Algebra 1 Practice Problems III

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2023-2024

The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

Contents

1 Functions

Throughout, the notation $f : D \to \mathbb{R}$ means that f is a function whose domain is D and whose output values are real numbers. By the "natural domain" of a formula, we mean the largest subset of the reals which can be the domain of a function given by that formula. For example, the natural domain of $1/x$ is the set of all real numbers other than 0.

1.1 Review problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 - 6$. Evaluate each of the following:
   (a) $f(0)$
   (b) $f(f(1))$
   (c) $f^6(3)$
   (d) $(f(3))^6$
   (e) $f(3+4) - (f(3) + f(4))$
2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 - 6$ and let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = 2x$. Evaluate each of the following:
   (a) $(f + g)(2)$
   (b) $(f \cdot g)(2)$
   (c) $(f \circ g)(2)$
   (d) $(g \circ f)(2)$
   (e) $f(g(f(2)))$
   (f) $(f^2 \circ g)(2)$
3. Determine the natural domains of each of the following formulas.
   (a) $x^3 - x$
   (b) $\frac{x+2}{(x-2)(x-3)}$
   (c) $\sqrt{2x - 8}$
   (d) $\sqrt[4]{x^2 - 5x - 14}$
4. For each of the following functions, determine the range. If the domain is unspecified and only a formula is given, assume that the corresponding function has the natural domain.
   (a) x^2
   (b) x^3
   (c) $\frac{1}{x-1} + 4$
   (d) $\sqrt{x+3} - 2$
   (e) x^2 with domain $(-2, 3]$
   (f) $\frac{2x}{x^2+1}$

5. For each of the following functions, determine whether the function is invertible. If so, find the inverse function (including domain specification as needed). If not, find two input values which produce the same output.

   (a) x^2

   (b) x^2 with domain $(-2, 3]$

   (c) $\frac{1}{x-1} + 4$

   (d) $\sqrt{x+3} - 2$

   (e) $x^2 - 6x + 8$ with domain $(3, +\infty)$

   (f) $\frac{2x}{x^2+1}$ with domain $[-1, 1]$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Describe a sequence of transformations that would transform the graph of f into the graph of the given equation.

   (a) $y = f(x) - 3$

   (b) $y = -2f(x)$

   (c) $y = f(x - 5)$

   (d) $y = f(x/4)$

   (e) $y = 3f(2x - 1) + 4$

   (f) $(y + 1)/2 = f(-x + 6)$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(4) = 7$ and let $g : \mathbb{R} \to \mathbb{R}$ be given by the formula $g(x) = 3f(x^2) - 4$.

   (a) Find two points on the graph of g.

   (b) Does g have an inverse?

1.2 Challenge problems

8. Let $f(x) = -1/(x + 1)$. Given that $f^6(x) = x$ for all x, compute $f^5(2024)$.

9. Let $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ be given by

$$\begin{array}{llll} f(1) = 6, & f(2) = 4, & f(3) = 2, & f(4) = 3, \\ f(5) = 7, & f(6) = 8, & f(7) = 1, & f(8) = 5. \end{array}$$

   Find the smallest positive integer n such that $f^n(k) = k$ for all $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

10. (Thomae's function) For any rational number q, let denom(q) be the least positive integer k for which kq is an integer. (In other words, denom(q) is the denominator of q when written as a fraction in lowest terms.) Sketch the graph of the function with domain $(0, 1)$

$$T(x) = \begin{cases} \frac{1}{\text{denom}(x)} & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

1.3 Answers

1. (a) $f(0) = -6$

   (b) $f(f(1)) = f(-5) = 19$

   (c) Since $f(3) = 3$, the function f doesn't do anything to 3, so $f^6(3) = 3$ as well.

   (d) $(f(3))^6 = 3^6 = 729$

   (e) $f(3 + 4) - (f(3) + f(4)) = f(7) - f(3) - f(4) = 43 - 3 - 10 = 30$

2. (a) $(f + g)(2) = f(2) + g(2) = -2 + 4 = 2$

   (b) $(f \cdot g)(2) = f(2) \cdot g(2) = (-2) \cdot 4 = -8$

   (c) $(f \circ g)(2) = f(g(2)) = f(4) = 10$

   (d) $(g \circ f)(2) = g(f(2)) = g(-2) = -4$

   (e) $f(g(f(2))) = f(-4) = 10$

   (f) $(f^2 \circ g)(2) = f^2(g(2)) = f(f(g(2))) = f(10) = 94$

3. (a) $\mathbb{R}$, or equivalently, $(-\infty, +\infty)$

   (b) $(-\infty, 2) \cup (2, 3) \cup (3, +\infty)$, or equivalently, $\mathbb{R} \backslash \{2, 3\}$

   (c) $[4, +\infty)$

   (d) We need $x^2 - 5x - 14 \geq 0$. The left hand side factors as $(x + 2)(x - 7)$, and this is non-negative when $x \leq -2$ (when both factors are non-positive) and when $x \geq 7$ (when both factors are non-negative). The domain is $(-\infty, -2] \cup [7, +\infty)$.

