Precalculus Practice Problems: Midterm 2

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The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. *SAS congruence.* Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Using the law of sines, or otherwise, find $\sin A$ and $\sin B$.
   (d) Show that $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$, and hence compute $\angle A$ and $\angle B$. (Hint: For which angles does $\arcsin(\sin\theta) = \theta$ hold?)

2. *SSS congruence.* Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Using the law of cosines, or otherwise, find $\cos A$, $\cos B$, and $\cos C$.
   (b) Compute $\angle A$, $\angle B$, and $\angle C$.
   (c) Find $[ABC]$.

3. *ASA/AAS congruence.* Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.

4. *SSA non-congruence.* Let ABC be a triangle with $\angle A = 30°$, $a = 6$, and $b = 9$.
   (a) Show that $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) Find all possible values of c.

5. *Extended law of sines.* If ABC is a triangle with circumradius R, then the **extended law of sines** states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

   (a) Express $\sin C$ in terms of a, b, and $[ABC]$.
   (b) Assuming the extended law of sines, show that $R = \dfrac{abc}{4[ABC]}$.
   (c) Given that $a = 13$, $b = 14$, and $c = 15$, compute R.

6. *Single-cevian computations.* Let ABC be a triangle and let D be a point on side $\overline{BC}$. (The line segment $\overline{AD}$ is a *cevian* from A.)
   (a) Express $\angle ADB$ in terms of $\angle CDA$.

(b) *Ratio lemma.* Using the law of sines, or otherwise, show that

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(c) *Angle bisector theorem.* Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

(d) *Stewart's theorem.* Let $\angle ADB = \theta$ and let $AD = d$, $BD = x$, and $DC = y$. Show that

$$c^2 = d^2 + x^2 - 2dx\cos\theta,$$
$$b^2 = d^2 + y^2 + 2dy\cos\theta,$$

and conclude that

$$b^2x + c^2y = a(d^2 + xy).$$

7. *Concurrent cevians.* Let ABC be a triangle and let D, E, F lie on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively.

(a) Show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}.$$

(b) *Ceva's theorem.* Show that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent if and only if the two sides of the above equation are equal to 1.

1.2 Challenge problems

8. Points O, A, B, and C are placed in the plane so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. Find all possible lengths of BC.

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Assuming that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. (Eisenstein triples) An *Eisenstein triple* is an unordered triple of positive integers $\{a, b, c\}$ for which a triangle with side lengths a, b, and c has an angle of measure either $60°$ or $120°$. If the Eisenstein triple corresponds to a triangle with an angle of measure $60°$, we will call it an Eisenstein triple of *acute type*, and otherwise, we call it an Eisenstein triple of *obtuse type*. (The "acute type" and "obtuse type" names are non-standard.)

(a) Suppose $a, b < c$ are such that $\{a, b, c\}$ is an Eisenstein triple of obtuse type. Show that $\{a, a + b, c\}$ and $\{a + b, b, c\}$ are Eisenstein triples of acute type.

(b) Conversely, show that every Eisenstein triple of acute type either corresponds to an equilateral triangle or arises from an Eisenstein triple of obtuse type in the above manner.

(c) Show that if $\{a, b, c\}$ is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that

$$\{a, b, c\} = \{2mn + n^2, m^2 - n^2, m^2 + mn + n^2\}.$$

(Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104^\circ) \approx 2.426$
   (b) $c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{26 - 10\cos(104^\circ)} \approx 5.331$
   (c) $\sin A = \frac{a\sin C}{c} \approx \frac{\sin(104^\circ)}{5.331} \approx 0.182$
   $\sin B = \frac{b\sin C}{c} \approx \frac{5\sin(104^\circ)}{5.331} \approx 0.910$
   (d) Since $\angle C$ is obtuse, $\angle A$ and $\angle B$ are acute. Since acute angles are included in the range of arcsin, we have $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$.
   $\angle A = \arcsin(\sin A) \approx \arcsin(0.182) \approx 10.49^\circ$
   $\angle B = \arcsin(\sin B) \approx \arcsin(0.910) \approx 65.51^\circ$

2. (a) $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3}{5}$
   $\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{33}{65}$
   $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{5}{13}$
   (b) The range of arccos is $[0^\circ, 180^\circ]$, so we can always use it to extract triangle angles.
   $\angle A = \arccos(\cos A) = \arccos(\frac{3}{5}) \approx 53.13^\circ$
   $\angle B = \arccos(\cos B) = \arccos(\frac{33}{65}) \approx 59.49^\circ$
   $\angle C = \arccos(\cos C) = \arccos(\frac{5}{13}) \approx 67.38^\circ$
   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7 \cdot 15 \cdot \frac{4}{5} = 84$

