Precalculus Practice Problems: Midterm 2

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2024-2025

The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. (SAS congruence) Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Find $\angle A$ and $\angle B$.
2. (SSS congruence) Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Find $\angle A$.
   (b) Find $\angle B$ and $\angle C$.
   (c) Find $[ABC]$.
3. (ASA/AAS congruence) Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.
4. (SSA non-congruence) Let ABC be a triangle with $\angle A = 20°$, $a = 6$, and $b = 9$.
   (a) Find all possible values of c.
   (b) For each possible value of c, find $\angle B$.
   (c) For what values of x does there exist exactly one triangle XYZ with $\angle X = 20°$, $XY = 9$, and $YZ = x$?
5. (Extended law of sines) If ABC is a triangle with circumradius R, then the *extended law of sines* states that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Prove that $R = \dfrac{abc}{4[ABC]}$.
   (b) Given that $a = 13$, $b = 14$, and $c = 15$, find R.
   (c) Prove the extended law of sines for acute triangles.
6. Let ABC be a triangle and let D be a point on side $\overline{BC}$.
   (a) (Ratio lemma) Prove that
$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(b) (Angle bisector theorem) Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

7. (Heron's formula) Let ABC be a triangle.

   (a) Show that
   $$[ABC]^2 = \frac{1}{4}a^2b^2(1 - \cos^2 C) = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{16}.$$

   (b) Conclude that
   $$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$
   where $s = (a+b+c)/2$ is the *semiperimeter* of triangle ABC.

1.2 Challenge problems

8. Points O, A, B, and C are placed in three-dimensional space so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. What are the shortest and longest possible lengths of BC?

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Given that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. (Eisenstein triples) An *Eisenstein triple* is a triple of positive integers (a, b, c) for which a triangle with side lengths a, b, and c has an angle of measure either $60°$ or $120°$. If the Eisenstein triple (a, b, c) corresponds to a triangle with an angle of measure $60°$, we will call it an Eisenstein triple of *acute type*, and otherwise, we call it an Eisenstein triple of *obtuse type*. (The "acute type" and "obtuse type" names are non-standard.)

    (a) Let (a, b, c) be an Eisenstein triple of obtuse type with $a < b < c$. Show that $(a, a+b, c)$ and $(a+b, b, c)$ are Eisenstein triples of acute type.

    (b) Conversely, show that every Eisenstein triple of acute type arises from an Eisenstein triple of obtuse type in the above manner.

    (c) Show that if (a, b, c) is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that
    $$\{a, b, c\} = \{m^2 + mn + n^2, 2mn + n^2, m^2 - n^2\}.$$
    (Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104^\circ) \approx 2.426$

   (b) $c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{26 - 10\cos(104^\circ)} \approx 5.331$

   (c) $\angle A = \arcsin(\frac{a\sin C}{c}) \approx 10.49^\circ$

   $\angle B = \arcsin(\frac{b\sin C}{c}) \approx 65.51^\circ$

   These angles can also be found with the law of cosines.

2. (a) $\angle A = \arccos(\frac{b^2+c^2-a^2}{2bc}) = \arccos(\frac{3}{5}) \approx 53.13^\circ$

   (b) $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) = \arccos(\frac{33}{65}) \approx 59.49^\circ$

   $\angle C = \arccos(\frac{a^2+b^2-c^2}{2ab}) = \arccos(\frac{5}{13}) \approx 67.38^\circ$

   These angles can also be found with the law of sines.

   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7 \cdot 15 \cdot \frac{4}{5} = 84$

3. (a) $\angle C = 91^\circ$

   (b) $a = \frac{c}{\sin C} \cdot \sin A = \frac{2\sin 12^\circ}{\sin 91^\circ} \approx 0.416$

   $b = \frac{c}{\sin C} \cdot \sin B = \frac{2\sin 77^\circ}{\sin 91^\circ} \approx 1.949$

   (c) $[ABC] = \frac{1}{2}ac\sin B = \frac{2\sin 12^\circ \sin 77^\circ}{\sin 91^\circ} \approx 0.405$

4. (a) By the law of cosines,

   $$a^2 = b^2 + c^2 - 2bc\cos A \implies 36 = 81 + c^2 - (18\cos 20^\circ)c.$$

   Solving the resulting quadratic yields

   $$c = \frac{18\cos 20^\circ \pm \sqrt{324\cos^2(20^\circ) - 180}}{2} = 9\cos 20^\circ \pm 3\sqrt{9\cos^2(20^\circ) - 5}.$$

   One solution is ≈ 3.307 and the other solution is ≈ 13.607.

   (b) When $c \approx 3.307$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 149.13^\circ$.

