Exercise 1

$$S = \{(x_i, f(x_i))\}_{i=1}^{m} \subseteq (\mathbb{R}^d \times \{0,1\})^m$$

Consider polynomial $p_s(x) = -(x-x_1)^2(x-x_2)^2 \cdots (x-x_m)^2$

Our $h_s(x)$ function is defined as

$$h_s(x) = \begin{cases} 1 & \text{if } \exists i \in \{1,2,\dots m\}, \text{ s.t. } x_i = x \\ 0 & \text{otherwise} \end{cases}$$

Let's show that $h_s(x) = 1$ if and only if $p_s(x) \geq 0$

$\boxed{\text{i}}$ first, let's show that $h_s(x)=1 \Rightarrow p_s(x) \geq 0$

Consider x, such that $h_s(x) = 1$, then it means that $\exists$ $i \in [m]$, s.t. $x_i = x$. Then, $(x-x_i)^2 = 0$ and, therefore, $p_s(x) = 0$. Hence, $p_s(x) \geq 0$.

$\boxed{\text{ii}}$ Now, let's show that $p_s(x) \geq 0 \Rightarrow h_s(x) = 1$

Consider x, such that $p_s(x) \geq 0$.

We know that for each $i \in [m]$, $(x-x_i)^2 \geq 0$ and equal to 0, if and only if $x = x_i$.

Assume in this case $x \neq x_i$ for all $i \in [m]$. Then, each of $(x-x_i)^2$ is greater than 0, hence $\prod_{i=1}^{m}(x-x_i)^2 > 0$.

Then, $p_s(x) = -(x-x_1)^2(x-x_2)^2 \cdots (x-x_m)^2 < 0$, ~~which~~ and that leads to contradiction.

Hence, our assumption was wrong and $\exists i \in [m]$, s.t. $x_i = x$

Therefore, $h_s(x) = 1$.

We have shown that $h_s(x) = 1 \Rightarrow p_s(x) \geq 0$ and $p_s(x) \geq 0 \Rightarrow h_s(x) = 1$

Hence, $h_s(x) = 1$, if and only if $p_s(x) \geq 0$.

q.e.d.