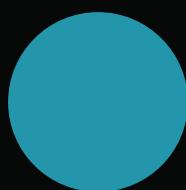


PEARSON NEW INTERNATIONAL EDITION



Linear Algebra with Applications  
Otto Bretscher  
Fifth Edition

# Pearson New International Edition

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Otto Bretscher  
Fifth Edition

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Edinburgh Gate

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**PEARSON**

ISBN 10: 1-292-02214-0  
ISBN 13: 978-1-292-02214-7

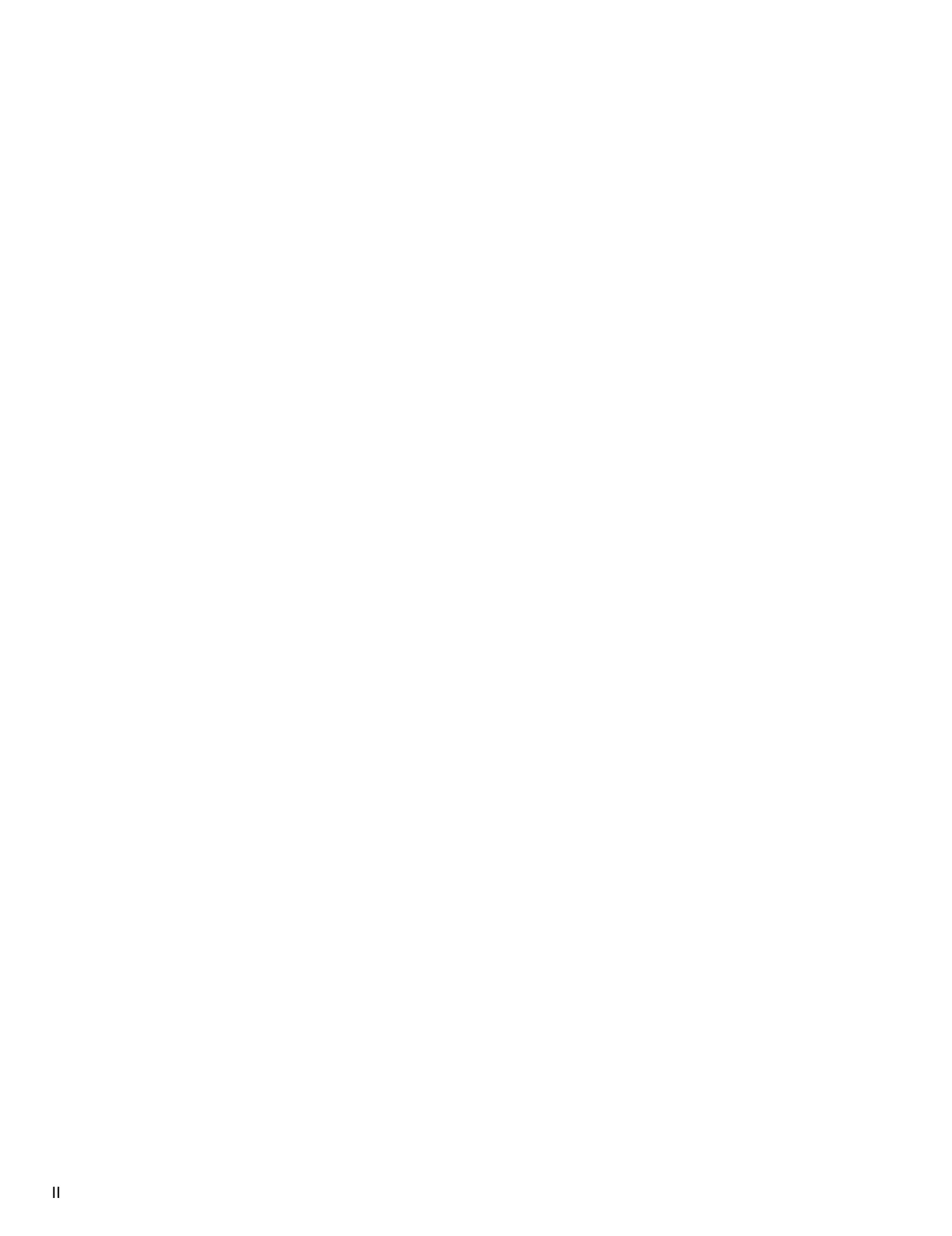
**British Library Cataloguing-in-Publication Data**

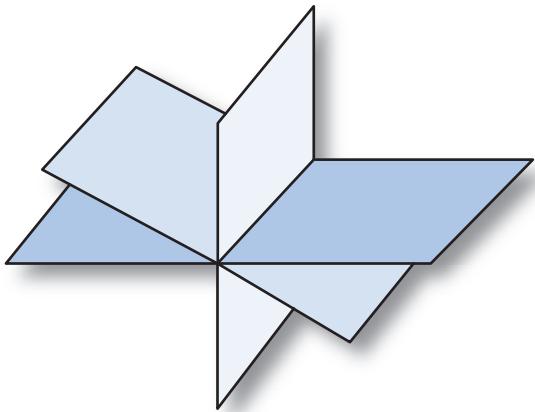
A catalogue record for this book is available from the British Library

Printed in the United States of America

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# Linear Equations

## I Introduction to Linear Systems

Traditionally, algebra was the art of solving equations and systems of equations. The word *algebra* comes from the Arabic *al-jabr* (الجبر), which means *restoration* (of broken parts).<sup>1</sup> The term was first used in a mathematical sense by Mohammed al-Khowarizmi (c. 780–850), who worked at the House of Wisdom, an academy established by Caliph al-Ma'mun in Baghdad. Linear algebra, then, is the art of solving systems of linear equations.

The need to solve systems of linear equations frequently arises in mathematics, statistics, physics, astronomy, engineering, computer science, and economics.

Solving systems of linear equations is not conceptually difficult. For small systems, ad hoc methods certainly suffice. Larger systems, however, require more systematic methods. The approach generally used today was beautifully explained 2,000 years ago in a Chinese text, the *Nine Chapters on the Mathematical Art* (*Jiuzhang Suanshu*, 九章算術).<sup>2</sup> Chapter 8 of that text, called *Method of Rectangular Arrays* (*Fang Cheng*, 方程), contains the following problem:

The yield of one bundle of inferior rice, two bundles of medium-grade rice, and three bundles of superior rice is 39 *dou* of grain.<sup>3</sup> The yield of one bundle of inferior rice, three bundles of medium-grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium-grade rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

In this problem the unknown quantities are the yields of one bundle of inferior, one bundle of medium-grade, and one bundle of superior rice. Let us denote these quantities by  $x$ ,  $y$ , and  $z$ , respectively. The problem can then be represented by the

<sup>1</sup>At one time, it was not unusual to see the sign *Algebrista y Sangrador* (bone setter and blood letter) at the entrance of a Spanish barber's shop.

<sup>2</sup>Shen Kangshen et al. (ed.), *The Nine Chapters on the Mathematical Art*, Companion and Commentary, Oxford University Press, 1999.

<sup>3</sup>The *dou* is a measure of volume, corresponding to about 2 liters at that time.

## Linear Equations

following system of linear equations:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right|.$$

To solve for  $x$ ,  $y$ , and  $z$ , we need to transform this system from the form

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right| \quad \text{into the form} \quad \left| \begin{array}{l} x = \dots \\ y = \dots \\ z = \dots \end{array} \right|.$$

In other words, we need to eliminate the terms that are off the diagonal, those circled in the following equations, and make the coefficients of the variables along the diagonal equal to 1:

$$\begin{aligned} x + \textcircled{2y} + \textcircled{3z} &= 39 \\ \textcircled{x} + 3y + \textcircled{2z} &= 34 \\ \textcircled{3x} + \textcircled{2y} + z &= 26. \end{aligned}$$

We can accomplish these goals step by step, one variable at a time. In the past, you may have simplified systems of equations by adding equations to one another or subtracting them. In this system, we can eliminate the variable  $x$  from the second equation by subtracting the first equation from the second:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right| \xrightarrow{-1\text{st equation}} \left| \begin{array}{l} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{array} \right|.$$

To eliminate the variable  $x$  from the third equation, we subtract the first equation from the third equation three times. We multiply the first equation by 3 to get

$$3x + 6y + 9z = 117 \quad (3 \times 1\text{st equation})$$

and then subtract this result from the third equation:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{array} \right| \xrightarrow{-3 \times 1\text{st equation}} \left| \begin{array}{l} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{array} \right|.$$

Similarly, we eliminate the variable  $y$  above and below the diagonal:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{array} \right| \xrightarrow{\begin{array}{l} -2 \times 2\text{nd equation} \\ +4 \times 2\text{nd equation} \end{array}} \left| \begin{array}{l} x + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{array} \right|.$$

Before we eliminate the variable  $z$  above the diagonal, we make the coefficient of  $z$  on the diagonal equal to 1, by dividing the last equation by  $-12$ :

$$\left| \begin{array}{l} x + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{array} \right| \xrightarrow{\div (-12)} \left| \begin{array}{l} x + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{array} \right|.$$

Finally, we eliminate the variable  $z$  above the diagonal:

$$\left| \begin{array}{l} x + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{array} \right| \xrightarrow{\begin{array}{l} -5 \times \text{third equation} \\ + \text{third equation} \end{array}} \left| \begin{array}{l} x = 2.75 \\ y = 4.25 \\ z = 9.25 \end{array} \right|.$$

The yields of inferior, medium-grade, and superior rice are 2.75, 4.25, and 9.25 *dou* per bundle, respectively.

## Linear Equations

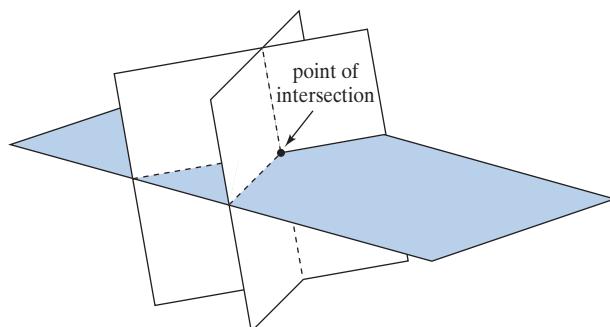
By substituting these values, we can check that  $x = 2.75$ ,  $y = 4.25$ ,  $z = 9.25$  is indeed the solution of the system:

$$\begin{aligned}2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\3 \times 2.75 + 2 \times 4.25 + 9.25 &= 26.\end{aligned}$$

Happily, in linear algebra, you are almost always able to check your solutions. It will help you if you get into the habit of checking now.

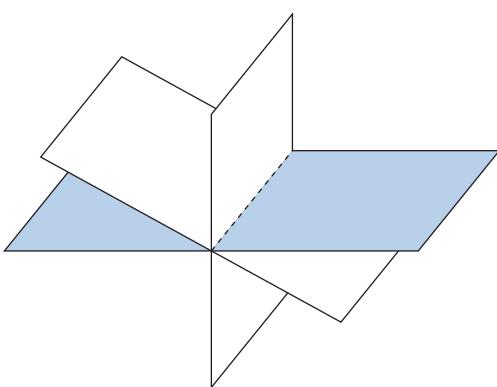
### Geometric Interpretation

How can we interpret this result geometrically? Each of the three equations of the system defines a plane in  $x$ - $y$ - $z$ -space. The solution set of the system consists of those points  $(x, y, z)$  that lie in all three planes (i.e., the intersection of the three planes). Algebraically speaking, the solution set consists of those ordered triples of numbers  $(x, y, z)$  that satisfy all three equations simultaneously. Our computations show that the system has only one solution  $(x, y, z) = (2.75, 4.25, 9.25)$ . This means that the planes defined by the three equations intersect at the point  $(x, y, z) = (2.75, 4.25, 9.25)$ , as shown in Figure 1.

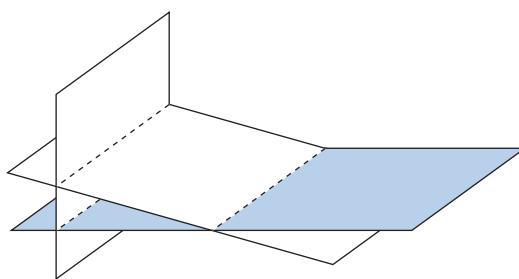


**Figure 1** Three planes in space, intersecting at a point.

While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.



**Figure 2(a)** Three planes having a line in common.



**Figure 2(b)** Three planes with no common intersection.

## Linear Equations

### A System with Infinitely Many Solutions

Next, let's consider a system of linear equations that has infinitely many solutions:

$$\begin{vmatrix} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{vmatrix}.$$

We can solve this system using the method of elimination as previously discussed. For simplicity, we label the equations with Roman numerals.

$$\begin{array}{lcl} \left| \begin{array}{l} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{array} \right. & \xrightarrow{\div 2} & \left| \begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{array} \right. \xrightarrow{-4 \text{ (I)}} \\ \left| \begin{array}{l} x + 2y + 3z = 0 \\ -3y - 6z = 3 \\ -6y - 12z = 6 \end{array} \right. & \xrightarrow{\div(-3)} & \left| \begin{array}{l} x + 2y + 3z = 0 \\ y + 2z = -1 \\ -6y - 12z = 6 \end{array} \right. \xrightarrow{-2 \text{ (II)}} \\ \left| \begin{array}{l} x - z = 2 \\ y + 2z = -1 \\ 0 = 0 \end{array} \right. & \xrightarrow{} & \left| \begin{array}{l} x - z = 2 \\ y + 2z = -1 \end{array} \right. \xrightarrow{+6 \text{ (II)}} \end{array}$$

After omitting the trivial equation  $0 = 0$ , we are left with only two equations with three unknowns. The solution set is the intersection of two nonparallel planes in space (i.e., a line). This system has infinitely many solutions.

The two foregoing equations can be written as follows:

$$\begin{vmatrix} x = z + 2 \\ y = -2z - 1 \end{vmatrix}.$$

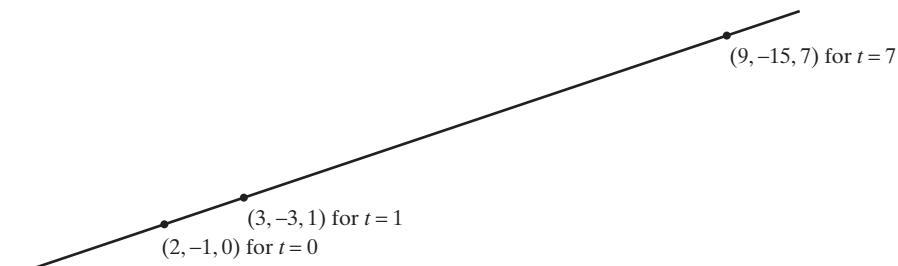
We see that both  $x$  and  $y$  are determined by  $z$ . We can freely choose a value of  $z$ , an arbitrary real number; then the two preceding equations give us the values of  $x$  and  $y$  for this choice of  $z$ . For example,

- Choose  $z = 1$ . Then  $x = z + 2 = 3$  and  $y = -2z - 1 = -3$ . The solution is  $(x, y, z) = (3, -3, 1)$ .
- Choose  $z = 7$ . Then  $x = z + 2 = 9$  and  $y = -2z - 1 = -15$ . The solution is  $(x, y, z) = (9, -15, 7)$ .

More generally, if we choose  $z = t$ , an arbitrary real number, we get  $x = t + 2$  and  $y = -2t - 1$ . Therefore, the general solution is

$$(x, y, z) = (t + 2, -2t - 1, t) = (2, -1, 0) + t(1, -2, 1).$$

This equation represents a line in space, as shown in Figure 3.



**Figure 3** The line  $(x, y, z) = (t + 2, -2t - 1, t)$ .

## A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{vmatrix} \rightarrow \begin{vmatrix} x - z = 2 \\ y + 2z = -1 \\ 0 = -6 \end{vmatrix}.$$

Whatever values we choose for  $x$ ,  $y$ , and  $z$ , the equation  $0 = -6$  cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

## EXERCISES I

**GOAL** Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. 
$$\begin{vmatrix} x + 2y = 1 \\ 2x + 3y = 1 \end{vmatrix}$$

2. 
$$\begin{vmatrix} 4x + 3y = 2 \\ 7x + 5y = 3 \end{vmatrix}$$

3. 
$$\begin{vmatrix} 2x + 4y = 3 \\ 3x + 6y = 2 \end{vmatrix}$$

4. 
$$\begin{vmatrix} 2x + 4y = 2 \\ 3x + 6y = 3 \end{vmatrix}$$

5. 
$$\begin{vmatrix} 2x + 3y = 0 \\ 4x + 5y = 0 \end{vmatrix}$$

6. 
$$\begin{vmatrix} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{vmatrix}$$

7. 
$$\begin{vmatrix} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{vmatrix}$$

8. 
$$\begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{vmatrix}$$

9. 
$$\begin{vmatrix} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{vmatrix}$$

10. 
$$\begin{vmatrix} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{vmatrix}$$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the  $x$ - $y$ -plane.

11. 
$$\begin{vmatrix} x - 2y = 2 \\ 3x + 5y = 17 \end{vmatrix}$$

12. 
$$\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 6 \end{vmatrix}$$

13. 
$$\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 8 \end{vmatrix}$$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. 
$$\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{vmatrix}$$

15. 
$$\begin{vmatrix} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{vmatrix}$$

16. 
$$\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{vmatrix}$$

17. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y = a \\ 3x + 5y = b \end{vmatrix},$$

where  $a$  and  $b$  are arbitrary constants.

18. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{vmatrix},$$

where  $a$ ,  $b$ , and  $c$  are arbitrary constants.

19. Consider the linear system

$$\begin{vmatrix} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{vmatrix},$$

where  $k$  is an arbitrary number.

- For which value(s) of  $k$  does this system have one or infinitely many solutions?
- For each value of  $k$  you found in part a, how many solutions does the system have?
- Find all solutions for each value of  $k$ .

20. Consider the linear system

$$\begin{vmatrix} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{vmatrix},$$

where  $k$  is an arbitrary constant. For which value(s) of  $k$  does this system have a unique solution? For which value(s) of  $k$  does the system have infinitely many solutions? For which value(s) of  $k$  is the system inconsistent?

21. The sums of any two of three real numbers are 24, 28, and 30. Find these three numbers.

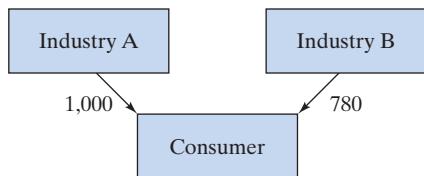
22. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

## Linear Equations

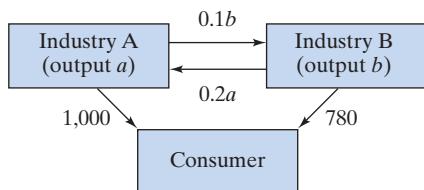
23. Consider a two-commodity market. When the unit prices of the products are  $P_1$  and  $P_2$ , the quantities demanded,  $D_1$  and  $D_2$ , and the quantities supplied,  $S_1$  and  $S_2$ , are given by

$$\begin{aligned} D_1 &= 70 - 2P_1 + P_2, & S_1 &= -14 + 3P_1, \\ D_2 &= 105 + P_1 - P_2, & S_2 &= -7 + 2P_2. \end{aligned}$$

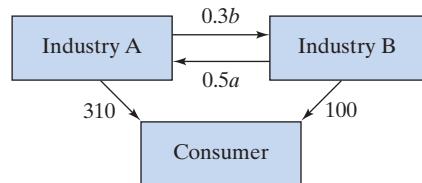
- a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?
  - b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.
24. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input–output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.



What outputs  $a$  and  $b$  (in millions of dollars per year) should the two industries generate to satisfy the demand? You may be tempted to say 1,000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢ worth of electricity for each \$1 of output B produces and that industry A needs 20¢ worth of B's products for each \$1 of output A produces. Find the outputs  $a$  and  $b$  needed to satisfy both consumer and interindustry demand.



25. Find the outputs  $a$  and  $b$  needed to satisfy the consumer and interindustry demands given in the following figure. See Exercise 24:



26. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t).$$

This equation could describe a forced damped oscillator. We are told that the differential equation has a solution of the form

$$x(t) = a \sin(t) + b \cos(t).$$

Find  $a$  and  $b$ , and graph the solution.

27. Find all solutions of the system

$$\begin{vmatrix} 7x - y &= \lambda x \\ -6x + 8y &= \lambda y \end{vmatrix}, \quad \text{for}$$

a.  $\lambda = 5$       b.  $\lambda = 10$ , and      c.  $\lambda = 15$ .

28. On a sunny summer day, you are taking the scenic boat ride from Stein am Rhein, Switzerland, to Schaffhausen, down the Rhein River. This nonstop trip takes 40 minutes, but the return trip to Stein, upstream, will take a full hour. Back in Stein, you decide to stay on the boat and continue on to Constance, Germany, now traveling on the still waters of Lake Constance. How long will this nonstop trip from Stein to Constance take? You may assume that the boat is traveling at a constant speed relative to the water throughout and that the Rhein River flows at a constant speed between Stein and Schaffhausen. The traveling distance from Stein to Schaffhausen is the same as from Stein to Constance.

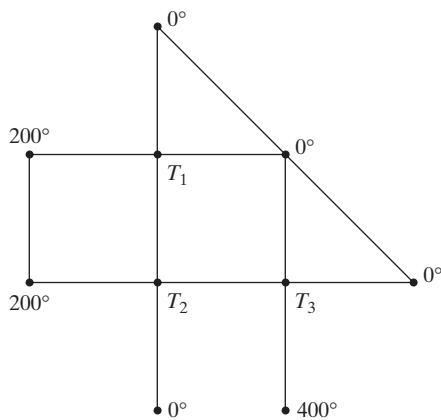
29. On your next trip to Switzerland, you should take the scenic boat ride from Rheinfall to Rheinau and back. The trip downstream from Rheinfall to Rheinau takes 20 minutes, and the return trip takes 40 minutes; the distance between Rheinfall and Rheinau along the river is 8 kilometers. How fast does the boat travel (relative to the water), and how fast does the river Rhein flow in this area? You may assume both speeds to be constant throughout the journey.

30. In a grid of wires, the temperature at exterior mesh points is maintained at constant values (in  $^{\circ}\text{C}$ ), as shown in the accompanying figure. When the grid is in thermal equilibrium, the temperature  $T$  at each interior mesh point is the average of the temperatures at the four adjacent points. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

## Linear Equations

Find the temperatures  $T_1$ ,  $T_2$ , and  $T_3$  when the grid is in thermal equilibrium.



31. Find the polynomial of degree 2 [a polynomial of the form  $f(t) = a + bt + ct^2$ ] whose graph goes through the points  $(1, -1)$ ,  $(2, 3)$ , and  $(3, 13)$ . Sketch the graph of this polynomial.
32. Find a polynomial of degree  $\leq 2$  [of the form  $f(t) = a + bt + ct^2$ ] whose graph goes through the points  $(1, p)$ ,  $(2, q)$ ,  $(3, r)$ , where  $p, q, r$  are arbitrary constants. Does such a polynomial exist for all values of  $p, q, r$ ?
33. Find all the polynomials  $f(t)$  of degree  $\leq 2$  [of the form  $f(t) = a + bt + ct^2$ ] whose graphs run through the points  $(1, 3)$  and  $(2, 6)$ , such that  $f'(1) = 1$  [where  $f'(t)$  denotes the derivative].
34. Find all the polynomials  $f(t)$  of degree  $\leq 2$  [of the form  $f(t) = a + bt + ct^2$ ] whose graphs run through the points  $(1, 1)$  and  $(2, 0)$ , such that  $\int_1^2 f(t) dt = -1$ .
35. Find all the polynomials  $f(t)$  of degree  $\leq 2$  [of the form  $f(t) = a + bt + ct^2$ ] whose graphs run through the points  $(1, 1)$  and  $(3, 3)$ , such that  $f'(2) = 1$ .
36. Find all the polynomials  $f(t)$  of degree  $\leq 2$  [of the form  $f(t) = a + bt + ct^2$ ] whose graphs run through the points  $(1, 1)$  and  $(3, 3)$ , such that  $f'(2) = 3$ .
37. Find the function  $f(t)$  of the form  $f(t) = ae^{3t} + be^{2t}$  such that  $f(0) = 1$  and  $f'(0) = 4$ .
38. Find the function  $f(t)$  of the form  $f(t) = a \cos(2t) + b \sin(2t)$  such that  $f''(t) + 2f'(t) + 3f(t) = 17 \cos(2t)$ . (This is the kind of differential equation you might have to solve when dealing with forced damped oscillators, in physics or engineering.)
39. Find the circle that runs through the points  $(5, 5)$ ,  $(4, 6)$ , and  $(6, 2)$ . Write your equation in the form  $a+bx+cy+x^2+y^2=0$ . Find the center and radius of this circle.
40. Find the ellipse centered at the origin that runs through the points  $(1, 2)$ ,  $(2, 2)$ , and  $(3, 1)$ . Write your equation in the form  $ax^2+bxy+cy^2=1$ .

41. Find all points  $(a, b, c)$  in space for which the system

$$\begin{vmatrix} x + 2y + 3z & = a \\ 4x + 5y + 6z & = b \\ 7x + 8y + 9z & = c \end{vmatrix}$$

has at least one solution.

42. Linear systems are particularly easy to solve when they are in triangular form (i.e., all entries above or below the diagonal are zero).

- a. Solve the lower triangular system

$$\begin{vmatrix} x_1 & = -3 \\ -3x_1 + x_2 & = 14 \\ x_1 + 2x_2 + x_3 & = 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 & = 33 \end{vmatrix}$$

by forward substitution, finding  $x_1$  first, then  $x_2$ , then  $x_3$ , and finally  $x_4$ .

- b. Solve the upper triangular system

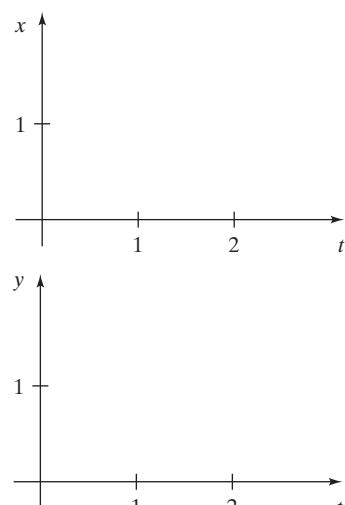
$$\begin{vmatrix} x_1 + 2x_2 - x_3 + 4x_4 & = -3 \\ x_2 + 3x_3 + 7x_4 & = 5 \\ x_3 + 2x_4 & = 2 \\ x_4 & = 0 \end{vmatrix}.$$

43. Consider the linear system

$$\begin{vmatrix} x + y & = 1 \\ x + \frac{t}{2}y & = t \end{vmatrix},$$

where  $t$  is a nonzero constant.

- a. Determine the  $x$ - and  $y$ -intercepts of the lines  $x+y=1$  and  $x+(t/2)y=t$ ; sketch these lines. For which values of the constant  $t$  do these lines intersect? For these values of  $t$ , the point of intersection  $(x, y)$  depends on the choice of the constant  $t$ ; that is, we can consider  $x$  and  $y$  as functions of  $t$ . Draw rough sketches of these functions.



## Linear Equations

Explain briefly how you found these graphs. Argue geometrically, without solving the system algebraically.

- b.** Now solve the system algebraically. Verify that the graphs you sketched in part (a) are compatible with your algebraic solution.
- 44.** Find a system of linear equations with three unknowns whose solutions are the points on the line through  $(1, 1, 1)$  and  $(3, 5, 0)$ .
- 45.** Find a system of linear equations with three unknowns  $x, y, z$  whose solutions are
- $$x = 6 + 5t, \quad y = 4 + 3t, \quad \text{and} \quad z = 2 + t,$$
- where  $t$  is an arbitrary constant.
- 46.** Boris and Marina are shopping for chocolate bars. Boris observes, “If I add half my money to yours, it will be enough to buy two chocolate bars.” Marina naively asks, “If I add half my money to yours, how many can we buy?” Boris replies, “One chocolate bar.” How much money did Boris have? (From Yuri Chernyak and Robert Rose, *The Chicken from Minsk*, Basic Books, 1995.)
- 47.** Here is another method to solve a system of linear equations: Solve one of the equations for one of the variables, and substitute the result into the other equations. Repeat this process until you run out of variables or equations. Consider the example of yields per bundle of rice discussed earlier:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right|$$

We can solve the first equation for  $x$ :

$$x = 39 - 2y - 3z.$$

Then we substitute this equation into the other equations:

$$\left| \begin{array}{l} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{array} \right|$$

We can simplify:

$$\left| \begin{array}{l} y - z = -5 \\ -4y - 8z = -91 \end{array} \right|$$

Now,  $y = z - 5$ , so that  $-4(z - 5) - 8z = -91$ , or

$$-12z = -111.$$

We find that  $z = \frac{111}{12} = 9.25$ . Then

$$y = z - 5 = 4.25,$$

and

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the same as the method discussed in this section; only the bookkeeping is different.

- 48.** A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates.
- a.** If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?
  - b.** If the hermit wants to take in  $P$  grams of protein and  $C$  grams of carbohydrates per day, how many servings of each item should he consume?
- 49.** I have 32 bills in my wallet, in the denominations of US\$ 1, 5, and 10, worth \$100 in total. How many do I have of each denomination?
- 50.** Some parking meters in Milan, Italy, accept coins in the denominations of 20¢, 50¢, and €2. As an incentive program, the city administrators offer a big reward (a brand new Ferrari Testarossa) to any meter maid who brings back exactly 1,000 coins worth exactly €1,000 from the daily rounds. What are the odds of this reward being claimed anytime soon?

## 2 Matrices, Vectors, and Gauss–Jordan Elimination

When mathematicians in ancient China had to solve a system of simultaneous linear equations such as<sup>4</sup>

$$\left| \begin{array}{l} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{array} \right|$$

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<sup>4</sup>This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*. Our source is George Gheverghese Joseph, *The Crest of the Peacock, Non-European Roots of Mathematics*, 3rd ed., Princeton University Press, 2010.

## Linear Equations

they took all the numbers involved in this system and arranged them in a rectangular pattern (*Fang Cheng* in Chinese), as follows:<sup>5</sup>

3	21	-3	0
-6	-2	-1	62
2	-3	8	32

All the information about this system is conveniently stored in this array of numbers.

The entries were represented by bamboo rods, as shown below; red and black rods stand for positive and negative numbers, respectively. (Can you detect how this number system works?) The equations were then solved in a hands-on fashion, by manipulating the rods. We leave it to the reader to find the solution.

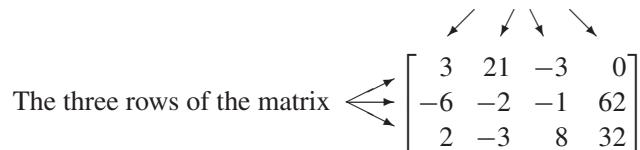
	=		
—			—

Today, such a rectangular array of numbers,

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix},$$

is called a *matrix*.<sup>6</sup> Since this particular matrix has three rows and four columns, it is called a  $3 \times 4$  matrix (“three by four”).

The four columns of the matrix



$$\begin{array}{c} \swarrow \quad \downarrow \quad \searrow \quad \nwarrow \\ \text{The three rows of the matrix} \quad \left[ \begin{array}{cccc} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{array} \right] \end{array}$$

Note that the first column of this matrix corresponds to the first variable of the system, while the first row corresponds to the first equation.

It is customary to label the entries of a  $3 \times 4$  matrix  $A$  with double subscripts as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

The first subscript refers to the row, and the second to the column: The entry  $a_{ij}$  is located in the  $i$ th row and the  $j$ th column.

Two matrices  $A$  and  $B$  are equal if they are the same size and if corresponding entries are equal:  $a_{ij} = b_{ij}$ .

If the number of rows of a matrix  $A$  equals the number of columns ( $A$  is  $n \times n$ ), then  $A$  is called a *square matrix*, and the entries  $a_{11}, a_{22}, \dots, a_{nn}$  form the (main) *diagonal* of  $A$ . A square matrix  $A$  is called *diagonal* if all its entries above and below

<sup>5</sup>Actually, the roles of rows and columns were reversed in the Chinese representation.

<sup>6</sup>It appears that the term *matrix* was first used in this sense by the English mathematician J. J. Sylvester, in 1850.

## Linear Equations

the main diagonal are zero; that is,  $a_{ij} = 0$  whenever  $i \neq j$ . A square matrix  $A$  is called *upper triangular* if all its entries below the main diagonal are zero; that is,  $a_{ij} = 0$  whenever  $i$  exceeds  $j$ . *Lower triangular* matrices are defined analogously. A matrix whose entries are all zero is called a *zero matrix* and is denoted by 0 (regardless of its size). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

The matrices  $B$ ,  $C$ ,  $D$ , and  $E$  are square,  $C$  is diagonal,  $C$  and  $D$  are upper triangular, and  $C$  and  $E$  are lower triangular.

Matrices with only one column or row are of particular interest.

### Vectors and vector spaces

A matrix with only one column is called a column vector, or simply a vector. The entries of a vector are called its components. The set of all column vectors with  $n$  components is denoted by  $\mathbb{R}^n$ ; we will refer to  $\mathbb{R}^n$  as a *vector space*.

A matrix with only one row is called a row vector.

In this text, the term *vector* refers to column vectors, unless otherwise stated. The reason for our preference for column vectors will become apparent in the next section.

Examples of vectors are

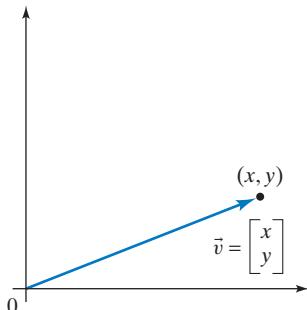
$$\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix},$$

a (column) vector in  $\mathbb{R}^4$ , and

$$\begin{bmatrix} 1 & 5 & 5 & 3 & 7 \end{bmatrix},$$

a row vector with five components. Note that the  $m$  columns of an  $n \times m$  matrix are vectors in  $\mathbb{R}^n$ .

In previous courses in mathematics or physics, you may have thought about vectors from a more geometric point of view. Let's establish some conventions regarding the geometric representation of vectors.



**Figure 1**

### Standard representation of vectors

The standard representation of a vector

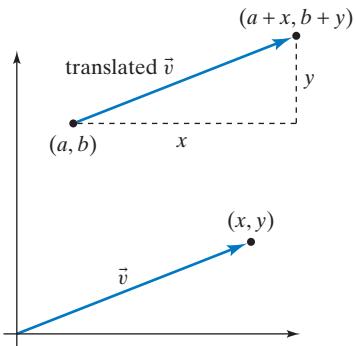
$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

in the Cartesian coordinate plane is as an *arrow* (a directed line segment) from the origin to the point  $(x, y)$ , as shown in Figure 1.

The standard representation of a vector in  $\mathbb{R}^3$  is defined analogously.

Here we will consider the standard representation of vectors, unless stated otherwise.

## Linear Equations

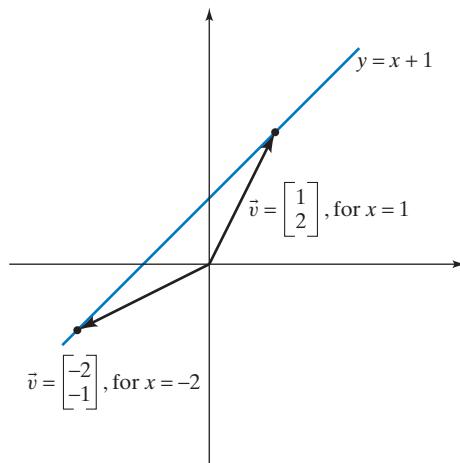


**Figure 2**

Occasionally, it is helpful to translate (or shift) a vector in the plane (preserving its direction and length), so that it will connect some point  $(a, b)$  to the point  $(a + x, b + y)$ , as shown in Figure 2.

When considering an infinite set of vectors, the arrow representation becomes impractical. In this case, it is sensible to represent the vector  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$  simply by the point  $(x, y)$ , the head of the standard arrow representation of  $\vec{v}$ .

For example, the set of all vectors  $\vec{v} = \begin{bmatrix} x \\ x+1 \end{bmatrix}$  (where  $x$  is arbitrary) can be represented as the line  $y = x + 1$ . For a few special values of  $x$  we may still use the arrow representation, as illustrated in Figure 3.



**Figure 3**

In this course, it will often be helpful to think about a vector numerically, as a list of numbers, which we will usually write in a column.

In our digital age, information is often transmitted and stored as a string of numbers (i.e., as a vector). A 10-second clip of music on a CD is stored as a vector with 440,000 components. A weather photograph taken by a satellite is transmitted to Earth as a string of numbers.

Consider the system

$$\begin{vmatrix} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{vmatrix}.$$

Sometimes we are interested in the matrix

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix},$$

which contains the coefficients of the variables in the system, called its *coefficient matrix*.

By contrast, the matrix

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix},$$

which displays all the numerical information contained in the system is called its *augmented matrix*. For the sake of clarity, we will often indicate the position of the equal signs in the equations by a dotted line:

## Linear Equations

$$\left[ \begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right].$$

To solve the system, it is more efficient to perform the elimination on the augmented matrix rather than on the equations themselves. Conceptually, the two approaches are equivalent, but working with the augmented matrix requires less writing yet is easier to read, with some practice. Instead of dividing an *equation* by a scalar,<sup>7</sup> you can divide a *row* by a scalar. Instead of adding a multiple of an equation to another equation, you can add a multiple of a row to another row.

As you perform elimination on the augmented matrix, you should always remember the linear system lurking behind the matrix. To illustrate this method, we perform the elimination both on the augmented matrix and on the linear system it represents:

$$\begin{array}{ll}
 \left[ \begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right] \div 2 & \left| \begin{array}{l} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{array} \right. \div 2 \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right] -2(\text{I}) & \left| \begin{array}{l} x + 4y + 2z = 1 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{array} \right. -2(\text{I}) \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array} \right] \div(-3) & \left| \begin{array}{l} x + 4y + 2z = 1 \\ -3y - 3z = 3 \\ -6y - 9z = -3 \end{array} \right. \div(-3) \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{array} \right] -4(\text{II}) & \left| \begin{array}{l} x + 4y + 2z = 1 \\ y + z = -1 \\ -6y - 9z = -3 \end{array} \right. -4(\text{II}) \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array} \right] \div(-3) & \left| \begin{array}{l} x - 2z = 5 \\ y + z = -1 \\ -3z = -9 \end{array} \right. \div(-3) \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] +2(\text{III}) & \left| \begin{array}{l} x - 2z = 5 \\ y + z = -1 \\ z = 3 \end{array} \right. +2(\text{III}) \\
 \downarrow & \downarrow \\
 \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array} \right] & \left| \begin{array}{l} x = 11 \\ y = -4 \\ z = 3 \end{array} \right.
 \end{array}$$

---

<sup>7</sup>In vector and matrix algebra, the term *scalar* is synonymous with (real) number.

## Linear Equations

The solution is often represented as a vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}.$$

Thus far we have been focusing on systems of three linear equations with three unknowns. Next we will develop a technique for solving systems of linear equations of arbitrary size.

Here is an example of a system of three linear equations with five unknowns:

$$\left| \begin{array}{ccc} x_1 - x_2 & + 4x_5 & = 2 \\ x_3 & - x_5 & = 2 \\ x_4 & - x_5 & = 3 \end{array} \right|.$$

We solve each equation for the leading variable:

$$\left| \begin{array}{ccc} x_1 & = 2 + x_2 - 4x_5 \\ x_3 & = 2 & + x_5 \\ x_4 & = 3 & + x_5 \end{array} \right|.$$

Now we can freely choose values for the nonleading variables,  $x_2 = t$  and  $x_5 = r$ , for example. The leading variables are then determined by these choices:

$$x_1 = 2 + t - 4r, \quad x_3 = 2 + r, \quad x_4 = 3 + r.$$

This system has infinitely many solutions; we can write the solutions in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & +t & -4r \\ & t & \\ 2 & & +r \\ 3 & & +r \\ & & r \end{bmatrix}.$$

Again, you can check this answer by substituting the solutions into the original equations, for example,  $x_3 - x_5 = (2 + r) - r = 2$ .

What makes this system so easy to solve? The following three properties are responsible for the simplicity of the solution, with the second property playing a key role:

- P1: The leading coefficient in each equation is 1. (The leading coefficient is the coefficient of the leading variable.)
- P2: The leading variable in each equation does not appear in any of the other equations. (For example, the leading variable  $x_3$  of the second equation appears neither in the first nor in the third equation.)
- P3: The leading variables appear in the “natural order,” with increasing indices as we go down the system ( $x_1, x_3, x_4$  as opposed to  $x_3, x_1, x_4$ , for example).

Whenever we encounter a linear system with these three properties, we can solve for the leading variables and then choose arbitrary values for the other, nonleading variables.

Now we are ready to tackle the case of an arbitrary system of linear equations. We will illustrate our approach by means of an example:

## Linear Equations

$$\left| \begin{array}{l} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array} \right|.$$

We wish to reduce this system to a system satisfying the three properties (P1, P2, and P3); this reduced system will then be easy to solve.

We will proceed from equation to equation, from top to bottom. The leading variable in the first equation is  $x_1$ , with leading coefficient 2. To satisfy property P1, we will divide this equation by 2. To satisfy property P2 for the variable  $x_1$ , we will then subtract suitable multiples of the first equation from the other three equations to eliminate the variable  $x_1$  from those equations. We will perform these operations both on the system and on the augmented matrix.

$$\begin{array}{c} \left| \begin{array}{l} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array} \right| \div 2 \quad \left[ \begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array} \right] \div 2 \\ \downarrow \qquad \qquad \qquad \downarrow \\ \left| \begin{array}{l} x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 1 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array} \right| - (I) \quad \left[ \begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array} \right] - (I) \\ -3(I) \quad -5(I) \quad -3(I) \quad -5(I) \\ \downarrow \qquad \qquad \qquad \downarrow \\ \left| \begin{array}{l} x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 1 \\ x_4 - 2x_5 = 3 \\ x_3 - 2x_4 + 3x_5 = -2 \\ x_3 - x_5 = 4 \end{array} \right| \quad \left[ \begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] \end{array}$$

Now on to the second equation, with leading variable  $x_4$ . Since the leading coefficient is 1 already, all we need to do is eliminate  $x_4$  from the other equations. Then we will proceed to the third equation and use the same approach.

$$\begin{array}{c} \left| \begin{array}{l} x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 1 \\ x_4 - 2x_5 = 3 \\ x_3 - 2x_4 + 3x_5 = -2 \\ x_3 - x_5 = 4 \end{array} \right| - (II) \quad \left[ \begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] - (II) \\ +2(II) \quad +2(II) \\ \downarrow \qquad \qquad \qquad \downarrow \\ \left| \begin{array}{l} x_1 + 2x_2 - x_3 + 4x_5 = -2 \\ x_4 - 2x_5 = 3 \\ x_3 - x_5 = 4 \\ x_3 - x_5 = 4 \end{array} \right| + (III) \quad \left[ \begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] + (III) \\ - (III) \quad - (III) \\ \downarrow \qquad \qquad \qquad \downarrow \\ \left| \begin{array}{l} x_1 + 2x_2 + 3x_5 = 2 \\ x_4 - 2x_5 = 3 \\ x_3 - x_5 = 4 \\ 0 = 0 \end{array} \right| \quad \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

There are no variables left in the fourth equation, and we are almost done. Our system now satisfies properties P1 and P2, but not P3: The leading variables  $x_3$  and

## Linear Equations

$x_4$  appear in reversed order. This is not a big problem, though: We can rearrange the order of the equations, swapping equations (II) and (III).

$$\left| \begin{array}{ccccc|c} x_1 + 2x_2 & & + 3x_5 & = & 2 \\ & x_3 & - x_5 & = & 4 \\ & & x_4 - 2x_5 & = & 3 \\ & & & 0 & = & 0 \end{array} \right| \quad \left| \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|$$

Our system now satisfies properties P1, P2, and P3. We can solve each equation for its leading variable:

$$\left| \begin{array}{l} x_1 = 2 - 2x_2 - 3x_5 \\ x_3 = 4 + x_5 \\ x_4 = 3 + 2x_5 \end{array} \right|.$$

If we let  $x_2 = t$  and  $x_5 = r$ , then the infinitely many solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & -2t & -3r \\ & t & \\ 4 & & + r \\ 3 & & + 2r \\ & & r \end{bmatrix}.$$

Let us summarize.

### Solving a system of linear equations

We proceed from equation to equation, from top to bottom.

Suppose we get to the  $i$ th equation, with leading variable  $x_j$  and leading (nonzero) coefficient  $c$ , so that the equation will be of the form  $cx_j + \dots = b$ .

Divide the  $i$ th equation by  $c$  to bring it into the form  $x_j + \dots = b/c$ .

Eliminate  $x_j$  from all the other equations, above and below the  $i$ th equation, by subtracting suitable multiples of the  $i$ th equation.

Now proceed to the next equation.

If an equation  $\text{zero} = \text{nonzero}$  emerges in this process, then the system is inconsistent, and there are no solutions.

If you get through the system without encountering an inconsistency, then rearrange the equations so that the leading variables appear in the “natural order” (see property P3; this can be accomplished by a sequence of swaps of equations).

Solve each equation for its leading variable. You may freely choose values for the nonleading variables; the leading variables are then determined by these choices.

Take another look at the preceding example to see how this algorithm works in practice.

This process can be performed on the augmented matrix. As you do so, just imagine the linear system lurking behind the matrix.

In the preceding example, we reduced the augmented matrix

$$M = \left[ \begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array} \right] \quad \text{to} \quad E = \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

We say that the final matrix  $E$  is the *reduced row-echelon form* of  $M$ , written  $E = \text{rref}(M)$ .

## Linear Equations

You can use computational software (Mathematica, MATLAB, Maple, etc.) or the computational search engine Wolfram Alpha to find the reduced row-echelon form of a given matrix.

### Reduced row-echelon form

A matrix is said to be in *reduced row-echelon form* (rref) if it satisfies all of the following conditions:

- If a row has nonzero entries, then the first nonzero entry is a 1, called the *leading 1* (or *pivot*) in this row.
- If a column contains a leading 1, then all the other entries in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Condition c implies that rows of 0's, if any, appear at the bottom of the matrix.

Conditions a, b, and c defining the reduced row-echelon form correspond to the conditions P1, P2, and P3 that we imposed on the system.

Note that the leading 1's in the matrix

$$E = \left[ \begin{array}{ccccc|c} (1) & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & (1) & 0 & -1 & 4 \\ 0 & 0 & 0 & (1) & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

correspond to the leading variables in the reduced system,

$$\left| \begin{array}{ccccc|c} (x_1) & + & 2x_2 & & & + & 3x_5 & = & 2 \\ & & & \xrightarrow{x_3} & & - & x_5 & = & 4 \\ & & & & \xrightarrow{x_4} & - & 2x_5 & = & 3 \end{array} \right|.$$

Here we draw the staircase formed by the leading variables. This is where the name *echelon form* comes from. According to Webster, an echelon is a formation “like a series of steps.”

The operations we perform when bringing a matrix into reduced row-echelon form are referred to as elementary row operations. Let’s review the three types of such operations.

### Types of elementary row operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

Consider the following system:

$$\left| \begin{array}{cccc|c} x_1 & - & 3x_2 & - & 5x_4 & = & -7 \\ 3x_1 & - & 12x_2 & - & 2x_3 & - & 27x_4 & = & -33 \\ -2x_1 & + & 10x_2 & + & 2x_3 & + & 24x_4 & = & 29 \\ -x_1 & + & 6x_2 & + & x_3 & + & 14x_4 & = & 17 \end{array} \right|.$$

## Linear Equations

The augmented matrix is

$$\left[ \begin{array}{cccc|c} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ -2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array} \right].$$

The reduced row-echelon form for this matrix is

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

(We leave it to you to perform the elimination.)

Since the last row of the echelon form represents the equation  $0 = 1$ , the system is inconsistent.

