Contents

1 Interest Theory

1.1 Interest and discount

Suppose a constant interest rate i.

- Discount factor $v = (1+i)^{-1}$
- Discount rate $d = 1 - v = i(1+i)^{-1} = iv$
- Nominal annual interest rate

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \implies i^{(m)} = m[(1+i)^{1/m} - 1]$$

- Nominal annual discount rate

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p \implies d^{(p)} = p[1 - (1-d)^{1/p}]$$

- Force of interest

$$\delta = \lim_{m\to\infty} i^{(m)} = \ln(1+i) = -\ln v$$

1.2 Mortality-free annuities

Constant payment discrete annuities:

- Annuity due

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}.$$

- Annuity immediate

$$a_{\overline{n}|} = v\ddot{a}_{\overline{n}|} = \frac{1-v^n}{i}$$

- Accumulated value of annuity due

$$\ddot{s}_{\overline{n}|} = (1+i)^n \ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

- Accumulated value of annuity immediate

$$s_{\overline{n}|} = v\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

Non-constant payment discrete annuities (due versions):

- Increasing annuity

$$\begin{aligned}(I\ddot{a})_{\overline{n}|} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \frac{\partial}{\partial v}\left(\frac{1-v^{n+1}}{1-v}\right) \\ &= \frac{(1-v)(-(n+1)v^n) - (1-v^{n+1})(-1)}{(1-v)^2} = \frac{nv^{n+1} - (n+1)v^n + 1}{(1-v)^2} \\ &= \frac{nv^n(v-1) + 1 - v^n}{d^2} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}\end{aligned}$$

- Decreasing annuity

$$\begin{aligned}(D\ddot{a})_{\overline{n}|} &= n + (n-1)v + (n-2)v^2 + \cdots + v^{n-1} = (n+1)\ddot{a}_{\overline{n}|} - (I\ddot{a})_{\overline{n}|} \\ &= \frac{d(n+1)\ddot{a}_{\overline{n}|} - (\ddot{a}_{\overline{n}|} - nv^n)}{d} = \frac{(n+1)(1-v^n) - \ddot{a}_{\overline{n}|} + nv^n}{d} \\ &= \frac{n + 1 - \ddot{a}_{\overline{n}|} - v^n}{d} = \frac{n - a_{\overline{n}|}}{d}.\end{aligned}$$

Perpetuities (due versions):

- Constant

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}$$

- Increasing

$$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty} (I\ddot{a})_{\overline{n}|} = \frac{1}{d^2}$$

Payments split mth-ly:

- Constant annuity due

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m}(1 + v^{1/m} + \cdots + v^{n-1/m}) = \frac{1}{m}\frac{1-v^n}{1-v^{1/m}} = \frac{1-v^n}{d^{(m)}}$$

- Constant annuity immediate

$$a^{(m)}_{\overline{n}|} = v^{1/m}\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}}$$

- Increasing annuity due

$$\begin{aligned}(I\ddot{a})^{(m)}_{\overline{n}|} &= \frac{1}{m} + \frac{2}{m}v^{1/m} + \cdots + nv^{n-1/m} \\ &= \frac{mnv^n(v^{1/m} - 1) + 1 - v^n}{m(1-v^{1/m})^2} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{d^{(m)}}\end{aligned}$$

Continuous annuities:

- Constant
$$\overline{a}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{\delta} = \int_0^n v^t\,dt$$
- Increasing
$$(I\overline{a})_{\overline{n}|} = \lim_{m\to\infty} (I\ddot{a})^{(m)}_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t\,dt$$

2 Mortality

2.1 Survival functions and continuous mortality

Let $X = T_0$ be the continuous random variable for the future life span of a newborn

- Distribution function $F_t = F_X(t) = \mathbb{P}[X \leq t]$
- Survival function $S_t = S_X(t) = \mathbb{P}[X > t] = 1 - F_X(t)$
- Probability density function $f_X(t) = \frac{d}{dt} F_X(t) = -\frac{d}{dt} S_X(t)$

For an individual aged x, denote by $T_x = X|_{X \geq x} - x$ the random variable for their future life span (additional years only)

- Distribution function
$$F_{x+t} = F_{T_x}(t) = \frac{\mathbb{P}[x \leq X \leq x+t]}{\mathbb{P}[X \geq x]}$$
- Survival function
$$S_{x+t} = 1 - F_{T_x}(t) = \frac{\mathbb{P}[X > x+t]}{\mathbb{P}[X \geq x]} = \frac{S_X(x+t)}{S_X(x)}$$
- Warning: $S_{30+10} \neq S_{40}$

