Algebra 1 Practice Problems II

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2023-2024

The focus of these review problems and the second midterm is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Graphing

1.1 Review problems

1. Draw a coordinate plane and label the origin and the four quadrants.

2. Let $A = (3, 1)$. Find the coordinates of each of the following:

   (a) the reflection of A across the x-axis
   (b) the reflection of A across the y-axis
   (c) the reflection of A across the line $y = x$
   (d) the rotation of A around the origin by $180°$
   (e) the rotation of A around the origin by $90°$ counterclockwise
   (f) the rotation of A around the point $(2, 2)$ by $90°$ clockwise

3. Quadrilateral $ABCD$ is positioned in the coordinate plane so that its vertices have coordinates

$$A = (5, 7); \quad B = (5, 6); \quad C = (3, 1); \quad D = (-4, -5).$$

   Points E, F, G, H are the midpoints of segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively.

   (a) Find the coordinates of E, F, G, and H.
   (b) Compute the midpoints of segments $\overline{EG}$ and $\overline{FH}$.

   To check your work, the two midpoints computed in part (b) should be the same. Doing this calculation in general (rather than with specific numbers) and finding that the midpoints of the diagonals of $EFGH$ coincide proves the following:

   The midpoints of the sides of any quadrilateral form a parallelogram.

4. Maurine needs to get from $(2, 3)$ to $(17, 11)$.

   (a) If they take the shortest path possible, how much distance would they cover?
   (b) Suppose Maurine gets distracted while pondering the meaning of life and goes from $(2, 3)$ to $(6, 6)$, then to $(11, 18)$, then to $(17, 10)$, and finally to $(17, 11)$. What is the minimum distance Maurine can cover which is consistent with this information?

5. Which of the following expressions correctly finds the slope between the points $(-1, 7)$ and $(3, -4)$? Circle all valid expressions.

$$\frac{3-(-1)}{-4-7} \qquad \frac{7-(-4)}{-1-3} \qquad \frac{-4-7}{3-(-1)} \qquad \frac{7-(-4)}{3-(-1)} \qquad \frac{-4-3}{7-(-1)}$$

6. A line is given by the point-slope form

$$y - 4 = \frac{1}{4}(x + 1).$$

   (a) Find the slope of the line and a point on the line.

(b) Put the equation in slope-intercept form and find the y-intercept of the line.

(c) Put the equation in standard form and find the x-intercept of the line.

7. (a) Of the equations

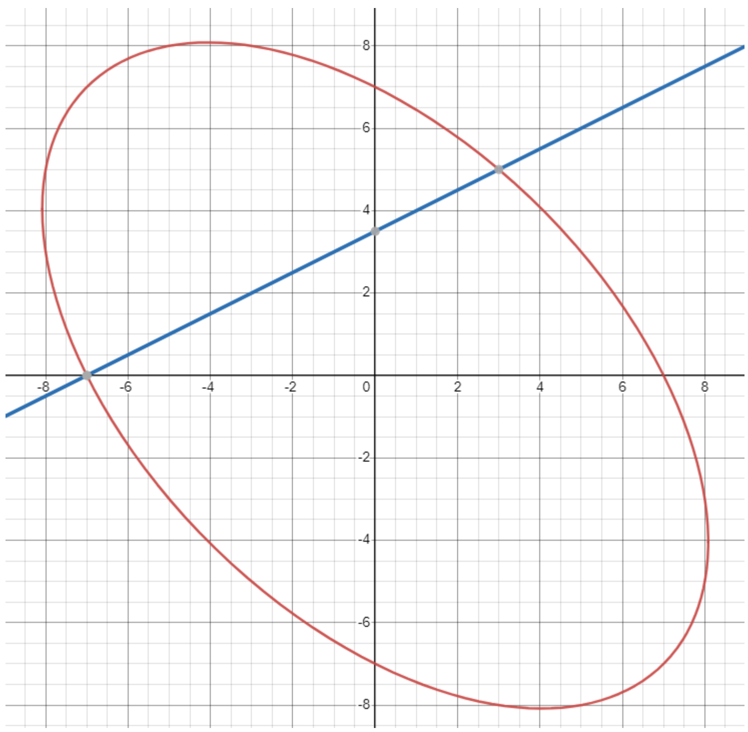
$$5x + 4y = 35; \quad (x+4)^2 + (y-1)^2 = 10; \quad x^2 + xy + y^2 = 49; \quad x - 2y = -7,$$

which one is an equation for the blue line below?

(b) Of the equations

$$5x + 4y = 35; \quad (x+4)^2 + (y-1)^2 = 10; \quad x^2 + xy + y^2 = 49; \quad x - 2y = -7,$$

which one is an equation for the red curve below?



