Precalculus Practice Problems: Midterm 2

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The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. (SAS congruence) Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Find $\angle A$ and $\angle B$.
2. (SSS congruence) Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Find $\angle A$.
   (b) Find $\angle B$ and $\angle C$.
   (c) Find $[ABC]$.
3. (ASA/AAS congruence) Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.
4. (SSA non-congruence) Let ABC be a triangle with $\angle A = 20°$, $a = 6$, and $b = 9$.
   (a) Find all possible values of c.
   (b) For each possible value of c, find $\angle B$.
   (c) For what values of x does there exist exactly one triangle XYZ with $\angle X = 20°$, $XY = 9$, and $YZ = x$?
5. (Extended law of sines) If ABC is a triangle with circumradius R, then the *extended law of sines* states that
   $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Prove that $R = \dfrac{abc}{4[ABC]}$.
   (b) Given that $a = 13$, $b = 14$, and $c = 15$, find R.
   (c) Prove the extended law of sines for acute triangles.
6. Let ABC be a triangle and let D be a point on side $\overline{BC}$.
   (a) (Ratio lemma) Prove that
   $$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(b) (Angle bisector theorem) Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

7. (Heron's formula) Let ABC be a triangle.

   (a) Show that $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

   (b) Show that
   $$[ABC]^2 = \frac{1}{4}a^2b^2(1-\cos^2 C) = \frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}.$$

   (c) Conclude that
   $$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$
   where $s = (a+b+c)/2$ is the *semiperimeter* of triangle ABC.

1.2 Challenge problems

8. Points O, A, B, and C are placed in three-dimensional space so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. What are the shortest and longest possible lengths of BC?

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Given that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. (Eisenstein triples) An *Eisenstein triple* is a triple of positive integers (a,b,c) for which a triangle with side lengths a, b, and c has an angle of measure either 60° or 120°. If the Eisenstein triple (a,b,c) corresponds to a triangle with an angle of measure 60°, we will call it an Eisenstein triple of *acute type*, and otherwise, we call it an Eisenstein triple of *obtuse type*. (The "acute type" and "obtuse type" names are non-standard.)

    (a) Let (a,b,c) be an Eisenstein triple of obtuse type with $a < b < c$. Show that $(a, a+b, c)$ and $(a+b, b, c)$ are Eisenstein triples of acute type.

    (b) Conversely, show that every Eisenstein triple of acute type arises from an Eisenstein triple of obtuse type in the above manner.

    (c) Show that if (a,b,c) is an Eisenstein triple of obtuse type with $\gcd(a,b,c) = 1$, then there are relatively prime positive integers m and n such that
    $$\{a,b,c\} = \{m^2+mn+n^2, 2mn+n^2, m^2-n^2\}.$$
    (Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104^\circ) \approx 2.426$

   (b) $c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{26 - 10\cos(104^\circ)} \approx 5.331$

   (c) $A = \arcsin(\frac{a\sin C}{c}) \approx 10.49^\circ$ and $B = \arcsin(\frac{b\sin C}{c}) \approx 65.51^\circ$

2.