This method of solving linear systems is sometimes referred to as *Gauss–Jordan elimination*, after the German mathematician Carl Friedrich Gauss (1777–1855; see Figure 4), perhaps the greatest mathematician of modern times, and the German engineer Wilhelm Jordan (1844–1899). Gauss himself called the method *eliminatio vulgaris*. Recall that the Chinese were using this method 2,000 years ago.



**Figure 4** Carl Friedrich Gauss appears on an old German 10-mark note. (In fact, this is the mirror image of a well-known portrait of Gauss.<sup>8</sup>)

How Gauss developed this method is noteworthy. On January 1, 1801, the Sicilian astronomer Giuseppe Piazzi (1746–1826) discovered a planet, which he named Ceres, in honor of the patron goddess of Sicily. Today, Ceres is called a dwarf planet, because it is only about 1,000 kilometers in diameter. Piazzi was able to observe Ceres for 40 nights, but then he lost track of it. Gauss, however, at the age of 24, succeeded in calculating the orbit of Ceres, even though the task seemed hopeless on the basis of a few observations. His computations were so accurate that the German astronomer W. Olbers (1758–1840) located the planet on December 31, 1801. In the course of his computations, Gauss had to solve systems of 17 linear equations.<sup>9</sup> In dealing with this problem, Gauss also used the method of least

<sup>8</sup>Reproduced by permission of the German Bundesbank.

<sup>9</sup>For the mathematical details, see D. Teets and K. Whitehead, “The Discovery of Ceres: How Gauss Became Famous,” *Mathematics Magazine*, 72, 2 (April 1999): 83–93.

## Linear Equations

squares, which he had developed around 1794. Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Gauss later described his methods of orbit computation in his book *Theoria Motus Corporum Coelestium* (1809).

The method of solving a linear system by Gauss–Jordan elimination is called an *algorithm*.<sup>10</sup> An algorithm can be defined as “a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete Steps, 1, 2, 3, . . . , whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end” (David Berlinski, *The Advent of the Algorithm: The Idea That Rules the World*, Harcourt Inc., 2000).

Gauss–Jordan elimination is well suited for solving linear systems on a computer, at least in principle. In practice, however, some tricky problems associated with roundoff errors can occur.

Numerical analysts tell us that we can reduce the proliferation of roundoff errors by modifying Gauss–Jordan elimination, employing more sophisticated reduction techniques.

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix  $A$  into a matrix  $B$  by a sequence of elementary row operations and if  $B$  is in reduced row-echelon form, is it necessarily true that  $B = \text{rref}(A)$ ? Fortunately (and perhaps surprisingly), this is indeed the case.

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<sup>10</sup> The word *algorithm* is derived from the name of the mathematician al-Khowarizmi, who introduced the term *algebra* into mathematics.

## EXERCISES 2

**GOAL** Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work.

1. 
$$\begin{vmatrix} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{vmatrix}$$

2. 
$$\begin{vmatrix} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{vmatrix}$$

3. 
$$x + 2y + 3z = 4$$

4. 
$$\begin{vmatrix} x + y &= 1 \\ 2x - y &= 5 \\ 3x + 4y &= 2 \end{vmatrix}$$

5. 
$$\begin{vmatrix} x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \\ x_1 + x_2 &= 0 \\ x_1 &+ x_4 = 0 \end{vmatrix}$$

6. 
$$\begin{vmatrix} x_1 - 7x_2 &+ x_5 &= 3 \\ &x_3 - 2x_5 &= 2 \\ &x_4 + x_5 &= 1 \end{vmatrix}$$

7. 
$$\begin{vmatrix} x_1 + 2x_2 && 2x_4 + 3x_5 &= 0 \\ &x_3 + 3x_4 + 2x_5 &= 0 \\ &x_3 + 4x_4 - x_5 &= 0 \\ &&&x_5 = 0 \end{vmatrix}$$

8. Solve this system for the variables  $x_1, x_2, x_3, x_4$ , and  $x_5$ .

$$\begin{vmatrix} x_2 + 2x_4 + 3x_5 &= 0 \\ 4x_4 + 8x_5 &= 0 \end{vmatrix}$$

9. 
$$\begin{vmatrix} x_4 + 2x_5 - x_6 &= 2 \\ x_1 + 2x_2 &+ x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 &- x_5 + x_6 = 2 \end{vmatrix}$$

10. 
$$\begin{vmatrix} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{vmatrix}$$

11. 
$$\begin{vmatrix} x_1 + 2x_3 + 4x_4 &= -8 \\ x_2 - 3x_3 - x_4 &= 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0 \\ -x_2 + 3x_3 + 4x_4 &= -12 \end{vmatrix}$$

## Linear Equations

12. 
$$\begin{array}{l} \left| \begin{array}{ccc|cc} 2x_1 & -3x_3 & +7x_5 & +7x_6 & =0 \\ -2x_1 & +x_2 & +6x_3 & -6x_5 & -12x_6 & =0 \\ & x_2 & -3x_3 & +x_5 & +5x_6 & =0 \\ & -2x_2 & +x_4 & +x_5 & +x_6 & =0 \\ 2x_1 & +x_2 & -3x_3 & +8x_5 & +7x_6 & =0 \end{array} \right| \\ \hline \end{array}$$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. 
$$\begin{array}{l} \left| \begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array} \right| \\ \hline \end{array}$$

14. 
$$\begin{array}{l} \left| \begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array} \right| \\ \hline \end{array}$$

15. 
$$\begin{array}{l} \left| \begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array} \right| \\ \hline \end{array}$$

16. 
$$\begin{array}{l} \left| \begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array} \right| \\ \hline \end{array}$$

17. 
$$\begin{array}{l} \left| \begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array} \right| \\ \hline \end{array}$$

18. Determine which of the matrices below are in reduced row-echelon form:

a. 
$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b. 
$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c. 
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

d. 
$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

19. Find all  $4 \times 1$  matrices in reduced row-echelon form.

20. For which values of  $a, b, c, d$ , and  $e$  is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 0 & a & 2 & 1 & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & e & 0 & 0 \end{bmatrix}$$

21. For which values of  $a, b, c, d$ , and  $e$  is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

22. We say that two  $n \times m$  matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are of the same type. How many types of  $2 \times 2$  matrices in reduced row-echelon form are there?

23. How many types of  $3 \times 2$  matrices in reduced row-echelon form are there? See Exercise 22.

24. How many types of  $2 \times 3$  matrices in reduced row-echelon form are there? See Exercise 22.

25. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

26. Suppose matrix  $A$  is transformed into matrix  $B$  by means of an elementary row operation. Is there an elementary row operation that transforms  $B$  into  $A$ ? Explain.

27. Suppose matrix  $A$  is transformed into matrix  $B$  by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms  $B$  into  $A$ ? Explain your answer. See Exercise 26.

28. Consider an  $n \times m$  matrix  $A$ . Can you transform  $\text{rref}(A)$  into  $A$  by a sequence of elementary row operations? See Exercise 27.

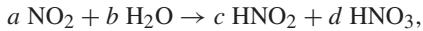
29. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

30. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

31. *Balancing a chemical reaction.* Consider the chemical reaction



where  $a, b, c$ , and  $d$  are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

$$2a + b = 2c + 3d.$$

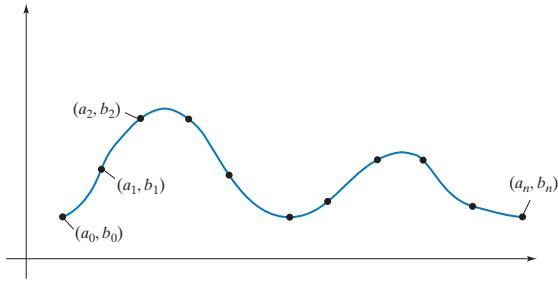
While there are many possible values for  $a, b, c$ , and  $d$  that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

32. Find the polynomial of degree 3 [a polynomial of the form  $f(t) = a + bt + ct^2 + dt^3$ ] whose graph goes through the points  $(0, 1), (1, 0), (-1, 0)$ , and  $(2, -15)$ . Sketch the graph of this cubic.

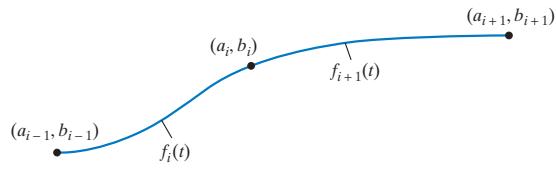
33. Find the polynomial of degree 4 whose graph goes through the points  $(1, 1), (2, -1), (3, -59), (-1, 5)$ , and  $(-2, -29)$ . Graph this polynomial.

## Linear Equations

- 34. Cubic splines.** Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points  $(a_i, b_i)$  are given to you, and your job is to connect the dots in a reasonably smooth way. Let  $a_{i+1} > a_i$ , for  $i = 0, \dots, n - 1$ .



One method often employed in such design problems is the technique of cubic splines. We choose  $f_i(t)$ , a polynomial of degree  $\leq 3$ , to define the shape of the ride between  $(a_{i-1}, b_{i-1})$  and  $(a_i, b_i)$ , for  $i = 1, \dots, n$ .



Obviously, it is required that  $f_i(a_i) = b_i$  and  $f_i(a_{i-1}) = b_{i-1}$ , for  $i = 1, \dots, n$ . To guarantee a smooth ride at the points  $(a_i, b_i)$ , we want the first and second derivatives of  $f_i$  and  $f_{i+1}$  to agree at these points:

$$\begin{aligned} f'_i(a_i) &= f'_{i+1}(a_i) && \text{and} \\ f''_i(a_i) &= f''_{i+1}(a_i), && \text{for } i = 1, \dots, n - 1. \end{aligned}$$

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

$$f'_1(a_0) = f'_n(a_n) = 0.$$

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (Note: It can be shown that this system has a unique solution.)

- 35.** Find the polynomial  $f(t)$  of degree 3 such that  $f(1) = 1$ ,  $f(2) = 5$ ,  $f'(1) = 2$ , and  $f'(2) = 9$ , where  $f'(t)$  is the derivative of  $f(t)$ . Graph this polynomial.

- 36.** The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in  $\mathbb{R}^n$  is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors  $\vec{x}$  and  $\vec{y}$  are *perpendicular* if  $\vec{x} \cdot \vec{y} = 0$ .

Find all vectors in  $\mathbb{R}^3$  perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

- 37.** Find all vectors in  $\mathbb{R}^4$  that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

See Exercise 36.

- 38.** Find all solutions  $x_1, x_2, x_3$  of the equation

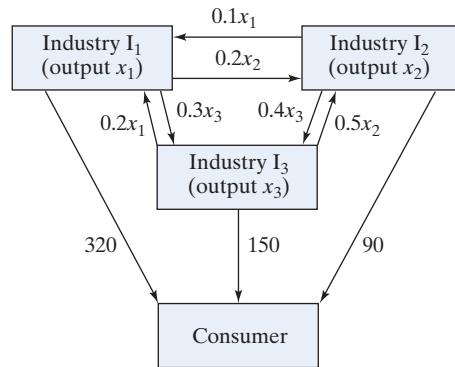
$$\vec{b} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

- 39.** For some background on this exercise, see Exercise 1.24.

Consider an economy with three industries,  $I_1$ ,  $I_2$ ,  $I_3$ . What outputs  $x_1, x_2, x_3$  should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



- 40.** If we consider more than three industries in an input-output model, it is cumbersome to represent all the demands in a diagram as in Exercise 39. Suppose we have the industries  $I_1, I_2, \dots, I_n$ , with outputs  $x_1, x_2, \dots, x_n$ . The *output vector* is

## Linear Equations

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The *consumer demand vector* is

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where  $b_i$  is the consumer demand on industry  $I_i$ . The demand vector for industry  $I_j$  is

$$\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

where  $a_{ij}$  is the demand industry  $I_j$  puts on industry  $I_i$ , for each \$1 of output industry  $I_j$  produces. For example,  $a_{32} = 0.5$  means that industry  $I_2$  needs 50¢ worth of products from industry  $I_3$  for each \$1 worth of goods  $I_2$  produces. The coefficient  $a_{ii}$  need not be 0: Producing a product may require goods or services from the same industry.

- a. Find the four demand vectors for the economy in Exercise 39.
- b. What is the meaning in economic terms of  $x_j \vec{v}_j$ ?
- c. What is the meaning in economic terms of  $x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n + \vec{b}$ ?
- d. What is the meaning in economic terms of the equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n + \vec{b} = \vec{x}?$$

41. Consider the economy of Israel in 1958.<sup>11</sup> The three industries considered here are

- $I_1$  : agriculture,
- $I_2$  : manufacturing,
- $I_3$  : energy.

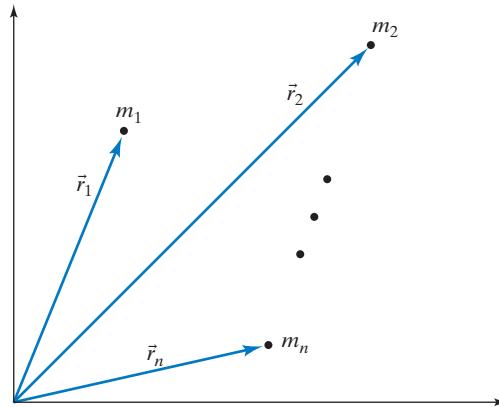
Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time. We are told that

$$\vec{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.$$

- a. Why do the first components of  $\vec{v}_2$  and  $\vec{v}_3$  equal 0?
- b. Find the outputs  $x_1, x_2, x_3$  required to satisfy demand.

42. Consider some particles in the plane with position vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  and masses  $m_1, m_2, \dots, m_n$ .



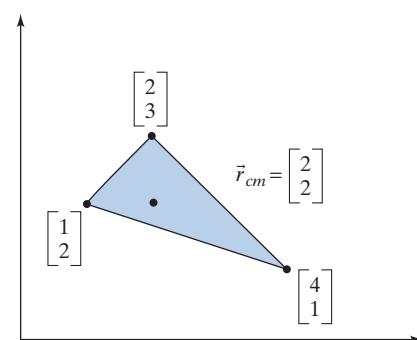
The position vector of the *center of mass* of this system is

$$\vec{r}_{cm} = \frac{1}{M} (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n),$$

where  $M = m_1 + m_2 + \dots + m_n$ .

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that

the plate can be supported at the point  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ; that is,  $\vec{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ? Assume that the mass of the plate itself is negligible.



43. The *momentum*  $\vec{P}$  of a system of  $n$  particles in space with masses  $m_1, m_2, \dots, m_n$  and velocities  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  is defined as

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n.$$

Now consider two elementary particles with velocities

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.$$

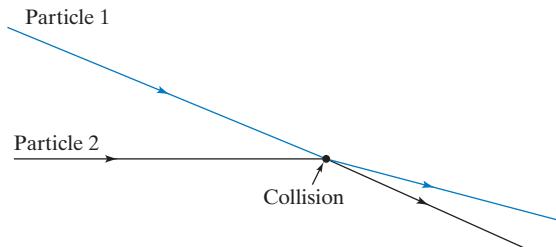
<sup>11</sup>W. Leontief, *Input–Output Economics*, Oxford University Press, 1966.

## Linear Equations

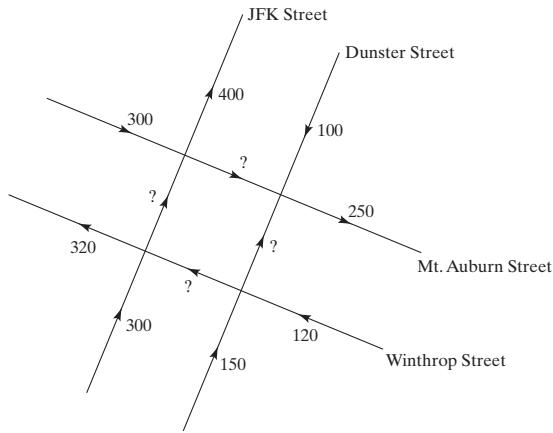
The particles collide. After the collision, their respective velocities are observed to be

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

Assume that the momentum of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? See the accompanying figure.



44. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

45. Let  $S(t)$  be the length of the  $t$ th day of the year 2013 in Mumbai (formerly known as Bombay), India (measured in hours, from sunrise to sunset). We are given the following values of  $S(t)$ :

$t$	$S(t)$
47	11.5
74	12
273	12

For example,  $S(47) = 11.5$  means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function  $S(t)$  is well approximated by a trigonometric function of the form

$$S(t) = a + b \cos\left(\frac{2\pi t}{365}\right) + c \sin\left(\frac{2\pi t}{365}\right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

46. Kyle is getting some flowers for Olivia, his Valentine. Being of a precise analytical mind, he plans to spend exactly \$24 on a bunch of exactly two dozen flowers. At the flower market they have lilies (\$3 each), roses (\$2 each), and daisies (\$0.50 each). Kyle knows that Olivia loves lilies; what is he to do?

47. Consider the equations

$$\begin{cases} x + 2y + & 3z = 4 \\ x + ky + & 4z = 6 \\ x + 2y + (k+2)z = 6 \end{cases},$$

where  $k$  is an arbitrary constant.

- a. For which values of the constant  $k$  does this system have a unique solution?
- b. When is there no solution?
- c. When are there infinitely many solutions?

48. Consider the equations

$$\begin{cases} y + 2kz = 0 \\ x + 2y + 6z = 2 \\ kx + 2z = 1 \end{cases},$$

where  $k$  is an arbitrary constant.

- a. For which values of the constant  $k$  does this system have a unique solution?
- b. When is there no solution?
- c. When are there infinitely many solutions?

49. a. Find all solutions  $x_1, x_2, x_3, x_4$  of the system  $x_2 = \frac{1}{2}(x_1 + x_3)$ ,  $x_3 = \frac{1}{2}(x_2 + x_4)$ .  
b. In part (a), is there a solution with  $x_1 = 1$  and  $x_4 = 13$ ?

50. For an arbitrary positive integer  $n \geq 3$ , find all solutions  $x_1, x_2, x_3, \dots, x_n$  of the simultaneous equations  $x_2 = \frac{1}{2}(x_1 + x_3)$ ,  $x_3 = \frac{1}{2}(x_2 + x_4)$ ,  $\dots$ ,  $x_{n-1} = \frac{1}{2}(x_{n-2} + x_n)$ . Note that we are asked to solve the simultaneous equations  $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$ , for  $k = 2, 3, \dots, n-1$ .

51. Consider the system

$$\begin{cases} 2x + y &= C \\ 3y + z &= C \\ x + 4z &= C \end{cases},$$

where  $C$  is a constant. Find the smallest positive integer  $C$  such that  $x$ ,  $y$ , and  $z$  are all integers.

## Linear Equations

- 52.** Find all the polynomials  $f(t)$  of degree  $\leq 3$  such that  $f(0) = 3$ ,  $f(1) = 2$ ,  $f(2) = 0$ , and  $\int_0^2 f(t) dt = 4$ . (If you have studied Simpson's rule in calculus, explain the result.)

*Exercises 53 through 62 are concerned with conics. A conic is a curve in  $\mathbb{R}^2$  that can be described by an equation of the form  $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$ , where at least one of the coefficients  $c_i$  is nonzero. Examples are circles, ellipses, hyperbolas, and parabolas. If  $k$  is any nonzero constant, then the equations  $f(x, y) = 0$  and  $kf(x, y) = 0$  describe the same conic. For example, the equation  $-4 + x^2 + y^2 = 0$  and  $-12 + 3x^2 + 3y^2 = 0$  both describe the circle of radius 2 centered at the origin. In Exercises 53 through 62, find all the conics through the given points. If there is a unique conic, make a rough sketch. If there are infinitely many conics, sketch two of them.*

**53.**  $(0, 0), (1, 0), (2, 0), (0, 1)$ , and  $(0, 2)$ .

**54.**  $(0, 0), (2, 0), (0, 2), (2, 2)$ , and  $(1, 3)$ .

**55.**  $(0, 0), (1, 0), (2, 0), (3, 0)$ , and  $(1, 1)$ .

**56.**  $(0, 0), (1, 1), (2, 2), (3, 3)$ , and  $(1, 0)$ .

**57.**  $(0, 0), (1, 0), (0, 1)$ , and  $(1, 1)$ .

**58.**  $(0, 0), (1, 0), (0, 1)$ , and  $(1, -1)$ .

**59.**  $(5, 0), (1, 2), (2, 1), (8, 1)$ , and  $(2, 9)$ .

**60.**  $(1, 0), (2, 0), (2, 2), (5, 2)$ , and  $(5, 6)$ .

**61.**  $(0, 0), (1, 0), (2, 0), (0, 1), (0, 2)$ , and  $(1, 1)$ .

**62.**  $(0, 0), (2, 0), (0, 2), (2, 2), (1, 3)$ , and  $(4, 1)$ .

- 63.** Students are buying books for the new semester. Eddie buys the environmental statistics book and the set theory book for \$178. Leah, who is buying books for herself and her friend, spends \$319 on two environmental statistics books, one set theory book, and one educational psychology book. Mehmet buys the educational psychology book and the set theory book for \$147 in total. How much does each book cost?

- 64.** Students are buying books for the new semester. Brigitte buys the German grammar book and the German novel, *Die Leiden des jungen Werther*, for €64 in total. Claude spends €98 on the linear algebra text and the German grammar book, while Denise buys the linear algebra text and *Werther*, for €76. How much does each of the three books cost?

- 65.** At the beginning of a political science class at a large university, the students were asked which term, *liberal* or *conservative*, best described their political views. They were asked the same question at the end of the course, to see what effect the class discussions had on their views. Of those that characterized themselves as "liberal" initially, 30% held conservative views at the end. Of those who were conservative initially, 40%

moved to the liberal camp. It turned out that there were just as many students with conservative views at the end as there had been liberal students at the beginning. Out of the 260 students in the class, how many held liberal and conservative views at the beginning of the course and at the end? (No students joined or dropped the class between the surveys, and they all participated in both surveys.)

- 66.** At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in Section A switch to Section B in the first few weeks of class, while 30% of the students in Section B switch to A, resulting in a net loss of 4 students for Section B. How large were the two sections at the beginning of the semester? No students dropped Linear Algebra (why would they?) or joined the course late.

### Historical Problems

- 67.** Five cows and two sheep together cost 10 *liang*<sup>12</sup> of silver. Two cows and five sheep together cost eight *liang* of silver. What is the cost of a cow and a sheep, respectively? (*Nine Chapters*,<sup>13</sup> Chapter 8, Problem 7)

- 68.** If you sell two cows and five sheep and you buy 13 pigs, you gain 1,000 coins. If you sell three cows and three pigs and buy nine sheep, you break even. If you sell six sheep and eight pigs and you buy five cows, you lose 600 coins. What is the price of a cow, a sheep, and a pig, respectively? (*Nine Chapters*, Chapter 8, Problem 8)

- 69.** You place five sparrows on one of the pans of a balance and six swallows on the other pan; it turns out that the sparrows are heavier. But if you exchange one sparrow and one swallow, the weights are exactly balanced. All the birds together weigh 1 *jin*. What is the weight of a sparrow and a swallow, respectively? [Give the answer in *liang*, with 1 *jin* = 16 *liang*.] (*Nine Chapters*, Chapter 8, Problem 9)

- 70.** Consider the task of pulling a weight of 40 *dan*<sup>14</sup> up a hill; we have one military horse, two ordinary horses, and three weak horses at our disposal to get the job done. It turns out that the military horse and one of the ordinary horses, pulling together, are barely able to pull

<sup>12</sup>A *liang* was about 16 grams at the time of the Han Dynasty.

<sup>13</sup>We present some of the problems from the *Nine Chapters on the Mathematical Art* in a free translation, with some additional explanations, since the scenarios discussed in a few of these problems are rather unfamiliar to the modern reader.

<sup>14</sup>1 *dan* = 120 *jin* = 1,920 *liang*. Thus, a *dan* was about 30 kilograms at that time.

## Linear Equations

the weight (but they could not pull any more). Likewise, the two ordinary horses together with one weak horse are just able to do the job, as are the three weak horses together with the military horse. How much weight can each of the horses pull alone? (*Nine Chapters*, Chapter 8, Problem 12)

- 71.** Five households share a deep well for their water supply. Each household owns a few ropes of a certain length, which varies only from household to household. The five households, A, B, C, D, and E, own 2, 3, 4, 5, and 6 ropes, respectively. Even when tying all their ropes together, none of the households alone is able to reach the water, but A's two ropes together with one of B's ropes just reach the water. Likewise, B's three ropes with one of C's ropes, C's four ropes with one of D's ropes, D's five ropes with one of E's ropes, and E's six ropes with one of A's ropes all just reach the water. How long are the ropes of the various households, and how deep is the well?

*Commentary:* As stated, this problem leads to a system of five linear equations in six variables; with the given information, we are unable to determine the depth of the well. The *Nine Chapters* gives one particular solution, where the depth of the well is 7 *zhang*,<sup>15</sup> 2 *chi*, 1 *cun*, or 721 *cun* (since 1 *zhang* = 10 *chi* and 1 *chi* = 10 *cun*). Using this particular value for the depth of the well, find the lengths of the various ropes.

- 72.** “A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?” (From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D.)

*Commentary:* This famous *Hundred Fowl Problem* has reappeared in countless variations in Indian, Arabic, and European texts (see Exercises 73 through 76); it has remained popular to this day. See Exercise 46 of this section.

- 73.** Pigeons are sold at the rate of 5 for 3 panas, sarasabirds at the rate of 7 for 5 panas, swans at the rate of 9 for 7 panas, and peacocks at the rate of 3 for 9 panas. A man was ordered to bring 100 birds for 100 panas for the amusement of the Prince. How much does he pay for each of the various kinds of birds he buys? (From the *Ganita-Sara-Sangraha* by Mahavira, India; 9th century A.D.) Find one solution to this problem.

- 74.** A duck costs four coins, five sparrows cost one coin, and a rooster costs one coin. Someone purchases 100 birds for 100 coins. How many birds of each kind can he buy? (From the *Key to Arithmetic* by Al-Kashi; 15th century)

- 75.** “A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him  $\frac{1}{2}$  a crown a-piece; the goats,  $1\frac{1}{3}$  crown; and the hogs  $3\frac{1}{2}$  crowns. How many had he of each?” (From the *Elements of Algebra* by Leonhard Euler, 1770)
- 76.** A gentleman runs a household of 100 people and orders that they be given 100 measures of grain. He directs that each man should receive three measures, each woman two measures, and each child half a measure. How many men, women, and children are there in this household? We are told that there is at least one man, one woman, and one child. (From the *Problems for Quickening a Young Mind* by Alcuin [c. 732–804], the Abbot of St. Martins at Tours. Alcuin was a friend and tutor to Charlemagne and his family at Aachen.)
- 77.** A dying father gave to his sons 30 barrels, of which 10 were full of wine, 10 were half-full, and the last 10 were empty. Divide the wine and flasks so that there will be equal division among the three sons of both wine and barrels. Find all the solutions of this problem. (From Alcuin)
- 78.** Make me a crown weighing 60 *mina* from a mixture of gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds of the weight, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must use. (From the *Greek Anthology* by Metrodorus, 6th century A.D.)
- 79.** Three merchants find a purse lying in the road. One merchant says, “If I keep the purse, I will have twice as much money as the two of you together.” “Give me the purse and I will have three times as much as the two of you together,” said the second merchant. The third merchant said, “I will be much better off than either of you if I keep the purse, I will have five times as much as the two of you together.” If there are 60 coins (of equal value) in the purse, how much money does each merchant have? (From Mahavira)
- 80.** 3 cows graze 1 field bare in 2 days,  
7 cows graze 4 fields bare in 4 days, and  
3 cows graze 2 fields bare in 5 days.  
It is assumed that each field initially provides the same amount,  $x$ , of grass; that the daily growth,  $y$ , of the fields remains constant; and that all the cows eat the same amount,  $z$ , each day. (Quantities  $x$ ,  $y$ , and  $z$  are measured by weight.) Find all the solutions of this problem. (This is a special case of a problem discussed by Isaac Newton in his *Arithmetica Universalis*, 1707.)

<sup>15</sup> 1 *zhang* was about 2.3 meters at that time.

### 3 On the Solutions of Linear Systems; Matrix Algebra

In this final section of this chapter, we will discuss two rather unrelated topics:

- First, we will examine how many solutions a system of linear equations can possibly have.
- Then, we will present some definitions and rules of matrix algebra.

#### The Number of Solutions of a Linear System

**EXAMPLE 1** The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?

$$\text{a. } \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{b. } \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{c. } \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

#### Solution

- a. The third row represents the equation  $0 = 1$ , so that there are no solutions. We say that this system is *inconsistent*.
- b. The given augmented matrix represents the system

$$\left| \begin{array}{l} x_1 + 2x_2 = 1 \\ x_3 = 2 \end{array} \right|, \quad \text{or} \quad \left| \begin{array}{l} x_1 = 1 - 2x_2 \\ x_3 = 2 \end{array} \right|.$$

We can assign an arbitrary value,  $t$ , to the free variable  $x_2$ , so that the system has infinitely many solutions,

$$\left[ \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[ \begin{array}{l} 1 - 2t \\ t \\ 2 \end{array} \right], \quad \text{where } t \text{ is an arbitrary constant.}$$

- c. Here there are no free variables, so that we have only one solution,  $x_1 = 1$ ,  $x_2 = 2$ ,  $x_3 = 3$ . ■

We can generalize our findings:<sup>16</sup>

#### Theorem 3.1

##### Number of solutions of a linear system

A system of equations is said to be *consistent* if there is at least one solution; it is *inconsistent* if there are no solutions.

A linear system is inconsistent if (and only if) the reduced row-echelon form of its augmented matrix contains the row  $[0 \ 0 \ \dots \ 0 \ | \ 1]$ , representing the equation  $0 = 1$ .

If a linear system is consistent, then it has either

- *infinitely many solutions* (if there is at least one free variable), or
- *exactly one solution* (if all the variables are leading).

<sup>16</sup>In this section, we will number the definitions we give and the theorems we derive. The  $n$ th theorem stated in Section  $q$  is labeled as Theorem  $q.n$ .

Example 1 illustrates what the number of leading 1's in the echelon form tells us about the number of solutions of a linear system. This observation motivates the following definition:

**Definition 3.2**
**The rank of a matrix<sup>17</sup>**

The rank of a matrix  $A$  is the number of leading 1's in  $\text{rref}(A)$ , denoted  $\text{rank}(A)$ .

**EXAMPLE 2**

$$\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 2, \text{ since } \text{rref} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

■

Note that we have defined the rank of a *matrix* rather than the rank of a linear system. When relating the concept of rank to a linear system, we must be careful to specify whether we consider the coefficient matrix or the augmented matrix of the system.

**EXAMPLE 3**

Consider a system of  $n$  linear equations with  $m$  variables, which has a coefficient matrix  $A$  of size  $n \times m$ . Show that

- a. The inequalities  $\text{rank}(A) \leq n$  and  $\text{rank}(A) \leq m$  hold.
- b. If the system is inconsistent, then  $\text{rank}(A) < n$ .
- c. If the system has exactly one solution, then  $\text{rank}(A) = m$ .
- d. If the system has infinitely many solutions, then  $\text{rank}(A) < m$ .

**Solution**

- a. By definition of the reduced row-echelon form, there is at most one leading 1 in each of the  $n$  rows and in each of the  $m$  columns of  $\text{rref}(A)$ .
- b. If the system is inconsistent, then the rref of the augmented matrix will contain a row of the form  $[0 \ 0 \dots \ 0 \ 1]$ , so that  $\text{rref}(A)$  will contain a row of zeros. Since there is no leading 1 in that row, we find that  $\text{rank}(A) < n$ , as claimed.
- c. For parts c and d, it is worth noting that

$$\binom{\text{number of free variables}}{\text{total number of variables}} = \binom{\text{number of leading variables}}{m} = m - \text{rank}(A).$$

If the system has exactly one solution, then there are no free variables (see Theorem 3.1), so that  $m - \text{rank}(A) = 0$  and  $\text{rank}(A) = m$  as claimed.

- d. If the system has infinitely many solutions, then there is at least one free variable, so that  $m - \text{rank}(A) > 0$  and  $\text{rank}(A) < m$ , as claimed. ■

**EXAMPLE 4**

It is useful to think about the *contrapositives* of the statements in parts b through d of Example 3.<sup>18</sup>

<sup>17</sup>This is a preliminary, rather technical definition.

<sup>18</sup>The *contrapositive* of the statement “if  $p$  then  $q$ ” is “if not- $q$  then not- $p$ .” A statement and its contrapositive are logically equivalent. For example, the contrapositive of “If you live in New York City, then you live in the United States” is “If you don’t live in the United States, then you don’t live in New York City.” Here is a more convoluted example: On the service truck of a plumbing company we read, “If we can’t fix it, then it ain’t broken.” The contrapositive of this claim is, “If it is broken, then we can fix it” (not quite as catchy!).

## Linear Equations

- b. If  $\text{rank}(A) = n$ , then the system is consistent.
- c. If  $\text{rank}(A) < m$ , then the system has no solution or infinitely many solutions.
- d. If  $\text{rank}(A) = m$ , then the system has no solution or exactly one solution. ■

In Theorems 3.3 and 3.4, we will discuss two important special cases of Example 3.

### Theorem 3.3

#### Number of equations vs. number of unknowns

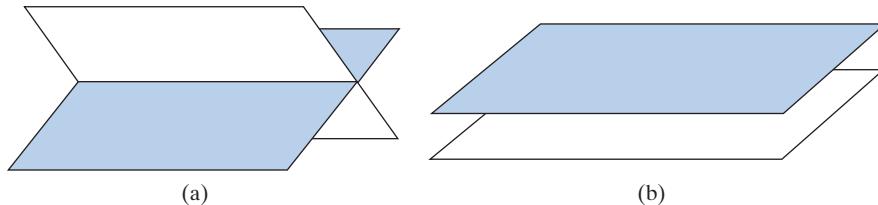
- a. If a linear system has exactly one solution, then there must be at least as many equations as there are variables ( $m \leq n$  with the notation from Example 3).

Equivalently, we can formulate the contrapositive:

- b. A linear system with fewer equations than unknowns ( $n < m$ ) has either no solutions or infinitely many solutions.

The proof of part (a) is based on parts (a) and (c) of Example 3:  $m = \text{rank}(A) \leq n$ , so that  $m \leq n$  as claimed.

To illustrate part (b) of Theorem 3.3, consider two linear equations in three variables, with each equation defining a plane. Two different planes in space either intersect in a line or are parallel (see Figure 1), but they will never intersect at a point! This means that a system of two linear equations with three unknowns cannot have a unique solution.



**Figure 1** (a) Two planes intersect in a line. (b) Two parallel planes.

### EXAMPLE 5

Consider a linear system of  $n$  equations with  $n$  variables. When does this system have exactly one solution? Give your answer in terms of the rank of the coefficient matrix  $A$ .

#### Solution

If the system has exactly one solution, then  $\text{rank}(A) = m = n$  by Example 3c.

Conversely, if  $\text{rank}(A) = n$ , then there will be a leading 1 in each row and in each column, and these leading 1's will be lined up along the diagonal:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

This system will have exactly one solution. ■

**Theorem 3.4**
**Systems of  $n$  equations in  $n$  variables**

A linear system of  $n$  equations in  $n$  variables has a unique solution if (and only if) the rank of its coefficient matrix  $A$  is  $n$ . In this case,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

the  $n \times n$  matrix with 1's along the diagonal and 0's everywhere else.

**Matrix Algebra**

We will now introduce some basic definitions and rules of matrix algebra.

Sums and scalar multiples of matrices are defined entry by entry, as for vectors.

**Definition 3.5**
**Sums of matrices**

The sum of two matrices of the same size is defined entry by entry:

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{nm} + b_{nm} \end{bmatrix}.$$

**Scalar multiples of matrices**

The product of a scalar with a matrix is defined entry by entry:

$$k \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \dots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \dots & ka_{nm} \end{bmatrix}.$$

**EXAMPLE 6**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{bmatrix}$$

**EXAMPLE 7**

$$3 \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$$

The definition of the product of matrices is less straightforward.

Because vectors are special matrices (with only one row or only one column), it makes sense to start with a discussion of products of vectors. The reader may be familiar with the dot product of vectors.

**Definition 3.6**
**Dot product of vectors**

Consider two vectors  $\vec{v}$  and  $\vec{w}$  with components  $v_1, \dots, v_n$  and  $w_1, \dots, w_n$ , respectively. Here  $\vec{v}$  and  $\vec{w}$  may be column or row vectors, and the two vectors need not be of the same type. The dot product of  $\vec{v}$  and  $\vec{w}$  is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n.$$

Note that our definition of the dot product isn't row-column-sensitive. The dot product does not distinguish between row and column vectors.

**EXAMPLE 8**

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 = 11$$



Now we are ready to define the product  $A\vec{x}$ , where  $A$  is a matrix and  $\vec{x}$  is a vector, in terms of the dot product.

**Definition 3.7**
**The product  $A\vec{x}$** 

If  $A$  is an  $n \times m$  matrix with row vectors  $\vec{w}_1, \dots, \vec{w}_n$ , and  $\vec{x}$  is a vector in  $\mathbb{R}^m$ , then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

In words, the  $i$ th component of  $A\vec{x}$  is the dot product of the  $i$ th row of  $A$  with  $\vec{x}$ .

Note that  $A\vec{x}$  is a column vector with  $n$  components, that is, a vector in  $\mathbb{R}^n$ .

**EXAMPLE 9**

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$


**EXAMPLE 10**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{for all vectors } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$



Note that the product  $A\vec{x}$  is defined only if the number of *columns* of matrix  $A$  matches the number of components of vector  $\vec{x}$ :

$$\underbrace{\begin{matrix} n \times m \\ A \\ n \times 1 \end{matrix}}_{\text{rows}} \underbrace{\begin{matrix} m \times 1 \\ \vec{x} \end{matrix}}_{\text{columns}}.$$

**EXAMPLE 11**

The product  $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  is undefined, because the number of columns of matrix  $A$  fails to match the number of components of vector  $\vec{x}$ .



In Definition 3.7, we express the product  $A\vec{x}$  in terms of the *rows* of the matrix  $A$ . Alternatively, the product can be expressed in terms of the *columns*.

## Linear Equations

Let's take another look at Example 9:

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + (-3) \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 \\ 1 \cdot 3 \end{bmatrix} + \begin{bmatrix} 2 \cdot 1 \\ 0 \cdot 1 \end{bmatrix} + \begin{bmatrix} -3 \cdot 2 \\ (-1) \cdot 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ -1 \end{bmatrix}. \end{aligned}$$

We recognize that the expression  $3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ -1 \end{bmatrix}$  involves the vectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ , the columns of  $A$ , and the scalars  $x_1 = 3$ ,  $x_2 = 1$ ,  $x_3 = 2$ , the components of  $\vec{x}$ . Thus, we can write

$$A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3.$$

We can generalize:

### Theorem 3.8

#### The product $A\vec{x}$ in terms of the columns of $A$

If the column vectors of an  $n \times m$  matrix  $A$  are  $\vec{v}_1, \dots, \vec{v}_m$  and  $\vec{x}$  is a vector in  $\mathbb{R}^m$  with components  $x_1, \dots, x_m$ , then

$$A\vec{x} = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

**Proof** As usual, we denote the rows of  $A$  by  $\vec{w}_1, \dots, \vec{w}_n$  and the entries by  $a_{ij}$ . It suffices to show that the  $i$ th component of  $A\vec{x}$  is equal to the  $i$ th component of  $x_1\vec{v}_1 + \dots + x_m\vec{v}_m$ , for  $i = 1, \dots, n$ . Now

$$\begin{aligned} (\text{$i$th component of } A\vec{x}) &\stackrel{\text{step 1}}{=} \underbrace{\vec{w}_i \cdot \vec{x}}_{\substack{= a_{i1}x_1 + \dots + a_{im}x_m \\ = x_1(\text{$i$th component of } \vec{v}_1) + \dots \\ + x_m(\text{$i$th component of } \vec{v}_m)}} \\ &\stackrel{\text{step 4}}{=} \underbrace{i\text{th component of } x_1\vec{v}_1 + \dots + x_m\vec{v}_m}_{\substack{}}. \end{aligned}$$

In Step 1 we are using Definition 3.7, and in step 4 we are using the fact that vector addition and scalar multiplication are defined component by component. ■

### EXAMPLE 12

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (-4) \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix} - \begin{bmatrix} 8 \\ 20 \\ 32 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Note that something remarkable is happening here: Although  $A$  isn't the zero matrix and  $\vec{x}$  isn't the zero vector, the product  $A\vec{x}$  is the zero vector. (By contrast, the product of any two nonzero scalars is nonzero.) ■

## Linear Equations

The formula for the product  $A\vec{x}$  in Theorem 3.8 involves the expression  $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$ , where  $\vec{v}_1, \dots, \vec{v}_m$  are vectors in  $\mathbb{R}^n$ , and  $x_1, \dots, x_m$  are scalars. Such expressions come up very frequently in linear algebra; they deserve a name.

### Definition 3.9

#### Linear combinations

A vector  $\vec{b}$  in  $\mathbb{R}^n$  is called a linear combination of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$  if there exist scalars  $x_1, \dots, x_m$  such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

**EXAMPLE 13** Is the vector  $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  a linear combination of the vectors  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ ?

#### Solution

According to Definition 3.9, we need to see whether we can find scalars  $x$  and  $y$  such that  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x+4y \\ 2x+5y \\ 3x+6y \end{bmatrix}$ . We need to solve the linear system  $\begin{cases} x+4y=1 \\ 2x+5y=1 \\ 3x+6y=1 \end{cases}$ , with augmented matrix  $M = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{bmatrix}$  and  $\text{rref}(M) = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$ .

It turns out that the system is consistent, with  $x = -1/3$  and  $y = 1/3$ . The vector  $\vec{b}$  is indeed a linear combination of  $\vec{v}$  and  $\vec{w}$ , with  $\vec{b} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}$ . ■

Note that the product  $A\vec{x}$  is the linear combination of the columns of  $A$  with the components of  $\vec{x}$  as the coefficients:

$$A\vec{x} = \left[ \begin{array}{ccc|c} | & & | & x_1 \\ \vec{v}_1 & \dots & \vec{v}_m & \vdots \\ | & & | & x_m \end{array} \right] = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Take a good look at this equation, because it is the most frequently used formula in this text. Particularly in theoretical work, it will often be useful to write the product  $A\vec{x}$  as the linear combination  $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$ . Conversely, when dealing with a linear combination  $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$ , it will often be helpful to introduce the matrix

$$A = \left[ \begin{array}{ccc|c} | & & | & x_1 \\ \vec{v}_1 & \dots & \vec{v}_m & \vdots \\ | & & | & x_m \end{array} \right] \quad \text{and the vector} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

and then write  $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = A\vec{x}$ .

Next we present two rules concerning the product  $A\vec{x}$ .

**Theorem 3.10**
**Algebraic rules for  $A\vec{x}$** 

If  $A$  is an  $n \times m$  matrix,  $\vec{x}$  and  $\vec{y}$  are vectors in  $\mathbb{R}^m$ , and  $k$  is a scalar, then

- a.  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ , and
- b.  $A(k\vec{x}) = k(A\vec{x})$ .

We will prove the first equation, leaving the second as Exercise 45.  
Denote the  $i$ th row of  $A$  by  $\vec{w}_i$ . Then

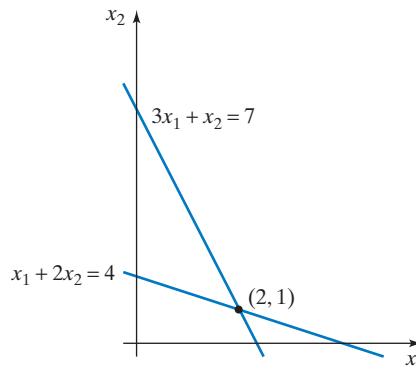
$$\begin{aligned} (\text{$i$th component of } A(\vec{x} + \vec{y})) &= \vec{w}_i \cdot (\vec{x} + \vec{y}) \xrightarrow{\text{step 2}} \vec{w}_i \cdot \vec{x} + \vec{w}_i \cdot \vec{y} \\ &= (\text{$i$th component of } A\vec{x}) + (\text{$i$th component of } A\vec{y}) \\ &= (\text{$i$th component of } A\vec{x} + A\vec{y}). \end{aligned}$$

Our new tools of matrix algebra allow us to see linear systems in a new light, as illustrated in the next example. The definition of the product  $A\vec{x}$  and the concept of a linear combination will be particularly helpful.

**EXAMPLE 14** Consider the linear system

$$\left| \begin{array}{l} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{array} \right|, \quad \text{with augmented matrix } \left[ \begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array} \right].$$

We can interpret the solution of this system as the intersection of two lines in the  $x_1x_2$ -plane, as illustrated in Figure 2.


**Figure 2**

Alternatively, we can write the system in vector form, as

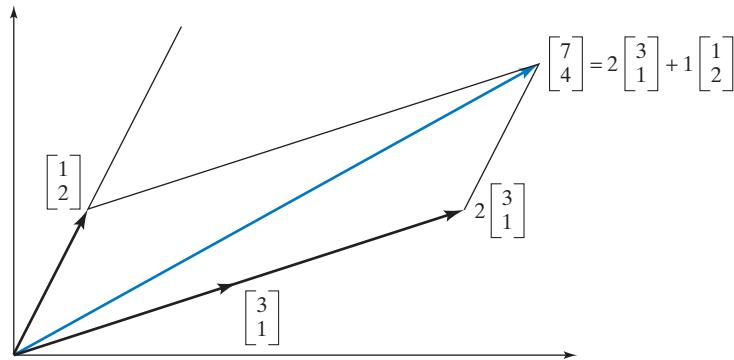
$$\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

We see that solving this system amounts to writing the vector  $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$  as a *linear combination* of the vectors  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . See Definition 3.9. The vector equation

## Linear Equations

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

and its solution can be represented geometrically, as shown in Figure 3. The problem amounts to resolving the vector  $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$  into two vectors parallel to  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , respectively, by means of a parallelogram.



**Figure 3**

We can go further and write the linear combination

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{as} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that the linear system

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{takes the form} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix},$$

the *matrix form* of the linear system.

Note that we started out with the augmented matrix

$$[A \mid \vec{b}] = \left[ \begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array} \right],$$

and we ended up writing the system as

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 4 \end{bmatrix}}_{\vec{b}}, \quad \text{or} \quad A\vec{x} = \vec{b}. \quad \blacksquare$$

We can generalize:

### Theorem 3.11

#### Matrix form of a linear system

We can write the linear system with augmented matrix  $[A \mid \vec{b}]$  in matrix form as

$$A\vec{x} = \vec{b}.$$

Note that the  $i$ th component of  $A\vec{x}$  is  $a_{1i}x_1 + \cdots + a_{mi}x_m$ , by Definition 3.7. Thus, the  $i$ th component of the equation  $A\vec{x} = \vec{b}$  is

$$a_{1i}x_1 + \cdots + a_{mi}x_m = b_i;$$

this is the  $i$ th equation of the system with augmented matrix  $[A \mid \vec{b}]$ .

## Linear Equations

Solving the linear system  $A\vec{x} = \vec{b}$  amounts to expressing vector  $\vec{b}$  as a linear combination of the column vectors of matrix  $A$ .

**EXAMPLE 15** Write the system

$$\begin{vmatrix} 2x_1 - 3x_2 + 5x_3 & 7 \\ 9x_1 + 4x_2 - 6x_3 & 8 \end{vmatrix}$$

in matrix form.

### Solution

The coefficient matrix is  $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix}$ , and  $\vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ . The matrix form is

$$A\vec{x} = \vec{b}, \quad \text{or} \quad \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}. \quad \blacksquare$$

Now that we can write a linear system as a *single equation*,  $A\vec{x} = \vec{b}$ , rather than a list of simultaneous equations, we can think about it in new ways.

