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# 1 Interest Theory

## 1.1 Interest and discount

Suppose a constant interest rate  $i$ .

- Discount factor  $v = (1 + i)^{-1}$
- Discount rate  $d = 1 - v = i(1 + i)^{-1} = iv$
- Nominal annual interest rate

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \implies i^{(m)} = m[(1 + i)^{1/m} - 1]$$

- Nominal annual discount rate

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p \implies d^{(p)} = p[1 - (1 - d)^{1/p}]$$

- Force of interest

$$\delta = \lim_{m \rightarrow \infty} i^{(m)} = \ln(1 + i) = -\ln v$$

## 1.2 Mortality-free annuities

Constant payment discrete annuities:

- Annuity due

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \dots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}.$$

- Annuity immediate

$$a_{\overline{n}|} = v\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{i}$$

- Accumulated value of annuity due

$$\ddot{s}_{\overline{n}|} = (1 + i)^n \ddot{a}_{\overline{n}|} = \frac{(1 + i)^n - 1}{d}$$

- Accumulated value of annuity immediate

$$s_{\overline{n}|} = v\ddot{s}_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$$

Non-constant payment discrete annuities (due versions):

- Increasing annuity

$$\begin{aligned}(I\ddot{a})_{\overline{n}|} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \frac{\partial}{\partial v} \left( \frac{1 - v^{n+1}}{1 - v} \right) \\ &= \frac{(1 - v)(-(n + 1)v^n) - (1 - v^{n+1})(-1)}{(1 - v)^2} = \frac{nv^{n+1} - (n + 1)v^n + 1}{(1 - v)^2} \\ &= \frac{nv^n(v - 1) + 1 - v^n}{d^2} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}\end{aligned}$$

- Decreasing annuity

$$\begin{aligned}(D\ddot{a})_{\overline{n}|} &= n + (n - 1)v + (n - 2)v^2 + \cdots + v^{n-1} = (n + 1)\ddot{a}_{\overline{n}|} - (I\ddot{a})_{\overline{n}|} \\ &= \frac{d(n + 1)\ddot{a}_{\overline{n}|} - (\ddot{a}_{\overline{n}|} - nv^n)}{d} = \frac{(n + 1)(1 - v^n) - \ddot{a}_{\overline{n}|} + nv^n}{d} \\ &= \frac{n + 1 - \ddot{a}_{\overline{n}|} - v^n}{d} = \frac{n - a_{\overline{n}|}}{d}.\end{aligned}$$

Perpetuities (due versions):

- Constant

$$\ddot{a}_{\infty|} = \lim_{n \rightarrow \infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}$$

- Increasing

$$(I\ddot{a})_{\infty|} = \lim_{n \rightarrow \infty} (I\ddot{a})_{\overline{n}|} = \frac{1}{d^2}$$

Payments split  $m$ th-ly:

- Constant annuity due

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}(1 + v^{1/m} + \cdots + v^{n-1/m}) = \frac{1}{m} \frac{1 - v^n}{1 - v^{1/m}} = \frac{1 - v^n}{d^{(m)}}$$

- Constant annuity immediate

$$a_{\overline{n}|}^{(m)} = v^{1/m} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}}$$

- Increasing annuity due

$$\begin{aligned}(I\ddot{a})_{\overline{n}|}^{(m)} &= \frac{1}{m} + \frac{2}{m}v^{1/m} + \cdots + nv^{n-1/m} \\ &= \frac{mnv^n(v^{1/m} - 1) + 1 - v^n}{m(1 - v^{1/m})^2} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{d^{(m)}}\end{aligned}$$

Continuous annuities:

- Constant

$$\bar{a}_{\overline{n}|} = \lim_{m \rightarrow \infty} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{\delta} = \int_0^n v^t dt$$

- Increasing

$$(I\bar{a})_{\overline{n}|} = \lim_{m \rightarrow \infty} (I\ddot{a})_{\overline{n}|}^{(m)} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t dt$$



## 2 Mortality

### 2.1 Survival functions and continuous mortality

Let  $X = T_0$  be the continuous random variable for the future life span of a newborn

- Distribution function  $F_t = F_X(t) = \mathbb{P}[X \leq t]$
- Survival function  $S_t = S_X(t) = \mathbb{P}[X > t] = 1 - F_X(t)$
- Probability density function  $f_X(t) = \frac{d}{dt}F_X(t) = -\frac{d}{dt}S_X(t)$

For an individual aged  $x$ , denote by  $T_x = X|_{X \geq x} - x$  the random variable for their future life span (additional years only)

- Distribution function

$$F_{x+t} = F_{T_x}(t) = \frac{\mathbb{P}[x \leq X \leq x+t]}{\mathbb{P}[X \geq x]}$$

- Survival function

$$S_{x+t} = 1 - F_{T_x}(t) = \frac{\mathbb{P}[X > x+t]}{\mathbb{P}[X \geq x]} = \frac{S_X(x+t)}{S_X(x)}$$

