MA582 Midterm

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2025-03-06

Problem 1

Suppose X has mgf $M(t) = (1 - 2t)^{-3/2}$ for t near 0.

(a) Compute m_3.

(b) Identify the distribution of X and justify your method.

Solution. (a) We compute

$$\begin{aligned} M'(t) &= -2 \cdot \frac{-3}{2} \cdot (1 - 2t)^{-5/2} = 3(1 - 2t)^{-5/2}, \\ M''(t) &= 3 \cdot -2 \cdot \frac{-5}{2} \cdot (1 - 2t)^{-7/2} = 15(1 - 2t)^{-7/2}, \\ M'''(t) &= 15 \cdot -2 \cdot \frac{-7}{2} \cdot (1 - 2t)^{-9/2} = 105(1 - 2t)^{-9/2}. \end{aligned}$$

Then, $m_3 = M'''(0) = \boxed{105}$.

(b) In class, we saw that a χ^2 distribution with r degrees of freedom has mgf equal to $(1-2t)^{-r/2}$ for $t < 1/2$. Our given mgf matches that of a χ^2 distribution with 3 degrees of freedom, so in fact, by mgf uniqueness, $X \sim \boxed{\chi^2_3}$.

□

Problem 2

Suppose in the Zev family $\{Z(\theta) : \theta \text{ real}\}$, it so happens that when $X \sim Z(\theta)$, we have $\theta = E(3X^5)$. Devise an estimator for θ and prove that it is UC for θ.

Solution. Let $Y = 3X^5$ and let $Y_i = 3X_i^5$ for each i, so that $E(Y) = E(Y_i) = \theta$ for all i. We consider the estimator

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} = \boxed{\frac{3X_1^5 + \cdots + 3X_n^5}{n}}.$$

It is unbiased for θ because

$$E(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \theta = \theta.$$

It is consistent for θ because the weak law of large numbers gives us

$$\hat{\theta}_n = \frac{Y_1 + \cdots + Y_n}{n} \xrightarrow{p} E(Y) = \theta.$$

□

Problem 3

Suppose $f_\theta(x) = 2x/\theta^2$ for $0 < x < \theta$, where $\theta > 0$. Using, as your estimator for θ, the sample maximum (call that Y_n here), prove it is consistent for θ.

Solution. If X has this distribution, then the cdf of X satisfies $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x \geq \theta$. In between, we calculate

$$F_X(x) = \int_0^x \frac{2t}{\theta^2}\, dt = \frac{x^2}{\theta^2}.$$

Then, as discussed in class, when $Y_n = \max(X_1, \ldots, X_n)$ we have

$$F_{Y_n}(y) = (F_X(y))^n = \begin{cases} 0 & y < 0, \\ (y/\theta)^{2n} & 0 \leq y < \theta, \\ 1 & y \geq \theta. \end{cases}$$

To see that Y_n is consistent for θ, it suffices to show for any $0 < \epsilon < \theta$ that

$$\lim_{n\to\infty} P(|Y_n - \theta| \geq \epsilon) = 0.$$

Since $Y_n \leq \theta$ always holds,

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n \leq \theta - \epsilon) = \left(\frac{\theta - \epsilon}{\theta}\right)^{2n} = \left(1 - \frac{\epsilon}{\theta}\right)^{2n}.$$

As $n \to \infty$, the right hand side tends to 0 as desired. □

Problem 4

In #3, show your estimator is biased and show how to correct for the bias, then show your new corrected estimator is also consistent for θ. (Call your corrected estimator T_n.)