4. (a) $[0, +\infty)$

   (b) $\mathbb{R}$, or equivalently, $(-\infty, +\infty)$

   (c) $(-\infty, 4) \cup (4, +\infty)$, or equivalently, $\mathbb{R} \backslash \{4\}$

   (d) $[-2, +\infty)$

   (e) $[0, 9]$

   (f) Let y be in the range, so there is a real number x satisfying

   $$y = \frac{2x}{x^2 + 1}.$$

   Clearing denominators (which is reversible since $x^2 + 1 \neq 0$ for real x) and rearranging,

   $$yx^2 - 2x + y = 0. \tag{$\dagger$}$$

   This has a real solution for x if and only if the discriminant

   $$(-2)^2 - 4 \cdot y \cdot y = 4 - 4y^2$$

   is non-negative. This occurs when $-1 \leq y \leq 1$, and when this condition is met, real solutions for x are given by the quadratic formula. Thus the range is $[-1, 1]$.

5. (a) No inverse, e.g. $(-1)^2 = 1^2$

(b) No inverse, e.g. $(-1)^2 = 1^2$

(c) Inverse given by $y \mapsto 1 + \frac{1}{y-4}$

(d) Inverse given by $y \mapsto (y+2)^2 - 3$ with domain $[-2, +\infty)$

(e) Inverse given by $y \mapsto \sqrt{y+1} + 3$ with domain $(-1, +\infty)$

(f) Solving (†) for x yields, when $y \neq 0$,

$$x = \frac{2 \pm \sqrt{4 - 4y^2}}{2y}.$$

The solution which falls within the required domain $[-1, 1]$ of the original function uses the $-$ sign in the $\pm$. When $y = 0$, we get $x = 0$, so the inverse function is

$$y \longmapsto \begin{cases} \frac{2-\sqrt{4-4y^2}}{2y} & y \neq 0, \\ 0 & y = 0. \end{cases}$$

6. We can swap the order in which we perform horizontal and vertical transformations (perhaps even intersperse them), but we have to keep horizontal transformations in the same order relative to each other, and similarly for vertical transformations.

   (a) Vertical translation by 3 down

   (b) Vertical dilation by -2 (i.e. reflect across x-axis and vertical dilation by 2)

   (c) Horizontal translation by 5 right

   (d) Horizontal dilation by 4

   (e) For the horizontal, (i) translate right by 1, then (ii) dilate by a factor of $1/2$.
   For the vertical, (i) dilate by a factor of 3, then (ii) translate up by 4.

   (f) For the horizontal, (i) translate left by 6, then (ii) reflect across the y-axis.
   For the vertical, (i) dilate by a factor of 2, then (ii) translate down by 1.

7. (a) $(-2, 17)$ and $(2, 17)$

   (b) The horizontal line $y = 17$ meets the graph of g in at least two points, namely the ones from part (a), so g has no inverse.

8. Let $y = f^5(2024)$. Then $f(y) = f^6(2024) = 2024$, so we need to solve the equation

$$\frac{-1}{1+y} = 2024.$$

   The solution is $y = -\frac{2025}{2024}$.

9. The numbers $\{2, 4, 3\}$ cycle every 3 iterations of f while the numbers $\{1, 6, 8, 5, 7\}$ cycle every 5 iterations of f. To get all of the numbers to cycle, we need $\operatorname{lcm}(3, 5) = \boxed{15}$ iterations.

10. Wikipedia page.

2 Special Functions

2.1 Review problems

1. Let $f(x) = 4x^3 - 6x^4 + 2x - 1$ and $g(x) = 3x^4 + x^2 + 9$ and $h(x) = \sqrt{x^2 + 1}$.
   (a) Which of f, g, and h are polynomial functions?
   (b) Evaluate $f(0)$ and $g(1)$.
   (c) Simplify $f(x) + g(x)$ and $f(x) \cdot g(x)$.
   (d) Compute $\deg f(x)$ and $\deg g(x)$.
   (e) Is there a constant a for which $f(x) + a \cdot g(x)$ has degree less than 4? If so, what is the value of a and the degree of $f(x) + a \cdot g(x)$ for that value? If not, explain why not.

2. Let $p(x) = x^4 + 9x^3 + 28x^2 + 39x + 21$. This polynomial has no integer roots, but there are positive integers a, b, c, d such that

   $$p(x) = (x^2 + ax + b)(x^2 + cx + d).$$

   Find all real roots of $p(x)$.