For example, if we have an equation  $ax = b$  of *numbers*, we can divide both sides by  $a$  to find the solution  $x$ :

$$x = \frac{b}{a} = a^{-1}b \quad (\text{if } a \neq 0).$$

It is natural to ask whether we can take an analogous approach in the case of the equation  $A\vec{x} = \vec{b}$ . Can we “divide by  $A$ ,” in some sense, and write

$$\vec{x} = \frac{\vec{b}}{A} = A^{-1}\vec{b}?$$

This is termed as the invertibility of a matrix.

## EXERCISES 3

**GOAL** Use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system. Apply the definition of the rank of a matrix. Compute the product  $A\vec{x}$  in terms of the rows or the columns of  $A$ . Represent a linear system in vector or matrix form.

1. The reduced row-echelon forms of the augmented matrices of three systems are given here. How many solutions does each system have?

a.  $\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$     b.  $\left[ \begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array} \right]$

c.  $\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$

Find the rank of the matrices in Exercises 2 through 4.

2.  $\left[ \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]$     3.  $\left[ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right]$     4.  $\left[ \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array} \right]$

5. a. Write the system

$$\begin{vmatrix} x + 2y & 7 \\ 3x + y & 11 \end{vmatrix}$$

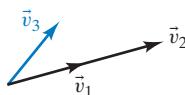
in vector form.

- b. Use your answer in part (a) to represent the system geometrically. Solve the system and represent the solution geometrically.

6. Consider the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^2$  (sketched in the accompanying figure). Vectors  $\vec{v}_1$  and  $\vec{v}_2$  are parallel. How many solutions  $x, y$  does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

have? Argue geometrically.

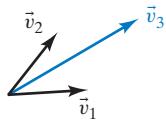


## Linear Equations

7. Consider the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^2$  shown in the accompanying sketch. How many solutions  $x, y$  does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

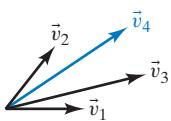
have? Argue geometrically.



8. Consider the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  in  $\mathbb{R}^2$  shown in the accompanying sketch. Arguing geometrically, find two solutions  $x, y, z$  of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has, in fact, infinitely many solutions?



9. Write the system

$$\begin{vmatrix} x + 2y + 3z & = 1 \\ 4x + 5y + 6z & = 4 \\ 7x + 8y + 9z & = 9 \end{vmatrix}$$

in matrix form.

*Compute the dot products in Exercises 10 through 12 (if the products are defined).*

$$10. \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$11. \begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$12. \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

*Compute the products  $A\vec{x}$  in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of  $A$  (Theorem 3.8) and in terms of the rows of  $A$  (Definition 3.7).*

$$13. \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

$$14. \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$15. \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

*Compute the products  $A\vec{x}$  in Exercises 16 through 19 using paper and pencil (if the products are defined).*

$$16. \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$17. \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$18. \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$19. \begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$20. \text{ a. Find } \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}.$$

$$\text{b. Find } 9 \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

21. Use technology to compute the product

$$\begin{bmatrix} 1 & 7 & 8 & 9 \\ 1 & 2 & 9 & 1 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 5 \\ 6 \end{bmatrix}.$$

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let  $A$  be a  $4 \times 4$  matrix, and let  $\vec{b}$  and  $\vec{c}$  be two vectors in  $\mathbb{R}^4$ . We are told that the system  $A\vec{x} = \vec{b}$  has a unique solution. What can you say about the number of solutions of the system  $A\vec{x} = \vec{c}$ ?

25. Let  $A$  be a  $4 \times 4$  matrix, and let  $\vec{b}$  and  $\vec{c}$  be two vectors in  $\mathbb{R}^4$ . We are told that the system  $A\vec{x} = \vec{b}$  is inconsistent. What can you say about the number of solutions of the system  $A\vec{x} = \vec{c}$ ?

26. Let  $A$  be a  $4 \times 3$  matrix, and let  $\vec{b}$  and  $\vec{c}$  be two vectors in  $\mathbb{R}^4$ . We are told that the system  $A\vec{x} = \vec{b}$  has a unique solution. What can you say about the number of solutions of the system  $A\vec{x} = \vec{c}$ ?

27. If the rank of a  $4 \times 4$  matrix  $A$  is 4, what is  $\text{rref}(A)$ ?

28. If the rank of a  $5 \times 3$  matrix  $A$  is 3, what is  $\text{rref}(A)$ ?

In Problems 29 through 32, let  $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ .

29. Find a *diagonal* matrix  $A$  such that  $A\vec{x} = \vec{y}$ .

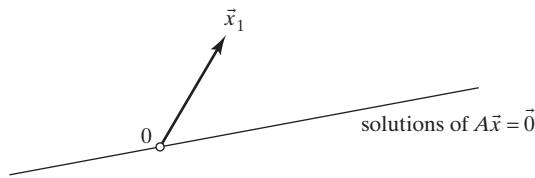
30. Find a matrix  $A$  of rank 1 such that  $A\vec{x} = \vec{y}$ .

31. Find an *upper triangular* matrix  $A$  such that  $A\vec{x} = \vec{y}$ ,

## Linear Equations

- where all the entries of  $A$  on and above the diagonal are nonzero.
- 32.** Find a matrix  $A$  with all nonzero entries such that  $A\vec{x} = \vec{y}$ .
- 33.** Let  $A$  be the  $n \times n$  matrix with all 1's on the diagonal and all 0's above and below the diagonal. What is  $A\vec{x}$ , where  $\vec{x}$  is a vector in  $\mathbb{R}^n$ ?
- 34.** We define the vectors
- $$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
- in  $\mathbb{R}^3$ .
- a.** For
- $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$
- compute  $A\vec{e}_1$ ,  $A\vec{e}_2$ , and  $A\vec{e}_3$ .
- b.** If  $B$  is an  $n \times 3$  matrix with columns  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$ , what are  $B\vec{e}_1$ ,  $B\vec{e}_2$ , and  $B\vec{e}_3$ ?
- 35.** In  $\mathbb{R}^m$ , we define
- $$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component.}$$
- If  $A$  is an  $n \times m$  matrix, what is  $A\vec{e}_i$ ?
- 36.** Find a  $3 \times 3$  matrix  $A$  such that
- $$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$
- and  $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ .
- 37.** Find all vectors  $\vec{x}$  such that  $A\vec{x} = \vec{b}$ , where
- $$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$
- 38. a.** Using technology, generate a random  $3 \times 3$  matrix  $A$ . (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find  $\text{rref}(A)$ . Repeat this experiment a few times.
- b.** What does the reduced row-echelon form of most  $3 \times 3$  matrices look like? Explain.
- 39.** Repeat Exercise 38 for  $3 \times 4$  matrices.
- 40.** Repeat Exercise 38 for  $4 \times 3$  matrices.
- 41.** How many solutions do most systems of three linear equations with three unknowns have? Explain in terms of your work in Exercise 38.
- 42.** How many solutions do most systems of three linear equations with four unknowns have? Explain in terms of your work in Exercise 39.
- 43.** How many solutions do most systems of four linear equations with three unknowns have? Explain in terms of your work in Exercise 40.
- 44.** Consider an  $n \times m$  matrix  $A$  with more rows than columns ( $n > m$ ). Show that there is a vector  $\vec{b}$  in  $\mathbb{R}^n$  such that the system  $A\vec{x} = \vec{b}$  is inconsistent.
- 45.** Consider an  $n \times m$  matrix  $A$ , a vector  $\vec{x}$  in  $\mathbb{R}^m$ , and a scalar  $k$ . Show that
- $$A(k\vec{x}) = k(A\vec{x}).$$
- 46.** Find the rank of the matrix
- $$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$
- where  $a, d$ , and  $f$  are nonzero, and  $b, c$ , and  $e$  are arbitrary numbers.
- 47.** A linear system of the form
- $$A\vec{x} = \vec{0}$$
- is called *homogeneous*. Justify the following facts:
- a.** All homogeneous systems are consistent.
- b.** A homogeneous system with fewer equations than unknowns has infinitely many solutions.
- c.** If  $\vec{x}_1$  and  $\vec{x}_2$  are solutions of the homogeneous system  $A\vec{x} = \vec{0}$ , then  $\vec{x}_1 + \vec{x}_2$  is a solution as well.
- d.** If  $\vec{x}$  is a solution of the homogeneous system  $A\vec{x} = \vec{0}$  and  $k$  is an arbitrary constant, then  $k\vec{x}$  is a solution as well.
- 48.** Consider a solution  $\vec{x}_1$  of the linear system  $A\vec{x} = \vec{b}$ . Justify the facts stated in parts (a) and (b):
- a.** If  $\vec{x}_h$  is a solution of the system  $A\vec{x} = \vec{0}$ , then  $\vec{x}_1 + \vec{x}_h$  is a solution of the system  $A\vec{x} = \vec{b}$ .
- b.** If  $\vec{x}_2$  is another solution of the system  $A\vec{x} = \vec{b}$ , then  $\vec{x}_2 - \vec{x}_1$  is a solution of the system  $A\vec{x} = \vec{0}$ .
- c.** Now suppose  $A$  is a  $2 \times 2$  matrix. A solution vector  $\vec{x}_1$  of the system  $A\vec{x} = \vec{b}$  is shown in the accompanying figure. We are told that the solutions of the system  $A\vec{x} = \vec{0}$  form the line shown in the sketch. Draw the line consisting of all solutions of the system  $A\vec{x} = \vec{b}$ .

## Linear Equations



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- 49.** Consider the accompanying table. For some linear systems  $A\vec{x} = \vec{b}$ , you are given either the rank of the coefficient matrix  $A$ , or the rank of the augmented matrix  $[A \mid \vec{b}]$ . In each case, state whether the system could have no solution, one solution, or infinitely many solutions. There may be more than one possibility for some systems. Justify your answers.

	Number of Equations	Number of Unknowns	Rank of $A$	Rank of $[A \mid \vec{b}]$
a.	3	4	—	2
b.	4	3	3	—
c.	4	3	—	4
d.	3	4	3	—

- 50.** Consider a linear system  $A\vec{x} = \vec{b}$ , where  $A$  is a  $4 \times 3$  matrix. We are told that  $\text{rank } [A \mid \vec{b}] = 4$ . How many solutions does this system have?
- 51.** Consider an  $n \times m$  matrix  $A$ , an  $r \times s$  matrix  $B$ , and a vector  $\vec{x}$  in  $\mathbb{R}^p$ . For which values of  $n, m, r, s$ , and  $p$  is the product

$$A(B\vec{x})$$

defined?

- 52.** Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Can you find a  $2 \times 2$  matrix  $C$  such that

$$A(B\vec{x}) = C\vec{x},$$

for all vectors  $\vec{x}$  in  $\mathbb{R}^2$ ?

- 53.** If  $A$  and  $B$  are two  $n \times m$  matrices, is

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

for all  $\vec{x}$  in  $\mathbb{R}^m$ ?

- 54.** Consider two vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^3$  that are not parallel. Which vectors in  $\mathbb{R}^3$  are linear combinations of  $\vec{v}_1$  and  $\vec{v}_2$ ? Describe the set of these vectors geometrically. Include a sketch in your answer.

- 55.** Is the vector  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$  a linear combination of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}?$$

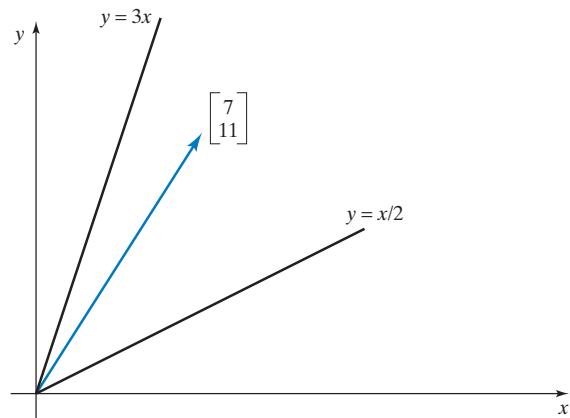
- 56.** Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

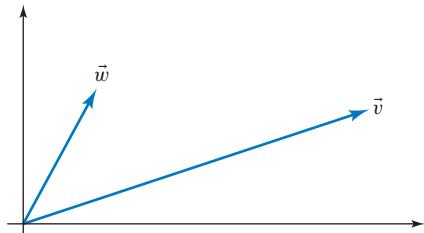
- 57.** Express the vector  $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$  as the sum of a vector on the line  $y = 3x$  and a vector on the line  $y = x/2$ .



- 58.** For which values of the constants  $b$  and  $c$  is the vector

$$\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} \quad \text{a linear combination of} \quad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \text{ and} \quad \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}?$$

## Linear Equations

- 59.** For which values of the constants  $c$  and  $d$  is  $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$  a linear combination of  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ?
- 60.** For which values of the constants  $a, b, c$ , and  $d$  is  $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$  a linear combination of  $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}$ ?
- 61.** For which values of the constant  $c$  is  $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$  a linear combination of  $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ ?
- 62.** For which values of the constant  $c$  is  $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$  a linear combination of  $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$ , where  $a$  and  $b$  are arbitrary constants?
- In Exercises 63 through 68, consider the vectors  $\vec{v}$  and  $\vec{w}$  in the accompanying figure.
- 
- 63.** Give a geometrical description of the set of all vectors of the form  $\vec{v} + c\vec{w}$ , where  $c$  is an arbitrary real number.
- 64.** Give a geometrical description of the set of all vectors of the form  $\vec{v} + c\vec{w}$ , where  $0 \leq c \leq 1$ .
- 65.** Give a geometrical description of the set of all vectors of the form  $a\vec{v} + b\vec{w}$ , where  $0 \leq a \leq 1$  and  $0 \leq b \leq 1$ .
- 66.** Give a geometrical description of the set of all vectors of the form  $a\vec{v} + b\vec{w}$ , where  $a + b = 1$ .
- 67.** Give a geometrical description of the set of all vectors of the form  $a\vec{v} + b\vec{w}$ , where  $0 \leq a, 0 \leq b$ , and  $a+b \leq 1$ .
- 68.** Give a geometrical description of the set of all vectors  $\vec{u}$  in  $\mathbb{R}^2$  such that  $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ .
- 69.** Solve the linear system
- $$\begin{cases} y + z = a \\ x + z = b \\ x + y = c \end{cases},$$
- where  $a, b$ , and  $c$  are arbitrary constants.
- 70.** Let  $A$  be the  $n \times n$  matrix with 0's on the main diagonal, and 1's everywhere else. For an arbitrary vector  $\vec{b}$  in  $\mathbb{R}^n$ , solve the linear system  $A\vec{x} = \vec{b}$ , expressing the components  $x_1, \dots, x_n$  of  $\vec{x}$  in terms of the components of  $\vec{b}$ . See Exercise 69 for the case  $n = 3$ .

## Exercises

### TRUE OR FALSE?<sup>19</sup>

Determine whether the statements that follow are true or false, and justify your answer.

- 1.** If  $A$  is an  $n \times n$  matrix and  $\vec{x}$  is a vector in  $\mathbb{R}^n$ , then the product  $A\vec{x}$  is a linear combination of the columns of matrix  $A$ .

- 2.** If vector  $\vec{u}$  is a linear combination of vectors  $\vec{v}$  and  $\vec{w}$ , then we can write  $\vec{u} = a\vec{v} + b\vec{w}$  for some scalars  $a$  and  $b$ .

- 3.** Matrix  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  is in reduced row-echelon form.

- 4.** A system of four linear equations in three unknowns is always inconsistent.

- 5.** There exists a  $3 \times 4$  matrix with rank 4.

- 6.** If  $A$  is a  $3 \times 4$  matrix and vector  $\vec{v}$  is in  $\mathbb{R}^4$ , then vector  $A\vec{v}$  is in  $\mathbb{R}^3$ .

- 7.** If the  $4 \times 4$  matrix  $A$  has rank 4, then any linear system with coefficient matrix  $A$  will have a unique solution.

<sup>19</sup>We will conclude this chapter with some true–false questions. We will start with a group of about 10 straightforward statements that refer directly to definitions and theorems given in the chapter. Then there may be some computational exercises, and the remaining ones are more conceptual, calling for independent reasoning. Don't expect a balanced coverage of all the topics; some concepts are better suited for this kind of questioning than others.

## Linear Equations

8. There exists a system of three linear equations with three unknowns that has exactly three solutions.
9. There exists a  $5 \times 5$  matrix  $A$  of rank 4 such that the system  $A\vec{x} = \vec{0}$  has only the solution  $\vec{x} = \vec{0}$ .
10. If matrix  $A$  is in reduced row-echelon form, then at least one of the entries in each column must be 1.

11. The system  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is inconsistent.

12. There exists a  $2 \times 2$  matrix  $A$  such that  $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

13. If  $A$  is a nonzero matrix of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , then the rank of  $A$  must be 2.

14.  $\text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = 3$

15. The system  $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  is inconsistent for all  $4 \times 3$  matrices  $A$ .

16. There exists a  $2 \times 2$  matrix  $A$  such that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

17.  $\text{rank} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 2$

18.  $\begin{bmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \\ 21 \end{bmatrix}$

19. There exists a matrix  $A$  such that  $A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ .

20. Vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is a linear combination of vectors

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

21. If the system  $A\vec{x} = \vec{b}$  has a unique solution, then  $A$  must be a square matrix.
22. If  $A$  is any  $4 \times 3$  matrix, then there exists a vector  $\vec{b}$  in  $\mathbb{R}^4$  such that the system  $A\vec{x} = \vec{b}$  is inconsistent.

23. There exist scalars  $a$  and  $b$  such that matrix

$$\begin{bmatrix} 0 & 1 & a \\ -1 & 0 & b \\ -a & -b & 0 \end{bmatrix}$$

has rank 3.

24. If  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^4$ , then  $\vec{v}$  must be a linear combination of  $\vec{v}$  and  $\vec{w}$ .

25. If  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are nonzero vectors in  $\mathbb{R}^2$ , then  $\vec{w}$  must be a linear combination of  $\vec{u}$  and  $\vec{v}$ .

26. If  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^4$ , then the zero vector in  $\mathbb{R}^4$  must be a linear combination of  $\vec{v}$  and  $\vec{w}$ .

27. If  $A$  and  $B$  are any two  $3 \times 3$  matrices of rank 2, then  $A$  can be transformed into  $B$  by means of elementary row operations.

28. If vector  $\vec{u}$  is a linear combination of vectors  $\vec{v}$  and  $\vec{w}$ , and  $\vec{v}$  is a linear combination of vectors  $\vec{p}$ ,  $\vec{q}$ , and  $\vec{r}$ , then  $\vec{u}$  must be a linear combination of  $\vec{p}$ ,  $\vec{q}$ ,  $\vec{r}$ , and  $\vec{w}$ .

29. A linear system with fewer unknowns than equations must have infinitely many solutions or none.

30. The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

31. There exists a  $4 \times 3$  matrix  $A$  of rank 3 such that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}.$$

32. The system  $A\vec{x} = \vec{b}$  is inconsistent if (and only if)  $\text{rref}(A)$  contains a row of zeros.

33. If  $A$  is a  $4 \times 3$  matrix of rank 3 and  $A\vec{v} = A\vec{w}$  for two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$ , then vectors  $\vec{v}$  and  $\vec{w}$  must be equal.

34. If  $A$  is a  $4 \times 4$  matrix and the system  $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$  has a

unique solution, then the system  $A\vec{x} = \vec{0}$  has only the solution  $\vec{x} = \vec{0}$ .

35. If vector  $\vec{u}$  is a linear combination of vectors  $\vec{v}$  and  $\vec{w}$ , then  $\vec{w}$  must be a linear combination of  $\vec{u}$  and  $\vec{v}$ .

36. If  $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$  and  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ , then the equation  $\vec{w} = 2\vec{u} + 3\vec{v}$  must hold.

37. If  $A$  and  $B$  are matrices of the same size, then the formula  $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$  must hold.

## Linear Equations

38. If  $A$  and  $B$  are any two  $n \times n$  matrices of rank  $n$ , then  $A$  can be transformed into  $B$  by means of elementary row operations.
39. If a vector  $\vec{v}$  in  $\mathbb{R}^4$  is a linear combination of  $\vec{u}$  and  $\vec{w}$ , and if  $A$  is a  $5 \times 4$  matrix, then  $A\vec{v}$  must be a linear combination of  $A\vec{u}$  and  $A\vec{w}$ .
40. If matrix  $E$  is in reduced row-echelon form, and if we omit a row of  $E$ , then the remaining matrix must be in reduced row-echelon form as well.
41. The linear system  $A\vec{x} = \vec{b}$  is consistent if (and only if)  $\text{rank}(A) = \text{rank} [A \mid \vec{b}]$ .
42. If  $A$  is a  $3 \times 4$  matrix of rank 3, then the system  $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  must have infinitely many solutions.
43. If two matrices  $A$  and  $B$  have the same reduced row-echelon form, then the equations  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  must have the same solutions.
44. If matrix  $E$  is in reduced row-echelon form, and if we omit a column of  $E$ , then the remaining matrix must be in reduced row-echelon form as well.
45. If  $A$  and  $B$  are two  $2 \times 2$  matrices such that the equations  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  have the same solutions, then  $\text{rref}(A)$  must be equal to  $\text{rref}(B)$ .
46. A lower triangular  $3 \times 3$  matrix has rank 3 if (and only if) the product of its diagonal entries is nonzero.
47. If  $ad - bc \neq 0$ , then the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  must have rank 2.
48. If vector  $\vec{w}$  is a linear combination of  $\vec{u}$  and  $\vec{v}$ , then  $\vec{u} + \vec{v} + \vec{w}$  must be a linear combination of  $\vec{u}$  and  $\vec{u} + \vec{v}$ .
49. If the linear system  $A\vec{x} = \vec{b}$  has a unique solution and the linear system  $A\vec{x} = \vec{c}$  is consistent, then the linear system  $A\vec{x} = \vec{b} + \vec{c}$  must have a unique solution.
50. A matrix is called a 0–1-matrix if all of its entries are ones and zeros. True or false: The majority of the 0–1-matrices of size  $3 \times 3$  have rank 3.

## Answers to Odd-Numbered Exercises

**1** Answers to more theoretical questions are omitted.

1.  $(x, y) = (-1, 1)$       3. No solutions

5.  $(x, y) = (0, 0)$       7. No solutions

9.  $(x, y, z) = (t, \frac{1}{2} - 2t, t)$ , where  $t$  is arbitrary

11.  $(x, y) = (4, 1)$       13. No solutions

15.  $(x, y, z) = (0, 0, 0)$

17.  $(x, y) = (-5a + 2b, 3a - b)$

19. a. If  $k = 7$   
 b. If  $k = 7$ , there are infinitely many solutions.  
 c. If  $k = 7$ , the solutions are  $(x, y, z) = (1 - t, 2t - 3, t)$ .

21. 11, 13, and 17

23. a. Products are competing.

b.  $P_1 = 26, P_2 = 46$

25.  $a = 400, b = 300$

27. a.  $(x, y) = (t, 2t)$ ;

b.  $(x, y) = (t, -3t)$ ;

c.  $(x, y) = (0, 0)$

31.  $f(t) = 1 - 5t + 3t^2$       33.  $f(t) = 2t^2 - 3t + 4$

35.  $f(t) = at^2 + (1 - 4a)t + 3a$ , for arbitrary  $a$

37.  $f(t) = 2e^{3t} - e^{2t}$

39.  $-20 - 2x - 4y + x^2 + y^2 = 0$ , the circle centered at  $(1, 2)$  with radius 5

41. If  $a - 2b + c = 0$

43. a. The intercepts of the line  $x + y = 1$  are  $(1, 0)$  and  $(0, 1)$ . The intercepts of the line  $x + \frac{t}{2}y = t$  are  $(t, 0)$  and  $(0, 2)$ . The lines intersect if  $t \neq 2$ .

b.  $x = -\frac{t}{t-2}, y = \frac{2t-2}{t-2}$

45. There are many correct answers. Example:

$$\left| \begin{array}{l} x - 5z = -4 \\ y - 3z = -2 \end{array} \right|$$

49. Twenty \$1 bills, eight \$5 bills, and four \$10 bills.

**2** Answers to more theoretical questions are omitted.

1.  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10t + 13 \\ -8t - 8 \\ t \end{bmatrix}$

3.  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}$

5.  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix}$

7.  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix}$

## Linear Equations

$$9. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} t-s-2r \\ r \\ -t+s+1 \\ t-2s+2 \\ s \\ t \end{bmatrix}$$

$$11. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 3t+4 \\ t \\ -2 \end{bmatrix}$$

13. No solutions

$$15. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$17. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

$$19. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

21.  $b = 0, c = 1, d = 0$ , with  $a$  being arbitrary

23. 4 types

27. Yes; perform the operations backwards.

29. No; you cannot make the last column zero by elementary row operations.

31.  $a = 2, b = c = d = 1$

$$33. f(t) = 1 - 5t + 4t^2 + 3t^3 - 2t^4$$

$$35. f(t) = -5 + 13t - 10t^2 + 3t^3$$

$$37. \begin{bmatrix} -t \\ 6t \\ -9t \\ 4t \end{bmatrix}, \text{ where } t \text{ is arbitrary.}$$

$$39. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 500 \\ 300 \\ 400 \end{bmatrix}$$

41. a. Neither the manufacturing nor the energy sector makes demands on agriculture.

b.  $x_1 \approx 18.67, x_2 \approx 22.60, x_3 \approx 3.63$

$$43. m_1 = \frac{2}{3}m_2$$

45.  $a \approx 12.17, b \approx -1.15, c \approx 0.18$ . The longest day is about 13.3 hours.

47. a. If  $k$  is neither 1 nor 2

b. If  $k = 1$

c. If  $k = 2$

49. a.  $x_1 = 3x_3 - 2x_4, x_2 = 2x_3 - x_4$ , for arbitrary  $x_3$  and  $x_4$

b. Yes,  $x_1 = 1, x_2 = 5, x_3 = 9, x_4 = 13$ .

$$51. C = 25$$

53.  $xy = 0$ , the union of the two coordinate axes

55.  $a(xy - y) + b(y^2 - y) = 0$ , where  $a \neq 0$  or  $b \neq 0$

57.  $a(x^2 - x) + b(y^2 - y) = 0$ , where  $a \neq 0$  or  $b \neq 0$

59.  $25 - 10x - 10y + x^2 + y^2 = 0$ , the circle of radius 5 centered at  $(5, 5)$

61. No solutions

63. Statistics: \$86; Set Theory: \$92; Psychology: \$55

65. Beginning: 120 liberal, 140 conservative.  
End: 140 liberal, 120 conservative.

67. Cow: 34/21 liang; sheep: 20/21 liang

69. Swallow: 24/19 liang; sparrow: 32/19 liang

71. A: 265; B: 191; C: 148; D: 129; E: 76

73. Gaussian elimination shows that

Pigeons =  $-250 + \frac{5}{9}$ (Swans) + 20(Peacocks) and

Sarasas =  $350 - \frac{14}{9}$ (Swans) - 21(Peacocks).

One solution (the one given by Mahavira) is 15 pigeons, 28 sarasabirds, 45 swans, and 12 peacocks (spending 9, 20, 35, and 36 panas, respectively).

75. 53 sheep, 42 goats, and 5 hogs

	Full	Half	Empty
1st Son	$p$	$10 - 2p$	$p$
2nd Son	$q$	$10 - 2q$	$q$
3rd Son	$10 - p - q$	$2p + 2q - 10$	$10 - p - q$

Here,  $p$  and  $q$  are integers between 0 and 5 such that  $p + q \geq 5$ .

3 Answers to more theoretical questions are omitted.

1. a. No solutions      b. One solution

c. Infinitely many solutions

3. Rank is 1.

$$5. a. x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

b.  $x = 3, y = 2$

7. One solution

$$9. \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

11. Undefined

### Linear Equations

13.  $\begin{bmatrix} 29 \\ 65 \end{bmatrix}$

15. 70

17. Undefined

19.  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

21.  $\begin{bmatrix} 158 \\ 70 \\ 81 \\ 123 \end{bmatrix}$

23.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

 25. The system  $A\vec{x} = \vec{c}$  has infinitely many solutions or none.

27.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

29.  $\begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$

31.  $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$ , for example

33.  $A\vec{x} = \vec{x}$

 35.  $Ae_i$  is the  $i$ th column of  $A$ .

37.  $\vec{x} = \begin{bmatrix} 2 - 2t \\ t \\ 1 \end{bmatrix}$ , where  $t$  is arbitrary

39.  $\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$

41. One solution      43. No solutions

 47. a.  $\vec{x} = 0$  is a solution.

b. By part (a) and Theorem 3.3

 c.  $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$ 

 d.  $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$ 

49. a. Infinitely many solutions or none

b. One solution or none

c. No solutions

d. Infinitely many solutions

 51. If  $m = r$  and  $s = p$ 

53. Yes

55. Yes;  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

57.  $\begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

59.  $c = 9, d = 11$

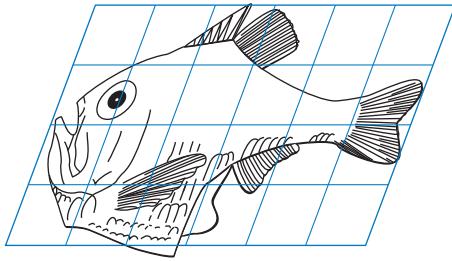
 61. For  $c = 2$  and for  $c = 3$ 

 63. Line through the endpoint of  $\vec{v}$  in the direction of  $\vec{w}$ 

 65. Parallelogram with its vertices at the origin and at the endpoints of vectors  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$ 

 67. Triangle with its vertices at the origin and at the endpoints of vectors  $\vec{v}$  and  $\vec{w}$ 

69.  $\left( \frac{b+c-a}{2}, \frac{c-b+a}{2}, \frac{b-c+a}{2} \right)$



# Linear Transformations

## I Introduction to Linear Transformations and Their Inverses

Imagine yourself cruising in the Mediterranean as a crew member on a French coast guard boat, looking for evildoers. Periodically, your boat radios its position to headquarters in Marseille. You expect that communications will be intercepted. So, before you broadcast anything, you have to transform the actual position of the boat,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

( $x_1$  for Eastern longitude,  $x_2$  for Northern latitude), into an encoded position

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

You use the following code:

$$\begin{aligned} y_1 &= x_1 + 3x_2 \\ y_2 &= 2x_1 + 5x_2. \end{aligned}$$

For example, when the actual position of your boat is  $5^\circ$  E,  $42^\circ$  N, or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 42 \end{bmatrix},$$

your encoded position will be

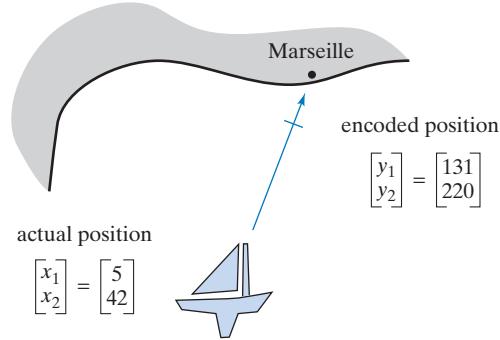
$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 5 + 3 \cdot 42 \\ 2 \cdot 5 + 5 \cdot 42 \end{bmatrix} = \begin{bmatrix} 131 \\ 220 \end{bmatrix}.$$

See Figure 1.

The coding transformation can be represented as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}},$$

## Linear Transformations



**Figure I**

or, more succinctly, as

$$\vec{y} = A\vec{x}.$$

The matrix  $A$  is called the (*coefficient*) *matrix* of the transformation.

A transformation of the form

$$\vec{y} = A\vec{x}$$

is called a *linear transformation*. We will discuss this important concept in greater detail later in this section and throughout this chapter.

As the ship reaches a new position, the sailor on duty at headquarters in Marseille receives the encoded message

$$\vec{b} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}.$$

He must determine the actual position of the boat. He will have to solve the linear system

$$A\vec{x} = \vec{b},$$

or, more explicitly,

$$\begin{vmatrix} x_1 + 3x_2 = 133 \\ 2x_1 + 5x_2 = 223 \end{vmatrix}.$$

Here is his solution. Is it correct?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 43 \end{bmatrix}$$

As the boat travels on and dozens of positions are radioed in, the sailor gets a little tired of solving all those linear systems, and he thinks there must be a general formula to simplify the task. He wants to solve the system

$$\begin{vmatrix} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = y_2 \end{vmatrix}$$

when  $y_1$  and  $y_2$  are arbitrary constants, rather than particular numerical values. He is looking for the *decoding transformation*

$$\vec{y} \rightarrow \vec{x},$$

## Linear Transformations

which is the *inverse*<sup>1</sup> of the coding transformation

$$\vec{x} \rightarrow \vec{y}.$$

The method of finding this solution is nothing new. We apply elimination as we have for a linear system with known values  $y_1$  and  $y_2$ :

$$\begin{array}{l|l} \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = y_2 \end{array} \right. & \xrightarrow{-2 \text{ (I)}} \quad \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ -x_2 = -2y_1 + y_2 \end{array} \right. \xrightarrow{\div(-1)} \\ \left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ x_2 = 2y_1 - y_2 \end{array} \right. & \xrightarrow{-3 \text{ (II)}} \quad \left| \begin{array}{l} x_1 = -5y_1 + 3y_2 \\ x_2 = 2y_1 - y_2 \end{array} \right. \end{array}$$

The formula for the decoding transformation is

$$\begin{aligned} x_1 &= -5y_1 + 3y_2, \\ x_2 &= 2y_1 - y_2, \end{aligned}$$

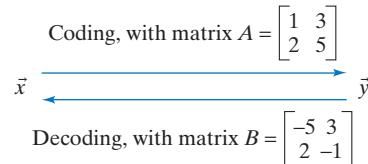
or

$$\vec{x} = B\vec{y}, \quad \text{where } B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Note that the decoding transformation is linear and that its coefficient matrix is

$$B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

The relationship between the two matrices  $A$  and  $B$  is shown in Figure 2.



**Figure 2**

Since the decoding transformation  $\vec{x} = B\vec{y}$  is the inverse of the coding transformation  $\vec{y} = A\vec{x}$ , we say that the matrix  $B$  is the *inverse* of the matrix  $A$ . We can write this as  $B = A^{-1}$ .

Not all linear transformations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

are invertible. Suppose some ignorant officer chooses the code

$$\begin{aligned} y_1 &= x_1 + 2x_2 & \text{with matrix } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \\ y_2 &= 2x_1 + 4x_2 \end{aligned}$$

for the French coast guard boats. When the sailor in Marseille has to decode a position, for example,

$$\vec{b} = \begin{bmatrix} 89 \\ 178 \end{bmatrix},$$

---

<sup>1</sup>We will discuss the concept of the inverse of a transformation more systematically in Section 4.

## Linear Transformations

he will be chagrined to discover that the system

$$\left| \begin{array}{l} x_1 + 2x_2 = 89 \\ 2x_1 + 4x_2 = 178 \end{array} \right|$$

has infinitely many solutions, namely,

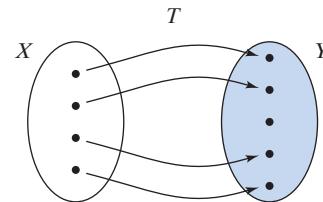
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 89 - 2t \\ t \end{bmatrix},$$

where  $t$  is an arbitrary number.

Because this system does not have a unique solution, it is impossible to recover the actual position from the encoded position: The coding transformation and the coding matrix  $A$  are *noninvertible*. This code is useless!

Now let us discuss the important concept of *linear transformations* in greater detail. Since linear transformations are a special class of functions, it may be helpful to review the concept of a *function* first.

Consider two sets  $X$  and  $Y$ . A function  $T$  from  $X$  to  $Y$  is a rule that associates with each element  $x$  of  $X$  a unique element  $y$  of  $Y$ . The set  $X$  is called the *domain* of the function, and  $Y$  is its *target space*. We will sometimes refer to  $x$  as the *input* of the function and to  $y$  as its *output*. Figure 3 shows an example where domain  $X$  and target space  $Y$  are finite.



**Figure 3** Domain  $X$  and target space  $Y$  of a function  $T$ .

In precalculus and calculus, you studied functions whose input and output are scalars (i.e., whose domain and target space are the real numbers  $\mathbb{R}$  or subsets of  $\mathbb{R}$ ); for example,

$$y = x^2, \quad f(x) = e^x, \quad g(t) = \frac{t^2 - 2}{t - 1}.$$

In multivariable calculus, you may have encountered functions whose input or output were vectors.

### EXAMPLE 1

$$y = x_1^2 + x_2^2 + x_3^2$$

This formula defines a function from the vector space  $\mathbb{R}^3$  to  $\mathbb{R}$ . The input is the vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ , and the output is the scalar  $y$ . ■

### EXAMPLE 2

$$\vec{r} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

This formula defines a function from  $\mathbb{R}$  to the vector space  $\mathbb{R}^3$ , with input  $t$  and output  $\vec{r}$ . ■

## Linear Transformations

We now return to the topic of linear transformations.

### Definition 1.1

#### Linear transformations<sup>2</sup>

A function  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  is called a *linear transformation* if there exists an  $n \times m$  matrix  $A$  such that

$$T(\vec{x}) = A\vec{x},$$

for all  $\vec{x}$  in the vector space  $\mathbb{R}^m$ .

It is important to note that a linear transformation is a special kind of *function*. The input and the output are both vectors. If we denote the output vector  $T(\vec{x})$  by  $\vec{y}$ , we can write

$$\vec{y} = A\vec{x}.$$

Let us write this equation in terms of its components:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix},$$

or

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots &= \vdots & \vdots & \vdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m. \end{aligned}$$

The output variables  $y_i$  are linear functions of the input variables  $x_j$ . In some branches of mathematics, a first-order function with a constant term, such as  $y = 3x_1 - 7x_2 + 5x_3 + 8$ , is called linear. Not so in linear algebra: The linear functions of  $m$  variables are those of the form  $y = c_1x_1 + c_2x_2 + \cdots + c_mx_m$ , for some coefficients  $c_1, c_2, \dots, c_m$ . By contrast, a function such as  $y = 3x_1 - 7x_2 + 5x_3 + 8$  is called *affine*.

### EXAMPLE 3

The linear transformation

$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 \\ y_3 &= 8x_1 + 4x_2 + 7x_4 \end{aligned}$$

(a function from  $\mathbb{R}^4$  to  $\mathbb{R}^3$ ) is represented by the  $3 \times 4$  matrix

$$A = \begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}.$$




---

<sup>2</sup>This is one of several possible definitions of a linear transformation; we could just as well have chosen the statement of Theorem 1.3 as the definition (as many texts do). Most of the central concepts of linear algebra can be characterized in two or more ways. Each of these characterizations can serve as a possible definition; the other characterizations will then be stated as theorems, since we need to prove that they are equivalent to the chosen definition. Among these multiple characterizations, there is no “correct” definition (although mathematicians may have their favorite). Each characterization will be best suited for certain purposes and problems, while it is inadequate for others.

## Linear Transformations

**EXAMPLE 4** The coefficient matrix of the *identity transformation*

$$\begin{aligned}y_1 &= x_1 \\y_2 &= x_2 \\&\vdots \quad \ddots \\y_n &= x_n\end{aligned}$$

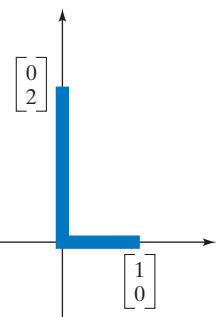
(a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  whose output equals its input) is the  $n \times n$  matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

All entries on the main diagonal are 1, and all other entries are 0. This matrix is called the *identity matrix* and is denoted by  $I_n$ :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.} \quad \blacksquare$$

**EXAMPLE 5** Consider the letter **L** (for Linear?) in Figure 4, made up of the vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ . Show the effect of the linear transformation



on this letter, and describe the transformation in words.

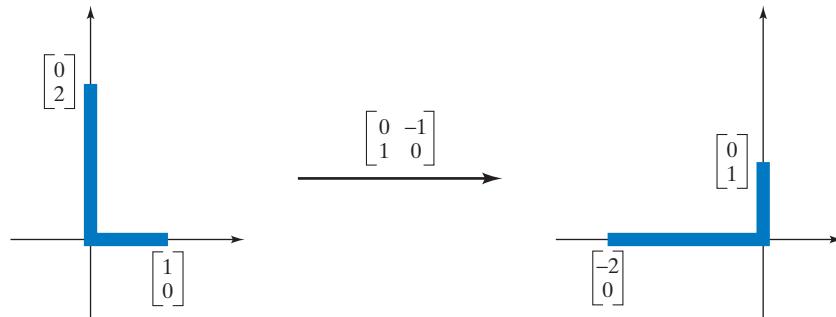
### Solution

We have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix},$$

as shown in Figure 5.

**Figure 4**



**Figure 5**

### Linear Transformations

The **L** is rotated through an angle of  $90^\circ$  in the counterclockwise direction.

Let's examine the effect of transformation  $T$  on an arbitrary vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ :

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

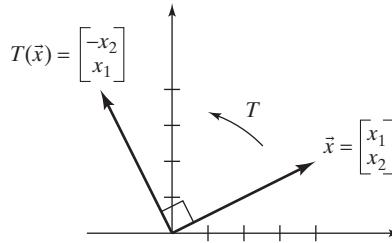
We observe that the vectors  $\vec{x}$  and  $T(\vec{x})$  have the same length,

$$\sqrt{x_1^2 + x_2^2} = \sqrt{(-x_2)^2 + x_1^2},$$

and that they are perpendicular to one another, since the dot product equals zero.

$$\vec{x} \cdot T(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -x_1 x_2 + x_2 x_1 = 0.$$

Paying attention to the signs of the components, we see that if  $\vec{x}$  is in the first quadrant (meaning that  $x_1$  and  $x_2$  are both positive), then  $T(\vec{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$  is in the second quadrant. See Figure 6.



**Figure 6**

We can conclude that  $T(\vec{x})$  is obtained by rotating vector  $\vec{x}$  through an angle of  $90^\circ$  in the counterclockwise direction, as in the special cases  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  considered earlier. (Check that the rotation is indeed counterclockwise when  $\vec{x}$  is in the second, third, or fourth quadrant.) ■

**EXAMPLE 6** Consider the linear transformation  $T(\vec{x}) = A\vec{x}$ , with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where for simplicity we write  $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  instead of  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ .

## Linear Transformations

### Solution

A straightforward computation shows that

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

and

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Note that  $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is the first column of the matrix  $A$  and that  $T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is its third column. ■

We can generalize this observation:

### Theorem 1.2

#### The columns of the matrix of a linear transformation

Consider a linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Then, the matrix of  $T$  is

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}, \quad \text{where } \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}.$$

To justify this result, write

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}.$$

Then

$$T(\vec{e}_i) = A\vec{e}_i = \begin{bmatrix} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_i & \cdots & \vec{v}_m \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i.$$

The vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$  in the vector space  $\mathbb{R}^m$  are sometimes referred to as the *standard vectors* in  $\mathbb{R}^m$ . The standard vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  in  $\mathbb{R}^3$  are often denoted by  $\vec{i}, \vec{j}, \vec{k}$ .

**EXAMPLE 7** Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .

- What is the relationship among  $T(\vec{v})$ ,  $T(\vec{w})$ , and  $T(\vec{v} + \vec{w})$ , where  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^m$ ?
- What is the relationship between  $T(\vec{v})$  and  $T(k\vec{v})$ , where  $\vec{v}$  is a vector in  $\mathbb{R}^m$  and  $k$  is a scalar?

### Solution

- Applying the algebraic rules for  $A\vec{x}$ , we find that

$$T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w}).$$

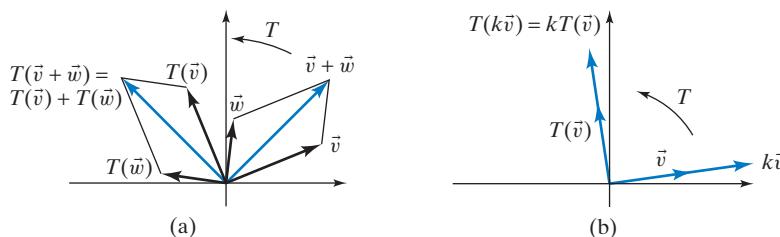
In words, the transform of the sum of two vectors equals the sum of the transforms.

- Again, by algebraic rules,

$$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}).$$

In words, the transform of a scalar multiple of a vector is the scalar multiple of the transform. ■

Figure 7 illustrates these two properties in the case of the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that rotates a vector through an angle of  $90^\circ$  in the counterclockwise direction. Compare this with Example 5.



**Figure 7** (a) Illustrating the property  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ .

(b) Illustrating the property  $T(k\vec{v}) = kT(\vec{v})$ .

In Example 7, we saw that a linear transformation satisfies the two equations  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  and  $T(k\vec{v}) = kT(\vec{v})$ . Now we will show that the converse is true as well: Any transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  that satisfies these two equations is a linear transformation.

### Theorem 1.3

#### Linear transformations

A transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  is linear if (and only if)

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ , for all vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^m$ , and
- $T(k\vec{v}) = kT(\vec{v})$ , for all vectors  $\vec{v}$  in  $\mathbb{R}^m$  and all scalars  $k$ .

**Proof** In Example 7, we saw that a linear transformation satisfies the equations in (a) and (b). To prove the converse, consider a transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  that satisfies equations (a) and (b). We must show that there exists a matrix  $A$  such that

## Linear Transformations

$T(\vec{x}) = A\vec{x}$ , for all  $\vec{x}$  in the vector space  $\mathbb{R}^m$ . Let  $\vec{e}_1, \dots, \vec{e}_m$  be the standard vectors introduced in Theorem 1.2.

$$\begin{aligned}
 T(\vec{x}) &= T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m) \\
 &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_m\vec{e}_m) \quad (\text{by property a}) \\
 &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m) \quad (\text{by property b}) \\
 &= \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = A\vec{x} \quad \blacksquare
 \end{aligned}$$

Here is an example illustrating Theorem 1.3.

**EXAMPLE 8** Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  such that  $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$  and  $T(\vec{v}_2) = 2\vec{v}_2$ , for the vectors  $\vec{v}_1$  and  $\vec{v}_2$  sketched in Figure 8. On the same axes, sketch  $T(\vec{x})$ , for the given vector  $\vec{x}$ . Explain your solution.

### Solution

Using a parallelogram, we can represent  $\vec{x}$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ , as shown in Figure 9:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2.$$

By Theorem 1.3,

$$T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = \frac{1}{2}c_1\vec{v}_1 + 2c_2\vec{v}_2.$$

The vector  $c_1\vec{v}_1$  is cut in half, and the vector  $c_2\vec{v}_2$  is doubled, as shown in Figure 10.

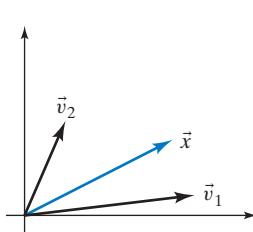


Figure 8

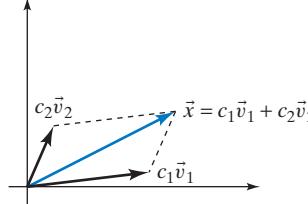


Figure 9

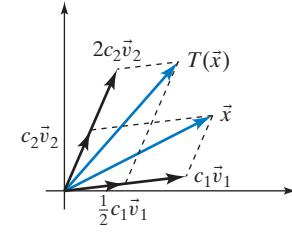


Figure 10

Imagine that vector  $\vec{x}$  is drawn on a rubber sheet. Transformation  $T$  expands this sheet by a factor of 2 in the  $\vec{v}_2$ -direction and contracts it by a factor of 2 in the  $\vec{v}_1$ -direction. (We prefer “contracts by a factor of 2” to the awkward “expands by a factor of  $\frac{1}{2}$ .”)  $\blacksquare$

We will conclude this section with one more example of a linear transformation, from computer science.

