Permutations

Definition: The *factorial* of a non-negative integer n is defined recursively by

$$0! = 1 \quad \text{and} \quad n! = n \cdot (n-1)! \text{ for all } n \geq 1.$$

When n is a positive integer, $n!$ is the product of all positive integers less than or equal to n, and it gives the number of orders in which n distinct items can be listed. The first several values are

$$\begin{aligned}
0! &= 1, \\
1! &= 1 \cdot 0! = 1 \cdot 1 = 1, \\
2! &= 2 \cdot 1! = 2 \cdot 1 = 2, \\
3! &= 3 \cdot 2! = 3 \cdot 2 = 6, \\
4! &= 4 \cdot 3! = 4 \cdot 6 = 24, \\
5! &= 5 \cdot 4! = 5 \cdot 24 = 120, \\
6! &= 6 \cdot 5! = 6 \cdot 120 = 720.
\end{aligned}$$

1. The library is giving one book to each student for free. Three friends show up at the library and find that there are 4 different books available, with only one copy left for each of the books. In how many ways can the friends choose their books?

2. How many positive 3-digit integers have 3 distinct digits?

3. How many 4-digit <u>odd</u> integers greater than 6000 can be formed from the digits $\{0, 1, 3, 5, 6, 8\}$ if no digit may be used more than once?

4. How many different 4-letter strings can be generated by using each of the letters A, O, P, and S exactly once? (A *string* in this context is a sequence of characters, such as $SOAP$. It need not be an actual English word, so for example, $SAOP$ would also be a valid string.)

Repeated Elements

1. If all the letters of the word $SYZYGY$[1] are used, in how many different ways can the six letters be arranged in a six-letter string?

2. (2002 School Sprint Problem #22) How many odd whole numbers are factors of 180?

3. (2012 National Sprint Problem #28) How many whole numbers n, such that $100 \leq n \leq 1000$, have the same number of odd factors as even factors?

4. (2006 State Team) Emma plays with her square unit tiles by arranging all of them into different shaped rectangular figures. (For example, a 5×7 rectangle would use 35 tiles and would be considered the same rectangle as a 7×5 rectangle.) Emma can form exactly ten different such rectangular figures that each use all of her tiles. What is the least number of tiles Emma could have?

[1] definition

Extensions

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________ (skippable, or feel free to be silly)

8. ______________________________

Extra Problems

1. A sequence of numbers has the property that the sum of the first n terms is given by n^3+n+1. What is the 100th term of the sequence?

2. An isosceles trapezoid has bases of lengths 6 and 18 and a height of length 4. In terms of π, what is the area of the circle passing through all four vertices of the trapezoid?

3. In how many ways can 2025 be written as a sum of 1s, 2s, and 3s? The order of the summands does not matter and not all numbers must appear in a given sum, so for instance, $2+2+1+1$ is a valid way to write 6 as a sum and it is considered to be the same as $1+2+1+2$.

4. For any positive integer n, let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Compute the sum of all fractions of the form $\varphi(k)/k^2$ that have terminating base fourteen representations. Express your answer as a common fraction.