Solution. The pdf of Y_n is

$$f_{Y_n}(y) = F'_{Y_n}(y) = \begin{cases} \frac{2n}{\theta^{2n}} \cdot y^{2n-1} & 0 < y < \theta, \\ 0 \text{ otherwise.} \end{cases}$$

The expected value of Y_n is then

$$\begin{aligned} E(Y_n) &= \int_{-\infty}^{\infty} y \cdot f_{Y_n}(y)\, dy = \int_0^\theta \frac{2n}{\theta^{2n}} y^{2n}\, dy \\ &= \frac{2n}{\theta^{2n}} \cdot \frac{\theta^{2n+1}}{2n+1} = \frac{2n}{2n+1} \cdot \theta. \end{aligned}$$

In particular, this is not θ itself, so Y_n is a biased estimator for θ. If we define $\boxed{T_n = \frac{2n+1}{2n} Y_n}$,

$$E(T_n) = \frac{2n+1}{2n} E(Y_n) = \frac{2n+1}{2n} \cdot \frac{2n}{2n+1} \cdot \theta = \theta,$$

so T_n is an unbiased estimator for θ. Also, since $\frac{2n+1}{2n} \to 1$ and $Y_n \xrightarrow{p} \theta$ as $n \to \infty$, one of our "preservation of convergence in probability" results gives us

$$T_n = \frac{2n+1}{2n} Y_n \xrightarrow{p} 1 \cdot \theta = \theta.$$

Thus T_n is consistent for θ as well. □

Problem 5

In #3, find the asymptotic distribution of $n(\theta - Y_n)$.

Solution. When $t < 0$, we have $P(n(\theta - Y_n) \le t) = 0$. Otherwise, for all n sufficiently large ($n\theta > t$),

$$\begin{aligned} P(n(\theta - Y_n) \le t) &= P\left(Y_n \ge \theta - \frac{t}{n}\right) = 1 - P\left(Y_n \le \theta - \frac{t}{n}\right) \\ &= 1 - \left(\frac{\theta - t/n}{\theta}\right)^{2n} = 1 - \left(1 - \frac{t/\theta}{n}\right)^{2n}. \end{aligned}$$

As $n \to \infty$, this tends to $1 - e^{-2t/\theta}$, so the asymptotic distribution of $n(\theta - Y_n)$ has cdf

$$t \longmapsto \begin{cases} 0 & t < 0, \\ 1 - e^{-2t/\theta} & t \ge 0. \end{cases}$$

This is the $\boxed{\text{Exp}(2/\theta)}$ distribution. □

Problem 6

In #3, find the asymptotic distribution of $n(\theta - T_n)$.

Solution. We start by writing

$$n(\theta - T_n) = n(\theta - Y_n) + n(Y_n - T_n) = n(\theta - Y_n) - \frac{1}{2} Y_n.$$

Since $n(\theta - Y_n) \xrightarrow{\mathcal{D}} \text{Exp}(2/\theta)$ and $Y_n \xrightarrow{p} \theta$, Slutsky's lemma tells us that

$$n(\theta - T_n) \xrightarrow{\mathcal{D}} \boxed{\text{Exp}\left(\frac{2}{\theta}\right) - \frac{\theta}{2}},$$

the $\text{Exp}(2/\theta)$ distribution shifted a constant amount $\theta/2$ to the left. That is, if $E_n = n(\theta - T_n)$ is the magnified error and F_n is its cdf, then

$$F_n(t) \approx \begin{cases} 0 & t < -\theta/2, \\ 1 - e^{-2(t+\theta/2)/\theta} & t \geq -\theta/2. \end{cases}$$

□

Problem 7

Compute the third moment of $X = 3Z + 2$ explicitly.

Solution. For the standard normal, $E(Z) = E(Z^3) = 0$ by symmetry while

$$E(Z^2) = \text{Var}(Z) + E(Z)^2 = 1 + 0 = 1.$$

Then, the third moment of X is

$$\begin{aligned} E(X^3) &= E((3Z+2)^3) = E(27Z^3 + 54Z^2 + 36Z + 8) \\ &= 27E(Z^3) + 54E(Z^2) + 36E(Z) + 8 = 54 + 8 = \boxed{62}. \end{aligned}$$

□