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1 Interest Theory

1.1 Interest and discount

Suppose a constant interest rate i .

- Discount factor $v = (1 + i)^{-1}$
- Discount rate $d = 1 - v = i(1 + i)^{-1} = iv$
- Nominal annual interest rate

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \implies i^{(m)} = m[(1 + i)^{1/m} - 1]$$

- Nominal annual discount rate

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p \implies d^{(p)} = p[1 - (1 - d)^{1/p}]$$

- Force of interest

$$\delta = \lim_{m \rightarrow \infty} i^{(m)} = \ln(1 + i) = -\ln v$$

1.2 Mortality-free annuities

Constant payment discrete annuities:

- Annuity due

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \dots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}.$$

- Annuity immediate

$$a_{\overline{n}|} = v\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{i}$$

- Accumulated value of annuity due

$$\ddot{s}_{\overline{n}|} = (1 + i)^n \ddot{a}_{\overline{n}|} = \frac{(1 + i)^n - 1}{d}$$

- Accumulated value of annuity immediate

$$s_{\overline{n}|} = v\ddot{s}_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$$

Non-constant payment discrete annuities (due versions):

- Increasing annuity

$$\begin{aligned}(I\ddot{a})_{\overline{n}|} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \frac{\partial}{\partial v} \left(\frac{1 - v^{n+1}}{1 - v} \right) \\ &= \frac{(1 - v)(-(n + 1)v^n) - (1 - v^{n+1})(-1)}{(1 - v)^2} = \frac{nv^{n+1} - (n + 1)v^n + 1}{(1 - v)^2} \\ &= \frac{nv^n(v - 1) + 1 - v^n}{d^2} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}\end{aligned}$$

- Decreasing annuity

$$\begin{aligned}(D\ddot{a})_{\overline{n}|} &= n + (n - 1)v + (n - 2)v^2 + \cdots + v^{n-1} = (n + 1)\ddot{a}_{\overline{n}|} - (I\ddot{a})_{\overline{n}|} \\ &= \frac{d(n + 1)\ddot{a}_{\overline{n}|} - (\ddot{a}_{\overline{n}|} - nv^n)}{d} = \frac{(n + 1)(1 - v^n) - \ddot{a}_{\overline{n}|} + nv^n}{d} \\ &= \frac{n + 1 - \ddot{a}_{\overline{n}|} - v^n}{d} = \frac{n - a_{\overline{n}|}}{d}.\end{aligned}$$

Perpetuities (due versions):

- Constant

$$\ddot{a}_{\infty|} = \lim_{n \rightarrow \infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}$$

- Increasing

$$(I\ddot{a})_{\infty|} = \lim_{n \rightarrow \infty} (I\ddot{a})_{\overline{n}|} = \frac{1}{d^2}$$

Payments split m th-ly:

- Constant annuity due

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}(1 + v^{1/m} + \cdots + v^{n-1/m}) = \frac{1}{m} \frac{1 - v^n}{1 - v^{1/m}} = \frac{1 - v^n}{d^{(m)}}$$

- Constant annuity immediate

$$a_{\overline{n}|}^{(m)} = v^{1/m} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}}$$

- Increasing annuity due

$$\begin{aligned}(I\ddot{a})_{\overline{n}|}^{(m)} &= \frac{1}{m} + \frac{2}{m}v^{1/m} + \cdots + nv^{n-1/m} \\ &= \frac{mnv^n(v^{1/m} - 1) + 1 - v^n}{m(1 - v^{1/m})^2} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{d^{(m)}}\end{aligned}$$

Continuous annuities:

- Constant

$$\bar{a}_{\overline{n}|} = \lim_{m \rightarrow \infty} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{\delta} = \int_0^n v^t dt$$

- Increasing

$$(I\bar{a})_{\overline{n}|} = \lim_{m \rightarrow \infty} (I\ddot{a})_{\overline{n}|}^{(m)} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t dt$$

2 Mortality

2.1 Survival functions and continuous mortality

Let $X = T_0$ be the continuous random variable for the future life span of a newborn

- Distribution function $F_t = F_X(t) = \mathbb{P}[X \leq t]$
- Survival function $S_t = S_X(t) = \mathbb{P}[X > t] = 1 - F_X(t)$
- Probability density function $f_X(t) = \frac{d}{dt}F_X(t) = -\frac{d}{dt}S_X(t)$

For an individual aged x , denote by $T_x = X|_{X \geq x} - x$ the random variable for their future life span (additional years only)

- Distribution function

$$F_{x+t} = F_{T_x}(t) = \frac{\mathbb{P}[x \leq X \leq x+t]}{\mathbb{P}[X \geq x]}$$

- Survival function

$$S_{x+t} = 1 - F_{T_x}(t) = \frac{\mathbb{P}[X > x+t]}{\mathbb{P}[X \geq x]} = \frac{S_X(x+t)}{S_X(x)}$$

- Warning: $S_{30+10} \neq S_{40}$

Mortality symbols

- ${}_t p_x = \mathbb{P}[T_x > t] = S_{x+t}$, special case $p_x = {}_1 p_x$
- ${}_t q_x = 1 - {}_t p_x$, special case $q_x = {}_1 q_x$
- $\ell_x = \ell_0 \cdot S_x$ is the number of people alive at time x
- Deferred death

$${}_t|u q_x = \mathbb{P}[t < T_x \leq t+u] = {}_t p_x \cdot {}_u q_{x+t} = {}_{t+u} q_x - {}_t q_x = {}_t p_x - {}_{t+u} p_x$$

Force of mortality

- “Probability of instant death” probability density

$$\begin{aligned} \mu_x dx &= \mathbb{P}[T_x \leq dx] = \mathbb{P}[x \leq X \leq x+dx \mid X \geq x] \\ \mu_x &= \lim_{dx \rightarrow 0} \frac{S_X(x) - S_X(x+dx)}{dx \cdot S_X(x)} = -\frac{S'_X(x)}{S_x} = -\frac{d}{dx}(\ln S_x) \end{aligned}$$

- Force of mortality and probability density

$$\begin{aligned} f_X(x) &= -S'_X(x) = S_x \mu_x = {}_x p_0 \mu_x, \\ f_{T_x}(t) &= -S'_{T_x}(t) = -S'_X(x+t)/S_x = \mu_{x+t} \frac{S_X(x+t)}{S_x} = {}_t p_x \mu_{x+t} \end{aligned}$$

2.2 Discrete mortality

Discrete mortality symbols

- Death symbol $d_x = \ell_x - \ell_{x+1}$
- First age in life table α
- Last age in life table ω , so that $p_\omega = 0$

Define random variable K_x for future *completed* years survived

- $T_x = K_x + s$ for a random variable s with values in $[0, 1)$
- Probability mass function

$$f_{K_x}(k) = \mathbb{P}[K_x = k] = \mathbb{P}[x \leq X < x + k \mid X \geq x] = {}_k|q_x = \frac{d_{x+k}}{\ell_x}$$

2.3 Life expectancy

- Complete life expectancy (continuous)

$$\begin{aligned} \hat{e}_x &= \mathbb{E}[T_x] = \int_0^\infty t \cdot f_{T_x}(t) dt = - \int_0^\infty t \cdot \frac{d}{dt} S'_{T_x}(t) dt \\ &= [-t \cdot S_{T_x}(t)]_0^\infty + \int_0^\infty S_{T_x}(t) dt = \int_0^\infty {}_t p_x dt = \frac{1}{S_x} \int_0^\infty S_X(x+t) dt \end{aligned}$$

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