Fiddler on the Proof - Sorting Hat

Alex Zhu

November 12, 2024

This is a solution to a Fiddler on the Proof puzzle.

To make things easier to follow, let us suppose that Logwarts' four houses are:

- Graphindor (hence referred to as $\mathcal{G}$)
- Riemannclaw (hence referred to as $\mathcal{R}$)
- Hexapuff (hence referred to as $\mathcal{H}$)
- Scalerin (hence referred to as $\mathcal{S}$)

Suppose that for some arbitrary round S, the probability of a student being sorted into $\mathcal{G}$ is P_s.

Now let us work through round $S+1$.
If the student S was placed into $\mathcal{G}$, then, by the rules, student $S+1$ cannot be a $\mathcal{G}$. So we can disregard this case.
On the other hand, suppose student S was placed into one of the other housees - let's denote that as $\mathcal{X}$, which may be any of $\mathcal{R}$, $\mathcal{H}$, $\mathcal{S}$.

- This student has a $^1/_4$ chance of choosing $\mathcal{G}$, which is allowed.
- However, this student also has a $^1/_4$ chance of choosing $\mathcal{X}$ again, which is not allowed. So that's s a $^1/_3$ chance of the hat placing them in $\mathcal{G}$ against their will, for a total of $^1/_4 * {}^1/_3 = {}^1/_{12}$ chance of $\mathcal{G}$ given that the previous student S was not placed into $\mathcal{G}$.
- The remaining $^1/_2$ chance is that the student picks one of the other two non-$\mathcal{G}$ and non-$\mathcal{X}$. This is allowed and doesn't contribute to the odds of student $S+1$ ending up in $\mathcal{G}$.

So, if student S is placed into $\mathcal{X}$, the chances of student $S+1$ being placed into $\mathcal{G}$ are $^1/_4 + {}^1/_{12} = {}^1/_3$.

We know the odds of student S being placed into $\mathcal{X}$ to be $1 - P_s$, and we can disregard the P_s case as impossible for student $S+1$ to be a $\mathcal{G}$, so that means that:

$$P_{s+1} = {}^1/_3(1 - P_s)$$

Let's fruther define B_s as the chances that our protagonist Barry Plotter gets his choice, given that he is S_{th} in line. Since Barry chooses $\mathcal{G}$, he will get his wish as long as student $S-1$ doesn't pick $\mathcal{G}$. So:

$$B_s = 1 - P_{s-1}$$

We can plug this into Excel (okay, Google Sheets) to get our results:

Round	P_s	B_s
1	1.0000	n/a (Barry can't be first)
2	0.0000	0.0000
3	0.3333	1.0000
4	0.2222	0.6667
5	0.2593	0.7778
6	0.2469	0.7407
7	0.2510	0.7531
8	0.2497	0.7490
9	0.2501	0.7503
10	0.2500	0.7499

So our solution is $B_{10} = 74.9886\%$.

Extra Credit

Define W_m as the odds that Barry gets into $\mathcal{G}$ given that he wakes up at student M getting sorted and E_n as Barry's overall chances to get into $\mathcal{G}$ given he is N^{th} in line.

If Barry wakes up when student M is being sorted, the problem is identical to the original prompt, except now we're starting from student M instead of student 1. So it follows that:

$$W_m = B_{n-m}$$

By the prompt, M is $[1..N-1]$. So W_m ranges from B_{n-1} to $B_{n-(n-1)} = B_1$.

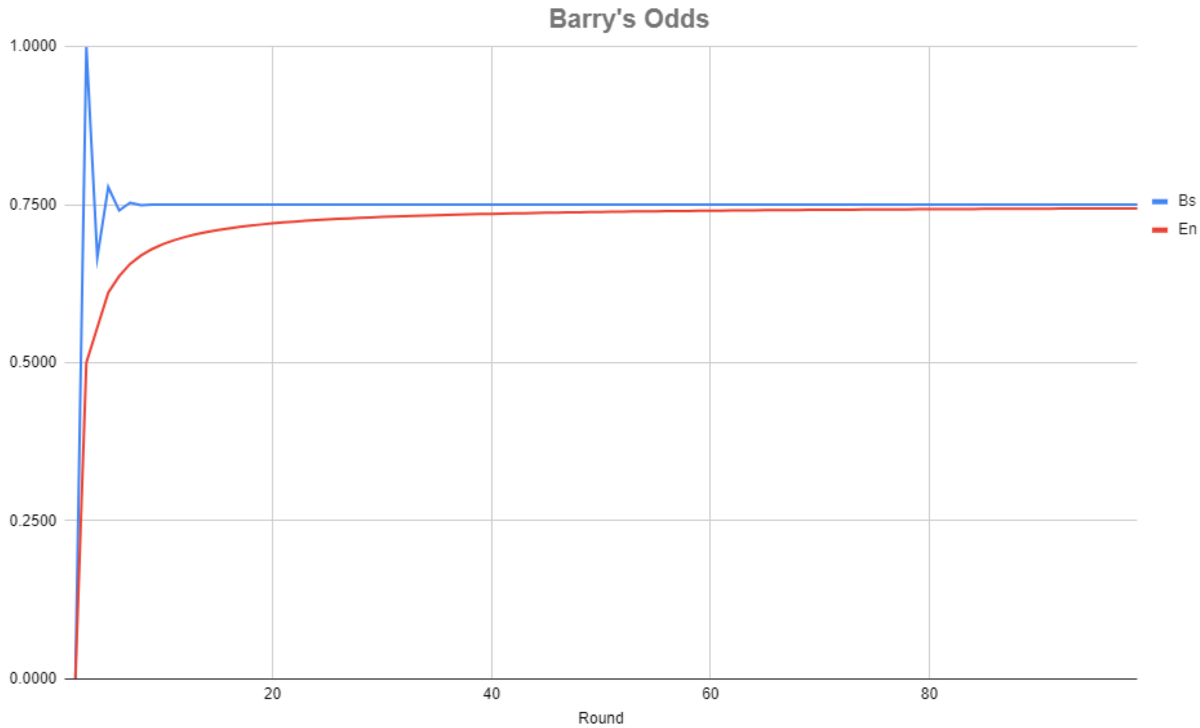
Every value in M has an equal chance to be chosen, so E_n is simply the average of every value $(B_1..B_{n-1})$. Plug this back into Google Sheets and the table starts:

Round	P_s	B_s	E_n
1	1.0000	n/a	n/a
2	0.0000	0.0000	0.0000
3	0.3333	1.0000	0.5000
4	0.2222	0.6667	0.5556
5	0.2593	0.7778	0.6111
6	0.2469	0.7407	0.6370
7	0.2510	0.7531	0.6564
8	0.2497	0.7490	0.6696
9	0.2501	0.7503	0.6797
10	0.2500	0.7499	0.6875

We continue a while until finally $E_n > p$ at $N = 4922$.

Graphs

We can see that Barry's overall odds quickly approach $^3/_4$, which makes sense for earlier rounds to matter less and less.



We can also see just how long it takes the extra credit scenario to reach just the 10th iteration if only Barry had gotten a better night's sleep.