   When $c \approx 13.607$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 30.87^\circ$.

   (c) Let $y = XZ$ be the missing side length. By the law of cosines,

   $$x^2 = 81 + y^2 - (18\cos 20^\circ)y \implies y^2 - (18\cos 20^\circ)y + (81 - x^2) = 0.$$

   For there to be only one triangle with the given properties, there must be exactly one positive solution for y. This can occur in two ways.

   Case 1 (exactly one real solution, which is positive). If there is exactly one real solution, then it must be $y = 9\cos 20^\circ$, which is positive as required. This situation occurs when $81 - x^2 = (9\cos 20^\circ)^2 = 81\cos^2(20^\circ)$, which holds when $x = 9\sin 20^\circ$. (This corresponds to "HL congruence.")

   Case 2 (two real solutions, only one of which is positive). The quadratic has a leading coefficient of 1, so this situation occurs precisely when the constant term is negative. Thus we need $81 - x^2 < 0$, and since x is a side length, we have $x > 9$.

5. (a) From $[ABC] = \frac{1}{2}ab\sin C$, we have $\sin C = \frac{2[ABC]}{ab}$. Then,

$$R = \frac{c}{2\sin C} = \frac{c}{\frac{4[ABC]}{ab}} = \frac{abc}{4[ABC]}.$$

(b) $R = 65/8$

(c) See this link.

6. (a) We have

$$[ABD] = \frac{1}{2} \cdot AB \cdot AD \cdot \sin(\angle BAD),$$
$$[ADC] = \frac{1}{2} \cdot AC \cdot AD \cdot \sin(\angle DAC),$$

and dividing the two equations yields

$$\frac{[ABD]}{[ADC]} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

The conclusion follows from the fact that triangles ABD and ADC share a height from A, so that then $\frac{[ABD]}{[ADC]} = \frac{BD}{DC}$.

(b) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (a).

7. (a) We compute

$$\begin{aligned}
[ABC]^2 &= \left(\frac{1}{2}ab\sin C\right)^2 = \frac{1}{4}a^2b^2\sin^2 C \\
&= \frac{1}{4}a^2b^2(1-\cos^2 C) \\
&= \frac{1}{4}a^2b^2\left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= \frac{1}{4}a^2b^2\left[\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}\right] \\
&= \frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}.
\end{aligned}$$

(b) From here, we observe some differences of squares to obtain

$$\begin{aligned}
[ABC]^2 &= \frac{[2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)]}{16} \\
&= \frac{[c^2-(a^2-2ab+b^2)][(a^2+2ab+b^2)-c^2]}{16} \\
&= \frac{[c-(a-b)][c+(a-b)][(a+b)-c][(a+b)+c]}{16} \\
&= \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \\
&= s(s-a)(s-b)(s-c).
\end{aligned}$$

8. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

The smallest possible value of $\angle BOC$ is $\angle AOB - \angle AOC$, so the smallest possible BC is

$$\begin{aligned} \min BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8}\cdot\frac{31}{32} + \frac{\sqrt{15}}{8}\cdot\frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

By a similar argument, the largest possible BC is

$$\max BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. From the law of cosines,

$$\cos(\angle BAD) = \frac{8^2 + 5^2 - 7^2}{2\cdot 8\cdot 5} = \frac{1}{2},$$

so $\cos(\angle DAC) = 1/2$ as well. Using the law of cosines at $\angle DAC$ gives us

$$(7x)^2 = (8x)^2 + 5^2 - 2\cdot 8x\cdot 5\cdot\frac{1}{2} \implies 15x^2 - 40x + 25 = 0.$$

This quadratic factors as $5(3x-5)(x-1)$, so there are two solutions, $x = 1$ or $x = 5/3$. When $x = 1$, we end up with $\triangle DAB \cong \triangle DAC$. However, this together with D lying on segment $\overline{BC}$ implies that $\angle ADB = 90^\circ$, a contradiction. Hence the only valid solution is that $x = 5/3$, in which case $CD = 35/3$.

