Precalculus Practice Problems: Midterm 2

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The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines **2**
1.1 Review problems 2
1.2 Challenge problems 3
1.3 Answers 4
2 Complex Numbers I: Without Trig **8**
2.1 Review problems 8
2.2 Challenge problems 9
2.3 Answers 12

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. *SAS congruence.* Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Using the law of sines, or otherwise, find $\sin A$ and $\sin B$.
   (d) Show that $\angle A = \arcsin(\sin A)$ and $\angle B = \arcsin(\sin B)$, and hence compute $\angle A$ and $\angle B$. (Hint: For which angles does $\arcsin(\sin\theta) = \theta$ hold?)
2. *SSS congruence.* Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Using the law of cosines, or otherwise, find $\cos A$, $\cos B$, and $\cos C$.
   (b) Compute $\angle A$, $\angle B$, and $\angle C$.
   (c) Find $[ABC]$.
3. *ASA/AAS congruence.* Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.
4. *SSA non-congruence.* Let ABC be a triangle with $\angle A = 30°$, $a = 6$, and $b = 9$.
   (a) Show that $c^2 - (9\sqrt{3})c + 45 = 0$.
   (b) Find all possible values of c.
5. *Extended law of sines.* If ABC is a triangle with circumradius R, then the **extended law of sines** states that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Prove that $R = \dfrac{abc}{4[ABC]}$.
   (b) Given that $a = 13$, $b = 14$, and $c = 15$, find R.
   (c) Prove the extended law of sines for acute triangles.
6. Let ABC be a triangle and let D be a point on side $\overline{BC}$.
   (a) *Ratio lemma.* Prove that
$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(b) *Angle bisector theorem.* Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

7. *Heron's formula.* Let ABC be a triangle.

   (a) Show that
   $$[ABC]^2 = \frac{1}{4}a^2b^2(1 - \cos^2 C) = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{16}.$$

   (b) Conclude that
   $$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$
   where $s = (a + b + c)/2$ is the *semiperimeter* of triangle ABC.

1.2 Challenge problems

8. Points O, A, B, and C are placed in three-dimensional space so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. What are the shortest and longest possible lengths of BC?

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Given that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. (Eisenstein triples) An *Eisenstein triple* is a triple of positive integers (a, b, c) for which a triangle with side lengths a, b, and c has an angle of measure either $60°$ or $120°$. If the Eisenstein triple (a, b, c) corresponds to a triangle with an angle of measure $60°$, we will call it an Eisenstein triple of *acute type*, and otherwise, we call it an Eisenstein triple of *obtuse type*. (The "acute type" and "obtuse type" names are non-standard.)

    (a) Let (a, b, c) be an Eisenstein triple of obtuse type with $a < b < c$. Show that $(a, a + b, c)$ and $(a + b, b, c)$ are Eisenstein triples of acute type.

    (b) Conversely, show that every Eisenstein triple of acute type arises from an Eisenstein triple of obtuse type in the above manner.

    (c) Show that if (a, b, c) is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that
    $$\{a, b, c\} = \{m^2 + mn + n^2, 2mn + n^2, m^2 - n^2\}.$$
    (Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104°) \approx 2.426$
   (b) $c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{26 - 10\cos(104°)} \approx 5.331$
   (c) $\angle A = \arcsin(\frac{a\sin C}{c}) \approx 10.49°$
   $\angle B = \arcsin(\frac{b\sin C}{c}) \approx 65.51°$
   These angles can also be found with the law of cosines.

2. (a) $\angle A = \arccos(\frac{b^2+c^2-a^2}{2bc}) = \arccos(\frac{3}{5}) \approx 53.13°$
   (b) $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) = \arccos(\frac{33}{65}) \approx 59.49°$
   $\angle C = \arccos(\frac{a^2+b^2-c^2}{2ab}) = \arccos(\frac{5}{13}) \approx 67.38°$
   These angles can also be found with the law of sines.
   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7 \cdot 15 \cdot \frac{4}{5} = 84$

