Contents

1 Interest Theory

1.1 Compound interest and rates

Definition 1.1 (Accumulation functions)**.** The *accumulation (amount) function* $A(t)$ is the value of an investment $P = A(0)$ at time t. The *accumulation (factor) function* $a(t)$ is the value at time t of an initial investment of 1,

$$A(t) = A(0)a(t) \quad \longleftrightarrow \quad a(t) = \frac{A(t)}{A(0)}. \tag{1.1}$$

Example 1.2. If an annual interest rate i is fixed, then after n whole years,

$$a(n) = (1+i)^n \quad \text{and} \quad A(n) = P(1+i)^n, \tag{1.2}$$

where $P = A(0)$ is the principal investment.

In practice, we often use a smaller compounding period, such as a quarter, month, or day. It is often helpful to be able to convert to another compounding period.

Definition 1.3 (Equivalence of rates)**.** Two interest rates are *equivalent* if their accumulation functions are equal at any time for which a whole number of compounding periods have taken place for both rates. For a given interest rate, the *effective (annual) interest rate* is the equivalent annual interest rate:

$$1 + i_{\text{eff}} = a(1) \implies i_{\text{eff}} = a(1) - 1, \tag{1.3}$$

where $a(1)$ is calculated using the given interest rate.

Example 1.4. A quarterly interest rate of 3% yields an effective annual interest rate

$$i_{\text{eff}} = (1.03)^4 - 1 \approx 12.55\%. \tag{1.4}$$

Carrying out the same calculation, the effective annual interest rate for a rate r compounded m times a year (at evenly spaced time intervals) is

$$i_{\text{eff}} = (1+r)^m - 1. \tag{1.5}$$

Definition 1.5 (Nominal annual interest rate)**.** Given an interest rate r compounded m times a year, the *nominal annual interest rate* is mr. When $m = 12$, so that compounding is monthly, r is the *periodic rate* and the nominal annual interest rate is the *annual percentage rate (APR)*.

We denote by $i^{(m)}$ the nominal annual interest rate for which compounding m times a year yields an effective annual interest rate of i. Thus

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1 \implies i^{(m)} = m[(1+i)^{1/m} - 1]. \tag{1.6}$$

In the limit $m \to \infty$ of *continuous compounding*, we obtain the *force of interest*

$$\begin{aligned}
\delta &= \lim_{m\to\infty} i^{(m)} \\
&= \lim_{m\to\infty} m[(1+i)^{1/m} - 1] \\
&= \lim_{t\to 0} \frac{(1+i)^t - 1}{t} \\
&= \frac{d}{dt}(1+i)^t \text{ at } t = 0 \\
&= \ln(1+i).
\end{aligned} \tag{1.7}$$