

Precalculus Practice Problems: Final

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The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

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1 Matrices in 2D

1.1 Review Problems

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector.

We also let $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.
 - (a) $\mathbf{u} + \mathbf{v}$
 - (b) $2\mathbf{v}$
 - (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$
 - (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$
 - (e) The angle between \mathbf{u} and \mathbf{v} (in terms of an inverse trig function)
 - (f) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{u}}(\mathbf{v})$
2. *Applying matrices to vectors.* Let $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.
 - (a) Compute $\mathbf{A}\mathbf{v}$
 - (b) Find a vector \mathbf{u} for which $\mathbf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.
3. *Matrix operations.* Let $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.
 - (a) $\mathbf{A} + \mathbf{B}$
 - (b) $-3\mathbf{A}$
 - (c) \mathbf{AB}
 - (d) \mathbf{BA}
 - (e) \mathbf{B}^T (the transpose of \mathbf{B})
4. *Geometric transformations.* Write down matrices for each of the following.
 - (a) Dilation about the origin by a factor of 4
 - (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2
 - (c) Rotation about the origin by $\pi/4$ counterclockwise
 - (d) Projection onto the line $y = (3/2)x$
 - (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

- (a) $\det A$ and $\det B$
- (b) $\det(AB)$
- (c) $\det(A^T)$
- (d) $\det(A + B)$
- (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

- (a) A^{-1} and B^{-1}
- (b) $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$
- (c) $(AB)^{-1}$
- (d) $(A^T)^{-1}$
- (e) $(A + B)^{-1}$
- (f) $\det(A^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

- (a) Describe the image of the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.
- (b) By what factor does a horizontal shear multiply areas?
- (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

(The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\operatorname{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

- (a) For the matrices \mathbf{A}, \mathbf{B} in problems 3, 5, 6, compute $\operatorname{tr} \mathbf{A}$, $\operatorname{tr} \mathbf{B}$, $\operatorname{tr}(\mathbf{A} + \mathbf{B})$, and $\operatorname{tr}(\mathbf{AB})$.
 - (b) Show that for any 2×2 matrices \mathbf{P} and \mathbf{Q} , we have $\operatorname{tr}(\mathbf{PQ}) = \operatorname{tr}(\mathbf{QP})$.
 - (c) In general, must it be true that $\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{ACB})$?
9. Two matrices \mathbf{A}, \mathbf{B} are **similar**, written $\mathbf{A} \sim \mathbf{B}$, if there is an invertible \mathbf{P} with $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.
- (a) Show that the only matrix similar to \mathbf{I} is \mathbf{I} .
 - (b) Show that if $\mathbf{A} \sim \mathbf{B}$, then $\det \mathbf{A} = \det \mathbf{B}$ and $\operatorname{tr} \mathbf{A} = \operatorname{tr} \mathbf{B}$.
 - (c) Let $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $\mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $\mathbf{D} \sim \mathbf{A}$. Find \mathbf{D} .
10. If \mathbf{A} is a square matrix, the **characteristic polynomial** of \mathbf{A} is defined by

$$f_{\mathbf{A}}(X) = \det(\mathbf{A} - X\mathbf{I}).$$

- (a) Compute the characteristic polynomial $f_{\mathbf{A}}(X)$ of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.
- (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_{\mathbf{A}}(X)$.
- (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $\mathbf{A}\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for $j = 1, 2$. (In general, if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call \mathbf{v} an **eigenvector** of \mathbf{A} corresponding to the **eigenvalue** λ .)
- (d) Let \mathbf{P} be the matrix whose columns are \mathbf{v}_1 and \mathbf{v}_2 . Compute $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.
- (e) Find \mathbf{A}^{100} .
- (f) *Cayley-Hamilton theorem.* Suppose $f_{\mathbf{A}}(X) = a_0 + a_1X + a_2X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$
(b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$
(c) Both are 5. In general, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
(d) $\|\mathbf{u}\| = \sqrt{13}$
 $\|\mathbf{v}\| = \sqrt{17}$
 $\|\mathbf{u} + \mathbf{v}\| = \sqrt{40} = 2\sqrt{10}$
(e) $\arccos\left(\frac{5}{\sqrt{221}}\right)$
(f) $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$
 $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$
2. (a) $\begin{pmatrix} 18 \\ 7 \end{pmatrix}$
(b) Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 4b \\ a + b \end{pmatrix},$$

so we require $2a + 4b = 5$ and $a + b = 2$. The solution to this system is that $a = 3/2$ and $b = 1/2$, so then $\mathbf{u} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.