8. A line is given by the slope-intercept form

$$y = -\frac{1}{4}x - 3.$$

(a) Find the slope of the line and the y-intercept of the line.

(b) Put the equation in standard form and find the x-intercept of the line.

(c) Find the point on the line with x-coordinate 2024, then put the equation in point-slope form using this point.

9. A line is given by the standard form

$$3x + y = -4.$$

   (a) Find the x-intercept and y-intercept of the line.
   (b) Put the equation in slope-intercept form and find the slope of the line.
   (c) Find the point on the line for which the sum of the coordinates is 10, then put the equation in point-slope form using this point.

10. A line passes through the point $(-5, 2)$ and has slope $1/2$.

   (a) Write down an equation for this line in point-slope form.
   (b) Find the slope-intercept form and the standard form of the line.

11. A line passes through the points $(-3, 4)$ and $(-3, -4)$. Find an equation for this line.

12. A line passes through the points $(-3, 3)$ and $(0, -4)$.

   (a) Find the slope of the line.
   (b) For each of the two given points, find the point-slope form of the line using that point.
   (c) Find the slope-intercept form and the standard form of the line.

   As a check of your answers, rearranging either of the point-slope equations from part (b) should give you the same slope-intercept form and standard form.

13. A line with equation $y = mx + b$ passes through the points $(5, 7)$ and $(8, -1)$. Find m and b.

14. Find the point at which the line with equation $2023x + 2022y = 2021$ intersects the line with equation $2024x + 2023y = 2022$.

15. Let ℓ be the line with equation $y = -4x - 2$ and let $P = (-4, 2)$. The point Q on line ℓ which is closest to P is the point for which $\overline{PQ} \perp \ell$. As such, if ℓ' is the line through P perpendicular to line ℓ, then Q is the intersection of ℓ and ℓ'.

   (a) Find the slope of ℓ'.
   (b) Find an equation for ℓ'.
   (c) Find the coordinates of Q.

1.2 Challenge problems

16. (Perpendicular bisector) Given any two points A and B, the *perpendicular bisector* of segment $\overline{AB}$ is the line passing through the midpoint of $\overline{AB}$ which is perpendicular to $\overline{AB}$.

   (a) Suppose $B = (14, 0)$ and $C = (5, 12)$. Find an equation for the perpendicular bisector of segment $\overline{BC}$.
   (b) Let $P = (x, y)$ be a point in the plane for which $BP = PC$. Find a linear equation relating x and y.

17. (Circumcenter) Given any triangle ABC, the perpendicular bisectors of the sides of ABC all pass through a single point O, called the *circumcenter* of triangle ABC. It is the center of the unique circle passing through all three of A, B, and C (called the *circumcircle* of ABC).

    (a) Let $A = (0, 0)$, $B = (14, 0)$, and $C = (5, 12)$. Find the circumcenter of ABC.

    (b) Find the radius of the circumcircle of ABC (i.e. the *circumradius* of ABC).

18. (Median and centroid) Given any triangle ABC, the *medians* are the lines that pass through one vertex and the midpoint of the side opposite that vertex. For example, the A-median is the line that passes through A and the midpoint of side $\overline{BC}$. The three medians of a triangle all pass through a single point G, called the *centroid* of triangle ABC.

    (a) Let $A = (0, 0)$, $B = (14, 0)$, and $C = (5, 12)$. Let D, E, and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Find the coordinates of D, E, and F.

    (b) Find equations for the medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

    (c) Find the coordinates of the centroid G of ABC. How do they relate to the coordinates of A, B, and C?

    (d) Compute the ratios AG/GD, BG/GE, and CG/GF.

19. (Altitude and orthocenter) Given any triangle ABC, the *altitudes* are the lines that pass through one vertex which are perpendicular to the opposite side. For example, the A-altitude is the line that passes through A and is perpendicular to line $\overline{BC}$. The three altitudes of a triangle all pass through a single point H, called the *orthocenter* of triangle ABC.

    (a) Let $A = (0, 0)$, $B = (14, 0)$, and $C = (5, 12)$. Find equations for the altitudes of ABC.

    (b) Find the coordinates of the orthocenter H of ABC.

20. Let $A = (1, 1)$, $B = (5, 2)$, and $C = (-4, 3)$. There are three parallelograms in the plane for which A, B, and C are three of the four vertices. What are the possible coordinates for the fourth vertex?

1.3 Answers

1. In your drawing, the origin should be at $(0,0)$, where the coordinate axes intersect. Quadrant I is in the upper right, quadrant II is in the upper left, quadrant III is in the lower left, and quadrant IV is in the lower right.

