Algebra 1 Practice Problems III

Alan Zhou

2023-2024

The focus of these review problems is on the material covered in Weeks 25 through 35, but keep in mind that prior material can still appear on the exam.

Contents

1 Functions

Throughout, the notation $f : D \to \mathbb{R}$ means that f is a function whose domain is D and whose output values are real numbers. By the "natural domain" of a formula, we mean the largest subset of the reals which can be the domain of a function given by that formula. For example, the natural domain of $1/x$ is the set of all real numbers other than 0.

1.1 Review problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 - 6$. Evaluate each of the following:
   (a) $f(0)$
   (b) $f(f(1))$
   (c) $f^6(3)$
   (d) $(f(3))^6$
   (e) $f(3+4) - (f(3) + f(4))$
2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 - 6$ and let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = 2x$. Evaluate each of the following:
   (a) $(f + g)(2)$
   (b) $(f \cdot g)(2)$
   (c) $(f \circ g)(2)$
   (d) $(g \circ f)(2)$
   (e) $f(g(f(2)))$
   (f) $(f^2 \circ g)(2)$
3. Determine the natural domains of each of the following formulas.
   (a) $x^3 - x$
   (b) $\frac{x+2}{(x-2)(x-3)}$
   (c) $\sqrt{2x - 8}$
   (d) $\sqrt[4]{x^2 - 5x - 14}$
4. For each of the following functions, determine the range. If the domain is unspecified and only a formula is given, assume that the corresponding function has the natural domain.
   (a) x^2
   (b) x^3
   (c) $\frac{1}{x-1} + 4$
   (d) $\sqrt{x+3} - 2$
   (e) x^2 with domain $(-2, 3]$
   (f) $\frac{2x}{x^2+1}$

5. For each of the following functions, determine whether the function is invertible. If so, find the inverse function (including domain specification as needed). If not, find two input values which produce the same output.

   (a) x^2

   (b) x^2 with domain $(-2, 3]$

   (c) $\frac{1}{x-1} + 4$

   (d) $\sqrt{x+3} - 2$

   (e) $x^2 - 6x + 8$ with domain $(3, +\infty)$

   (f) $\frac{2x}{x^2+1}$ with domain $[-1, 1]$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Describe a sequence of transformations that would transform the graph of f into the graph of the given equation.

   (a) $y = f(x) - 3$

   (b) $y = -2f(x)$

   (c) $y = f(x - 5)$

   (d) $y = f(x/4)$

   (e) $y = 3f(2x - 1) + 4$

   (f) $(y + 1)/2 = f(-x + 6)$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(4) = 7$ and let $g : \mathbb{R} \to \mathbb{R}$ be given by the formula $g(x) = 3f(x^2) - 4$.

   (a) Find two points on the graph of g.

   (b) Does g have an inverse?

1.2 Challenge problems

8. Let $f(x) = -1/(x + 1)$. Given that $f^6(x) = x$ for all x, compute $f^5(2024)$.

9. Let $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ be given by

$$\begin{array}{llll} f(1) = 6, & f(2) = 4, & f(3) = 2, & f(4) = 3, \\ f(5) = 7, & f(6) = 8, & f(7) = 1, & f(8) = 5. \end{array}$$

   Find the smallest positive integer n such that $f^n(k) = k$ for all $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

10. (Thomae's function) For any rational number q, let denom(q) be the least positive integer k for which kq is an integer. (In other words, denom(q) is the denominator of q when written as a fraction in lowest terms.) Sketch the graph of the function with domain $(0, 1)$

$$T(x) = \begin{cases} \frac{1}{\text{denom}(x)} & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

1.3 Answers

1. (a) $f(0) = -6$
   (b) $f(f(1)) = f(-5) = 19$
   (c) Since $f(3) = 3$, the function f doesn't do anything to 3, so $f^6(3) = 3$ as well.
   (d) $(f(3))^6 = 3^6 = 729$
   (e) $f(3 + 4) - (f(3) + f(4)) = f(7) - f(3) - f(4) = 43 - 3 - 10 = 30$