3. Solve each of the following equations for real solutions for x.
   (a) $\sqrt[6]{625} = 5^x$
   (b) $9^{1+x} = 27^{1+1/x}$
   (c) $4^x - 2^x = 56$

4. (Calculator allowed) Janelle puts \$10,000 into an account that earns 4% interest compounded annually. How much money will there be in the account after 18 years?

5. Compute each of the following logarithms without a calculator (some are undefined).
   (a) $\log_7(49)$
   (b) $\log_2(64)$
   (c) $\log_{27}(9)$
   (d) $\log_5(1)$
   (e) $\log_4(0)$
   (f) $\log_6(-6)$
   (g) $\log_8(1/4)$
   (h) $\log_{12}\left(2\sqrt[3]{18}\right)$

6. Find the domain and range of the function $f(x) = \log_{10}\left(\sqrt{100 - x^2}\right)$.

7. Find the values of a, b, and c which make the following function continuous:

   $$f(x) = \begin{cases} x^2 - 3 & x \leq -1; \\ ax + b & -1 < x \leq 2; \\ \frac{2x^2-3x-9}{x^2-4x+3} & x > 2 \text{ and } x \neq 3; \\ c & x = 3. \end{cases}$$

2.2 Challenge problems

8. Let $p(x) = (1 + x)^{20}$.

    (a) Find the sum of the coefficients of $p(x)$.

    (b) Find the sum of the coefficients of $p(x^2 - 3x + 2)$.

    (c) (Very challenging) Find the sum of the coefficients of the terms of $p(x)$ whose degrees are multiples of 4.

9. (Calculator allowed) In this problem, i is *not* the imaginary unit.

    (a) Let $E(i, m)$ denote the *effective annual interest rate* for a nominal annual interest rate i compounded m times per year. That is, for any whole number of years, compounding annually at a rate of $E(i, m)$ results in the same balance as compounding m times per year with a nominal annual interest rate i.

    Write down a formula for $E(i, m)$ in terms of i and m, then compute $E(4\%, 12)$.

    (b) Let $F(i, m)$ be the solution to the equation $E(F(i, m), m) = i$. Compute $F(4\%, 12)$.

    (c) If we hold i fixed, then as m gets larger and larger, $F(i, m)$ approaches a value called the *force of interest*. For the interest rate $i = 4\%$, compute the force of interest as a percentage to one decimal place.

10. Compute $(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$ without a calculator.

2.3 Answers

1. (a) f and g are polynomials, h is not
   (b) $f(0) = -1$ and $g(1) = 13$
   (c)
   $$\begin{aligned} f(x) + g(x) &= -3x^4 + 4x^3 + x^2 + 2x + 8 \\ f(x) \cdot g(x) &= -18x^8 + 12x^7 - 6x^6 + 10x^5 - 57x^4 + 38x^3 - x^2 + 18x - 9 \end{aligned}$$
   (d) $\deg f(x) = \deg g(x) = 4$
   (e) When $a = 2$,
   $$f(x) + a \cdot g(x) = 4x^3 + 2x^2 + 2x + 17$$
   so $\deg(f(x) + a \cdot g(x)) = 3$.

2. We can suppose that $b \leq d$, as otherwise we simply swap the two factors of $p(x)$. Expanding,
   $$\begin{aligned} a + c &= 9, \\ b + ac + d &= 28, \\ ad + bc &= 39, \\ bd &= 21. \end{aligned}$$
   Since a, b, c, d are positive integers, either $b = 1$ and $d = 21$ or $b = 3$ and $d = 7$. In the first case, we have $a + c = 9$ and $21a + b = 39$, but this is not satisfied by positive integers. Thus we are in the second case, so $a + c = 9$ and $7a + 3b = 39$. This system has solution $a = 3$ and $c = 6$, and we can check the other equations (or by expanding from scratch) to see that
   $$p(x) = (x^2 + 3x + 3)(x^2 + 6x + 7).$$
   If r is a root of $p(x)$, then $p(r) = 0$, so either $r^2 + 3r + 3 = 0$ or $r^2 + 6r + 7 = 0$. By the quadratic formula, the first case gives us
   $$r = \frac{-3 \pm \sqrt{-3}}{2},$$
   which are non-real roots, while the second case gives us
   $$r = \frac{-6 \pm \sqrt{8}}{2} = \boxed{-3 \pm \sqrt{2}}.$$

3. (a) $\sqrt[6]{625} = (5^4)^{1/6} = 5^{4/6} = 5^{2/3} \implies x = 2/3$
   (b) We have $9^{1+x} = (3^2)^{1+x} = 3^{2x+2}$ and $27^{1+1/x} = (3^3)^{1+1/x} = 3 + 3/x$, so
   $$2x + 2 = 3 + \frac{3}{x}.$$
   Multiplying through by x gives $2x^2 + 2x = 3x + 3$, or $2x^2 - x - 3 = 0$. Factoring the left hand side yields
   $$(2x - 3)(x + 1) = 0,$$
   so the possible solutions are $x = 3/2$ and $x = -1$. Both solutions work.