2. (a) $(3,-1)$
   (b) $(-3,1)$
   (c) $(1,3)$
   (d) $(-3,-1)$
   (e) $(-1,3)$
   (f) $(1,1)$

3. (a) $E=(5,13/2)$; $F=(4,7/2)$; $G=(-1/2,-2)$; $H=(1/2,1)$
   (b) The midpoint of $\overline{EG}$ is $(9/4,9/4)$, which is also the midpoint of $\overline{FH}$.

4. (a) $\boxed{17}$
   (b) $5+13+10+1=\boxed{29}$

5. The second and third expressions are valid.

6. (a) The slope is $\boxed{1/4}$, and one of the points on the line is $\boxed{(-1,4)}$.
   (b) Distributing the right hand side gives us

$$y-4=\frac{1}{4}x+\frac{1}{4}.$$

   Adding 4 to both sides gives us the slope-intercept form

$$\boxed{y=\frac{1}{4}x+\frac{17}{4}}.$$

   The y-intercept is $\boxed{(0,17/4)}$.

   (c) Subtracting y from both sides, then subtracting $17/4$ from both sides,

$$\frac{1}{4}x-y=-\frac{17}{4}.$$

   To clear denominators, we multiply both sides by 4 to get the standard form

$$\boxed{x-4y=-17}.$$

   When $y=0$, we have $x=-17$, so the x-intercept is $\boxed{(-17,0)}$.

7. (a) The blue line passes through $(0,3.5)$, which only satisfies the equation $\boxed{x-2y=-7}$.

(b) The first and last equation describe lines, so the answer cannot be those. Of the remaining two equations, the point $(7, 0)$ on the red curve only satisfies $\boxed{x^2 + xy + y^2 = 49}$.

8. (a) The slope is $\boxed{-1/4}$ and the y-intercept is $\boxed{(0, -3)}$.

   (b) To get standard form, we start by moving all of the variables to one side so that x has a positive coefficient and moving all of the constants to the other. Adding $\frac{1}{4}x$ to both sides gives us

$$\frac{1}{4}x + y = -3.$$

   Multiplying both sides by 4 gives the standard form

$$\boxed{x + 4y = -12}.$$

   To get the x-intercept, we plug in $y = 0$ to get $x = -12$, so the x-intercept is $\boxed{(-12, 0)}$.

   (c) Substituting $x = 2024$ into the slope-intercept form given to us,

$$y = \left(-\frac{1}{4}\right) \cdot 2024 - 3 = -509.$$

   The point-slope form with slope $-1/4$ and point $(2024, -509)$ is

$$\boxed{y + 509 = -\frac{1}{4}(x - 2024)}.$$

9. (a) Letting $x = 0$, the y-intercept is $\boxed{(0, -4)}$.

   Letting $y = 0$, the x-intercept is $\boxed{(-4/3, 0)}$.

   (b) Solving for y in terms of x, the slope-intercept form is

$$\boxed{y = -3x - 4}.$$

   The slope of the line is $\boxed{-3}$.

   (c) We want the point (x, y) to lie on the line, i.e. to satisfy $3x + y = -4$, and to have sum of coordinates 10, i.e. $x + y = 10$. Solving the system of equations, $(x, y) = \boxed{(-7, 17)}$. The desired point-slope form is then

$$\boxed{y - 17 = -3(x + 7)}.$$

10. (a) $y - 2 = \frac{1}{2}(x + 5)$

    (b) Slope-intercept form: $y = \frac{1}{2}x + \frac{9}{2}$

    Standard form: $x - 2y = -9$

11. The line through the two points is the vertical line $\boxed{x = -3}$.

12. (a) $\dfrac{3-(-4)}{(-3)-0} = \boxed{-\dfrac{7}{3}}$

(b) Using $(-3,3)$, we get $\boxed{y-3=-\dfrac{7}{3}(x+3)}$.

Using $(0,-4)$, we get $\boxed{y+4=-\dfrac{7}{3}(x-0)}$.

(c) Slope-intercept form: $\boxed{y=-\dfrac{7}{3}x-4}$

Standard form: $\boxed{7x+3y=-12}$

13. Substituting the two given points, we get the system of equations

$$5m+b=7 \qquad \text{and} \qquad 8m+b=-1.$$

Solving yields $\boxed{(m,b)=\left(-\dfrac{8}{3},\dfrac{61}{3}\right)}$.

14. The point of intersection is given by the solution to the system of equations

$$2023x+2022y=2021, \tag{1}$$
$$2024x+2023y=2022. \tag{2}$$

Taking the difference of equations $(2)-(1)$ gives us

$$x+y=1. \tag{3}$$

Then, $(1)-2022\cdot(3)$ gives us $x=-1$. Substituting back into (3) gives us $y=2$, so the point of intersection is $\boxed{(-1,2)}$.

