MA582 Final

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Problem 1

Consider the family of pdfs $f(x,\theta) = 4x^3/\theta^4$ for $0 < x < \theta$.

(a) Find a sufficient statistic Y for θ.

(b) Show that Y is complete for θ.

(c) Compute $E(Y)$.

(d) Use (c) to find and justify an MVUE for θ.

(e) Find the MLE for θ^2. (Just the answer, no need for derivation here!)

Solution. (a) We compute the likelihood function

$$L(\mathbf{x},\theta) = \prod_{i=1}^n f(x_i,\theta) = \prod_{i=1}^n \frac{4x_i^3 \mathbf{1}_{0<x_i<\theta}}{\theta^4} = \frac{4^n}{\theta^{4n}} \left(\prod_{i=1}^n x_i^3 \right) \mathbf{1}_{0<x_1,\ldots,x_n<\theta}.$$

This factors as $L(\mathbf{x},\theta) = \alpha(y,\theta)\cdot\beta(\mathbf{x})$, where $y = \max(x_1,\ldots,x_n)$, as

$$\alpha(y,\theta) = \frac{\mathbf{1}_{0<y<\theta}}{\theta^{4n}}, \qquad \beta(\mathbf{x}) = 4^n \prod_i x_i^3.$$

By Neymann's factorization lemma, $Y = \max(X_1,\ldots,X_n)$ is sufficient for θ.

(b) Suppose $E_\theta(\delta(Y)) = 0$ for all θ. We compute

$$E_\theta(\delta(Y)) = \int L(\mathbf{x},\theta)\delta(y)\,d\mathbf{x} = \frac{4^n}{\theta^{4n}} \int_{\mathbf{0}}^{\theta\mathbf{1}} x_1^3 \cdots x_n^3 \delta(\max(x_1 \cdots x_n))\,d\mathbf{x}.$$

By symmetry, we can just consider the integral in the case that $y = x_n$ and write

$$\begin{aligned} E_\theta(\delta(Y)) &= \frac{4^n \cdot n}{\theta^{4n}} \int_0^\theta \left(\int_{(0,\ldots,0)}^{(y,\ldots,y)} x_1^3 \cdots x_{n-1}^3 \,dx_1 \,\ldots\, dx_n \right) y^3 \delta(y)\,dy \\ &= \frac{4n}{\theta^{4n}} \int_0^\theta y^{4n-1}\delta(y)\,dy. \end{aligned}$$

This means

$$D(\theta) = \int_0^\theta y^{4n-1}\delta(y)\,dy = 0$$

for all θ. Assuming δ continuous, we can invoke the fundamental theorem of calculus to say that $D'(\theta) = \theta^{4n-1}\delta(\theta) = 0$, so then $\delta(\theta) = 0$ everywhere. (I believe as long as $E_\theta(\delta(Y))$ exists this should still hold as an almost-surely statement.) Thus Y is consistent for θ.

(c) Borrowing the previous calculation with $\delta(Y) = Y$,

$$E_\theta(Y) = \frac{4n}{\theta^{4n}} \int_0^\theta y^{4n}\,dy = \frac{4n}{4n+1} \cdot \theta.$$

(d) Let $Z = \frac{4n+1}{4n} \cdot Y$. Then $E(Z) = \theta$, so Z is unbiased for θ. Since Y is sufficient and complete for θ, by the Lehmann-Scheffe theorem, $E(Z \mid Y) = Z$ is an MVUE for θ.

(e) $MLE(\theta^2) = Y^2$.

$\square$

Problem 2

Suppose $X \sim N(0, \theta)$, where $\mathrm{var}(X) = \theta > 0$.

1. Find Fisher's information $I(\theta)$.
2. Find $MLE(\theta)$.
3. Show that your MLE is CAN for θ and specify the ANV explicitly.

Solution.

1. With
$$f(x, \theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta},$$
we have
$$\log f(\mathbf{x}, \theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\theta - \sum_{i=1}^{n} \frac{x_i^2}{2\theta}.$$
Therefore,
$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log f(\mathbf{X}, \theta)\right) = -E\left(\frac{n}{2\theta^2} - \sum_{i=1}^{n}\frac{X_i^2}{\theta^3}\right) = -\frac{n}{2\theta^2} + \frac{\sum_i E(X_i^2)}{\theta^3}.$$
Since $E(X^2) = E(X)^2 + \mathrm{var}(X) = \theta$, we find $I(\theta) = n/2\theta^2$.

2. For n trials, the log likelihood is maximized when
$$\frac{\partial}{\partial\theta}\left[\sum_{i=1}^{n}\log f(x_i, \theta)\right] = -\frac{n}{2\theta} + \frac{\sum_i x_i^2}{2\theta^2} = \frac{\sum_i x_i^2 - n\theta}{2\theta^2} = 0.$$
Thus $MLE(\theta) = \frac{1}{n}\sum_i x_i^2$.

3. The normal distribution meets all regularity conditions discussed, so by the theorem from class with the very long initialism name, $MLE(\theta)$ is CAN for θ with ANV $n/I(\theta) = 2\theta^2$.

□

Problem 3

Let $X \sim \text{Exp}(\lambda)$, where $\lambda > 0$ and $E(X) = 1/\lambda$.

1. Find $MLE(\lambda)$.
2. Find Fisher's information $I(\lambda)$.
3. Show that your MLE is CAN for λ.
4. Find the ANV for your MLE.
5. Show that your MLE is biased for λ.
6. Find $MLE(\lambda^2)$.

Solution.

1. We are taking the pdf of an individual X to be $f(x, \lambda) = \lambda e^{-\lambda x}$, so then

$$\log f(\mathbf{x}, \lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i.$$

   Differentiating with respect to λ, the equation $n/\lambda - \sum_i x_i = 0$ gives $MLE(\lambda) = n/\sum_i x_i$.

2. The Fisher's information $I(\lambda)$ is

$$I(\lambda) = -E\left(\frac{\partial^2}{\partial \lambda^2} \log f(\mathbf{X}, \lambda)\right) = \frac{n}{\lambda^2}.$$

3. The exponential distribution meets all regularity conditions discussed, so by the theorem from class with the very long initialism name, $MLE(\lambda)$ is CAN for λ.
4. By the same theorem, the ANV is $n/I(\theta) = \lambda^2$.
5. Since $x \mapsto 1/x$ is strictly convex on $(0, \infty)$, Jensen's inequality tells us that since $\sum_i x_i/n$ has expectation $E(X) = 1/\lambda$, the expectation of $MLE(\lambda) = n/\sum_i x_i$ is strictly less than λ, in particular unequal to λ.
6. Writing $\ell = \lambda^2$, our log likelihood is $(n/2)\log \ell - \ell^{1/2} \sum_{i=1}^{n}$, and differentiating gives us $(n/2\ell) - 1/2\sqrt{\ell} \sum_i x_i = 0$. This means $n - \sqrt{\ell} \sum_i x_i = 0$, so then $MLE(\lambda^2) = (n/\sum_i x_i)^2$.

$\square$

Problem 4

Name a parametric family whose parameter has its MLE being the sample median.

Solution. □