(c) If $y = 2^x$, then $y^2 - y = 56$. This has solutions $y = 8$ and $y = -7$. There is no real x for which $2^x = -7$, while $2^x = 8$ is satisfied for $x = 3$.

4. $\$10{,}000 \cdot (1 + 0.04)^{18} \approx \$20{,}258.17$

5. (a) 2
   (b) 6
   (c) 2/3
   (d) 0
   (e) Undefined
   (f) Undefined
   (g) −2/3
   (h) 2/3

6. Domain: For the logarithm to be defined, we need $\sqrt{100 - x^2}$ to be defined and positive. This occurs for $-10 < x < 10$, or $x \in \boxed{(-10, 10)}$.

   Range: For $x \in \boxed{(-10, 10)}$, the quantity $\sqrt{100 - x^2}$ can take any value in the interval $(0, 10]$, and no value outside this interval. Applying the base 10 logarithm to this, the interval of possible outputs is $\boxed{(-\infty, 1]}$.

7. First, we find that
$$\frac{2x^2 - 3x - 9}{x^2 - 4x + 3} = \frac{(x-3)(2x+3)}{(x-3)(x-1)} = \frac{2x+3}{x-1}$$
for $x \neq 3$, so the third line of the function definition simplifies. With this, to ensure continuity,
$$\begin{aligned} x^2 - 3 &= ax + b && \text{at } x = -1, \\ ax + b &= \frac{2x+3}{x-1} && \text{at } x = 2, \\ \frac{2x+3}{x-1} &= c && \text{at } x = 3. \end{aligned}$$
The last equation gives us $c = 9/2$, while the first two equations give us the system
$$\begin{aligned} -a + b &= -2, \\ 2a + b &= 7. \end{aligned}$$
This system is solved by $a = 3$ and $b = 1$.

8. (a) The sum of the coefficients of $p(x)$ is $p(1) = \boxed{2^{20}} = 1\,048\,576$.
   (b) The sum of the coefficients of $q(x) = p(x^2 - 3x + 2)$ is $q(1) = p(0) = \boxed{1}$.

(c) Let $p(x) = a_{20}x^{20} + a_{19}x^{19} + \cdots + a_1x + a_0$. Since the powers of i go in the cycle $1, i, -1, -i$, it turns out that

$$a_0 + a_4 + a_8 + a_{12} + a_{16} + a_{20} = \frac{p(1) + p(i) + p(-1) + p(-i)}{4}.$$

We found $p(1) = 2^{20}$ before, while $p(-1) = (1 + (-1))^{20} = 0$. For the other two terms,

$$p(\pm i) = (1 \pm i)^{20} = (\pm 2i)^{10} = 2^{10}i^{10} = -2^{10},$$

so our final answer is

$$\frac{2^{20} - 2^{10} + 0 - 2^{10}}{4} = \boxed{2^{18} - 2^9} = 261\,632.$$

9. (a) $E(i, m) = \left(1 + \dfrac{i}{m}\right)^m - 1$ and $E(4\%, 12) \approx 4.074\%$

(b) For ease of notation, let $j = F(i, m)$, so that we need $E(j, m) = i$, or

$$\left(1 + \frac{j}{m}\right)^m - 1 = i.$$

This is solved by $j = m\left[(1 + i)^{1/m} - 1\right]$. Then $F(4\%, 12) \approx 3.928\%$.

(c) Trying larger and larger values of m, we find $F(4\%, m)$ approaches $\approx 3.9\%$.

10. We will first prove that for all a, b, c (for which the involved logarithms are defined),

$$(\log_a b)(\log_b c) = \log_a c. \tag{$\star$}$$

Let $x = \log_a b$ and $y = \log_b c$. Then $a^x = b$ and $b^y = c$, so

$$c = b^y = (a^x)^y = a^{xy}.$$

This means $\log_a c = xy$, as required.

Applying ($\star$) repeatedly to the given problem,

$$\begin{aligned}(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8) &= (\log_4 6)(\log_6 7)(\log_7 8) \\ &= (\log_4 7)(\log_7 8) \\ &= \log_4 8 = \boxed{\frac{3}{2}}.\end{aligned}$$

3 Sequences and Series

3.1 Review problems

1.

3.2 Challenge problems

2.

3.3 Answers

1.