15. (a) Given a line with slope $m \neq 0$, the slope of any line perpendicular to it is given by $-1/m$. Applying this formula with $m=-4$, the slope of ℓ, tells us the slope of ℓ' must be $\boxed{1/4}$.

(b) Using point-slope form, since ℓ' passes through P, one equation for ℓ' is

$$\boxed{y-2=\frac{1}{4}(x+4)}.$$

(c) Solving the system of equations

$$y=-4x-2 \qquad \text{and} \qquad y-2=\frac{1}{4}(x+4),$$

we find that $Q=\boxed{(-20/17,46/17)}$.

16. (a) The midpoint of $\overline{BC}$ is $(\frac{14+5}{2},\frac{0+12}{2})=(19/2,6)$ and the slope is $\frac{0-12}{14-5}=-4/3$. Thus the slope of the perpendicular bisector is $3/4$, so an equation for it is

$$\boxed{y-6=\frac{3}{4}\left(x-\frac{19}{2}\right)}.$$

(b) Using the distance formula, $BP = PC$ becomes

$$\begin{aligned}\sqrt{(x-5)^2+(y-12)^2} &= \sqrt{(x-14)^2+y^2},\\ (x-5)^2+(y-12)^2 &= (x-14)^2+y^2,\\ x^2-10x+25+y^2-24y+144 &= x^2-28x+196+y^2,\\ 18x-24y &= 27.\end{aligned}$$

Dividing by 3 gives us the standard form equation $\boxed{6x-8y=9}$.

Remark: The equations found in parts (a) and (b) are equivalent, which tells us that any point which is equidistant from B and C lies on the perpendicular bisector of $\overline{BC}$. Going in the other direction, one can show that every point on the perpendicular bisector of $\overline{BC}$ is equidistant from B and C.

17. (a) To get the circumcenter, we intersect two of the three perpendicular bisectors (we can use the third one to check our work). We found one of the perpendicular bisectors in the previous problem: the perpendicular bisector of $\overline{BC}$ is given by

$$y-6=\frac{3}{4}\left(x-\frac{19}{2}\right).$$

The perpendicular bisector of $\overline{AB}$ is the vertical line $x = 7$, so at the circumcenter,

$$y-6=\frac{3}{4}\left(7-\frac{19}{2}\right)=-\frac{15}{8}.$$

Hence $y = 33/8$ and the circumcenter is $O = \boxed{(7, 33/8)}$.

(b) The circumradius is the distance from O to any of A, B, or C. Using A,

$$OA=\sqrt{7^2+\left(\frac{33}{8}\right)^2}=\boxed{\frac{65}{8}}.$$

18. (a) $D = (19/2, 6)$; $E = (5/2, 6)$; $F = (7, 0)$

(b) AD: $y = \frac{12}{19}x$
BE: $y = -\frac{12}{23}(x-14)$
CF: $y = -6(x-7)$

(c) $G = (19/3, 4)$; the x-coordinate of G is the average of the x-coordinates of A, B, and C, and a similar statement holds for the y-coordinates.

(d) $AG/GD = BG/GE = CG/GF = 2$

19. (a) A-altitude: $y = \frac{3}{4}x$
B-altitude: $y = -\frac{5}{12}(x-14)$
C-altitude: $x = 5$

(b) $(5, 15/4)$

20. Each possible fourth vertex D comes from picking two of the three points A, B, and C to be opposite each other, and then D will be opposite the third point. We will show how to find D in the case that A and C are opposite each other, i.e. the parallelogram is $ABCD$. We will then provide answers for the case that A and B are opposite each other (so the parallelogram is $ACBD$) and the case that B and C are opposite each other (so the parallelogram is $BACD$).

    Solution 1: If $ABCD$ is a parallelogram, then $\overline{AB} \parallel \overline{CD}$, so $\overline{AB}$ and $\overline{CD}$ have the same slope. The slope of $\overline{AB}$ is $\frac{1-2}{1-5} = \frac{1}{4}$, so letting $D = (x, y)$,

    $$\frac{y-3}{x+4} = \frac{1}{4}.$$

    Multiplying both sides by $4(x+4)$ (i.e. cross multiplying) turns this into the linear equation $4y - 12 = x + 4$. To get another equation, $\overline{BC} \parallel \overline{AD}$, so $\overline{BC}$ and $\overline{AD}$ have the same slope. The slope of $\overline{BC}$ is $\frac{2-3}{5-(-4)} = -\frac{1}{9}$, so

    $$\frac{y-1}{x-1} = -\frac{1}{9}.$$

    Cross multiplying gives us $9y - 9 = 1 - x$. We now have two linear equations for the variables x and y, and solving the resulting system of equations gives us $(x, y) = \boxed{(-8, 2)}$. By similar reasoning, the point which makes $ACBD$ a parallelogram is $D = \boxed{(10, 0)}$, and the point which makes $BACD$ a parallelogram is $D = \boxed{(0, 4)}$.

