Fiddler on the Proof - Sorting Hat

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This is a solution to a Fiddler on the Proof puzzle.

To make things easier to follow, let us suppose that Logwarts' four houses are:

- Graphindor (hence referred to as $\mathcal{G}$)
- Riemannclaw (hence referred to as $\mathcal{R}$)
- Hexapuff (hence referred to as $\mathcal{H}$)
- Scalerin (hence referred to as $\mathcal{S}$)

Suppose that for some arbitrary round S, the probability of a student being sorted into $\mathcal{G}$ is P_s.

Now let us work through round $S+1$.
If the student S was placed into $\mathcal{G}$, then, by the rules, student $S+1$ cannot be a $\mathcal{G}$. So we can disregard this case.
On the other hand, suppose student S was placed into one of the other housees - let's denote that as $\mathcal{X}$, which may be any of $\mathcal{R}$, $\mathcal{H}$, $\mathcal{S}$.

- This student has a $1/4$ chance of choosing $\mathcal{G}$, which is allowed.
- However, this student also has a $1/4$ chance of choosing $\mathcal{X}$ again, which is not allowed. So that's s a $1/3$ chance of the hat placing them in $\mathcal{G}$ against their will, for a total of $1/4 * 1/3 = 1/12$ chance of $\mathcal{G}$ given that the previous student S was not placed into $\mathcal{G}$.
- The remaining $1/2$ chance is that the student picks one of the other two non-$\mathcal{G}$ and non-$\mathcal{X}$. This is allowed and doesn't contribute to the odds of student $S+1$ ending up in $\mathcal{G}$.

So, if student S is placed into $\mathcal{X}$, the chances of student $S+1$ being placed into $\mathcal{G}$ are $1/4 + 1/12 = 1/3$.

We know the odds of student S being placed into $\mathcal{X}$ to be $1 - P_s$, and we can disregard the P_s case as impossible for student $S+1$ to be a $\mathcal{G}$, so that means that:

$$P_{s+1} = 1/3(1 - P_s)$$

Let's fruther define B_s as the chances that our protagonist Barry Plotter gets his choice, given that he is S_{th} in line. Since Barry chooses $\mathcal{G}$, he will get his wish as long as student $S-1$ doesn't pick $\mathcal{G}$. So:

$$B_s = 1 - P_{s-1}$$

We can plug this into Excel (okay, Google Sheets) to get our results:

Round	P_s	B_s
1	1.0000	n/a (Barry can't be first)
2	0.0000	0.0000
3	0.3333	1.0000
4	0.2222	0.6667
5	0.2593	0.7778
6	0.2469	0.7407
7	0.2510	0.7531
8	0.2497	0.7490
9	0.2501	0.7503
10	0.2500	0.7499

So our solution is $B_{10} = 74.9886\%$.

Extra Credit

Define W_m as the odds that Barry gets into $\mathcal{G}$ given that he wakes up at student M getting sorted and E_n as Barry's overall chances to get into $\mathcal{G}$ given he is N^{th} in line.

If Barry wakes up when student M is being sorted, the problem is identical to the original prompt, except now we're starting from student M instead of student 1. So it follows that:

$$W_m = B_{n-m}$$

By the prompt, M is $[1..N-1]$. So W_m ranges from B_{n-1} to $B_{n-(n-1)} = B_1$.

Every value in M has an equal chance to be chosen, so E_n is simply the average of every value $(B_1..B_{n-1})$. Plug this back into Google Sheets and the table starts:

Round	P_s	B_s	E_n
1	1.0000	n/a	n/a
2	0.0000	0.0000	0.0000
3	0.3333	1.0000	0.5000
4	0.2222	0.6667	0.5556
5	0.2593	0.7778	0.6111
6	0.2469	0.7407	0.6370
7	0.2510	0.7531	0.6564
8	0.2497	0.7490	0.6696
9	0.2501	0.7503	0.6797
10	0.2500	0.7499	0.6875

We continue a while until finally $E_n > p$ at $N = 4922$.