MA582 Homework 4

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1 Problems 1-2

Suppose for the family $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ we have a CAN estimator $\hat{\theta}_n$ with an ANV of $v(\theta)$ which is continuous with respect to θ.

1. Show step-by-step how to find a reparameterization function $g : \Theta \to \mathbb{R}$ such that the new ANV for the new estimator of the new parameter is completely parameter-free.
2. Show how, for your new parameter/estimator's CI, the margin of error is quite simplified.

Solution.

1. By the reparameterization lemma (Δ method), if $g : \Theta \to \mathbb{R}$ is 1-1 and differentiable, $\hat{\delta}_n = g(\hat{\theta}_n)$ is a CAN estimator of $\delta = g(\theta)$ with ANV $v(\theta) \cdot g'(\theta)^2$. Without loss of generality, we can aim for this to be constantly 1, which is achieved if $g'(\theta) = v(\theta)^{-1/2}$ for all θ. Therefore, we can take as a candidate reparameterization function any of the antiderivatives specified by
$$g(\theta) = \int v(\theta)^{-1/2}\, d\theta.$$
To see that any such g meets the hypotheses of the reparameterization lemma, observe that v being continuous tells us that g is differentiable by the fundamental theorem of calculus, and $v > 0$ tells us that g is a strictly increasing function, hence 1-1.
2. From homework 3, since $\hat{\delta}_n$ is a CAN estimator for δ with ANV 1, our CI for δ is
$$\hat{\delta}_n - \frac{z_{\alpha/2}}{\sqrt{n}} \le \delta \le \hat{\delta}_n + \frac{z_{\alpha/2}}{\sqrt{n}}.$$

$\square$

2 Problem 3

For Exp(λ), where $\lambda > 0$, we found a CAN estimator in lectures. Apply variance stabilization.

Solution. The CAN estimator we found in lectures is $\hat{\lambda}_n = 1/\overline{X_n}$ with ANV $v(\lambda) = \lambda^2$, so we let
$$g(\lambda) = \int v(\lambda)^{-1/2}\, d\lambda = \int d\lambda/\lambda = \log \lambda.$$
That is, $\log(\hat{\lambda}_n) = -\log \overline{X_n}$ is a CAN estimator for $\log \lambda = -\log(\text{mean})$ with constant ANV 1. $\square$