3. (a) $\angle C = 91°$
   (b) $a = \frac{c}{\sin C} \cdot \sin A = \frac{2\sin 12°}{\sin 91°} \approx 0.416$
   $b = \frac{c}{\sin C} \cdot \sin B = \frac{2\sin 77°}{\sin 91°} \approx 1.949$
   (c) $[ABC] = \frac{1}{2}ac\sin B = \frac{2\sin 12°\sin 77°}{\sin 91°} \approx 0.405$

4. (a) By the law of cosines,
   $$a^2 = b^2 + c^2 - 2bc\cos A \implies 36 = 81 + c^2 - (18\cos 20°)c.$$
   Solving the resulting quadratic yields
   $$c = \frac{18\cos 20° \pm \sqrt{324\cos^2(20°) - 180}}{2} = 9\cos 20° \pm 3\sqrt{9\cos^2(20°) - 5}.$$
   One solution is ≈ 3.307 and the other solution is ≈ 13.607.
   (b) When $c \approx 3.307$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 149.13°$.
   When $c \approx 13.607$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 30.87°$.
   (c) Let $y = XZ$ be the missing side length. By the law of cosines,
   $$x^2 = 81 + y^2 - (18\cos 20°)y \implies y^2 - (18\cos 20°)y + (81 - x^2) = 0.$$
   For there to be only one triangle with the given properties, there must be exactly one positive solution for y. This can occur in two ways.

   Case 1 (exactly one real solution, which is positive). If there is exactly one real solution, then it must be $y = 9\cos 20°$, which is positive as required. This situation occurs when $81 - x^2 = (9\cos 20°)^2 = 81\cos^2(20°)$, which holds when $x = 9\sin 20°$. (This corresponds to "HL congruence.")

   Case 2 (two real solutions, only one of which is positive). The quadratic has a leading coefficient of 1, so this situation occurs precisely when the constant term is negative. Thus we need $81 - x^2 < 0$, and since x is a side length, we have $x > 9$.

5. (a) From $[ABC] = \frac{1}{2}ab\sin C$, we have $\sin C = \frac{2[ABC]}{ab}$. Then,

$$R = \frac{c}{2\sin C} = \frac{c}{\frac{4[ABC]}{ab}} = \frac{abc}{4[ABC]}.$$

   (b) $R = 65/8$

   (c) See this link.

6. (a) We have

$$[ABD] = \frac{1}{2} \cdot AB \cdot AD \cdot \sin(\angle BAD),$$
$$[ADC] = \frac{1}{2} \cdot AC \cdot AD \cdot \sin(\angle DAC),$$

   and dividing the two equations yields

$$\frac{[ABD]}{[ADC]} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

   The conclusion follows from the fact that triangles ABD and ADC share a height from A, so that then $\frac{[ABD]}{[ADC]} = \frac{BD}{DC}$.

   (b) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (a).

7. (a) We compute

$$\begin{aligned}
[ABC]^2 &= \left(\frac{1}{2}ab\sin C\right)^2 = \frac{1}{4}a^2b^2\sin^2 C \\
&= \frac{1}{4}a^2b^2(1-\cos^2 C) \\
&= \frac{1}{4}a^2b^2\left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= \frac{1}{4}a^2b^2\left[\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}\right] \\
&= \frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}.
\end{aligned}$$

   (b) From here, we observe some differences of squares to obtain

$$\begin{aligned}
[ABC]^2 &= \frac{[2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)]}{16} \\
&= \frac{[c^2-(a^2-2ab+b^2)][(a^2+2ab+b^2)-c^2]}{16} \\
&= \frac{[c-(a-b)][c+(a-b)][(a+b)-c][(a+b)+c]}{16} \\
&= \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \\
&= s(s-a)(s-b)(s-c).
\end{aligned}$$

8. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

The smallest possible value of $\angle BOC$ is $\angle AOB - \angle AOC$, so the smallest possible BC is

$$\begin{aligned} \min BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8}\cdot\frac{31}{32} + \frac{\sqrt{15}}{8}\cdot\frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