2. (a) $(f + g)(2) = f(2) + g(2) = -2 + 4 = 2$
   (b) $(f \cdot g)(2) = f(2) \cdot g(2) = (-2) \cdot 4 = -8$
   (c) $(f \circ g)(2) = f(g(2)) = f(4) = 10$
   (d) $(g \circ f)(2) = g(f(2)) = g(-2) = -4$
   (e) $f(g(f(2))) = f(-4) = 10$
   (f) $(f^2 \circ g)(2) = f^2(g(2)) = f(f(g(2))) = f(10) = 94$

3. (a) $\mathbb{R}$, or equivalently, $(-\infty, +\infty)$
   (b) $(-\infty, 2) \cup (2, 3) \cup (3, +\infty)$, or equivalently, $\mathbb{R}\backslash\{2, 3\}$
   (c) $[4, +\infty)$
   (d) We need $x^2 - 5x - 14 \geq 0$. The left hand side factors as $(x + 2)(x - 7)$, and this is non-negative when $x \leq -2$ (when both factors are non-positive) and when $x \geq 7$ (when both factors are non-negative). The domain is $(-\infty, -2] \cup [7, +\infty)$.

4. (a) $[0, +\infty)$
   (b) $\mathbb{R}$, or equivalently, $(-\infty, +\infty)$
   (c) $(-\infty, 4) \cup (4, +\infty)$, or equivalently, $\mathbb{R}\backslash\{4\}$
   (d) $[-2, +\infty)$
   (e) $[0, 9]$
   (f) Let y be in the range, so there is a real number x satisfying

   $$y = \frac{2x}{x^2 + 1}.$$

   Clearing denominators (which is reversible since $x^2 + 1 \neq 0$ for real x) and rearranging,

   $$yx^2 - 2x + y = 0. \tag{†}$$

   This has a real solution for x if and only if the discriminant

   $$(-2)^2 - 4 \cdot y \cdot y = 4 - 4y^2$$

   is non-negative. This occurs when $-1 \leq y \leq 1$, and when this condition is met, real solutions for x are given by the quadratic formula. Thus the range is $[-1, 1]$.

5. (a) No inverse, e.g. $(-1)^2 = 1^2$

(b) No inverse, e.g. $(-1)^2 = 1^2$

(c) Inverse given by $y \mapsto 1 + \frac{1}{y-4}$

(d) Inverse given by $y \mapsto (y+2)^2 - 3$ with domain $[-2, +\infty)$

(e) Inverse given by $y \mapsto \sqrt{y+1} + 3$ with domain $(-1, +\infty)$

(f) Solving (†) for x yields, when $y \neq 0$,

$$x = \frac{2 \pm \sqrt{4 - 4y^2}}{2y}.$$

The solution which falls within the required domain $[-1, 1]$ of the original function uses the $-$ sign in the $\pm$. When $y = 0$, we get $x = 0$, so the inverse function is

$$y \longmapsto \begin{cases} \frac{2-\sqrt{4-4y^2}}{2y} & y \neq 0, \\ 0 & y = 0. \end{cases}$$

6. We can swap the order in which we perform horizontal and vertical transformations (perhaps even intersperse them), but we have to keep horizontal transformations in the same order relative to each other, and similarly for vertical transformations.

   (a) Vertical translation by 3 down

   (b) Vertical dilation by -2 (i.e. reflect across x-axis and vertical dilation by 2)

   (c) Horizontal translation by 5 right

   (d) Horizontal dilation by 4

   (e) For the horizontal, (i) translate right by 1, then (ii) dilate by a factor of $1/2$.
   For the vertical, (i) dilate by a factor of 3, then (ii) translate up by 4.

   (f) For the horizontal, (i) translate left by 6, then (ii) reflect across the y-axis.
   For the vertical, (i) dilate by a factor of 2, then (ii) translate down by 1.

7. (a) $(-2, 17)$ and $(2, 17)$

   (b) The horizontal line $y = 17$ meets the graph of g in at least two points, namely the ones from part (a), so g has no inverse.

8. Let $y = f^5(2024)$. Then $f(y) = f^6(2024) = 2024$, so we need to solve the equation

$$\frac{-1}{1+y} = 2024.$$

   The solution is $y = -\frac{2025}{2024}$.

