Precalculus Practice Problems: Final

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2024-2025

The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

Contents

1 Matrices in 2D

1.1 Review Problems

Throughout, $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the standard unit vectors while $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector. We also let $\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the (2×2) identity matrix and $\mathsf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

1. *Vector calculations.* Let $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Compute each of the following.
   (a) $\mathbf{u} + \mathbf{v}$
   (b) $2\mathbf{v}$
   (c) $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$
   (d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$
   (e) The angle between $\mathbf{u}$ and $\mathbf{v}$ (in terms of an inverse trig function)
   (f) $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{u}}(\mathbf{v})$
2. *Applying matrices to vectors.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.
   (a) Compute $\mathsf{A}\mathbf{v}$
   (b) Find a vector $\mathbf{u}$ for which $\mathsf{A}\mathbf{u} = \mathbf{v}$, or show that none exists.
3. *Matrix operations.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.
   (a) $\mathsf{A} + \mathsf{B}$
   (b) $-3\mathsf{A}$
   (c) AB
   (d) BA
   (e) B^T (the transpose of B)
4. *Geometric transformations.* Write down matrices for each of the following.
   (a) Dilation about the origin by a factor of 4
   (b) Horizontal dilation by a factor of 3 and vertical dilation by a factor of 2
   (c) Rotation about the origin by $\pi/4$ counterclockwise
   (d) Projection onto the line $y = (3/2)x$
   (e) Reflection across the line $y = (3/2)x$

5. *Matrix determinants.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) $\det \mathsf{A}$ and $\det \mathsf{B}$
   (b) $\det(\mathsf{AB})$
   (c) $\det(\mathsf{A}^T)$
   (d) $\det(\mathsf{A} + \mathsf{B})$
   (e) The area of the ellipse formed by applying A to the unit circle

6. *Matrix inverses.* Let $\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -7 \end{pmatrix}$. Compute each of the following.

   (a) A^{-1} and B^{-1}
   (b) $\mathsf{A}^{-1}\mathsf{B}^{-1}$ and $\mathsf{B}^{-1}\mathsf{A}^{-1}$
   (c) $(\mathsf{AB})^{-1}$
   (d) $(\mathsf{A}^T)^{-1}$
   (e) $(\mathsf{A} + \mathsf{B})^{-1}$
   (f) $\det(\mathsf{A}^{-1})$

7. *Shear transformations.* A **horizontal shear** is given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

   (a) Describe the image of the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ when the horizontal shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied.
   (b) By what factor does a horizontal shear multiply areas?
   (c) Find real constants a, b, k, θ for which

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

   (The constant θ can be expressed in terms of an inverse trig function.)

1.2 Challenge Problems

8. The **trace** of a square matrix is the sum of its main diagonal entries,

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

   (a) For the matrices A, B in problems 3, 5, 6, compute $\operatorname{tr}\mathsf{A}$, $\operatorname{tr}\mathsf{B}$, $\operatorname{tr}(\mathsf{A}+\mathsf{B})$, and $\operatorname{tr}(\mathsf{AB})$.

   (b) Show that for any 2×2 matrices P and Q, we have $\operatorname{tr}(\mathsf{PQ}) = \operatorname{tr}(\mathsf{QP})$.

   (c) In general, must it be true that $\operatorname{tr}(\mathsf{ABC}) = \operatorname{tr}(\mathsf{ACB})$?

9. Two matrices A, B are **similar**, written $\mathsf{A} \sim \mathsf{B}$, if there is an invertible P with $\mathsf{B} = \mathsf{P}^{-1}\mathsf{AP}$.

   (a) Show that the only matrix similar to I is I.

   (b) Show that if $\mathsf{A} \sim \mathsf{B}$, then $\det \mathsf{A} = \det \mathsf{B}$ and $\operatorname{tr}\mathsf{A} = \operatorname{tr}\mathsf{B}$.

   (c) Let $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. There is exactly one diagonal matrix $\mathsf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1 \geq d_2$ for which $\mathsf{D} \sim \mathsf{A}$. Find D.

10. If A is a square matrix, the **characteristic polynomial** of A is defined by

$$f_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I}).$$

   (a) Compute the characteristic polynomial $f_{\mathsf{A}}(X)$ of the matrix $\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find the two roots $\lambda_1 \geq \lambda_2$ of $f_{\mathsf{A}}(X)$.

   (c) Find non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$ for which $\mathsf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, 2$. (In general, if $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, we call $\mathbf{v}$ an **eigenvector** of A corresponding to the **eigenvalue** λ.)

   (d) Let P be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Compute $\mathsf{P}^{-1}\mathsf{AP}$.

   (e) Find A^{100}.

