Contents

1 Quadratics

A *quadratic (polynomial) in* X is an expression of the form aX^2+bX+c, where a, b, c are independent of X and $a \neq 0$. Collectively a, b, c are the *coefficients* of the quadratic while aX^2, bX, c are the *terms* of the quadratic.

1.1 Factoring quadratics whose leading coefficient is 1

Some quadratics arise as a product of two linear expressions, such as

$$(X-2)(X+3) = X(X+3) - 2(X+3) = X^2 + 3X - 2X - 6 = X^2 + X - 6.$$

Factoring refers to the reverse process of finding, for a given quadratic, two linear expressions which multiply to that quadratic. Here we focus on factoring over the integers.

Example 1.1. Factor $X^2 + 8X + 12$.

Solution. A reasonable guess is that a factorization has the form $(X + A)(X + B)$. Expanding,

$$(X + A)(X + B) = X(X + B) + A(X + B) = X^2 + (A + B)X + AB.$$

Matching coefficients, we want $A+B = 8$ and $AB = 12$. Listing out pairs of integers which multiply to 12, or pairs of integers which sum to 8, we find that if A and B are 2 and 6 (in either order), both equations hold. Therefore, $X^2 + 8X + 12 = \boxed{(X + 2)(X + 6)}$. □

Example 1.2. Factor $X^2 - 10X + 21$.

Solution. Setting up a factorization $(X+A)(X+B)$ as before, this time $A+B = -10$ and $AB = 21$. We find A and B are -3 and -7 in some order, so $X^2 - 10X + 21 = \boxed{(X - 3)(X - 7)}$. □

To make guess-and-check easier, we can start with some sign analysis. This allows us to narrow our search to positive integers instead of all integers.

Example 1.3. Factor $X^2 - 19X + 48$.

Solution. Since the constant term 48 is positive, we know that the constants of the factors are both positive or both negative. Since the linear coefficient -19 is negative, those constants have to be negative. Therefore, we can set up a factorization of the form

$$(X - A)(X - B) = X^2 - (A + B)X + AB.$$

We need two positive integers whose sum is 19 and whose product is 48. These turn out to be 3 and 16, so $X^2 - 19X + 48 = \boxed{(X - 3)(X - 16)}$. □

Example 1.4. Factor $X^2 + 7X - 44$.

Solution. Since the constant term -44 is negative, we can set up a factorization of the form

$$(X + A)(X - B) = X^2 + (A - B)X - AB.$$

We need two positive integers whose product is 44 and whose difference is 7. These turn out to be 4 and 11. As $A - B = 7$, we require $A = 11$ and $B = 4$, so $X^2 + 7X - 44 = \boxed{(X + 11)(X - 4)}$. □

1.2 Factoring quadratics whose leading coefficient is not 1

So far, all of the examples we considered are *monic*, meaning the leading coefficient (coefficient of X^2) is 1. When the leading coefficient is not 1, the task becomes more challenging.

Example 1.5. Factor $3X^2 + 10X + 8$.

Proof. □

1.3 Finding roots by factoring

One way that quadratic expressions arise is as a product of two linear expressions,

For a given quadratic, if we can find linear factors, identifying roots becomes straightforward.

Example 1.6. Find the roots of $X^2 + X - 6$.

Solution. Let r be a root, so by definition, we need $r^2 + r - 6 = 0$. By our calculation above, the left hand side is equal to $(r-2)(r+3)$. For a product of two (or more) factors to be equal to 0, at least one of them must be 0. Therefore, any root r must satisfy $r - 2 = 0$ or $r + 3 = 0$, and if r satisfies at least one of these two equations, it is a root. Hence the roots of $X^2 + X - 6$ are 2 and -3. □

Example 1.7. Find the roots of $X^2 - 8X + 12$.

Solution. This time, we need to find a factorisation of $X^2 - 8X + 12$ first. A reasonable guess is that the factorisation has the form $(X+A)(X+B)$, where A and B are constants to be determined. Expanding,

$$\begin{aligned}(X+A)(X+B) &= X(X+B) + A(X+B)\\ &= X^2 + BX + AX + AB\\ &= X^2 + (A+B)X + AB.\end{aligned}$$

The coefficients must match, □