Precalculus Practice Problems: Final

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The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

Contents

1 Matrices in 2D

1.1 Review Problems

Review problems are meant to cover "standard" definitions and calculations as well as the use of some important results.

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector. We also let $\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathsf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.

   (a) $\mathbf{u} + \mathbf{v}$

   (b) $2\mathbf{v}$

   (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

   (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$

   (e) The angle between $\mathbf{u}$ and $\mathbf{v}$

   (f) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{u}}(\mathbf{v})$

2. *Applying matrices to vectors.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

   (a) Compute $\mathsf{A}\mathbf{v}$

   (b) Find a vector $\mathbf{u}$ for which $\mathsf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.

3. *Matrix operations.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\mathsf{A} + \mathsf{B}$

   (b) $-3\mathsf{A}$

   (c) AB

   (d) BA

   (e) B^T (the transpose of B)

4. *Geometric transformations.* Write down matrices for each of the following.

   (a) Dilation about the origin by a factor of 4

   (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2

   (c) Rotation about the origin by $\pi/4$ counterclockwise

   (d) Projection onto the line $y = (3/2)x$

   (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\det \mathsf{A}$ and $\det \mathsf{B}$
   (b) $\det(\mathsf{AB})$
   (c) $\det(\mathsf{A}^T)$
   (d) $\det(\mathsf{A} + \mathsf{B})$
   (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) A^{-1} and B^{-1}
   (b) $\mathsf{A}^{-1}\mathsf{B}^{-1}$ and $\mathsf{B}^{-1}\mathsf{A}^{-1}$
   (c) $(\mathsf{AB})^{-1}$
   (d) $(\mathsf{A}^T)^{-1}$
   (e) $(\mathsf{A} + \mathsf{B})^{-1}$
   (f) $\det(\mathsf{A}^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

   (a) Describe the image of the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.
   (b) By what factor does a horizontal shear multiply areas?
   (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

   (The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

Challenge problems are meant to provide optional extensions of the ideas from class.

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

   (a) For the matrices A and B in problems 3, 5, and 6, compute $\operatorname{tr}\mathsf{A}$, $\operatorname{tr}\mathsf{B}$, and $\operatorname{tr}(\mathsf{AB})$.

   (b) Show that for any 2×2 matrices P and Q, we have $\operatorname{tr}(\mathsf{PQ}) = \operatorname{tr}(\mathsf{QP})$.

   (c) In general, must it be true that $\operatorname{tr}(\mathsf{ABC}) = \operatorname{tr}(\mathsf{ACB})$?

9. Two matrices A, B are **similar**, written $\mathsf{A} \sim \mathsf{B}$, if there is an invertible P with $\mathsf{B} = \mathsf{P}^{-1}\mathsf{AP}$.

   (a) Show that the only matrix similar to I is I.

   (b) Show that if $\mathsf{A} \sim \mathsf{B}$, then $\det \mathsf{A} = \det \mathsf{B}$ and $\operatorname{tr}\mathsf{A} = \operatorname{tr}\mathsf{B}$.

   (c) Let $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $\mathsf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $\mathsf{D} \sim \mathsf{A}$. Find D.

10. If A is a square matrix, the **characteristic polynomial** of A is defined by

$$f_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I}).$$

   (a) Compute the characteristic polynomial $f_{\mathsf{A}}(X)$ of the matrix $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_{\mathsf{A}}(X)$.

   (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $\mathsf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, 2$. (In general, if $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call $\mathbf{v}$ an **eigenvector** of A corresponding to the **eigenvalue** λ.)

   (d) Let P be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Compute $\mathsf{P}^{-1}\mathsf{AP}$.

   (e) *Cayley-Hamilton theorem.* Suppose $f_{\mathsf{A}}(X) = a_0 + a_1 X + a_2 X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0\mathsf{I} + a_1\mathsf{A} + a_2\mathsf{A}^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

2. (a) $\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -9 \end{pmatrix}$ (there are many other options)

   (b) Let p and q be the value of the parameters for $\mathbf{x}_1$ and $\mathbf{x}_2$ at the point of intersection, so $\mathbf{x}_1(p) = \mathbf{x}_2(q)$. Then
   $$\begin{pmatrix} 0 \\ 3 \end{pmatrix} + p\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + q\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
   which means $p = -4 + 2q$ and $3 - p = 2 + 3q$. Solving the system, $q = 1$ and $p = -2$, and the point of intersection is $(-2, 5)$.

3. (a) $\|\mathbf{u}\| = \sqrt{13}$
   $\mathbf{u} \cdot \mathbf{v} = 5$

   (b) $\hat{\mathbf{v}} = \dfrac{1}{\sqrt{17}}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{pmatrix}$

   (c) $\cos\theta = \frac{5}{\sqrt{221}}$

   (d) $\pm\dfrac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

4. (a) If $a\mathbf{u} + b\mathbf{v} = 0$, then $2a + 4b = 0$ and $3a - b = 0$. Adding 4 times the second equation to the first, $14a = 0$, so $a = 0$ and hence $b = 0$ as well.

