Precalculus Practice Problems: Midterm 2

Alan Zhou

2024-2025

The focus of these review problems is on the material covered in Weeks 13 through 23, but keep in mind that prior material can still appear on the exam.

Contents

1 Laws of Sines and Cosines

1.1 Review problems

Calculators are recommended for this section. Throughout, if ABC is a triangle, then we use a, b, and c to denote the side lengths BC, CA, and AB, respectively. (That is, a is the length of the side opposite A, etc.) The notation $[ABC]$ denotes the area of ABC.

1. (SAS congruence) Let ABC be a triangle with $a = 1$, $b = 5$, and $\angle C = 104°$.
   (a) Find $[ABC]$.
   (b) Find c.
   (c) Find $\angle A$ and $\angle B$.
2. (SSS congruence) Let ABC be a triangle with $a = 13$, $b = 14$, and $c = 15$.
   (a) Find $\angle A$.
   (b) Find $\angle B$ and $\angle C$.
   (c) Find $[ABC]$.
3. (ASA/AAS congruence) Let ABC be a triangle with $c = 2$, $\angle A = 12°$, and $\angle B = 77°$.
   (a) Find $\angle C$.
   (b) Find a and b.
   (c) Find $[ABC]$.
4. (SSA non-congruence) Let ABC be a triangle with $\angle A = 20°$, $a = 6$, and $b = 9$.
   (a) Find all possible values of c.
   (b) For each possible value of c, find $\angle B$.
   (c) For what values of x does there exist exactly one triangle XYZ with $\angle X = 20°$, $XY = 9$, and $YZ = x$?
5. (Extended law of sines) If ABC is a triangle with circumradius R, then the *extended law of sines* states that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
   (a) Prove that $R = \dfrac{abc}{4[ABC]}$.
   (b) Given that $a = 13$, $b = 14$, and $c = 15$, find R.
   (c) Prove the extended law of sines for acute triangles.
6. Let ABC be a triangle and let D be a point on side $\overline{BC}$.
   (a) (Ratio lemma) Prove that
$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

(b) (Angle bisector theorem) Show that if $\overline{AD}$ bisects $\angle BAC$, then $\dfrac{AB}{BD} = \dfrac{AC}{DC}$.

7. (Heron's formula) Let ABC be a triangle.

   (a) Show that
   $$[ABC]^2 = \frac{1}{4}a^2b^2(1-\cos^2 C) = \frac{4a^2b^2 - (a^2+b^2-c^2)^2}{16}.$$

   (b) Conclude that
   $$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$
   where $s = (a+b+c)/2$ is the *semiperimeter* of triangle ABC.

1.2 Challenge problems

8. Points O, A, B, and C are placed in three-dimensional space so that $AO = BO = CO = 4$, $AB = 2$, and $AC = 1$. What are the shortest and longest possible lengths of BC?

9. In triangle ABC, point D lies on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. Given that $BD = 7$, $BA = 8$, and $AD = 5$, find CD.

10. (Eisenstein triples) An *Eisenstein triple* is a triple of positive integers (a, b, c) for which a triangle with side lengths a, b, and c has an angle of measure either 60° or 120°. If the Eisenstein triple (a, b, c) corresponds to a triangle with an angle of measure 60°, we will call it an Eisenstein triple of *acute type*, and otherwise, we call it an Eisenstein triple of *obtuse type*. (The "acute type" and "obtuse type" names are non-standard.)

    (a) Let (a, b, c) be an Eisenstein triple of obtuse type with $a < b < c$. Show that $(a, a+b, c)$ and $(a+b, b, c)$ are Eisenstein triples of acute type.

    (b) Conversely, show that every Eisenstein triple of acute type arises from an Eisenstein triple of obtuse type in the above manner.

    (c) Show that if (a, b, c) is an Eisenstein triple of obtuse type with $\gcd(a, b, c) = 1$, then there are relatively prime positive integers m and n such that
    $$\{a, b, c\} = \{m^2 + mn + n^2, 2mn + n^2, m^2 - n^2\}.$$
    (Hint: See Section 1 Problem 10 from the Midterm 1 review.)

1.3 Answers

1. (a) $[ABC] = \frac{1}{2}ab\sin C = \frac{5}{2}\sin(104°) \approx 2.426$
   (b) $c = \sqrt{a^2+b^2-2ab\cos C} = \sqrt{26-10\cos(104°)} \approx 5.331$
   (c) $\angle A = \arcsin(\frac{a\sin C}{c}) \approx 10.49°$
   $\angle B = \arcsin(\frac{b\sin C}{c}) \approx 65.51°$
   These angles can also be found with the law of cosines.