3. (a) $\angle C = 180^\circ - \angle A - \angle B = 91^\circ$
   (b) $a = \frac{c}{\sin C} \cdot \sin A = \frac{2\sin 12^\circ}{\sin 91^\circ} \approx 0.416$
   $b = \frac{c}{\sin C} \cdot \sin B = \frac{2\sin 77^\circ}{\sin 91^\circ} \approx 1.949$
   (c) $[ABC] = \frac{1}{2}ac\sin B \approx 0.416\sin 77^\circ \approx 0.405$

4. (a) By the law of cosines,
   $$a^2 = b^2 + c^2 - 2bc\cos A \implies 36 = 81 + c^2 - 18\cos(30^\circ)c.$$
   Evaluating $\cos(30^\circ) = \sqrt{3}/2$ and rearranging gives us $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) By the quadratic formula,
   $$c = \frac{9\sqrt{3} \pm \sqrt{(9\sqrt{3})^2 - 4 \cdot 1 \cdot 45}}{2} = \frac{9\sqrt{3} \pm 3\sqrt{7}}{2}.$$

5. (a) $\sin C = \frac{2[ABC]}{ab}$.
   (b) $R = \dfrac{c}{2\sin C} = \dfrac{c}{\frac{4[ABC]}{ab}} = \dfrac{abc}{4[ABC]}$
   (c) $R = 65/8$

6. (a) $\angle ADB = 180^\circ - \angle CDA$

(b) By the law of sines,

$$\frac{AB}{\sin(\angle ADB)} = \frac{BD}{\sin(\angle BAD)} \quad \text{and} \quad \frac{AC}{\sin(\angle CDA)} = \frac{DC}{\sin(\angle DAC)}.$$

Dividing one equation by the other and using the fact that $\sin(\angle ADB) = \sin(\angle CDA)$ gets us the desired result after rearranging.

(c) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (b).

(d) The law of cosines in triangle ADB, using $\angle ADB$, gives us

$$(AB)^2 = (AD)^2 + (BD)^2 - 2(AD)(BD)\cos(\angle ADB) \implies c^2 = d^2 + x^2 - 2dx\cos\theta,$$

while the law of cosines in triangle ADC, using $\angle CDA$, gives us

$$b^2 = d^2 + y^2 - 2dy\cos(180^\circ - \theta) = d^2 + y^2 + 2dy\cos\theta.$$

Adding y times the first equation to x times the second equation, so as to eliminate $\cos\theta$,

$$\begin{aligned} b^2x + c^2y &= (d^2x + y^2 + 2dy\cos\theta)x + (d^2y + x^2 - 2dx\cos\theta)y \\ &= d^2(x+y) + xy(x+y) = a(d^2 + xy). \end{aligned}$$

7. (a) By the ratio lemma (problem 6b),

$$\begin{aligned} \frac{AF}{FB} &= \frac{CA}{CB} \cdot \frac{\sin(\angle ACF)}{\sin(\angle FCB)}, \\ \frac{BD}{DC} &= \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}, \\ \frac{CE}{EA} &= \frac{BC}{BA} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)}. \end{aligned}$$

Multiplying these equations together gives us the desired result.

(b) First suppose $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur at point P. Then by the law of sines,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle DAC)} &= \frac{\sin(\angle ACP)}{\sin(\angle PAC)} = \frac{AP}{CP}, \\ \frac{\sin(\angle BAD)}{\sin(\angle EBA)} &= \frac{\sin(\angle BAP)}{\sin(\angle PBA)} = \frac{BP}{AP}, \\ \frac{\sin(\angle CBE)}{\sin(\angle FCB)} &= \frac{\sin(\angle CBP)}{\sin(\angle PCB)} = \frac{CP}{BP}. \end{aligned}$$

Multiplying these equations,

$$\begin{aligned} \frac{\sin(\angle ACF)}{\sin(\angle FCB)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle EBA)} &= \frac{\sin(\angle ACF)}{\sin(\angle DAC)} \cdot \frac{\sin(\angle BAD)}{\sin(\angle EBA)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle FCB)} \\ &= \frac{AP}{CP} \cdot \frac{BP}{AP} \cdot \frac{CP}{BP} = 1. \end{aligned}$$

Conversely, suppose both sides of the equation from part (a) are 1, so in particular

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Let $\overline{AD}$ and $\overline{BE}$ intersect at point Q, and let line $\overline{CQ}$ intersect side $\overline{AB}$ at point F'. Then using what we just showed,

$$\frac{AF'}{F'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

This means $AF/FB = AF'/F'B$. With F and F' both interior to segment $\overline{AB}$, this can only happen if $F = F'$, which means that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur (at Q) as desired.

