Precalculus Practice Problems: Final

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The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

Contents

1 Matrices in 2D

1.1 Review Problems

Review problems are meant to cover "standard" definitions and calculations as well as the use of some important results.

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector. We also let $\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathsf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.

   (a) $\mathbf{u} + \mathbf{v}$

   (b) $2\mathbf{v}$

   (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

   (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$

   (e) The angle between $\mathbf{u}$ and $\mathbf{v}$ (in terms of an inverse trig function)

   (f) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{u}}(\mathbf{v})$

2. *Applying matrices to vectors.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

   (a) Compute $\mathsf{A}\mathbf{v}$

   (b) Find a vector $\mathbf{u}$ for which $\mathsf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.

3. *Matrix operations.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\mathsf{A} + \mathsf{B}$

   (b) $-3\mathsf{A}$

   (c) AB

   (d) BA

   (e) B^T (the transpose of B)

4. *Geometric transformations.* Write down matrices for each of the following.

   (a) Dilation about the origin by a factor of 4

   (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2

   (c) Rotation about the origin by $\pi/4$ counterclockwise

   (d) Projection onto the line $y = (3/2)x$

   (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

    (a) $\det A$ and $\det B$

    (b) $\det(AB)$

    (c) $\det(A^T)$

    (d) $\det(A + B)$

    (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

    (a) A^{-1} and B^{-1}

    (b) $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$

    (c) $(AB)^{-1}$

    (d) $(A^T)^{-1}$

    (e) $(A + B)^{-1}$

    (f) $\det(A^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

    (a) Describe the image of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.

    (b) By what factor does a horizontal shear multiply areas?

    (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

    (The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

Challenge problems are meant to provide optional extensions of the ideas from class.

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\operatorname{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

    (a) For the matrices A, B in problems 3, 5, 6, compute $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr}(A + B)$, and $\operatorname{tr}(AB)$.

    (b) Show that for any 2×2 matrices P and Q, we have $\operatorname{tr}(PQ) = \operatorname{tr}(QP)$.

    (c) In general, must it be true that $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$?

9. Two matrices A, B are **similar**, written $A \sim B$, if there is an invertible P with $B = P^{-1}AP$.

    (a) Show that the only matrix similar to I is I.

    (b) Show that if $A \sim B$, then $\det A = \det B$ and $\operatorname{tr} A = \operatorname{tr} B$.

    (c) Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $D \sim A$. Find D.

10. If A is a square matrix, the **characteristic polynomial** of A is defined by

$$f_A(X) = \det(A - XI).$$

    (a) Compute the characteristic polynomial $f_A(X)$ of the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

    (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_A(X)$.

    (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, 2$. (In general, if $A\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call $\mathbf{v}$ an **eigenvector** of A corresponding to the **eigenvalue** λ.)

    (d) Let P be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Compute $P^{-1}AP$.

    (e) *Cayley-Hamilton theorem.* Suppose $f_A(X) = a_0 + a_1 X + a_2 X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0 I + a_1 A + a_2 A^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

   (c) Both are 5. In general, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

   (d) $\|\mathbf{u}\| = \sqrt{13}$
   $\|\mathbf{v}\| = \sqrt{17}$
   $\|\mathbf{u} + \mathbf{v}\| = \sqrt{40} = 2\sqrt{10}$

   (e) $\arccos\left(\frac{5}{\sqrt{221}}\right)$

   (f) $\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

   $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

2. (a) $\begin{pmatrix} 18 \\ 7 \end{pmatrix}$

   (b) Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then
   $$\mathsf{A}\mathbf{u} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 4b \\ a + b \end{pmatrix},$$
   so we require $2a + 4b = 5$ and $a + b = 2$. The solution to this system is that $a = 3/2$ and $b = 1/2$, so then $\mathbf{u} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.

   Remark: We can also compute $\mathbf{u} = \mathsf{A}^{-1}\mathbf{v}$ once we have A^{-1} (see Problem 6).