By a similar argument, the largest possible BC is

$$\max BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. From the law of cosines,

$$\cos(\angle BAD) = \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} = \frac{1}{2},$$

so $\cos(\angle DAC) = 1/2$ as well. Using the law of cosines at $\angle DAC$ gives us

$$(7x)^2 = (8x)^2 + 5^2 - 2 \cdot 8x \cdot 5 \cdot \frac{1}{2} \implies 15x^2 - 40x + 25 = 0.$$

This quadratic factors as $5(3x-5)(x-1)$, so there are two solutions, $x = 1$ or $x = 5/3$. When $x = 1$, we end up with $\triangle DAB \cong \triangle DAC$. However, this together with D lying on segment $\overline{BC}$ implies that $\angle ADB = 90^\circ$, a contradiction. Hence the only valid solution is that $x = 5/3$, in which case $CD = 35/3$.

10. Suppose without loss of generality that c is the side opposite the 120° angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

Graphing the conic, we see that it is an ellipse passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$, and that every point with positive rational coordinates can be connected to $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$x^2 + x \cdot (tx - 1) + (tx - 1)^2 = 1,$$
$$(t^2 + t + 1)x^2 - (2t + 1)x = 0.$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m, n) = 1$,

$$\begin{aligned}
\gcd(2mn + n^2, m^2 + mn + n^2) &= \gcd(n \cdot (2m + n), m^2 + mn + n^2) \\
&= \gcd(2m + n, m^2 + mn + n^2) \\
&= \gcd(2m + n, m^2 + mn + n^2 - n \cdot (2m + n)) \\
&= \gcd(2m + n, m^2 - mn) = \gcd(2m + n, m \cdot (m - n)) \\
&= \gcd(2m + n, m - n) = \gcd(3n, m - n).
\end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n + 3k$. Then $\gcd(n, k) = 1$ and

$$\gcd(3n, m - n) = \gcd(3n, 3k) = 3 \gcd(n, k) = 3.$$

Otherwise,

$$\gcd(3n, m - n) = \gcd(n, m - n) = \gcd(n, m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn + n^2}{3}, \quad b = \frac{m^2 - n^2}{3}, \quad c = \frac{m^2 + mn + n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r - s$ and $m = r + 2s$. Then

$$\begin{aligned}
a &= \frac{2(r + 2s)(r - s) + (r - s)^2}{3} = r^2 - s^2, \\
b &= \frac{(r + 2s)^2 - (r - s)^2}{3} = 2rs + s^2, \\
c &= \frac{(r + 2s)^2 + (r + 2s)(r - s) + (r - s)^2}{3} = r^2 + rs + s^2,
\end{aligned}$$

so the result still holds with r and s in place of m and n.

2 Complex Numbers I: Without Trig

Throughout, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

2.1 Review problems

1. Let $z = -3 + 3i$ and $w = -4 - 2i$. Compute each of the following:
   (a) $z + w$
   (b) $z - w$
   (c) zw
   (d) $\overline{w}$
   (e) z/w
   (f) $|z|$

2. Find all complex solutions to the equation $z^2 + 5 = 4z$.

3. Identify each of the following complex numbers.
   (a) The complex number corresponding to the point $(-5, -1)$.
   (b) The two complex numbers of magnitude 2 whose real and imaginary parts are equal.
   (c) The three complex numbers z for which 0, $3 - 2i$, $5 + 2i$, and z are the vertices of a parallelogram (in some order).

4. (a) Find a complex number w for which $w^2 = -16 + 30i$.
   (b) Find the two complex numbers z satisfying $2z^2 - (8 + 4i)z + (14 - 7i) = 0$.
   (c) Prove that for every complex number z, there is a complex number w for which $w^2 = z$.
   Remark: It follows from this that every quadratic polynomial with complex coefficients has complex roots (with roots given by the familiar quadratic formula).