8. Fix points A and B on a circle of radius 4 centered at O so that $AB = 2$. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

Since $CO = 4$, we know C lies on this circle as well, and since $AC = 1$, there are two possible locations for C, one on either side of $\overline{OA}$. When C and B lie on the same side of $\overline{OA}$, we have $\angle BOC = \angle AOB - \angle AOC$, which gives us

$$\begin{aligned} BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8} \cdot \frac{31}{32} + \frac{\sqrt{15}}{8} \cdot \frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

When C and B lie on opposite sides of $\overline{OA}$, we instead have $\angle BOC = \angle BOA + \angle AOC$. A similar calculation to the first case yields

$$BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. By Stewart's theorem,

$$\begin{aligned} (AB)^2 \cdot (DC) + (AC)^2 \cdot (BD) &= (BC)[(AD)^2 + (BD)(DC)] \\ 8^2 \cdot 7x + (8x)^2 \cdot 7 &= (7 + 7x)(5^2 + 7 \cdot 7x) \\ (8^2 \cdot 7)(x + x^2) &= 7(1 + x)(25 + 49x). \end{aligned}$$

Since $x > 0$, we can safely divide by $7(1 + x)$ on both sides to get

$$64x = 25 + 49x \implies x = \frac{5}{3}.$$

Therefore, $CD = 7x = 35/3$.

10. (a) Let ABC be the corresponding triangle, with $\angle C = 120°$ opposite side $AB = c$. Extend ray $\overrightarrow{AC}$ to point D so that $CD = CB$. Then triangle BCD is equilateral, so triangle ADB has side lengths $AD = a + b$, $DB = a$, and $AB = c$ with $\angle D = 60°$. This means $\{a, a+b, c\}$ is an Eisenstein triple of acute type. A similar construction where we extend $\overrightarrow{BC}$ tells us that $\{a + b, b, c\}$ is also an Eisenstein triple of acute type.

(b) Let a, b, c be such that $\{a, b, c\}$ is an Eisenstein triple of acute type with $AB = c$ opposite the $60°$ angle in the corresponding triangle ABC. If $a = b$, then the triangle is equilateral. Otherwise, suppose without loss of generality that $a < b$. Let point D on side $\overline{AC}$ be such that $CD = CB$ and let $a' = AD = b - a$. Then triangle ADB has $\angle ADB = 120°$, so $\{a', a, c\}$ is an Eisenstein triple of obtuse type. As $b = a' + a$, the original Eisenstein triple $\{a, b, c\}$ can be constructed from $\{a', a, c\}$ according to part (a).

(c) Suppose without loss of generality that c is the side opposite the $120°$ angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

This equation describes an ellipse in the plane passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$.[1] Moreover, every point with positive rational coordinates can be connected to the point $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$\begin{aligned} x^2 + x \cdot (tx - 1) + (tx - 1)^2 &= 1, \\ (t^2 + t + 1)x^2 - (2t + 1)x &= 0. \end{aligned}$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

[1] More specifically, we can rewrite the equation as $\frac{u^2}{2/3} + \frac{v^2}{2} = 1$ where $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$. This is an ellipse centered at $(0, 0)$ whose semimajor axis has length $\sqrt{2}$ lying along the v-axis, i.e. the line $u = 0$ or equivalently $y = -x$. The semiminor axis has length $\sqrt{2/3}$ lying along the u-axis, i.e. the line $v = 0$ or equivalently $y = x$.

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m,n)=1$,

$$\begin{aligned}\gcd(2mn+n^2, m^2+mn+n^2) &= \gcd(n\cdot(2m+n), m^2+mn+n^2)\\ &= \gcd(2m+n, m^2+mn+n^2)\\ &= \gcd(2m+n, m^2+mn+n^2-n\cdot(2m+n))\\ &= \gcd(2m+n, m^2-mn) = \gcd(2m+n, m\cdot(m-n))\\ &= \gcd(2m+n, m-n) = \gcd(3n, m-n).\end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n+3k$. Then $\gcd(n,k)=1$ and

$$\gcd(3n, m-n) = \gcd(3n, 3k) = 3\gcd(n,k) = 3.$$

Otherwise,

$$\gcd(3n, m-n) = \gcd(n, m-n) = \gcd(n,m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn+n^2}{3}, \quad b = \frac{m^2-n^2}{3}, \quad c = \frac{m^2+mn+n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r-s$ and $m = r+2s$. Then

$$\begin{aligned}a &= \frac{2(r+2s)(r-s)+(r-s)^2}{3} = r^2-s^2,\\ b &= \frac{(r+2s)^2-(r-s)^2}{3} = 2rs+s^2,\\ c &= \frac{(r+2s)^2+(r+2s)(r-s)+(r-s)^2}{3} = r^2+rs+s^2,\end{aligned}$$

so the result still holds with r and s in place of m and n.

