Precalculus Practice Problems: Midterm 1

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Contents

1 Trig (I): Right Triangle and Unit Circle

1.1 Review problems

1. *Unit conversions for angles.*

   (a) 360 degrees to radians

   (b) π radians to degrees

   (c) 60 degrees to radians

   (d) $3\pi/4$ radians to degrees

   (e) $\pi/5$ degrees to radians

2. *Trig functions as ratios of lengths.* Let ABC be a triangle with a right angle at B. Suppose $AB = 8$ and $BC = 15$.

   (a) Evaluate $\tan A$ and $\cot A$.

   (b) Find the length of AC.

   (c) Evaluate $\sin A$, $\cos A$, $\sec A$, and $\csc A$.

3. *Using one trig function to compute another.* Throughout, assume θ is acute.

   (a) If $\sin\theta = 1/3$, what is $\cos\theta$?

   (b) If $\sec\theta = \sqrt{10}$, what is $\tan\theta$?

   (c) If $\tan\theta = 2/5$, what is $\csc\theta$?

4. *Important acute angles.* Fill in the table below.

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
30							
45							
	$\pi/3$						

5. *Unit circle calculations.* Compute the following.

   (a) $\cos(0)$
   (b) $\sin(150^\circ)$
   (c) $\cos(-3\pi/4)$
   (d) $\sin(7\pi/3)$
   (e) $\cos(330^\circ)$
   (f) $\sin(-\pi/4)$

6. *Unit circle identities.* Express each of the following in terms of $\sin\theta$ and/or $\cos\theta$.

   (a) $\sin(\pi-\theta)$
   (b) $\cos(\pi-\theta)$
   (c) $\sin(\pi+\theta)$
   (d) $\cos(\pi+\theta)$
   (e) $\sin(-\theta)$
   (f) $\cos(-\theta)$
   (g) $\sin(\frac{\pi}{2}-\theta)$
   (h) $\cos(\frac{\pi}{2}-\theta)$
   (i) $\sin(\frac{\pi}{2}+\theta)$
   (j) $\cos(\frac{\pi}{2}+\theta)$

7. *Some triangle geometry.* In acute triangle ABC, it is given that $AB = 13$, that $BC = 14$, and that $\sin B = 12/13$.

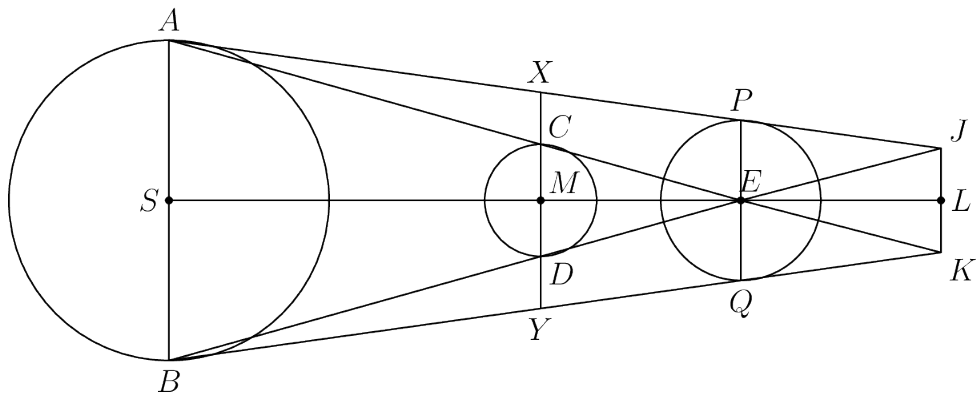
   (a) Find the area of triangle ABC.
   (b) Find the length of AC.
   (c) Find $\sin A$.

1.2 Challenge problems

8. Much of classical trigonometry was done in terms of the *chord function*, which is defined for angles $0 < \theta < 180^\circ$ as follows. Let O be the center of a circle of radius 1, and let A and B be points on the circle so that $\angle AOB = \theta$. Then $\operatorname{chd}\theta$ is defined to be the length AB.

   (a) Compute $\operatorname{chd} 90^\circ$, $\operatorname{chd} 60^\circ$, and $\operatorname{chd} 30^\circ$.
   (b) Express $\operatorname{chd}\theta$ in terms of the sine function.
   (c) Prove that $\operatorname{chd}^2\theta + \operatorname{chd}^2(180^\circ-\theta) = 4$.

9. A method for estimating the distances to the Sun and the Moon using measurements of solar and lunar eclipses dates back to the Greek astronomer Hipparchus (c.$\tilde{1}$90 BC to c.$\tilde{1}$20 BC). In this problem, we work through the relevant geometric argument.



In the diagram, points S, M, E, and L lie on a line, with S, M, and E representing the centers of the Sun, Moon, and Earth, respectively. Segments $\overline{AB}$, $\overline{CD}$, and $\overline{PQ}$ are diameters of their respective circles, all perpendicular to $\overline{SL}$. Segment $\overline{JK}$ is also perpendicular to $\overline{SL}$. Points A, X, P, J are collinear, points B, Y, Q, K are collinear, points X, C, D, Y are collinear, points A, C, E are collinear, and points B, D, E are collinear. Finally, E is the midpoint of $\overline{LM}$. Let $PE = 1$ (so lengths are in terms of Earth radii) and define

$$\ell = EM = EL, \quad s = ES, \quad \theta = \angle CEM, \quad \phi = \angle JEL.$$

(a) Show that

$$s = \frac{\ell}{(\tan\theta + \tan\phi)\ell - 1}$$

(b) (Calculator recommended) The measurements used by Hipparchus were

$$\ell \approx 67\tfrac{1}{3}, \quad \theta \approx 0.277^\circ, \quad \phi \approx 0.693^\circ.$$

Given these measurements, what value do we get for s?

(c) (Calculator recommended) Currently, our measurements for the same quantities are

$$\ell = 60.268, \quad \theta \approx 0.267^\circ, \quad \phi \approx 0.746^\circ.$$

Given these measurements, what value do we get for s?

For reference, the true value of s is $s \approx 23{,}455$, so some of the approximations made in order to set up the diagram turn out to be a substantial source of error.

10. A *Pythagorean triple* is a triple (X, Y, Z) of positive integers for which $X^2 + Y^2 = Z^2$. Note that if (X, Y, Z) is a Pythagorean triple, then $(x, y) = (X/Z, Y/Z)$ is a point on the unit circle whose coordinates are rational numbers.

    (a) Let $O = (-1, 0)$. If $P \neq O$ has rational coordinates and lies on the unit circle, then the slope of $\overline{OP}$ is rational. Conversely, show that if ℓ is a line passing through O which has rational slope, then the other point $P \neq O$ at which ℓ intersects the unit circle must have rational coordinates.

    (b) Use part (a) to show that aside from O, every point on the unit circle with rational coordinates can be written in the form $\left(\dfrac{n^2 - m^2}{n^2 + m^2}, \dfrac{2mn}{n^2 + m^2}\right)$ for integers m and n.

    A result going back to Euclid states that every *primitive* Pythagorean triple, meaning a Pythagorean triple (X, Y, Z) where $\gcd(X, Y, Z) = 1$, can be written as either

    $$X = n^2 - m^2, \quad Y = 2mn, \quad Z = n^2 + m^2$$

    with $\gcd(m, n) = 1$ and $m + n$ odd, or in the same form with X and Y swapped. The result of part (b) gets most of the way to proving this, with a subtle detail or two to fill in.