Mortality symbols

- ${}_tp_x = \mathbb{P}[T_x > t] = S_{x+t}$, special case $p_x = {}_1p_x$
- ${}_tq_x = 1 - {}_tp_x$, special case $q_x = {}_1q_x$
- $\ell_x = \ell_0 \cdot S_x$ is the number of people alive at time x
- Deferred death
$${}_{t|u}q_x = \mathbb{P}[t < T_x \leq t+u] = {}_tp_x \cdot {}_uq_{x+t} = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$$

Force of mortality

- "Probability of instant death" probability density
$$\mu_x \, dx = \mathbb{P}[T_x \leq dx] = \mathbb{P}[x \leq X \leq x + dx \mid X \geq x]$$
$$\mu_x = \lim_{dx \to 0} \frac{S_X(x) - S_X(x+dx)}{dx \cdot S_X(x)} = -\frac{S'_X(x)}{S_x} = -\frac{d}{dx}(\ln S_x)$$
- Force of mortality and probability density
$$f_X(x) = -S'_X(x) = S_x \mu_x = {}_xp_0 \mu_x,$$
$$f_{T_x}(t) = -S'_{T_x}(t) = -S'_X(x+t)/S_x = \mu_{x+t} \frac{S_X(x+t)}{S_x} = {}_tp_x \mu_{x+t}$$

2.2 Discrete mortality

Discrete mortality symbols

- Death symbol $d_x = \ell_x - \ell_{x+1}$
- First age in life table α
- Last age in life table ω, so that $p_\omega = 0$

Define random variable K_x for future *completed* years survived

- $T_x = K_x + s$ for a random variable s with values in $[0, 1)$
- Probability mass function

$$f_{K_x}(k) = \mathbb{P}[K_x = k] = \mathbb{P}[x \leq X < x + k \mid X \geq x] = {}_{k|}q_x = \frac{d_{x+k}}{\ell_x}$$

2.3 Life expectancy

- Complete life expectancy (continuous)

$$\begin{aligned}
\mathring{e}_x = \mathbb{E}[T_x] &= \int_0^\infty t \cdot f_{T_x}(t)\,dt = -\int_0^\infty t \cdot \frac{d}{dt} S'_{T_x}(t)\,dt \\
&= [-t \cdot S_{T_x}(t)]_0^\infty + \int_0^\infty S_{T_x}(t)\,dt = \int_0^\infty {}_tp_x\,dt = \frac{1}{S_x}\int_0^\infty S_X(x+t)\,dt
\end{aligned}$$

- Curtate life expectancy (discrete)

$$\begin{aligned}
e_x = \mathbb{E}[K_x] &= \sum_{t=1}^\infty t \cdot f_{K_x}(t) \\
&= \sum_{t=1}^\infty \mathbb{P}[K_x \geq t] = \sum_{t=1}^\infty {}_tp_x
\end{aligned}$$

- Since $\mathbb{E}[T_x] = \mathbb{E}[K_x] + \mathbb{E}[s]$, can estimate $\mathbb{E}[s] \approx 1/2$ and

$$\mathring{e}_x \approx e_x + \frac{1}{2}$$

- Temporary life expectancy random variables $T_{x:\overline{n}|} = \min(T_x, n)$ and $K_{x:\overline{n}|} = \min(K_x, n)$

$$\begin{aligned}
\mathring{e}_{x:\overline{n}|} &= \mathbb{E}[T_{x:\overline{n}|}] = \int_0^n {}_tp_x\,dt \\
e_{x:\overline{n}|} &= \mathbb{E}[K_{x:\overline{n}|}] = \sum_{t=1}^n {}_tp_x
\end{aligned}$$

- $T_{x:\overline{n}|} = K_{x:\overline{n}|} + s_n$ with $s_n = s$ if $T_x < n$ and $s_n = 0$ otherwise, so

$$\mathbb{E}[s_n] = \mathbb{P}[T_x < n] \cdot \mathbb{E}[s \mid T_x < n] \approx \frac{{}_nq_x}{2}$$

- Backward recurrences

$$\begin{aligned} e_x &= \sum_{t=1}^{n} {}_tp_x + \sum_{t=n+1}^{\infty} {}_tp_x = e_{x:\overline{n}|} + {}_np_x \cdot e_{x+n}, & e_\omega = 0 \\ \mathring{e}_x &= \int_0^n {}_tp_x \, dt + \int_n^\infty {}_tp_x \, dt = \mathring{e}_{x:\overline{n}|} + {}_np_x \cdot \mathring{e}_{x+n}, & \mathring{e}_\omega = 0 \end{aligned}$$