10. Suppose without loss of generality that c is the side opposite the 120° angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

Graphing the conic, we see that it is an ellipse passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$, and that every point with positive rational coordinates can be connected to $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$x^2 + x \cdot (tx - 1) + (tx - 1)^2 = 1,$$
$$(t^2 + t + 1)x^2 - (2t + 1)x = 0.$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m, n) = 1$,

$$\begin{aligned}
\gcd(2mn + n^2, m^2 + mn + n^2) &= \gcd(n \cdot (2m + n), m^2 + mn + n^2) \\
&= \gcd(2m + n, m^2 + mn + n^2) \\
&= \gcd(2m + n, m^2 + mn + n^2 - n \cdot (2m + n)) \\
&= \gcd(2m + n, m^2 - mn) = \gcd(2m + n, m \cdot (m - n)) \\
&= \gcd(2m + n, m - n) = \gcd(3n, m - n).
\end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n + 3k$. Then $\gcd(n, k) = 1$ and

$$\gcd(3n, m - n) = \gcd(3n, 3k) = 3 \gcd(n, k) = 3.$$

Otherwise,

$$\gcd(3n, m - n) = \gcd(n, m - n) = \gcd(n, m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn + n^2}{3}, \quad b = \frac{m^2 - n^2}{3}, \quad c = \frac{m^2 + mn + n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r - s$ and $m = r + 2s$. Then

$$\begin{aligned}
a &= \frac{2(r + 2s)(r - s) + (r - s)^2}{3} = r^2 - s^2, \\
b &= \frac{(r + 2s)^2 - (r - s)^2}{3} = 2rs + s^2, \\
c &= \frac{(r + 2s)^2 + (r + 2s)(r - s) + (r - s)^2}{3} = r^2 + rs + s^2,
\end{aligned}$$

so the result still holds with r and s in place of m and n.

2 Complex Numbers

Throughout, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

2.1 Review problems

1. Let $z = -3 + 3i$ and $w = -4 - 2i$. Compute each of the following:
   (a) $z + w$
   (b) $z - w$
   (c) zw
   (d) z/w
   (e) $|z|$
   (f) $\overline{w}$
2. Find all complex solutions to the equation $z^2 + 5 = 4z$.
3. Identify each of the following complex numbers.
   (a) The complex number corresponding to the point $(-5, -1)$.
   (b) The two complex numbers of magnitude 2 whose real and imaginary parts are equal.
   (c) The three complex numbers z for which 0, $3 - 2i$, $5 + 2i$, and z are the vertices of a parallelogram (in some order).
4. (a) Find a complex number w for which $w^2 = -16 + 30i$.
   (b) Find the two complex numbers z satisfying $2z^2 - (8 + 4i)z + (14 - 7i) = 0$.
   (c) Prove that for every complex number z, there is a complex number w for which $w^2 = z$.
   Remark: It follows from this that every quadratic polynomial with complex coefficients has complex roots (with roots given by the familiar quadratic formula).
5. (a) Let ℓ_1 be the line through $a = -4 - 3i$ and $b = 4 + i$, and let ℓ_2 be the line through $c = -4i$ and $d = -3 + 2i$.
      i. By considering slopes, or otherwise, show that ℓ_1 and ℓ_2 are perpendicular.
      ii. Compute $\frac{d-c}{b-a}$.
   (b) Show that in general, the line through $p \neq q$ is perpendicular to the line through $r \neq s$ if and only if $\frac{r-s}{p-q}$ is purely imaginary.
   (c) Given two distinct complex numbers a and b, the *perpendicular bisector* of the line segment connecting a and b is the line perpendicular to this segment passing through the midpoint $m = \frac{a+b}{2}$.
   Show that z lies on the perpendicular bisector of the line segment connecting a and b if and only if $|z - a| = |z - b|$.
   Hint: Consider squared magnitudes and use the fact that a complex number α is purely imaginary if and only if $\alpha = -\overline{\alpha}$.

6. (a) Show that $i^4 = 1$, and that conversely, if $z^4 = 1$, then $z = i^k$ for some positive integer k.

   (b) Let $z_1, z_2, z_3, \dots$ be a 4-periodic sequence of complex numbers, meaning that $z_{n+4} = z_n$ for all positive integers n. Show that there exist complex numbers a, b, c, d such that

$$z_n = a + b \cdot i^n + c \cdot i^{2n} + d \cdot i^{3n}$$

   for all n.

7. If ℓ is a line in the complex plane, then *reflection across* ℓ is the function $f_\ell : \mathbb{C} \to \mathbb{C}$ defined the property that for any complex number z, line ℓ is the perpendicular bisector of the line segment connecting z and $f_\ell(z)$. (When z already lies on ℓ, then we define $f(z) = z$.)

   (a) What complex number operation is equivalent to reflection across the x-axis?

   (b) Let ℓ be the line passing through 0 and $4 + 2i$. Find the reflection of -3 across ℓ.

   (c) More generally, let ℓ be the line passing through 0 and d, where d is a non-zero complex number. Find the reflection of z across ℓ, i.e. determine the function $f_\ell(z)$.

   (d) Even more generally, let ℓ be the line passing through a and b, where a and b are two distinct complex numbers. Find the reflection of z across ℓ.