5. (a) Let ℓ_1 be the line through $a = -4 - 3i$ and $b = 4 + i$, and let ℓ_2 be the line through $c = -4i$ and $d = -3 + 2i$.
      i. By considering slopes, or otherwise, show that ℓ_1 and ℓ_2 are perpendicular.
      ii. Compute $\frac{d-c}{b-a}$.
   (b) Show that in general, the line through $p \neq q$ is perpendicular to the line through $r \neq s$ if and only if $\frac{r-s}{p-q}$ is purely imaginary.
   (c) Given two distinct complex numbers a and b, the *perpendicular bisector* of the line segment connecting a and b is the line perpendicular to this segment passing through the midpoint $m = \frac{a+b}{2}$.
   Show that z lies on the perpendicular bisector of the line segment connecting a and b if and only if $|z - a| = |z - b|$.
   Hint: Consider squared magnitudes and use the fact that a complex number α is purely imaginary if and only if $\alpha = -\overline{\alpha}$.

6. (a) Show that $i^4 = 1$, and that conversely, if $z^4 = 1$, then $z = i^k$ for some positive integer k.
   (b) Let $z_1, z_2, z_3, \dots$ be a 4-periodic sequence of complex numbers, meaning that $z_{n+4} = z_n$ for all positive integers n. Show that there exist complex numbers a, b, c, d such that

$$z_n = a + b \cdot i^n + c \cdot i^{2n} + d \cdot i^{3n}$$

   for all n.

7. If ℓ is a line in the complex plane, then *reflection across* ℓ is the function $f_\ell : \mathbb{C} \to \mathbb{C}$ defined the property that for any complex number z, line ℓ is the perpendicular bisector of the line segment connecting z and $f_\ell(z)$. (When z already lies on ℓ, then we define $f(z) = z$.)
   (a) What complex number operation is equivalent to reflection across the x-axis?
   (b) Let ℓ be the line passing through 0 and $4 + 2i$. Find the reflection of -5 across ℓ.
   (c) More generally, let ℓ be the line passing through 0 and d, where d is a non-zero complex number. Find the reflection of z across ℓ, i.e. determine the function $f_\ell(z)$.
   (d) Even more generally, let ℓ be the line passing through a and b, where a and b are two distinct complex numbers. Find the reflection of z across ℓ.
   (e) From the previous two parts, every reflection has the form $f(z) = \alpha\overline{z} + \beta$ where $|\alpha| = 1$. Conversely, show that any such function is a reflection composed with a translation where the translation is parallel to the line of reflection. (When the translation is non-zero, we call the overall transformation a *glide reflection*.)

2.2 Challenge problems

8. An *isometry* of the complex plane is a function $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$|f(z) - f(w)| = |z - w|$$

   for all complex numbers z and w. In other words, f preserves distances between points.
   (a) Show that every translation and every reflection is an isometry.
   (b) Let f be an isometry satisfying $f(0) = 0$, $f(1) = 1$, and $f(i) = i$. Show that f must be the identity map, i.e. $f(z) = z$ for all z.
   (c) Prove that every isometry can be written as a composition of at most three reflections.
   (d) Show that the composition of three reflections is either a reflection or a glide reflection.

9. In this problem, we work through one formal construction of the complex numbers.
   Let $\mathcal{C}$ be the set of all ordered pairs of real numbers, and define operations $\oplus$ and $\otimes$ on $\mathcal{C}$ by

$$(a, b) \oplus (c, d) = (a + c, b + d),$$
$$(a, b) \otimes (c, d) = (ac - bd, ad + bc).$$

   We call $\oplus$ and $\otimes$ the addition and multiplication on $\mathcal{C}$, respectively.
   (a) The first task is to show that $\mathcal{C}$, with these operations, satisfies the "usual rules" of algebra. In fancy language, we would say that $\mathcal{C}$ is a *field*.

i. (Associative rules) Show that for any $u, v, w \in \mathcal{C}$,

$$u \oplus (v \oplus w) = (u \oplus v) \oplus w \quad \text{and} \quad u \otimes (v \otimes w) = (u \otimes v) \otimes w.$$

ii. (Commutative rules) Show that for any $z, w \in \mathcal{C}$,

$$z \oplus w = w \oplus z \quad \text{and} \quad z \otimes w = w \otimes z.$$

iii. (Distributive rule) Show that for any $u, v, w \in \mathcal{C}$,

$$u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w).$$

iv. (Identity rules) Show that for any $z \in \mathcal{C}$,

$$z \oplus (0,0) = (0,0) \oplus z = z \quad \text{and} \quad z \otimes (1,0) = (1,0) \otimes z = z.$$