9. The numbers $\{2, 4, 3\}$ cycle every 3 iterations of f while the numbers $\{1, 6, 8, 5, 7\}$ cycle every 5 iterations of f. To get all of the numbers to cycle, we need $\operatorname{lcm}(3, 5) = \boxed{15}$ iterations.

10. Wikipedia page.

2 Special Functions

2.1 Review problems

1. Let $f(x) = 4x^3 - 6x^4 + 2x - 1$ and $g(x) = 3x^4 + x^2 + 9$ and $h(x) = \sqrt{x^2 + 1}$.
   (a) Which of f, g, and h are polynomial functions?
   (b) Evaluate $f(0)$ and $g(1)$.
   (c) Simplify $f(x) + g(x)$ and $f(x) \cdot g(x)$.
   (d) Compute $\deg f(x)$ and $\deg g(x)$.
   (e) Is there a constant a for which $f(x) + a \cdot g(x)$ has degree less than 4? If so, what is the value of a and the degree of $f(x) + a \cdot g(x)$ for that value? If not, explain why not.

2. Let $p(x) = x^4 + 9x^3 + 28x^2 + 39x + 21$. This polynomial has no integer roots, but there are positive integers a, b, c, d such that

$$p(x) = (x^2 + ax + b)(x^2 + cx + d).$$

   Find all real roots of $p(x)$.

3. Solve each of the following equations for real solutions for x.
   (a) $\sqrt[6]{625} = 5^x$
   (b) $9^{1+x} = 27^{1+1/x}$
   (c) $4^x - 2^x = 56$

4. (Calculator allowed) Janelle puts \$10,000 into an account that earns 4% interest compounded annually. How much money will there be in the account after 18 years?

5. Compute each of the following logarithms without a calculator (some are undefined).
   (a) $\log_7(49)$
   (b) $\log_2(64)$
   (c) $\log_{27}(9)$
   (d) $\log_5(1)$
   (e) $\log_4(0)$
   (f) $\log_6(-6)$
   (g) $\log_8(1/4)$
   (h) $\log_{12}\left(2\sqrt[3]{18}\right)$

6. Find the domain and range of the function $f(x) = \log_{10}\left(\sqrt{100 - x^2}\right)$.

7. Find the values of a, b, and c which make the following function continuous:

$$f(x) = \begin{cases} x^2 - 3 & x \leq -1; \\ ax + b & -1 < x \leq 2; \\ \frac{2x^2-3x-9}{x^2-4x+3} & x > 2 \text{ and } x \neq 3; \\ c & x = 3. \end{cases}$$

2.2 Challenge problems

8. Let $p(x) = (1 + x)^{20}$.
   (a) Find the sum of the coefficients of $p(x)$.
   (b) Find the sum of the coefficients of $p(x^2 - 3x + 2)$.
   (c) (Very challenging) Find the sum of the coefficients of the terms of $p(x)$ whose degrees are multiples of 4.
9. (Calculator allowed) In this problem, i is *not* the imaginary unit.
   (a) Let $E(i, m)$ denote the *effective annual interest rate* for a nominal annual interest rate i compounded m times per year. That is, for any whole number of years, compounding annually at a rate of $E(i, m)$ results in the same balance as compounding m times per year with a nominal annual interest rate i.
   Write down a formula for $E(i, m)$ in terms of i and m, then compute $E(4\%, 12)$.
   (b) Let $F(i, m)$ be the solution to the equation $E(F(i, m), m) = i$. Compute $F(4\%, 12)$.
   (c) If we hold i fixed, then as m gets larger and larger, $F(i, m)$ approaches a value called the *force of interest*. For the interest rate $i = 4\%$, compute the force of interest as a percentage to one decimal place.
10. Compute $(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$ without a calculator.

2.3 Answers

1. (a) f and g are polynomials, h is not
   (b) $f(0) = -1$ and $g(1) = 13$
   (c)
   $$\begin{aligned} f(x) + g(x) &= -3x^4 + 4x^3 + x^2 + 2x + 8 \\ f(x) \cdot g(x) &= -18x^8 + 12x^7 - 6x^6 + 10x^5 - 57x^4 + 38x^3 - x^2 + 18x - 9 \end{aligned}$$
   (d) $\deg f(x) = \deg g(x) = 4$
   (e) When $a = 2$,
   $$f(x) + a \cdot g(x) = 4x^3 + 2x^2 + 2x + 17$$
   so $\deg(f(x) + a \cdot g(x)) = 3$.

