{and} \quad c \le \sqrt{a^2 + ac + c^2} = 4.$$

We cannot simultaneously have $b \le 3$, $c \le 4$, and $b + c \ge 8$, so there are no solutions in positive real numbers to the given system of equations.