2 Complex Numbers I: Without Trig

Throughout, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

2.1 Review problems

1. Let $z = -3 + 3i$ and $w = -4 - 2i$. Compute each of the following:
   (a) $z + w$
   (b) $z - w$
   (c) zw
   (d) $\overline{w}$
   (e) z/w
   (f) $|z|$
2. Find all complex solutions to the equation $z^2 + 5 = 4z$.
3. Identify each of the following complex numbers.
   (a) The complex number corresponding to the point $(-5, -1)$.
   (b) The two complex numbers of magnitude 2 whose real and imaginary parts are equal.
   (c) The three complex numbers z for which 0, $3 - 2i$, $5 + 2i$, and z are the vertices of a parallelogram (in some order).
4. (a) Find a complex number w for which $w^2 = -16 + 30i$.
   (b) Find the two complex numbers z satisfying $2z^2 - (8 + 4i)z + (14 - 7i) = 0$.
   (c) Prove that for every complex number z, there is a complex number w for which $w^2 = z$.
   Remark: It follows from this that every quadratic polynomial with complex coefficients has complex roots (with roots given by the familiar quadratic formula).
5. (a) Let ℓ_1 be the line through $a = -4 - 3i$ and $b = 4 + i$, and let ℓ_2 be the line through $c = -4i$ and $d = -3 + 2i$.
      i. By considering slopes, or otherwise, show that ℓ_1 and ℓ_2 are perpendicular.
      ii. Compute $\frac{d-c}{b-a}$.
   (b) Show that in general, the line through $p \neq q$ is perpendicular to the line through $r \neq s$ if and only if $\frac{r-s}{p-q}$ is purely imaginary.
   (c) Given two distinct complex numbers a and b, the *perpendicular bisector* of the line segment connecting a and b is the line perpendicular to this segment passing through the midpoint $m = \frac{a+b}{2}$.
   Show that z lies on the perpendicular bisector of the line segment connecting a and b if and only if $|z - a| = |z - b|$.
   Hint: Consider squared magnitudes and use the fact that a complex number α is purely imaginary if and only if $\alpha = -\overline{\alpha}$.

6. (a) Show that $i^4 = 1$, and that conversely, if $z^4 = 1$, then $z = i^k$ for some positive integer k.
   (b) Let $z_1, z_2, z_3, \ldots$ be a 4-periodic sequence of complex numbers, meaning that $z_{n+4} = z_n$ for all positive integers n. Show that there exist complex numbers a, b, c, d such that

$$z_n = a + b \cdot i^n + c \cdot i^{2n} + d \cdot i^{3n}$$

   for all n.

7. If ℓ is a line in the complex plane, then *reflection across* ℓ is the function $f_\ell : \mathbb{C} \to \mathbb{C}$ defined the property that for any complex number z, line ℓ is the perpendicular bisector of the line segment connecting z and $f_\ell(z)$. (When z already lies on ℓ, then we define $f(z) = z$.)
   (a) What complex number operation is equivalent to reflection across the x-axis?
   (b) Let ℓ be the line passing through 0 and $4 + 2i$. Find the reflection of -5 across ℓ.
   (c) More generally, let ℓ be the line passing through 0 and d, where d is a non-zero complex number. Find the reflection of z across ℓ, i.e. determine the function $f_\ell(z)$.
   (d) Even more generally, let ℓ be the line passing through a and b, where a and b are two distinct complex numbers. Find the reflection of z across ℓ.
   (e) From the previous two parts, every reflection has the form $f(z) = \alpha\overline{z} + \beta$ where $|\alpha| = 1$. Conversely, show that any such function is a reflection composed with a translation where the translation is parallel to the line of reflection. (When the translation is non-zero, we call the overall transformation a *glide reflection.*)

2.2 Challenge problems

8. An *isometry* of the complex plane is a function $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$|f(z) - f(w)| = |z - w|$$

   for all complex numbers z and w. In other words, f preserves distances between points.
   (a) Show that every translation and every reflection is an isometry.
   (b) Let f be an isometry satisfying $f(0) = 0$, $f(1) = 1$, and $f(i) = i$. Show that f must be the identity map, i.e. $f(z) = z$ for all z.
   (c) Prove that every isometry can be written as a composition of at most three reflections.
   (d) Show that the composition of three reflections is either a reflection or a glide reflection.