2.4 Fractional age assumptions

Uniform distribution of deaths (UDD)

- $\mathbb{P}[s \leq t] = t$ when $0 \leq t < 1$
- ${}_tq_x = \mathbb{P}[0 \leq T_x < 1] \cdot \mathbb{P}[s < t] = tq_x$
- $\mu_{x+t} = {}_tp_x^{-1} f_{T_x}(t) = q_x / {}_tp_x$ when $0 \leq t < 1$
- Linear interpolation: $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$
- $\mathbb{E}[s] = 1/2$, so $\mathring{e}_x = e_x + 1/2$ exactly

Constant form of mortality (CFM fractional)

- $\mu_{x+t} = \mu_x$ when $0 \leq t < 1$
- $\ln(S_{x+t}/S_x) = \int_0^t (-\mu_{x+t}) \, dt = -t\mu_x$, so ${}_tp_x = e^{-t\mu_x}$
- Exponential interpolation: $\ell_{x+t} = \ell_x^{1-t} \cdot \ell_{x+1}^t$

2.5 Mortality modifications

Select and ultimate probabilities

- ${}_kq_{[x]+t}$ is the probability that someone who was issued a policy at age x and is currently age $x+t$ will die in the next k years
- Similar bracket notation for ℓ, p, etc.

Change in mortality over time

- $q(x,t)$ is the probability that in year t, someone at age x dies within the next year
- Relative change in mortality within the t-th year: $\varphi(x,t) = 1 - q(x,t)/q(x,t-1)$
- If $\varphi(x,t)$ is independent of t, write φ_x

2.6 Analytic laws

de Moivre's law (DML), straight-line mortality

- $S_x = 1 - \frac{x}{\omega}$
- ${}_tq_x = \frac{t}{\omega - x}$
- $\mu_x = -\frac{S'_x}{S_x} = \frac{1}{\omega - x}$
- $\mathring{e}_x = \frac{\omega - x}{2}$
- DML implies UDD

Modified DML

- $\mu_x = \alpha \cdot \mu_x^{DML} = \frac{\alpha}{\omega - x}$
- $S_x = \exp\left[\int_0^x (-\mu_t)\, dt\right] = (S_x^{DML})^\alpha = (1 - \frac{x}{\omega})^\alpha$

Constant force of mortality (CFM law)

- $\mu_x = \mu$ constant
- $S_x = e^{-\mu x}$
- ${}_tp_x = e^{-\mu t}$ independent of x
- $\mathring{e}_x = \int_0^\infty {}_tp_x\, dt = 1/\mu$
- CFM law implies CFM fractional

Makeham's law

- $\mu_x = A + Bc^x$ for parameters A, B, c
- Gompertz's law when $A = 0$

3 Life Insurance

3.1 Discrete life insurance

Assumptions and notation

- Benefit b_{K+1} paid at the end of the year of death
- Constant interest rate
- Payout random variable Z

Whole life insurance

- $Z = b_{K+1} v^{K+1}$
- Expected value $\mathbb{E}[Z] = \sum_{k=0}^{\infty} b_{k+1} v^{k+1} {}_{k|}q_x$
- Second moment $\mathbb{E}[Z^2] = \sum_{k=0}^{\infty} b_{k+1}^2 v^{2(k+1)} {}_{k|}q_x$ equivalent to expected value of Z for benefit $b_{k+1}^* = b_{k+1}^2$ and interest $i^* = (1+i)^2 - 1$, i.e. $v^* = v^2$ or $\delta^* = 2\delta$
- If $b_{k+1} = 1$ throughout, define $A_x = \mathbb{E}[Z]$ and ${}^2A_x = \mathbb{E}[Z^2]$
- DML mortality law

$$\begin{aligned} A_x &= \sum_{k=0}^{\omega-x-1} \frac{v^{k+1}}{\omega - x} = \frac{a_{\overline{\omega-x|}}}{\omega - x} \\ {}^2A_x &= \sum_{k=0}^{\omega-x-1} \frac{v^{2(k+1)}}{\omega - x} = \frac{a_{\overline{\omega-x|}i^*}}{\omega - x} \end{aligned}$$