2.2 Challenge problems

8. An *isometry* of the complex plane is a function $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$|f(z) - f(w)| = |z - w|$$

   for all complex numbers z and w. In other words, f preserves distances between points.

   (a) Show that every translation and every reflection is an isometry.

   (b) Let a, b, c be distinct complex numbers. Show that there is at most one isometry f satisfying $f(0) = a$, $f(1) = b$, and $f(i) = c$.

   (c) Prove that every isometry can be written as a composition of at most three reflections.

   (d) Show that the composition of three reflections is equivalent to a reflection followed by a translation. (If the translation is non-zero, then the isometry is a *glide reflection*.)

9. In this problem, we work through one formal construction of the complex numbers.

   Let $\mathcal{C}$ be the set of all ordered pairs of real numbers, and define operations $\oplus$ and $\otimes$ on $\mathcal{C}$ by

$$\begin{aligned}(a, b) \oplus (c, d) &= (a + c, b + d),\\ (a, b) \otimes (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

   We call $\oplus$ and $\otimes$ the addition and multiplication on $\mathcal{C}$, respectively.

   (a) The first task is to show that $\mathcal{C}$, with these operations, satisfies the "usual rules" of algebra. In fancy language, we would say that $\mathcal{C}$ is a *field*.

      i. (Associative rules) Show that for any $u, v, w \in \mathcal{C}$,

$$u \oplus (v \oplus w) = (u \oplus v) \oplus w \quad \text{and} \quad u \otimes (v \otimes w) = (u \otimes v) \otimes w.$$

ii. (Commutative rules) Show that for any $z, w \in \mathcal{C}$,

$$z \oplus w = w \oplus z \quad \text{and} \quad z \otimes w = w \otimes z.$$

iii. (Distributive rule) Show that for any $u, v, w \in \mathcal{C}$,

$$u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w).$$

iv. (Identity rules) Show that for any $z \in \mathcal{C}$,

$$z \oplus (0,0) = (0,0) \oplus z = z \quad \text{and} \quad z \otimes (1,0) = (1,0) \otimes z = z.$$

This makes $(0,0)$ and $(1,0)$ the *additive identity* and *multiplicative identity* in $\mathcal{C}$.

v. (Additive inverse rule) Show that for any $z \in \mathcal{C}$, there exists $a_z \in \mathcal{C}$ such that

$$z \oplus a_z = (0,0).$$

The element a_z is the *additive inverse* of z in $\mathcal{C}$, and we denote it by $-z$.

vi. (Multiplicative inverse rule) Show that for any $z \in \mathcal{C}$ other than $(0,0)$, there exists $m_z \in \mathcal{C}$ such that

$$z \otimes m_z = (1,0).$$

The element m_z is the *multiplicative inverse* of z in $\mathcal{C}$, and we denote it by z^{-1}.

From these properties, all of the familiar algebraic rules can be shown to hold, such as the zero product property and certain common factorisations. Next, for this to reasonably be called an extension of the real numbers, we need to show that $\mathcal{C}$, with these operations, "contains" $\mathbb{R}$ with its usual addition and multiplication. This is made precise in the next part.

(b) Prove that for any two real numbers x and y,

$$(x,0) \oplus (y,0) = (x+y,0) \quad \text{and} \quad (x,0) \otimes (y,0) = (xy,0).$$

This shows that the elements $(r,0)$ for $r \in \mathbb{R}$, with operations $\oplus$ and $\otimes$, "act like" the real numbers with the usual addition and multiplication operations $+$ and $\times$.

With "$\mathcal{C}$ extends $\mathbb{R}$" shown, when r is a real number we simply write r instead of $(r,0)$, and we write $+$ and $\times$ (or $\cdot$) instead of $\oplus$ and $\otimes$. We also introduce the subtraction and division operations as $z - w = z + (-w)$ and $z/w = z \cdot w^{-1}$.

Finally, the complex numbers should have a square root of -1.

(c) Show that $(0,1) \times (0,1) = -1$ and $(0,-1) \times (0,-1) = -1$.

We can now recover the usual notation, replacing $\mathcal{C}$ with $\mathbb{C}$ and forever forgetting the initial definitions, by defining $i = (0,1)$ and then observing that $(x,y) = x + y \cdot i$.

10. A function $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-*automorphism of* $\mathbb{C}$ if

$$f(z+w) = f(z) + f(w) \quad \text{and} \quad f(zw) = f(z) \cdot f(w)$$

for all $z, w \in \mathbb{C}$ and $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$, then $f(i) = i$ or $f(i) = -i$.

(b) Show that the only two $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity function $f(z) = z$ and the conjugation function $f(z) = \overline{z}$.

2.3 Answers

1.