2. We can suppose that $b \le d$, as otherwise we simply swap the two factors of $p(x)$. Expanding,
   $$\begin{aligned} a + c &= 9, \\ b + ac + d &= 28, \\ ad + bc &= 39, \\ bd &= 21. \end{aligned}$$
   Since a, b, c, d are positive integers, either $b = 1$ and $d = 21$ or $b = 3$ and $d = 7$. In the first case, we have $a + c = 9$ and $21a + b = 39$, but this is not satisfied by positive integers. Thus we are in the second case, so $a + c = 9$ and $7a + 3b = 39$. This system has solution $a = 3$ and $c = 6$, and we can check the other equations (or by expanding from scratch) to see that
   $$p(x) = (x^2 + 3x + 3)(x^2 + 6x + 7).$$
   If r is a root of $p(x)$, then $p(r) = 0$, so either $r^2 + 3r + 3 = 0$ or $r^2 + 6r + 7 = 0$. By the quadratic formula, the first case gives us
   $$r = \frac{-3 \pm \sqrt{-3}}{2},$$
   which are non-real roots, while the second case gives us
   $$r = \frac{-6 \pm \sqrt{8}}{2} = \boxed{-3 \pm \sqrt{2}}.$$

3. (a) $\sqrt[6]{625} = (5^4)^{1/6} = 5^{4/6} = 5^{2/3} \implies x = 2/3$
   (b) We have $9^{1+x} = (3^2)^{1+x} = 3^{2x+2}$ and $27^{1+1/x} = (3^3)^{1+1/x} = 3 + 3/x$, so
   $$2x + 2 = 3 + \frac{3}{x}.$$
   Multiplying through by x gives $2x^2 + 2x = 3x + 3$, or $2x^2 - x - 3 = 0$. Factoring the left hand side yields
   $$(2x - 3)(x + 1) = 0,$$
   so the possible solutions are $x = 3/2$ and $x = -1$. Both solutions work.

(c) If $y = 2^x$, then $y^2 - y = 56$. This has solutions $y = 8$ and $y = -7$. There is no real x for which $2^x = -7$, while $2^x = 8$ is satisfied for $x = 3$.

4. $\$10{,}000 \cdot (1 + 0.04)^{18} \approx \$20{,}258.17$

5. (a) 2
   (b) 6
   (c) 2/3
   (d) 0
   (e) Undefined
   (f) Undefined
   (g) $-2/3$
   (h) 2/3

6. Domain: For the logarithm to be defined, we need $\sqrt{100 - x^2}$ to be defined and positive. This occurs for $-10 < x < 10$, or $x \in \boxed{(-10, 10)}$.

   Range: For $x \in \boxed{(-10, 10)}$, the quantity $\sqrt{100 - x^2}$ can take any value in the interval $(0, 10]$, and no value outside this interval. Applying the base 10 logarithm to this, the interval of possible outputs is $\boxed{(-\infty, 1]}$.

7. First, we find that
$$\frac{2x^2 - 3x - 9}{x^2 - 4x + 3} = \frac{(x-3)(2x+3)}{(x-3)(x-1)} = \frac{2x+3}{x-1}$$
   for $x \neq 3$, so the third line of the function definition simplifies. With this, to ensure continuity,
$$\begin{aligned} x^2 - 3 &= ax + b && \text{at } x = -1, \\ ax + b &= \frac{2x+3}{x-1} && \text{at } x = 2, \\ \frac{2x+3}{x-1} &= c && \text{at } x = 3. \end{aligned}$$
   The last equation gives us $c = 9/2$, while the first two equations give us the system
$$\begin{aligned} -a + b &= -2, \\ 2a + b &= 7. \end{aligned}$$
   This system is solved by $a = 3$ and $b = 1$.

8. (a) The sum of the coefficients of $p(x)$ is $p(1) = \boxed{2^{20}} = 1\,048\,576$.
   (b) The sum of the coefficients of $q(x) = p(x^2 - 3x + 2)$ is $q(1) = p(0) = \boxed{1}$.