This makes $(0,0)$ and $(1,0)$ the *additive identity* and *multiplicative identity* in $\mathcal{C}$.

v. (Additive inverse rule) Show that for any $z \in \mathcal{C}$, there exists $a_z \in \mathcal{C}$ such that

$$z \oplus a_z = (0,0).$$

The element a_z is the *additive inverse* of z in $\mathcal{C}$, and we denote it by $-z$.

vi. (Multiplicative inverse rule) Show that for any $z \in \mathcal{C}$ other than $(0,0)$, there exists $m_z \in \mathcal{C}$ such that

$$z \otimes m_z = (1,0).$$

The element m_z is the *multiplicative inverse* of z in $\mathcal{C}$, and we denote it by z^{-1}.

From these properties, all of the familiar algebraic rules can be shown to hold, such as the zero product property and certain common factorisations. Next, for this to reasonably be called an extension of the real numbers, we need to show that $\mathcal{C}$, with these operations, "contains" $\mathbb{R}$ with its usual addition and multiplication. This is made precise in the next part.

(b) Prove that for any two real numbers x and y,

$$(x,0) \oplus (y,0) = (x+y,0) \quad \text{and} \quad (x,0) \otimes (y,0) = (xy,0).$$

This shows that the elements $(r,0)$ for $r \in \mathbb{R}$, with operations $\oplus$ and $\otimes$, "act like" the real numbers with the usual addition and multiplication operations $+$ and $\times$.

With "$\mathcal{C}$ extends $\mathbb{R}$" shown, when r is a real number we simply write r instead of $(r,0)$, and we write $+$ and $\times$ (or $\cdot$) instead of $\oplus$ and $\otimes$. We also introduce the subtraction and division operations as $z - w = z + (-w)$ and $z/w = z \cdot w^{-1}$.

Finally, the complex numbers should have a square root of -1.

(c) Show that $(0,1) \times (0,1) = -1$ and $(0,-1) \times (0,-1) = -1$.

We can now recover the usual notation, replacing $\mathcal{C}$ with $\mathbb{C}$ and forever forgetting the initial definitions, by defining $i = (0,1)$ and then observing that $(x,y) = x + y \cdot i$.

10. A function $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-*automorphism of* $\mathbb{C}$ if

$$f(z + w) = f(z) + f(w) \quad \text{and} \quad f(zw) = f(z) \cdot f(w)$$

for all $z, w \in \mathbb{C}$ and $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$, then $f(i) = i$ or $f(i) = -i$.

(b) Show that the only two $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity function $f(z) = z$ and the conjugation function $f(z) = \overline{z}$.

2.3 Answers

1. (a) $-7+i$

   (b) $1+5i$

   (c) $18-6i$

   (d) $-4+2i$

   (e) $\frac{3}{10}-\frac{9}{10}i$

   (f) $3\sqrt{2}$

2. $2+i$ and $2-i$

3. (a) $-5-i$

   (b) $\sqrt{2}+\sqrt{2}i$ and $-\sqrt{2}-\sqrt{2}i$

   (c) 8, $-2-4i$, and $2+4i$

4. (a) Let $w=a+bi$, so then $w^2=(a^2-b^2)+(2ab)i$. This yields the system of equations

$$a^2-b^2=-16 \quad \text{and} \quad 2ab=30.$$

   From the second equation, $b=15/a$. Substituting into the first equation,

$$\begin{aligned} a^2-\frac{225}{a^2}&=-16,\\ a^4-225&=-16a^2,\\ a^4+16a^2-225&=0,\\ (a^2+25)(a^2-9)&=0. \end{aligned}$$

   Since a is real, $a^2\geq 0$, so we must take $a^2=9$ and hence $a=\pm 3$. If $a=3$, then $b=5$, and if $a=-3$, then $b=-5$, so the two square roots are $\pm(3+5i)$.