- Warning:  $S_{30+10} \neq S_{40}$

Mortality symbols

- ${}_t p_x = \mathbb{P}[T_x > t] = S_{x+t}$ , special case  $p_x = {}_1 p_x$
- ${}_t q_x = 1 - {}_t p_x$ , special case  $q_x = {}_1 q_x$
- $\ell_x = \ell_0 \cdot S_x$  is the number of people alive at time  $x$
- Deferred death

$${}_t|u q_x = \mathbb{P}[t < T_x \leq t+u] = {}_t p_x \cdot {}_u q_{x+t} = {}_{t+u} q_x - {}_t q_x = {}_t p_x - {}_{t+u} p_x$$

Force of mortality

- “Probability of instant death” probability density

$$\begin{aligned} \mu_x dx &= \mathbb{P}[T_x \leq dx] = \mathbb{P}[x \leq X \leq x+dx \mid X \geq x] \\ \mu_x &= \lim_{dx \rightarrow 0} \frac{S_X(x) - S_X(x+dx)}{dx \cdot S_X(x)} = -\frac{S'_X(x)}{S_x} = -\frac{d}{dx}(\ln S_x) \end{aligned}$$

- Force of mortality and probability density

$$\begin{aligned} f_X(x) &= -S'_X(x) = S_x \mu_x = {}_x p_0 \mu_x, \\ f_{T_x}(t) &= -S'_{T_x}(t) = -S'_X(x+t)/S_x = \mu_{x+t} \frac{S_X(x+t)}{S_x} = {}_t p_x \mu_{x+t} \end{aligned}$$

## 2.2 Discrete mortality

Discrete mortality symbols

- Death symbol  $d_x = \ell_x - \ell_{x+1}$
- First age in life table  $\alpha$
- Last age in life table  $\omega$ , so that  $p_\omega = 0$

Define random variable  $K_x$  for future *completed* years survived

- $T_x = K_x + s$  for a random variable  $s$  with values in  $[0, 1)$
- Probability mass function

$$f_{K_x}(k) = \mathbb{P}[K_x = k] = \mathbb{P}[x \leq X < x + k \mid X \geq x] = {}_k|q_x = \frac{d_{x+k}}{\ell_x}$$

## 2.3 Life expectancy

- Complete life expectancy (continuous)

$$\begin{aligned} \dot{e}_x &= \mathbb{E}[T_x] = \int_0^\infty t \cdot f_{T_x}(t) dt = - \int_0^\infty t \cdot \frac{d}{dt} S'_{T_x}(t) dt \\ &= [-t \cdot S_{T_x}(t)]_0^\infty + \int_0^\infty S_{T_x}(t) dt = \int_0^\infty {}_t p_x dt = \frac{1}{S_x} \int_0^\infty S_X(x+t) dt \end{aligned}$$

- Curtate life expectancy (discrete)

$$\begin{aligned} e_x &= \mathbb{E}[K_x] = \sum_{t=1}^\infty t \cdot f_{K_x}(t) \\ &= \sum_{t=1}^\infty \mathbb{P}[K_x \geq t] = \sum_{t=1}^\infty {}_t p_x \end{aligned}$$

- Since  $\mathbb{E}[T_x] = \mathbb{E}[K_x] + \mathbb{E}[s]$ , can estimate  $\mathbb{E}[s] \approx 1/2$  and

$$\dot{e}_x \approx e_x + \frac{1}{2}$$

- Temporary life expectancy random variables  $T_{x:\overline{n}|} = \min(T_x, n)$  and  $K_{x:\overline{n}|} = \min(K_x, n)$

$$\begin{aligned} \dot{e}_{x:\overline{n}|} &= \mathbb{E}[T_{x:\overline{n}|}] = \int_0^n {}_t p_x dt \\ e_{x:\overline{n}|} &= \mathbb{E}[K_{x:\overline{n}|}] = \sum_{t=1}^n {}_t p_x \end{aligned}$$



- $T_{x:\overline{n}} = K_{x:\overline{n}} + s_n$  with  $s_n = s$  if  $T_x < n$  and  $s_n = 0$  otherwise, so

$$\mathbb{E}[s_n] = \mathbb{P}[T_x < n] \cdot \mathbb{E}[s \mid T_x < n] \approx \frac{nq_x}{2}$$

- Backward recurrences

$$\begin{aligned} e_x &= \sum_{t=1}^n {}_t p_x + \sum_{t=n+1}^{\infty} {}_t p_x = e_{x:\overline{n}} + {}_n p_x \cdot e_{x+n}, & e_{\omega} &= 0 \\ \dot{e}_x &= \int_0^n {}_t p_x dt + \int_n^{\infty} {}_t p_x dt = \dot{e}_{x:\overline{n}} + {}_n p_x \cdot \dot{e}_{x+n}, & \dot{e}_{\omega} &= 0 \end{aligned}$$

## 2.4 Fractional age assumptions

Uniform distribution of deaths (UDD)

