Permutations

Definition: The *factorial* of a non-negative integer n is defined recursively by

$$0! = 1 \quad \text{and} \quad n! = n \cdot (n-1)! \text{ for all } n \geq 1.$$

When n is a positive integer, $n!$ is the product of all positive integers less than or equal to n, and it gives the number of orders in which n distinct items can be listed. The first several values are

$$\begin{aligned}
0! &= 1, \\
1! &= 1 \cdot 0! = 1 \cdot 1 = 1, \\
2! &= 2 \cdot 1! = 2 \cdot 1 = 2, \\
3! &= 3 \cdot 2! = 3 \cdot 2 = 6, \\
4! &= 4 \cdot 3! = 4 \cdot 6 = 24, \\
5! &= 5 \cdot 4! = 5 \cdot 24 = 120, \\
6! &= 6 \cdot 5! = 6 \cdot 120 = 720.
\end{aligned}$$

1. The library is giving one book to each student for free. Three friends show up at the library and find that there are 4 different books available, with only one copy left for each of the books. In how many ways can the friends choose their books?

2. How many positive 3-digit integers have 3 distinct digits?

3. How many 4-digit <u>odd</u> integers greater than 6000 can be formed from the digits $\{0, 1, 3, 5, 6, 8\}$ if no digit may be used more than once?

4. How many different 4-letter strings can be generated by using each of the letters A, O, P, and S exactly once? (A *string* in this context is a sequence of characters, such as $SOAP$. It need not be an actual English word, so for example, $SAOP$ would also be a valid string.)

Repeated Elements

1. If all the letters of the word $SYZYGY$[1] are used, in how many different ways can the six letters be arranged in a six-letter string?

2. In how many different ways can the letters in the word $PEOPLE$ be scrambled, including the original spelling $PEOPLE$?

3. (2006 State Sprint Problem 28) Derek's phone number, 336-7624, has the property that the three-digit prefix, 336, equals the product of the last four digits, $7 \times 6 \times 2 \times 4$. How many seven-digit phone numbers beginning with 336 have this property?

[1] "In astronomy, a roughly straight-line configuration of three or more celestial bodies."

Extensions

1. _____ (1998 Chapter Countdown)

2. _____

3. _____

4. _____

5. _____ (2011 National Sprint Problem 12)

6. _____

7. _____

8. _____

Extra Problems

1. (*Calculator permitted.*) Two positive real numbers x and y satisfy the equation

$$x^2 + 4xy + 5y^2 - 14x - 32y + 52 = 0.$$

What is the maximum possible value of y/x? Round your answer to two decimal places.

2. All 7-digit positive integers which have each of the digits from 1 to 7 (inclusive) exactly once are written in increasing order. What is the 2025th number in the list?

3. In triangle ABC, points P and Q are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Points X and Y lie on lines $\overline{AB}$ and $\overline{AC}$, respectively, so that the areas of triangle APQ, quadrilateral $PQYX$, and quadrilateral $XYCB$ form an arithmetic sequence in that order. Find the ratio XY/BC, expressing your answer as a common fraction in simplest radical form.

4. If a, b, and c are positive integers such that

$$\gcd(a, b) = 12, \quad \gcd(b, c) = 45, \quad \gcd(c, a) = 21,$$

what is the minimum possible value of $a + b + c$?