   (f) *Cayley-Hamilton theorem.* Suppose $f_{\mathsf{A}}(X) = a_0 + a_1 X + a_2 X^2$. (The values of a_0, a_1, a_2 are known from part (a).) Compute

$$a_0\mathsf{I} + a_1\mathsf{A} + a_2\mathsf{A}^2.$$

1.3 Answers

1. (a) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

   (c) Both are 5. In general, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

   (d) $\|\mathbf{u}\| = \sqrt{13}$
   $\|\mathbf{v}\| = \sqrt{17}$
   $\|\mathbf{u} + \mathbf{v}\| = \sqrt{40} = 2\sqrt{10}$

   (e) $\arccos\left(\frac{5}{\sqrt{221}}\right)$

   (f) $\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \begin{pmatrix} 20/17 \\ -5/17 \end{pmatrix}$

   $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 10/13 \\ 15/13 \end{pmatrix}$

2. (a) $\begin{pmatrix} 18 \\ 7 \end{pmatrix}$

   (b) Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

   $$\mathsf{A}\mathbf{u} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 4b \\ a + b \end{pmatrix},$$

   so we require $2a + 4b = 5$ and $a + b = 2$. The solution to this system is that $a = 3/2$ and $b = 1/2$, so then $\mathbf{u} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.

   Remark: We can also compute $\mathbf{u} = \mathsf{A}^{-1}\mathbf{v}$ once we have A^{-1} (see Problem 6).

3. (a) $\begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix}$

   (b) $\begin{pmatrix} -6 & -12 \\ -3 & -3 \end{pmatrix}$

   (c) $\begin{pmatrix} 14 & -20 \\ 2 & -3 \end{pmatrix}$

   (d) $\begin{pmatrix} -2 & -8 \\ 3 & 13 \end{pmatrix}$

   (e) $\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$

4. (a) $4\mathsf{I} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{pmatrix}$

(e) $2\mathsf{P} - \mathsf{I} = \begin{pmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{pmatrix}$

5. (a) $\det \mathsf{A} = -2$
   $\det \mathsf{B} = 1$

   (b) $\det(\mathsf{AB}) = \det(\mathsf{A}) \cdot \det(\mathsf{B}) = -2$

   (c) $\det(\mathsf{A}^T) = \det \mathsf{A} = -2$

   (d) $\det(\mathsf{A} + \mathsf{B}) = \det \begin{pmatrix} -1 & 8 \\ 6 & -6 \end{pmatrix} = -42$

   (e) $|\det \mathsf{A}| \cdot (\text{unit circle area}) = 2\pi$

6. (a) $\mathsf{A}^{-1} = \dfrac{1}{\det \mathsf{A}} \begin{pmatrix} 1 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 1/2 & -1 \end{pmatrix}$

   $\mathsf{B}^{-1} = \dfrac{1}{\det \mathsf{B}} \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ -5 & -3 \end{pmatrix}$

   (b) $\mathsf{A}^{-1}\mathsf{B}^{-1} = \begin{pmatrix} -13/2 & -4 \\ 3/2 & 1 \end{pmatrix}$

   $\mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

   (c) $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} 3/2 & -10 \\ 1 & -7 \end{pmatrix}$

   (d) $(\mathsf{A}^T)^{-1} = (\mathsf{A}^{-1})^T = \begin{pmatrix} -1/2 & 1/2 \\ 2 & -1 \end{pmatrix}$

   (e) $(\mathsf{A} + \mathsf{B})^{-1} = \dfrac{1}{\det(\mathsf{A} + \mathsf{B})} \begin{pmatrix} -6 & -8 \\ -6 & -1 \end{pmatrix} = \begin{pmatrix} 1/7 & 4/21 \\ 1/7 & 1/42 \end{pmatrix}$

   (f) $\det(\mathsf{A}^{-1}) = 1/\det \mathsf{A} = -1/2$

7. (a) A parallelogram with vertices $(0,0), (1,0), (3,1), (2,1)$

   (b) $\det \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = 1$

   (c) Multiplying the right two matrices, $\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & ak \\ 0 & b \end{pmatrix}$. Looking at the image of vector $\hat{\mathbf{i}}$, we need $\begin{pmatrix} a \\ 0 \end{pmatrix}$ to rotate to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. This can be achieved with a rotation by $\theta = \arccos(4/5)$ and $a = 5$. To find b, taking the determinant on both sides and noting that rotations have determinant 1, we require $ab = 25$, so $b = 5$. Finally, to get k, we need $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ to rotate to $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$. Comparing lengths and noting that $\begin{pmatrix} 5k \\ 5 \end{pmatrix}$ must be in the first quadrant, $k = 1$.

