Contents

1 Interest Theory

1.1 Interest and discount

Suppose a constant interest rate i.

- Discount factor $v = (1+i)^{-1}$
- Discount rate $d = 1 - v = i(1+i)^{-1} = iv$
- Nominal annual interest rate

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \implies i^{(m)} = m[(1+i)^{1/m} - 1]$$

- Nominal annual discount rate

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p \implies d^{(p)} = p[1 - (1-d)^{1/p}]$$

- Force of interest

$$\delta = \lim_{m \to \infty} i^{(m)} = \ln(1+i) = -\ln v$$

1.2 Mortality-free annuities

Constant payment discrete annuities:

- Annuity due

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}.$$

- Annuity immediate

$$a_{\overline{n}|} = v\ddot{a}_{\overline{n}|} = \frac{1-v^n}{i}$$

- Accumulated value of annuity due

$$\ddot{s}_{\overline{n}|} = (1+i)^n \ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

- Accumulated value of annuity immediate

$$s_{\overline{n}|} = v\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

Non-constant payment discrete annuities (due versions):

- Increasing annuity

$$
\begin{aligned}
(I\ddot{a})_{\overline{n}|} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \frac{\partial}{\partial v}\left(\frac{1-v^{n+1}}{1-v}\right) \\
&= \frac{(1-v)(-(n+1)v^n) - (1-v^{n+1})(-1)}{(1-v)^2} = \frac{nv^{n+1} - (n+1)v^n + 1}{(1-v)^2} \\
&= \frac{nv^n(v-1) + 1 - v^n}{d^2} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}
\end{aligned}
$$

- Decreasing annuity

$$
\begin{aligned}
(D\ddot{a})_{\overline{n}|} &= n + (n-1)v + (n-2)v^2 + \cdots + v^{n-1} = (n+1)\ddot{a}_{\overline{n}|} - (I\ddot{a})_{\overline{n}|} \\
&= \frac{d(n+1)\ddot{a}_{\overline{n}|} - (\ddot{a}_{\overline{n}|} - nv^n)}{d} = \frac{(n+1)(1-v^n) - \ddot{a}_{\overline{n}|} + nv^n}{d} \\
&= \frac{n + 1 - \ddot{a}_{\overline{n}|} - v^n}{d} = \frac{n - a_{\overline{n}|}}{d}.
\end{aligned}
$$

Perpetuities (due versions):

- Constant

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}$$

- Increasing

$$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty} (I\ddot{a})_{\overline{n}|} = \frac{1}{d^2}$$

Payments split mth-ly:

- Constant annuity due

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m}(1 + v^{1/m} + \cdots + v^{n-1/m}) = \frac{1}{m}\frac{1-v^n}{1-v^{1/m}} = \frac{1-v^n}{d^{(m)}}$$

- Constant annuity immediate

$$a^{(m)}_{\overline{n}|} = v^{1/m}\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}}$$

- Increasing annuity due

$$
\begin{aligned}
(I\ddot{a})^{(m)}_{\overline{n}|} &= \frac{1}{m} + \frac{2}{m}v^{1/m} + \cdots + nv^{n-1/m} \\
&= \frac{mnv^n(v^{1/m} - 1) + 1 - v^n}{m(1-v^{1/m})^2} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{d^{(m)}}
\end{aligned}
$$

Continuous annuities:

- Constant
$$\overline{a}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \int_0^n v^t \, dt$$
- Increasing
$$(I\overline{a})_{\overline{n}|} = \lim_{m\to\infty} (I\ddot{a})^{(m)}_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t \, dt$$

2 Mortality

2.1 Survival functions and continuous mortality

Let $X = T_0$ be the continuous random variable for the future life span of a newborn

- Distribution function $F_t = F_X(t) = \mathbb{P}[X \leq t]$
- Survival function $S_t = S_X(t) = \mathbb{P}[X > t] = 1 - F_X(t)$
- Probability density function $f_X(t) = \frac{d}{dt} F_X(t) = -\frac{d}{dt} S_X(t)$

For an individual aged x, denote by $T_x = X|_{X \geq x} - x$ the random variable for their future life span (additional years only)

- Distribution function
$$F_{x+t} = F_{T_x}(t) = \frac{\mathbb{P}[x \leq X \leq x+t]}{\mathbb{P}[X \geq x]}$$
- Survival function
$$S_{x+t} = 1 - F_{T_x}(t) = \frac{\mathbb{P}[X > x+t]}{\mathbb{P}[X \geq x]} = \frac{S_X(x+t)}{S_X(x)}$$
- Warning: $S_{30+10} \neq S_{40}$