9. In this problem, we work through one formal construction of the complex numbers.

   Let $\mathcal{C}$ be the set of all ordered pairs of real numbers, and define operations $\oplus$ and $\otimes$ on $\mathcal{C}$ by

$$(a, b) \oplus (c, d) = (a + c, b + d),$$
$$(a, b) \otimes (c, d) = (ac - bd, ad + bc).$$

   We call $\oplus$ and $\otimes$ the addition and multiplication on $\mathcal{C}$, respectively.
   (a) The first task is to show that $\mathcal{C}$, with these operations, satisfies the "usual rules" of algebra. In fancy language, we would say that $\mathcal{C}$ is a *field.*

i. (Associative rules) Show that for any $u, v, w \in \mathcal{C}$,

$$u \oplus (v \oplus w) = (u \oplus v) \oplus w \quad \text{and} \quad u \otimes (v \otimes w) = (u \otimes v) \otimes w.$$

ii. (Commutative rules) Show that for any $z, w \in \mathcal{C}$,

$$z \oplus w = w \oplus z \quad \text{and} \quad z \otimes w = w \otimes z.$$

iii. (Distributive rule) Show that for any $u, v, w \in \mathcal{C}$,

$$u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w).$$

iv. (Identity rules) Show that for any $z \in \mathcal{C}$,

$$z \oplus (0,0) = (0,0) \oplus z = z \quad \text{and} \quad z \otimes (1,0) = (1,0) \otimes z = z.$$

This makes $(0,0)$ and $(1,0)$ the *additive identity* and *multiplicative identity* in $\mathcal{C}$.

v. (Additive inverse rule) Show that for any $z \in \mathcal{C}$, there exists $a_z \in \mathcal{C}$ such that

$$z \oplus a_z = (0,0).$$

The element a_z is the *additive inverse* of z in $\mathcal{C}$, and we denote it by $-z$.

vi. (Multiplicative inverse rule) Show that for any $z \in \mathcal{C}$ other than $(0,0)$, there exists $m_z \in \mathcal{C}$ such that

$$z \otimes m_z = (1,0).$$

The element m_z is the *multiplicative inverse* of z in $\mathcal{C}$, and we denote it by z^{-1}.

From these properties, all of the familiar algebraic rules can be shown to hold, such as the zero product property and certain common factorisations. Next, for this to reasonably be called an extension of the real numbers, we need to show that $\mathcal{C}$, with these operations, "contains" $\mathbb{R}$ with its usual addition and multiplication. This is made precise in the next part.

(b) Prove that for any two real numbers x and y,

$$(x,0) \oplus (y,0) = (x+y,0) \quad \text{and} \quad (x,0) \otimes (y,0) = (xy,0).$$

This shows that the elements $(r,0)$ for $r \in \mathbb{R}$, with operations $\oplus$ and $\otimes$, "act like" the real numbers with the usual addition and multiplication operations $+$ and $\times$.

With "$\mathcal{C}$ extends $\mathbb{R}$" shown, when r is a real number we simply write r instead of $(r,0)$, and we write $+$ and $\times$ (or $\cdot$) instead of $\oplus$ and $\otimes$. We also introduce the subtraction and division operations as $z - w = z + (-w)$ and $z/w = z \cdot w^{-1}$.

Finally, the complex numbers should have a square root of -1.

(c) Show that $(0,1) \times (0,1) = -1$ and $(0,-1) \times (0,-1) = -1$.

We can now recover the usual notation, replacing $\mathcal{C}$ with $\mathbb{C}$ and forever forgetting the initial definitions, by defining $i = (0,1)$ and then observing that $(x,y) = x + y \cdot i$.

10. A function $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-*automorphism of* $\mathbb{C}$ if

$$f(z+w) = f(z) + f(w) \quad \text{and} \quad f(zw) = f(z) \cdot f(w)$$

for all $z, w \in \mathbb{C}$ and $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$, then $f(i) = i$ or $f(i) = -i$.

(b) Show that the only two $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity function $f(z) = z$ and the conjugation function $f(z) = \overline{z}$.