    Solution 2: A geometry theorem alluded to in Problem 3 states that a quadrilateral $WXYZ$ is a parallelogram if and only if the midpoints of $\overline{WY}$ and $\overline{XZ}$ coincide. Thus for $ABCD$ to be a parallelogram with $D = (x, y)$, we need

    $$\text{midpoint of } \overline{AC} = \text{midpoint of } \overline{BD},$$
    $$\left(\frac{1+(-4)}{2}, \frac{1+3}{2}\right) = \left(\frac{5+x}{2}, \frac{2+y}{2}\right).$$

    Matching the first coordinate gives $x = -8$, and matching the second coordinate gives $y = 2$, so $D = \boxed{(-8, 2)}$. Similarly, $ACBD$ is a parallelogram when $D = \boxed{(10, 0)}$, and $BACD$ is a parallelogram when $D = \boxed{(0, 4)}$.

2 Linear Inequalities

2.1 Review problems

1. Solve each of the following inequalities, expressing the solution set in interval notation and graphing the solution set on a number line.

   (a) $x > -1$

   (b) $2x + 5 \leq 7$

   (c) $7 - 3x < 5x - 9$

   (d) $(x + 4)^2 + x^2 \geq (x + 2)^2 + (x + 1)^2$

2. For each of the following combinations of inequalities, write the solution set in interval notation and graph the solution set on a number line.

   (a) $2 \leq 3 + x < 6$

   (b) $x \geq 2 - x$ or $5 - 3x \geq x + 9$

   (c) $-9x + 4 < 8x + 2 < 2x + 7$

   (d) $-6x - 8 \leq 4$ and $-x - 10 \leq -7$

   (e) $-6x - 8 \leq 4$ or $-x - 10 \leq -7$

3. (a) Write down a pair of linear inequalities, each involving a single variable x, for which no real number satisfies both of the inequalities simultaneously. In this case, the solution set for "[first inequality] and [second inequality]" is the *empty set*, denoted $\emptyset$.

   (b) Write down a pair of linear inequalities, each involving a single variable x, for which every real number satisfies at least one of the inequalities. In this case, the solution set for "[first inequality] or [second inequality]" is the entire set of real numbers, denoted $\mathbb{R}$. We could also write $(-\infty, \infty)$.

   (c) Write down a pair of linear inequalities, each involving a single variable x, for which the only real number which does not satisfy at least one of the inequalities is 6.626. Express this solution set in interval notation as a union ($\cup$) of two intervals.

4. Solve each of the following inequalities, expressing the solution set in interval notation and graphing the solution set on a number line.

   (a) $|x| < 4$

   (b) $|x - 4| \geq 2$

5. Graph the solution set of each of the following inequalities.

   (a) $x + y < -2$

   (b) $3x - y \geq 7$

   (c) $(x - 1)^2 + y^2 \leq (x - 5)^2 + (y - 2)^2$

   (d) $|x| + |y| < 1$

6. Graph the solution set of each of the following combinations of inequalities.

   (a) $x + y \leq -2$ and $3x - y > 7$

   (b) $x + y \leq -2$ or $3x - y > 7$

   (c) $2x + 3y \geq 4$ and $6y \leq 12 - 4x$

   (d) $3x + 3y < -5x < 5y$

7. Supposing $a \leq b$ and $c \leq d$, which of the following must also be true?

   (I) $a + c \leq b + d$

   (II) $a - c \leq b - d$

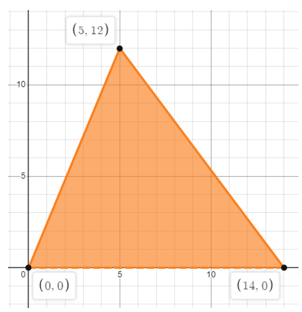
   (III) $ac \leq bd$

   (IV) $a/c \leq b/d$

   For the statements that were not guaranteed to be true, which ones are guaranteed to be true when we also assume $a \geq 0$ and $c \geq 0$?

2.2 Challenge problems

8. Write down three inequalities for which the set of all points (x, y) satisfying all of the three inequalities is the set shown below.