**EXAMPLE 9**

Let's develop a simple model of how people might surf the World Wide Web, by following links. To keep things manageable, we consider a "mini-Web" with only four pages, labeled 1, 2, 3, 4, linked as shown in the diagram below:

$$\begin{array}{ccc} 1 & \rightleftharpoons & 2 \\ \downarrow & \swarrow & \uparrow \\ 3 & \rightarrow & 4. \end{array}$$

Let  $x_1, x_2, x_3$ , and  $x_4$  be the proportions of the surfers who find themselves on each of the four pages initially; we can collect this information in the *distribution vector*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \text{ For example, the initial distribution might be } \vec{x} = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix},$$

meaning that 40% of the surfers are on page 1, and so forth. The components of a distribution vector add up to 1, or 100%, of course.

At a predetermined time, at the blow of a whistle, each surfer will randomly follow one of the links: If several links are available out of a web page, then an equal proportion of the surfers will follow each of them. For example, the proportion of surfers taking each of the two links out of page 1 (to pages 2 and 3) will be  $\frac{x_1}{2}$ .

Let the vector  $\vec{y}$ , with components  $y_1, y_2, y_3, y_4$ , represent the distribution of the surfers after the transition. According to the rules stated above, we will have

$$\begin{aligned} y_1 &= \frac{1}{2}x_2 \\ y_2 &= \frac{1}{2}x_1 + x_4 \\ y_3 &= \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ y_4 &= x_3 \end{aligned},$$

or  $\vec{y} = A\vec{x}$  in vector form, where

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

showing that  $\vec{y} = T(\vec{x})$  is a linear transformation. Matrix  $A$  is referred to as the *transition matrix* of this transformation.

For example, the equation  $y_2 = \frac{1}{2}x_1 + x_4$  reflects the fact that half of the surfers from page 1 and all of the surfers from page 4 will follow the links to page 2.

Alternatively, we can construct the matrix  $A$  column by column. The  $j$ th column of  $A$  tells us where the surfers go from page  $j$ . For example, the second column of  $A$  is

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix},$$

indicating that half of the surfers from page 2 take the link to page 1, and the other half go to page 3. Thus, the entries in each column of  $A$  must add up to 1.

Let's discuss the structure of matrix  $A$  more formally. Let  $c_j$  be the number of links going out of page  $j$ . For example, we have  $c_2 = 2$  and  $c_3 = 1$  for our

## Linear Transformations

mini-Web. Then the proportion of the surfers taking a link from page  $j$  to page  $i$  will be  $\frac{x_j}{c_j}$ , since the initial population  $x_j$  of page  $j$  gets distributed equally among the  $c_j$  outgoing links. We see that the  $ij$ th entry of  $A$  is  $\frac{1}{c_j}$  if there is a link from page  $j$  to page  $i$ ; the other entries are 0. Thus, we have

$$a_{ij} = \begin{cases} 1/c_j & \text{if there is a link } j \rightarrow i \\ 0 & \text{otherwise.} \end{cases}$$

We might wonder whether this system has any *equilibrium distributions*, that is, distributions  $\vec{x}$  such that  $A\vec{x} = \vec{x}$ . The distribution after the transition is required to be exactly the same as before. To find out, we have to solve the linear system

$$\left| \begin{array}{l} \frac{1}{2}x_2 = x_1 \\ \frac{1}{2}x_1 + x_4 = x_2 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 = x_3 \\ x_3 = x_4 \end{array} \right| \quad \text{or} \quad \left| \begin{array}{l} -x_1 + \frac{1}{2}x_2 = 0 \\ \frac{1}{2}x_1 - x_2 + x_4 = 0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - x_3 = 0 \\ x_3 - x_4 = 0 \end{array} \right|$$

with augmented matrix  $M = \left[ \begin{array}{ccccc} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$  and  $\text{rref}(M) = \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -4/3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$ . The solutions are  $x_1 = \frac{2t}{3}$ ,  $x_2 = \frac{4t}{3}$ ,  $x_3 = x_4 = t$ ,

where  $t$  is an arbitrary real number. Since we are looking for a distribution vector, we want  $x_1 + x_2 + x_3 + x_4 = 4t = 1$ , so  $t = \frac{1}{4}$ . The equilibrium distribution is

$$\vec{x}_{equ} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} \approx \begin{bmatrix} 16.7\% \\ 33.3\% \\ 25\% \\ 25\% \end{bmatrix}.$$

In this context, an interesting question arises: If we iterate our transition, letting the surfers move to a new page over and over again, following links at random, will the system eventually approach this equilibrium state  $\vec{x}_{equ}$ , *regardless of the initial distribution*? Perhaps surprisingly, the answer is affirmative for the mini-Web considered in this example, as well as for many others: *The equilibrium distribution represents the distribution of the surfers in the long run, for any initial distribution.* We will further discuss these important issues in Theorem 3.11.

In 1998, Sergey Brin and Lawrence Page, then at Stanford University, published a landmark paper, “The Anatomy of a Large-Scale Hypertextual Search Engine,” where they present a prototype of the search engine Google. The key feature of this search engine is a “quality ranking” of each web page, which measures its “link popularity.” They call this ranking PageRank, after the second author of the paper.

The basic idea is to define the popularity of a web page as the likelihood that random surfers find themselves on that web page in the long run. But this is exactly the corresponding component of the equilibrium vector  $\vec{x}_{equ}$  we found above. In our example, page 2 is the most popular, with a PageRank of  $1/3$ , while page 1 is only half as popular, with a PageRank of  $1/6$ .

Actually, the model developed by Brin and Page in their 1998 paper is a bit more complicated, involving a “jumping rate” as well. They assume that surfers will not always follow links, but sometimes randomly jump to a new page, even if there is no link available to that page. The “jumping rate,” often set at 0.15 or 0.2, represents the proportion of transitions where a surfer jumps rather than following a link. See Exercise 53. In our example, we set the jumping rate to be 0 in order to simplify the computation. The resulting popularity ranking (with jumping rate 0) is sometimes referred to as the *naïve PageRank*. ■

Example 9 motivates the following definitions:

#### Definition 1.4

#### Distribution vectors and transition matrices

A vector  $\vec{x}$  in  $\mathbb{R}^n$  is said to be a *distribution vector* if its components add up to 1 and all the components are positive or zero.

A square matrix  $A$  is said to be a *transition matrix* (or stochastic matrix) if all its columns are distribution vectors. This means that all the entries of a transition matrix are positive or zero, and the entries in each column add up to 1.

If  $A$  is a transition matrix and  $\vec{x}$  is a distribution vector, then  $A\vec{x}$  will be a distribution vector as well. See Exercise 49.

In Example 9, the vector  $\vec{x}_{equ}$  is a distribution vector and the matrix  $A$  is a transition matrix. Simple examples of transition matrices are  $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ , while  $\begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 1.1 \end{bmatrix}$  fails to be a transition matrix since one of its entries is negative.

## EXERCISES |

**GOAL** Use the concept of a linear transformation in terms of the formula  $\vec{y} = A\vec{x}$ , and interpret simple linear transformations geometrically. Find the inverse of a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  (if it exists). Find the matrix of a linear transformation column by column.

Consider the transformations from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined in Exercises 1 through 3. Which of these transformations are linear?

$$\begin{array}{lll} 1. \quad y_1 = 2x_2 & 2. \quad y_1 = 2x_2 & 3. \quad y_1 = x_2 - x_3 \\ y_2 = x_2 + 2 & y_2 = 3x_3 & y_2 = x_1x_3 \\ y_3 = 2x_2 & y_3 = x_1 & y_3 = x_1 - x_2 \end{array}$$

4. Find the matrix of the linear transformation

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3. \end{aligned}$$

5. Consider the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^2$  with

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix},$$

$$\text{and } T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}.$$

Find the matrix  $A$  of  $T$ .

6. Consider the transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^3$  given by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

7. Suppose  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are arbitrary vectors in  $\mathbb{R}^n$ . Consider the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m.$$

Is this transformation linear? If so, find its matrix  $A$  in terms of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ .

## Linear Transformations

8. Find the inverse of the linear transformation

$$\begin{aligned}y_1 &= x_1 + 7x_2 \\y_2 &= 3x_1 + 20x_2.\end{aligned}$$

In Exercises 9 through 12, decide whether the given matrix is invertible. Find the inverse if it exists. In Exercise 12, the constant  $k$  is arbitrary.

9.  $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

10.  $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$

12.  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

13. Prove the following facts:

- a. The  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if  $ad - bc \neq 0$ . Hint: Consider the cases  $a \neq 0$  and  $a = 0$  separately.

- b. If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

[The formula in part (b) is worth memorizing.]

14. a. For which values of the constant  $k$  is the matrix  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$  invertible?

- b. For which values of the constant  $k$  are all entries of  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$  integers?

See Exercise 13.

15. For which values of the constants  $a$  and  $b$  is the matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

invertible? What is the inverse in this case? See Exercise 13.

*Give a geometric interpretation of the linear transformations defined by the matrices in Exercises 16 through 23. Show the effect of these transformations on the letter L considered in Example 5. In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically. See Exercise 13.*

16.  $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

17.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

18.  $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

19.  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

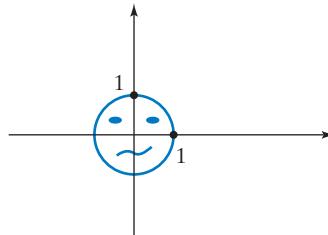
20.  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

21.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

22.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

23.  $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Consider the circular face in the accompanying figure. For each of the matrices  $A$  in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation  $T(\vec{x}) = A\vec{x}$  on this face.

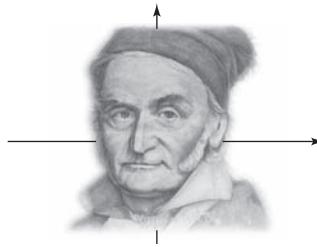


24.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$     25.  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$     26.  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

27.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$     28.  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$     29.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

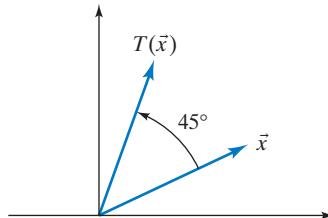
30.  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

31. An old German bill shows the mirror image of Gauss's likeness. What linear transformation  $T$  can you apply to get the actual picture back?



32. Find an  $n \times n$  matrix  $A$  such that  $A\vec{x} = 3\vec{x}$ , for all  $\vec{x}$  in  $\mathbb{R}^n$ .

33. Consider the transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that rotates any vector  $\vec{x}$  through an angle of  $45^\circ$  in the counterclockwise direction, as shown in the following figure:



You are told that  $T$  is a linear transformation. (This will be shown in the next section.) Find the matrix of  $T$ .

34. Consider the transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that rotates any vector  $\vec{x}$  through a given angle  $\theta$  in the counterclockwise direction. Compare this with Exercise 33. You are told that  $T$  is linear. Find the matrix of  $T$  in terms of  $\theta$ .

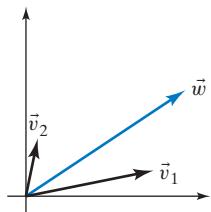
## Linear Transformations

- 35.** In the example about the French coast guard in this section, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

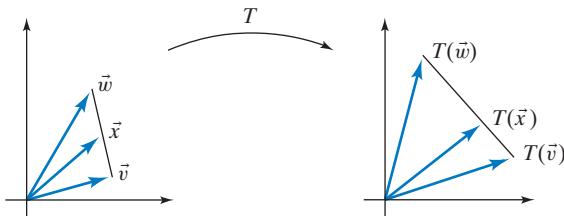
- When the actual position is  $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$ , they radio  $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$ .
- When the actual position is  $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$ , they radio  $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$ .

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

- 36.** Let  $T$  be a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Let  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{w}$  be three vectors in  $\mathbb{R}^2$ , as shown below. We are told that  $T(\vec{v}_1) = \vec{v}_1$  and  $T(\vec{v}_2) = 3\vec{v}_2$ . On the same axes, sketch  $T(\vec{w})$ .



- 37.** Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Suppose that  $\vec{v}$  and  $\vec{w}$  are two arbitrary vectors in  $\mathbb{R}^2$  and that  $\vec{x}$  is a third vector whose endpoint is on the line segment connecting the endpoints of  $\vec{v}$  and  $\vec{w}$ . Is the endpoint of the vector  $T(\vec{x})$  necessarily on the line segment connecting the endpoints of  $T(\vec{v})$  and  $T(\vec{w})$ ? Justify your answer.



*Hint:* We can write  $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$ , for some scalar  $k$  between 0 and 1.

We can summarize this exercise by saying that a linear transformation maps a line onto a line.

- 38.** The two column vectors  $\vec{v}_1$  and  $\vec{v}_2$  of a  $2 \times 2$  matrix  $A$  are shown in the accompanying sketch. Consider the linear transformation  $T(\vec{x}) = A\vec{x}$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Sketch the vector

$$T \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$



- 39.** Show that if  $T$  is a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , then

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m),$$

where  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$  are the standard vectors in  $\mathbb{R}^m$ .

- 40.** Describe all linear transformations from  $\mathbb{R}$  ( $= \mathbb{R}^1$ ) to  $\mathbb{R}$ . What do their graphs look like?

- 41.** Describe all linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}$  ( $= \mathbb{R}^1$ ). What do their graphs look like?

- 42.** When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a computer screen), you have to transform spatial coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

into plane coordinates,  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ . The simplest choice is a linear transformation, for example, the one given by the matrix

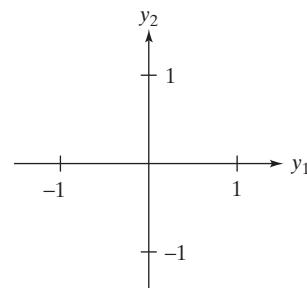
$$\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

- a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Include the images of the  $x_1$ ,  $x_2$ , and  $x_3$  axes in your sketch:



- b. Represent the image of the point  $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$  in your figure in part (a).

(part c on next page)

## Linear Transformations

- c. Find all the points

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{in } \mathbb{R}^3$$

that are transformed to  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . Explain.

43. a. Consider the vector  $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ . Is the transformation  $T(\vec{x}) = \vec{v} \cdot \vec{x}$  (the dot product) from  $\mathbb{R}^3$  to  $\mathbb{R}$  linear? If so, find the matrix of  $T$ .

- b. Consider an arbitrary vector  $\vec{v}$  in  $\mathbb{R}^3$ . Is the transformation  $T(\vec{x}) = \vec{v} \cdot \vec{x}$  linear? If so, find the matrix of  $T$  (in terms of the components of  $\vec{v}$ ).

- c. Conversely, consider a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}$ . Show that there exists a vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $T(\vec{x}) = \vec{v} \cdot \vec{x}$ , for all  $\vec{x}$  in  $\mathbb{R}^3$ .

44. The cross product of two vectors in  $\mathbb{R}^3$  is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector  $\vec{v}$  in  $\mathbb{R}^3$ . Is the transformation  $T(\vec{x}) = \vec{v} \times \vec{x}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  linear? If so, find its matrix in terms of the components of the vector  $\vec{v}$ .

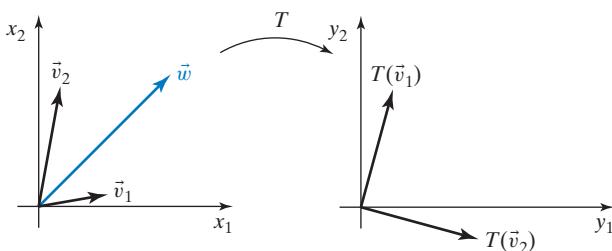
45. Consider two linear transformations  $\vec{y} = T(\vec{x})$  and  $\vec{z} = L(\vec{y})$ , where  $T$  goes from  $\mathbb{R}^m$  to  $\mathbb{R}^p$  and  $L$  goes from  $\mathbb{R}^p$  to  $\mathbb{R}^n$ . Is the transformation  $\vec{z} = L(T(\vec{x}))$  linear as well? [The transformation  $\vec{z} = L(T(\vec{x}))$  is called the *composite* of  $T$  and  $L$ .]

46. Let

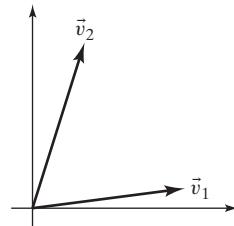
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation  $T(\vec{x}) = B(A\vec{x})$ . See Exercise 45. Hint: Find  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$ .

47. Let  $T$  be a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Three vectors  $\vec{v}_1, \vec{v}_2, \vec{w}$  in  $\mathbb{R}^2$  and the vectors  $T(\vec{v}_1), T(\vec{v}_2)$  are shown in the accompanying figure. Sketch  $T(\vec{w})$ . Explain your answer.



48. Consider two linear transformations  $T$  and  $L$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . We are told that  $T(\vec{v}_1) = L(\vec{v}_1)$  and  $T(\vec{v}_2) = L(\vec{v}_2)$  for the vectors  $\vec{v}_1$  and  $\vec{v}_2$  sketched below. Show that  $T(\vec{x}) = L(\vec{x})$ , for all vectors  $\vec{x}$  in  $\mathbb{R}^2$ .



49. Prove that if  $A$  is a transition matrix and  $\vec{x}$  is a distribution vector, then  $A\vec{x}$  is a distribution vector as well.

*For each of the mini-Webs in Exercises 50 through 52,*

- a. *find the transition matrix  $A$  as defined in Example 9,*  
 b. *find the equilibrium distribution, and*  
 c. *find the web page(s) with the highest (naïve) Page-Rank.*

*Feel free to use technology throughout.*

- 50.

$$\begin{array}{c} 1 \rightleftharpoons 2 \\ \downarrow \nearrow \uparrow \\ 3 \rightarrow 4 \end{array}$$

- 51.

$$\begin{array}{c} 1 \rightarrow 2 \\ \uparrow \nearrow \downarrow \\ 3 \rightleftharpoons 4 \end{array}$$

- 52.

$$\begin{array}{c} 1 \rightleftharpoons 2 \\ \downarrow \nearrow \\ 3 \end{array}$$

53. As in Example 9, consider the mini-Web with the link structure shown in the diagram below:

$$\begin{array}{c} 1 \rightleftharpoons 2 \\ \downarrow \swarrow \uparrow \\ 3 \rightarrow 4. \end{array}$$

Again, let vector  $\vec{x}$  represent the distribution of surfers among the various pages at a given time. Following the 1998 paper by Brin and Page (see Example 9), we consider a more sophisticated model of transition: At the blow of a whistle, 80% of the surfers on a given page will randomly follow a link, while the remaining 20% will randomly “jump” to a web page (even if no link is available). The jumpers have the option to stay on the current page. For example, of the surfers that are initially on page 2, 40% will follow the link to page 1, 40% will follow the link to page 3, and 5% will jump to any of the four webpages. If we add up the jumpers and the link-followers, we see that 45% will move to page 1; another 45% go to page 3; 5% will jump to page 4; and 5% will stay put on page 2. Thus, the second column of the transition matrix  $B$  will be

## Linear Transformations

$$\begin{bmatrix} 0.4 + 0.05 \\ 0.05 \\ 0.4 + 0.05 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.05 \\ 0.45 \\ 0.05 \end{bmatrix}.$$

↑      ↗  
 link      jumpers  
 followers

Let  $\vec{y}$  be the distribution vector after the transition; we have  $y_2 = 0.45x_1 + 0.05x_2 + 0.05x_3 + 0.85x_4$ , for example.

- a. Find the transition matrix  $B$  such that  $\vec{y} = B\vec{x}$ .
- b. Explain why  $B = \frac{0.2}{N}E + 0.8A = 0.05E + 0.8A$ , where  $A$  is the matrix we found in Example 9,  $E$  is the matrix whose entries are all 1, and  $N$  is the total number of web pages.
- c. Find the equilibrium distribution  $\vec{x}_{equ}$ , such that  $B\vec{x}_{equ} = \vec{x}_{equ}$ . Feel free to use technology. (The components of this equilibrium solution represent the PageRank of the various web pages, as defined in the 1998 paper by Brin and Page.)

*For each of the mini-Webs in Exercises 54 through 56,*

- a. *find the transition matrix  $B$  as defined in Exercise 53 (with a jumping rate of 20%),*
- b. *find the equilibrium distribution for the transition matrix  $B$ , and*
- c. *find the web page(s) with the highest PageRank (with a jumping rate of 20%).*

*Feel free to use technology throughout.*

54.

$$\begin{array}{c} 1 \rightleftharpoons 2 \\ \downarrow \nearrow \uparrow \\ 3 \rightarrow 4 \end{array}$$

55.

$$\begin{array}{c} 1 \rightarrow 2 \\ \uparrow \nearrow \downarrow \\ 3 \rightleftharpoons 4 \end{array}$$

56.

$$\begin{array}{c} 1 \rightleftharpoons 2 \\ \downarrow \nearrow \\ 3 \end{array}$$

57. Some parking meters in downtown Geneva, Switzerland, accept 2 Franc and 5 Franc coins.

- a. A parking officer collects 51 coins worth 144 Francs. How many coins are there of each kind?
- b. Find the matrix  $A$  that transforms the vector

$$\begin{bmatrix} \text{number of 2 Franc coins} \\ \text{number of 5 Franc coins} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix}.$$

- c. Is the matrix  $A$  in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

58. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

- a. King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?

- b. Find the matrix  $A$  that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix},$$

for any piece of jewelry this goldsmith makes.

- c. Is the matrix  $A$  in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

59. The conversion formula  $C = \frac{5}{9}(F - 32)$  from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

- a. Find the  $2 \times 2$  matrix  $A$  that transforms the vector  $\begin{bmatrix} F \\ 1 \end{bmatrix}$  into the vector  $\begin{bmatrix} C \\ 1 \end{bmatrix}$ . (The second row of  $A$  will be  $[0 \ 1]$ .)
- b. Is the matrix  $A$  in part (a) invertible? If so, find the inverse (use Exercise 13). Use the result to write a formula expressing  $F$  in terms of  $C$ .

60. In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (C\$) and the South African Rand (ZAR). Consider the matrix

$$A = \begin{bmatrix} \text{C\$} & \text{ZAR} \\ 1 & 1/8 \\ 8 & 1 \end{bmatrix} \begin{matrix} \text{C\$} \\ \text{ZAR} \end{matrix}$$

## Linear Transformations

representing the fact that C\$1 is worth ZAR8 (as of September 2012).

- a. After a trip you have C\$100 and ZAR1,600 in your pocket. We represent these two values in the vector  $\vec{x} = \begin{bmatrix} 100 \\ 1,600 \end{bmatrix}$ . Compute  $A\vec{x}$ . What is the practical significance of the two components of the vector  $A\vec{x}$ ?
  - b. Verify that matrix  $A$  fails to be invertible. For which vectors  $\vec{b}$  is the system  $A\vec{x} = \vec{b}$  consistent? What is the practical significance of your answer? If the system  $A\vec{x} = \vec{b}$  is consistent, how many solutions  $\vec{x}$  are there? Again, what is the practical significance of the answer?
61. Consider a larger currency exchange matrix (see Exercise 60), involving four of the world's leading currencies: Euro (€), U.S. dollar (\$), Chinese yuan (¥), and British pound (£).

$$A = \begin{bmatrix} \text{€} & \$ & ¥ & £ \\ * & 0.8 & * & * \\ * & * & * & * \\ * & * & * & 10 \\ 0.8 & * & * & * \end{bmatrix}$$

The entry  $a_{ij}$  gives the value of one unit of the  $j$ th currency, expressed in terms of the  $i$ th currency. For example,  $a_{34} = 10$  means that £1 = ¥10 (as of August 2012). Find the exact values of the 13 missing entries of  $A$  (expressed as fractions).

62. Consider an arbitrary currency exchange matrix  $A$ . See Exercises 60 and 61.

- a. What are the diagonal entries  $a_{ii}$  of  $A$ ?
- b. What is the relationship between  $a_{ij}$  and  $a_{ji}$ ?
- c. What is the relationship among  $a_{ik}$ ,  $a_{kj}$ , and  $a_{ij}$ ?
- d. What is the rank of  $A$ ? What is the relationship between  $A$  and  $\text{rref}(A)$ ?

63. Solving a linear system  $A\vec{x} = \vec{0}$  by Gaussian elimination amounts to writing the vector of leading variables as a linear transformation of the vector of free variables. Consider the linear system

$$\begin{array}{rl} x_1 - x_2 & + 4x_5 = 0 \\ x_3 & - x_5 = 0 \\ x_4 - 2x_5 & = 0. \end{array}$$

Find the matrix  $B$  such that  $\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = B \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$ .

64. Consider the linear system

$$\begin{array}{l} x_1 + 2x_2 + x_3 + 7x_4 = 0 \\ x_1 + 2x_2 + 2x_3 + 11x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 15x_4 = 0 \\ x_1 + 2x_2 + 4x_3 + 19x_4 = 0. \end{array}$$

Find the matrix  $B$  such that  $\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = B \begin{bmatrix} x_2 \end{bmatrix}$ . See Exercise 63.

## 2 Linear Transformations in Geometry

In Example 1.5 we saw that the matrix  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents a counterclockwise rotation through  $90^\circ$  in the coordinate plane. Many other  $2 \times 2$  matrices define simple geometrical transformations as well; this section is dedicated to a discussion of some of those transformations.

**EXAMPLE I** Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

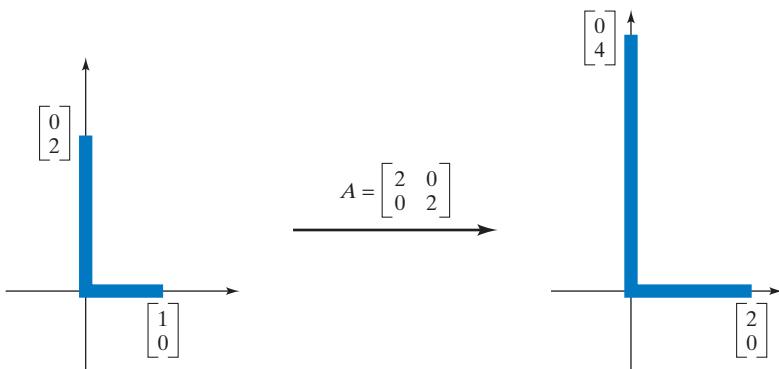
$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Show the effect of each of these matrices on our standard letter L,<sup>3</sup> and describe each transformation in words.

<sup>3</sup>See Example 1.5. Recall that vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is the foot of our standard L, and  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  is its back.

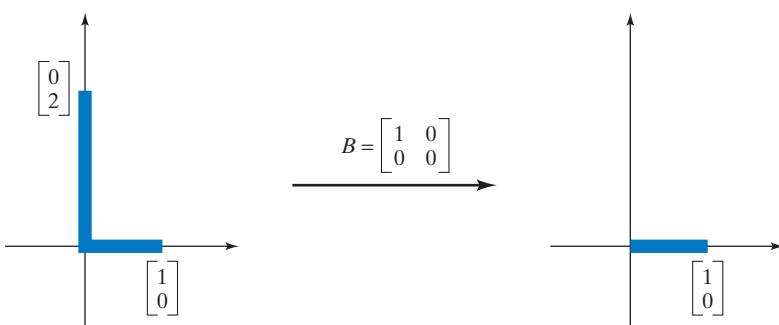
### Linear Transformations

a.



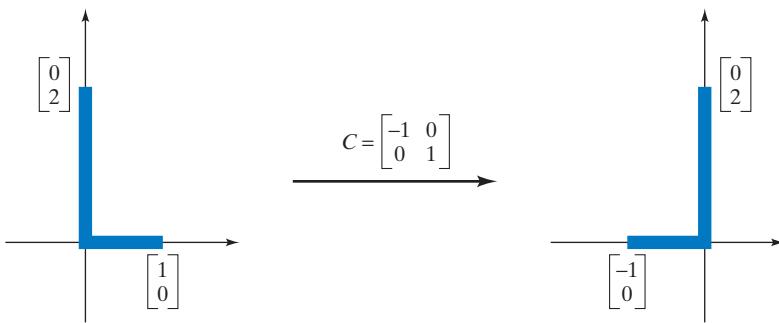
The L gets enlarged by a factor of 2; we will call this transformation a *scaling* by 2.

b.



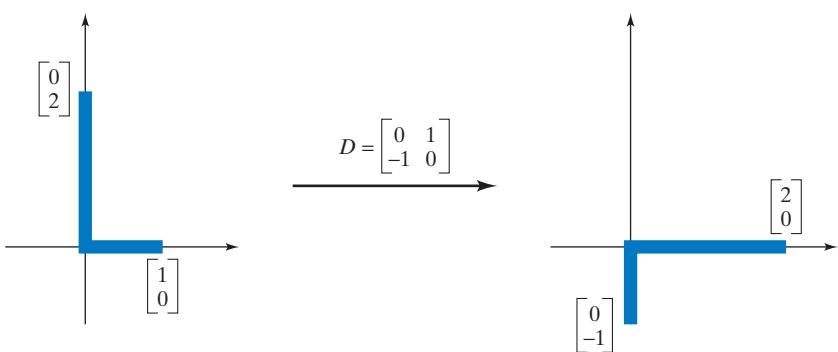
The L gets smashed into the horizontal axis. We will call this transformation the *orthogonal projection onto the horizontal axis*.

c.



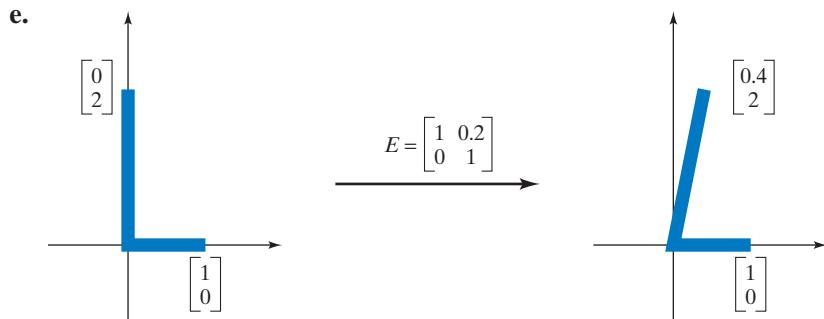
The L gets flipped over the vertical axis. We will call this the *reflection about the vertical axis*.

d.

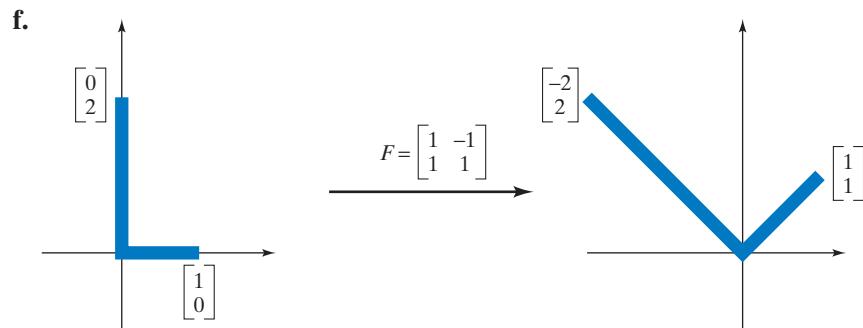


### Linear Transformations

The L is *rotated* through  $90^\circ$ , in the clockwise direction (this amounts to a rotation through  $-90^\circ$ ). The result is the opposite of what we got in Example 1.5.



The foot of the L remains unchanged, while the back is shifted horizontally to the right; the L is italicized, becoming *L*. We will call this transformation a *horizontal shear*.



There are two things going on here: The L is rotated through  $45^\circ$  and also enlarged (scaled) by a factor of  $\sqrt{2}$ . This is a *rotation combined with a scaling* (you may perform the two transformations in either order). Among all the possible composites of the transformations considered in parts (a) through (e), this one is particularly important in applications as well as in pure mathematics. ■

We will now take a closer look at the six types of transformations we encountered in Example 1.

### Scalings

For any positive constant  $k$ , the matrix  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$  defines a scaling by  $k$ , since

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

This is a *dilation* (or enlargement) if  $k$  exceeds 1, and it is a *contraction* (or shrinking) for values of  $k$  between 0 and 1. (What happens when  $k$  is negative or zero?)

## Orthogonal Projections<sup>4</sup>

Consider a line  $L$  in the plane, running through the origin. Any vector  $\vec{x}$  in  $\mathbb{R}^2$  can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where  $\vec{x}^{\parallel}$  is parallel to line  $L$ , and  $\vec{x}^{\perp}$  is perpendicular to  $L$ . See Figure 1.

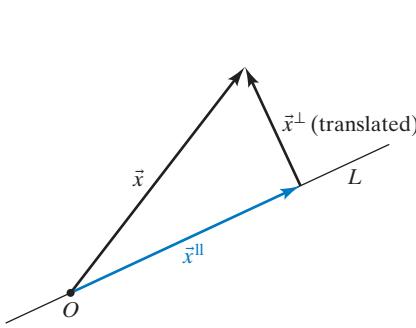


Figure 1

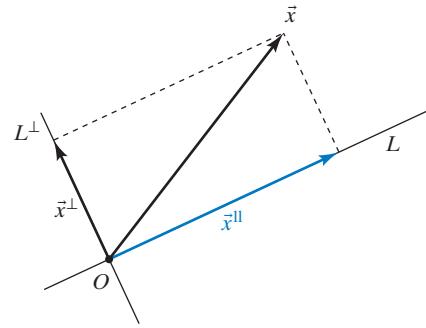


Figure 2

The transformation  $T(\vec{x}) = \vec{x}^{\parallel}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  is called the *orthogonal projection of  $\vec{x}$  onto  $L$* , often denoted by  $\text{proj}_L(\vec{x})$ :

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}.$$

You can think of  $\text{proj}_L(\vec{x})$  as the shadow that vector  $\vec{x}$  casts on  $L$  if you shine a light straight down on  $L$ .

Let  $L^{\perp}$  be the line through the origin perpendicular to  $L$ . Note that  $\vec{x}^{\perp}$  is parallel to  $L^{\perp}$ , and we can interpret  $\vec{x}^{\perp}$  as the orthogonal projection of  $\vec{x}$  onto  $L^{\perp}$ , as illustrated in Figure 2.

We can use the dot product to write a formula for an orthogonal projection.

To find a formula for  $\vec{x}^{\parallel}$ , let  $\vec{w}$  be a nonzero vector parallel to  $L$ . Since  $\vec{x}^{\parallel}$  is parallel to  $\vec{w}$ , we can write

$$\vec{x}^{\parallel} = k\vec{w},$$

for some scalar  $k$  about to be determined. Now  $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w}$  is perpendicular to line  $L$ , that is, perpendicular to  $\vec{w}$ , meaning that

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0.$$

It follows that

$$\vec{x} \cdot \vec{w} - k(\vec{w} \cdot \vec{w}) = 0, \quad \text{or} \quad k = \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}.$$

We can conclude that

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = k\vec{w} = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}.$$

---

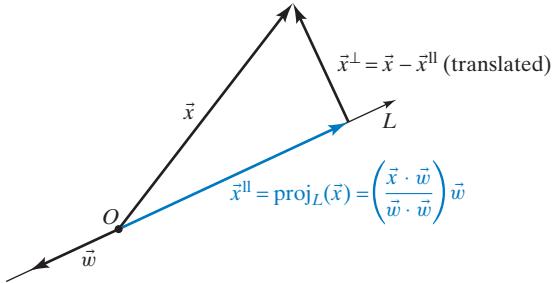
<sup>4</sup>The term *orthogonal* is synonymous with perpendicular. For a more general discussion of projections, see Exercise 33.

### Linear Transformations

See Figure 3. Consider the special case of a *unit* vector  $\vec{u}$  parallel to  $L$ . Then the formula for projection simplifies to

$$\text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u} = (\vec{x} \cdot \vec{u}) \vec{u}$$

since  $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1$  for a unit vector  $\vec{u}$ .



**Figure 3**

Is the transformation  $T(\vec{x}) = \text{proj}_L(\vec{x})$  linear? If so, what is its matrix? If we write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

then

$$\begin{aligned} \text{proj}_L(\vec{x}) &= (\vec{x} \cdot \vec{u}) \vec{u} = \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= (x_1 u_1 + x_2 u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 x_1 + u_1 u_2 x_2 \\ u_1 u_2 x_1 + u_2^2 x_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \vec{x}. \end{aligned}$$

It turns out that  $T(\vec{x}) = \text{proj}_L(\vec{x})$  is indeed a linear transformation, with matrix  $\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$ . More generally, if  $\vec{w}$  is a nonzero vector parallel to  $L$ , then the matrix is  $P = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$ . See Exercise 12.

**EXAMPLE 2** Find the matrix  $P$  of the orthogonal projection onto the line  $L$  spanned by  $\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

**Solution**

$$P = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

■

Let us summarize our findings.

**Definition 2.1**
**Orthogonal Projections**

Consider a line  $L$  in the coordinate plane, running through the origin. Any vector  $\vec{x}$  in  $\mathbb{R}^2$  can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where  $\vec{x}^{\parallel}$  is parallel to line  $L$ , and  $\vec{x}^{\perp}$  is perpendicular to  $L$ .

The transformation  $T(\vec{x}) = \vec{x}^{\parallel}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  is called the *orthogonal projection of  $\vec{x}$  onto  $L$* , often denoted by  $\text{proj}_L(\vec{x})$ . If  $\vec{w}$  is a nonzero vector parallel to  $L$ , then

$$\text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}.$$

In particular, if  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  is a *unit* vector parallel to  $L$ , then

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}.$$

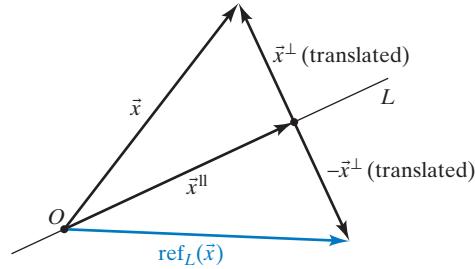
The transformation  $T(\vec{x}) = \text{proj}_L(\vec{x})$  is linear, with matrix

$$P = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}.$$

**Reflections**

Again, consider a line  $L$  in the coordinate plane, running through the origin, and let  $\vec{x}$  be a vector in  $\mathbb{R}^2$ . The reflection  $\text{ref}_L(\vec{x})$  of  $\vec{x}$  about  $L$  is shown in Figure 4: We are flipping vector  $\vec{x}$  over the line  $L$ . The line segment joining the tips of vectors  $\vec{x}$  and  $\text{ref}_L(\vec{x})$  is perpendicular to line  $L$  and bisected by  $L$ . In previous math courses you have surely seen examples of reflections about the horizontal and vertical axes [when comparing the graphs of  $y = f(x)$ ,  $y = -f(x)$ , and  $y = f(-x)$ , for example].

We can use the representation  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$  to write a formula for  $\text{ref}_L(\vec{x})$ . See Figure 4.



**Figure 4**

We can see that

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

Adding up the equations  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$  and  $\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ , we find that  $\vec{x} + \text{ref}_L(\vec{x}) = 2\vec{x}^{\parallel} = 2\text{proj}_L(\vec{x})$ , so

$$\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2P\vec{x} - \vec{x} = (2P - I_2)\vec{x},$$

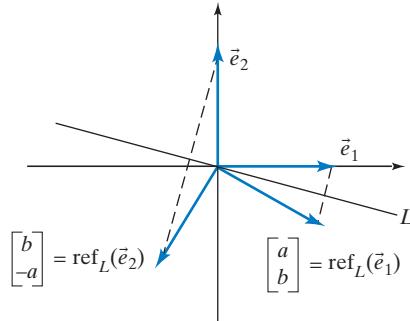
## Linear Transformations

where  $P$  is the matrix representing the orthogonal projection onto the line  $L$ . See Definition 2.1. Thus, the matrix  $S$  of the reflection is

$$S = 2P - I_2 = \begin{bmatrix} 2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix},$$

It turns out that this matrix  $S$  is of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$  (we leave the straightforward verification as Exercise 13). Conversely, any matrix of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , with  $a^2 + b^2 = 1$ , represents a reflection about a line. See Exercise 17.

We are not surprised to see that the column vectors  $\begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} b \\ -a \end{bmatrix}$  of a reflection matrix are unit vectors, with  $a^2 + b^2 = b^2 + (-a)^2 = 1$ . Indeed, the column vectors are the reflections of the standard vectors,  $\begin{bmatrix} a \\ b \end{bmatrix} = \text{ref}_L(\vec{e}_1)$  and  $\begin{bmatrix} b \\ -a \end{bmatrix} = \text{ref}_L(\vec{e}_2)$ , by Theorem 1.2. Since the standard vectors  $\vec{e}_1$  and  $\vec{e}_2$  are unit vectors and a reflection preserves length, these column vectors will be unit vectors as well. Also, it makes sense that the column vectors are perpendicular, with dot product  $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab + b(-a) = 0$ , since the reflection preserves the right angle between  $\vec{e}_1$  and  $\vec{e}_2$ . See Figure 5.



**Figure 5**

### Definition 2.2

#### Reflections

Consider a line  $L$  in the coordinate plane, running through the origin, and let  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$  be a vector in  $\mathbb{R}^2$ . The linear transformation  $T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$  is called the *reflection of  $\vec{x}$  about  $L$* , often denoted by  $\text{ref}_L(\vec{x})$ :

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

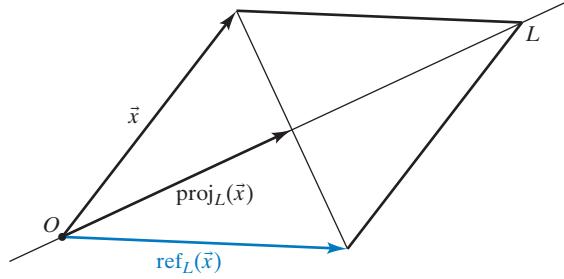
We have a formula relating  $\text{ref}_L(\vec{x})$  to  $\text{proj}_L(\vec{x})$ :

$$\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}.$$

The matrix of  $T$  is of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ . Conversely, any matrix of this form represents a reflection about a line.

## Linear Transformations

Use Figure 6 to explain the formula  $\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$  geometrically.



**Figure 6**

### Orthogonal Projections and Reflections in Space

Although this section is mostly concerned with linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , we will take a quick look at orthogonal projections and reflections in space, since this theory is analogous to the case of two dimensions.

Let  $L$  be a line in coordinate space, running through the origin. Any vector  $\vec{x}$  in  $\mathbb{R}^3$  can be written uniquely as  $\vec{x} = \vec{x}^\parallel + \vec{x}^\perp$ , where  $\vec{x}^\parallel$  is parallel to  $L$ , and  $\vec{x}^\perp$  is perpendicular to  $L$ . We define

$$\text{proj}_L(\vec{x}) = \vec{x}^\parallel,$$

and we have the formula

$$\text{proj}_L(\vec{x}) = \vec{x}^\parallel = (\vec{x} \cdot \vec{u})\vec{u},$$

where  $\vec{u}$  is a unit vector parallel to  $L$ . See Definition 2.1.

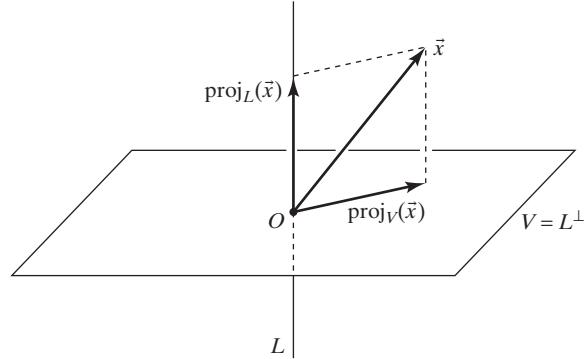
Let  $L^\perp = V$  be the plane through the origin perpendicular to  $L$ ; note that the vector  $\vec{x}^\perp$  will be parallel to  $L^\perp = V$ . We can give formulas for the orthogonal projection onto  $V$ , as well as for the reflections about  $V$  and  $L$ , in terms of the orthogonal projection onto  $L$ :

$$\text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u},$$

$$\text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}, \quad \text{and}$$

$$\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x}) = -\text{ref}_L(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}.$$

See Figure 7, and compare with Definition 2.2.



**Figure 7**

**EXAMPLE 3** Let  $V$  be the plane defined by  $2x_1 + x_2 - 2x_3 = 0$ , and let  $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$ . Find  $\text{ref}_V(\vec{x})$ .

**Solution**

Note that the vector  $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$  is perpendicular to plane  $V$  (the components of  $\vec{v}$  are the coefficients of the variables in the given equation of the plane: 2, 1, and  $-2$ ). Thus,

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

is a unit vector perpendicular to  $V$ , and we can use the formula we derived earlier:

$$\begin{aligned} \text{ref}_V(\vec{x}) &= \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \frac{2}{9} \left( \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}. \end{aligned}$$
■

### Rotations

Consider the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that rotates any vector  $\vec{x}$  through a fixed angle  $\theta$  in the counterclockwise direction,<sup>5</sup> as shown in Figure 8. Recall Example 1.5, where we studied a rotation through  $\theta = \pi/2$ .

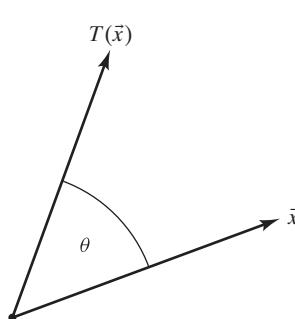


Figure 8

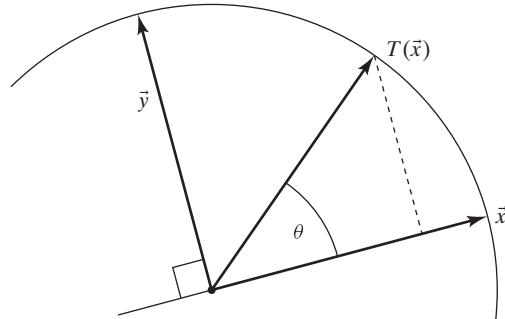


Figure 9

---

<sup>5</sup>We can define a rotation more formally in terms of the polar coordinates of  $\vec{x}$ . The length of  $T(\vec{x})$  equals the length of  $\vec{x}$ , and the polar angle (or argument) of  $T(\vec{x})$  exceeds the polar angle of  $\vec{x}$  by  $\theta$ .

## Linear Transformations

Now consider Figure 9, where we introduce the auxiliary vector  $\vec{y}$ , obtained by rotating  $\vec{x}$  through  $\pi/2$ . From Example 1.5 we know that if  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ , then  $\vec{y} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ . Using basic trigonometry, we find that

$$\begin{aligned} T(\vec{x}) &= (\cos \theta)\vec{x} + (\sin \theta)\vec{y} = (\cos \theta) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (\sin \theta) \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} (\cos \theta)x_1 - (\sin \theta)x_2 \\ (\sin \theta)x_1 + (\cos \theta)x_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}. \end{aligned}$$

This computation shows that a rotation through  $\theta$  is indeed a linear transformation, with the matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

### Theorem 2.3

**Rotations** The matrix of a counterclockwise rotation in  $\mathbb{R}^2$  through an angle  $\theta$  is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Note that this matrix is of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ . Conversely, any matrix of this form represents a rotation.

**EXAMPLE 4** The matrix of a counterclockwise rotation through  $\pi/6$  (or  $30^\circ$ ) is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

■

### Rotations Combined with a Scaling

**EXAMPLE 5** Examine how the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

affects our standard letter L. Here  $a$  and  $b$  are arbitrary constants.

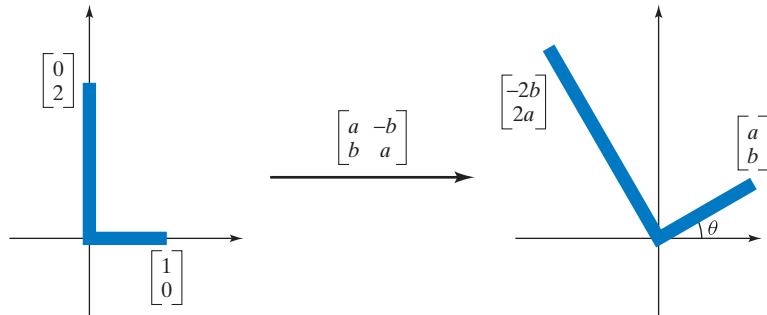
### Solution

Figure 10 suggests that  $T$  represents a *rotation combined with a scaling*. Think polar coordinates: This is a rotation through the polar angle  $\theta$  of vector  $\begin{bmatrix} a \\ b \end{bmatrix}$ , combined

## Linear Transformations

with a scaling by the magnitude  $r = \sqrt{a^2 + b^2}$  of vector  $\begin{bmatrix} a \\ b \end{bmatrix}$ . To verify this claim algebraically, we can write the vector  $\begin{bmatrix} a \\ b \end{bmatrix}$  in polar coordinates, as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix},$$

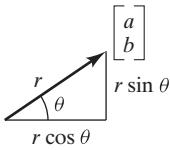


**Figure 10**

as illustrated in Figure 11. Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

It turns out that matrix  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  is a scalar multiple of a rotation matrix, as claimed. ■



**Figure 11**

### Theorem 2.4

#### Rotations combined with a scaling

A matrix of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  represents a rotation combined with a scaling.