- $\mathbb{P}[s \leq t] = t$  when  $0 \leq t < 1$
- ${}_t q_x = \mathbb{P}[0 \leq T_x < 1] \cdot \mathbb{P}[s < t] = t q_x$
- $\mu_{x+t} = {}_t p_x^{-1} f_{T_x}(t) = q_x / {}_t p_x$  when  $0 \leq t < 1$
- Linear interpolation:  $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$
- $\mathbb{E}[s] = 1/2$ , so  $\dot{e}_x = e_x + 1/2$  exactly

Constant form of mortality (CFM fractional)

- $\mu_{x+t} = \mu_x$  when  $0 \leq t < 1$
- $\ln(S_{x+t}/S_x) = \int_0^t (-\mu_{x+t}) dt = -t\mu_x$ , so  ${}_t p_x = e^{-t\mu_x}$
- Exponential interpolation:  $\ell_{x+t} = \ell_x^{1-t} \cdot \ell_{x+1}^t$

## 2.5 Mortality modifications

Select and ultimate probabilities

- ${}_k q_{[x]+t}$  is the probability that someone who was issued a policy at age  $x$  and is currently age  $x+t$  will die in the next  $k$  years
- Similar bracket notation for  $\ell$ ,  $p$ , etc.

Change in mortality over time

- $q(x, t)$  is the probability that in year  $t$ , someone at age  $x$  dies within the next year
- Relative change in mortality within the  $t$ -th year:  $\varphi(x, t) = 1 - q(x, t)/q(x, t-1)$
- If  $\varphi(x, t)$  is independent of  $t$ , write  $\varphi_x$

## 2.6 Analytic laws

de Moivre's law (DML), straight-line mortality

- $S_x = 1 - \frac{x}{\omega}$
- ${}_tq_x = \frac{t}{\omega-x}$
- $\mu_x = -\frac{S'_x}{S_x} = \frac{1}{\omega-x}$
- $\hat{e}_x = \frac{\omega-x}{2}$
- DML implies UDD

Modified DML

- $\mu_x = \alpha \cdot \mu_x^{DML} = \frac{\alpha}{\omega-x}$
- $S_x = \exp \left[ \int_0^x (-\mu_t) dt \right] = (S_x^{DML})^\alpha = \left(1 - \frac{x}{\omega}\right)^\alpha$

Constant force of mortality (CFM law)

- $\mu_x = \mu$  constant
- $S_x = e^{-\mu x}$
- ${}_tp_x = e^{-\mu t}$  independent of  $x$
- $\hat{e}_x = \int_0^\infty {}_tp_x dt = 1/\mu$
- CFM law implies CFM fractional

Makeham's law

- $\mu_x = A + Bc^x$  for parameters  $A, B, c$
- Gompertz's law when  $A = 0$

### 3 Life Insurance

#### 3.1 Discrete life insurance

Assumptions and notation

- Benefits  $b_{K+1}$  paid at the end of the year
- Constant interest rate
- Payout random variable  $Z$

Whole life insurance

- $Z = b_{K+1}v^{K+1}$
- Expected value  $\mathbb{E}[Z] = \sum_{k=0}^{\infty} b_{k+1}v^{k+1}{}_k|q_x$
- Second moment  $\mathbb{E}[Z^2] = \sum_{k=0}^{\infty} b_{k+1}^2 v^{2(k+1)}{}_k|q_x$  equivalent to expected value of  $Z$  for benefit  $b_{k+1}^* = b_{k+1}^2$  and interest  $i^* = (1+i)^2 - 1$ , i.e.  $v^* = v^2$  or  $\delta^* = 2\delta$
- If  $b_{k+1} = 1$  throughout, define  $A_x = \mathbb{E}[Z]$  and  ${}^2A_x = \mathbb{E}[Z^2]$
- DML mortality law

$$A_x = \sum_{k=0}^{\omega-x-1} \frac{v^{k+1}}{\omega-x} = \frac{a_{\overline{\omega-x}|}}{\omega-x}$$

$${}^2A_x = \sum_{k=0}^{\omega-x-1} \frac{v^{2(k+1)}}{\omega-x} = \frac{a_{\overline{\omega-x}|i^*}}{\omega-x}$$

- CFM mortality law

$$A_x = \sum_{k=0}^{\infty} v^{k+1}e^{-\mu k}(1-e^{-\mu}) = v(1-e^{-\mu}) \sum_{k=0}^{\infty} (ve^{-\mu})^k$$

$$= e^{-\delta}(1-e^{-\mu}) \cdot \frac{1}{1-e^{-(\delta+\mu)}} = \frac{e^{-\delta}(1-e^{-\mu})}{1-e^{-(\delta+\mu)}}$$

$${}^2A_x = \sum_{k=0}^{\infty} v^{2(k+1)}e^{-\mu k}(1-e^{-\mu}) = \frac{e^{-\delta^*}(1-e^{-\mu})}{1-e^{-(\delta^*+\mu)}}$$

$n$ -year term insurance