MA582 Homework UC

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1 Problem 1

Suppose a parameter family has parameter $\theta = E(X^3)$, where θ is real. Devise an estimator of θ and show it is UC for θ.

Solution. Let $Y_n = X_n^3$ and consider

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} = \frac{X_1^3 + \cdots + X_n^3}{n}.$$

This estimator is unbiased for θ because

$$E(\hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} \sum_{i=1}^{n} \theta = \theta.$$

This estimator is consistent for θ because the weak law of large numbers gives

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} \xrightarrow{p} \theta.$$

□

2 Problem 2

We say the rv X has the Weiner distribution with parameter $\theta > 0$ if X has pdf $f(x) = 2x/\theta^2$ for $0 < x < \theta$ and $f(x) = 0$ elsewhere.

Consider the parameterized Weiner family $\{W(\theta) : \theta > 0\}$.

(a) Let Y_n be the maximum of the random sample of size n from $W(\theta)$. Show that Y_n is a consistent estimator of θ.

(b) Find the pdf of Y_n.

(c) Show that Y_n is NOT an unbiased estimator of θ.

(d) Show how to "correct" Y_n here to make a new unbiased estimator T_n. Show T_n is also consistent, hence UC.

(e) Obtain the asymptotic distribution of $n(Y_n - \theta)$.

(f) Obtain the asymptotic distribution of $n(T_n - \theta)$.

Solution. (a) We must show $Y_n \xrightarrow{p} \theta$, meaning that for any $\epsilon > 0$, we have $P(|Y_n - \theta| \geq \epsilon) \to 0$ as $n \to \infty$. As $Y_n \leq \theta$, this probability equals $P(Y_n \leq \theta - \epsilon)$. Since $Y_n = \max(X_1, \ldots, X_n)$, the condition $Y_n \leq \theta - \epsilon$ holds precisely when $X_i \leq \theta - \epsilon$ for all i. As the X_is are independent,

$$\begin{aligned} P(Y_n \leq \theta - \epsilon) &= \prod_{i=1}^{n} P(X_i \leq \theta - \epsilon) \\ &= \left(\int_0^{\theta - \epsilon} \frac{2x}{\theta^2} \, dx \right)^n \\ &= \left(\frac{(\theta - \epsilon)^2}{\theta^2} \right)^n \\ &= \left(1 - \frac{\epsilon}{\theta} \right)^{2n}. \end{aligned}$$

As $n \to \infty$, this probability does indeed tend to 0.

(b) By the same reasoning as before, the cdf of Y_n is

$$F_{Y_n}(y) = P(Y_n \leq y) = \left(\int_0^y \frac{2x}{\theta^2} \, dx \right)^n = \frac{y^{2n}}{\theta^{2n}}$$

for $0 < y < \theta$, with $F_{Y_n}(y) = 0$ for $y \leq 0$ and $F_{Y_n}(y) = 1$ for $y \geq \theta$. The pdf of Y_n is then

$$f_{Y_n}(y) = F'_{Y_n}(y) = \frac{2n}{\theta^{2n}} \cdot y^{2n-1}$$

when $0 < y < \theta$, with $f_{Y_n}(y) = 0$ for all other y.

(c) The expected value of Y_n is

$$\begin{aligned} E(Y_n) &= \int_0^\theta y \cdot f_{Y_n}(y) \, dy = \int_0^\theta \frac{2n}{\theta^{2n}} y^{2n} \, dy \\ &= \frac{2n}{(2n+1)\theta^{2n}} \cdot \theta^{2n+1} = \frac{2n}{2n+1} \cdot \theta. \end{aligned}$$

In particular, this is not θ itself, so Y_n is not an unbiased estimator for θ.

(d) If we define $T_n = \frac{2n+1}{2n} Y_n$, then $E(T_n) = \frac{2n+1}{2n} E(Y_n) = \theta$, so T_n is an unbiased estimator for θ. Also, since $\frac{2n+1}{2n} \to 1$ and $Y_n \xrightarrow{p} \theta$ as $n \to \infty$, we have $T_n \xrightarrow{p} 1 \cdot \theta = \theta$ as $n \to \infty$, so T_n is consistent for θ. Hence T_n is UC for θ.

(e)

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