More precisely, if  $r$  and  $\theta$  are the polar coordinates of vector  $\begin{bmatrix} a \\ b \end{bmatrix}$ , then

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  represents a rotation through  $\theta$  combined with a scaling by  $r$ .

## Shears

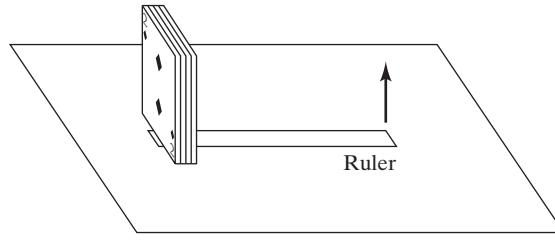
We will introduce shears by means of some simple experiments involving a ruler and a deck of cards.<sup>6</sup>

In the first experiment, we place the deck of cards on the ruler, as shown in Figure 12. Note that the 2 of diamonds is placed on one of the short edges of the ruler. That edge will stay in place throughout the experiment. Now we lift the other short edge of the ruler up, keeping the cards in vertical position at all times. The cards will slide up, being “fanned out,” without any horizontal displacement.

<sup>6</sup>Two hints for instructors:

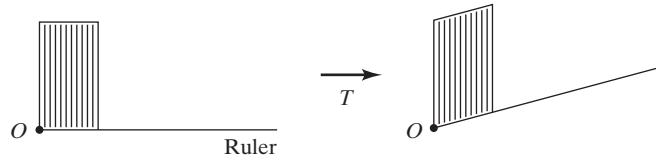
- Use several decks of cards for dramatic effect.
- Hold the decks together with a rubber band to avoid embarrassing accidents.

## Linear Transformations



**Figure 12**

Figure 13 shows a side view of this transformation. The origin represents the ruler's short edge that is staying in place.



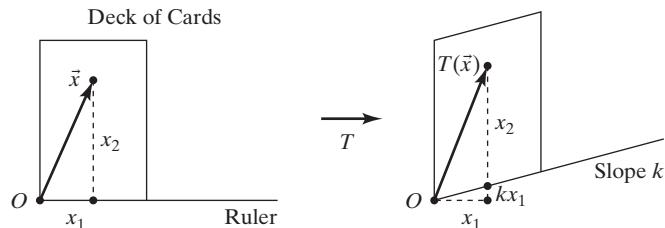
**Figure 13**

Such a transformation  $T$  is called a *vertical shear*. If we focus on the side view only, we have a vertical shear in  $\mathbb{R}^2$  (although in reality the experiment takes place in 3-space).

Now let's draw a vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  on the side of our deck of cards, and let's find

a formula for the sheared vector  $T(\vec{x})$ , using Figure 14 as a guide. Here,  $k$  denotes the slope of the ruler after the transformation:

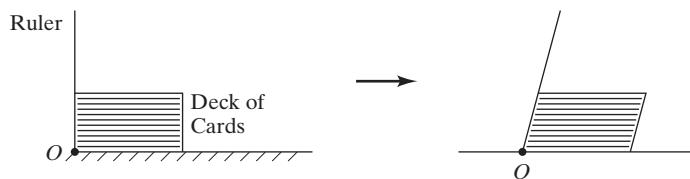
$$T(\vec{x}) = T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \vec{x}.$$



**Figure 14**

We find that the matrix of a vertical shear is of the form  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ , where  $k$  is an arbitrary constant.

*Horizontal shears* are defined analogously; consider Figure 15.



**Figure 15**

## Linear Transformations

We leave it as an exercise for the reader to verify that the matrix of a horizontal shear is of the form  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ . Take another look at part (e) of Example 1.

Oblique shears are far less important in applications, and we will not consider them here.

### Theorem 2.5

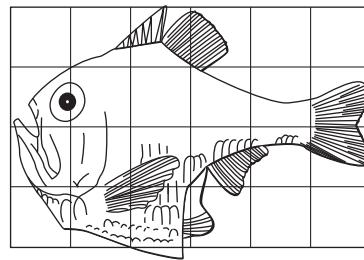
#### Horizontal and vertical shears

The matrix of a *horizontal shear* is of the form  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ , and the matrix of a *vertical shear* is of the form  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ , where  $k$  is an arbitrary constant.

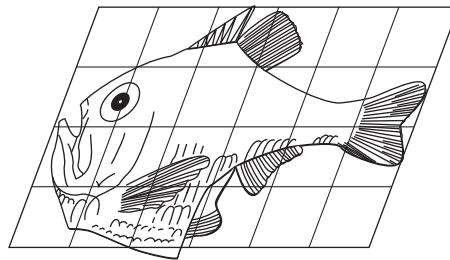
Let us summarize the main definitions of this section in a table.

Transformation	Matrix
Scaling by $k$	$kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Orthogonal projection onto line $L$	$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ , where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector parallel to $L$
Reflection about a line	$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where $a^2 + b^2 = 1$
Rotation through angle $\theta$	$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , where $a^2 + b^2 = 1$
Rotation through angle $\theta$ combined with scaling by $r$	$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
Horizontal shear	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

The Scottish scholar d'Arcy Thompson showed how the shapes of related species of plants and animals can often be transformed into one another, using linear as well as nonlinear transformations.<sup>7</sup> In Figure 16 he uses a horizontal shear to transform the shape of one species of fish into another.



*Argyropelecus olfersi.*



*Sternopyx diaphana.*

**Figure 16**

<sup>7</sup> Thompson, d'Arcy W., *On Growth and Form*, Cambridge University Press, 1917. P. B. Medawar calls this "the finest work of literature in all the annals of science that have been recorded in the English tongue."

**EXERCISES 2**

**GOAL** Use the matrices of orthogonal projections, reflections, and rotations. Apply the definitions of shears, orthogonal projections, and reflections.

1. Sketch the image of the standard L under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}.$$

See Example 1.

2. Find the matrix of a rotation through an angle of  $60^\circ$  in the counterclockwise direction.
3. Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ . Use  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$  to describe the image of the unit square geometrically.
4. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

5. The matrix

$$\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

represents a rotation. Find the angle of rotation (in radians).

6. Let  $L$  be the line in  $\mathbb{R}^3$  that consists of all scalar multiples of the vector  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ . Find the orthogonal projection of the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  onto  $L$ .

7. Let  $L$  be the line in  $\mathbb{R}^3$  that consists of all scalar multiples of  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ . Find the reflection of the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  about the line  $L$ .

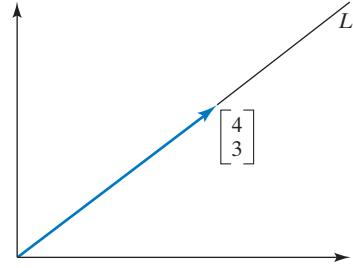
8. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}.$$

9. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}.$$

10. Find the matrix of the orthogonal projection onto the line  $L$  in  $\mathbb{R}^2$  shown in the accompanying figure:



11. Refer to Exercise 10. Find the matrix of the reflection about the line  $L$ .

12. Consider a reflection matrix  $A$  and a vector  $\vec{x}$  in  $\mathbb{R}^2$ . We define  $\vec{v} = \vec{x} + A\vec{x}$  and  $\vec{w} = \vec{x} - A\vec{x}$ .

- a. Using the definition of a reflection, express  $A(A\vec{x})$  in terms of  $\vec{x}$ .
- b. Express  $A\vec{v}$  in terms of  $\vec{v}$ .
- c. Express  $A\vec{w}$  in terms of  $\vec{w}$ .
- d. If the vectors  $\vec{v}$  and  $\vec{w}$  are both nonzero, what is the angle between  $\vec{v}$  and  $\vec{w}$ ?
- e. If the vector  $\vec{v}$  is nonzero, what is the relationship between  $\vec{v}$  and the line  $L$  of reflection?

Illustrate all parts of this exercise with a sketch showing  $\vec{x}$ ,  $A\vec{x}$ ,  $A(A\vec{x})$ ,  $\vec{v}$ ,  $\vec{w}$ , and the line  $L$ .

13. Suppose a line  $L$  in  $\mathbb{R}^2$  contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Find the matrix  $A$  of the linear transformation  $T(\vec{x}) = \text{ref}_L(\vec{x})$ . Give the entries of  $A$  in terms of  $u_1$  and  $u_2$ . Show that  $A$  is of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ .

14. Suppose a line  $L$  in  $\mathbb{R}^3$  contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

- a. Find the matrix  $A$  of the linear transformation  $T(\vec{x}) = \text{proj}_L(\vec{x})$ . Give the entries of  $A$  in terms of the components  $u_1, u_2, u_3$  of  $\vec{u}$ .
- b. What is the sum of the diagonal entries of the matrix  $A$  you found in part (a)?

15. Suppose a line  $L$  in  $\mathbb{R}^3$  contains the unit vector

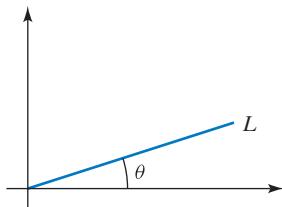
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Find the matrix  $A$  of the linear transformation  $T(\vec{x}) = \text{ref}_L(\vec{x})$ . Give the entries of  $A$  in terms of the components  $u_1, u_2, u_3$  of  $\vec{u}$ .

## Linear Transformations

16. Let  $T(\vec{x}) = \text{ref}_L(\vec{x})$  be the reflection about the line  $L$  in  $\mathbb{R}^2$  shown in the accompanying figure.

- a. Draw sketches to illustrate that  $T$  is linear.  
 b. Find the matrix of  $T$  in terms of  $\theta$ .



17. Consider a matrix  $A$  of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ . Find two nonzero perpendicular vectors  $\vec{v}$  and  $\vec{w}$  such that  $A\vec{v} = \vec{v}$  and  $A\vec{w} = -\vec{w}$  (write the entries of  $\vec{v}$  and  $\vec{w}$  in terms of  $a$  and  $b$ ). Conclude that  $T(\vec{x}) = A\vec{x}$  represents the reflection about the line  $L$  spanned by  $\vec{v}$ .

18. The linear transformation  $T(\vec{x}) = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\vec{x}$  is a reflection about a line  $L$ . See Exercise 17. Find the equation of line  $L$  (in the form  $y = mx$ ).

*Find the matrices of the linear transformations from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  given in Exercises 19 through 23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.*

19. The orthogonal projection onto the  $x$ - $y$ -plane.  
 20. The reflection about the  $x$ - $z$ -plane.  
 21. The rotation about the  $z$ -axis through an angle of  $\pi/2$ , counterclockwise as viewed from the positive  $z$ -axis.  
 22. The rotation about the  $y$ -axis through an angle  $\theta$ , counterclockwise as viewed from the positive  $y$ -axis.  
 23. The reflection about the plane  $y = z$ .  
 24. Rotations and reflections have two remarkable properties: They preserve the length of vectors and the angle between vectors. (Draw figures illustrating these properties.) We will show that, conversely, any linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that preserves length and angles is either a rotation or a reflection (about a line).

- a. Show that if  $T(\vec{x}) = A\vec{x}$  preserves length and angles, then the two column vectors  $\vec{v}$  and  $\vec{w}$  of  $A$  must be perpendicular unit vectors.

- b. Write the first column vector of  $A$  as  $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ ; note that  $a^2 + b^2 = 1$ , since  $\vec{v}$  is a unit vector. Show that for a given  $\vec{v}$  there are two possibilities for  $\vec{w}$ , the second column vector of  $A$ . Draw a sketch showing  $\vec{v}$  and the two possible vectors  $\vec{w}$ . Write the components of  $\vec{w}$  in terms of  $a$  and  $b$ .

- c. Show that if a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  preserves length and angles, then  $T$  is either a rotation or a reflection (about a line). See Exercise 17.

25. Find the inverse of the matrix  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ , where  $k$  is an arbitrary constant. Interpret your result geometrically.

26. a. Find the scaling matrix  $A$  that transforms  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  into  $\begin{bmatrix} 8 \\ -4 \end{bmatrix}$ .

- b. Find the orthogonal projection matrix  $B$  that transforms  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$  into  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ .

- c. Find the rotation matrix  $C$  that transforms  $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$  into  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

- d. Find the shear matrix  $D$  that transforms  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  into  $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ .

- e. Find the reflection matrix  $E$  that transforms  $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$  into  $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ .

27. Consider the matrices  $A$  through  $E$  below.

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{bmatrix}, \quad D = \begin{bmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Fill in the blanks in the sentences below.

We are told that there is a solution in each case.

Matrix \_\_\_\_\_ represents a scaling.

Matrix \_\_\_\_\_ represents an orthogonal projection.

Matrix \_\_\_\_\_ represents a shear.

Matrix \_\_\_\_\_ represents a reflection.

Matrix \_\_\_\_\_ represents a rotation.

28. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices  $A$  through  $J$ . Match them up.

- a. Scaling      b. Shear      c. Rotation

- d. Orthogonal projection      e. Reflection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix},$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix},$$

## Linear Transformations

$$G = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix}, \quad H = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

29. Let  $T$  be a function from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , and let  $L$  be a function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Suppose that  $L(T(\vec{x})) = \vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^m$  and  $T(L(\vec{y})) = \vec{y}$  for all  $\vec{y}$  in  $\mathbb{R}^n$ . If  $T$  is a linear transformation, show that  $L$  is linear as well. Hint:  $\vec{v} + \vec{w} = T(L(\vec{v})) + T(L(\vec{w})) = T(L(\vec{v}) + L(\vec{w}))$  since  $T$  is linear. Now apply  $L$  on both sides.

30. Find a nonzero  $2 \times 2$  matrix  $A$  such that  $A\vec{x}$  is parallel to the vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , for all  $\vec{x}$  in  $\mathbb{R}^2$ .

31. Find a nonzero  $3 \times 3$  matrix  $A$  such that  $A\vec{x}$  is perpendicular to  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , for all  $\vec{x}$  in  $\mathbb{R}^3$ .

32. Consider the rotation matrix  $D = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$  and the vector  $\vec{v} = \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix}$ , where  $\alpha$  and  $\beta$  are arbitrary angles.

- a. Draw a sketch to explain why  $D\vec{v} = \begin{bmatrix} \cos(\alpha + \beta) \\ \sin(\alpha + \beta) \end{bmatrix}$ .  
 b. Compute  $D\vec{v}$ . Use the result to derive the addition theorems for sine and cosine:

$$\cos(\alpha + \beta) = \dots, \quad \sin(\alpha + \beta) = \dots$$

33. Consider two nonparallel lines  $L_1$  and  $L_2$  in  $\mathbb{R}^2$ . Explain why a vector  $\vec{v}$  in  $\mathbb{R}^2$  can be expressed uniquely as

$$\vec{v} = \vec{v}_1 + \vec{v}_2,$$

where  $\vec{v}_1$  is on  $L_1$  and  $\vec{v}_2$  on  $L_2$ . Draw a sketch. The transformation  $T(\vec{v}) = \vec{v}_1$  is called the *projection onto  $L_1$  along  $L_2$* . Show algebraically that  $T$  is linear.

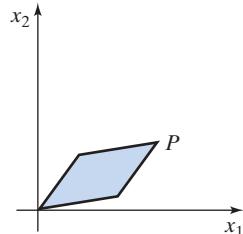
34. One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.

$$A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

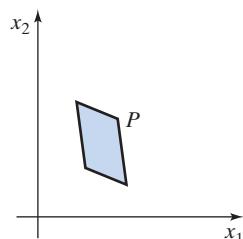
$$C = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad D = -\frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$E = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

35. Let  $T$  be an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Let  $P$  be a parallelogram in  $\mathbb{R}^2$  with one vertex at the origin. Is the image of  $P$  a parallelogram as well? Explain. Draw a sketch of the image.



36. Let  $T$  be an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Let  $P$  be a parallelogram in  $\mathbb{R}^2$ . Is the image of  $P$  a parallelogram as well? Explain.



37. The *trace* of a matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is the sum  $a + d$  of its diagonal entries. What can you say about the trace of a  $2 \times 2$  matrix that represents a(n)

- a. orthogonal projection    b. reflection about a line  
 c. rotation    d. (horizontal or vertical) shear.

In three cases, give the exact value of the trace, and in one case, give an interval of possible values.

38. The *determinant* of a matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $ad - bc$  (we have seen this quantity in Exercise 1.13 already). Find the determinant of a matrix that represents a(n)

- a. orthogonal projection    b. reflection about a line  
 c. rotation    d. (horizontal or vertical) shear.

What do your answers tell you about the invertibility of these matrices?

39. Describe each of the linear transformations defined by the matrices in parts (a) through (c) geometrically, as a well-known transformation combined with a scaling. Give the scaling factor in each case.

a.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$     b.  $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$

c.  $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

40. Let  $P$  and  $Q$  be two perpendicular lines in  $\mathbb{R}^2$ . For a vector  $\vec{x}$  in  $\mathbb{R}^2$ , what is  $\text{proj}_P(\vec{x}) + \text{proj}_Q(\vec{x})$ ? Give

## Linear Transformations

your answer in terms of  $\vec{x}$ . Draw a sketch to justify your answer.

41. Let  $P$  and  $Q$  be two perpendicular lines in  $\mathbb{R}^2$ . For a vector  $\vec{x}$  in  $\mathbb{R}^2$ , what is the relationship between  $\text{ref}_P(\vec{x})$  and  $\text{ref}_Q(\vec{x})$ ? Draw a sketch to justify your answer.
42. Let  $T(\vec{x}) = \text{proj}_L(\vec{x})$  be the orthogonal projection onto a line in  $\mathbb{R}^2$ . What is the relationship between  $T(\vec{x})$  and  $T(T(\vec{x}))$ ? Justify your answer carefully.
43. Use the formula derived in Exercise 1.13 to find the inverse of the rotation matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Interpret the linear transformation defined by  $A^{-1}$  geometrically. Explain.

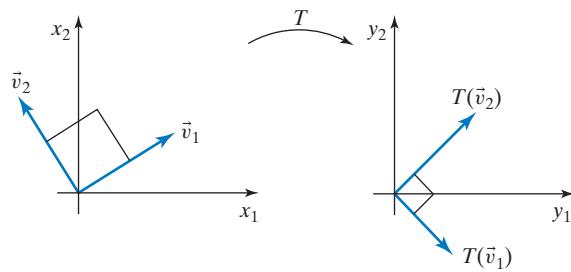
44. A nonzero matrix of the form  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  represents a rotation combined with a scaling. Use the formula derived in Exercise 1.13 to find the inverse of  $A$ . Interpret the linear transformation defined by  $A^{-1}$  geometrically. Explain.
45. A matrix of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ , represents a reflection about a line. See Exercise 17. Use the formula derived in Exercise 1.13 to find the inverse of  $A$ . Explain.
46. A nonzero matrix of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$  represents a reflection about a line  $L$  combined with a scaling. (Why? What is the scaling factor?) Use the formula derived in Exercise 1.13 to find the inverse of  $A$ . Interpret the linear transformation defined by  $A^{-1}$  geometrically. Explain.
47. In this exercise we will prove the following remarkable theorem: If  $T(\vec{x}) = A\vec{x}$  is any linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , then there exist perpendicular unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^2$  such that the vectors  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular as well (see the accompanying figure), in the sense that  $T(\vec{v}_1) \cdot T(\vec{v}_2) = 0$ . This is not intuitively obvious: Think about the case of a shear, for example.

For any real number  $t$ , the vectors  $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$  and  $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$  will be perpendicular unit vectors. Now we can consider the function

$$\begin{aligned} f(t) &= \left( T \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \right) \cdot \left( T \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right) \\ &= \left( A \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \right) \cdot \left( A \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right). \end{aligned}$$

It is our goal to show that there exists a number  $c$  such that  $f(c) = \left( T \begin{bmatrix} \cos c \\ \sin c \end{bmatrix} \right) \cdot \left( T \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix} \right) = 0$ . Then the vectors  $\vec{v}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix}$  will have the required property that they are perpendicular unit vectors such that  $T(\vec{v}_1) \cdot T(\vec{v}_2) = 0$ .

- a. Show that the function  $f(t)$  is continuous. You may assume that  $\cos t$ ,  $\sin t$ , and constant functions are continuous. Also, sums and products of continuous functions are continuous. Hint: Write  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .
- b. Show that  $f\left(\frac{\pi}{2}\right) = -f(0)$ .
- c. Show that there exists a number  $c$ , with  $0 \leq c \leq \frac{\pi}{2}$ , such that  $f(c) = 0$ . Hint: Use the intermediate value theorem: If a function  $f(t)$  is continuous for  $a \leq t \leq b$  and if  $L$  is any number between  $f(a)$  and  $f(b)$ , then there exists a number  $c$  between  $a$  and  $b$  with  $f(c) = L$ .



48. If a  $2 \times 2$  matrix  $A$  represents a rotation, find perpendicular unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^2$  such that the vectors  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular as well. See Exercise 47.

*For the linear transformations  $T$  in Exercises 49 through 52, do the following:*

- a. Find the function  $f(t)$  defined in Exercise 47 and graph it for  $0 \leq t \leq \frac{\pi}{2}$ . You may use technology.
- b. Find a number  $c$ , with  $0 \leq c \leq \frac{\pi}{2}$ , such that  $f(c) = 0$ . (In Problem 50, approximate  $c$  to three significant digits, using technology.)
- c. Find perpendicular unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^2$  such that the vectors  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular as well. Draw a sketch showing  $\vec{v}_1$ ,  $\vec{v}_2$ ,  $T(\vec{v}_1)$ , and  $T(\vec{v}_2)$ .

49.  $T(\vec{x}) = \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix} \vec{x}$

50.  $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

51.  $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$

52.  $T(\vec{x}) = \begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix} \vec{x}$

53. Sketch the image of the unit circle under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}.$$

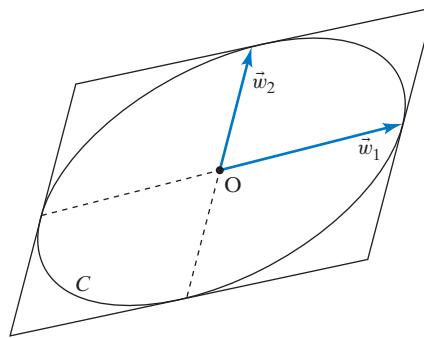
54. Let  $T$  be an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Show that the image of the unit circle is an ellipse centered at the origin.<sup>8</sup> Hint: Consider two perpendicular unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  such that  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular. See Exercise 47. The unit circle consists of all vectors of the form

$$\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2,$$

where  $t$  is a parameter.

55. Let  $\vec{w}_1$  and  $\vec{w}_2$  be two nonparallel vectors in  $\mathbb{R}^2$ . Consider the curve  $C$  in  $\mathbb{R}^2$  that consists of all vectors of the form  $\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$ , where  $t$  is a parameter.

Show that  $C$  is an ellipse. Hint: You can interpret  $C$  as the image of the unit circle under a suitable linear transformation; then use Exercise 54.



56. Consider an invertible linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Let  $C$  be an ellipse in  $\mathbb{R}^2$ . Show that the image of  $C$  under  $T$  is an ellipse as well. Hint: Use the result of Exercise 55.

### 3 Matrix Products

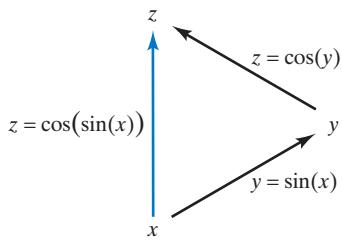


Figure 1

Recall the *composition* of two functions: The composite of the functions  $y = \sin(x)$  and  $z = \cos(y)$  is  $z = \cos(\sin(x))$ , as illustrated in Figure 1. Similarly, we can compose two linear transformations.

To understand this concept, let's return to the coding example discussed in Section 1. Recall that the position  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  of your boat is encoded and that you radio the encoded position  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$  to Marseille. The coding transformation is

$$\vec{y} = A\vec{x}, \quad \text{with } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

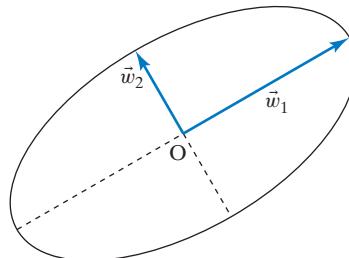
In Section 1, we left out one detail: Your position is radioed on to Paris, as you would expect in a centrally governed country such as France. Before broadcasting to Paris, the position  $\vec{y}$  is again encoded, using the linear transformation

<sup>8</sup>An ellipse in  $\mathbb{R}^2$  centered at the origin may be defined as a curve that can be parametrized as

$$\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2,$$

for two perpendicular vectors  $\vec{w}_1$  and  $\vec{w}_2$ . Suppose the length of  $\vec{w}_1$  exceeds the length of  $\vec{w}_2$ . Then we call the vectors  $\pm\vec{w}_1$  the semimajor axes of the ellipse and  $\pm\vec{w}_2$  the semiminor axes.

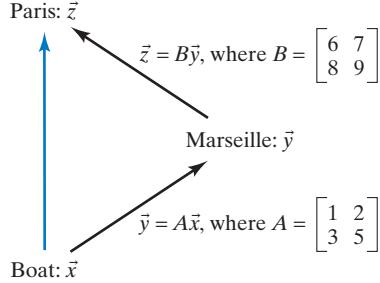
*Convention:* All ellipses considered in this chapter are centered at the origin unless stated otherwise.



## Linear Transformations

$$\vec{z} = B\vec{y}, \quad \text{with} \quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

this time, and the sailor in Marseille radios the encoded position  $\vec{z}$  to Paris. See Figure 2.



**Figure 2**

We can think of the message  $\vec{z}$  received in Paris as a function of the actual position  $\vec{x}$  of the boat,

$$\vec{z} = B(A\vec{x}),$$

the composite of the two transformations  $\vec{y} = A\vec{x}$  and  $\vec{z} = B\vec{y}$ . Is this transformation  $\vec{z} = T(\vec{x})$  linear, and, if so, what is its matrix? We will show two approaches to these important questions: (a) using brute force, and (b) using some theory.

- a.** We write the components of the two transformations and substitute:

$$\begin{aligned} z_1 &= 6y_1 + 7y_2 & \text{and} & \quad y_1 = x_1 + 2x_2 \\ z_2 &= 8y_1 + 9y_2 & & \quad y_2 = 3x_1 + 5x_2 \end{aligned}$$

so that

$$\begin{aligned} z_1 &= 6(x_1 + 2x_2) + 7(3x_1 + 5x_2) = (6 \cdot 1 + 7 \cdot 3)x_1 + (6 \cdot 2 + 7 \cdot 5)x_2 \\ &= 27x_1 + 47x_2, \\ z_2 &= 8(x_1 + 2x_2) + 9(3x_1 + 5x_2) = (8 \cdot 1 + 9 \cdot 3)x_1 + (8 \cdot 2 + 9 \cdot 5)x_2 \\ &= 35x_1 + 61x_2. \end{aligned}$$

This shows that the composite is indeed linear, with matrix

$$\begin{bmatrix} 6 \cdot 1 + 7 \cdot 3 & 6 \cdot 2 + 7 \cdot 5 \\ 8 \cdot 1 + 9 \cdot 3 & 8 \cdot 2 + 9 \cdot 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}.$$

- b.** We show that the transformation  $T(\vec{x}) = B(A\vec{x})$  is linear:

$$\begin{aligned} T(\vec{v} + \vec{w}) &= B(A(\vec{v} + \vec{w})) = B(A\vec{v} + A\vec{w}) \\ &= B(A\vec{v}) + B(A\vec{w}) = T(\vec{v}) + T(\vec{w}), \\ T(k\vec{v}) &= B(A(k\vec{v})) = B(k(A\vec{v})) = k(B(A\vec{v})) = kT(\vec{v}). \end{aligned}$$

Once we know that  $T$  is linear, we can find its matrix by computing the vectors  $T(\vec{e}_1) = B(A\vec{e}_1)$  and  $T(\vec{e}_2) = B(A\vec{e}_2)$ ; the matrix of  $T$  is then  $[T(\vec{e}_1) \quad T(\vec{e}_2)]$ , by Theorem 1.2:

$$T(\vec{e}_1) = B(A\vec{e}_1) = B(\text{first column of } A) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 27 \\ 35 \end{bmatrix},$$

$$T(\vec{e}_2) = B(A\vec{e}_2) = B(\text{second column of } A) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ 61 \end{bmatrix}.$$

## Linear Transformations

We find that the matrix of the linear transformation  $T(\vec{x}) = B(A\vec{x})$  is

$$\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}.$$

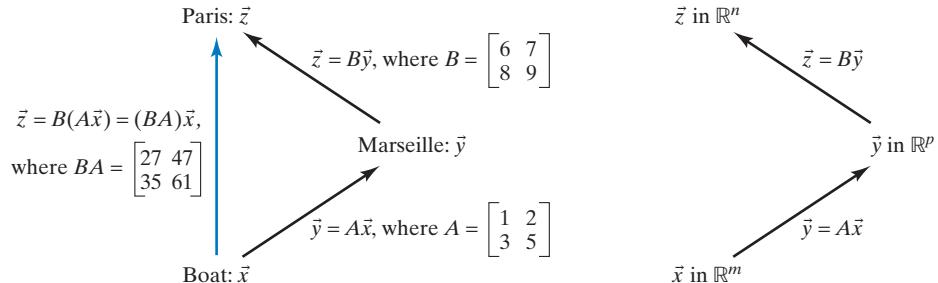
This result agrees with the result in (a), of course.

The matrix of the linear transformation  $T(\vec{x}) = B(A\vec{x})$  is called the *product* of the matrices  $B$  and  $A$ , written as  $BA$ . This means that

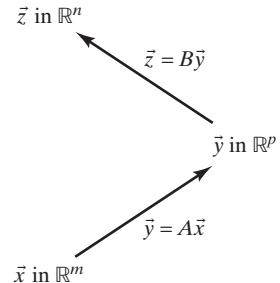
$$T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x},$$

for all vectors  $\vec{x}$  in  $\mathbb{R}^2$ . See Figure 3.

Now let's look at the product of larger matrices. Let  $B$  be an  $n \times p$  matrix and  $A$  a  $p \times m$  matrix. These matrices represent linear transformations, as shown in Figure 4.



**Figure 3**

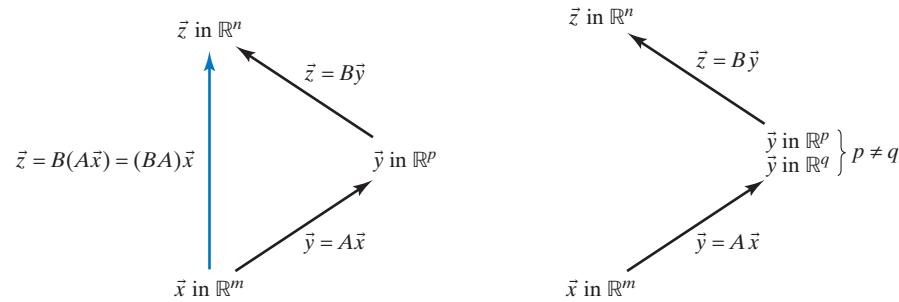


**Figure 4**

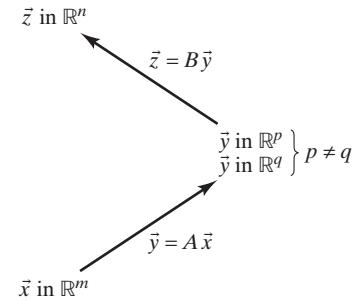
Again, the composite transformation  $\vec{z} = B(A\vec{x})$  is linear. [Part (b) of the foregoing justification applies in this more general case as well.] The matrix of the linear transformation  $\vec{z} = B(A\vec{x})$  is called the *product* of the matrices  $B$  and  $A$ , written as  $BA$ . Note that  $BA$  is an  $n \times m$  matrix (as it represents a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ). As in the case of  $\mathbb{R}^2$ , the equation

$$\vec{z} = B(A\vec{x}) = (BA)\vec{x}$$

holds for all vectors  $\vec{x}$  in  $\mathbb{R}^m$ , by definition of the product  $BA$ . See Figure 5.



**Figure 5**



**Figure 6**

In the definition of the matrix product  $BA$ , the number of columns of  $B$  matches the number of rows of  $A$ . What happens if these two numbers are different? Suppose  $B$  is an  $n \times p$  matrix and  $A$  is a  $q \times m$  matrix, with  $p \neq q$ .

In this case, the transformations  $\vec{z} = B\vec{y}$  and  $\vec{y} = A\vec{x}$  cannot be composed, since the target space of  $\vec{y} = A\vec{x}$  is different from the domain of  $\vec{z} = B\vec{y}$ . See

## Linear Transformations

Figure 6. To put it more plainly: The output of  $\vec{y} = A\vec{x}$  is not an acceptable input for the transformation  $\vec{z} = B\vec{y}$ . In this case, the matrix product  $BA$  is undefined.

### Definition 3.1

#### Matrix multiplication

- a. Let  $B$  be an  $n \times p$  matrix and  $A$  a  $q \times m$  matrix. The product  $BA$  is defined if (and only if)  $p = q$ .
- b. If  $B$  is an  $n \times p$  matrix and  $A$  a  $p \times m$  matrix, then the product  $BA$  is defined as the matrix of the linear transformation  $T(\vec{x}) = B(A\vec{x})$ . This means that  $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$ , for all  $\vec{x}$  in the vector space  $\mathbb{R}^m$ . The product  $BA$  is an  $n \times m$  matrix.

Although this definition of matrix multiplication does not give us concrete instructions for computing the product of two numerically given matrices, such instructions can be derived easily from the definition.

As in Definition 3.1, let  $B$  be an  $n \times p$  matrix and  $A$  a  $p \times m$  matrix. Let's think about the columns of the matrix  $BA$ :

$$\begin{aligned}(i\text{th column of } BA) &= (BA)\vec{e}_i \\ &= B(A\vec{e}_i) \\ &= B(i\text{th column of } A).\end{aligned}$$

If we denote the columns of  $A$  by  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ , we can write

$$BA = B \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & | \end{bmatrix}.$$

### Theorem 3.2

#### The columns of the matrix product

Let  $B$  be an  $n \times p$  matrix and  $A$  a  $p \times m$  matrix with columns  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ . Then, the product  $BA$  is

$$BA = B \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & | \end{bmatrix}.$$

To find  $BA$ , we can multiply  $B$  by the columns of  $A$  and combine the resulting vectors.

This is exactly how we computed the product

$$BA = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

using approach (b).

For practice, let us multiply the same matrices in the reverse order. The first column of  $AB$  is  $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 58 \end{bmatrix}$ ; the second is  $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 25 \\ 66 \end{bmatrix}$ . Thus,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 22 & 25 \\ 58 & 66 \end{bmatrix}.$$

Compare the two previous displays to see that  $AB \neq BA$ : Matrix multiplication is *noncommutative*. This should come as no surprise, in view of the fact that

the matrix product represents a composite of transformations. Even for functions of one variable, the order in which we compose matters. Refer to the first example in this section and note that the functions  $\cos(\sin(x))$  and  $\sin(\cos(x))$  are different.

**Theorem 3.3**
**Matrix multiplication is noncommutative**

$AB \neq BA$ , in general. However, at times it does happen that  $AB = BA$ ; then we say that the matrices  $A$  and  $B$  *commute*.

It is useful to have a formula for the  $ij$ th entry of the product  $BA$  of an  $n \times p$  matrix  $B$  and a  $p \times m$  matrix  $A$ .

Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  be the columns of  $A$ . Then, by Theorem 3.2,

$$BA = B \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_j & \cdots & \vec{v}_m \\ | & | & & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_j & \cdots & B\vec{v}_m \\ | & | & & | & & | \end{bmatrix}.$$

The  $ij$ th entry of the product  $BA$  is the  $i$ th component of the vector  $B\vec{v}_j$ , which is the dot product of the  $i$ th row of  $B$  and  $\vec{v}_j$ .

**Theorem 3.4**
**The entries of the matrix product**

Let  $B$  be an  $n \times p$  matrix and  $A$  a  $p \times m$  matrix. The  $ij$ th entry of  $BA$  is the dot product of the  $i$ th row of  $B$  with the  $j$ th column of  $A$ .

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ip} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \end{bmatrix}$$

is the  $n \times m$  matrix whose  $ij$ th entry is

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ip}a_{pj} = \sum_{k=1}^p b_{ik}a_{kj}.$$

**EXAMPLE 1**

$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 7 \cdot 3 & 6 \cdot 2 + 7 \cdot 5 \\ 8 \cdot 1 + 9 \cdot 3 & 8 \cdot 2 + 9 \cdot 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

We have done these computations before. (Where?) ■

**EXAMPLE 2**

Compute the products  $BA$  and  $AB$  for  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Interpret your answers geometrically, as composites of linear transformation. Draw composition diagrams.

**Solution**

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that in this special example it turns out that  $BA = -AB$ .

## Linear Transformations

From Section 2 we recall the following geometrical interpretations:

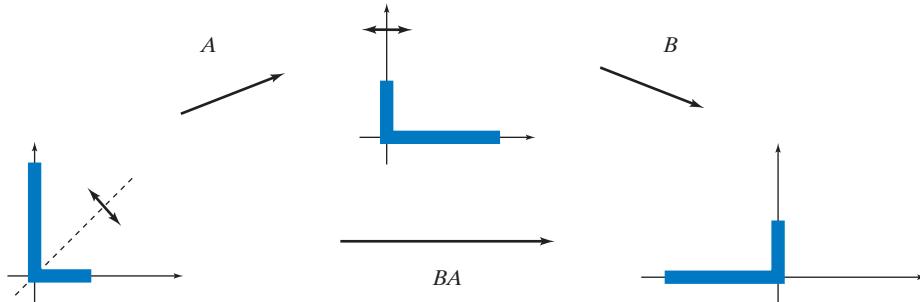
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  represents the reflection about the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ;

$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  represents the reflection about  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ;

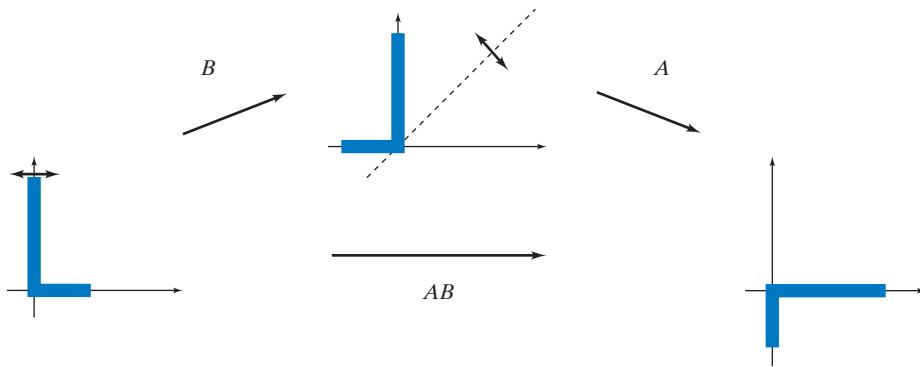
$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents the rotation through  $\frac{\pi}{2}$ ; and

$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  represents the rotation through  $-\frac{\pi}{2}$ .

Let's use our standard L to show the effect of these transformations. See Figures 7 and 8.



**Figure 7**



**Figure 8**

## Matrix Algebra

Next let's discuss some algebraic rules for matrix multiplication.

- Composing a linear transformation with the identity transformation, on either side, leaves the transformation unchanged. See Example 1.4.

### Theorem 3.5

#### Multiplying with the identity matrix

For an  $n \times m$  matrix  $A$ ,

$$AI_m = I_n A = A.$$

## Linear Transformations

- If  $A$  is an  $n \times p$  matrix,  $B$  a  $p \times q$  matrix, and  $C$  a  $q \times m$  matrix, what is the relationship between  $(AB)C$  and  $A(BC)$ ?

One way to think about this problem (although perhaps not the most elegant one) is to write  $C$  in terms of its columns:  $C = [\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_m]$ . Then

$$(AB)C = (AB) [\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_m] = [(AB)\vec{v}_1 \quad (AB)\vec{v}_2 \quad \cdots \quad (AB)\vec{v}_m],$$

and

$$A(BC) = A [B\vec{v}_1 \quad B\vec{v}_2 \quad \cdots \quad B\vec{v}_m] = [A(B\vec{v}_1) \quad A(B\vec{v}_2) \quad \cdots \quad A(B\vec{v}_m)].$$

Since  $(AB)\vec{v}_i = A(B\vec{v}_i)$ , by definition of the matrix product, we find that  $(AB)C = A(BC)$ .

### Theorem 3.6

#### Matrix multiplication is associative

$$(AB)C = A(BC)$$

We can simply write  $ABC$  for the product  $(AB)C = A(BC)$ .

A more conceptual proof is based on the fact that the composition of functions is associative. The two linear transformations

$$T(\vec{x}) = ((AB)C)\vec{x} \quad \text{and} \quad L(\vec{x}) = (A(BC))\vec{x}$$

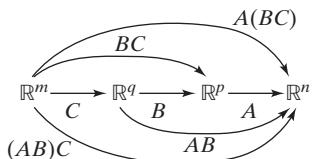
are identical because, by the definition of matrix multiplication,

$$T(\vec{x}) = ((AB)C)\vec{x} = (AB)(C\vec{x}) = A(B(C\vec{x}))$$

and

$$L(\vec{x}) = (A(BC))\vec{x} = A((BC)\vec{x}) = A(B(C\vec{x})).$$

The domains and target spaces of the linear transformations defined by the matrices  $A$ ,  $B$ ,  $C$ ,  $BC$ ,  $AB$ ,  $A(BC)$ , and  $(AB)C$  are shown in Figure 9.



**Figure 9**

### Theorem 3.7

#### Distributive property for matrices

If  $A$  and  $B$  are  $n \times p$  matrices, and  $C$  and  $D$  are  $p \times m$  matrices, then

$$\begin{aligned} A(C + D) &= AC + AD, \quad \text{and} \\ (A + B)C &= AC + BC. \end{aligned}$$

You will be asked to verify this property in Exercise 27.

### Theorem 3.8

If  $A$  is an  $n \times p$  matrix,  $B$  is a  $p \times m$  matrix, and  $k$  is a scalar, then

$$(kA)B = A(kB) = k(AB).$$

You will be asked to verify this property in Exercise 28.

### Block Matrices (Optional)

In the popular puzzle Sudoku, one considers a  $9 \times 9$  matrix  $A$  that is subdivided into nine  $3 \times 3$  matrices called *blocks*. The puzzle setter provides some of the 81 entries of matrix  $A$ , and the objective is to fill in the remaining entries so that each row of  $A$ , each column of  $A$ , and each block contains each of the digits 1 through 9 exactly once.

## Linear Transformations

5	3			7				
6			1	9	5			
	9	8				6		
8			6				3	
4		8		3				1
7			2				6	
	6				2	8		
		4	1	9			5	
			8		7	9		

This Sudoku puzzle is an example of a *block matrix* (or *partitioned matrix*), that is, a matrix that is partitioned into rectangular submatrices, called blocks, by means of horizontal and vertical lines that go all the way through the matrix.

The blocks need not be of equal size.

For example, we can partition the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \quad \text{as} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where  $B_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ ,  $B_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ ,  $B_{21} = [6 \ 7]$ , and  $B_{22} = [9]$ .

A useful property of block matrices is the following:

### Theorem 3.9

#### Multiplying block matrices

Block matrices can be multiplied as though the blocks were scalars (i.e., using the formula in Theorem 3.4):

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ip} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{np} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{p1} & B_{p2} & \cdots & B_{pj} & \cdots & B_{pm} \end{bmatrix}$$

is the block matrix whose  $ij$ th block is the matrix

$$A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^p A_{ik}B_{kj},$$

provided that all the products  $A_{ik}B_{kj}$  are defined.

Verifying this fact is left as an exercise. A numerical example follows.

### EXAMPLE 3

$$\begin{aligned} & \left[ \begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array} \right] \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right] \\ &= \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} [7 \ 8] \quad \left| \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} [9] \right. \right] \\ &= \begin{bmatrix} -3 & -3 & -3 \\ 8 & 10 & 12 \end{bmatrix}. \end{aligned}$$

## Linear Transformations

Compute this product without using a partition, and see whether you find the same result. ■

In this simple example, using blocks is somewhat pointless. Example 3 merely illustrates Theorem 3.9. In Example 4.7, we will see a more sensible usage of the concept of block matrices.

### Powers of Transition Matrices

We will conclude this section with an example on transition matrices. See Definition 1.4.

**EXAMPLE 4** Let's revisit the mini-Web we considered in Example 9 of Section 1:

$$\begin{array}{ccc} 1 & \rightleftharpoons & 2 \\ \downarrow & \swarrow & \uparrow \\ 3 & \rightarrow & 4 \end{array}$$

with the transition matrix

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

At a predetermined time, all the surfers will follow one of the available links, in the manner described in Example 1.9. If the initial distribution of the surfers among the four pages is given by the vector  $\vec{x}$ , then the distribution after this transition will be  $A\vec{x}$ . Now, let's iterate this process: Imagine an event of “speed surfing,” where, every few minutes, at the blow of a whistle, each surfer will follow an available link. After two transitions, the distribution will be  $A(A\vec{x}) = A^2\vec{x}$ , and after  $m$  transitions the distribution will be given by the vector  $A^m\vec{x}$ . Let's use technology to compute some of the powers  $A^m$  of matrix  $A$ :

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad A^{10} \approx \begin{bmatrix} 0.173 & 0.172 & 0.172 & 0.150 \\ 0.344 & 0.345 & 0.301 & 0.344 \\ 0.247 & 0.247 & 0.270 & 0.236 \\ 0.236 & 0.236 & 0.258 & 0.270 \end{bmatrix},$$

$$A^{20} \approx \begin{bmatrix} 0.16697 & 0.16697 & 0.16650 & 0.16623 \\ 0.33347 & 0.33347 & 0.33246 & 0.33393 \\ 0.25008 & 0.25008 & 0.25035 & 0.24948 \\ 0.24948 & 0.24948 & 0.25068 & 0.25035 \end{bmatrix}.$$

These powers  $A^m$  will be transition matrices as well; see Exercise 68.

In Exercises 69 through 72, you will have a chance to explore the significance of the entries of these matrices  $A^m$ , in terms of our mini-Web and its graph.

As we take a closer look at the matrix  $A^{20}$ , our attention may be drawn to the fact that the four column vectors are all close to the vector

$$\begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix},$$

## Linear Transformations

which happens to be the equilibrium distribution  $\vec{x}_{equ}$  for the matrix  $A$ , as discussed in Example 1.9. We might conjecture that the *limit* of the column vectors of  $A^m$  is  $\vec{x}_{equ}$  as  $m$  goes to infinity. ■

Before we address this issue, let's introduce some terminology.

### Definition 3.10

#### Regular transition matrices

A transition matrix is said to be *positive* if all its entries are positive (meaning that all the entries are greater than 0).

A transition matrix is said to be *regular* (or *eventually positive*) if the matrix  $A^m$  is positive for some positive integer  $m$ .

For example, the transition matrix  $\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$  is positive (and therefore also regular; let  $m = 1$  in Definition 3.10). The transition matrix  $A = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}$  fails to be positive, but it is regular since  $A^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$ .