   (b) $(a, b, c, d) = (1/14, 3/14, 2/7, -1/7)$

   (c) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. If $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$, then
   $$x_1 a + y_1 b = 0, \tag{1}$$
   $$x_2 a + y_2 b = 0. \tag{2}$$
   Taking the combination $y_2 \cdot (1) - y_1 \cdot (2)$,
   $$(x_1 y_2 - x_2 y_1)a = 0.$$
   If $x_1 y_2 - x_2 y_1 = 0$, then we consider further subcases.

   - If $\mathbf{x} = \mathbf{0}$, then taking $(a, b) = (1, 0)$ gives a non-zero solution, so $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Moreover, linear combinations $c\mathbf{x} + d\mathbf{y} = d\mathbf{y}$ can only produce multiples of $\mathbf{y}$, hence will not span all of $\mathbb{R}^2$.
   - If $\mathbf{x} \neq \mathbf{0}$, either $x_1 \neq 0$ or $x_2 \neq 0$. If $x_1 \neq 0$, then let $\lambda = y_1/x_1$, so $y_1 = \lambda x_1$. Substituting, we find $y_2 = \lambda x_2$, so $\mathbf{y} = \lambda\mathbf{x}$. This means that $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Also, any linear combination $c\mathbf{x} + d\mathbf{y} = (c + d\lambda)\mathbf{x}$ will be a multiple of $\mathbf{x}$, hence will not span all of $\mathbb{R}^2$.

If $x_1y_2 - x_2y_1 \neq 0$, then $a = 0$. At least one of y_1 and y_2 is non-zero in this case, and substituting into the relevant equation gives $b = 0$ as well. Therefore, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. To see that they span $\mathbb{R}^2$, let $\begin{pmatrix} p \\ q \end{pmatrix}$ be arbitrary. We look for coefficients c, d such that $c\mathbf{x} + d\mathbf{y} = \begin{pmatrix} p \\ q \end{pmatrix}$, or

$$\begin{aligned} x_1c + y_1d &= p, \\ x_2c + y_2d &= q. \end{aligned}$$

This has a solution, namely $(c, d) = \left(\dfrac{py_2 - qy_1}{x_1y_2 - x_2y_1}, \dfrac{qx_1 - px_2}{x_1y_2 - x_2y_1} \right)$.

(d) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be given. If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are as well. Otherwise, by part (c), there exist a, b for which $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. Then $a\mathbf{x} + b\mathbf{y} - \mathbf{z}$ is a non-trivial linear combination of the three which equals $\mathbf{0}$, so they are linearly dependent.

5. (a) $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

(b) It suffices to show $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Suppose $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$. Then

$$0 = \mathbf{x} \cdot (a\mathbf{x} + b\mathbf{y}) = a(\mathbf{x} \cdot \mathbf{x}) + b(\mathbf{x} \cdot \mathbf{y}) = a(1) + b(0) = a.$$

By a similar argument, $b = 0$.

(c) Translating up by 1 unit, it suffices to find the distance between the head of $\mathbf{u} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and the line $y = 2x$, which is spanned by the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We compute

$$\| \operatorname{proj}_{\mathbf{v}}(\mathbf{u}) \| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{v}\|} = \frac{16}{\sqrt{5}}.$$

6. (a) $y = (-2/3)x + (4/3)$

(b) $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $d = 5$ (there are many choices that work)

(c) If $\mathbf{x}_1, \mathbf{x}_2$ are two position vectors for points on the line, then $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$ points along the line. Since

$$\hat{\mathbf{n}} \cdot \mathbf{v} = \hat{\mathbf{n}} \cdot (\mathbf{x}_2 - \mathbf{x} - 1) = \hat{\mathbf{n}} \cdot \mathbf{x}_2 - \hat{\mathbf{n}} \cdot \mathbf{x}_1 = d - d = 0,$$

$\hat{\mathbf{n}}$ is perpendicular to the line. Then, for any position vector $\mathbf{x}$ on the line, we compute the distance from the origin to the line as

$$\| \operatorname{proj}_{\hat{\mathbf{n}}}(\mathbf{x}) \| = \frac{|\hat{\mathbf{n}} \cdot \mathbf{x}|}{\|\hat{\mathbf{n}}\|} = d.$$

7. (a) By the angle bisector theorem, $BD/DC = AB/AC = 1/2$. Therefore, $\mathbf{D} = \frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}$.

   (b) The points on line $\overline{AD}$ are those with position vector of the form

$$t\mathbf{A} + (1-t)\mathbf{D} = t\mathbf{A} + (1-t)\left(\frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}\right)$$

   for a real number t. Setting $t = 5/14$ gives us $\mathbf{I}$.