2. (a) $\angle A = \arccos(\frac{b^2+c^2-a^2}{2bc}) = \arccos(\frac{3}{5}) \approx 53.13°$
   (b) $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) = \arccos(\frac{33}{65}) \approx 59.49°$
   $\angle C = \arccos(\frac{a^2+b^2-c^2}{2ab}) = \arccos(\frac{5}{13}) \approx 67.38°$
   These angles can also be found with the law of sines.
   (c) $[ABC] = \frac{1}{2}bc\sin A = \frac{14\cdot 15}{2}\sin(\arccos(\frac{3}{5})) = 7\cdot 15\cdot \frac{4}{5} = 84$

3. (a) $\angle C = 91°$
   (b) $a = \frac{c}{\sin C}\cdot \sin A = \frac{2\sin 12°}{\sin 91°} \approx 0.416$
   $b = \frac{c}{\sin C}\cdot \sin B = \frac{2\sin 77°}{\sin 91°} \approx 1.949$
   (c) $[ABC] = \frac{1}{2}ac\sin B = \frac{2\sin 12°\sin 77°}{\sin 91°} \approx 0.405$

4. (a) By the law of cosines,
   $$a^2 = b^2+c^2-2bc\cos A \implies 36 = 81+c^2-(18\cos 20°)c.$$
   Solving the resulting quadratic yields
   $$c = \frac{18\cos 20° \pm \sqrt{324\cos^2(20°)-180}}{2} = 9\cos 20° \pm 3\sqrt{9\cos^2(20°)-5}.$$
   One solution is ≈ 3.307 and the other solution is ≈ 13.607.
   (b) When $c \approx 3.307$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 149.13°$.
   When $c \approx 13.607$, we have $\angle B = \arccos(\frac{a^2+c^2-b^2}{2ac}) \approx 30.87°$.
   (c) Let $y = XZ$ be the missing side length. By the law of cosines,
   $$x^2 = 81+y^2-(18\cos 20°)y \implies y^2-(18\cos 20°)y+(81-x^2) = 0.$$
   For there to be only one triangle with the given properties, there must be exactly one positive solution for y. This can occur in two ways.

   Case 1 (exactly one real solution, which is positive). If there is exactly one real solution, then it must be $y = 9\cos 20°$, which is positive as required. This situation occurs when $81 - x^2 = (9\cos 20°)^2 = 81\cos^2(20°)$, which holds when $x = 9\sin 20°$. (This corresponds to "HL congruence.")

   Case 2 (two real solutions, only one of which is positive). The quadratic has a leading coefficient of 1, so this situation occurs precisely when the constant term is negative. Thus we need $81 - x^2 < 0$, and since x is a side length, we have $x > 9$.

5. (a) From $[ABC] = \frac{1}{2}ab\sin C$, we have $\sin C = \frac{2[ABC]}{ab}$. Then,

$$R = \frac{c}{2\sin C} = \frac{c}{\frac{4[ABC]}{ab}} = \frac{abc}{4[ABC]}.$$

   (b) $R = 65/8$

   (c) See this link.

6. (a) We have

$$[ABD] = \frac{1}{2} \cdot AB \cdot AD \cdot \sin(\angle BAD),$$
$$[ADC] = \frac{1}{2} \cdot AC \cdot AD \cdot \sin(\angle DAC),$$

   and dividing the two equations yields

$$\frac{[ABD]}{[ADC]} = \frac{AB}{AC} \cdot \frac{\sin(\angle BAD)}{\sin(\angle DAC)}.$$

   The conclusion follows from the fact that triangles ABD and ADC share a height from A, so that then $\frac{[ABD]}{[ADC]} = \frac{BD}{DC}$.

   (b) When $\overline{AD}$ bisects $\angle BAC$, we have $\angle BAD = \angle DAC$, so the sines cancel in part (a).

7. (a) We compute

$$\begin{aligned}
[ABC]^2 &= \left(\frac{1}{2}ab\sin C\right)^2 = \frac{1}{4}a^2b^2\sin^2 C \\
&= \frac{1}{4}a^2b^2(1-\cos^2 C) \\
&= \frac{1}{4}a^2b^2\left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= \frac{1}{4}a^2b^2\left[\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}\right] \\
&= \frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}.
\end{aligned}$$

   (b) From here, we observe some differences of squares to obtain

$$\begin{aligned}
[ABC]^2 &= \frac{[2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)]}{16} \\
&= \frac{[c^2-(a^2-2ab+b^2)][(a^2+2ab+b^2)-c^2]}{16} \\
&= \frac{[c-(a-b)][c+(a-b)][(a+b)-c][(a+b)+c]}{16} \\
&= \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \\
&= s(s-a)(s-b)(s-c).
\end{aligned}$$

8. By the law of cosines, we can find

$$\cos(\angle AOB) = \frac{7}{8} \quad \text{and} \quad \cos(\angle AOC) = \frac{31}{32},$$

from which we find

$$\sin(\angle AOB) = \frac{\sqrt{15}}{8} \quad \text{and} \quad \sin(\angle AOC) = \frac{3\sqrt{7}}{32}.$$

