MA582 Homework 2

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1 Main Problem

Consider the Weiner parametric family $\{W(\theta) : \theta \text{ real}\}$, where it so happens that $\theta = E[\log X]$, if the random variable $X > 0$ obeys $X \sim W(\theta)$. Devise a UC estimator of θ.

Solution. Let $Y_i = \log X_i$ for $i = 1, 2, 3, \ldots$, where $X_i \sim W(\theta)$ for all i. Then $\theta = E[Y_i]$ for all i and we can consider the estimator

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} = \frac{\log(X_1 \cdots X_n)}{n}.$$

To see that $\hat{\theta}_n$ is unbiased, we compute

$$E[\hat{\theta}_n] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = \frac{1}{n}\sum_{i=1}^{n} \theta = \theta.$$

To see that $\hat{\theta}_n$ is consistent, the weak law of large numbers[1] tells us that since the Y_i's are i.i.d. with finite expectation θ,

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} \xrightarrow{p} \theta.$$

□

[1] I assume "This family is meant to be boring" means that the Y_is have finite variances, since our version of the WLLN in class invokes this assumption.

2 Additional Problem

Prove that for any constant $0 < \alpha < 1/2$ there is a constant $C = C(\alpha) > 0$ such that for all $t > 0$,

$$P(|Z| \geq t) \leq C \exp(-\alpha t^2),$$

which implies an extremely rapid decay for the survival function of $|Z|$ as $t \to \infty$.

Solution. Squaring both sides, for $t > 0$ the statement $|Z| \geq t$ is equivalent to $Z^2 \geq t^2$. For any $\alpha > 0$ satisfying $M_{Z^2}(\alpha) < \infty$, we apply Markov's inequality to the random variable Z^2 with the increasing function $u \mapsto e^{\alpha u}$ to get

$$P(Z^2 \geq t^2) = \frac{E[e^{\alpha Z^2}]}{e^{\alpha t^2}} = M_{Z^2}(\alpha) \exp(-\alpha t^2).$$

In class, we computed that $M_{Z^2}(\alpha) = (1 - 2\alpha)^{-1/2}$ for $0 < \alpha < 1/2$, so we can take our constant to be $C(\alpha) = (1 - 2\alpha)^{-1/2}$ for $0 < \alpha < 1/2$. $\square$