9. Miko and Yuu plan to sell alms bowls and cowrie shell necklaces at their school's culture festival. Each bowl sold earns them \$30 while each necklace sold earns them \$23. Their stand will allow them to stock up to 120 items in total, and the amounts of these two items cannot differ by more than 40. (Otherwise, Miko and Yuu will start arguing so much that they drive away all potential customers.) To make the items, they need to borrow materials from the art club. Each bowl uses up 2 units of a 170-unit material allowance that the art club gives them, while each necklace uses up 1 unit of that allowance. Given all of these restrictions, what is the maximum possible amount of revenue that Miko and Yuu could bring in?

10. Find all values of x for which

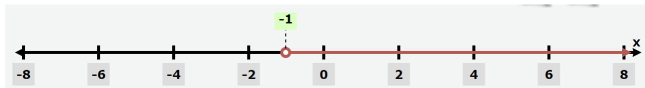
$$||x - 1| - |2x - 8|| \leq 3/2.$$

   Express your answer in interval notation.

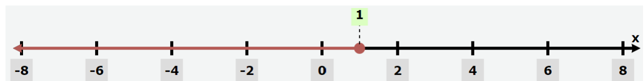
2.3 Answers

In all instances of interval notation, writing ∞ in place of $+\infty$ would be valid (but writing ∞ in place of $-\infty$ would not).

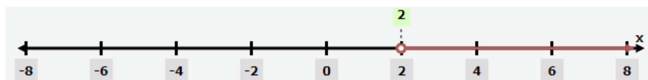
1. (a) $(-1, +\infty)$



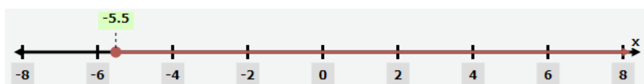
(b) $(-\infty, 1]$



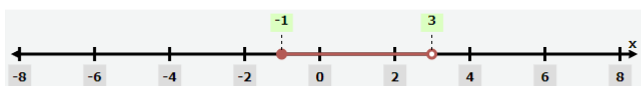
(c) $(2, +\infty)$



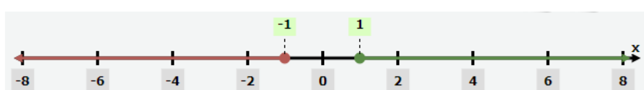
(d) $[-11/2, +\infty)$



2. (a) $[-1, 3)$

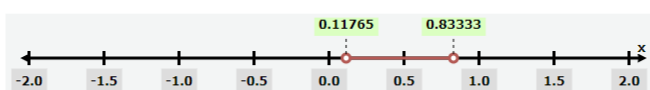


(b) $(-\infty, -1] \cup [1, +\infty)$



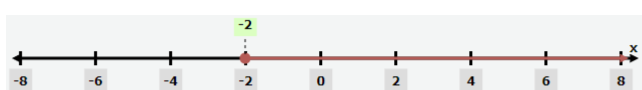
(The colors on the graph are irrelevant.)

(c) $(2/17, 5/6)$

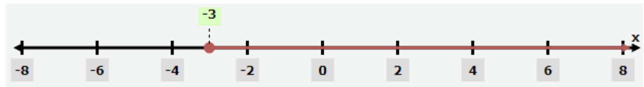


(The decimals labeling the endpoints are approximations.)

(d) $[-2, +\infty)$



(e) $[-3, +\infty)$



3. There are several possible options for each of these questions.

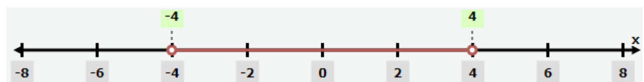
   (a) $x > 0$ and $x < 0$

   (b) $x > 0$ or $x \leq 0$

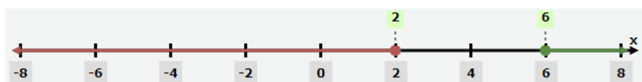
   (c) $x > 6.626$ or $x < 6.626$

4. The *absolute value* of a real number x, denoted $|x|$, is the distance between x and 0 on the number line. A useful corollary of this is that $|a - b|$ is the distance between a and b.

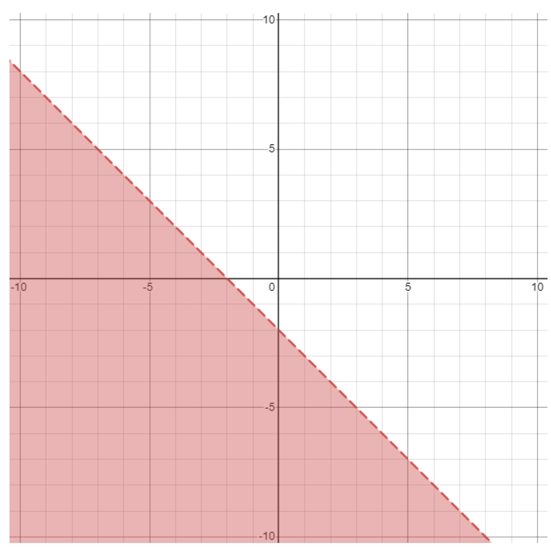
   (a) We want the numbers whose distance from 0 is less than 4, so the answer is $\boxed{(-4, 4)}$.