(c) Let $p(x) = a_{20}x^{20} + a_{19}x^{19} + \cdots + a_1 x + a_0$. Since the powers of i go in the cycle $1, i, -1, -i$, it turns out that

$$a_0 + a_4 + a_8 + a_{12} + a_{16} + a_{20} = \frac{p(1) + p(i) + p(-1) + p(-i)}{4}.$$

We found $p(1) = 2^{20}$ before, while $p(-1) = (1 + (-1))^{20} = 0$. For the other two terms,

$$p(\pm i) = (1 \pm i)^{20} = (\pm 2i)^{10} = 2^{10} i^{10} = -2^{10},$$

so our final answer is

$$\frac{2^{20} - 2^{10} + 0 - 2^{10}}{4} = \boxed{2^{18} - 2^9} = 261\,632.$$

9. (a) $E(i, m) = \left(1 + \dfrac{i}{m}\right)^m - 1$ and $E(4\%, 12) \approx 4.074\%$

(b) For ease of notation, let $j = F(i, m)$, so that we need $E(j, m) = i$, or

$$\left(1 + \frac{j}{m}\right)^m - 1 = i.$$

This is solved by $j = m\left[(1 + i)^{1/m} - 1\right]$. Then $F(4\%, 12) \approx 3.928\%$.

(c) Trying larger and larger values of m, we find $F(4\%, m)$ approaches $\approx 3.9\%$.

10. We will first prove that for all a, b, c (for which the involved logarithms are defined),

$$(\log_a b)(\log_b c) = \log_a c. \qquad (\star)$$

Let $x = \log_a b$ and $y = \log_b c$. Then $a^x = b$ and $b^y = c$, so

$$c = b^y = (a^x)^y = a^{xy}.$$

This means $\log_a c = xy$, as required.

Applying $(\star)$ repeatedly to the given problem,

$$\begin{aligned}(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8) &= (\log_4 6)(\log_6 7)(\log_7 8) \\ &= (\log_4 7)(\log_7 8) \\ &= \log_4 8 = \boxed{\frac{3}{2}}.\end{aligned}$$

3 Sequences and Series

3.1 Review problems

1. Identify, for each of the following sequences, whether they could be arithmetic, geometric, both, or neither. For any arithmetic sequence, identify the common difference. For any geometric sequence, identify the common ratio.
   (a) $1, 1, 1, 1, 1, \ldots$
   (b) $2, -2, 2, -2, 2, \ldots$
   (c) $1, 1, 2, 3, 5, \ldots$
   (d) $5, 11, 17, 23, 29, \ldots$
   (e) $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{2}\sqrt{2}, \sqrt{8}, \frac{5\sqrt{2}}{2}, \ldots$
2. Compute the arithmetic series
$$3 + 5 + 7 + 9 + \cdots + 89.$$
3. The sum of the first 6 terms of an arithmetic sequence is 114, and the sum of the next 5 terms is -15. What is the least positive integer N for which the sum of the first N terms of the series is (strictly) negative?
4. (Calculator permitted) Jo puts \$1000 into a savings account at the beginning of every year starting at the beginning of 2024. The account earns 2% nominal annual interest, compounded quarterly. How much money will there be in her account at the end of 2050?
5. When
$$0.304\,878\,048\,780\,487\,804\,878\,\ldots = 0.3\overline{04878}$$
is expressed as a fraction in lowest terms, the numerator is 25. What is the denominator?
6. (a) Find constants A and B such that
$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$
for all positive integers n.
   (b) Evaluate the series
$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \cdots + \frac{1}{9800}.$$
7. For each positive integer n, we define $n!$ (read as: "n factorial") to be the product of the first n positive integers. The first few factorials are
$$1! = 1, \quad 2! = 2 \cdot 1 = 2, \quad 3! = 3 \cdot 2 \cdot 1 = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$
   (a) For which positive integers n is it true that $n! > 2^n$?
   (b) When the infinite series
$$\frac{10}{1!} + \frac{10}{2!} + \frac{10}{3!} + \frac{10}{4!} + \frac{10}{5!} + \frac{10}{6!} + \cdots$$
is computed, between what two consecutive positive integers does the value lie?