Remark: We can also compute $\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$ once we have \mathbf{A}^{-1} (see Problem 6).

3. (a) $\begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix}$
(b) $\begin{pmatrix} -6 & -12 \\ -3 & -3 \end{pmatrix}$
(c) $\begin{pmatrix} 14 & -20 \\ 2 & -3 \end{pmatrix}$
(d) $\begin{pmatrix} -2 & -8 \\ 3 & 13 \end{pmatrix}$
(e) $\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$
4. (a) $4\mathbf{I} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
(b) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

- (c) $\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$
- (d) $P = \begin{pmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{pmatrix}$
- (e) $2P - I = \begin{pmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{pmatrix}$
5. (a) $\det A = -2$
 $\det B = 1$
- (b) $\det(AB) = \det(A) \cdot \det(B) = -2$
- (c) $\det(A^T) = \det A = -2$
- (d) $\det(A + B) = \det \begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix} = -42$
- (e) $|\det A| \cdot (\text{unit circle area}) = 2\pi$
6. (a) $A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 1/2 & -1 \end{pmatrix}$
 $B^{-1} = \frac{1}{\det B} \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix}$
- (b) $A^{-1}B^{-1} = \begin{pmatrix} -13/2 & -4 \\ 3/2 & 1 \end{pmatrix}$
 $B^{-1}A^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$
- (c) $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$
- (d) $(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} -1/2 & 1/2 \\ 2 & -1 \end{pmatrix}$
- (e) $(A + B)^{-1} = \frac{1}{\det(A + B)} \begin{pmatrix} -6 & -8 \\ -6 & -1 \end{pmatrix} = \begin{pmatrix} 1/7 & 4/21 \\ 1/7 & 1/42 \end{pmatrix}$
- (f) $\det(A^{-1}) = 1/\det A = -1/2$
7. (a) A parallelogram with vertices $(0, 0), (1, 0), (3, 1), (2, 1)$
- (b) $\det \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = 1$
- (c) Multiplying the right two matrices, $\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & ak \\ 0 & b \end{pmatrix}$. Looking at the image of vector $\hat{\mathbf{i}}$, we need $\begin{pmatrix} a \\ 0 \end{pmatrix}$ to rotate to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. This can be achieved with a rotation by $\theta = \arccos(4/5)$ and $a = 5$. To find b , taking the determinant on both sides and noting that rotations have determinant 1, we require $ab = 25$, so $b = 5$. Finally, to get k , we need $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ to rotate to $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$. Comparing lengths and noting that $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ must be in the first quadrant, $k = 1$.

8. (a) $\text{tr } \mathbf{A} = 3$
 $\text{tr } \mathbf{B} = -10$
 $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr } \mathbf{A} + \text{tr } \mathbf{B} = -7$
 $\text{tr}(\mathbf{AB}) = 11$

- (b) Let $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$\mathbf{PQ} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \quad \text{and} \quad \mathbf{QP} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix},$$

so $\text{tr}(\mathbf{PQ}) = \text{tr}(\mathbf{QP}) = ae + bg + cf + dh$.

- (c) In general, the answer is **no**. For example, let

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{ABC} &= \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}, \\ \mathbf{ACB} &= \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 3 \end{pmatrix}, \end{aligned}$$

so $\text{tr}(\mathbf{ABC}) = 6$ while $\text{tr}(\mathbf{ACB}) = 7$.

9. (a) Suppose $\mathbf{I} \sim \mathbf{B}$. Then there is an invertible matrix \mathbf{P} such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{I}\mathbf{P}$, but the right hand side simplifies to $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$.
(b) If $\mathbf{A} \sim \mathbf{B}$ with $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, then

$$\det \mathbf{B} = \det(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \det(\mathbf{P})^{-1} \cdot \det \mathbf{A} \cdot \det \mathbf{P} = \det \mathbf{A}.$$

For the trace, Problem 8b gives us

$$\text{tr } \mathbf{B} = \text{tr}(\mathbf{P}^{-1}(\mathbf{A}\mathbf{P})) = \text{tr}((\mathbf{A}\mathbf{P})\mathbf{P}^{-1}) = \text{tr } \mathbf{A}.$$

- (c) We have $\det \mathbf{A} = 4$ and $\text{tr } \mathbf{A} = 5$, so

$$\det \mathbf{D} = d_1 d_2 = 4 \quad \text{and} \quad \text{tr } \mathbf{D} = d_1 + d_2 = 5.$$

This is satisfied by $d_1 = 4$ and $d_2 = 1$, so $\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

10. (a) We compute

$$f_{\mathbf{A}}(X) = \det(\mathbf{A} - X\mathbf{I}) = \det \begin{pmatrix} 3-X & 1 \\ 2 & 2-X \end{pmatrix} = (3-X)(2-X) - 2 = X^2 - 5X + 4.$$

- (b) The roots are $\lambda_1 = 4$ and $\lambda_2 = 1$.