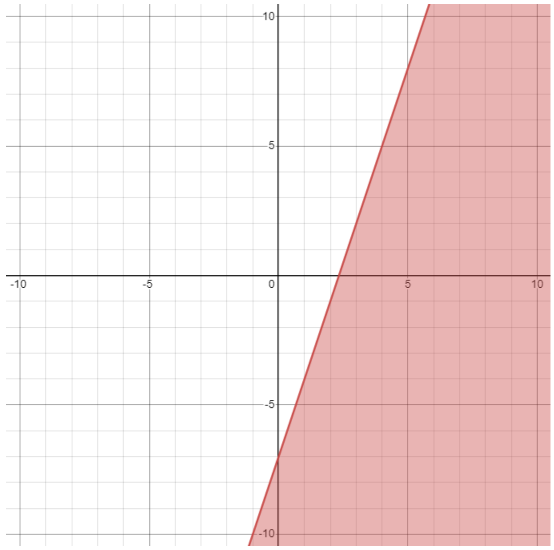
   (b) We want the numbers whose distance from 4 is greater than or equal to 2, so the answer is $\boxed{(-\infty, 2] \cup [6, +\infty)}$.



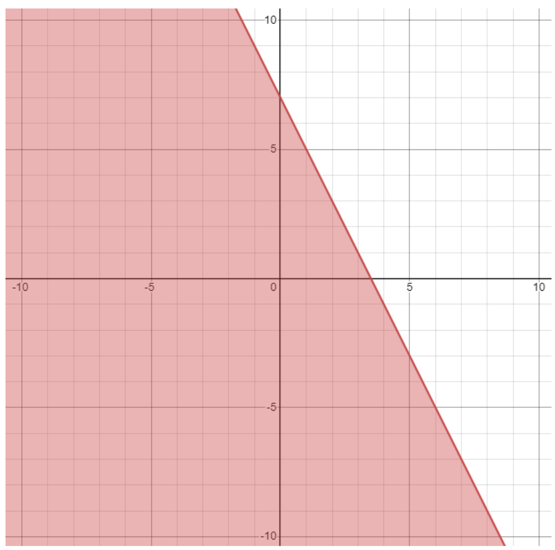
5. (a)



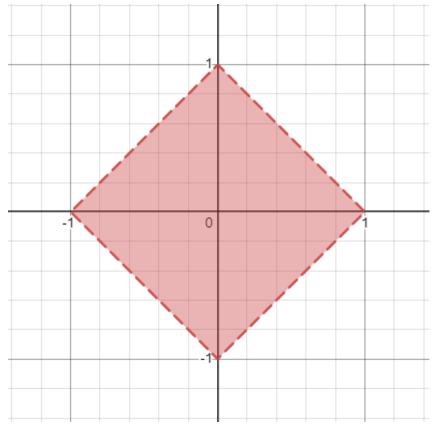
(b)



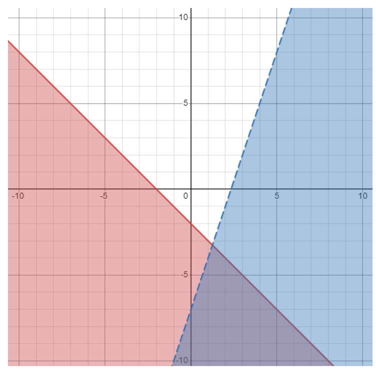
(c)



(d) In the first quadrant, $|x| + |y| = x + y$, so the first-quadrant points in the solution set are the ones satisfying $x + y < 1$. In the second quadrant, $|x| + |y| = -x + y$, so the second-quadrant points in the solution set are the ones satisfying $-x+y < 1$. In the third quadrant, $|x|+|y| = -x-y$, so the third-quadrant points in the solution set are the ones satisfying $-x - y < 1$. In the fourth quadrant, $|x| + |y| = x - y$, so the fourth-quadrant points in the solution set are the ones satisfying $x - y < 1$. Putting these together gives us a square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$, with the boundaries not included.

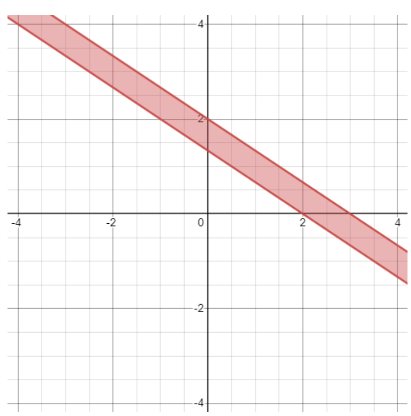


6. (a) In the image below, the solution set to this problem is the region covered by both shadings. The intersection point of the lines is not included since it does not satisfy the strict inequality.