3.2 Challenge problems

8. Let $0 < x < 1$ be a real number.

   (a) Evaluate the infinite series

   $$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots \qquad (\dagger)$$

   in terms of x.

   (b) As x gets closer and closer to 1, what value does the infinite series ($\dagger$) approach?

   (c) When we compute the sum of the first n terms of

   $$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

   for larger and larger values of n, do these sums approach your answer to part (b)?

9. For each positive integer n, let A_n be the sum of the first n squares and let T_n be the sum of the first n positive integers.

   (a) Show that $T_n = \frac{n(n+1)}{2}$.

   (b) Show that for each positive integer n,

   $$(n+1)^3 - 1 = 3A_n + 3T_n + n.$$

   Hint: Expand $(k+1)^3 - k^3$.

   (c) Evaluate T_{100} and A_{100}.

10. (Basel problem and related sums) It was found by Leonhard Euler in 1734 that

    $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

    (a) Compute

    $$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \cdots.$$

    (b) Compute

    $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots.$$

3.3 Answers

1. (a) Arithmetic (common difference 0) and geometric (common difference 1)
   (b) Geometric with common ratio -1
   (c) Neither arithmetic nor geometric
   (d) Arithmetic with common difference 6
   (e) Arithmetic with common difference $\sqrt{2}/2$

2. Let n be the number of terms in the series. The common difference is 2, so since there are $n-1$ "steps" from the first term to the last term,

$$3 + 2(n-1) = 89 \implies n = 44.$$

   Using the formula from class, the value of the series is

$$\frac{1}{2} \cdot 44 \cdot (3 + 89) = \boxed{2024}.$$

3. Let a be the first term and d be the common difference. Denoting the n-th term of the sequence by a_n, the relevant terms for the two given series are

$$\begin{aligned} a_1 &= a, & a_6 &= a + 5d; \\ a_7 &= a + 6d, & a_{11} &= a + 10d. \end{aligned}$$

   The sum of the first 6 terms gives us the equation

$$114 = \frac{1}{2} \cdot 6 \cdot (a_1 + a_6) = 6a + 15d,$$

   while the sum of the next 5 terms gives us the equation

$$-15 = \frac{1}{2} \cdot 5 \cdot (a_7 + a_{11}) = 5a + 40d.$$

   Solving this system, we get $a = 29$ and $d = -3$.

   From here, we want to find the least N for which the sum of the first N terms,

$$\frac{1}{2} \cdot N \cdot (29 + [29 - 3(N-1)]),$$

   is negative. Simplifying, we have the inequality

$$\frac{N}{2}(61 - 3N) < 0,$$

   which is first satisfied when $N = \boxed{21}$.

4. After a year passes, the amount of money in the account is multiplied by

$$r = \left(1 + \frac{0.02}{4}\right)^4 = 1.005^4.$$

The first \$1000 is multiplied by r^{27}, since its contribution to the account is present for 27 years (the *beginning* of 2024 to the *end* of 2050). The next \$1000 is multiplied by r^{26}, then the next \$1000 is multiplied by r^{25}, and so on. We stop at the last deposit of \$1000, which is multiplied by r^1. The total in the account by the end of 2050 is then

$$\begin{aligned}\$1000 \cdot r^{27} + \$1000 \cdot r^{26} + \cdots + \$1000 \cdot r^1 &= \$1000 \cdot (r^1 + r^2 + \cdots + r^{27}) \\ &= \$1000 \cdot \frac{r^{28} - r^1}{r - 1} \\ &\approx \$36\,132.\end{aligned}$$

5. First, the infinite geometric series formula gives us

$$\begin{aligned}0.\overline{04878} &= \frac{4878}{10^5} + \frac{4878}{10^{10}} + \frac{4878}{10^{15}} + \cdots = \frac{4878}{10^5} \cdot \frac{1}{1 - \frac{1}{10^5}} \\ &= \frac{4878}{10^5 - 1} = \frac{4878}{99999} = \frac{542}{11111}.\end{aligned}$$