- (c) Note that the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, which has a non-zero solution if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Moreover, we can use this version of the equation to find solutions more easily.

For $\lambda_1 = 4$, we have $\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$, so we can take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v} = \mathbf{0}$.

For $\lambda_2 = 1$, we have $\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, so we can take $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = \mathbf{0}$.

- (d) Here $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, so then $\mathbf{P}^{-1} = -\frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. We compute

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 12 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Remark 1: If we produced different valid choices of \mathbf{v}_1 and \mathbf{v}_2 from part (c), \mathbf{P} and \mathbf{P}^{-1} would change, but the end result would be the same. If we swapped the order of the columns of \mathbf{P} , then we would swap the order of the diagonal entries correspondingly.

Remark 2: The fact that we got a diagonal matrix with entries λ_1, λ_2 , the same one as in Problem 9c, is not a coincidence. The process we went through in this problem is called **diagonalisation**. (Not all $n \times n$ matrices are diagonalisable, but one sufficient condition for diagonalisability is that the characteristic polynomial has n distinct roots.)

- (e) Let $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, so then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then

$$\begin{aligned} \mathbf{A}^{100} &= \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdot \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdot \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdot \dots \cdot \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \cdot \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\ &= \mathbf{P}\mathbf{D} \cdot \mathbf{D} \cdot \mathbf{D} \cdot \dots \cdot \mathbf{D} \cdot \mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}^{100}\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4^{100} & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{-1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 4^{100} & 1 \\ 4^{100} & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 \cdot 4^{100} + 1 & 4^{100} - 1 \\ 2 \cdot 4^{100} - 2 & 4^{100} + 2 \end{pmatrix}. \end{aligned}$$

- (f) Here $(a_0, a_1, a_2) = (4, -5, 1)$, so

$$\begin{aligned} a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -15 & -5 \\ -10 & -10 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -5 \\ -10 & -6 \end{pmatrix} + \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

2 Vectors in 3D

Problems and solutions can be found at <https://azhou5849.github.io/teaching/>

2.1 Review Problems

1. *Operations.* Let

$$\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}.$$

Compute each of the following. (Write “Err” or similar for any undefined expressions.)

- | | |
|---|--|
| (a) $2\mathbf{a} + \mathbf{b} - \mathbf{c}$ | (d) $\mathbf{a} \times \mathbf{b}$ |
| (b) $\ \mathbf{a}\ + \ \mathbf{b}\ - \ \mathbf{a} + \mathbf{b}\ $ | (e) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ |
| (c) $\mathbf{b} \cdot \mathbf{c}$ | (f) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ |

2. *Distances and spheres.*

- (a) Find the distance between the points $(2, -5, -2)$ and $(1, -5, 0)$.
- (b) Write down an equation for the sphere with center $(5, -1, 0)$ and radius 5.
- (c) Find the center and radius of the sphere with equation

$$x^2 + y^2 + z^2 - 2x + 8y + 8z + 17 = 0.$$

3. *Angles.* Let $A = (-20, -2, 1)$, $B = (-15, 3, 21)$, and $C = (-16, 14, 5)$. Compute $\angle BAC$.

4. *Cross products.* Let $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$.

- (a) Find all vectors orthogonal to both \mathbf{u} and \mathbf{v} with norm 1.
- (b) Find the area of the parallelogram with vertices at $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$.
- (c) Let θ be the angle between \mathbf{u} and \mathbf{v} . Compute $\sin \theta$.

5. *Planes.* Let $A = (4, -5, 5)$, $B = (-2, 5, -5)$, and $C = (3, -3, -3)$. Find an equation for the plane passing through A , B , and C

- (a) in parametric form;
- (b) in cartesian form $ax + by + cz = d$.

Then find a parametric form for the intersection of this plane and the plane $x + 2y + 3z = 4$.