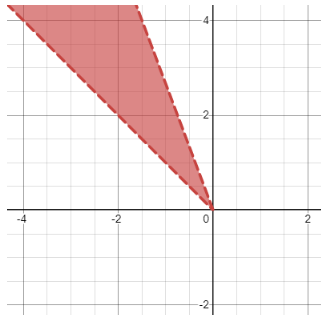


(b) In the image above, the solution set to this problem is the region covered by at least one shading. The intersection point is included since it satisfies the non-strict inequality.

(c)



(d)



7. Statement (I) is the only one that is guaranteed to hold. If we introduce the additional condition that $a, c \geq 0$, so that all variables are non-negative, then statement (III) is also guaranteed to hold.

8. The bottom boundary is a dashed line segment along the x-axis, so to get the region above it, we can use the inequality $\boxed{y > 0}$.

   The left boundary is a full line segment along $y = \frac{12}{5}x$, so to get a region below it, we can use the inequality $\boxed{y \leq \dfrac{12}{5}x}$.

   The right boundary is a full line segment along $y = -\frac{4}{3}x + \frac{56}{3}$, so to get a region below it, we can use the inequality $\boxed{y \leq -\dfrac{4}{3}x + \dfrac{56}{3}}$.

9. Let x be the number of alms bowls and y be the number of cowrie shell necklaces. The problem statement imposes the following constraints on x and y:

$$\begin{aligned} x &\geq 0, \\ y &\geq 0, \\ x + y &\leq 120, && \text{(maximum stock)} \\ x - y &\leq 40, && \text{(cannot have too many more bowls)} \\ y - x &\leq 40, && \text{(cannot have too many more necklaces)} \\ 2x + y &\leq 170. && \text{(materials allowance)} \end{aligned}$$

   The revenue we wish to maximise is $30x + 23y$. Our key observation for solving this type of problem (a linear program) is that a revenue R can be achieved if and only if the line $30x + 23y = R$ intersects the region defined by the constraints above, so we want to find the maximum such R. As we increase R, our line moves up and to the right while retaining its slope, and when R is at the desired maximum, the line will pass through a vertex of the region. (You can experiment with this here.) By drawing such lines, we can identify that the vertex of interest is the intersection of the lines $x + y = 120$ and $2x + y = 170$. This intersection point is $(50, 70)$, and the corresponding revenue is

$$30x + 23y = 30 \cdot 50 + 23 \cdot 70 = \boxed{3110}.$$

Remark: In general, the maximum or minimum for a linear program is achieved at a vertex of the feasible region, so even without drawing lines to figure out the right vertex, we can test all of the vertices and see which one results in the largest revenue.

10. A useful tool for handling complicated expressions involving absolute values is to consider graphs. We can interpret the given inequality as saying that we want the values of x for which the vertical distance between the graphs $y = |x-1|$ and $y = |2x-8|$ should be at most $3/2$. (For reference, the graphs are here.)

    From the graphs, we can see two intersection points whose x coordinates will be part of the solution set, as the vertical distance at those values is just 0. For the left intersection point, the lines intersecting are $y = x - 1$ and $y = -(2x - 8)$, and the intersection point is $(3, 2)$. For the right intersection point, the lines intersecting are $y = x - 1$ and $y = 2x - 8$, and the intersection point is $(7, 6)$.

    As we move leftward from $(3, 2)$, the blue line $y = -(2x - 8)$ moves up 2 units for every 1 unit left while the red line $y = x - 1$ moves down 1 unit for every 1 unit left, so the vertical distance between them grows by 3 units for every 1 unit left we move. Therefore, the graphs stay within $3/2$ of each other until we have moved $1/2$ units to the left and reached $x = 5/2$. Similarly, they stay within $3/2$ of each other as we move rightward from $(3, 2)$ until we have moved $1/2$ units to the right and reached $x = 7/2$.

    As we move from $(7, 6)$, we can use similar reasoning to see that the vertical distance between the graphs grows by 1 unit for every 1 unit left or right we move. Thus we can move $3/2$ units left to reach $x = 11/2$ or $3/2$ units right to reach $x = 17/2$.

    The final answer is $\boxed{[5/2, 7/2] \cup [11/2, 17/2]}$.

3 Quadratic Equations (I)

3.1 Review problems

1. Solve each of the following equations for all real solutions.

   (a)

3.2 Challenge problems

2.

3.3 Answers

1.