The matrix  $A$  in Example 4 fails to be positive, but it is regular since  $A^{10}$  is positive. The reflection matrix  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  fails to be regular since  $A^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  for even  $m$  and  $A^m = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  for odd  $m$ .

Now we can address the conjecture we made at the end of Example 4.

### Theorem 3.11

#### Equilibria for regular transition matrices

Let  $A$  be a regular transition matrix of size  $n \times n$ .

- a. There exists exactly one distribution vector  $\vec{x}$  in  $\mathbb{R}^n$  such that  $A\vec{x} = \vec{x}$ . This is called the *equilibrium distribution* for  $A$ , denoted  $\vec{x}_{equ}$ . All the components of  $\vec{x}_{equ}$  are positive.
- b. If  $\vec{x}$  is any distribution vector in  $\mathbb{R}^n$ , then  $\lim_{m \rightarrow \infty} (A^m \vec{x}) = \vec{x}_{equ}$ .
- c.  $\lim_{m \rightarrow \infty} A^m = \begin{bmatrix} | & | \\ \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | \end{bmatrix}$ , which is the matrix whose columns are all  $\vec{x}_{equ}$ .<sup>9</sup>

Part (b) states that in the long run the system will approach the equilibrium distribution  $\vec{x}_{equ}$ , *regardless of the initial distribution*; we say that  $\vec{x}_{equ}$  is the *globally stable equilibrium distribution*.

Parts (b) and (c) are easily seen to be equivalent. If we assume that part (b) holds, then  $\lim_{m \rightarrow \infty} (j\text{th column of } A^m) = \lim_{m \rightarrow \infty} (A^m \vec{e}_j) = \vec{x}_{equ}$  since  $\vec{e}_j$  is a distribution vector. In Exercise 73, you are asked to derive part (b) from part (c).

---

<sup>9</sup>This limit is defined entry by entry. We claim that any entry of  $A^m$  converges to the corresponding entry of the matrix  $\begin{bmatrix} | & | \\ \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | \end{bmatrix}$  as  $m$  goes to infinity.

## EXERCISES 3

**GOAL** Compute matrix products column by column and entry by entry. Interpret matrix multiplication in terms of the underlying linear transformations. Use the rules of matrix algebra. Multiply block matrices.

If possible, compute the matrix products in Exercises 1 through 13, using paper and pencil.

1.  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2.  $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

3.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$

5.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6.  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

8.  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

9.  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix}$

10.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12.  $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$

13.  $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

14. For the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E = [5],$$

determine which of the 25 matrix products  $AA$ ,  $AB$ ,  $AC, \dots, ED, EE$  are defined, and compute those that are defined.

Use the given partitions to compute the products in Exercises 15 and 16. Check your work by computing the same products without using a partition. Show all your work.

15. 
$$\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \begin{array}{c|c} 0 \\ \hline 4 \end{array} \quad \begin{array}{c|c} 1 & 0 \\ \hline 2 & 0 \end{array} \begin{array}{c|c} 3 & 4 \end{array}$$

16. 
$$\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array} \begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline 0 & 1 \end{array} \quad \begin{array}{c|c} 1 & 2 \\ \hline 3 & 4 \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array} \begin{array}{c|c} 2 & 3 \\ \hline 4 & 5 \\ \hline 1 & 2 \\ \hline 3 & 4 \end{array}$$

In the Exercises 17 through 26, find all matrices that commute with the given matrix  $A$ .

17.  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

18.  $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

19.  $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

20.  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

21.  $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

22.  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

23.  $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

24.  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

25.  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

26.  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

27. Prove the distributive laws for matrices:

$$A(C + D) = AC + AD$$

and

$$(A + B)C = AC + BC.$$

28. Consider an  $n \times p$  matrix  $A$ , a  $p \times m$  matrix  $B$ , and a scalar  $k$ . Show that

$$(kA)B = A(kB) = k(AB).$$

29. Consider the matrix

$$D_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

We know that the linear transformation  $T(\vec{x}) = D_\alpha \vec{x}$  is a counterclockwise rotation through an angle  $\alpha$ .

a. For two angles,  $\alpha$  and  $\beta$ , consider the products  $D_\alpha D_\beta$  and  $D_\beta D_\alpha$ . Arguing geometrically, describe the linear transformations  $\vec{y} = D_\alpha D_\beta \vec{x}$  and  $\vec{y} = D_\beta D_\alpha \vec{x}$ . Are the two transformations the same?

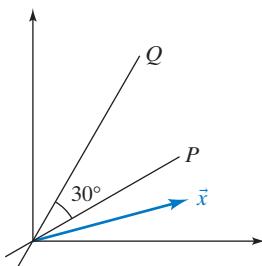
b. Now compute the products  $D_\alpha D_\beta$  and  $D_\beta D_\alpha$ . Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta.$$

30. Consider the lines  $P$  and  $Q$  in  $\mathbb{R}^2$  in the accompanying figure. Consider the linear transformation  $T(\vec{x}) = \text{ref}_Q(\text{ref}_P(\vec{x}))$ ; that is, we first reflect  $\vec{x}$  about  $P$  and then we reflect the result about  $Q$ .

## Linear Transformations



- For the vector  $\vec{x}$  given in the figure, sketch  $T(\vec{x})$ . What angle do the vectors  $\vec{x}$  and  $T(\vec{x})$  enclose? What is the relationship between the lengths of  $\vec{x}$  and  $T(\vec{x})$ ?
  - Use your answer in part (a) to describe the transformation  $T$  geometrically, as a reflection, rotation, shear, or projection.
  - Find the matrix of  $T$ .
  - Give a geometrical interpretation of the linear transformation  $L(\vec{x}) = \text{ref}_P(\text{ref}_Q(\vec{x}))$ , and find the matrix of  $L$ .
- 31.** Consider two matrices  $A$  and  $B$  whose product  $AB$  is defined. Describe the  $i$ th row of the product  $AB$  in terms of the rows of  $A$  and the matrix  $B$ .
- 32.** Find all  $2 \times 2$  matrices  $X$  such that  $AX = XA$  for all  $2 \times 2$  matrices  $A$ .

For the matrices  $A$  in Exercises 33 through 42, compute  $A^2 = AA$ ,  $A^3 = AAA$ , and  $A^4$ . Describe the pattern that emerges, and use this pattern to find  $A^{1001}$ . Interpret your answers geometrically, in terms of rotations, reflections, shears, and orthogonal projections.

- 33.**  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$     **34.**  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$     **35.**  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- 36.**  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$     **37.**  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$     **38.**  $\frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$
- 39.**  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$     **40.**  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- 41.**  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$     **42.**  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

In Exercises 43 through 48, find a  $2 \times 2$  matrix  $A$  with the given properties. Hint: It helps to think of geometrical examples.

- 43.**  $A \neq I_2$ ,  $A^2 = I_2$     **44.**  $A^2 \neq I_2$ ,  $A^4 = I_2$
- 45.**  $A^2 \neq I_2$ ,  $A^3 = I_2$
- 46.**  $A^2 = A$ , all entries of  $A$  are nonzero.
- 47.**  $A^3 = A$ , all entries of  $A$  are nonzero.
- 48.**  $A^{10} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**In Exercises 49 through 54, consider the matrices**

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Compute the indicated products. Interpret these products geometrically, and draw composition diagrams, as in Example 2.

- 49.**  $AF$  and  $FA$     **50.**  $CG$  and  $GC$
- 51.**  $FJ$  and  $JF$     **52.**  $JH$  and  $HJ$
- 53.**  $CD$  and  $DC$     **54.**  $BE$  and  $EB$ .

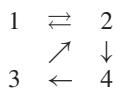
**In Exercises 55 through 64, find all matrices  $X$  that satisfy the given matrix equation.**

- 55.**  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- 56.**  $X \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = I_2$     **57.**  $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} X = I_2$
- 58.**  $X \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$     **59.**  $X \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = I_2$
- 60.**  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = I_2$     **61.**  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} X = I_2$
- 62.**  $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} X = I_3$     **63.**  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} X = I_3$
- 64.**  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} X = I_2$

- 65.** Find all upper triangular  $2 \times 2$  matrices  $X$  such that  $X^2$  is the zero matrix.
- 66.** Find all lower triangular  $3 \times 3$  matrices  $X$  such that  $X^3$  is the zero matrix.
- 67.**
  - If  $A$  is any  $3 \times 3$  transition matrix (see Definition 1.4), find the matrix product  $[1 \ 1 \ 1] A$ .
  - For a fixed  $n$ , let  $\vec{e}$  be the row vector  $\vec{e} = [\underbrace{1 \ 1 \ \dots \ 1}_{n \ 1's}]$ . Show that an  $n \times n$  matrix  $A$  with nonnegative entries is a transition matrix  $A$  if (and only if)  $\vec{e}A = \vec{e}$ .
- 68.** Show that if  $A$  and  $B$  are  $n \times n$  transition matrices, then  $AB$  will be a transition matrix as well. Hint: Use Exercise 67b.
- 69.** Consider the matrix  $A^2$  in Example 4 of Section 3.
  - The third component of the first column of  $A^2$  is 1/4. What does this entry mean in practical terms, that is, in terms of surfers following links in our mini-Web?

## Linear Transformations

- b.** When is the  $ij$ th entry of  $A^2$  equal to 0? Give your answer both in terms of paths of length 2 in the graph of the mini-Web and also in terms of surfers being able to get from page  $j$  to page  $i$  by following two consecutive links.
- 70.** **a.** Compute  $A^3$  for the matrix  $A$  in Example 3.4.  
**b.** The fourth component of the first column of  $A^3$  is 1/4. What does this entry mean in practical terms, that is, in terms of surfers following links in our mini-Web?  
**c.** When is the  $ij$ th entry of  $A^3$  equal to 0? Give your answer both in terms of paths in the graph of the mini-Web and also in terms of surfers being able to get from page  $j$  to page  $i$  by following consecutive links.  
**d.** How many paths of length 3 are there in the graph of the mini-Web from page 1 to page 2? How many surfers are taking each of these paths, expressed as a proportion of the initial population of page 1?
- 71.** For the mini-Web in Example 3.4, find pages  $i$  and  $j$  such that it is impossible to get from page  $j$  to page  $i$  by following exactly four consecutive links. What does the answer tell you about the entries of  $A^4$ ?
- 72.** For the mini-Web in Example 3.4, find the smallest positive integer  $m$  such that all the entries of  $A^m$  are positive; you may use technology. What does your answer tell you in terms of paths in the graph of the mini-Web and also in terms of surfers following consecutive links?
- 73.** Use part (c) of Theorem 3.11 to prove part (b): If  $A$  is a regular transition matrix of size  $n \times n$  with equilibrium distribution  $\vec{x}_{equ}$ , and if  $\vec{x}$  is any distribution vector in  $\mathbb{R}^n$ , then  $\lim_{m \rightarrow \infty} (A^m \vec{x}) = \vec{x}_{equ}$ .
- 74.** Suppose  $A$  is a transition matrix and  $B$  is a positive transition matrix (see Definition 3.10), where  $A$  and  $B$  are of the same size. Is  $AB$  necessarily a positive transition matrix? What about  $BA$ ?
- 75.** Prove the following: If  $A$  is a transition matrix and  $A^m$  is positive, then  $A^{m+1}$  is positive as well.
- 76.** For the mini-Web graphed below, find the equilibrium distribution in the following way: Write the transition matrix  $A$ , test high powers of  $A$  to come up with a conjecture for the equilibrium distribution  $\vec{x}_{equ}$ , and then verify that  $A\vec{x}_{equ} = \vec{x}_{equ}$ . (This method, based on Theorem 3.11, is referred to as the *power method* for finding the equilibrium distribution of a regular transition matrix.) Also, find the page with the highest naïve PageRank. You may use technology.



- 77.** Consider the transition matrix

$$A = \begin{bmatrix} 0.4 & 0.2 & 0.7 \\ 0 & 0.6 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}.$$

Verify that  $A$  is a regular transition matrix and then use the power method (see Exercise 76) to find the equilibrium distribution. You may use technology.

- 78.** Let's revisit the mini-Web with the graph

$$\begin{array}{ccc} 1 & \rightleftharpoons & 2 \\ \downarrow & \swarrow & \uparrow \\ 3 & \rightarrow & 4, \end{array}$$

but here we consider the surfing model with a “jumping rate” of 20%, as discussed in Exercise 1.53. The corresponding transition matrix is

$$B = \begin{bmatrix} 0.05 & 0.45 & 0.05 & 0.05 \\ 0.45 & 0.05 & 0.05 & 0.85 \\ 0.45 & 0.45 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.85 & 0.05 \end{bmatrix}.$$

This transition matrix is positive and therefore regular, so that Theorem 3.11 applies. Use the power method (see Exercise 76) to find the equilibrium distribution. You may use technology. Write the components of  $\vec{x}_{equ}$  as rational numbers.

- 79.** Give an example of a transition matrix  $A$  such that there exists more than one distribution vector  $\vec{x}$  with  $A\vec{x} = \vec{x}$ .
- 80.** Give an example of a transition matrix  $A$  such that  $\lim_{m \rightarrow \infty} A^m$  fails to exist.
- 81.** If  $A\vec{v} = 5\vec{v}$ , express  $A^2\vec{v}$ ,  $A^3\vec{v}$ , and  $A^m\vec{v}$  as scalar multiples of the vector  $\vec{v}$ .
- 82.** In this exercise we will verify part (b) of Theorem 3.11 in the special case when  $A$  is the transition matrix  $\begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$  and  $\vec{x}$  is the distribution vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . [We will not be using parts (a) and (c) of Theorem 3.11.] The general proof of Theorem 3.11 runs along similar lines.
- a.** Compute  $A \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . Write  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  as a scalar multiple of the vector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
  - b.** Write the distribution vector  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  as a linear combination of the vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
  - c.** Use your answers in parts (a) and (b) to write  $A\vec{x}$  as a linear combination of the vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . More generally, write  $A^m\vec{x}$  as a linear combination

## Linear Transformations

of the vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , for any positive integer  $m$ . See Exercise 81.

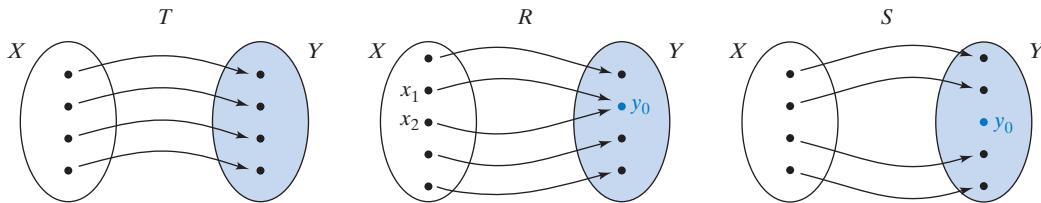
- d.** In your equation in part (c), let  $m$  go to infinity to find  $\lim_{m \rightarrow \infty} (A^m \vec{x})$ . Verify that your answer is the equilibrium distribution for  $A$ .
- 83.** If  $A\vec{x} = \vec{x}$  for a regular transition matrix  $A$  and a distribution vector  $\vec{x}$ , show that all components of  $\vec{x}$  must be

positive. (Here you are proving the last claim of Theorem 3.11a.)

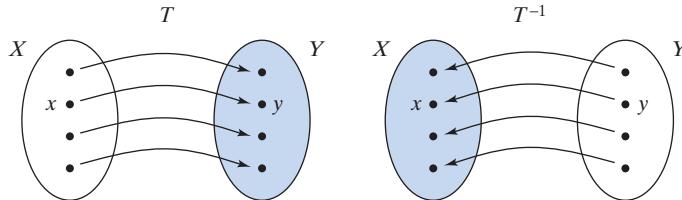
- 84.** Consider an  $n \times m$  matrix  $A$  of rank  $n$ . Show that there exists an  $m \times n$  matrix  $X$  such that  $AX = I_n$ . If  $n < m$ , how many such matrices  $X$  are there?
- 85.** Consider an  $n \times n$  matrix  $A$  of rank  $n$ . How many  $n \times n$  matrices  $X$  are there such that  $AX = I_n$ ?

## 4 The Inverse of a Linear Transformation

Let's first review the concept of an invertible function. As you read these abstract definitions, consider the examples in Figures 1 and 2, where  $X$  and  $Y$  are finite sets.



**Figure 1**  $T$  is invertible.  $R$  is not invertible: The equation  $R(x) = y_0$  has two solutions,  $x_1$  and  $x_2$ .  $S$  is not invertible: There is no  $x$  such that  $S(x) = y_0$ .



**Figure 2** A function  $T$  and its inverse  $T^{-1}$ .

### Definition 4.1

#### Invertible functions

A function  $T$  from  $X$  to  $Y$  is called invertible if the equation  $T(x) = y$  has a unique solution  $x$  in  $X$  for each  $y$  in  $Y$ .

In this case, the inverse  $T^{-1}$  from  $Y$  to  $X$  is defined by

$$T^{-1}(y) = (\text{the unique } x \text{ in } X \text{ such that } T(x) = y).$$

To put it differently, the equation

$$x = T^{-1}(y) \quad \text{means that} \quad y = T(x).$$

Note that

$$T^{-1}(T(x)) = x \quad \text{and} \quad T(T^{-1}(y)) = y$$

for all  $x$  in  $X$  and for all  $y$  in  $Y$ .

Conversely, if  $L$  is a function from  $Y$  to  $X$  such that

$$L(T(x)) = x \quad \text{and} \quad T(L(y)) = y$$

for all  $x$  in  $X$  and for all  $y$  in  $Y$ , then  $T$  is invertible and  $T^{-1} = L$ .

If a function  $T$  is invertible, then so is  $T^{-1}$  and  $(T^{-1})^{-1} = T$ .

## Linear Transformations

If a function is given by a formula, we may be able to find the inverse by solving the formula for the input variable(s). For example, the inverse of the function

$$y = \frac{x^3 - 1}{5} \quad (\text{from } \mathbb{R} \text{ to } \mathbb{R})$$

is

$$x = \sqrt[3]{5y + 1}.$$

Now consider the case of a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  given by

$$\vec{y} = T(\vec{x}) = A\vec{x},$$

where  $A$  is an  $n \times n$  matrix. (The case of an  $n \times m$  matrix will be discussed in Exercise 48.)

According to Definition 4.1, the linear transformation  $\vec{y} = T(\vec{x}) = A\vec{x}$  is invertible if the *linear system*

$$A\vec{x} = \vec{y}$$

has a unique solution  $\vec{x}$  in  $\mathbb{R}^n$  for all  $\vec{y}$  in the vector space  $\mathbb{R}^n$ . This is the case if (and only if)  $\text{rank}(A) = n$  or, equivalently, if

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = I_n.$$

### Definition 4.2

#### Invertible matrices

A square matrix  $A$  is said to be *invertible* if the linear transformation  $\vec{y} = T(\vec{x}) = A\vec{x}$  is invertible. In this case, the matrix<sup>10</sup> of  $T^{-1}$  is denoted by  $A^{-1}$ . If the linear transformation  $\vec{y} = T(\vec{x}) = A\vec{x}$  is invertible, then its inverse is  $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$ .

### Theorem 4.3

#### Invertibility

An  $n \times n$  matrix  $A$  is invertible if (and only if)

$$\text{rref}(A) = I_n$$

or, equivalently, if

$$\text{rank}(A) = n.$$

### Theorem 4.4

#### Invertibility and linear systems

Let  $A$  be an  $n \times n$  matrix.

- Consider a vector  $\vec{b}$  in  $\mathbb{R}^n$ . If  $A$  is invertible, then the system  $A\vec{x} = \vec{b}$  has the unique solution  $\vec{x} = A^{-1}\vec{b}$ . If  $A$  is noninvertible, then the system  $A\vec{x} = \vec{b}$  has infinitely many solutions or none.

<sup>10</sup>The inverse transformation is linear. See Exercise 2.29.

**Theorem 4.4**
**Invertibility and linear systems (*Continued*)**

- b. Consider the special case when  $\vec{b} = \vec{0}$ . The system  $A\vec{x} = \vec{0}$  has  $\vec{x} = \vec{0}$  as a solution. If  $A$  is invertible, then this is the only solution. If  $A$  is noninvertible, then the system  $A\vec{x} = \vec{0}$  has infinitely many solutions.

**EXAMPLE 1** Is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible?

**Solution**

$$\begin{array}{lcl} \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{array} \right] -2(I) \rightarrow \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 8 & 2 \end{array} \right] -3(I) \rightarrow \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{array} \right] -5(II) \rightarrow \\ \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \div(-1) \rightarrow \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] - (III) \rightarrow \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = I_3 = \text{rref}(A) \end{array}$$

Matrix  $A$  is invertible since  $\text{rref}(A) = I_3$ . ■

Let's find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

in Example 1 or, equivalently, the inverse of the linear transformation

$$\vec{y} = A\vec{x} \quad \text{or} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 2x_3 \end{bmatrix}.$$

To find the inverse transformation, we solve this system for the input variables  $x_1$ ,  $x_2$ , and  $x_3$ :

$$\begin{array}{l} \left| \begin{array}{l} x_1 + x_2 + x_3 = y_1 \\ 2x_1 + 3x_2 + 2x_3 = y_2 \\ 3x_1 + 8x_2 + 2x_3 = y_3 \end{array} \right| \xrightarrow{-2(I)} \\ \left| \begin{array}{l} x_1 + x_2 + x_3 = y_1 \\ x_2 = -2y_1 + y_2 \\ 5x_2 - x_3 = -3y_1 + y_3 \end{array} \right| \xrightarrow{-5(II)} \\ \left| \begin{array}{l} x_1 + x_3 = 3y_1 - y_2 \\ x_2 = -2y_1 + y_2 \\ -x_3 = 7y_1 - 5y_2 + y_3 \end{array} \right| \xrightarrow{\div(-1)} \\ \left| \begin{array}{l} x_1 + x_3 = 3y_1 - y_2 \\ x_2 = -2y_1 + y_2 \\ x_3 = -7y_1 + 5y_2 - y_3 \end{array} \right| \xrightarrow{-(III)} \\ \left| \begin{array}{l} x_1 = 10y_1 - 6y_2 + y_3 \\ x_2 = -2y_1 + y_2 \\ x_3 = -7y_1 + 5y_2 - y_3 \end{array} \right|. \end{array}$$

## Linear Transformations

We have found the inverse transformation; its matrix is

$$B = A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}.$$

We can write the preceding computations in matrix form:

$$\begin{array}{c|ccccc} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 2 & | & 0 & 1 & 0 \\ 3 & 8 & 2 & | & 0 & 0 & 1 \end{array} \xrightarrow{-2(I)} \begin{array}{c|ccccc} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 5 & -1 & | & -3 & 0 & 1 \end{array} \xrightarrow{-3(I)} \begin{array}{c|ccccc} 1 & 0 & 1 & | & 3 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & 7 & -5 & 1 \end{array} \xrightarrow{\div(-1)} \begin{array}{c|ccccc} 1 & 0 & 1 & | & 3 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{array} \xrightarrow{-5(II)} \begin{array}{c|ccccc} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{array} \xrightarrow{- (III)}$$

This process can be described succinctly as follows.

### Theorem 4.5

#### Finding the inverse of a matrix

To find the *inverse* of an  $n \times n$  matrix  $A$ , form the  $n \times (2n)$  matrix  $[A \mid I_n]$  and compute rref  $[A \mid I_n]$ .

- If rref  $[A \mid I_n]$  is of the form  $[I_n \mid B]$ , then  $A$  is invertible, and  $A^{-1} = B$ .
- If rref  $[A \mid I_n]$  is of another form (i.e., its left half fails to be  $I_n$ ), then  $A$  is not invertible. Note that the left half of rref  $[A \mid I_n]$  is rref( $A$ ).

Next let's discuss some algebraic rules for matrix inversion.

- Consider an invertible linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . By Definition 4.1, the equation  $T^{-1}(T(\vec{x})) = \vec{x}$  holds for all  $\vec{x}$  in  $\mathbb{R}^n$ . Written in matrix form, this equation reads  $A^{-1}A\vec{x} = \vec{x} = I_n\vec{x}$ . It follows that  $A^{-1}A = I_n$ . Likewise, we can show that  $AA^{-1} = I_n$ .

### Theorem 4.6

#### Multiplying with the inverse

For an invertible  $n \times n$  matrix  $A$ ,

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n.$$

- If  $A$  and  $B$  are invertible  $n \times n$  matrices, is  $BA$  invertible as well? If so, what is its inverse?

To find the inverse of the linear transformation

$$\vec{y} = BA\vec{x},$$

we solve the equation for  $\vec{x}$  in two steps. First, we multiply both sides of the equation by  $B^{-1}$  from the left:

$$B^{-1}\vec{y} = B^{-1}BA\vec{x} = I_nA\vec{x} = A\vec{x}.$$

Now, we multiply by  $A^{-1}$  from the left:

$$A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x} = \vec{x}.$$

## Linear Transformations

This computation shows that the linear transformation

$$\vec{y} = BA\vec{x}$$

is invertible and that its inverse is

$$\vec{x} = A^{-1}B^{-1}\vec{y}.$$

### Theorem 4.7

#### The inverse of a product of matrices

If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $BA$  is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}.$$

Pay attention to the order of the matrices. (Order matters!)

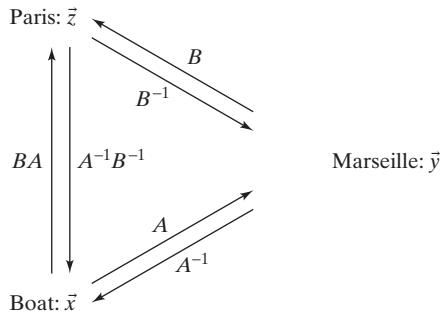
To verify this result, we can multiply  $A^{-1}B^{-1}$  by  $BA$  (in either order), and check that the result is  $I_n$ :

$$\begin{aligned} BAA^{-1}B^{-1} &= BI_nB^{-1} = BB^{-1} = I_n, \text{ and} \\ A^{-1}B^{-1}BA &= A^{-1}A = I_n. \end{aligned}$$

Everything works out!

To understand the order of the factors in the formula  $(BA)^{-1} = A^{-1}B^{-1}$ , think about our French coast guard story again.

To recover the actual position  $\vec{x}$  from the doubly encoded position  $\vec{z}$ , you *first* apply the decoding transformation  $\vec{y} = B^{-1}\vec{z}$  and *then* the decoding transformation  $\vec{x} = A^{-1}\vec{y}$ . The inverse of  $\vec{z} = BA\vec{x}$  is therefore  $\vec{x} = A^{-1}B^{-1}\vec{z}$ , as illustrated in Figure 3.



**Figure 3**

The following result is often useful in finding inverses:

### Theorem 4.8

#### A criterion for invertibility

Let  $A$  and  $B$  be two  $n \times n$  matrices such that

$$BA = I_n.$$

Then

- a.  $A$  and  $B$  are both invertible,
- b.  $A^{-1} = B$  and  $B^{-1} = A$ , and
- c.  $AB = I_n$ .

### Linear Transformations

It follows from the definition of an invertible function that if  $AB = I_n$  and  $BA = I_n$ , then  $A$  and  $B$  are inverses; that is,  $A = B^{-1}$  and  $B = A^{-1}$ . Theorem 4.8 makes the point that the equation  $BA = I_n$  alone guarantees that  $A$  and  $B$  are inverses. Exercise 107 illustrates the significance of this claim.

**Proof** To demonstrate that  $A$  is invertible, it suffices to show that the linear system  $A\vec{x} = \vec{0}$  has only the solution  $\vec{x} = \vec{0}$  (by Theorem 4.4b). If we multiply the equation  $A\vec{x} = \vec{0}$  by  $B$  from the left, we find that  $BA\vec{x} = B\vec{0} = \vec{0}$ . It follows that  $\vec{x} = I_n\vec{x} = BA\vec{x} = \vec{0}$ , as claimed. Therefore,  $A$  is invertible. If we multiply the equation  $BA = I_n$  by  $A^{-1}$  from the right, we find that  $B = A^{-1}$ . Matrix  $B$ , being the inverse of  $A$ , is itself invertible, and  $B^{-1} = (A^{-1})^{-1} = A$ . See Definition 4.1. Finally,  $AB = AA^{-1} = I_n$ .

You can use Theorem 4.8 to check your work when computing the inverse of a matrix. Earlier in this section we claimed that

$$B = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \quad \text{is the inverse of} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}.$$

Let's use Theorem 4.8b to check our work:

$$BA = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \quad \blacksquare$$

**EXAMPLE 2** Suppose  $A$ ,  $B$ , and  $C$  are three  $n \times n$  matrices such that  $ABC = I_n$ . Show that  $B$  is invertible, and express  $B^{-1}$  in terms of  $A$  and  $C$ .

#### Solution

Write  $ABC = (AB)C = I_n$ . We have  $C(AB) = I_n$ , by Theorem 4.8c. Since matrix multiplication is associative, we can write  $(CA)B = I_n$ . Applying Theorem 4.8 again, we conclude that  $B$  is invertible, and  $B^{-1} = CA$ .  $\blacksquare$

**EXAMPLE 3** For an arbitrary  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , compute the product  $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . When is  $A$  invertible? If so, what is  $A^{-1}$ ?

#### Solution

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

$$\text{If } ad - bc \neq 0, \text{ we can write } \underbrace{\left( \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)}_B \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_A = I_2.$$

It now follows from Theorem 4.8 that  $A$  is invertible, with  $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ . Conversely, if  $A$  is invertible, then we can multiply the equation  $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - bc)I_2$  with  $A^{-1}$  from the right, finding  $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad - bc)A^{-1}$ . Since some of the scalars  $a$ ,  $b$ ,  $c$ ,  $d$  are nonzero (being the entries of the invertible matrix  $A$ ), it follows that  $ad - bc \neq 0$ .  $\blacksquare$

**Theorem 4.9**
**Inverse and determinant of a  $2 \times 2$  matrix**

a. The  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if (and only if)  $ad - bc \neq 0$ .

Quantity  $ad - bc$  is called the *determinant* of  $A$ , written  $\det(A)$ :

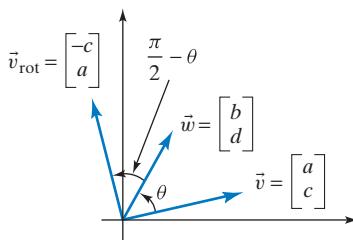
$$\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

b. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



**Figure 4**

What is the geometrical interpretation of the determinant of a  $2 \times 2$  matrix  $A$ ? Write  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , and consider the column vectors  $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} b \\ d \end{bmatrix}$ . It turns out to be helpful to introduce the auxiliary vector  $\vec{v}_{\text{rot}} = \begin{bmatrix} -c \\ a \end{bmatrix}$ , obtained by rotating vector  $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$  through an angle of  $\frac{\pi}{2}$ . Let  $\theta$  be the (oriented) angle from  $\vec{v}$  to  $\vec{w}$ , with  $-\pi < \theta \leq \pi$ . See Figure 4. Then

$$\det A = ad - bc \underset{\text{step 2}}{=} \vec{v}_{\text{rot}} \cdot \vec{w} \underset{\text{step 3}}{=} \|\vec{v}_{\text{rot}}\| \cos\left(\frac{\pi}{2} - \theta\right) \|\vec{w}\| = \|\vec{v}\| \sin \theta \|\vec{w}\|.$$

In steps 2 and 3 we use the definition of the dot product and its geometrical interpretation.

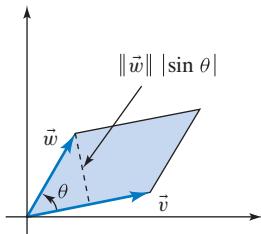
**Theorem 4.10**
**Geometrical interpretation of the determinant of a  $2 \times 2$  matrix**

If  $A = [\vec{v} \ \vec{w}]$  is a  $2 \times 2$  matrix with nonzero columns  $\vec{v}$  and  $\vec{w}$ , then

$$\det A = \det [\vec{v} \ \vec{w}] = \|\vec{v}\| \sin \theta \|\vec{w}\|,$$

where  $\theta$  is the oriented angle from  $\vec{v}$  to  $\vec{w}$ , with  $-\pi < \theta \leq \pi$ . It follows that

- $|\det A| = \|\vec{v}\| |\sin \theta| \|\vec{w}\|$  is the *area of the parallelogram* spanned by  $\vec{v}$  and  $\vec{w}$ . See Figure 5,
- $\det A = 0$  if  $\vec{v}$  and  $\vec{w}$  are *parallel*, meaning that  $\theta = 0$  or  $\theta = \pi$ ,
- $\det A > 0$  if  $0 < \theta < \pi$ , and
- $\det A < 0$  if  $-\pi < \theta < 0$ .



**Figure 5**

**EXAMPLE 4**

Is the matrix  $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$  invertible? If so, find the inverse. Interpret  $\det A$  geometrically.

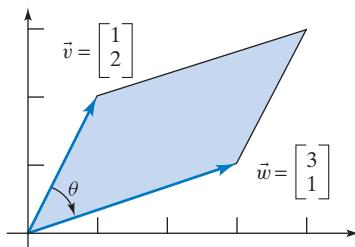
**Solution**

We find the determinant  $\det(A) = 1 \cdot 1 - 3 \cdot 2 = -5 \neq 0$ , so that  $A$  is indeed invertible, by Theorem 4.9a. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(-5)} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix},$$

by Theorem 4.9b.

Furthermore,  $|\det A| = 5$  is the area of the shaded parallelogram in Figure 6, and  $\det A$  is negative since the angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is negative. ■



**Figure 6**

**EXAMPLE 5**

For which values of the constant  $k$  is the matrix  $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$  invertible?

**Solution**

By Theorem 4.9a, the matrix  $A$  fails to be invertible if  $\det A = 0$ . Now

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix} = (1-k)(3-k) - 2 \cdot 4 \\ &= k^2 - 4k - 5 = (k-5)(k+1) = 0 \end{aligned}$$

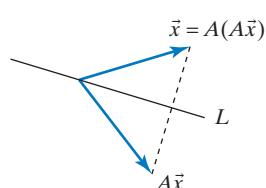
when  $k = 5$  or  $k = -1$ . Thus,  $A$  is invertible for all values of  $k$  except  $k = 5$  and  $k = -1$ . ■

**EXAMPLE 6**

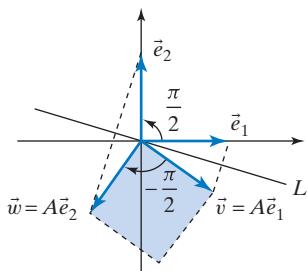
Consider a matrix  $A$  that represents the reflection about a line  $L$  in the plane. Use the determinant to verify that  $A$  is invertible. Find  $A^{-1}$ . Explain your answer conceptually, and interpret the determinant geometrically.

**Solution**

By Definition 2.2, a reflection matrix is of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ . Now  $\det A = \det \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = -a^2 - b^2 = -1$ . It turns out that  $A$  is invertible, and  $A^{-1} = \frac{1}{(-1)} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = A$ . It makes good sense that  $A$  is its own inverse, since  $A(A\vec{x}) = \vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^2$ , by definition of a reflection. See Figure 7.



**Figure 7**



**Figure 8**

To interpret the determinant geometrically, recall that  $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = A\vec{e}_1$  and  $\vec{w} = \begin{bmatrix} b \\ -a \end{bmatrix} = A\vec{e}_2$ . The parallelogram spanned by  $\vec{v}$  and  $\vec{w}$  is actually a unit square, with area  $1 = |\det A|$ , and  $\theta$  is  $-\frac{\pi}{2}$  since the reflection about  $L$  reverses the orientation of an angle. See Figure 8. ■

## The Inverse of a Block Matrix (Optional)

We will conclude this chapter with two examples involving block matrices. To refresh your memory, take another look at Theorem 3.9.

**EXAMPLE 7** Let  $A$  be a block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where  $A_{11}$  is an  $n \times n$  matrix,  $A_{22}$  is an  $m \times m$  matrix, and  $A_{12}$  is an  $n \times m$  matrix.

- a. For which choices of  $A_{11}$ ,  $A_{12}$ , and  $A_{22}$  is  $A$  invertible?
- b. If  $A$  is invertible, what is  $A^{-1}$  (in terms of  $A_{11}$ ,  $A_{12}$ ,  $A_{22}$ )?

### Solution

We are looking for an  $(n+m) \times (n+m)$  matrix  $B$  such that

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Let us partition  $B$  in the same way as  $A$ :

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where  $B_{11}$  is  $n \times n$ ,  $B_{22}$  is  $m \times m$ , and so on. The fact that  $B$  is the inverse of  $A$  means that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix},$$

or, using Theorem 3.9,

$$\left| \begin{array}{l} B_{11}A_{11} = I_n \\ B_{11}A_{12} + B_{12}A_{22} = 0 \\ B_{21}A_{11} = 0 \\ B_{21}A_{12} + B_{22}A_{22} = I_m \end{array} \right|.$$

We have to solve for the blocks  $B_{ij}$ . Applying Theorem 4.8 to the equation  $B_{11}A_{11} = I_n$ , we find that  $A_{11}$  is invertible, and  $B_{11} = A_{11}^{-1}$ . Equation 3 now implies that  $B_{21} = 0A_{11}^{-1} = 0$ . Next, Equation 4 simplifies to  $B_{22}A_{22} = I_m$ . By Theorem 4.8,  $A_{22}$  is invertible, and  $B_{22} = A_{22}^{-1}$ . Lastly, Equation 2 becomes  $A_{11}^{-1}A_{12} + B_{12}A_{22} = 0$ , or  $B_{12}A_{22} = -A_{11}^{-1}A_{12}$ , or  $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$ . We conclude that

- a.  $A$  is invertible if (and only if) both  $A_{11}$  and  $A_{22}$  are invertible (no condition is imposed on  $A_{12}$ ), and
- b. If  $A$  is invertible, then its inverse is

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

■

Verify this result for the following example:

**EXAMPLE 8**

$$\left[ \begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]^{-1} = \left[ \begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]. \quad \blacksquare$$

**EXERCISES 4**

**GOAL** Apply the concept of an invertible function. Determine whether a matrix (or a linear transformation) is invertible, and find the inverse if it exists.

Decide whether the matrices in Exercises 1 through 15 are invertible. If they are, find the inverse. Do the computations with paper and pencil. Show all your work.

1.  $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

2.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

5.  $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

6.  $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

8.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

9.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10.  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12.  $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

13.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

14.  $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

15.  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}$

Decide whether the linear transformations in Exercises 16 through 20 are invertible. Find the inverse transformation if it exists. Do the computations with paper and pencil. Show all your work.

16.  $y_1 = 3x_1 + 5x_2$   
 $y_2 = 5x_1 + 8x_2$

17.  $y_1 = x_1 + 2x_2$   
 $y_2 = 4x_1 + 8x_2$

18.  $y_1 = x_2$   
 $y_2 = x_3$   
 $y_3 = x_1$

19.  $y_1 = x_1 + x_2 + x_3$   
 $y_2 = x_1 + 2x_2 + 3x_3$   
 $y_3 = x_1 + 4x_2 + 9x_3$

20.  $y_1 = x_1 + 3x_2 + 3x_3$   
 $y_2 = x_1 + 4x_2 + 8x_3$   
 $y_3 = 2x_1 + 7x_2 + 12x_3$

Which of the functions  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  in Exercises 21 through 24 are invertible?

21.  $f(x) = x^2$

22.  $f(x) = 2^x$

23.  $f(x) = x^3 + x$

24.  $f(x) = x^3 - x$

Which of the (nonlinear) transformations from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  in Exercises 25 through 27 are invertible? Find the inverse if it exists.

25.  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^3 \\ x_2^2 \end{bmatrix}$

26.  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^3 + x_2 \end{bmatrix}$

27.  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 \cdot x_2 \end{bmatrix}$

28. Find the inverse of the linear transformation

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

from  $\mathbb{R}^4$  to  $\mathbb{R}^4$ .

29. For which values of the constant  $k$  is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

## Linear Transformations

30. For which values of the constants  $b$  and  $c$  is the following matrix invertible?

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

31. For which values of the constants  $a$ ,  $b$ , and  $c$  is the following matrix invertible?

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

32. Find all matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $ad - bc = 1$  and  $A^{-1} = A$ .

33. Consider the matrices of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a$  and  $b$  are arbitrary constants. For which values of  $a$  and  $b$  is  $A^{-1} = A$ ?

34. Consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

- a. For which values of  $a$ ,  $b$ , and  $c$  is  $A$  invertible? If it is invertible, what is  $A^{-1}$ ?  
 b. For which values of the diagonal elements is a diagonal matrix (of arbitrary size) invertible?

35. a. Consider the upper triangular  $3 \times 3$  matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

- For which values of  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  is  $A$  invertible?  
 b. More generally, when is an upper triangular matrix (of arbitrary size) invertible?  
 c. If an upper triangular matrix is invertible, is its inverse an upper triangular matrix as well?  
 d. When is a lower triangular matrix invertible?

36. To determine whether a square matrix  $A$  is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce  $A$  to (upper or lower) triangular form, using elementary row operations. Show that  $A$  is invertible if (and only if) all entries on the diagonal of this triangular form are nonzero.

37. If  $A$  is an invertible matrix and  $c$  is a nonzero scalar, is the matrix  $cA$  invertible? If so, what is the relationship between  $A^{-1}$  and  $(cA)^{-1}$ ?

38. Find  $A^{-1}$  for  $A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ .

39. Consider a square matrix that differs from the identity matrix at just one entry, off the diagonal, for example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

In general, is a matrix  $M$  of this form invertible? If so, what is the  $M^{-1}$ ?

40. Show that if a square matrix  $A$  has two equal columns, then  $A$  is not invertible.

41. Which of the following linear transformations  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  are invertible? Find the inverse if it exists.

- a. Reflection about a plane  
 b. Orthogonal projection onto a plane  
 c. Scaling by a factor of 5 [i.e.,  $T(\vec{v}) = 5\vec{v}$ , for all vectors  $\vec{v}$ ]  
 d. Rotation about an axis

42. A square matrix is called a *permutation matrix* if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are  $I_n$  and

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

43. Consider two invertible  $n \times n$  matrices  $A$  and  $B$ . Is the linear transformation  $\vec{y} = A(B\vec{x})$  invertible? If so, what is the inverse? Hint: Solve the equation  $\vec{y} = A(B\vec{x})$  first for  $B\vec{x}$  and then for  $\vec{x}$ .

44. Consider the  $n \times n$  matrix  $M_n$ , with  $n \geq 2$ , that contains all integers  $1, 2, 3, \dots, n^2$  as its entries, written in sequence, column by column; for example,

$$M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

- a. Determine the rank of  $M_4$ .  
 b. Determine the rank of  $M_n$ .  
 c. For which  $n$  is  $M_n$  invertible?  
 45. To gauge the complexity of a computational task, mathematicians and computer scientists count the number of elementary operations (additions, subtractions, multiplications, and divisions) required. For a rough count, we will sometimes consider multiplications and divisions only, referring to those jointly as *multiplicative operations*. As an example, we examine the process of inverting a  $2 \times 2$  matrix by elimination.

## Linear Transformations

$$\begin{array}{c}
 \left[ \begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] \quad \div a, \text{ requires 2 multiplicative operations: } b/a \text{ and } 1/a \\
 \downarrow \\
 \left[ \begin{array}{cc|cc} 1 & b' & e & 0 \\ c & d & 0 & 1 \end{array} \right] \quad (\text{where } b' = b/a, \text{ and } e = 1/a) \\
 -c \text{ (I), requires 2 multiplicative operations: } cb' \text{ and } ce \\
 \downarrow \\
 \left[ \begin{array}{cc|cc} 1 & b' & e & 0 \\ 0 & d' & g & 1 \end{array} \right] \quad \div d', \text{ requires 2 multiplicative operations} \\
 \downarrow \\
 \left[ \begin{array}{cc|cc} 1 & b' & e & 0 \\ 0 & 1 & g' & h \end{array} \right] \quad -b' \text{ (II), requires 2 multiplicative operations} \\
 \downarrow \\
 \left[ \begin{array}{cc|cc} 1 & 0 & e' & f \\ 0 & 1 & g' & h \end{array} \right]
 \end{array}$$

The whole process requires eight multiplicative operations. Note that we do not count operations with predictable results, such as  $1a$ ,  $0a$ ,  $a/a$ ,  $0/a$ .

- a. How many multiplicative operations are required to invert a  $3 \times 3$  matrix by elimination?
- b. How many multiplicative operations are required to invert an  $n \times n$  matrix by elimination?
- c. If it takes a slow hand-held calculator 1 second to invert a  $3 \times 3$  matrix, how long will it take the same calculator to invert a  $12 \times 12$  matrix? Assume that the matrices are inverted by Gauss–Jordan elimination and that the duration of the computation is proportional to the number of multiplications and divisions involved.

- 46.** Consider the linear system

$$A\vec{x} = \vec{b},$$

where  $A$  is an invertible matrix. We can solve this system in two different ways:

- By finding the reduced row-echelon form of the augmented matrix  $[A \mid \vec{b}]$ ,
- By computing  $A^{-1}$  and using the formula  $\vec{x} = A^{-1}\vec{b}$ .

In general, which approach requires fewer multiplicative operations? See Exercise 45.

- 47.** Give an example of a noninvertible function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  and a number  $b$  such that the equation

$$f(x) = b$$

has a unique solution.

- 48.** Consider an invertible linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , with inverse  $L = T^{-1}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . In Exercise 2.29 we show that  $L$  is a linear transformation, so that  $L(\vec{y}) = B\vec{y}$  for some  $m \times n$  matrix  $B$ . Use the equations  $BA = I_m$  and  $AB = I_n$  to show

that  $n = m$ . Hint: Think about the number of solutions of the linear systems  $A\vec{x} = \vec{0}$  and  $B\vec{y} = \vec{0}$ .

- 49. Input–Output Analysis.** Consider the industries  $J_1, J_2, \dots, J_n$  in an economy. Suppose the consumer demand vector is  $\vec{b}$ , the output vector is  $\vec{x}$ , and the demand vector of the  $j$ th industry is  $\vec{v}_j$ . (The  $i$ th component  $a_{ij}$  of  $\vec{v}_j$  is the demand industry  $J_j$  puts on industry  $J_i$ , per unit of output of  $J_j$ .) The output  $\vec{x}$  just meets the aggregate demand if

$$\underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}}_{\text{aggregate demand}} = \underbrace{\vec{x}}_{\text{output}}.$$

This equation can be written more succinctly as

$$\left[ \begin{array}{cccc|c} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & | \\ | & | & & | & | \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \vec{b} = \vec{x},$$

or  $A\vec{x} + \vec{b} = \vec{x}$ . The matrix  $A$  is called the *technology matrix* of this economy; its coefficients  $a_{ij}$  describe the interindustry demand, which depends on the technology used in the production process. The equation

$$A\vec{x} + \vec{b} = \vec{x}$$

describes a linear system, which we can write in the customary form:

$$\begin{aligned}
 \vec{x} - A\vec{x} &= \vec{b} \\
 I_n\vec{x} - A\vec{x} &= \vec{b} \\
 (I_n - A)\vec{x} &= \vec{b}.
 \end{aligned}$$

If we want to know the output  $\vec{x}$  required to satisfy a given consumer demand  $\vec{b}$ , we can solve this linear system, preferably via the augmented matrix.

In economics, however, we often ask other questions: If  $\vec{b}$  changes, how will  $\vec{x}$  change in response? If the consumer demand on one industry increases by 1 unit and the consumer demand on the other industries remains unchanged, how will  $\vec{x}$  change?<sup>11</sup> If we ask questions like these, we think of the output  $\vec{x}$  as a *function* of the consumer demand  $\vec{b}$ .

<sup>11</sup> The relevance of questions like these became particularly clear during World War II, when the demand on certain industries suddenly changed dramatically. When U.S. President F. D. Roosevelt asked for 50,000 airplanes to be built, it was easy enough to predict that the country would have to produce more aluminum. Unexpectedly, the demand for copper dramatically increased (why?). A copper shortage then occurred, which was solved by borrowing silver from Fort Knox. People realized that input–output analysis can be effective in modeling and predicting chains of increased demand like this. After World War II, this technique rapidly gained acceptance and was soon used to model the economies of more than 50 countries.