8. (a) We show that $\mathbf{y} = 2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}$ has the defining properties of reflection, namely that the line ℓ spanned by $\mathbf{v}$ is orthogonal to $\mathbf{x} - \mathbf{y}$ and passes through the midpoint of the segment connecting $\mathbf{x}$ and $\mathbf{y}$. First, $(\mathbf{x} + \mathbf{y})/2 = \operatorname{proj}_{\mathbf{v}}(\mathbf{x})$, so ℓ passes through the midpoint. For orthogonality,

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{y}) = 2\mathbf{v} \cdot (\mathbf{x} - \operatorname{proj}_{\mathbf{v}}(\mathbf{x})) = 0.$$

   (b) We compute

$$\begin{aligned}
\operatorname{refl}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) &= 2\operatorname{proj}_{\mathbf{v}}(a\mathbf{x} + b\mathbf{y}) - (a\mathbf{x} + b\mathbf{y}) \\
&= 2(a\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{proj}_{\mathbf{v}}(\mathbf{y})) - (a\mathbf{x} + b\mathbf{y}) \\
&= a(2\operatorname{proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{x}) + b(2\operatorname{proj}_{\mathbf{v}}(\mathbf{y}) - \mathbf{y}) \\
&= a\operatorname{refl}_{\mathbf{v}}(\mathbf{x}) + b\operatorname{refl}_{\mathbf{v}}(\mathbf{y}).
\end{aligned}$$

9. (a) Linearity of $\operatorname{proj}_{\mathbf{v}}$ follows from (bi)linearity of the dot product.
   To see that $\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}} = \operatorname{proj}_{\mathbf{v}}$, we compute

$$\begin{aligned}
(\operatorname{proj}_{\mathbf{v}} \circ \operatorname{proj}_{\mathbf{v}})(\mathbf{x}) &= \operatorname{proj}_{\mathbf{v}}\left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v}\right) \\
&= \left(\left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
&= \left[\mathbf{x} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right]\mathbf{v} = \operatorname{proj}_{\mathbf{v}}(\mathbf{x}).
\end{aligned}$$

   Finally, orthogonality is part of how $\operatorname{proj}_{\mathbf{v}}$ was defined.

   (b) Let $\mathbf{x}$ be a vector for which $P(\mathbf{x}) \neq \mathbf{0}$ and let $\mathbf{y}$ be a vector for which $P(\mathbf{y}) \neq \mathbf{y}$. (These conditions imply $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.) Let $\mathbf{u} = P(\mathbf{x})$ and $\mathbf{v} = \mathbf{y} - P(\mathbf{y})$. These are non-zero, and

$$\begin{aligned}
&P(\mathbf{u}) = P(P(\mathbf{x})) = P(\mathbf{x}) = \mathbf{u}, \\
&P(\mathbf{v}) = P(\mathbf{y} - P(\mathbf{y})) = P(\mathbf{y}) - P(P(\mathbf{y})) = P(\mathbf{y}) - P(\mathbf{y}) = \mathbf{0}.
\end{aligned}$$

   For linear independence, suppose $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$. Applying P,

$$\mathbf{0} = P(\mathbf{0}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u},$$

   so $a = 0$. Then, $b\mathbf{v} = \mathbf{0}$, so $b = 0$ as well.

(c) Since $\mathbf{u}, \mathbf{v}$ are linearly independent in $\mathbb{R}^2$, they also span $\mathbb{R}^2$. Given any vector $\mathbf{x} \in \mathbb{R}^2$, let $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$. Then

$$P(\mathbf{x}) = P(a\mathbf{u} + b\mathbf{v}) = aP(\mathbf{u}) + bP(\mathbf{v}) = a\mathbf{u}.$$

As a and b range over all real numbers, we find the range of P is the span of $\mathbf{u}$.

10. (a) We proceed by induction. By definition, $s_0 = ax_0 + by_0$ and $s_1 = ax_1 + by_1$. Then, if $s_n = ax_n + by_n$ and $s_{n+1} = ax_{n+1} + by_{n+1}$,

$$\begin{aligned} s_{n+2} &= s_{n+1} + 2s_n \\ &= (ax_{n+1} + by_{n+1}) + 2(ax_n + by_n) \\ &= a(x_{n+1} + 2x_n) + b(y_{n+1} + 2y_n) \\ &= ax_{n+2} + by_{n+2}. \end{aligned}$$

(b) We need $\lambda^n(\lambda^2 - \lambda - 2) = 0$ for all $n \geq 0$. This holds when $\lambda = -1$ and when $\lambda = 2$.

(c) The sequence $e_n = (-1)^n$ corresponds to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, while the sequence $f_n = 2^n$ corresponds to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We can find

$$\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{11}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so

$$x_n = \frac{1}{3}e_n + \frac{11}{3}f_n = \frac{1}{3} \cdot (-1)^n + \frac{11}{3} \cdot 2^n.$$