2.3 Answers

1. (a) $-7+i$
   (b) $1+5i$
   (c) $18-6i$
   (d) $-4+2i$
   (e) $\frac{3}{10}-\frac{9}{10}i$
   (f) $3\sqrt{2}$
2. $2+i$ and $2-i$
3. (a) $-5-i$
   (b) $\sqrt{2}+\sqrt{2}i$ and $-\sqrt{2}-\sqrt{2}i$
   (c) 8, $-2-4i$, and $2+4i$
4. (a) Let $w=a+bi$, so then $w^2=(a^2-b^2)+(2ab)i$. This yields the system of equations

$$a^2-b^2=-16 \quad \text{and} \quad 2ab=30.$$

From the second equation, $b=15/a$. Substituting into the first equation,

$$\begin{aligned} a^2-\frac{225}{a^2}&=-16,\\ a^4-225&=-16a^2,\\ a^4+16a^2-225&=0,\\ (a^2+25)(a^2-9)&=0. \end{aligned}$$

Since a is real, $a^2\geq 0$, so we must take $a^2=9$ and hence $a=\pm 3$. If $a=3$, then $b=5$, and if $a=-3$, then $b=-5$, so the two square roots are $\pm(3+5i)$.

   (b) Applying the quadratic formula and using the result from part (a), the solutions are

$$\begin{aligned} z&=\frac{(8+4i)\pm\sqrt{(8+4i)^2-4\cdot 2\cdot(14-7i)}}{2\cdot 2}\\ &=\frac{(8+4i)\pm\sqrt{(64-16+64i)-(112-56i)}}{4}\\ &=\frac{(8+4i)\pm\sqrt{-64+120i}}{4}\\ &=\frac{(8+4i)\pm 2\sqrt{-16+30i}}{4}\\ &=\frac{(4+2i)\pm(3+5i)}{2}. \end{aligned}$$

Simplifying in each case, we get $\frac{7}{2}+\frac{7}{2}i$ and $\frac{1}{2}-\frac{3}{2}i$.

(c) Let $z = x + yi$ and $w = a + bi$. Setting up as in part (a), we get the system

$$a^2 - b^2 = x \quad \text{and} \quad 2ab = y,$$

and substituting $b = \frac{y}{2a}$ yields

$$a^2 - \frac{y^2}{4a^2} = x \qquad \Longleftrightarrow \qquad 4a^4 - 4xa^2 - y^2 = 0.$$

This is a real quadratic in a^2 for which the product of the roots is non-negative, so there is a non-negative solution for a^2. Taking either square root of this value gives us a real value of a, hence a corresponding real value of b, and $w = a + bi$ is the desired solution.

5. (a) i. The slope of ℓ_1 is $\frac{1-(-3)}{4-(-4)} = \frac{1}{2}$ and the slope of ℓ_2 is $\frac{2-(-4)}{-3-0} = -2$. The product of their slopes is -1, so the lines are perpendicular.

ii. $\frac{d-c}{b-a} = \frac{-3+6i}{8+4i} = \frac{3}{4} \cdot \frac{-1+2i}{2+i} \cdot \frac{2-i}{2-i} = \frac{3}{4} \cdot \frac{5i}{5} = \frac{3}{4}i$

(b) Translating each line individually does not impact perpendicularity, so without loss of generality, we can translate so that $q = s = 0$ and show that for any two non-zero complex numbers r and p, the line through 0 and $r = a + bi$ is perpendicular to the line through 0 and $p = c + di$ if and only if r/p is purely imaginary.

The slope of the line through 0 and r is b/a and the slope of the line through 0 and p is d/c, so the two lines are perpendicular if and only if $\frac{bd}{ac} = -1$, or $ac + bd = 0$. (The latter condition also detects when one line is horizontal and the other line is vertical.)

The quotient r/p is purely imaginary if and only if it is equal to the negative of its conjugate, so we compute

$$\frac{r}{p} + \frac{\overline{r}}{\overline{p}} = \frac{r\overline{p} + \overline{r}p}{p\overline{p}} = \frac{2\,\mathrm{Re}(r\overline{p})}{|p|^2} = \frac{2(ac + bd)}{|p|^2}.$$

Hence r/p is purely imaginary if and only if $ac + bd = 0$, which is exactly the condition we found for perpendicularity.