The smallest possible value of $\angle BOC$ is $\angle AOB - \angle AOC$, so the smallest possible BC is

$$\begin{aligned} \min BC &= \sqrt{32 - 32\cos(\angle AOB - \angle AOC)} \\ &= 4\sqrt{2 - 2\left(\frac{7}{8}\cdot\frac{31}{32} + \frac{\sqrt{15}}{8}\cdot\frac{3\sqrt{7}}{32}\right)} \\ &= 4\sqrt{2 - \frac{217 + 3\sqrt{105}}{128}} \approx 1.016. \end{aligned}$$

By a similar argument, the largest possible BC is

$$\max BC = 4\sqrt{2 - \frac{217 - 3\sqrt{105}}{128}} \approx 2.953.$$

9. Let $CD = 7x$, so that $AC = 8x$ by the angle bisector theorem. From the law of cosines,

$$\cos(\angle BAD) = \frac{8^2 + 5^2 - 7^2}{2\cdot 8\cdot 5} = \frac{1}{2},$$

so $\cos(\angle DAC) = 1/2$ as well. Using the law of cosines at $\angle DAC$ gives us

$$(7x)^2 = (8x)^2 + 5^2 - 2\cdot 8x\cdot 5\cdot\frac{1}{2} \implies 15x^2 - 40x + 25 = 0.$$

This quadratic factors as $5(3x-5)(x-1)$, so there are two solutions, $x = 1$ or $x = 5/3$. When $x = 1$, we end up with $\triangle DAB \cong \triangle DAC$. However, this together with D lying on segment $\overline{BC}$ implies that $\angle ADB = 90^\circ$, a contradiction. Hence the only valid solution is that $x = 5/3$, in which case $CD = 35/3$.

10. Suppose without loss of generality that c is the side opposite the 120° angle, so that by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + ab + b^2.$$

Dividing through by c^2 and letting $x = a/c$ and $y = b/c$, finding Eisenstein triples is equivalent to finding points with positive rational coordinates on the conic

$$x^2 + xy + y^2 = 1.$$

Graphing the conic, we see that it is an ellipse passing through the points $(\pm 1, 0)$ and $(0, \pm 1)$, and that every point with positive rational coordinates can be connected to $(0, -1)$ by a line of rational slope greater than 1.

Let $t = m/n$ be a rational number greater than 1, where m and n are relatively prime positive integers. The line of slope t through $(0, -1)$ is $y = tx - 1$, so to find the other point where the line intersects the conic, we substitute to get the equation

$$\begin{aligned} x^2 + x \cdot (tx - 1) + (tx - 1)^2 &= 1, \\ (t^2 + t + 1)x^2 - (2t + 1)x &= 0. \end{aligned}$$

One solution is $x = 0$, corresponding to $y = -1$, and the other solution is

$$x = \frac{2t + 1}{t^2 + t + 1},$$

corresponding to

$$y = tx - 1 = \frac{t^2 - 1}{t^2 + t + 1}.$$

Substituting $t = m/n$ and clearing nested denominators gives us

$$(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn + n^2}{m^2 + mn + n^2}, \frac{m^2 - n^2}{m^2 + mn + n^2}\right).$$

To finish, we need to check whether the fractions on the right hand side are fully reduced. To start, since $\gcd(m, n) = 1$,

$$\begin{aligned} \gcd(2mn + n^2, m^2 + mn + n^2) &= \gcd(n \cdot (2m + n), m^2 + mn + n^2) \\ &= \gcd(2m + n, m^2 + mn + n^2) \\ &= \gcd(2m + n, m^2 + mn + n^2 - n \cdot (2m + n)) \\ &= \gcd(2m + n, m^2 - mn) = \gcd(2m + n, m \cdot (m - n)) \\ &= \gcd(2m + n, m - n) = \gcd(3n, m - n). \end{aligned}$$

If $m \equiv n \pmod 3$, then let $m = n + 3k$. Then $\gcd(n, k) = 1$ and

$$\gcd(3n, m - n) = \gcd(3n, 3k) = 3 \gcd(n, k) = 3.$$

Otherwise,

$$\gcd(3n, m - n) = \gcd(n, m - n) = \gcd(n, m) = 1.$$

Thus we are done in the case that $m \not\equiv n \pmod 3$, while in the case that $m \equiv n \pmod 3$,

$$a = \frac{2mn + n^2}{3}, \quad b = \frac{m^2 - n^2}{3}, \quad c = \frac{m^2 + mn + n^2}{3}.$$

Let $r = \frac{m+2n}{3}$ and $s = \frac{m-n}{3}$, so that $n = r - s$ and $m = r + 2s$. Then

$$\begin{aligned} a &= \frac{2(r + 2s)(r - s) + (r - s)^2}{3} = r^2 - s^2, \\ b &= \frac{(r + 2s)^2 - (r - s)^2}{3} = 2rs + s^2, \\ c &= \frac{(r + 2s)^2 + (r + 2s)(r - s) + (r - s)^2}{3} = r^2 + rs + s^2, \end{aligned}$$

so the result still holds with r and s in place of m and n.