   (b) Applying the quadratic formula and using the result from part (a), the solutions are

$$\begin{aligned} z&=\frac{(8+4i)\pm\sqrt{(8+4i)^2-4\cdot 2\cdot(14-7i)}}{2\cdot 2}\\ &=\frac{(8+4i)\pm\sqrt{(64-16+64i)-(112-56i)}}{4}\\ &=\frac{(8+4i)\pm\sqrt{-64+120i}}{4}\\ &=\frac{(8+4i)\pm 2\sqrt{-16+30i}}{4}\\ &=\frac{(4+2i)\pm(3+5i)}{2}. \end{aligned}$$

   Simplifying in each case, we get $\frac{7}{2}+\frac{7}{2}i$ and $\frac{1}{2}-\frac{3}{2}i$.

(c) Let $z = x + yi$ and $w = a + bi$. Setting up as in part (a), we get the system

$$a^2 - b^2 = x \quad \text{and} \quad 2ab = y,$$

and substituting $b = \frac{y}{2a}$ yields

$$a^2 - \frac{y^2}{4a^2} = x \qquad \Longleftrightarrow \qquad 4a^4 - 4xa^2 - y^2 = 0.$$

This is a real quadratic in a^2 for which the product of the roots is non-negative, so there is a non-negative solution for a^2. Taking either square root of this value gives us a real value of a, hence a corresponding real value of b, and $w = a + bi$ is the desired solution.

5. (a) i. The slope of ℓ_1 is $\frac{1-(-3)}{4-(-4)} = \frac{1}{2}$ and the slope of ℓ_2 is $\frac{2-(-4)}{-3-0} = -2$. The product of their slopes is -1, so the lines are perpendicular.

ii. $\frac{d-c}{b-a} = \frac{-3+6i}{8+4i} = \frac{3}{4} \cdot \frac{-1+2i}{2+i} \cdot \frac{2-i}{2-i} = \frac{3}{4} \cdot \frac{5i}{5} = \frac{3}{4}i$

(b) Translating each line individually does not impact perpendicularity, so without loss of generality, we can translate so that $q = s = 0$ and show that for any two non-zero complex numbers r and p, the line through 0 and $r = a + bi$ is perpendicular to the line through 0 and $p = c + di$ if and only if r/p is purely imaginary.

The slope of the line through 0 and r is b/a and the slope of the line through 0 and p is d/c, so the two lines are perpendicular if and only if $\frac{bd}{ac} = -1$, or $ac + bd = 0$. (The latter condition also detects when one line is horizontal and the other line is vertical.)

The quotient r/p is purely imaginary if and only if it is equal to the negative of its conjugate, so we compute

$$\frac{r}{p} + \frac{\overline{r}}{\overline{p}} = \frac{r\overline{p} + \overline{r}p}{p\overline{p}} = \frac{2\,\mathrm{Re}(r\overline{p})}{|p|^2} = \frac{2(ac + bd)}{|p|^2}.$$

Hence r/p is purely imaginary if and only if $ac + bd = 0$, which is exactly the condition we found for perpendicularity.

(c) From part (b), z lies on the perpendicular bisector of the segment connecting a and b if and only if $\frac{z-m}{a-b}$ is purely imaginary. In terms of conjugates, this is equivalent to

$$\begin{aligned}
\frac{z-m}{a-b} + \frac{\overline{z} - \overline{m}}{\overline{a} - \overline{b}} &= 0,\\
(\overline{a} - \overline{b})(z - m) + (a - b)(\overline{z} - \overline{m}) &= 0,\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} - [(\overline{a} - \overline{b})m + (a - b)\overline{m}] &= 0,\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} &= \frac{(\overline{a} - \overline{b})(a + b)}{2} + \frac{(a - b)(\overline{a} + \overline{b})}{2},\\
(\overline{a} - \overline{b})z + (a - b)\overline{z} &= a\overline{a} - b\overline{b}.
\end{aligned}$$

The condition $|z - a| = |z - b|$ is equivalent to $|z - a|^2 = |z - b|^2$, or

$$\begin{aligned}
(z - a)(\overline{z} - \overline{a}) &= (z - b)(\overline{z} - \overline{b}),\\
z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} &= z\overline{z} - \overline{b}z - b\overline{z} + b\overline{b},\\
a\overline{a} - b\overline{b} &= (\overline{a} - \overline{b})z + (a - b)\overline{z}.
\end{aligned}$$

Thus the two statements are equivalent, as desired.