(c) From part (b), z lies on the perpendicular bisector of the segment connecting a and b if and only if $\frac{z-m}{a-b}$ is purely imaginary. In terms of conjugates, this is equivalent to

$$\begin{aligned}
\frac{z-m}{a-b} + \frac{\overline{z} - \overline{m}}{\overline{a} - \overline{b}} &= 0,\\
(\overline{a} - \overline{b})(z - m) + (a - b)(\overline{z} - \overline{m}) &= 0,\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} - [(\overline{a} - \overline{b})m + (a - b)\overline{m}] &= 0,\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} &= \frac{(\overline{a} - \overline{b})(a + b)}{2} + \frac{(a - b)(\overline{a} + \overline{b})}{2},\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} &= a\overline{a} - b\overline{b}.
\end{aligned}$$

The condition $|z - a| = |z - b|$ is equivalent to $|z - a|^2 = |z - b|^2$, or

$$\begin{aligned}
(z - a)(\overline{z} - \overline{a}) &= (z - b)(\overline{z} - \overline{b}),\\
z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} &= z\overline{z} - \overline{b}z - b\overline{z} + b\overline{b},\\
a\overline{a} - b\overline{b} &= (\overline{a} - \overline{b})z + (a - b)\overline{z}.
\end{aligned}$$

Thus the two statements are equivalent, as desired.

6. (a) For the first statement, $i^2 = -1$ and $i^3 = -i$ and $i^4 = -i^2 = -(-1) = 1$.
For the second statement, $z^4 - 1$ factors as $(z-1)(z+1)(z^2+1)$. The last factor has roots $i = i^1$ and $-i = i^3$, while $1 = i^4$ and $-1 = i^2$ are the other two roots of $z^4 - 1$.

   (b) By part (a), any sequence of the form specified on the right hand side is 4-periodic, so it suffices to show that there exist complex numbers a, b, c, d such that

$$\begin{aligned}
z_1 &= a + b \cdot i^1 + c \cdot i^2 + d \cdot i^3 = a + ib - c - id, \\
z_2 &= a + b \cdot i^2 + c \cdot i^4 + d \cdot i^6 \qquad = a - b + c - d, \\
z_3 &= a + b \cdot i^3 + c \cdot i^6 + d \cdot i^9 \qquad = a - ib - c + id, \\
z_4 &= a + b \cdot i^4 + c \cdot i^8 + d \cdot i^{12} \qquad = a + b + c + d.
\end{aligned}$$

This amounts to solving a system of linear equations. Adding the first and third equations gives $z_1 + z_3 = 2a - 2c$ while adding the second and fourth gives $z_2 + z_4 = 2a + 2c$. Subtracting the first and third equations gives $z_1 - z_3 = 2ib - 2id$, while subtracting the second and fourth gives $z_4 - z_2 = 2b + 2d$. The new system

$$\begin{aligned}
2a - 2c &= z_1 + z_3 & \qquad 2ib - 2id &= z_1 - z_3 \\
2a + 2c &= z_2 + z_4 & \qquad 2b + 2d &= -z_2 + z_4
\end{aligned}$$

has as a solution for (a, b, c, d) the 4-tuple

$$\left(\frac{z_1 + z_2 + z_3 + z_4}{4}, \frac{z_1 - iz_2 - z_3 + iz_4}{4i}, \frac{-z_1 + z_2 - z_3 + z_4}{4}, \frac{-z_1 - iz_2 + z_3 + iz_4}{4i} \right),$$

and it can be checked that this satisfies the original system as well.

7. (a) Complex conjugation

   (b) $-3 - 4i$

   (c) First, the midpoint of the segment connecting z and $w = f_\ell(z)$ has to lie on ℓ, so $\frac{z+w}{2}$ is a real multiple of d. That is, $\frac{z+w}{2d}$ is real, so

$$\begin{aligned}
\frac{z + w}{2d} &= \frac{\overline{z} + \overline{w}}{2\overline{d}}, \\
\overline{d}(z + w) &= d(\overline{z} + \overline{w}), \\
\overline{d}w - d\overline{w} &= -\overline{d}z + d\overline{z}. \qquad (1)
\end{aligned}$$

Second, the segment has to be perpendicular to ℓ, so $\frac{z-w}{d}$ is purely imaginary. Therefore,

$$\begin{aligned}
\frac{z - w}{d} &= -\frac{\overline{z} - \overline{w}}{\overline{d}}, \\
\overline{d}(z - w) &= -d(\overline{z} - \overline{w}), \\
-\overline{d}w - d\overline{w} &= -\overline{d}z - d\overline{z}. \qquad (2)
\end{aligned}$$

Taking the difference $(1) - (2)$ gives us $f_\ell(z) = w = \frac{d}{\overline{d}} \cdot \overline{z}$.