(If you are familiar with the Euclidean algorithm, then instead of what follows, it is recommended to use it to find the greatest common divisor and simplify the fraction further that way.) This means that

$$\begin{aligned}0.3\overline{04878} = 0.3 + 0.0\overline{04878} &= \frac{3}{10} + \frac{1}{10} \cdot \frac{542}{11111} \\ &= \frac{33333 + 542}{111110} = \frac{33875}{111110}.\end{aligned}$$

The problem tells us that when this fraction is fully simplified, the numerator is 25. This means the greatest common divisor of the numerator and denominator is $33875/25 = 1355$, and the denominator is

$$\frac{111110}{1355} = \frac{22222}{271} = \boxed{82}.$$

6. (a) Clearing denominators by multiplying both sides by $n(n+2)$, we need

$$1 = A(n+2) + Bn = (A+B)n + 2A$$

for all n. This means $2A = 1$, so $A = 1/2$, and $A + B = 0$, so $B = -1/2$.

(b) The n-th term of the series is $\frac{1}{n(n+2)}$, so

$$\begin{aligned}\frac{1}{3} + \frac{1}{8} + \cdots + \frac{1}{9800} &= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{98} - \frac{1}{100}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{99} - \frac{1}{100}\right] \\ &= \frac{1}{2} \cdot \frac{9900 + 4950 - 100 - 99}{9900} = \frac{14651}{19800}.\end{aligned}$$

7. (a) We have $n! > 2^n$ for $n \geq 4$.

   (b) For an upper bound,

$$\begin{aligned}
&\frac{10}{1!}+\frac{10}{2!}+\frac{10}{3!}+\frac{10}{4!}+\frac{10}{5!}+\frac{10}{6!}+\cdots\\
&\qquad = 10\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots\right)\\
&\qquad < 10\left(1+\frac{1}{2}+\frac{1}{6}+\left[\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\cdots\right]\right)\\
&\qquad = 10\left(\frac{5}{3}+\frac{1/2^4}{1-1/2}\right)\\
&\qquad = 17\frac{11}{12}.
\end{aligned}$$

   For a lower bound, we can compute partial sums, and eventually we find that

$$10\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots\right) > 10\left(1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}\right) = 17\frac{1}{12}.$$

   Thus the value of the series lies between $\boxed{\text{17 and 18}}$.
   The actual value is roughly 17.182 818.

8. (a) This is an infinite geometric series with common ratio $r = -x$. Since $|r| < 1$, the series has a finite value, which is $\dfrac{1}{1-r} = \boxed{\dfrac{1}{1+x}}$.

   (b) 1/2

   (c) The partial sums go $1, 0, 1, 0, 1, 0, \ldots$, alternating without actually getting closer to 1/2.

9. (a) The series $1 + 2 + \cdots + n$ is arithmetic with n terms, first term 1, and last term n, so

$$T_n = \frac{1}{2} \cdot n \cdot (1+n) = \frac{n(n+1)}{2}.$$

   (b) Following the hint, we can "open" a telescoping series and write

$$(n+1)^3 - 1 = [(n+1)^3 - n^3] + [n^3 - (n-1)^3] + \cdots + [2^3 - 1^3].$$

   For each positive integer k, we have $(k+1)^3 - k^3 = 3k^2 + 3k + 1$, so the above sum is

$$[3n^2 + 3n + 1] + [3(n-1)^2 + 3(n-1) + 1] + \cdots + [3 \cdot 1^2 + 3 \cdot 1 + 1].$$

   Regrouping, we get

$$3[n^2 + (n-1)^2 + \cdots + 1^2] + 3[n + (n-1) + \cdots + 1] + [1 + 1 + \cdots + 1] = 3A_n + 3T_n + n.$$

   (c) For $n = 100$, we get $T_{100} = 5050$ and $A_n = 338\,350$.
   In general, $A_n = \frac{n(n+1)(2n+1)}{6}$.

10. (a) For each k,

$$\frac{1}{(2k)^2} = \frac{1}{4k^2} = \frac{1}{4} \cdot \frac{1}{k^2}.$$

Therefore,

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \cdots = \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) = \boxed{\frac{\pi^2}{24}}.$$

(b) The two series in parts (a) and (b) add up to form the original series given to us, so

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$