6. *Projections and reflections.*

- (a) What point on the line through $(-5, 0, -2)$ and $(2, 5, 2)$ is closest to $(3, 1, -4)$?
- (b) Find the reflection of the point $P = (4, -4, 5)$ across the plane $-4x + 4y + 3z = 3$.
- (c) (*) Let \mathcal{C} be the circle centered at $(0, 0, 1)$ of radius 1 lying in the plane $z = 1$ and let \mathcal{P} be the plane passing through the origin as well as the points $(5, 1, 1)$ and $(1, 3, 1)$. What is the shortest possible distance between a point on \mathcal{C} and a point on \mathcal{P} ?

7. *Using cross products in 2D problems.*

- (a) Let ABC be a triangle in the xy -plane with area 14. If the points A, B, C are listed in clockwise order going around the triangle, what is $\vec{AB} \times \vec{AC}$?
- (b) (*) Let $ABCD$ be a convex quadrilateral and let points P and Q lie on segments \overline{AB} and \overline{CD} respectively so that $AP/AB = CQ/CD$. Let R be the intersection of \overline{AQ} and \overline{PD} and let S be the intersection of \overline{BQ} and \overline{PC} . Show that

$$[PSQR] = [ARD] + [BCS].$$

2.2 Challenge Problems

- 8.
- 9.
- 10.

2.3 Answers

1. (a) $2\mathbf{a} + \mathbf{b} - \mathbf{c} = \begin{pmatrix} 2(-2) + 3 - (-1) \\ 2(-1) + 0 - 1 \\ 2(2) + (-4) - 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -5 \end{pmatrix}$
- (b) $\|\mathbf{a}\| + \|\mathbf{b}\| - \|\mathbf{a} + \mathbf{b}\| = 3 + 5 - \left\| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\| = 8 - \sqrt{6}$
- (c) $\mathbf{b} \cdot \mathbf{c} = 3 \cdot (-1) + 0 \cdot 1 + (-4) \cdot 5 = -23$
- (d) $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (-1) \cdot (-4) - 2 \cdot 0 \\ 2 \cdot 3 - (-2) \cdot (-4) \\ (-2) \cdot 0 - (-1) \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$
- (e) Err ($\mathbf{b} \cdot \mathbf{c}$ produces a real number, which cannot be dotted with \mathbf{a})
- (f) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} (-2) \cdot 5 - 3 \cdot 1 \\ 3 \cdot (-1) - 4 \cdot 5 \\ 4 \cdot 1 - (-2) \cdot (-1) \end{pmatrix} = \begin{pmatrix} -13 \\ -23 \\ 2 \end{pmatrix}$
2. (a) $\sqrt{(2-1)^2 + ((-5) - (-5))^2 - ((-2) - 0)^2} = \sqrt{5}$
- (b) $(x-5)^2 + (y+1)^2 + z^2 = 25$
- (c) We complete the square:

$$\begin{aligned}
 x^2 + y^2 + z^2 - 2x + 8y + 8z + 17 &= 0, \\
 (x^2 - 2x) + (y^2 + 8y) + (z^2 + 8z) &= -17, \\
 (x^2 - 2x + 1) + (y^2 + 8y + 16) + (z^2 + 8z + 16) &= -17 + 1 + 16 + 16 \\
 (x-1)^2 + (y+4)^2 + (z+4)^2 &= 16.
 \end{aligned}$$

This is a sphere with center $(1, -4, -4)$ and radius $\sqrt{16} = 4$.

3. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$, so that $\theta = \angle BAC$ is the angle between \mathbf{u} and \mathbf{v} . We compute

$$\begin{aligned}
 \mathbf{u} &= \begin{pmatrix} (-15) - (-20) \\ 3 - (-2) \\ 21 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 20 \end{pmatrix}, \\
 \mathbf{v} &= \begin{pmatrix} (-16) - (-20) \\ 14 - (-2) \\ 5 - 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ 4 \end{pmatrix},
 \end{aligned}$$

so then

$$\begin{aligned}
 \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{5 \cdot 4 + 5 \cdot 16 + 20 \cdot 4}{\sqrt{5^2 + 5^2 + 20^2} \cdot \sqrt{4^2 + 16^2 + 4^2}} \\
 &= \frac{180}{5\sqrt{1^2 + 1^2 + 4^2} \cdot 4\sqrt{1^2 + 4^2 + 1^2}} = \frac{9}{18} = \frac{1}{2}.
 \end{aligned}$$

This means that $\theta = \pi/3 = 60^\circ$.