6. (a) For the first statement, $i^2 = -1$ and $i^3 = -i$ and $i^4 = -i^2 = -(-1) = 1$.
   For the second statement, $z^4 - 1$ factors as $(z-1)(z+1)(z^2+1)$. The last factor has roots $i = i^1$ and $-i = i^3$, while $1 = i^4$ and $-1 = i^2$ are the other two roots of $z^4 - 1$.

   (b) By part (a), any sequence of the form specified on the right hand side is 4-periodic, so it suffices to show that there exist complex numbers a, b, c, d such that

$$\begin{aligned}
z_1 &= a + b \cdot i^1 + c \cdot i^2 + d \cdot i^3 = a + ib - c - id, \\
z_2 &= a + b \cdot i^2 + c \cdot i^4 + d \cdot i^6 \qquad = a - b + c - d, \\
z_3 &= a + b \cdot i^3 + c \cdot i^6 + d \cdot i^9 \qquad = a - ib - c + id, \\
z_4 &= a + b \cdot i^4 + c \cdot i^8 + d \cdot i^{12} \qquad = a + b + c + d.
\end{aligned}$$

   This amounts to solving a system of linear equations. Adding the first and third equations gives $z_1 + z_3 = 2a - 2c$ while adding the second and fourth gives $z_2 + z_4 = 2a + 2c$. Subtracting the first and third equations gives $z_1 - z_3 = 2ib - 2id$, while subtracting the second and fourth gives $z_4 - z_2 = 2b + 2d$. The new system

$$\begin{aligned}
2a - 2c &= z_1 + z_3 & \qquad 2ib - 2id &= z_1 - z_3 \\
2a + 2c &= z_2 + z_4 & \qquad 2b + 2d &= -z_2 + z_4
\end{aligned}$$

   has as a solution for (a, b, c, d) the 4-tuple

$$\left(\frac{z_1 + z_2 + z_3 + z_4}{4}, \frac{z_1 - iz_2 - z_3 + iz_4}{4i}, \frac{-z_1 + z_2 - z_3 + z_4}{4}, \frac{-z_1 - iz_2 + z_3 + iz_4}{4i} \right),$$

   and it can be checked that this satisfies the original system as well.

7. (a) Complex conjugation

   (b) $-3 - 4i$

   (c) First, the midpoint of the segment connecting z and $w = f_\ell(z)$ has to lie on ℓ, so $\frac{z+w}{2}$ is a real multiple of d. That is, $\frac{z+w}{2d}$ is real, so

$$\begin{aligned}
\frac{z+w}{2d} &= \frac{\overline{z} + \overline{w}}{2\overline{d}}, \\
\overline{d}(z+w) &= d(\overline{z} + \overline{w}), \\
\overline{d}w - d\overline{w} &= -\overline{d}z + d\overline{z}. \qquad (1)
\end{aligned}$$

   Second, the segment has to be perpendicular to ℓ, so $\frac{z-w}{d}$ is purely imaginary. Therefore,

$$\begin{aligned}
\frac{z-w}{d} &= -\frac{\overline{z} - \overline{w}}{\overline{d}}, \\
\overline{d}(z-w) &= -d(\overline{z} - \overline{w}), \\
-\overline{d}w - d\overline{w} &= -\overline{d}z - d\overline{z}. \qquad (2)
\end{aligned}$$

   Taking the difference $(1) - (2)$ gives us $f_\ell(z) = w = \frac{d}{\overline{d}} \cdot \overline{z}$.