- CFM mortality law

$$\begin{aligned} A_x &= \sum_{k=0}^{\infty} v^{k+1} e^{-\mu k}(1 - e^{-\mu}) = v(1 - e^{-\mu}) \sum_{k=0}^{\infty} (ve^{-\mu})^k \\ &= e^{-\delta}(1 - e^{-\mu}) \cdot \frac{1}{1 - e^{-(\delta+\mu)}} = \frac{e^{-\delta}(1 - e^{-\mu})}{1 - e^{-(\delta+\mu)}} \\ {}^2A_x &= \sum_{k=0}^{\infty} v^{2(k+1)} e^{-\mu k}(1 - e^{-\mu}) = \frac{e^{-\delta^*}(1 - e^{-\mu})}{1 - e^{-(\delta^*+\mu)}} \end{aligned}$$

n-year term insurance

- $Z = \begin{cases} b_{K+1} v^{K+1} & K < n \\ 0 & K \geq n \end{cases}$
- Expected value $\mathbb{E}[Z] = \sum_{k=0}^{n-1} b_{k+1} v^{k+1} {}_{k|}q_x$

- Second moment $\mathbb{E}[Z^2] = \sum_{k=0}^{n-1} b_{k+1}^2 v^{2(k+1)} {}_{k|}q_x$
- If $b_{k+1} = 1$ for $k < n$, define $A^1_{x:\overline{n}|} = \mathbb{E}[Z]$
- DML mortality law: if $n' = \min\{n, \omega - x\}$, then

$$A^1_{x:\overline{n}|} = \frac{a_{\overline{n'}|}}{\omega - x}$$
$${}^2A^1_{x:\overline{n}|} = \frac{a_{\overline{n'}|i^*}}{\omega - x}$$

- CFM mortality law

$$A^1_{x:\overline{n}|} = v(1 - e^{-\mu}) \sum_{k=0}^{n-1} (ve^{-\mu})^k = e^{-\delta}(1 - e^{-\mu}) \cdot \frac{1 - e^{-n(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}$$
$${}^2A^1_{x:\overline{n}|} = e^{-\delta^*}(1 - e^{-\mu}) \cdot \frac{1 - e^{-n(\delta^*+\mu)}}{1 - e^{-(\delta^*+\mu)}}$$

n-year pure endowment

- $Z = 0$ if $K < n$ and $Z = bv^n$ if $K \geq n$, where b is benefit paid out for surviving
- Expected value $\mathbb{E}[Z] = bv^n \mathbb{P}[K \geq n] = bv^n {}_n p_x$
- Second moment $\mathbb{E}[Z^2] = b^2 v^{2n} {}_n p_x$
- If $b = 1$, define $A_{x:\overset{1}{\overline{n}|}} = {}_nE_x = \mathbb{E}[Z] = v^n {}_n p_x$

n-year (endowment) insurance

- $Z = Z_{n\text{-year terminsurance}} + Z_{n\text{-year pure endowment}} = \begin{cases} b_{K+1}v^{K+1} & K < n \\ bv^n & K \geq n \end{cases}$
- If $b_{k+1} = 1$ for $k < n$ and $b = 1$, define $A_{x:\overline{n}|} = \mathbb{E}[Z] = A^1_{x:\overline{n}|} + {}_nE_x$

Recurrences

- $A_x = A^1_{x:\overline{n}|} + {}_nE_x \cdot A_{x+n}$
- $A^1_{x:\overline{n}|} = A^1_{x:\overline{k}|} + {}_kE_x \cdot A^{\ 1}_{x+k:\overline{n-k}|}$ for $0 \leq k \leq n$

3.2 Continuous life insurance

Assumptions and notation

- Benefit b_T paid at the moment of death
- Constant force of interest

- Payout random variable Z or $\overline{Z}$

Whole life insurance

- $\overline{Z} = b_T v^T = b_T e^{-\delta T}$
- Expected value $\mathbb{E}[\overline{Z}] = \int_0^\infty b_t e^{-\delta t} \cdot f_{T_x}(t)\,dt$
- Second moment $\mathbb{E}[\overline{Z}^2] = \int_0^\infty b_t^2 e^{-2\delta t} \cdot f_{T_x}(t)\,dt$
- If $b_t = 1$ throughout, define $\overline{A}_x = \mathbb{E}[\overline{Z}]$
- DML mortality law

$$\overline{A}_x = \int_0^{\omega-x} e^{-\delta t} \cdot \frac{1}{\omega - x}\,dt = \frac{1-e^{-\delta(\omega-x)}}{\delta(\omega-x)} = \frac{\overline{a}_{\overline{\omega-x}|}}{\omega-x}$$
$$^2\overline{A}_x = \int_0^{\omega-x} e^{-2\delta t} \cdot \frac{1}{\omega - x}\,dt = \frac{\overline{a}_{\overline{\omega-x}|i^*}}{\omega-x}$$