## Linear Transformations

If the matrix  $(I_n - A)$  is invertible,<sup>12</sup> we can express  $\vec{x}$  as a function of  $\vec{b}$  (in fact, as a linear transformation):

$$\vec{x} = (I_n - A)^{-1}\vec{b}.$$

- a. Consider the economy of Israel in 1958. The three industries considered here are

- I<sub>1</sub>: agriculture,
- I<sub>2</sub>: manufacturing,
- I<sub>3</sub>: energy.

Find the technology matrix  $A$ , the matrix  $(I_n - A)$ , and its inverse  $(I_n - A)^{-1}$ .

- b. In the example discussed in part (a), suppose the consumer demand on agriculture (Industry 1) is 1 unit (1 million pounds), and the demands on the other two industries are zero. What output  $\vec{x}$  is required in this case? How does your answer relate to the matrix  $(I_n - A)^{-1}$ ?
- c. Explain, in terms of economics, why the diagonal elements of the matrix  $(I_n - A)^{-1}$  you found in part (a) must be at least 1.
- d. If the consumer demand on manufacturing increases by 1 (from whatever it was), and the consumer demand on the other two industries remains the same, how will the output have to change? How does your answer relate to the matrix  $(I_n - A)^{-1}$ ?
- e. Using your answers in parts (a) through (d) as a guide, explain in general (not just for this example) what the columns and the entries of the matrix  $(I_n - A)^{-1}$  tell you, in terms of economics. Those who have studied multivariable calculus may wish to consider the partial derivatives

$$\frac{\partial x_i}{\partial b_j}.$$

50. This exercise refers to Exercise 49a. Consider the entry  $k = a_{11} = 0.293$  of the technology matrix  $A$ . Verify that the entry in the first row and the first column of  $(I_n - A)^{-1}$  is the value of the geometrical series  $1 + k + k^2 + \dots$ . Interpret this observation in terms of economics.

51. a. Consider an  $n \times m$  matrix  $A$  with  $\text{rank}(A) < n$ . Show that there exists a vector  $\vec{b}$  in  $\mathbb{R}^n$  such that the system  $A\vec{x} = \vec{b}$  is inconsistent. Hint: For  $E = \text{rref}(A)$ , show that there exists a vector  $\vec{c}$  in  $\mathbb{R}^n$  such that the system  $E\vec{x} = \vec{c}$  is inconsistent; then, “work backward.”
- b. Consider an  $n \times m$  matrix  $A$  with  $n > m$ . Show that there exists a vector  $\vec{b}$  in  $\mathbb{R}^n$  such that the system  $A\vec{x} = \vec{b}$  is inconsistent.

52. For

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{bmatrix},$$

<sup>12</sup>This will always be the case for a “productive” economy. See Exercise 103.

find a vector  $\vec{b}$  in  $\mathbb{R}^4$  such that the system  $A\vec{x} = \vec{b}$  is inconsistent. See Exercise 51.

53. Let  $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$  in all parts of this problem.
- a. Find a scalar  $\lambda$  (lambda) such that the matrix  $A - \lambda I_2$  fails to be invertible. There are two solutions; choose one and use it in parts (b) and (c).
  - b. For the  $\lambda$  you chose in part (a), find the matrix  $A - \lambda I_2$ ; then find a nonzero vector  $\vec{x}$  such that  $(A - \lambda I_2)\vec{x} = \vec{0}$ . (This can be done, since  $A - \lambda I_2$  fails to be invertible.)
  - c. Note that the equation  $(A - \lambda I_2)\vec{x} = \vec{0}$  can be written as  $A\vec{x} - \lambda\vec{x} = \vec{0}$ , or  $A\vec{x} = \lambda\vec{x}$ . Check that the equation  $A\vec{x} = \lambda\vec{x}$  holds for your  $\lambda$  from part (a) and your  $\vec{x}$  from part (b).

54. Let  $A = \begin{bmatrix} 1 & 10 \\ -3 & 12 \end{bmatrix}$ . Using Exercise 53 as a guide, find a scalar  $\lambda$  and a nonzero vector  $\vec{x}$  such that  $A\vec{x} = \lambda\vec{x}$ .

*In Exercises 55 through 65, show that the given matrix  $A$  is invertible, and find the inverse. Interpret the linear transformation  $T(\vec{x}) = A\vec{x}$  and the inverse transformation  $T^{-1}(\vec{y}) = A^{-1}\vec{y}$  geometrically. Interpret  $\det A$  geometrically. In your figure, show the angle  $\theta$  and the vectors  $\vec{v}$  and  $\vec{w}$  introduced in Theorem 4.10.*

55.  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$       56.  $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

57.  $\begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix}$       58.  $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

59.  $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$       60.  $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

61.  $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$       62.  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

63.  $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$       64.  $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

65.  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

66. Consider two  $n \times n$  matrices  $A$  and  $B$  such that the product  $AB$  is invertible. Show that the matrices  $A$  and  $B$  are both invertible. Hint:  $AB(AB)^{-1} = I_n$  and  $(AB)^{-1}AB = I_n$ . Use Theorem 4.8.

*For two invertible  $n \times n$  matrices  $A$  and  $B$ , determine which of the formulas stated in Exercises 67 through 75 are necessarily true.*

67.  $(A + B)^2 = A^2 + 2AB + B^2$

68.  $(A - B)(A + B) = A^2 - B^2$

69.  $A + B$  is invertible, and  $(A + B)^{-1} = A^{-1} + B^{-1}$

70.  $A^2$  is invertible, and  $(A^2)^{-1} = (A^{-1})^2$

71.  $ABB^{-1}A^{-1} = I_n$

## Linear Transformations

72.  $ABA^{-1} = B$   
 73.  $(ABA^{-1})^3 = AB^3A^{-1}$   
 74.  $(I_n + A)(I_n + A^{-1}) = 2I_n + A + A^{-1}$   
 75.  $A^{-1}B$  is invertible, and  $(A^{-1}B)^{-1} = B^{-1}A$   
 76. Find all linear transformations  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  such that

$$T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*Hint:* We are looking for the  $2 \times 2$  matrices  $A$  such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

These two equations can be combined to form the matrix equation

$$A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

77. Using the last exercise as a guide, justify the following statement:

Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  be vectors in  $\mathbb{R}^m$  such that the matrix

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

is invertible. Let  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$  be arbitrary vectors in  $\mathbb{R}^n$ . Then there exists a unique linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  such that  $T(\vec{v}_i) = \vec{w}_i$ , for all  $i = 1, \dots, m$ . Find the matrix  $A$  of this transformation in terms of  $S$  and

$$B = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \\ | & | & & | \end{bmatrix}.$$

78. Find the matrix  $A$  of the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^3$  with

$$T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

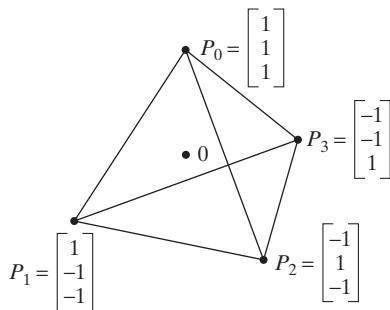
Compare with Exercise 77.

79. Find the matrix  $A$  of the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  with

$$T \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Compare with Exercise 77.

80. Consider the regular tetrahedron sketched below, whose center is at the origin.



Let  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  be the rotation about the axis through the points 0 and  $P_2$  that transforms  $P_1$  into  $P_3$ . Find the images of the four corners of the tetrahedron under this transformation.

$$\begin{array}{l} P_0 \xrightarrow{T} \\ P_1 \rightarrow P_3 \\ P_2 \rightarrow \\ P_3 \rightarrow \end{array}$$

Let  $L$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  be the reflection about the plane through the points 0,  $P_0$ , and  $P_3$ . Find the images of the four corners of the tetrahedron under this transformation.

$$\begin{array}{l} P_0 \xrightarrow{L} \\ P_1 \rightarrow \\ P_2 \rightarrow \\ P_3 \rightarrow \end{array}$$

Describe the transformations in parts (a) through (c) geometrically.

- a.  $T^{-1}$       b.  $L^{-1}$   
 c.  $T^2 = T \circ T$  (the composite of  $T$  with itself)  
 d. Find the images of the four corners under the transformations  $T \circ L$  and  $L \circ T$ . Are the two transformations the same?

$$\begin{array}{ll} P_0 \xrightarrow{T \circ L} & P_0 \xrightarrow{L \circ T} \\ P_1 \rightarrow & P_1 \rightarrow \\ P_2 \rightarrow & P_2 \rightarrow \\ P_3 \rightarrow & P_3 \rightarrow \end{array}$$

- e. Find the images of the four corners under the transformation  $L \circ T \circ L$ . Describe this transformation geometrically.

81. Find the matrices of the transformations  $T$  and  $L$  defined in Exercise 80.

82. Consider the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and an arbitrary  $3 \times 3$  matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

## Linear Transformations

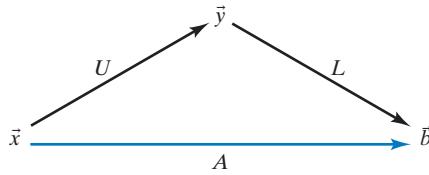
- a.** Compute  $EA$ . Comment on the relationship between  $A$  and  $EA$ , in terms of the technique of elimination.
- b.** Consider the matrix
- $$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- and an arbitrary  $3 \times 3$  matrix  $A$ . Compute  $EA$ . Comment on the relationship between  $A$  and  $EA$ .
- c.** Can you think of a  $3 \times 3$  matrix  $E$  such that  $EA$  is obtained from  $A$  by swapping the last two rows (for any  $3 \times 3$  matrix  $A$ )?
- d.** The matrices of the forms introduced in parts (a), (b), and (c) are called *elementary*: An  $n \times n$  matrix  $E$  is elementary if it can be obtained from  $I_n$  by performing one of the three elementary row operations on  $I_n$ . Describe the format of the three types of elementary matrices.
- 83.** Are elementary matrices invertible? If so, is the inverse of an elementary matrix elementary as well? Explain the significance of your answers in terms of elementary row operations.
- 84. a.** Justify the following: If  $A$  is an  $n \times m$  matrix, then there exist elementary  $n \times n$  matrices  $E_1, E_2, \dots, E_p$  such that
- $$\text{rref}(A) = E_1 E_2 \cdots E_p A.$$
- b.** Find such elementary matrices  $E_1, E_2, \dots, E_p$  for
- $$A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}.$$
- 85. a.** Justify the following: If  $A$  is an  $n \times m$  matrix, then there exists an invertible  $n \times n$  matrix  $S$  such that
- $$\text{rref}(A) = SA.$$
- b.** Find such an invertible matrix  $S$  for
- $$A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}.$$
- 86. a.** Justify the following: Any invertible matrix is a product of elementary matrices.
- b.** Write  $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$  as a product of elementary matrices.
- 87.** Write all possible forms of elementary  $2 \times 2$  matrices  $E$ . In each case, describe the transformation  $\vec{y} = E\vec{x}$  geometrically.
- 88.** Consider an invertible  $n \times n$  matrix  $A$  and an  $n \times n$  matrix  $B$ . A certain sequence of elementary row operations transforms  $A$  into  $I_n$ .
- a.** What do you get when you apply the same row operations in the same order to the matrix  $AB$ ?
- b.** What do you get when you apply the same row operations to  $I_n$ ?
- 89.** Is the product of two lower triangular matrices a lower triangular matrix as well? Explain your answer.
- 90.** Consider the matrix
- $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}.$$
- a.** Find lower triangular elementary matrices  $E_1, E_2, \dots, E_m$  such that the product
- $$E_m \cdots E_2 E_1 A$$
- is an upper triangular matrix  $U$ . Hint: Use elementary row operations to eliminate the entries below the diagonal of  $A$ .
- b.** Find lower triangular elementary matrices  $M_1, M_2, \dots, M_m$  and an upper triangular matrix  $U$  such that
- $$A = M_1 M_2 \cdots M_m U.$$
- c.** Find a lower triangular matrix  $L$  and an upper triangular matrix  $U$  such that
- $$A = LU.$$
- Such a representation of an invertible matrix is called an *LU-factorization*. The method outlined in this exercise to find an *LU-factorization* can be streamlined somewhat, but we have seen the major ideas. An *LU-factorization* (as introduced here) does not always exist. See Exercise 92.
- d.** Find a lower triangular matrix  $L$  with 1's on the diagonal, an upper triangular matrix  $U$  with 1's on the diagonal, and a diagonal matrix  $D$  such that  $A = LDU$ . Such a representation of an invertible matrix is called an *LDU-factorization*.
- 91.** Knowing an *LU-factorization* of a matrix  $A$  makes it much easier to solve a linear system
- $$A\vec{x} = \vec{b}.$$
- Consider the *LU-factorization*
- $$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -3 & -5 & 6 & -5 \\ 1 & 4 & 6 & 20 \\ -1 & 6 & 20 & 43 \end{bmatrix}$$
- $$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 8 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- $$= LU.$$

## Linear Transformations

Suppose we have to solve the system  $A\vec{x} = \vec{b}$ , where

$$\vec{b} = \begin{bmatrix} -3 \\ 14 \\ 9 \\ 33 \end{bmatrix}.$$

- a. Set  $\vec{y} = U\vec{x}$ , and solve the system  $L\vec{y} = \vec{b}$ , by forward substitution (finding first  $y_1$ , then  $y_2$ , etc.). Do this using paper and pencil. Show all your work.
- b. Solve the system  $U\vec{x} = \vec{y}$ , using back substitution, to find the solution  $\vec{x}$  of the system  $A\vec{x} = \vec{b}$ . Do this using paper and pencil. Show all your work.



92. Show that the matrix  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  cannot be written in the form  $A = LU$ , where  $L$  is lower triangular and  $U$  is upper triangular.
93. In this exercise we will examine which invertible  $n \times n$  matrices  $A$  admit an  $LU$ -factorization  $A = LU$ , as discussed in Exercise 90. The following definition will be useful: For  $m = 1, \dots, n$ , the *principal submatrix*  $A^{(m)}$  of  $A$  is obtained by omitting all rows and columns of  $A$  past the  $m$ th. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \end{bmatrix}$$

has the principal submatrices

$$A^{(1)} = [1], A^{(2)} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, A^{(3)} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \end{bmatrix}.$$

We will show that an invertible  $n \times n$  matrix  $A$  admits an  $LU$ -factorization  $A = LU$  if (and only if) all its principal submatrices are invertible.

- a. Let  $A = LU$  be an  $LU$ -factorization of an  $n \times n$  matrix  $A$ . Use block matrices to show that  $A^{(m)} = L^{(m)}U^{(m)}$  for  $m = 1, \dots, n$ .
- b. Use part (a) to show that if an invertible  $n \times n$  matrix  $A$  has an  $LU$ -factorization, then all its principal submatrices  $A^{(m)}$  are invertible.
- c. Consider an  $n \times n$  matrix  $A$  whose principal submatrices are all invertible. Show that  $A$  admits an  $LU$ -factorization. Hint: By induction, you can assume that  $A^{(n-1)}$  has an  $LU$ -factorization  $A^{(n-1)} = L'U'$ . Use block matrices to find an  $LU$ -factorization for  $A$ . Alternatively, you can explain this result in terms of Gauss–Jordan elimination (if

all principal submatrices are invertible, then no row swaps are required).

94. a. Show that if an invertible  $n \times n$  matrix  $A$  admits an  $LU$ -factorization, then it admits an  $LDU$ -factorization. See Exercise 90d.

- b. Show that if an invertible  $n \times n$  matrix  $A$  admits an  $LDU$ -factorization, then this factorization is unique. Hint: Suppose that  $A = L_1D_1U_1 = L_2D_2U_2$ . Then  $U_2U_1^{-1} = D_2^{-1}L_2^{-1}L_1D_1$  is diagonal (why?). Conclude that  $U_2 = U_1$ .

95. Consider a block matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

where  $A_{11}$  and  $A_{22}$  are square matrices. For which choices of  $A_{11}$  and  $A_{22}$  is  $A$  invertible? In these cases, what is  $A^{-1}$ ?

96. Consider a block matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where  $A_{11}$  and  $A_{22}$  are square matrices. For which choices of  $A_{11}$ ,  $A_{21}$ , and  $A_{22}$  is  $A$  invertible? In these cases, what is  $A^{-1}$ ?

97. Consider the block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & 0 & A_{23} \end{bmatrix},$$

where  $A_{11}$  is an invertible matrix. Determine the rank of  $A$  in terms of the ranks of the blocks  $A_{11}$ ,  $A_{12}$ ,  $A_{13}$ , and  $A_{23}$ .

98. Consider the block matrix

$$A = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix},$$

where  $\vec{v}$  is a vector in  $\mathbb{R}^n$ , and  $\vec{w}$  is a row vector with  $n$  components. For which choices of  $\vec{v}$  and  $\vec{w}$  is  $A$  invertible? In these cases, what is  $A^{-1}$ ?

99. Find all invertible  $n \times n$  matrices  $A$  such that  $A^2 = A$ .

100. Find a nonzero  $n \times n$  matrix  $A$  with identical entries such that  $A^2 = A$ .

101. Consider two  $n \times n$  matrices  $A$  and  $B$  whose entries are positive or zero. Suppose that all entries of  $A$  are less than or equal to  $s$ , and all column sums of  $B$  are less than or equal to  $r$  (the  $j$ th column sum of a matrix is the sum of all the entries in its  $j$ th column). Show that all entries of the matrix  $AB$  are less than or equal to  $sr$ .

102. (This exercise builds on Exercise 101.) Consider an  $n \times n$  matrix  $A$  whose entries are positive or zero. Suppose that all column sums of  $A$  are less than 1. Let  $r$  be the largest column sum of  $A$ .

- a. Show that the entries of  $A^m$  are less than or equal to  $r^m$ , for all positive integers  $m$ .

## Linear Transformations

- b.** Show that

$$\lim_{m \rightarrow \infty} A^m = 0$$

(meaning that all entries of  $A^m$  approach zero).

- c.** Show that the infinite series

$$I_n + A + A^2 + \cdots + A^m + \cdots$$

converges (entry by entry).

- d.** Compute the product

$$(I_n - A)(I_n + A + A^2 + \cdots + A^m).$$

Simplify the result. Then let  $m$  go to infinity, and thus show that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots$$

- 103.** (This exercise builds on Exercises 49, 101, and 102.)

- a.** Consider the industries  $J_1, \dots, J_n$  in an economy. We say that industry  $J_j$  is *productive* if the  $j$ th column sum of the technology matrix  $A$  is less than 1. What does this mean in terms of economics?
- b.** We say that an economy is productive if all of its industries are productive. Exercise 102 shows that if  $A$  is the technology matrix of a productive economy, then the matrix  $I_n - A$  is invertible. What does this result tell you about the ability of a productive economy to satisfy consumer demand?
- c.** Interpret the formula

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots$$

derived in Exercise 102d in terms of economics.

- 104.** The color of light can be represented in a vector

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix},$$

where  $R$  = amount of red,  $G$  = amount of green, and  $B$  = amount of blue. The human eye and the brain transform the incoming signal into the signal

$$\begin{bmatrix} I \\ L \\ S \end{bmatrix},$$

where

$$\text{intensity } I = \frac{R + G + B}{3}$$

$$\text{long-wave signal } L = R - G$$

$$\text{short-wave signal } S = B - \frac{R + G}{2}.$$

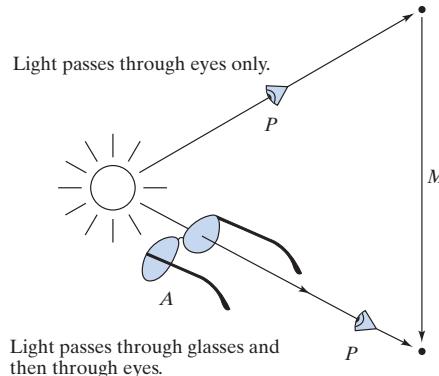
- a.** Find the matrix  $P$  representing the transformation from

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} I \\ L \\ S \end{bmatrix}.$$

- b.** Consider a pair of yellow sunglasses for water sports that cuts out all blue light and passes all red and green light. Find the  $3 \times 3$  matrix  $A$  that represents the transformation incoming light undergoes as it passes through the sunglasses. All the entries of your matrix  $A$  will be 0's and 1's.

- c.** Find the matrix for the composite transformation that light undergoes as it first passes through the sunglasses and then the eye.

- d.** As you put on the sunglasses, the signal you receive (intensity, long- and short-wave signals) undergoes a transformation. Find the matrix  $M$  of this transformation. Feel free to use technology.



- 105.** A village is divided into three mutually exclusive groups called *clans*. Each person in the village belongs to a clan, and this identification is permanent. There are rigid rules concerning marriage: A person from one clan can only marry a person from one other clan. These rules are encoded in the matrix  $A$  below. The fact that the 2–3 entry is 1 indicates that marriage between a man from clan III and a woman from clan II is allowed. The clan of a child is determined by the mother's clan, as indicated by the matrix  $B$ . According to this scheme, siblings belong to the same clan.

	Husband's clan		
	I	II	III
$A =$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	I	Wife's clan
	II	III	

	Mother's clan		
	I	II	III
$B =$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	I	Child's clan
	II	III	

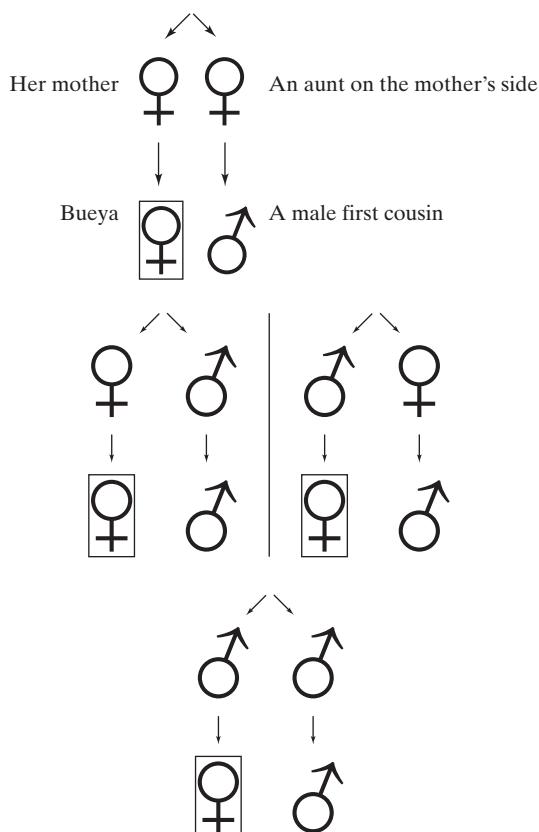
The identification of a person with clan I can be represented by the vector

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

## Linear Transformations

and likewise for the two other clans. Matrix  $A$  transforms the husband's clan into the wife's clan (if  $\vec{x}$  represents the husband's clan, then  $A\vec{x}$  represents the wife's clan).

- Are the matrices  $A$  and  $B$  invertible? Find the inverses if they exist. What do your answers mean, in practical terms?
- What is the meaning of  $B^2$ , in terms of the rules of the community?
- What is the meaning of  $AB$  and  $BA$ , in terms of the rules of the community? Are  $AB$  and  $BA$  the same?
- Bueya is a young woman who has many male first cousins, both on her mother's and on her father's sides. The kinship between Bueya and each of her male cousins can be represented by one of the four diagrams below:



In each of the four cases, find the matrix that gives you the cousin's clan in terms of Bueya's clan.

- According to the rules of the village, could Bueya marry a first cousin? (We do not know Bueya's clan.)
- 106.** As background to this exercise, see Exercise 45.
- If you use Theorem 3.4, how many multiplications of scalars are necessary to multiply two  $2 \times 2$  matrices?
  - If you use Theorem 3.4, how many multiplications are needed to multiply an  $n \times p$  and a  $p \times m$  matrix?

In 1969, the German mathematician Volker Strassen surprised the mathematical community by showing that two  $2 \times 2$  matrices can be multiplied with only seven multiplications of numbers. Here is his trick: Suppose you have to find  $AB$  for  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ . First compute

$$\begin{aligned} h_1 &= (a+d)(p+s) \\ h_2 &= (c+d)p \\ h_3 &= a(q-s) \\ h_4 &= d(r-p) \\ h_5 &= (a+b)s \\ h_6 &= (c-a)(p+q) \\ h_7 &= (b-d)(r+s). \end{aligned}$$

Then

$$AB = \begin{bmatrix} h_1 + h_4 - h_5 + h_7 & h_3 + h_5 \\ h_2 + h_4 & h_1 + h_3 - h_2 + h_6 \end{bmatrix}.$$

- 107.** Let  $\mathbb{N}$  be the set of all positive integers,  $1, 2, 3, \dots$ . We define two functions  $f$  and  $g$  from  $\mathbb{N}$  to  $\mathbb{N}$ :

$$f(x) = 2x, \quad \text{for all } x \text{ in } \mathbb{N}$$

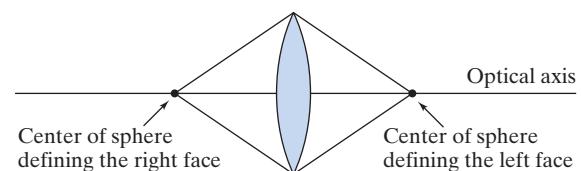
$$g(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

Find formulas for the composite functions  $g(f(x))$  and  $f(g(x))$ . Is one of them the identity transformation from  $\mathbb{N}$  to  $\mathbb{N}$ ? Are the functions  $f$  and  $g$  invertible?

- 108.** *Geometrical optics.* Consider a thin biconvex lens with two spherical faces.



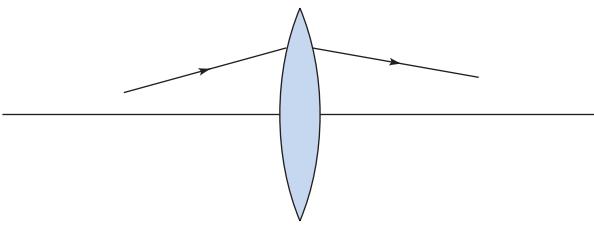
This is a good model for the lens of the human eye and for the lenses used in many optical instruments, such as reading glasses, cameras, microscopes, and telescopes. The line through the centers of the spheres defining the two faces is called the *optical axis* of the lens.



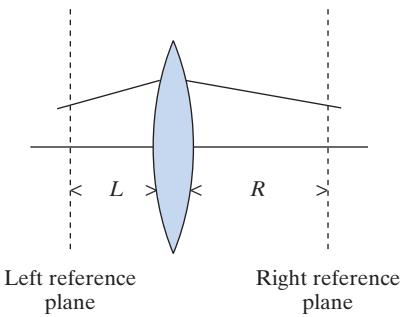
In this exercise, we learn how we can track the path of a ray of light as it passes through the lens, provided that the following conditions are satisfied:

- The ray lies in a plane with the optical axis.
- The angle the ray makes with the optical axis is small.

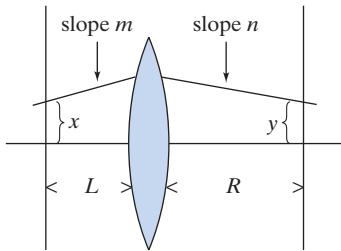
## Linear Transformations



To keep track of the ray, we introduce two *reference planes* perpendicular to the optical axis, to the left and to the right of the lens.



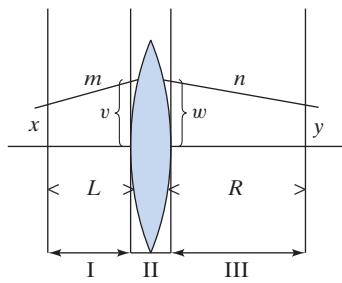
We can characterize the incoming ray by its slope  $m$  and its intercept  $x$  with the left reference plane. Likewise, we characterize the outgoing ray by slope  $n$  and intercept  $y$ .



We want to know how the outgoing ray depends on the incoming ray; that is, we are interested in the transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad \begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix}.$$

We will see that  $T$  can be approximated by a linear transformation provided that  $m$  is small, as we assumed. To study this transformation, we divide the path of the ray into three segments, as shown in the following figure:



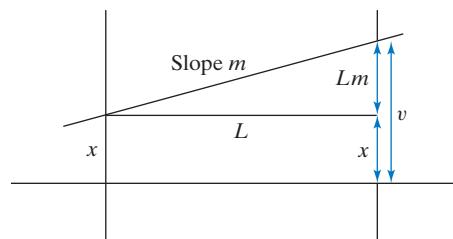
We have introduced two auxiliary reference planes, directly to the left and to the right of the lens. Our transformation

$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix}$$

can now be represented as the composite of three simpler transformations:

$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} v \\ m \end{bmatrix} \rightarrow \begin{bmatrix} w \\ n \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix}.$$

From the definition of the slope of a line, we get the relations  $v = x + Lm$  and  $y = w + Rn$ .



$$\begin{bmatrix} v \\ m \end{bmatrix} = \begin{bmatrix} x + Lm \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$$

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ n \end{bmatrix}$$

$$\begin{bmatrix} x \\ m \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} v \\ m \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} w \\ n \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} y \\ n \end{bmatrix}$$

It would lead us too far into physics to derive a formula for the transformation

$$\begin{bmatrix} v \\ m \end{bmatrix} \rightarrow \begin{bmatrix} w \\ n \end{bmatrix}$$

here.<sup>13</sup> Under the assumptions we have made, the transformation is well approximated by

$$\begin{bmatrix} w \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} v \\ m \end{bmatrix},$$

for some positive constant  $k$  (this formula implies that  $w = v$ ).

$$\begin{bmatrix} x \\ m \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} v \\ m \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}} \begin{bmatrix} w \\ n \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} y \\ n \end{bmatrix}$$

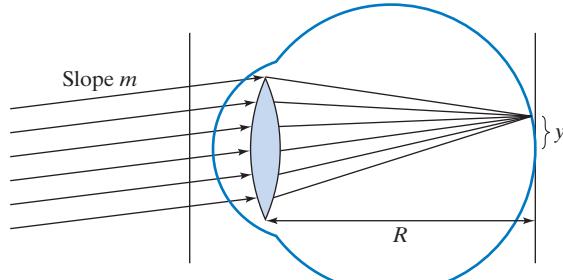
<sup>13</sup> See, for example, Paul Bamberg and Shlomo Sternberg, *A Course in Mathematics for Students of Physics 1*, Cambridge University Press, 1991.

## Linear Transformations

The transformation  $\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix}$  is represented by the matrix product

$$\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - Rk & L + R - kLR \\ -k & 1 - kL \end{bmatrix}.$$

- a. *Focusing parallel rays.* Consider the lens in the human eye, with the retina as the right reference plane. In an adult, the distance  $R$  is about 0.025 meters (about 1 inch). The ciliary muscles allow you to vary the shape of the lens and thus the lens constant  $k$ , within a certain range. What value of  $k$  enables you to focus parallel incoming rays, as shown in the figure? This value of  $k$  will allow you to see a distant object clearly. (The customary unit of measurement for  $k$  is 1 diopter =  $\frac{1}{1}$  meter.)

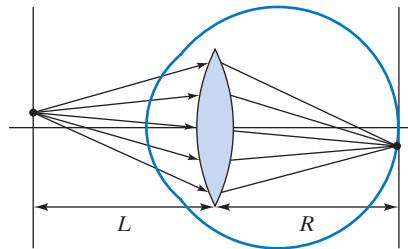


*Hint:* In terms of the transformation

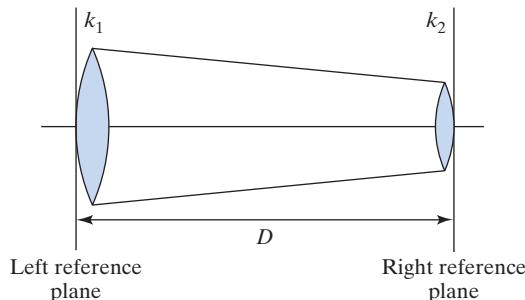
$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix},$$

you want  $y$  to be independent of  $x$  ( $y$  must depend on the slope  $m$  alone). Explain why  $1/k$  is called the *focal length* of the lens.

- b. What value of  $k$  enables you to read this text from a distance of  $L = 0.3$  meters? Consider the following figure (which is not to scale).



- c. *The telescope.* An astronomical telescope consists of two lenses with the same optical axis.



Find the matrix of the transformation

$$\begin{bmatrix} x \\ m \end{bmatrix} \rightarrow \begin{bmatrix} y \\ n \end{bmatrix},$$

in terms of  $k_1$ ,  $k_2$ , and  $D$ . For given values of  $k_1$  and  $k_2$ , how do you choose  $D$  so that parallel incoming rays are converted into parallel outgoing rays? What is the relationship between  $D$  and the focal lengths of the two lenses,  $1/k_1$  and  $1/k_2$ ?

## Exercises

### TRUE OR FALSE?

1. The matrix  $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$  represents a rotation combined with a scaling.
2. If  $A$  is any invertible  $n \times n$  matrix, then  $A$  commutes with  $A^{-1}$ .
3. The function  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$  is a linear transformation.
4. Matrix  $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$  represents a rotation.
5. If  $A$  is any invertible  $n \times n$  matrix, then  $\text{rref}(A) = I_n$ .
6. The formula  $(A^2)^{-1} = (A^{-1})^2$  holds for all invertible matrices  $A$ .
7. The formula  $AB = BA$  holds for all  $n \times n$  matrices  $A$  and  $B$ .
8. If  $AB = I_n$  for two  $n \times n$  matrices  $A$  and  $B$ , then  $A$  must be the inverse of  $B$ .
9. If  $A$  is a  $3 \times 4$  matrix and  $B$  is a  $4 \times 5$  matrix, then  $AB$  will be a  $5 \times 3$  matrix.
10. The function  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$  is a linear transformation.

## Linear Transformations

11. Matrix  $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$  is invertible for all real numbers  $k$ .
12. There exists a real number  $k$  such that the matrix  $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$  fails to be invertible.
13. There exists a real number  $k$  such that the matrix  $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$  fails to be invertible.
14.  $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$  is a regular transition matrix.
15. The formula  $\det(2A) = 2\det(A)$  holds for all  $2 \times 2$  matrices  $A$ .
16. There exists a matrix  $A$  such that  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .
17. Matrix  $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  is invertible.
18. Matrix  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  is invertible.
19. There exists an upper triangular  $2 \times 2$  matrix  $A$  such that  $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .
20. The function  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$  is a linear transformation.
21. There exists an invertible  $n \times n$  matrix with two identical rows.
22. If  $A^2 = I_n$ , then matrix  $A$  must be invertible.
23. There exists a matrix  $A$  such that  $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ .
24. There exists a matrix  $A$  such that  $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .
25. The matrix  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  represents a reflection about a line.
26. For every regular transition matrix  $A$  there exists a transition matrix  $B$  such that  $AB = B$ .
27. The matrix product  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  is always a scalar multiple of  $I_2$ .
28. There exists a nonzero upper triangular  $2 \times 2$  matrix  $A$  such that  $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .
29. There exists a positive integer  $n$  such that  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$ .
30. There exists an invertible  $2 \times 2$  matrix  $A$  such that  $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .
31. There exists a regular transition matrix  $A$  of size  $3 \times 3$  such that  $A^2 = A$ .
32. If  $A$  is any transition matrix and  $B$  is any positive transition matrix, then  $AB$  must be a positive transition matrix.
33. If matrix  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  is invertible, then matrix  $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$  must be invertible as well.
34. If  $A^2$  is invertible, then matrix  $A$  itself must be invertible.
35. If  $A^{17} = I_2$ , then matrix  $A$  must be  $I_2$ .
36. If  $A^2 = I_2$ , then matrix  $A$  must be either  $I_2$  or  $-I_2$ .
37. If matrix  $A$  is invertible, then matrix  $5A$  must be invertible as well.
38. If  $A$  and  $B$  are two  $4 \times 3$  matrices such that  $A\vec{v} = B\vec{v}$  for all vectors  $\vec{v}$  in  $\mathbb{R}^3$ , then matrices  $A$  and  $B$  must be equal.
39. If matrices  $A$  and  $B$  commute, then the formula  $A^2B = BA^2$  must hold.
40. If  $A^2 = A$  for an invertible  $n \times n$  matrix  $A$ , then  $A$  must be  $I_n$ .
41. If  $A$  is any transition matrix such that  $A^{100}$  is positive, then  $A^{101}$  must be positive as well.
42. If a transition matrix  $A$  is invertible, then  $A^{-1}$  must be a transition matrix as well.
43. If matrices  $A$  and  $B$  are both invertible, then matrix  $A + B$  must be invertible as well.
44. The equation  $A^2 = A$  holds for all  $2 \times 2$  matrices  $A$  representing a projection.
45. The equation  $A^{-1} = A$  holds for all  $2 \times 2$  matrices  $A$  representing a reflection.
46. The formula  $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$  holds for all invertible  $2 \times 2$  matrices  $A$  and for all vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^2$ .
47. There exist a  $2 \times 3$  matrix  $A$  and a  $3 \times 2$  matrix  $B$  such that  $AB = I_2$ .
48. There exist a  $3 \times 2$  matrix  $A$  and a  $2 \times 3$  matrix  $B$  such that  $AB = I_3$ .
49. If  $A^2 + 3A + 4I_3 = 0$  for a  $3 \times 3$  matrix  $A$ , then  $A$  must be invertible.
50. If  $A$  is an  $n \times n$  matrix such that  $A^2 = 0$ , then matrix  $I_n + A$  must be invertible.

51. If matrix  $A$  commutes with  $B$ , and  $B$  commutes with  $C$ , then matrix  $A$  must commute with  $C$ .

52. If  $T$  is any linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , then  $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$  for all vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$ .

53. There exists an invertible  $10 \times 10$  matrix that has 92 ones among its entries.

54. The formula  $\text{rref}(AB) = \text{rref}(A) \text{ rref}(B)$  holds for all  $n \times p$  matrices  $A$  and for all  $p \times m$  matrices  $B$ .

55. There exists an invertible matrix  $S$  such that  $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$  is a diagonal matrix.

56. If the linear system  $A^2\vec{x} = \vec{b}$  is consistent, then the system  $A\vec{x} = \vec{b}$  must be consistent as well.

57. There exists an invertible  $2 \times 2$  matrix  $A$  such that  $A^{-1} = -A$ .

58. There exists an invertible  $2 \times 2$  matrix  $A$  such that  $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

59. If a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  represents the orthogonal projection onto a line  $L$ , then the equation  $a^2 + b^2 + c^2 + d^2 = 1$  must hold.

60. If  $A$  is an invertible  $2 \times 2$  matrix and  $B$  is any  $2 \times 2$  matrix, then the formula  $\text{rref}(AB) = \text{rref}(B)$  must hold.

61. There is a transition matrix  $A$  such that  $\lim_{m \rightarrow \infty} A^m$  fails to exist.

62. For every transition matrix  $A$  there exists a nonzero vector  $\vec{x}$  such that  $A\vec{x} = \vec{x}$ .

## Answers to Odd-Numbered Exercises

**1** Answers to more theoretical questions are omitted.

# Linear Transformations

59. a.  $\begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}$

b. Yes.  $\begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}$ ,  $F = \frac{9}{5}C + 32$

61.  $A = \begin{bmatrix} 1 & 4/5 & 1/8 & 5/4 \\ 5/4 & 1 & 5/32 & 25/16 \\ 8 & 32/5 & 1 & 10 \\ 4/5 & 16/25 & 1/10 & 1 \end{bmatrix}$

63.  $B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$

Answers to more theoretical questions are omitted.

1. The image consists of the vectors  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

3. The parallelogram in  $\mathbb{R}^3$  defined by the vectors  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$

5. About 2.5 radians

7.  $\frac{1}{9} \begin{bmatrix} 11 \\ 1 \\ 11 \end{bmatrix}$

9. A vertical shear

11.  $\frac{1}{25} \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$

13.  $\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

15.  $a_{ii} = 2u_i^2 - 1$ , and  $a_{ij} = 2u_i u_j$  when  $i \neq j$

17. If  $b \neq 0$ , then let  $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -b \\ 1+a \end{bmatrix}$ , for example.

19.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

21.  $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

23.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

25.  $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ ; you shear back.

27.  $A$  is a reflection;  $B$  is a scaling;  $C$  is a projection;  $D$  is a rotation;  $E$  is a shear.

31.  $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

33. Use a parallelogram. 35. Yes

37. a. 1 b. 0  
c.  $-2 \leq \text{tr } A \leq 2$  d. 2

39. a. Projection with scaling  
b. Shear with scaling  
c. Reflection with scaling

41.  $\text{ref}_Q \vec{x} = -\text{ref}_P \vec{x}$

43.  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ , a clockwise rotation through the angle  $\theta$

45.  $A^{-1} = A$ . The inverse of a reflection is the reflection itself.

47. Write  $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$ . Express  $f(t)$  in terms of  $a, b, c, d$ .

49.  $c = 0$  (or  $c = \frac{\pi}{2}$ ),  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

51.  $c = \frac{\pi}{4}$ ,  $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and  $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

53. The image is an ellipse with semimajor axes  $\pm 5\vec{e}_1$  and semiminor axes  $\pm 2\vec{e}_2$ .

55. The curve  $C$  is the image of the unit circle under the transformation with matrix  $[\vec{w}_1 \quad \vec{w}_2]$ .

3 Answers to more theoretical questions are omitted.

1.  $\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$  3. Undefined

5.  $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$  7.  $\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

9.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  11. [10] 13. [h]

15.  $\begin{array}{|c|c|} \hline 1 & 0 \\ 2 & 0 \\ \hline 19 & 16 \\ \hline \end{array}$

17. All matrices of the form  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

19. All matrices of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

21. All matrices of the form  $\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$

23. All matrices of the form  $\begin{bmatrix} a & b \\ \frac{2}{3}b & \frac{5}{3}b+a \end{bmatrix}$

25. All matrices of the form  $\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$

29. a. The matrices  $D_\alpha D_\beta$  and  $D_\beta D_\alpha$  both represent the counterclockwise rotation through the angle  $\alpha + \beta$ .  
b.  $D_\alpha D_\beta = D_\beta D_\alpha$   
 $= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$

31.  $(i\text{th row of } AB) = (i\text{th row of } A)B$

33.  $A^n = I_2$  for even  $n$  and  $A^n = -I_2$  for odd  $n$ .  $A$  represents the rotation through  $\pi$ .

## Linear Transformations

35.  $A^n = I_2$  for even  $n$  and  $A^n = A$  for odd  $n$ . A represents the reflection about the line spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .
37.  $A^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$ .  $A$  represents a vertical shear.
39.  $A^n$  represents the rotation through  $n\pi/4$  in the clockwise direction, so that  $A^8 = I_2$ . Now  $A^{1001} = (A^8)^{125}A = A$ .
41.  $A^n = I_2$  for even  $n$  and  $A^n = A$  for odd  $n$ .  $A$  represents the reflection about a line.
43.  $A = -I_2$ , for example, or any matrix representing a reflection about a line
45. The matrix representing the rotation through  $2\pi/3$ , for example,  $A = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$
47. Any projection or reflection matrix with nonzero entries will do, for example,  $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ .
49.  $AF = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $FA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . We compose a rotation with a reflection to obtain a reflection.
51.  $FJ = JF = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ . Note that  $F$  represents the rotation through  $\pi/2$  while  $J$  represents the rotation through  $\pi/4$  combined with a scaling by  $\sqrt{2}$ . The products  $FJ$  and  $JF$  both represent the rotation through  $3\pi/4$  combined with a scaling by  $\sqrt{2}$ .
53.  $CD = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $DC = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . We compose two reflections to obtain a rotation.
55.  $X = \begin{bmatrix} -2s & -2t \\ s & t \end{bmatrix}$ , where  $s$  and  $t$  are arbitrary constants
57.  $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$
59. No such matrix  $X$  exists.
61.  $X = \begin{bmatrix} 1+s & -2+t \\ -2s & 1-2t \\ s & t \end{bmatrix}$ , where  $s$  and  $t$  are arbitrary constants
63. No such matrix  $X$  exists.
65.  $X = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ , where  $t$  is an arbitrary constant
- 67a.  $[1 \ 1 \ 1]A = [1 \ 1 \ 1]$
- 69a. Of the surfers who are on Page 1 initially, 25% will be on Page 3 after following two consecutive links.
- 69b. The  $ij$ th entry of  $A^2$  is 0 if there is no path of length 2 from vertex  $j$  to vertex  $i$  in the graph of the mini-Web, meaning that a surfer cannot get from Page  $j$  to Page  $i$  by following two consecutive links.
71. A straightforward computation shows that the only nonzero entry of  $A^4$  is the first component of the third column. Thus there is no path of length 4 from Page 3 to Page 1 in the graph of the mini-Web, while there is a path of length 4 linking any other two pages. We cannot get from Page 3 to Page 1 by following four consecutive links.
73.  $\lim_{m \rightarrow \infty} (A^m \vec{x}) = \left( \lim_{m \rightarrow \infty} A^m \right) \vec{x} = \begin{bmatrix} | & | \\ \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | \end{bmatrix} \vec{x} = \underbrace{(x_1 + \cdots + x_n)}_{\vec{x}_{equ}} \vec{x}_{equ} = \text{components of } \vec{x}$
75. The  $ij$ th entry of  $A^{m+1} = A^m A$  is the dot product of the  $i$ th row  $\vec{w}$  or  $A^m$  with the  $j$ th column  $\vec{v}$  of  $A$ . Since all components of  $\vec{w}$  and some components of  $\vec{v}$  are positive (with all components of  $\vec{v}$  being non-negative), this dot product is positive, as claimed.
77. We find  $A^{10} \approx \begin{bmatrix} 0.5003 & 0.4985 & 0.5000 \\ 0.0996 & 0.1026 & 0.0999 \\ 0.4002 & 0.3989 & 0.4001 \end{bmatrix}$ . This shows that  $A$  is regular and it suggests then  $\vec{x}_{equ} = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix}$ . To be sure, we verify that  $A \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix}$ .
79. An extreme example is the identity matrix  $I_n$ . In this case,  $I_n \vec{x} = \vec{x}$  for all distribution vectors.
81.  $A^m \vec{v} = 5^m \vec{v}$
83. By Definition 3.10, there exists a positive integer  $m$  such that  $A^m$  is a positive transition matrix. Note that  $A^m \vec{x} = \vec{x}$ . Thus, the  $i$ th component of  $\vec{x}$  is the dot product of the  $i$ th row  $\vec{w}$  of  $A^m$  with  $\vec{x}$ . Since all components of  $\vec{w}$  and some components of  $\vec{x}$  are positive (with all components of  $\vec{x}$  being nonnegative), this dot product is positive, as claimed.
85. There is one and only one such  $X$ .
- 4** Answers to more theoretical questions are omitted.
1.  $\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$
3.  $\begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$
5. Not invertible
7. Not invertible
9. Not invertible
11.  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

## Linear Transformations

13. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

15. 
$$\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}$$

17. Not invertible

19.  $x_1 = 3y_1 - 2.5y_2 + 0.5y_3$   
 $x_2 = -3y_1 + 4y_2 - y_3$   
 $x_3 = y_1 - 1.5y_2 + 0.5y_3$

21. Not invertible

23. Invertible

25. Invertible

27. Not invertible

 29. For all  $k$  except  $k = 1$  and  $k = 2$ 

31. It's never invertible.

 33. If  $a^2 + b^2 = 1$ 

35. a. Invertible if  $a, d, f$  are all nonzero  
 b. Invertible if all diagonal entries are nonzero  
 c. Yes; use Theorem 4.5.  
 d. Invertible if all diagonal entries are nonzero

37.  $(cA)^{-1} = \frac{1}{c}A^{-1}$

 39.  $M$  is invertible; if  $m_{ij} = k$  (where  $i \neq j$ ), then the  $ij$ th entry of  $M^{-1}$  is  $-k$ ; all other entries are the same.