4. (a) Any vector orthogonal to both \mathbf{u} and \mathbf{v} must be parallel to

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \cdot (-5) - (-5) \cdot 2 \\ (-5) \cdot (-3) - 3 \cdot (-5) \\ 3 \cdot 2 - 3 \cdot (-3) \end{pmatrix} = \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix}.$$

The two vectors parallel to \mathbf{n} of length 1 are

$$\frac{\pm 1}{\|\mathbf{n}\|} \mathbf{n} = \frac{\pm 1}{\sqrt{(-5)^2 + 30^2 + 15^2}} \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix} = \frac{\pm 1}{5\sqrt{46}} \begin{pmatrix} -5 \\ 30 \\ 15 \end{pmatrix} = \frac{\pm 1}{\sqrt{46}} \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

- (b) Since $\mathbf{u} = \mathbf{v} + (\mathbf{u} - \mathbf{v})$, this parallelogram is the one defined by \mathbf{v} and $\mathbf{u} - \mathbf{v}$. Its area is

$$\|\mathbf{v} \times (\mathbf{u} - \mathbf{v})\| = \|\mathbf{v} \times \mathbf{u} - \mathbf{v} \times \mathbf{v}\| = \|\mathbf{v} \times \mathbf{u}\| = \|\mathbf{n}\| = 5\sqrt{46}.$$

- (c) We compute

$$\sin \theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{5\sqrt{46}}{\sqrt{3^2 + 3^2 + (-5)^2} \cdot \sqrt{(-3)^2 + 2^2 + (-5)^2}} = \frac{5\sqrt{23}}{\sqrt{817}}.$$

5. (a) If $P = (x, y, z)$ is an arbitrary point in the plane, then there exist s and t for which

$$\vec{P} = \vec{A} + s(\vec{AB}) + t(\vec{AC}) = \begin{pmatrix} 4 \\ -5 \\ 5 \end{pmatrix} + s \begin{pmatrix} -6 \\ 10 \\ -10 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}.$$

- (b) A normal vector to the plane is given by

$$\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} 10 \cdot (-8) - (-10) \cdot 2 \\ (-10) \cdot (-1) - (-6) \cdot (-8) \\ (-6) \cdot 2 - 10 \cdot (-1) \end{pmatrix} = \begin{pmatrix} -60 \\ -38 \\ -2 \end{pmatrix}.$$

Therefore, an equation for the plane is

$$0 = \mathbf{n} \cdot (\vec{P} - \vec{A}) = -60(x - 4) - 38(y + 5) - 2(z - 5),$$

which can be rearranged to $30x + 19y + z = 30$.

To find the intersection of this plane with $x + 2y + 3z = 4$, we eliminate x to get

$$\begin{aligned} 30(x + 2y + 3z) - (30x + 19y + z) &= 30 \cdot 4 - 30, \\ 41y + 89z &= 90. \end{aligned}$$

If $z = t$, then $y = \frac{90}{41} - \frac{89}{41}t$ and

$$x = 4 - 2y - 3z = 4 - \left(\frac{90}{41} - \frac{89}{41}t\right) - 3t = -\frac{16}{41} + \frac{55}{41}t.$$

In vector parametric form,

$$\vec{P} = \begin{pmatrix} -16/41 \\ 90/41 \\ 0 \end{pmatrix} + t \begin{pmatrix} 55/41 \\ -89/41 \\ 1 \end{pmatrix}.$$

6. (a) Translating $(-5, 0, -2)$ to the origin, the problem is equivalent to finding the point on the line generated by $\mathbf{u} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}$ closest to $\mathbf{v} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$. This is

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{8 \cdot 7 + 1 \cdot 5 + (-2) \cdot 4}{7 \cdot 7 + 5 \cdot 5 + 4 \cdot 4} \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} = \frac{53}{90} \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}.$$

Translating back, the desired point is $\left(\frac{53}{90} \cdot 7 - 5, \frac{53}{90} \cdot 5, \frac{53}{90} \cdot 4 - 2 \right) = \left(\frac{-79}{90}, \frac{53}{18}, \frac{16}{45} \right)$.

- (b) Let Q be the desired reflection. Since \overrightarrow{PQ} is normal to the plane, we can write

$$\overrightarrow{Q} = \overrightarrow{P} + t \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 - 4t \\ -4 + 4t \\ 5 + 3t \end{pmatrix}$$

for some value of t . The midpoint of \overline{PQ} lies on the plane, so

$$(-4) \cdot \frac{4 + (4 - 4t)}{2} + 4 \cdot \frac{-4 + (-4 + 4t)}{2} + 3 \cdot \frac{5 + (5 + 3t)}{2} = 3.$$

Solving this equation yields $t = 40/41$ and $Q = \left(\frac{4}{41}, \frac{-4}{41}, \frac{325}{41} \right)$.