- CFM mortality law

$$\overline{A}_x = \int_0^\infty e^{-\delta t} \cdot \mu e^{-\mu t}\,dt = \frac{\mu}{\delta+\mu}$$
$$^2\overline{A}_x = \int_0^\infty e^{-2\delta t} \cdot \mu e^{-\mu t}\,dt = \frac{\mu}{\delta^*+\mu}$$

n-year term insurance

- $\overline{Z} = \begin{cases} b_T e^{-\delta T} & T < n, \\ 0 & T \geq n \end{cases}$
- Expected value $\mathbb{E}[\overline{Z}] = \int_0^n b_t e^{-\delta t} \cdot f_{T_x}(t)\,dt$
- Second moment $\mathbb{E}[\overline{Z}^2] = \int_0^n b_t^2 e^{-2\delta t} \cdot f_{T_x}(t)\,dt$
- If $b_t = 1$ for $t < n$, define $\overline{A}^1_{x:\overline{n}|} = \mathbb{E}[\overline{Z}]$
- DML mortality law: if $n' = \min\{n, \omega - x\}$, then

$$\overline{A}^1_{x:\overline{n}|} = \int_0^{n'} e^{-\delta t} \cdot \frac{1}{\omega - x}\,dt = \frac{1-e^{-\delta n'}}{\delta(\omega-x)} = \frac{\overline{a}_{\overline{n'}|}}{\omega-x}$$
$$^2\overline{A}^1_{x:\overline{n}|} = \int_0^{n'} e^{-2\delta t} \cdot \frac{1}{\omega - x}\,dt = \frac{\overline{a}_{\overline{n'}|i^*}}{\omega-x}$$

- CFM mortality law

$$\overline{A}^1_{x:\overline{n}|} = \int_0^n e^{-\delta t} \cdot \mu e^{-\mu t}\,dt = \frac{\mu}{\delta+\mu}\left(1-e^{-(\delta+\mu)n}\right)$$
$$^2\overline{A}^1_{x:\overline{n}|} = \int_0^n e^{-2\delta t} \cdot \mu e^{-\mu t}\,dt = \frac{\mu}{\delta^*+\mu}\left(1-e^{-(\delta^*+\mu)n}\right)$$

Pure endowment and endowment insurance

- Defined the same way as for discrete insurance
- Notations ${}_nE_x$ and $\overline{A}_{x:\overline{n}|}$

Recurrences

- $\overline{A}_x = \overline{A}^1_{x:\overline{n}|} + {}_nE_x \cdot \overline{A}_{x+n}$
- $\overline{A}^1_{x:\overline{n}|} = \overline{A}^1_{x:\overline{k}|} + {}_kE_x \cdot \overline{A}^{\ 1}_{x+k:\overline{n-k}|}$ for $0 \leq k \leq n$

3.3 Insurance payable more often per year

Assumptions and notation

- Benefit paid at end of period is constant through any given year
- Payout random variable $Z^{(m)}$, expectations $A^{(m)}_x$, etc.

Claims acceleration

- Time of payout on average approximately $\frac{m+1}{2m}$ through the year
- $\mathbb{E}[Z^{(m)}] \approx (1+i)^{(m-1)/2m}\mathbb{E}[Z]$
- In $m \to \infty$ limit, $\mathbb{E}[\overline{Z}] \approx (1+i)^{1/2}\mathbb{E}[Z]$

UDD assumption

- $K^{(m)} = K + S^{(m)}$ with $S^{(m)} \sim U(\{1/m, 2/m, \ldots, 1\})$ independent of K
- Expected payout

$$\begin{aligned}
\mathbb{E}[Z^(m)] &= \mathbb{E}[b_{K+1}v^K]\mathbb{E}[v^{S^{(m)}}] = v^{-1}\mathbb{E}[Z]\mathbb{E}[v^{S^{(m)}}] \\
&= \frac{v^{-1}}{m}(v^{1/m} + v^{2/m} + \cdots + v)\mathbb{E}[Z] = \frac{v^{-1}-1}{m(v^{-1/m}-1)}\mathbb{E}[Z] \\
&= \frac{i}{i^{(m)}}\mathbb{E}[Z]
\end{aligned}$$

- If $b = 1$ throughout, $A^{(m)}_x = \frac{i}{i^{(m)}}A_x$, etc.
- In limit $m \to \infty$,

$$\mathbb{E}[\overline{Z}] = \frac{i}{\delta}\mathbb{E}[Z], \quad \overline{A}_x = \frac{i}{\delta}A_x, \quad \text{etc.}$$

agreeing with result derived from $T = K + S$ with $S \sim U([0,1])$ independent on K

4 Life Annuities

4.1 Discrete life annuities