8. (a) $\operatorname{tr} \mathsf{A} = 3$
$\operatorname{tr} \mathsf{B} = -10$
$\operatorname{tr}(\mathsf{A}+\mathsf{B}) = \operatorname{tr} \mathsf{A} + \operatorname{tr} \mathsf{B} = -7$
$\operatorname{tr}(\mathsf{AB}) = 11$

(b) Let $\mathsf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathsf{Q} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$\mathsf{PQ} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \quad \text{and} \quad \mathsf{QP} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix},$$

so $\operatorname{tr}(\mathsf{PQ}) = \operatorname{tr}(\mathsf{QP}) = ae+bg+cf+dh$.

(c) In general, the answer is **no**. For example, let

$$\mathsf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\mathsf{ABC} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix},$$
$$\mathsf{ACB} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 3 \end{pmatrix},$$

so $\operatorname{tr}(\mathsf{ABC}) = 6$ while $\operatorname{tr}(\mathsf{ACB}) = 7$.

9. (a) Suppose $\mathsf{I} \sim \mathsf{B}$. Then there is an invertible matrix P such that $\mathsf{B} = \mathsf{P}^{-1}\mathsf{IP}$, but the right hand side simplifies to $\mathsf{P}^{-1}\mathsf{P} = \mathsf{I}$.

(b) If $\mathsf{A} \sim \mathsf{B}$ with $\mathsf{B} = \mathsf{P}^{-1}\mathsf{AP}$, then

$$\det \mathsf{B} = \det(\mathsf{P}^{-1}\mathsf{AP}) = \det(\mathsf{P})^{-1} \cdot \det \mathsf{A} \cdot \det \mathsf{P} = \det \mathsf{A}.$$

For the trace, Problem 8b gives us

$$\operatorname{tr} \mathsf{B} = \operatorname{tr}(\mathsf{P}^{-1}(\mathsf{AP})) = \operatorname{tr}((\mathsf{AP})\mathsf{P}^{-1}) = \operatorname{tr} \mathsf{A}.$$

(c) We have $\det \mathsf{A} = 4$ and $\operatorname{tr} \mathsf{A} = 5$, so

$$\det \mathsf{D} = d_1 d_2 = 4 \quad \text{and} \quad \operatorname{tr} \mathsf{D} = d_1 + d_2 = 5.$$

This is satisfied by $d_1 = 4$ and $d_2 = 1$, so $\mathsf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

10. (a) We compute

$$f_{\mathsf{A}}(X) = \det(\mathsf{A} - X\mathsf{I}) = \det \begin{pmatrix} 3-X & 1 \\ 2 & 2-X \end{pmatrix} = (3-X)(2-X) - 2 = X^2 - 5X + 4.$$

(b) The roots are $\lambda_1 = 4$ and $\lambda_2 = 1$.

(c) Note that the equation $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, which has a non-zero solution if and only if $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$. Moreover, we can use this version of the equation to find solutions more easily.

For $\lambda_1 = 4$, we have $\mathsf{A} - \lambda_1\mathsf{I} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$, so we can take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathsf{A} - \lambda_1\mathsf{I})\mathbf{v} = \mathbf{0}$.

For $\lambda_2 = 1$, we have $\mathsf{A} - \lambda_2\mathsf{I} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, so we can take $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any non-zero scalar multiple) as a solution to $(\mathsf{A} - \lambda_2\mathsf{I})\mathbf{v} = \mathbf{0}$.

(d) Here $\mathsf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, so then $\mathsf{P}^{-1} = -\frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. We compute

$$\begin{aligned} \mathsf{P}^{-1}\mathsf{A}\mathsf{P} &= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 12 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Remark 1: If we produced different valid choices of $\mathbf{v}_1$ and $\mathbf{v}_2$ from part (c), P and P^{-1} would change, but the end result would be the same. If we swapped the order of the columns of P, then we would swap the order of the diagonal entries correspondingly.

Remark 2: The fact that we got a diagonal matrix with entries λ_1, λ_2, the same one as in Problem 9c, is not a coincidence. The process we went through in this problem is called **diagonalisation**. (Not all $n \times n$ matrices are diagonalisable, but one sufficient condition for diagonalisability is that the characteristic polynomial has n distinct roots.)