Mortality symbols

- ${}_tp_x = \mathbb{P}[T_x > t] = S_{x+t}$, special case $p_x = {}_1p_x$
- ${}_tq_x = 1 - {}_tp_x$, special case $q_x = {}_1q_x$
- $\ell_x = \ell_0 \cdot S_x$ is the number of people alive at time x
- Deferred death

$${}_{t|u}q_x = \mathbb{P}[t < T_x \leq t+u] = {}_tp_x \cdot {}_uq_{x+t} = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$$

Force of mortality

- "Probability of instant death" probability density
$$\mu_x \, dx = \mathbb{P}[T_x \leq dx] = \mathbb{P}[x \leq X \leq x+dx \mid X \geq x]$$
$$\mu_x = \lim_{dx \to 0} \frac{S_X(x) - S_X(x+dx)}{dx \cdot S_X(x)} = -\frac{S'_X(x)}{S_x} = -\frac{d}{dx}(\ln S_x)$$
- Force of mortality and probability density
$$f_X(x) = -S'_X(x) = S_x \mu_x = {}_xp_0 \mu_x,$$
$$f_{T_x}(t) = -S'_{T_x}(t) = -S'_X(x+t)/S_x = \mu_{x+t} \frac{S_X(x+t)}{S_x} = {}_tp_x \mu_{x+t}$$

2.2 Discrete mortality

Discrete mortality symbols

- Death symbol $d_x = \ell_x - \ell_{x+1}$
- First age in life table α
- Last age in life table ω, so that $p_\omega = 0$

Define random variable K_x for future *completed* years survived

- $T_x = K_x + s$ for a random variable s with values in $[0, 1)$
- Probability mass function

$$f_{K_x}(k) = \mathbb{P}[K_x = k] = \mathbb{P}[x \leq X < x + k \mid X \geq x] = {}_{k|}q_x = \frac{d_{x+k}}{\ell_x}$$

2.3 Life expectancy

- Complete life expectancy (continuous)

$$\begin{aligned} \mathring{e}_x = \mathbb{E}[T_x] &= \int_0^\infty t \cdot f_{T_x}(t)\,dt = -\int_0^\infty t \cdot \frac{d}{dt} S'_{T_x}(t)\,dt \\ &= [-t \cdot S_{T_x}(t)]_0^\infty + \int_0^\infty S_{T_x}(t)\,dt = \int_0^\infty {}_tp_x\,dt = \frac{1}{S_x}\int_0^\infty S_X(x+t)\,dt \end{aligned}$$

- Curtate life expectancy (discrete)

$$\begin{aligned} e_x = \mathbb{E}[K_x] &= \sum_{t=1}^\infty t \cdot f_{K_x}(t) \\ &= \sum_{t=1}^\infty \mathbb{P}[K_x \geq t] = \sum_{t=1}^\infty {}_tp_x \end{aligned}$$

- Since $\mathbb{E}[T_x] = \mathbb{E}[K_x] + \mathbb{E}[s]$, can estimate $\mathbb{E}[s] \approx 1/2$ and

$$\mathring{e}_x \approx e_x + \frac{1}{2}$$

- Temporary life expectancy random variables $T_{x:\overline{n}|} = \min(T_x, n)$ and $K_{x:\overline{n}|} = \min(K_x, n)$

$$\begin{aligned} \mathring{e}_{x:\overline{n}|} &= \mathbb{E}[T_{x:\overline{n}|}] = \int_0^n {}_tp_x\,dt \\ e_{x:\overline{n}|} &= \mathbb{E}[K_{x:\overline{n}|}] = \sum_{t=1}^n {}_tp_x \end{aligned}$$

- $T_{x:\overline{n}|} = K_{x:\overline{n}|} + s_n$ with $s_n = s$ if $T_x < n$ and $s_n = 0$ otherwise, so

$$\mathbb{E}[s_n] = \mathbb{P}[T_x < n] \cdot \mathbb{E}[s \mid T_x < n] \approx \frac{{}_nq_x}{2}$$

- Backward recurrences

$$e_x = \sum_{t=1}^{n} {}_tp_x + \sum_{t=n+1}^{\infty} {}_tp_x = e_{x:\overline{n}|} + {}_np_x \cdot e_{x+n}, \qquad e_\omega = 0$$

$$\mathring{e}_x = \int_0^n {}_tp_x \, dt + \int_n^\infty {}_tp_x \, dt = \mathring{e}_{x:\overline{n}|} + {}_np_x \cdot \mathring{e}_{x+n}, \qquad \mathring{e}_\omega = 0$$

2.4 Fractional age assumptions