MA582 Homework 1

Alan Zhou

Due 2025-02-06

Problem 1

Let $X \sim N(3, 25)$ (thus $\text{var}(X) = 25$). Compute m_4 exactly.

Solution. We know from class that the mgf of X is

$$M_X(t) = \exp\left[\mu t + \frac{1}{2}\sigma^2 t^2\right],$$

and we wish to find $m_4 = M_X^{(4)}(0)$ when $\mu = 3$ and $\sigma^2 = 25$.
Several uses of the product and chain rule yield

$$\begin{aligned}
M_X^{(1)}(t) &= [\mu + \sigma^2 t] M_X(t), \\
M_X^{(2)}(t) &= \sigma^2 M_X(t) + [\mu + \sigma^2 t] M_X^{(1)}(t), \\
M_X^{(3)}(t) &= \sigma^2 M_X^{(1)}(t) + \sigma^2 M_X^{(1)}(t) + [\mu + \sigma^2 t] M_X^{(2)}(t) \\
&= 2\sigma^2 M_X^{(1)}(t) + [\mu + \sigma^2 t] M_X^{(2)}(t), \\
M_X^{(4)}(t) &= 2\sigma^2 M_X^{(2)}(t) + \sigma^2 M_X^{(2)}(t) + [\mu + \sigma^2 t] M_X^{(3)}(t) \\
&= 3\sigma^2 M_X^{(2)}(t) + [\mu + \sigma^2 t] M_X^{(3)}(t).
\end{aligned}$$

Then, letting $t = 0$,

$$\begin{aligned}
m_1 &= M_X^{(1)}(0) = \mu M_X(0) = \mu, \\
m_2 &= M_X^{(2)}(0) = \sigma^2 M_X(0) + \mu M_X^{(1)}(0) = \mu^2 + \sigma^2, \\
m_3 &= M_X^{(3)}(0) = 2\sigma^2 M_X^{(1)}(0) + \mu M_X^{(2)}(0) \\
&= 2\sigma^2 \cdot \mu + \mu \cdot (\mu^2 + \sigma^2) = \mu^3 + 3\mu\sigma^2, \\
m_4 &= M_X^{(4)}(0) = 3\sigma^2 M_X^{(2)}(0) + \mu M_X^{(3)}(0) \\
&= 3\sigma^2(\mu^2 + \sigma^2) + \mu(\mu^3 + 3\mu\sigma^2) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.
\end{aligned}$$

When $\mu = 3$ and $\sigma^2 = 25$ this evaluates to $\boxed{3306}$. □

Problem 2

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2$, and suppose X_1 and X_2 are independent.
Prove that $T = X_1 + X_2$ is normally distributed, and determine the mean and variance of T.

Solution. Since X_1 and X_2 are independent, the mgf of T is the product of the mgfs of X_1 and X_2. From class, we know the mgfs of normal random variables, and we multiply to find that

$$\begin{aligned} M_T(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \\ &= \exp\left[\mu_1 t + \frac{1}{2}\sigma_1^2 t^2\right] \cdot \exp\left[\mu_2 t + \frac{1}{2}\sigma_2^2 t^2\right] \\ &= \exp\left[(\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2\right]. \end{aligned}$$

This is the mgf of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore, by the uniqueness of moment generating functions, $\boxed{T \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)}$. □