- (c) A parameterization for \mathcal{C} is given by $\mathbf{v}(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$. The shortest possible distance between an arbitrary point on \mathcal{C} and a point on \mathcal{P} can be found by projecting $\mathbf{v}(\theta)$ onto a normal vector for \mathcal{P} . One such normal vector is

$$\mathbf{n} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 14 \end{pmatrix},$$

so then the distance between $\mathbf{v}(\theta)$ and the plane \mathcal{P} is

$$\|\text{proj}_{\mathbf{n}}(\mathbf{v}(\theta))\| = \frac{|\mathbf{v}(\theta) \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-2 \cos \theta - 4 \sin \theta + 14|}{6\sqrt{6}}.$$

We can write

$$2 \cos \theta + 4 \sin \theta = 2\sqrt{5} \left(\frac{1}{\sqrt{5}} \cos \theta + \frac{2}{\sqrt{5}} \sin \theta \right) = 2\sqrt{5} \sin(\phi + \theta),$$

where $\sin \phi = 1/\sqrt{5}$ and $\cos \phi = 2/\sqrt{5}$. Therefore,

$$\|\text{proj}_{\mathbf{n}}(\mathbf{v}(\theta))\| = \frac{|14 - 2\sqrt{5} \sin(\phi + \theta)|}{6\sqrt{6}} \geq \boxed{\frac{14 - 2\sqrt{5}}{6\sqrt{6}}},$$

with equality when $\sin(\phi + \theta) = 1$.

7. (a) Since ABC lies in the xy -plane, the cross product is parallel to $\hat{\mathbf{k}}$. By the right-hand rule, $\overrightarrow{AB} \times \overrightarrow{AC}$ points downward since A, B, C go around the triangle in clockwise order. The norm of $\overrightarrow{AB} \times \overrightarrow{AC}$ is twice the area of triangle ABC . Therefore, $\overrightarrow{AB} \times \overrightarrow{AC} = (0, 0, -28)$.
- (b) Without loss of generality, suppose that the vertices of $ABCD$ are in counterclockwise order and that the quadrilateral lies in the xy -plane. Then the z -coordinate of the vector

$$\overrightarrow{AP} \times \overrightarrow{AD} + \overrightarrow{QB} \times \overrightarrow{QA} + \overrightarrow{BC} \times \overrightarrow{BP} \quad (*)$$

is precisely $2([ARD] - [PSQR] + [BSC])$, so it suffices to show that $(*)$ is $\mathbf{0}$. Let $\mathbf{a} = \overrightarrow{A}$, $\mathbf{b} = \overrightarrow{B}$, etc., so then $(*)$ becomes

$$(\mathbf{p} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) + (\mathbf{b} - \mathbf{q}) \times (\mathbf{a} - \mathbf{q}) + (\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}).$$

Let $r = AP/AB = CQ/CD$. Then

$$\overrightarrow{P} = (1 - r)\mathbf{a} + r\mathbf{b} \quad \text{and} \quad \overrightarrow{Q} = (1 - r)\mathbf{c} + r\mathbf{d},$$

so

$$\begin{aligned} (\mathbf{p} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) &= r(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) \\ &= r(\mathbf{b} - \mathbf{a}) \times \mathbf{d} - r\mathbf{b} \times \mathbf{a} \\ (\mathbf{b} - \mathbf{q}) \times (\mathbf{a} - \mathbf{q}) &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{q} - \mathbf{q} \times \mathbf{a} \\ &= \mathbf{b} \times \mathbf{a} + \mathbf{q} \times (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \times \mathbf{a} + ((1 - r)\mathbf{c} + r\mathbf{d}) \times (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \times \mathbf{a} + (1 - r)\mathbf{c} \times (\mathbf{b} - \mathbf{a}) - r(\mathbf{b} - \mathbf{a}) \times \mathbf{d} \\ (\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}) &= (\mathbf{c} - \mathbf{b}) \times (1 - r)(\mathbf{a} - \mathbf{b}) \\ &= -(1 - r)\mathbf{c} \times (\mathbf{b} - \mathbf{a}) - (1 - r)\mathbf{b} \times \mathbf{a}. \end{aligned}$$

Adding these up gives $\mathbf{0}$, as required.