The Multiplication Principle

1. Josh has an orange hat, a blue hat, and a green hat. He has a blue shirt, a green shirt, a red shirt, and a magenta shirt. He also has a pair of red pants and a pair of blue pants. How many different outfits can Josh make that consist of one hat, one shirt, and one pair of pants?

2. How many odd five-digit counting numbers can be formed by choosing digits from the set $\{1, 2, 3, 4, 5, 6, 7\}$ if digits <u>can</u> be repeated?

3. (2011 National Sprint Problem #2) A local restaurant boasts that they have 240 different dinner combinations. A dinner combination consists of an appetizer, entree, and dessert. If the restaurant offers 10 appetizer choices and 6 entree choices, how many different dessert choices does it have?

4. A sandwich restaurant offers 6 different meats and 5 different vegetables. For each sandwich, customers can choose at most one meat and up to 5 vegetables. How many different sandwiches does the restaurant offer?

Counting Factors (Divisors)

Throughout, only positive factors (divisors) are considered.

1. How many factors does 3600 have?

2. (2002 School Sprint Problem #22) How many odd whole numbers are factors of 180?

3. (2012 National Sprint Problem #28) How many whole numbers n, such that $100 \leq n \leq 1000$, have the same number of odd factors as even factors?

4. (2006 State Team) Emma plays with her square unit tiles by arranging all of them into different shaped rectangular figures. (For example, a 5×7 rectangle would use 35 tiles and would be considered the same rectangle as a 7×5 rectangle.) Emma can form exactly ten different such rectangular figures that each use all of her tiles. What is the least number of tiles Emma could have?

Additional: For any k, the *sum of k-th powers of divisors function*, σ_k, takes a positive integer n and returns the sum of the k-th powers of all (positive) divisors of n. For example,

$$\sigma_3(12) = 1^3 + 2^3 + 3^3 + 4^3 + 6^3 + 12^3 = 2044.$$

If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is the prime factorization of n, then

$$\boxed{\sigma_k(n) = (1 + p_1^k + p_1^{2k} + \cdots + p_1^{e_1 k})(1 + p_2^k + p_2^{2k} + \cdots + p_1^{e_2 k}) \cdots (1 + p_r^k + p_r^{2k} + \cdots + p_r^{e_r k}).}$$

In our example, $12 = 2^2 \cdot 3^1$, so $\sigma_3(12) = (1 + 2^3 + (2^2)^3)(1 + 3^3)$. The most relevant special cases are that σ_0 is the number-of-divisors function and σ_1 is the sum-of-divisors function.

Extensions

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________ (skippable, or feel free to be silly)

8. ______________________________

Extra Problems

1. A sequence of numbers has the property that the sum of the first n terms is given by n^3+n+1. What is the 100th term of the sequence?

2. An isosceles trapezoid has bases of lengths 6 and 18 and a height of length 4. In terms of π, what is the area of the circle passing through all four vertices of the trapezoid?

3. In how many ways can 2025 be written as a sum of 1s, 2s, and 3s? The order of the summands does not matter and not all numbers must appear in a given sum, so for instance, $2+2+1+1$ is a valid way to write 6 as a sum and it is considered to be the same as $1+2+1+2$.

4. For any positive integer n, let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Compute the sum of all fractions of the form $\varphi(k)/k^2$ that have terminating base fourteen representations. Express your answer as a common fraction.