(d) We can translate a to the origin, reflect, then translate back, so

$$f_\ell(z) = f_{\ell-a}(z-a) + a = \frac{b-a}{\overline{b}-\overline{a}} \cdot (\overline{z}-\overline{a}) + a.$$

(e)

8. (a) First let f be a translation, $f(z) = z + a$. Then

$$|f(z) - f(w)| = |(z+a) - (w+a)| = |z-w|.$$

Now we consider reflections. The composition of isometries is also an isometry, so since general reflections can be obtained by composing reflections across lines through the origin with translations, it suffices to consider reflections across lines through 0 and another point d. Then

$$|f(z) - f(w)| = \left|\frac{d}{\overline{d}}\overline{z} - \frac{d}{\overline{d}}\overline{w}\right| = \frac{|d|}{|d|}|\overline{z-w}| = |z-w|.$$

(b) Let z be arbitrary and let $w = f(z)$. By the isometry property,

$$\begin{aligned} |f(z) - f(0)| &= |z-0| & w\overline{w} &= z\overline{z}, \\ |f(z) - f(1)| &= |z-1| & = (w-1)(\overline{w}-1) &= (z-1)(\overline{z}-1), \\ |f(z) - f(i)| &= |z-i| & = (w-i)(\overline{w}+i) &= (z-i)(\overline{z}+i). \end{aligned}$$

Expanding the latter two equations gives us

$$\begin{aligned} w\overline{w} - w - \overline{w} + 1 &= z\overline{z} - z - \overline{z} + 1, \\ w\overline{w} + iw - i\overline{w} + 1 &= z + iz - i\overline{z} + 1. \end{aligned}$$

Then, substituting $w\overline{w} = z\overline{z}$ and rearranging yields the system of equations

$$\begin{aligned} w + \overline{w} &= z + \overline{z}, \\ iw - i\overline{w} &= iz - i\overline{z}. \end{aligned}$$

Eliminating $\overline{w}$ gives us $2iw = 2iz$, so $w = z$ as required.

(c) Let f be a given isometry and suppose $a = f(0)$, $b = f(1)$, and $c = f(i)$. We define three reflections and compositions as follows:

i. If $a = 0$, then let r_1 be the identity map. Otherwise, let r_1 be reflection across the perpendicular bisector of the segment connecting 0 and a. Then $f_1 = r_1 \circ f$ is an isometry with $f_1(0) = 0$.

ii. Let $b' = r_1(b) = f_1(1)$ and $c' = r_1(c) = f_1(i)$. If $b' = 1$, then let r_2 be the identity map. Otherwise, as f_1 is an isometry, $0 = f_1(0)$ lies on the perpendicular bisector of the segment connecting 1 and $b' = f_1(1)$. As such, we can let r_2 be reflection across this perpendicular bisector, so $f_2 = r_2 \circ f_1$ satisfies $f_2(0) = 0$ and $f_2(1) = 1$.

iii. Let $c'' = r_2(c') = f_2(i)$. If $c'' = i$, then let r_3 be the identity map. Otherwise, as f_2 is an isometry, both $0 = f_2(0)$ and $1 = f_2(1)$ are equidistant from i and $c'' = f_2(i)$, so the reflection across the real axis sends c'' to i. Let this reflection be r_3, so then $f_3 = r_3 \circ f_2$ satisfies $f_3(0) = 0$, $f_3(1) = 1$, and $f_3(i) = i$.

By part (b), $f_3 = (r_3 \circ r_2 \circ r_1) \circ f$ must be the identity map. Composing on the left with r_3, then r_2, then r_1, we get $f = r_1 \circ r_2 \circ r_3 \circ f_3 = r_1 \circ r_2 \circ r_3$ as required.

(d) Note that the formula we found for general reflections takes the form $f(z) = \alpha(\overline{z} - \overline{a}) + a$, where $|\alpha| = 1$. Let our three reflections be

$$\begin{aligned} r_1(z) &= \alpha(\overline{z} - \overline{a}) + a, \\ r_2(z) &= \beta\overline{z} + b, \\ r_3(z) &= \gamma\overline{z} + c. \end{aligned}$$

Then

$$\begin{aligned} (r_1 \circ r_2 \circ r_3)(z) &= \alpha\overline{(\beta\overline{[\gamma\overline{z} + c]} + b)} + a \\ &= \alpha\overline{(\beta[\overline{\gamma}z + \overline{c}] + b)} + a \\ &= \alpha \end{aligned}$$