Geometric Probability

1. A real number u between 0 and 5 is chosen at random. What is the probability that u is within $1/5$ of an integer?

2. (2004 National Team) A point P is randomly placed in the interior of right triangle ABC with right angle at C. What is the probability that the area of triangle PBC is less than half of the area of triangle ABC?

3. (2004 National Target) Given that a and b are real numbers chosen independent at random so that $-3 \leq a \leq 1$ and $-2 \leq b \leq 4$, what is the probability that the product $a \cdot b$ is positive?

4. Randomly picking two points independently on the circumference of a unit circle, what is the probability that the straight-line distance between the two points is shorter than 1?

5. If we choose two numbers between 0 and 2 independently at random, what is the probability that their sum is smaller than 1?

Extensions

1. _____

2. _____ [common fraction, in terms of π]

3. _____

4. _____ [common fraction, in terms of π]

5. _____ (2009 National Countdown)

6. _____ [simplest radical form]

7. _____ [common fraction]

8. _____ [common fraction, in terms of π]

Extra Problems ($\star$)

1. Define a sequence $a_0, a_1, a_2, \ldots$ by $a_1 = 6$ and

$$a_n = \frac{-1}{a_{n-1} + 1}$$

for all integers $n \geq 2$. What is the value of a_{2025}? Express your answer as a common fraction.

2. Three real numbers are chosen independently and uniformly at random between 0 and 1. What is the probability that these three real numbers can be the side lengths of a triangle? Express your answer as a common fraction.

3. Let ABC be a triangle with $AB = 12$, $\angle A = 15°$, and $\angle B = 30°$. Find the length of BC, expressing your answer in simplest radical form.

4. For each positive integer n, let $d(n)$ denote the number of positive integer divisors of n. For example, $d(6) = 4$ and $d(16) = 5$. Given that

$$\left\lfloor \frac{100}{1} \right\rfloor + \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \cdots + \left\lfloor \frac{100}{10} \right\rfloor = 291,$$

what is the value of

$$d(1) + d(2) + d(3) + \cdots + d(99) + d(100)?$$