3. (a) $\begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix}$

   (b) $\begin{pmatrix} -6 & -12 \\ -3 & -3 \end{pmatrix}$

   (c) $\begin{pmatrix} 14 & -20 \\ 2 & -3 \end{pmatrix}$

   (d) $\begin{pmatrix} -2 & -8 \\ 3 & 13 \end{pmatrix}$

   (e) $\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$

4. (a) $4\mathsf{I} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{pmatrix}$

(e) $2\mathsf{P} - \mathsf{I} = \begin{pmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{pmatrix}$

5. (a) $\det \mathsf{A} = -2$

   $\det \mathsf{B} = 1$

   (b) $\det(\mathsf{AB}) = \det(\mathsf{A}) \cdot \det(\mathsf{B}) = -2$

   (c) $\det(\mathsf{A}^T) = \det \mathsf{A} = -2$

   (d) $\det(\mathsf{A} + \mathsf{B}) = \det \begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix} = -42$

   (e) $|\det \mathsf{A}| \cdot (\text{unit circle area}) = 2\pi$

6. (a) $\mathsf{A}^{-1} = \dfrac{1}{\det \mathsf{A}} \begin{pmatrix} 1 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 1/2 & -1 \end{pmatrix}$

   $\mathsf{B}^{-1} = \dfrac{1}{\det \mathsf{B}} \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix}$

   (b) $\mathsf{A}^{-1}\mathsf{B}^{-1} = \begin{pmatrix} -13/2 & -4 \\ 3/2 & 1 \end{pmatrix}$

   $\mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

   (c) $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

   (d) $(\mathsf{A}^T)^{-1} = (\mathsf{A}^{-1})^T = \begin{pmatrix} -1/2 & 1/2 \\ 2 & -1 \end{pmatrix}$

   (e) $(\mathsf{A} + \mathsf{B})^{-1} = \dfrac{1}{\det(\mathsf{A} + \mathsf{B})} \begin{pmatrix} -6 & -8 \\ -6 & -1 \end{pmatrix} = \begin{pmatrix} 1/7 & 4/21 \\ 1/7 & 1/42 \end{pmatrix}$

   (f) $\det(\mathsf{A}^{-1}) = 1/\det \mathsf{A} = -1/2$

7. (a) A parallelogram with vertices $(0,0), (1,0), (3,1), (2,1)$

   (b) $\det \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = 1$

   (c) Multiplying the right two matrices, $\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & ak \\ 0 & b \end{pmatrix}$. Looking at the image of vector $\hat{\mathbf{i}}$, we need $\begin{pmatrix} a \\ 0 \end{pmatrix}$ to rotate to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. This can be achieved with a rotation by $\theta = \arccos(4/5)$ and $a = 5$. To find b, taking the determinant on both sides and noting that rotations have determinant 1, we require $ab = 25$, so $b = 5$. Finally, to get k, we need $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ to rotate to $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$. Comparing lengths and noting that $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ must be in the first quadrant, $k = 1$.

8. (a) $\operatorname{tr} A = 3$

$\operatorname{tr} B = -10$

$\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B = -7$

$\operatorname{tr}(AB) = 11$

(b) Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$PQ = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \quad \text{and} \quad QP = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix},$$

so $\operatorname{tr}(PQ) = \operatorname{tr}(QP) = ae + bg + cf + dh$.

(c) In general, the answer is **no**. For example, let

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$ABC = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix},$$

$$ACB = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 3 \end{pmatrix},$$

so $\operatorname{tr}(ABC) = 6$ while $\operatorname{tr}(ACB) = 7$.

9. (a) Suppose $I \sim B$. Then there is an invertible matrix P such that $B = P^{-1}IP$, but the right hand side simplifies to $P^{-1}P = I$.

(b) If $A \sim B$ with $B = P^{-1}AP$, then

$$\det B = \det(P^{-1}AP) = \det(P)^{-1} \cdot \det A \cdot \det P = \det A.$$

For the trace, Problem 8b gives us

$$\operatorname{tr} B = \operatorname{tr}(P^{-1}(AP)) = \operatorname{tr}((AP)P^{-1}) = \operatorname{tr} A.$$

(c) We have $\det A = 4$ and $\operatorname{tr} A = 5$, so

$$\det D = d_1 d_2 = 4 \quad \text{and} \quad \operatorname{tr} D = d_1 + d_2 = 5.$$

This is satisfied by $d_1 = 4$ and $d_2 = 1$, so $D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

10. placeholder