MA582 Homework UC

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1 Problem 1

Suppose a parameter family has parameter $\theta = E(X^3)$, where θ is real. Devise an estimator of θ and show it is UC for θ.

Solution. Let $Y_n = X_n^3$ and consider

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} = \frac{X_1^3 + \cdots + X_n^3}{n}.$$

This estimator is unbiased for θ because

$$E(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \theta = \theta.$$

This estimator is consistent for θ because the weak law of large numbers gives

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} \xrightarrow{p} \theta.$$

□

2 Problem 2

We say the rv X has the Weiner distribution with parameter $\theta > 0$ if X has pdf $f(x) = 2x/\theta^2$ for $0 < x < \theta$ and $f(x) = 0$ elsewhere.

Consider the parameterized Weiner family $\{W(\theta) : \theta > 0\}$.

(a) Let Y_n be the maximum of the random sample of size n from $W(\theta)$. Show that Y_n is a consistent estimator of θ.

(b) Find the pdf of Y_n.

(c) Show that Y_n is NOT an unbiased estimator of θ.

(d) Show how to "correct" Y_n here to make a new unbiased estimator T_n. Show T_n is also consistent, hence UC.

(e) Obtain the asymptotic distribution of $n(\theta - Y_n)$. [Originally: $n(Y_n - \theta)$]

(f) Obtain the asymptotic distribution of $n(\theta - T_n)$. [Originally: $n(T_n - \theta)$]

Solution. If $X \sim W(\theta)$, then the cdf of X satisfies $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x \geq \theta$. In between, we calculate

$$F_X(x) = \int_0^x \frac{2t}{\theta^2}\,dt = \frac{x^2}{\theta^2}.$$

Then, as discussed in class, when $Y_n = \max(X_1, \ldots, X_n)$ we have

$$F_{Y_n}(y) = (F_X(y))^n = \begin{cases} 0 & y < 0, \\ (y/\theta)^{2n} & 0 \leq y < \theta, \\ 1 & y \geq \theta. \end{cases}$$

(a) Since $Y_n \leq \theta$, for any small $\epsilon > 0$,

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n \leq \theta - \epsilon) = \left(\frac{\theta - \epsilon}{\theta}\right)^{2n} = \left(1 - \frac{\epsilon}{\theta}\right)^{2n}.$$

As $n \to \infty$, the right hand side tends to 0. As $\epsilon > 0$ was arbitrary, $Y_n \xrightarrow{p} \theta$.

(b) The pdf of Y_n is

$$f_{Y_n}(y) = F'_{Y_n}(y) = \frac{2n}{\theta^{2n}} \cdot y^{2n-1}$$

when $0 < y < \theta$, with $f_{Y_n}(y) = 0$ for all other y.

(c) The expected value of Y_n is

$$\begin{aligned} E(Y_n) &= \int_0^\theta y \cdot f_{Y_n}(y)\,dy = \int_0^\theta \frac{2n}{\theta^{2n}} y^{2n}\,dy \\ &= \frac{2n}{(2n+1)\theta^{2n}} \cdot \theta^{2n+1} = \frac{2n}{2n+1} \cdot \theta. \end{aligned}$$

In particular, this is not θ itself, so Y_n is not an unbiased estimator for θ.

(d) If we define $T_n = \frac{2n+1}{2n} Y_n$, then

$$E(T_n) = \frac{2n+1}{2n} E(Y_n) = \theta,$$

so T_n is an unbiased estimator for θ. Also, since $\frac{2n+1}{2n} \to 1$ and $Y_n \xrightarrow{p} \theta$ as $n \to \infty$, another "preservation of convergence in probability" result gives us

$$T_n = \frac{2n+1}{2n} Y_n \xrightarrow{p} 1 \cdot \theta = \theta.$$

Thus T_n is consistent for θ and hence UC for θ.

(e) When $t < 0$, we have $P(n(\theta - Y_n) \leq t) = 0$. Otherwise, for all n sufficiently large ($n\theta > t$),

$$\begin{aligned} P(n(\theta - Y_n) \leq t) &= P\left(Y_n \geq \theta - \frac{t}{n}\right) = 1 - P\left(Y_n \leq \theta - \frac{t}{n}\right) \\ &= 1 - \left(\frac{\theta - t/n}{\theta}\right)^{2n} = 1 - \left(1 - \frac{t/\theta}{n}\right)^{2n}. \end{aligned}$$

As $n \to \infty$, this tends to $1 - e^{-2t/\theta}$, so the asymptotic distribution of $n(\theta - Y_n)$ has CDF

$$t \longmapsto \begin{cases} 0 & t < 0, \\ 1 - e^{-2t/\theta} & t \geq 0. \end{cases}$$

This is the $\mathrm{Exp}(2/\theta)$ distribution.

(f) We start by writing

$$n(\theta - T_n) = n(\theta - Y_n) + n(Y_n - T_n) = n(\theta - Y_n) - \frac{1}{2}Y_n.$$

Since $n(\theta - Y_n) \xrightarrow{\mathcal{D}} \mathrm{Exp}(2/\theta)$ and $Y_n \xrightarrow{p} \theta$, Slutsky's lemma tells us that

$$n(\theta - T_n) \xrightarrow{\mathcal{D}} \mathrm{Exp}\left(\frac{2}{\theta}\right) - \frac{\theta}{2},$$

the $\mathrm{Exp}(2/\theta)$ distribution shifted a constant amount $\theta/2$ to the left.

$\square$