(d) We can translate a to the origin, reflect, then translate back, so

$$f_\ell(z) = f_{\ell-a}(z-a) + a = \frac{b-a}{\overline{b}-\overline{a}} \cdot (\overline{z}-\overline{a}) + a.$$

(e)

8. (a) First let f be a translation, $f(z) = z + a$. Then

$$|f(z) - f(w)| = |(z+a) - (w+a)| = |z-w|.$$

Now we consider reflections. The composition of isometries is also an isometry, so since general reflections can be obtained by composing reflections across lines through the origin with translations, it suffices to consider reflections across lines through 0 and another point d. Then

$$|f(z) - f(w)| = \left| \frac{d}{\overline{d}}\overline{z} - \frac{d}{\overline{d}}\overline{w} \right| = \frac{|d|}{|d|}|\overline{z-w}| = |z-w|.$$

(b) Let z be arbitrary and let $w = f(z)$. By the isometry property,

$$\begin{aligned} |f(z) - f(0)| &= |z-0| & w\overline{w} &= z\overline{z}, \\ |f(z) - f(1)| &= |z-1| & &= (w-1)(\overline{w}-1) = (z-1)(\overline{z}-1), \\ |f(z) - f(i)| &= |z-i| & &= (w-i)(\overline{w}+i) = (z-i)(\overline{z}+i). \end{aligned}$$

Expanding the latter two equations gives us

$$\begin{aligned} w\overline{w} - w - \overline{w} + 1 &= z\overline{z} - z - \overline{z} + 1, \\ w\overline{w} + iw - i\overline{w} + 1 &= z + iz - i\overline{z} + 1. \end{aligned}$$

Then, substituting $w\overline{w} = z\overline{z}$ and rearranging yields the system of equations

$$\begin{aligned} w + \overline{w} &= z + \overline{z}, \\ iw - i\overline{w} &= iz - i\overline{z}. \end{aligned}$$

Eliminating $\overline{w}$ gives us $2iw = 2iz$, so $w = z$ as required.

(c) Let f be a given isometry and suppose $a = f(0)$, $b = f(1)$, and $c = f(i)$. We define three reflections and compositions as follows:

i. If $a = 0$, then let r_1 be the identity map. Otherwise, let r_1 be reflection across the perpendicular bisector of the segment connecting 0 and a. Then $f_1 = r_1 \circ f$ is an isometry with $f_1(0) = 0$.

ii. Let $b' = r_1(b) = f_1(1)$ and $c' = r_1(c) = f_1(i)$. If $b' = 1$, then let r_2 be the identity map. Otherwise, as f_1 is an isometry, $0 = f_1(0)$ lies on the perpendicular bisector of the segment connecting 1 and $b' = f_1(1)$. As such, we can let r_2 be reflection across this perpendicular bisector, so $f_2 = r_2 \circ f_1$ satisfies $f_2(0) = 0$ and $f_2(1) = 1$.

iii. Let $c'' = r_2(c') = f_2(i)$. If $c'' = i$, then let r_3 be the identity map. Otherwise, as f_2 is an isometry, both $0 = f_2(0)$ and $1 = f_2(1)$ are equidistant from i and $c'' = f_2(i)$, so the reflection across the real axis sends c'' to i. Let this reflection be r_3, so then $f_3 = r_3 \circ f_2$ satisfies $f_3(0) = 0$, $f_3(1) = 1$, and $f_3(i) = i$.

By part (b), $f_3 = (r_3 \circ r_2 \circ r_1) \circ f$ must be the identity map. Composing on the left with r_3, then r_2, then r_1, we get $f = r_1 \circ r_2 \circ r_3 \circ f_3 = r_1 \circ r_2 \circ r_3$ as required.

(d) Note that the formula we found for general reflections takes the form $f(z) = \alpha(\overline{z} - \overline{a}) + a$, where $|\alpha| = 1$. Let our three reflections be

$$\begin{aligned} r_1(z) &= \alpha(\overline{z} - \overline{a}) + a, \\ r_2(z) &= \beta\overline{z} + b, \\ r_3(z) &= \gamma\overline{z} + c. \end{aligned}$$

Then

$$\begin{aligned} (r_1 \circ r_2 \circ r_3)(z) &= \alpha\overline{(\beta\overline{[\gamma\overline{z} + c]} + b)} + a \\ &= \alpha\overline{(\beta[\overline{\gamma}z + \overline{c}] + b)} + a \\ &= \alpha \end{aligned}$$