MA582 Homework 3

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Due 2025-03-30

1 Problem 1

Suppose for the family $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ we have a CAN estimator $\hat{\theta}_n$ with an ANV of $v(\theta) > 0$ which is continuous with respect to θ.

Derive a confidence interval formula for $CI(\theta)$ similar to the $CI(\mu)$ derived in recent lectures. It should be data-ready, since of course θ is unknown so $v(\theta)$ is also unknown and must also be estimated suitably.

Solution. Asymptotic normality says that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, v(\theta))$, so rescaling,

$$\frac{\hat{\theta}_n - \theta}{\sqrt{v(\theta)/n}} \xrightarrow{\mathcal{D}} Z \sim N(0,1).$$

Consistency says that $\hat{\theta}_n \xrightarrow{p} \theta$, so with v and the square root function continuous, our results on properties of convergence in probability give us

$$\frac{\sqrt{v(\theta)}}{\sqrt{v(\hat{\theta}_n)}} \xrightarrow{p} 1.$$

We can now apply Slutsky's lemma to say that

$$\frac{\hat{\theta}_n - \theta}{\sqrt{v(\hat{\theta}_n)/n}} \xrightarrow{\mathcal{D}}= \frac{\sqrt{v(\theta)}}{\sqrt{v(\hat{\theta}_n)}} \cdot \frac{\hat{\theta}_n - \theta}{\sqrt{v(\theta)/n}} \xrightarrow{\mathcal{D}} 1 \cdot Z = Z \sim N(0,1).$$

Following our derivation for $CI(\mu)$ in class, we let α be the desired significance level. Then with $z_{\alpha/2} = \Phi^{-1}(1 - \alpha/2)$, where Φ is the cdf of Z, we have

$$1 - \alpha = P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) \approx P\left(-z_{\alpha/2} \leq \frac{\hat{\theta}_n - \theta}{\sqrt{v(\hat{\theta}_n)/n}} \leq z_{\alpha/2}\right)$$

for sufficiently large n. Rearranging to get upper and lower bounds on θ,

$$\hat{\theta}_n - z_{\alpha/2}\sqrt{\frac{v(\hat{\theta}_n)}{n}} \leq \theta \leq \hat{\theta}_n + z_{\alpha/2}\sqrt{\frac{v(\hat{\theta}_n)}{n}}.$$

$\square$

2 Problem 2

Suppose in the geometric family with parameter p, a sample of size 100 has an average of 7.22. Using your results in Problem 1, construct a 95% CI for p.

Solution. The mean and variance of a geometric random variable with parameter p are $1/p$ and $(1-p)/p^2$, respectively. Therefore, if we set $\theta = 1/p$, the central limit theorem tells us that the sample mean $\hat{X}_n$ is a CAN estimator $\hat{\theta}_n$ for θ with ANV $v(\theta) = \theta^2 - \theta$, which is continuous. Part (a) gives us the 95% confidence interval ($\alpha = 0.05$)

$$\hat{\theta}_n - z_{\alpha/2}\sqrt{\frac{v(\hat{\theta}_n)}{n}} \leq \theta \leq \hat{\theta}_n + z_{\alpha/2}\sqrt{\frac{v(\hat{\theta}_n)}{n}},$$

and when we substitute the values

$$n = 100, \quad \hat{\theta}_n = 7.22, \quad z_{\alpha/2} \approx 1.96,$$

we get the confidence interval $5.91 \lesssim \theta \lesssim 8.53$, corresponding to

$$0.117 \lesssim p \lesssim 0.169.$$

□