(e) Let $\mathsf{D} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, so then $\mathsf{A} = \mathsf{P}\mathsf{D}\mathsf{P}^{-1}$. Then

$$\begin{aligned} \mathsf{A}^{100} &= \mathsf{PDP}^{-1} \cdot \mathsf{PDP}^{-1} \cdot \mathsf{PDP}^{-1} \cdot \ldots \cdot \mathsf{PDP}^{-1} \cdot \mathsf{PDP}^{-1} \\ &= \mathsf{PD} \cdot \mathsf{D} \cdot \mathsf{D} \cdot \ldots \cdot \mathsf{D} \cdot \mathsf{DP}^{-1} = \mathsf{PD}^{100}\mathsf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4^{100} & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{-1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 4^{100} & 1 \\ 4^{100} & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 \cdot 4^{100} + 1 & 4^{100} - 1 \\ 2 \cdot 4^{100} - 2 & 4^{100} + 2 \end{pmatrix}. \end{aligned}$$

(f) Here $(a_0, a_1, a_2) = (4, -5, 1)$, so

$$\begin{aligned} a_0\mathsf{I} + a_1\mathsf{A} + a_2\mathsf{A}^2 &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -15 & -5 \\ -10 & -10 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -5 \\ -10 & -6 \end{pmatrix} + \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix} = \mathsf{0}. \end{aligned}$$

2 Vectors in 3D

Problems and solutions can be found at https://azhou5849.github.io/teaching/

2.1 Review Problems

1. *Operations.* Let
$$\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}.$$
Compute each of the following. (Write "Err" or similar for any undefined expressions.)

   (a) $2\mathbf{a} + \mathbf{b} - \mathbf{c}$
   (b) $\|\mathbf{a}\| + \|\mathbf{b}\| - \|\mathbf{a} + \mathbf{b}\|$
   (c) $\mathbf{b} \cdot \mathbf{c}$
   (d) $\mathbf{a} \times \mathbf{b}$
   (e) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
   (f) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

2. *Distances and spheres.*

   (a) Find the distance between the points $(2, -5, -2)$ and $(1, -5, 0)$.
   (b) Write down an equation for the sphere with center $(5, -1, 0)$ and radius 5.
   (c) Find the center and radius of the sphere with equation
$$x^2 + y^2 + z^2 - 2x + 8y + 8z + 17 = 0.$$

3. *Angles.* Let $A = (-20, -2, 1)$, $B = (-15, 3, 21)$, and $C = (-16, 14, 5)$. Compute $\angle BAC$.

4. *Cross products.* Let $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$.

   (a) Find all vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ with norm 1.
   (b) Find the area of the parallelogram with vertices at $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$.
   (c) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Compute $\sin\theta$.

5. *Planes.* Let $A = (4, -5, 5)$, $B = (-2, 5, -5)$, and $C = (3, -3, -3)$. Find an equation for the plane passing through A, B, and C

   (a) in parametric form;
   (b) in cartesian form $ax + by + cz = d$.

   Then find a parametric form for the intersection of this plane and the plane $x + 2y + 3z = 4$.

6. *Projections and reflections.*

   (a) What point on the line through $(-5, 0, -2)$ and $(2, 5, 2)$ is closest to $(3, 1, -4)$?

   (b) Find the reflection of the point $P = (4, -4, 5)$ across the plane $-4x + 4y + 3z = 3$.

   (c) $(*)$ Let $\mathcal{C}$ be the circle centered at $(0, 0, 1)$ of radius 1 lying in the plane $z = 1$ and let ℓ be the line passing through the origin and the point $(1, 3, 1)$. What is the shortest possible distance between a point on $\mathcal{C}$ and a point on ℓ?

7. *Using cross products in 2D problems.*

   (a) Let ABC be a triangle in the xy-plane with area 14. If the points A, B, C are listed in clockwise order going around the triangle, what is $\overrightarrow{AB} \times \overline{AC}$?

   (b) $(*)$ Let $ABCD$ be a convex quadrilateral and let points P and Q lie on segments $\overline{AB}$ and $\overline{CD}$ respectively so that $AP/AB = CQ/CD$. Let R be the intersection of $\overline{AQ}$ and $\overline{PD}$ and let S be the intersection of $\overline{BQ}$ and $\overline{PC}$. Show that

$$[PSQR] = [ARD] + [BSC].$$

2.2 Challenge Problems

8.

9.

10.

2.3 Answers

1. (a) $2\mathbf{a}+\mathbf{b}-\mathbf{c}=\begin{pmatrix}2(-2)+3-(-1)\\2(-1)+0-1\\2(2)+(-4)-5\end{pmatrix}=\begin{pmatrix}0\\-3\\-5\end{pmatrix}$

   (b) $\|\mathbf{a}\|+\|\mathbf{b}\|-\|\mathbf{a}+\mathbf{b}\|=3+5-\left\|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right\|=8-\sqrt{6}$

   (c) $\mathbf{b}\cdot\mathbf{c}=3\cdot(-1)+0\cdot1+(-4)\cdot5=-23$

   (d) $\mathbf{a}\times\mathbf{b}=$

   (e) Err ($\mathbf{b}\cdot\mathbf{c}$ produces a real number, which cannot be dotted with $\mathbf{a}$)

   (f)