41. The transformations in parts a, c, and d are invertible, while the projection in part b is not.

 43. Yes;  $\vec{x} = B^{-1}(A^{-1}\vec{y})$ 

45. a.  $3^3 = 27$       b.  $n^3$

c.  $\frac{12^3}{3^3} = 64$  (seconds)

 47.  $f(x) = x^2$  is not invertible, but the equation  $f(x) = 0$  has the unique solution  $x = 0$ .

51. a. Since  $\text{rank } A < n$ , all the entries in the last row of  $E$  will be zero. We can let  $\vec{c} = \vec{e}_n$ ; the system  $E\vec{x} = \vec{e}_n$  will be inconsistent. Reversing the row reduction from  $A$  to  $E$ , we can transform the system  $E\vec{x} = \vec{e}_n$  into an inconsistent system  $A\vec{x} = \vec{b}$ .

 b. We have  $\text{rank } A \leq m < n$ . Now use part a.

53. a.  $\lambda_1 = 2, \lambda_2 = 6$

b. We chose  $\lambda = 2$ .  $A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

c. Check that  $\begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \vec{x} = 2\vec{x}$ .

55.  $\det A = \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\| \sin(\pi/2) \left\| \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\| = 4$ , and  $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ , a scaling by  $1/2$

57.  $\det A = \left\| \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} \right\| \sin(-\pi/2) \left\| \begin{bmatrix} \sin \alpha \\ -\cos \alpha \end{bmatrix} \right\| = -1$ , and  $A^{-1} = A$ , a reflection

59.  $\det A = \left\| \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \right\| \sin(\pi/2) \left\| \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix} \right\| = 1$ , and  $A^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$ , a rotation

61.  $\det A = \left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| \sin(\pi/2) \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = 2$ , and  $A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ , a rotation through  $\pi/4$  combined with a scaling by  $\sqrt{2}/2$

63.  $\det A = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\| \sin(-\pi/2) \left\| \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\| = -25$ , and  $A^{-1} = \frac{1}{25} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$ , a reflection combined with a scaling by  $1/5$

65.  $\det A = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| \sin(\pi/4) \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = 1$ , and  $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ , a vertical shear

67. False      69. False      71. True

73. True      75. True

77.  $A = BS^{-1}$       79.  $A = \frac{1}{5} \begin{bmatrix} 9 & 3 \\ -2 & 16 \end{bmatrix}$

81. 
$$\left\{ \begin{array}{l} \text{matrix of } T: \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ \text{matrix of } L: \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right.$$

83. Yes; yes; each elementary row operation can be "undone" by an elementary row operation.

 85. a. Use Exercise 84; let  $S = E_1 E_2 \cdots E_p$ .

b.  $S = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix}$

87.  $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , where  $k$  is nonzero and  $c$  is arbitrary. Cases 3 and 4 represent shears, and case 5 is a reflection.

89. Yes; use Theorem 3.4.

### Linear Transformations

91. a.  $\vec{y} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$    b.  $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

93. a. Write  $A = \begin{bmatrix} A^{(m)} & A_2 \\ A_3 & A_4 \end{bmatrix}$ ,  $L = \begin{bmatrix} L^{(m)} & 0 \\ L_3 & L_4 \end{bmatrix}$ ,  
 $U = \begin{bmatrix} U^{(m)} & U_2 \\ 0 & U_4 \end{bmatrix}$ . Use Theorem 3.9.

c. Solve the equation

$$A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{w} & k \end{bmatrix} = \begin{bmatrix} L' & 0 \\ \vec{x} & t \end{bmatrix} \begin{bmatrix} U' & \vec{y} \\ 0 & 1 \end{bmatrix}$$

for  $\vec{x}$ ,  $\vec{y}$ , and  $t$ .

95.  $A$  is invertible if both  $A_{11}$  and  $A_{22}$  are invertible. In this case,

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

97.  $\text{rank } A = \text{rank}(A_{11}) + \text{rank}(A_{23})$

99. Only  $A = I_n$

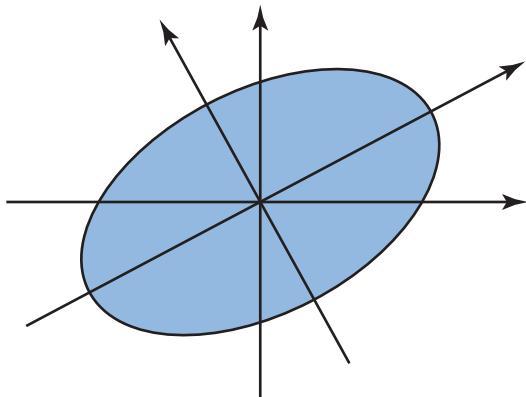
101.  $(ij\text{th entry of } AB) = \sum_{k=1}^n a_{ik}b_{kj} \leq s \sum_{k=1}^n b_{kj} \leq sr$

107.  $g(f(x)) = x$ , for all  $x$

$$f(g(x)) = \begin{cases} x & \text{if } x \text{ is even} \\ x+1 & \text{if } x \text{ is odd} \end{cases}$$

The functions  $f$  and  $g$  are not invertible.





# Subspaces of $\mathbb{R}^n$ and Their Dimensions

## I Image and Kernel of a Linear Transformation

You may be familiar with the notion of the *image*<sup>1</sup> of a function.

**Definition I.1**

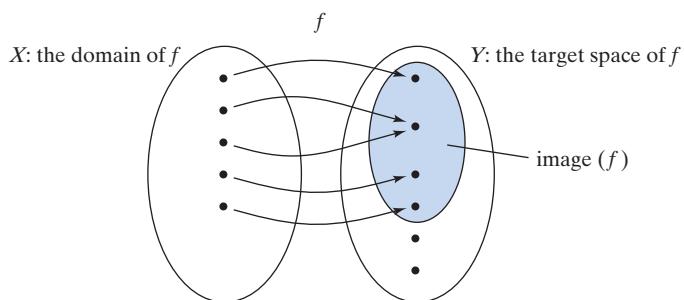
### Image of a function

The *image* of a function consists of all the values the function takes in its target space. If  $f$  is a function from  $X$  to  $Y$ , then

$$\begin{aligned}\text{image}(f) &= \{f(x): x \text{ in } X\} \\ &= \{b \text{ in } Y: b = f(x), \text{ for some } x \text{ in } X\}.\end{aligned}$$

### EXAMPLE I

A group  $X$  of students and a group  $Y$  of professors stand in the yard. Each student throws a snowball at one of the professors (and each snowball hits its intended target). Consider the function  $y = f(x)$  from  $X$  to  $Y$  that associates with each student  $x$  the target  $y$  of his or her snowball. The image of  $f$  consists of those professors that are hit. See Figure 1. ■



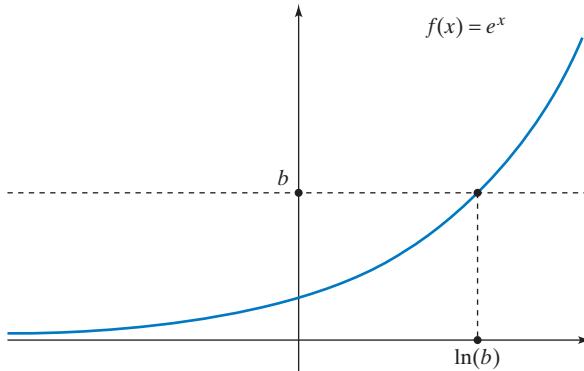
**Figure 1**

<sup>1</sup>Some authors use the term *range* for what we call the image, while others use the term *range* for what we call the target space. Because of this ambiguity, we will not use the term *range* at all.

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

#### EXAMPLE 2

The image of the exponential function  $f(x) = e^x$  from  $\mathbb{R}$  to  $\mathbb{R}$  consists of all positive real numbers. Indeed,  $f(x) = e^x$  is positive for all  $x$ , and every positive number  $b$  can be written as  $b = e^{\ln b} = f(\ln b)$ . See Figure 2. ■



**Figure 2**

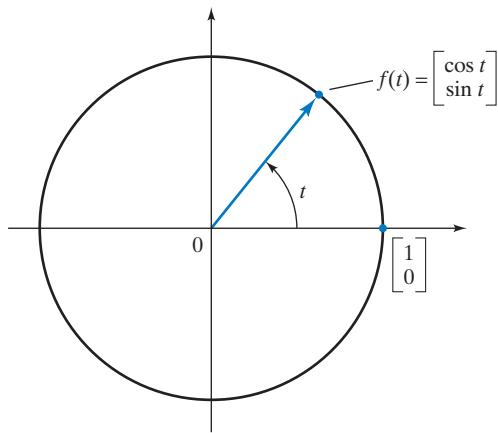
More generally, the image of a function  $f(x)$  from  $\mathbb{R}$  to  $\mathbb{R}$  consists of all real numbers  $b$  such that the line  $y = b$  intersects the graph of  $f$  (take another look at Figure 2). The image of  $f$  is the orthogonal projection of the graph of  $f$  onto the vertical axis.

#### EXAMPLE 3

The image of the function

$$f(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \quad \text{from } \mathbb{R} \text{ to } \mathbb{R}^2$$

is the unit circle centered at the origin. See Figure 3. Indeed,  $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$  is a unit vector for all  $t$ , since  $\cos^2 t + \sin^2 t = 1$ , and, conversely, any unit vector  $\vec{u}$  in  $\mathbb{R}^2$  can be written in polar coordinates as  $\vec{u} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ , where  $t$  is its polar angle. ■



**Figure 3**

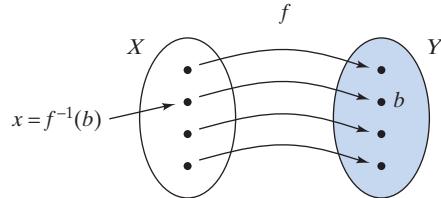
The function  $f$  in Example 3 is called a *parametrization* of the unit circle. More generally, a parametrization of a curve  $C$  in  $\mathbb{R}^2$  is a function  $g$  from  $\mathbb{R}$  to  $\mathbb{R}^2$  whose image is  $C$ .

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

**EXAMPLE 4** If the function  $f$  from  $X$  to  $Y$  is invertible, then the image of  $f$  is  $Y$ . Indeed, for every  $b$  in  $Y$ , there exists one (and only one)  $x$  in  $X$  such that  $b = f(x)$ , namely,  $x = f^{-1}(b)$ :

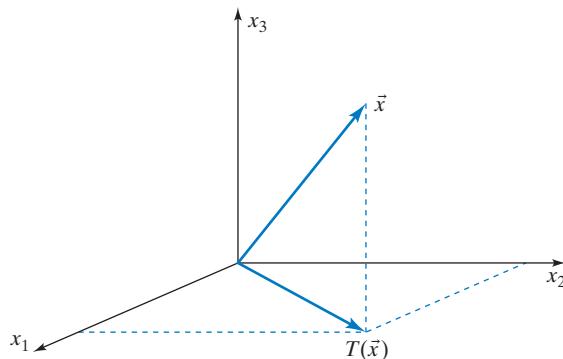
$$b = f(f^{-1}(b)).$$

See Figure 4. ■



**Figure 4**

**EXAMPLE 5** Consider the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  that projects a vector  $\vec{x}$  orthogonally into the  $x_1$ - $x_2$ -plane, meaning that  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ . See Figure 5.



**Figure 5**

The image of  $T$  is the  $x_1$ - $x_2$ -plane in  $\mathbb{R}^3$ , consisting of all vectors of the form  $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ . ■

**EXAMPLE 6** Describe the image of the linear transformation

$$T(\vec{x}) = A\vec{x} \quad \text{from } \mathbb{R}^2 \text{ to } \mathbb{R}^2, \text{ where } A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

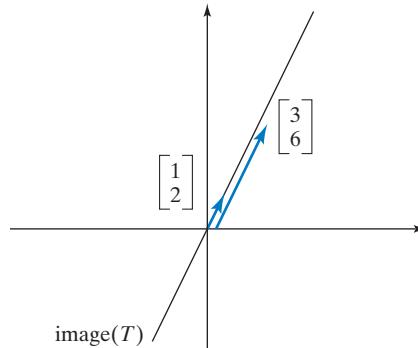
#### Solution

The image of  $T$  consists of all the values of  $T$ , that is, all vectors of the form

$$\begin{aligned} T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \end{aligned}$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

Since the vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$  are parallel, the image of  $T$  is the line of all scalar multiples of the vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , as illustrated in Figure 6. ■



**Figure 6**

**EXAMPLE 7** Describe the image of the linear transformation

$$T(\vec{x}) = A\vec{x} \quad \text{from } \mathbb{R}^2 \text{ to } \mathbb{R}^3, \text{ where } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

#### Solution

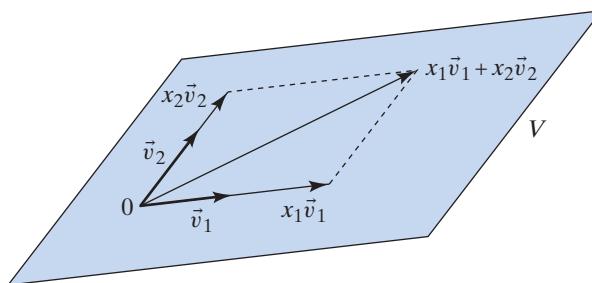
The image of  $T$  consists of all vectors of the form

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

that is, all linear combinations of the column vectors of  $A$ ,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The image of  $T$  is the plane  $V$  “spanned” by the two vectors  $\vec{v}_1$  and  $\vec{v}_2$ , that is, the plane through the origin and the endpoints  $(1, 1, 1)$  and  $(1, 2, 3)$  of  $\vec{v}_1$  and  $\vec{v}_2$ , respectively. See Figure 7. ■



**Figure 7**

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

The observations we made in Examples 6 and 7 motivate the following definition.

### Definition 1.2

**Span**  
Consider the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . The set of all linear combinations  $c_1\vec{v}_1 + \dots + c_m\vec{v}_m$  of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  is called their *span*:

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \{c_1\vec{v}_1 + \dots + c_m\vec{v}_m : c_1, \dots, c_m \text{ in } \mathbb{R}\}.$$

### Theorem 1.3

**Image of a linear transformation**  
The image of a linear transformation  $T(\vec{x}) = A\vec{x}$  is the span of the column vectors of  $A$ .<sup>2</sup> We denote the image of  $T$  by  $\text{im}(T)$  or  $\text{im}(A)$ .

To prove this theorem, we write the transformation  $T$  in vector form as in Examples 6 and 7:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

This shows that the image of  $T$  consists of all linear combinations of the column vectors  $\vec{v}_1, \dots, \vec{v}_m$  of matrix  $A$ . Thus,  $\text{im}(T)$  is the span of the vectors  $\vec{v}_1, \dots, \vec{v}_m$ .

The image of a linear transformation has some noteworthy properties.

### Theorem 1.4

#### Some properties of the image

The image of a linear transformation  $T$  (from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ) has the following properties:

- a. The zero vector  $\vec{0}$  in  $\mathbb{R}^n$  is in the image of  $T$ .
- b. The image of  $T$  is *closed under addition*: If  $\vec{v}_1$  and  $\vec{v}_2$  are in the image of  $T$ , then so is  $\vec{v}_1 + \vec{v}_2$ .
- c. The image of  $T$  is *closed under scalar multiplication*: If  $\vec{v}$  is in the image of  $T$  and  $k$  is an arbitrary scalar, then  $k\vec{v}$  is in the image of  $T$  as well.

#### Proof

a.  $\vec{0} = A\vec{0} = T(\vec{0})$ .

b. There exist vectors  $\vec{w}_1$  and  $\vec{w}_2$  in  $\mathbb{R}^m$  such that  $\vec{v}_1 = T(\vec{w}_1)$  and  $\vec{v}_2 = T(\vec{w}_2)$ . Then  $\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) = T(\vec{w}_1 + \vec{w}_2)$ , so that  $\vec{v}_1 + \vec{v}_2$  is in the image of  $T$  as well.

c. If  $\vec{v} = T(\vec{w})$ , then  $k\vec{v} = kT(\vec{w}) = T(k\vec{w})$ . ■

It follows from properties (b) and (c) that the image of  $T$  is *closed under linear combinations*: If some vectors  $\vec{v}_1, \dots, \vec{v}_p$  are in the image and  $c_1, \dots, c_p$  are arbitrary scalars, then  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p$  is in the image as well. In Figure 8 we illustrate this property in the case when  $p = 2$  and  $n = 3$  (that is, the target space of  $T$  is  $\mathbb{R}^3$ ).

### EXAMPLE 8

Consider an  $n \times n$  matrix  $A$ . Show that  $\text{im}(A^2)$  is a subset of  $\text{im}(A)$ ; that is, each vector in  $\text{im}(A^2)$  is also in  $\text{im}(A)$ .

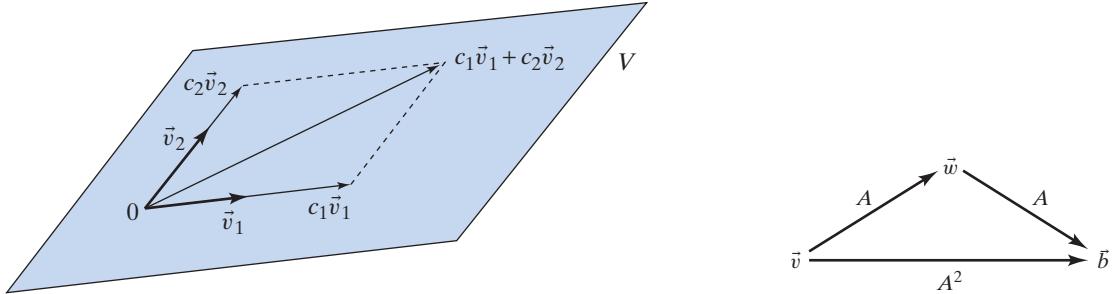
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<sup>2</sup>The image of  $T$  is also called the *column space* of  $A$ .

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

### Solution

Consider a vector  $\vec{b} = A^2\vec{v} = AA\vec{v}$  in the image of  $A^2$ . We can write  $\vec{b} = A(A\vec{v}) = A\vec{w}$ , where  $\vec{w} = A\vec{v}$ . The equation  $\vec{b} = A\vec{w}$  shows that  $\vec{b}$  is in the image of  $A$ . See Figure 9. ■



**Figure 8** If  $\vec{v}_1$  and  $\vec{v}_2$  are in the image, then so are all vectors in the plane  $V$  in  $\mathbb{R}^3$  spanned by  $\vec{v}_1$  and  $\vec{v}_2$ .

**Figure 9**

### The Kernel of a Linear Transformation

When you study functions  $y = f(x)$  of one variable, you are often interested in the *zeros* of  $f(x)$ , that is, the solutions of the equation  $f(x) = 0$ . For example, the function  $y = \sin(x)$  has infinitely many zeros, namely, all integer multiples of  $\pi$ .

The zeros of a linear transformation are of interest as well.

#### Definition 1.5

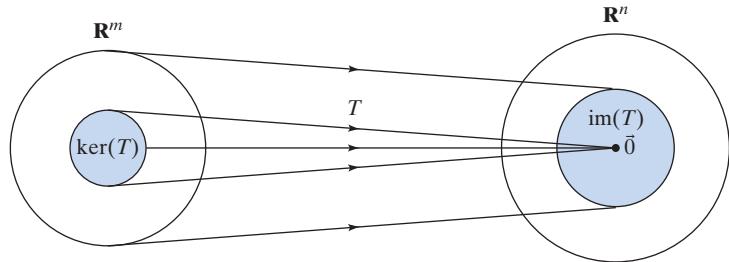
#### Kernel

The *kernel*<sup>3</sup> of a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  consists of all zeros of the transformation, that is, the solutions of the equation  $T(\vec{x}) = A\vec{x} = \vec{0}$ . See Figure 10, where we show the kernel along with the image.

In other words, the kernel of  $T$  is the solution set of the linear system

$$A\vec{x} = \vec{0}.$$

We denote the kernel of  $T$  by  $\ker(T)$  or  $\ker(A)$ .



**Figure 10**

For a linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ,

- $\text{im}(T) = \{T(\vec{x}) : \vec{x} \text{ in } \mathbb{R}^m\}$  is a subset of the *target space*  $\mathbb{R}^n$  of  $T$ , and
- $\ker(T) = \{\vec{x} \text{ in } \mathbb{R}^m : T(\vec{x}) = \vec{0}\}$  is a subset of the *domain*  $\mathbb{R}^m$  of  $T$ .

<sup>3</sup>The kernel of  $T$  is also called the *null space* of  $A$ .

**EXAMPLE 9**

Consider the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  that projects a vector orthogonally into the  $x_1$ - $x_2$ -plane. See Example 5 and Figure 5.

The kernel of  $T$  consists of the solutions of the equation  $T(\vec{x}) = \vec{0}$ , that is, the vectors whose orthogonal projection onto the  $x_1$ - $x_2$ -plane is zero. Those are the vectors on the  $x_3$ -axis, that is, the scalar multiples of  $\vec{e}_3$ . See Figure 11. ■

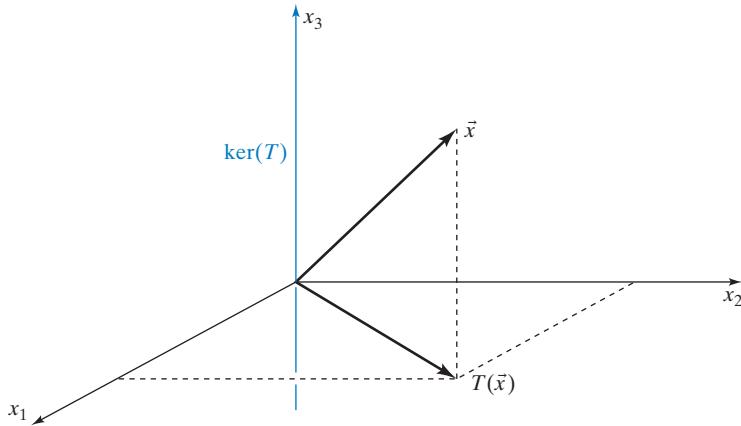


Figure 11

**EXAMPLE 10** Find the kernel of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} \quad \text{from } \mathbb{R}^3 \text{ to } \mathbb{R}^2.$$

**Solution**

We have to solve the linear system

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \vec{0}.$$

$$\text{rref} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left| \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array} \right|, \quad \text{or} \quad \left| \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \end{array} \right|$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{where } t \text{ is an arbitrary constant.}$$

The kernel of  $T$  is the line spanned by  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$  in  $\mathbb{R}^3$ . ■

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , where  $m$  exceeds  $n$  (as in Example 10, where  $m = 3$  and  $n = 2$ ). There will be free variables for the equation  $T(\vec{x}) = A\vec{x} = \vec{0}$ ; that is, this system has infinitely many solutions. Therefore, the kernel of  $T$  consists of infinitely many vectors. This agrees with our intuition: We expect some collapsing to take place as we transform the “large” vector space  $\mathbb{R}^m$  into the “smaller”  $\mathbb{R}^n$ . (Recall that the kernel consists of everything that “collapses to zero.”)

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

**EXAMPLE 11** Find the kernel of the linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^5$  to  $\mathbb{R}^4$ , where

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}.$$

### Solution

We have to solve the linear system

$$T(\vec{x}) = A\vec{x} = \vec{0}.$$

We leave it to the reader to verify that

$$\text{rref}[A \mid \vec{0}] = \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The kernel of  $T$  consists of the solutions of the system

$$\begin{vmatrix} x_1 + 2x_2 & + 3x_4 - 4x_5 = 0 \\ x_3 - 4x_4 + 5x_5 = 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} x_1 = -2x_2 - 3x_4 + 4x_5 \\ x_3 = 4x_4 - 5x_5 \end{vmatrix}.$$

The solutions are the vectors of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s & -3t & +4r \\ s & & \\ & 4t & -5r \\ & t & \\ & & r \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix},$$

where  $s$ ,  $t$ , and  $r$  are arbitrary constants. Using the concept of the *span* introduced in Definition 1.2, we can write

$$\ker(T) = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right). \quad \blacksquare$$

The kernel has some remarkable properties, analogous to the properties of the image listed in Theorem 1.4.

### Theorem 1.6

#### Some properties of the kernel

Consider a linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .

- a. The zero vector  $\vec{0}$  in  $\mathbb{R}^m$  is in the kernel of  $T$ .
- b. The kernel is closed under addition.
- c. The kernel is closed under scalar multiplication.

The verification of Theorem 1.6 is left as Exercise 49.

**EXAMPLE 12** For an invertible  $n \times n$  matrix  $A$ , find  $\ker(A)$ .

### Solution

The system

$$A\vec{x} = \vec{0}$$

has the sole solution  $\vec{x} = \vec{0}$ , so that  $\ker(A) = \{\vec{0}\}$ . ■

Conversely, if  $A$  is a noninvertible  $n \times n$  matrix, then  $\ker(A) \neq \{\vec{0}\}$ , meaning that the kernel consists of more than just the zero vector.

**EXAMPLE 13** For which  $n \times m$  matrices  $A$  is  $\ker(A) = \{\vec{0}\}$ ? Give your answer in terms of the rank of  $A$ .

### Solution

It is required that there be no free variables for the system  $A\vec{x} = \vec{0}$ , meaning that all  $m$  variables are leading variables. Thus, we want  $\text{rank}(A) = m$ , since the rank is the number of leading variables. ■

Let's summarize the results of Examples 12 and 13.

### Theorem 1.7

#### When is $\ker(A) = \{\vec{0}\}$ ?

- a. Consider an  $n \times m$  matrix  $A$ . Then  $\ker(A) = \{\vec{0}\}$  if (and only if)  $\text{rank}(A) = m$ .
- b. Consider an  $n \times m$  matrix  $A$ . If  $\ker(A) = \{\vec{0}\}$ , then  $m \leq n$ . Equivalently, if  $m > n$ , then there are nonzero vectors in the kernel of  $A$ .
- c. For a *square* matrix  $A$ , we have  $\ker(A) = \{\vec{0}\}$  if (and only if)  $A$  is invertible.

We conclude this section with a summary that relates many concepts we have introduced thus far.

### SUMMARY 1.8

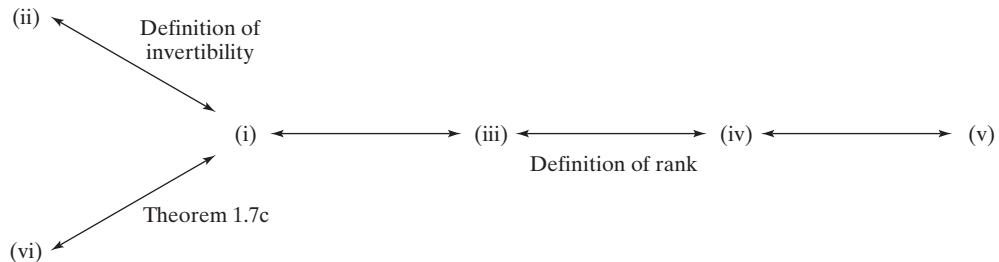
#### Various characterizations of invertible matrices

For an  $n \times n$  matrix  $A$ , the following statements are equivalent; that is, for a given  $A$ , they are either all true or all false.

- i.  $A$  is invertible.
- ii. The linear system  $A\vec{x} = \vec{b}$  has a unique solution  $\vec{x}$ , for all  $\vec{b}$  in  $\mathbb{R}^n$ .
- iii.  $\text{rref}(A) = I_n$ .
- iv.  $\text{rank}(A) = n$ .
- v.  $\text{im}(A) = \mathbb{R}^n$ .
- vi.  $\ker(A) = \{\vec{0}\}$ .

In Figure 12 we briefly recall the justification for these equivalences.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions



**Figure 12**

### EXERCISES I

**GOAL** Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.

For each matrix  $A$  in Exercises 1 through 13, find vectors that span the kernel of  $A$ . Use paper and pencil.

$$1. A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$2. A = [1 \ 2 \ 3]$$

$$3. A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$9. A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$$

$$11. A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

$$12. A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$13. A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For each matrix  $A$  in Exercises 14 through 16, find vectors that span the image of  $A$ . Give as few vectors as possible. Use paper and pencil.

$$14. A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$15. A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad 16. A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

For each matrix  $A$  in Exercises 17 through 22, describe the image of the transformation  $T(\vec{x}) = A\vec{x}$  geometrically (as a line, plane, etc. in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ).

$$17. A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad 18. A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$$

$$19. A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$$

$$20. A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix} \quad 21. A = \begin{bmatrix} 4 & 7 & 3 \\ 1 & 9 & 2 \\ 5 & 6 & 8 \end{bmatrix}$$

$$22. A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line  $y = x/3$  in  $\mathbb{R}^2$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

24. Orthogonal projection onto the plane  $x + 2y + 3z = 0$  in  $\mathbb{R}^3$

25. Rotation through an angle of  $\pi/4$  in the counterclockwise direction (in  $\mathbb{R}^2$ )

26. What is the image of a function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  given by

$$f(t) = t^3 + at^2 + bt + c,$$

where  $a, b, c$  are arbitrary scalars?

27. Give an example of a noninvertible function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  with  $\text{im}(f) = \mathbb{R}$ .

28. Give an example of a parametrization of the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

in  $\mathbb{R}^2$ . See Example 3.

29. Give an example of a function whose image is the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in  $\mathbb{R}^3$ .

30. Give an example of a matrix  $A$  such that  $\text{im}(A)$  is spanned by the vector  $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .

31. Give an example of a matrix  $A$  such that  $\text{im}(A)$  is the plane with normal vector  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$  in  $\mathbb{R}^3$ .

32. Give an example of a linear transformation whose image is the line spanned by

$$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

in  $\mathbb{R}^3$ .

33. Give an example of a linear transformation whose kernel is the plane  $x + 2y + 3z = 0$  in  $\mathbb{R}^3$ .

34. Give an example of a linear transformation whose kernel is the line spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in  $\mathbb{R}^3$ .

35. Consider a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$ . Arguing geometrically, describe the image and the kernel of the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}$  given by

$$T(\vec{x}) = \vec{v} \cdot \vec{x}.$$

36. Consider a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$ . Using a geometric argument, describe the kernel of the linear

transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  given by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

37. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

describe the images and kernels of the matrices  $A$ ,  $A^2$ , and  $A^3$  geometrically.

38. Consider a square matrix  $A$ .

- a. What is the relationship among  $\ker(A)$  and  $\ker(A^2)$ ? Are they necessarily equal? Is one of them necessarily contained in the other? More generally, what can you say about  $\ker(A)$ ,  $\ker(A^2)$ ,  $\ker(A^3)$ , ...?

- b. What can you say about  $\text{im}(A)$ ,  $\text{im}(A^2)$ ,  $\text{im}(A^3)$ , ...?

*Hint:* Exercise 37 is helpful.

39. Consider an  $n \times p$  matrix  $A$  and a  $p \times m$  matrix  $B$ .

- a. What is the relationship between  $\ker(AB)$  and  $\ker(B)$ ? Are they always equal? Is one of them always contained in the other?

- b. What is the relationship between  $\text{im}(A)$  and  $\text{im}(AB)$ ?

40. Consider an  $n \times p$  matrix  $A$  and a  $p \times m$  matrix  $B$ . If  $\ker(A) = \text{im}(B)$ , what can you say about the product  $AB$ ?

41. Consider the matrix  $A = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$ .

- a. Describe  $\ker(A)$  and  $\text{im}(A)$  geometrically.

- b. Find  $A^2$ . If  $\vec{v}$  is in the image of  $A$ , what can you say about  $A\vec{v}$ ?

- c. Describe the transformation  $T(\vec{x}) = A\vec{x}$  geometrically.

42. Express the image of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 2 \\ 1 & 4 & 7 & 0 \end{bmatrix}$$

as the kernel of a matrix  $B$ . *Hint:* The image of  $A$  consists of all vectors  $\vec{y}$  in  $\mathbb{R}^4$  such that the system  $A\vec{x} = \vec{y}$  is consistent. Write this system more explicitly:

$$\left| \begin{array}{l} x_1 + x_2 + x_3 + 6x_4 = y_1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = y_2 \\ x_1 + 3x_2 + 5x_3 + 2x_4 = y_3 \\ x_1 + 4x_2 + 7x_3 = y_4 \end{array} \right|.$$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

Now, reduce rows:

$$\left| \begin{array}{ccc|c} x_1 & - & x_3 + 8x_4 & = & 4y_3 - 3y_4 \\ x_2 & + & 2x_3 - 2x_4 & = & -y_3 + y_4 \\ 0 & & = y_1 & - 3y_3 + 2y_4 \\ 0 & & = y_2 - 2y_3 + y_4 \end{array} \right|.$$

For which vectors  $\vec{y}$  is this system consistent? The answer allows you to express  $\text{im}(A)$  as the kernel of a  $2 \times 4$  matrix  $B$ .

- 43. Using your work in Exercise 42 as a guide, explain how you can write the image of any matrix  $A$  as the kernel of some matrix  $B$ .
- 44. Consider a matrix  $A$ , and let  $B = \text{rref}(A)$ .
  - a. Is  $\ker(A)$  necessarily equal to  $\ker(B)$ ? Explain.
  - b. Is  $\text{im}(A)$  necessarily equal to  $\text{im}(B)$ ? Explain.
- 45. Consider an  $n \times m$  matrix  $A$  with  $\text{rank}(A) = r < m$ . Explain how you can write  $\ker(A)$  as the span of  $m - r$  vectors.
- 46. Consider a  $3 \times 4$  matrix  $A$  in reduced row-echelon form. What can you say about the image of  $A$ ? Describe all cases in terms of  $\text{rank}(A)$ , and draw a sketch for each.
- 47. Let  $T$  be the projection along a line  $L_1$  onto a line  $L_2$ . Describe the image and the kernel of  $T$  geometrically.
- 48. Consider a  $2 \times 2$  matrix  $A$  with  $A^2 = A$ .
  - a. If  $\vec{w}$  is in the image of  $A$ , what is the relationship between  $\vec{w}$  and  $A\vec{w}$ ?
  - b. What can you say about  $A$  if  $\text{rank}(A) = 2$ ? What if  $\text{rank}(A) = 0$ ?
  - c. If  $\text{rank}(A) = 1$ , show that the linear transformation  $T(\vec{x}) = A\vec{x}$  is the projection onto  $\text{im}(A)$  along  $\ker(A)$ .
- 49. Verify that the kernel of a linear transformation is closed under addition and scalar multiplication. See Theorem 1.6.
- 50. Consider a square matrix  $A$  with  $\ker(A^2) = \ker(A^3)$ . Is  $\ker(A^3) = \ker(A^4)$ ? Justify your answer.
- 51. Consider an  $n \times p$  matrix  $A$  and a  $p \times m$  matrix  $B$  such that  $\ker(A) = \{\vec{0}\}$  and  $\ker(B) = \{\vec{0}\}$ . Find  $\ker(AB)$ .
- 52. Consider a  $p \times m$  matrix  $A$  and a  $q \times m$  matrix  $B$ , and form the block matrix

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

What is the relationship among  $\ker(A)$ ,  $\ker(B)$ , and  $\ker(C)$ ?

- 53. In Exercises 53 and 54, we will work with the binary digits (or bits) 0 and 1, instead of the real numbers  $\mathbb{R}$ . Addition and multiplication in this system are defined as usual, except for the rule  $1 + 1 = 0$ . We denote this number system with  $\mathbb{F}_2$ , or simply  $\mathbb{F}$ . The set of all vec-

tors with  $n$  components in  $\mathbb{F}$  is denoted by  $\mathbb{F}^n$ ; note that  $\mathbb{F}^n$  consists of  $2^n$  vectors. (Why?) In information technology, a vector in  $\mathbb{F}^8$  is called a *byte*. (A byte is a string of eight binary digits.)

The basic ideas of linear algebra introduced so far (for the real numbers) apply to  $\mathbb{F}$  without modifications.

A *Hamming matrix* with  $n$  rows is a matrix that contains all nonzero vectors in  $\mathbb{F}^n$  as its columns (in any order). Note that there are  $2^n - 1$  columns. Here is an example:

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{matrix} 3 \text{ rows} \\ 2^3 - 1 = 7 \text{ columns.} \end{matrix}$$

- a. Express the kernel of  $H$  as the span of four vectors in  $\mathbb{F}^7$  of the form

$$\vec{v}_1 = \begin{bmatrix} * \\ * \\ * \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

- b. Form the  $7 \times 4$  matrix

$$M = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

Explain why  $\text{im}(M) = \ker(H)$ . If  $\vec{x}$  is an arbitrary vector in  $\mathbb{F}^4$ , what is  $H(M\vec{x})$ ?

- 54. See Exercise 53 for some background. When information is transmitted, there may be some errors in the communication. We present a method of adding extra information to messages so that most errors that occur during transmission can be detected and corrected. Such methods are referred to as *error-correcting codes*. (Compare these with codes whose purpose is to conceal information.) The pictures of man's first landing on the Moon (in 1969) were televised just as they had been received and were not very clear, since they contained many errors induced during transmission. On later missions, much clearer error-corrected pictures were obtained.

In computers, information is stored and processed in the form of strings of binary digits, 0 and 1. This stream of binary digits is often broken up into "blocks" of eight binary digits (bytes). For the sake of simplicity, we will work with blocks of only four binary digits (i.e., with vectors in  $\mathbb{F}^4$ ), for example,

$$\dots | \quad 1 \ 0 \ 1 \ 1 \ | \ 1 \ 0 \ 0 \ 1 \ | \ 1 \ 0 \ 1 \ 0 \ | \ 1 \ 0 \ 1 \ 1 \ | \\ \quad 1 \ 0 \ 0 \ 0 \ | \ \dots .$$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

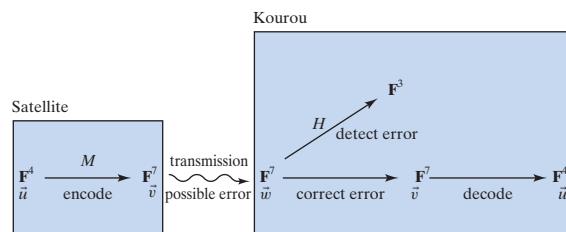
Suppose these vectors in  $\mathbb{F}^4$  have to be transmitted from one computer to another, say, from a satellite to ground control in Kourou, French Guiana (the station of the European Space Agency). A vector  $\vec{u}$  in  $\mathbb{F}^4$  is first transformed into a vector  $\vec{v} = M\vec{u}$  in  $\mathbb{F}^7$ , where  $M$  is the matrix you found in Exercise 53. The last four entries of  $\vec{v}$  are just the entries of  $\vec{u}$ ; the first three entries of  $\vec{v}$  are added to detect errors. The vector  $\vec{v}$  is now transmitted to Kourou. We assume that at most one error will occur during transmission; that is, the vector  $\vec{w}$  received in Kourou will be either  $\vec{v}$  (if no error has occurred) or  $\vec{w} = \vec{v} + \vec{e}_i$  (if there is an error in the  $i$ th component of the vector).

- a. Let  $H$  be the Hamming matrix introduced in Exercise 53. How can the computer in Kourou use  $H\vec{w}$  to determine whether there was an error in the transmission? If there was no error, what is  $H\vec{w}$ ? If there was an error, how can the computer determine in which component the error was made?

- b. Suppose the vector

$$\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is received in Kourou. Determine whether an error was made in the transmission and, if so, correct it. (That is, find  $\vec{v}$  and  $\vec{u}$ .)



## 2 Subspaces of $\mathbb{R}^n$ ; Bases and Linear Independence

In the last section, we saw that both the image and the kernel of a linear transformation contain the zero vector (of the target space and the domain, respectively), are closed under addition, and are closed under scalar multiplication. Subsets of the vector space  $\mathbb{R}^n$  with these three properties are called (linear) *subspaces* of  $\mathbb{R}^n$ .

### Definition 2.1

#### Subspaces of $\mathbb{R}^n$

A subset  $W$  of the vector space  $\mathbb{R}^n$  is called a (linear) *subspace* of  $\mathbb{R}^n$  if it has the following three properties:

- a.  $W$  contains the zero vector in  $\mathbb{R}^n$ .
- b.  $W$  is closed under addition: If  $\vec{w}_1$  and  $\vec{w}_2$  are both in  $W$ , then so is  $\vec{w}_1 + \vec{w}_2$ .
- c.  $W$  is closed under scalar multiplication: If  $\vec{w}$  is in  $W$  and  $k$  is an arbitrary scalar, then  $k\vec{w}$  is in  $W$ .

Properties (b) and (c) together mean that  $W$  is *closed under linear combinations*: If vectors  $\vec{w}_1, \dots, \vec{w}_m$  are in  $W$  and  $k_1, \dots, k_m$  are scalars, then the linear combination  $k_1\vec{w}_1 + \dots + k_m\vec{w}_m$  is in  $W$  as well.

Theorems 1.4 and 1.6 tell us the following:

### Theorem 2.2

#### Image and kernel are subspaces

If  $T(\vec{x}) = A\vec{x}$  is a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , then

- $\ker(T) = \ker(A)$  is a subspace of  $\mathbb{R}^m$ , and
- $\text{image}(T) = \text{im}(A)$  is a subspace of  $\mathbb{R}^n$ .

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

**EXAMPLE 1** Is  $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\}$  a subspace of  $\mathbb{R}^2$ ?

#### Solution

Note that  $W$  consists of all vectors in the first quadrant of the  $x$ - $y$ -plane, including the positive axes and the origin, as illustrated in Figure 1.

$W$  contains the zero vector and is closed under addition, but it is not closed under multiplication with a *negative* scalar. See Figure 2. Thus,  $W$  fails to be a subspace of  $\mathbb{R}^2$ . ■

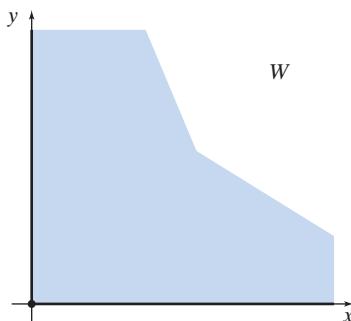


Figure 1

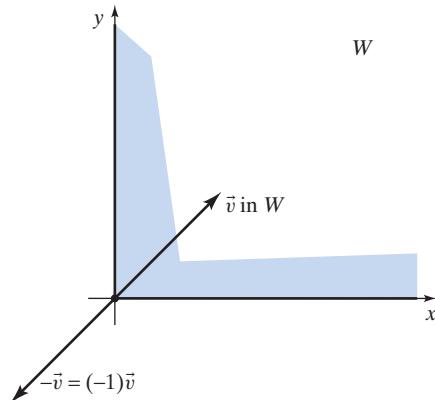


Figure 2

**EXAMPLE 2** Show that the only subspaces of  $\mathbb{R}^2$  are  $\mathbb{R}^2$  itself, the set  $\{\vec{0}\}$ , and any of the lines through the origin.

#### Solution

Suppose  $W$  is a subspace of  $\mathbb{R}^2$  that is neither  $\{\vec{0}\}$  nor a line through the origin. We have to show that  $W$  must equal  $\mathbb{R}^2$ . Consider a nonzero vector  $\vec{v}_1$  in  $W$ . (We can find such a vector since  $W \neq \{\vec{0}\}$ .) The line  $L$  spanned by  $\vec{v}_1$  is a subset of  $W$ , since  $W$  is closed under scalar multiplication; but  $W$  does not equal  $L$ , since  $W$  isn't a line. Consider a vector  $\vec{v}_2$  in  $W$  that isn't on  $L$ . See Figure 3. Using a parallelogram, we can express any vector  $\vec{v}$  in  $\mathbb{R}^2$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ . Therefore,  $\vec{v}$  belongs to  $W$ , since  $W$  is closed under linear combinations. This shows that  $W = \mathbb{R}^2$ , as claimed. ■

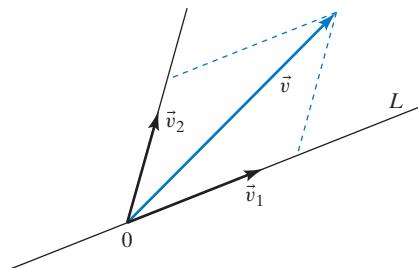


Figure 3

Similarly, the only subspaces of  $\mathbb{R}^3$  are  $\mathbb{R}^3$  itself, the planes through the origin, the lines through the origin, and the set  $\{\vec{0}\}$ . See Exercise 5. Note the hierarchy of

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

subspaces, arranged according to their dimensions. (The concept of dimension will be made precise in the next section.)

	Subspaces of $\mathbb{R}^2$	Subspaces of $\mathbb{R}^3$
Dimension 3		$\mathbb{R}^3$
Dimension 2	$\mathbb{R}^2$	Planes through $\vec{0}$
Dimension 1	Lines through $\vec{0}$	Lines through $\vec{0}$
Dimension 0	$\{\vec{0}\}$	$\{\vec{0}\}$

We have seen that both the kernel and the image of a linear transformation are subspaces (of the domain and target space, respectively). Conversely, can we express any subspace  $V$  of  $\mathbb{R}^n$  as the kernel or the image of a linear transformation (or, equivalently, of a matrix)?

Let us consider an example.

**EXAMPLE 3** Consider the plane  $V$  in  $\mathbb{R}^3$  given by the equation  $x_1 + 2x_2 + 3x_3 = 0$ .

- a. Find a matrix  $A$  such that  $V = \ker(A)$ .
- b. Find a matrix  $B$  such that  $V = \text{im}(B)$ .

#### Solution

- a. We can write the equation  $x_1 + 2x_2 + 3x_3 = 0$  as  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ , so that  $V = \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ .
- b. Since the image of a matrix is the span of its columns, we need to describe  $V$  as the span of some vectors. For the plane  $V$ , any two nonparallel vectors will do, for example,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ . Thus,  $V = \text{im} \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ . ■

A subspace of  $\mathbb{R}^n$  is usually given either as the solution set of a homogeneous linear system (that is, as a kernel), as in Example 3, or as the span of some vectors (that is, as an image). Sometimes, a subspace that has been defined as a kernel must be given as an image (as in part b of Example 3), or vice versa. The transition from kernel to image is straightforward: Using Gaussian elimination, we can represent the solution set as the span of some vectors. See Examples 10 and 11 of Section 1. A method of writing the image of a matrix as a kernel is discussed in Exercises 1.42 and 1.43.

### Bases and Linear Independence

**EXAMPLE 4** Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

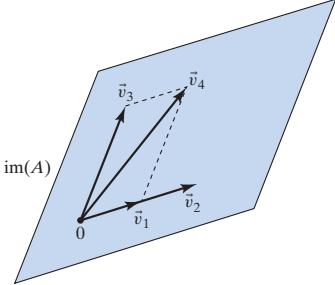
Find vectors in  $\mathbb{R}^3$  that span the image of  $A$ . What is the smallest number of vectors needed to span the image of  $A$ ?

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

### Solution

We know from Theorem 1.3 that the image of  $A$  is spanned by the four column vectors of  $A$ ,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$



**Figure 4**

Figure 4 illustrates that the image of  $A$  is a plane; we don't need all four vectors to span  $\text{im}(A)$ . We observe that  $\vec{v}_2 = 2\vec{v}_1$  and  $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$ , so that the vectors  $\vec{v}_2$  and  $\vec{v}_4$  are “redundant” as far as the span is concerned:

$$\text{im} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_3).$$

The image of  $A$  can be spanned by two vectors, but not by one vector alone.

Let us verify the equation  $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_3)$  algebraically. If a vector  $\vec{v}$  is in  $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ , then

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\ &= c_1\vec{v}_1 + c_2(2\vec{v}_1) + c_3\vec{v}_3 + c_4(\vec{v}_1 + \vec{v}_3) \\ &= (c_1 + 2c_2 + c_4)\vec{v}_1 + (c_3 + c_4)\vec{v}_3, \end{aligned}$$

showing that  $\vec{v}$  is in  $\text{span}(\vec{v}_1, \vec{v}_3)$ , as claimed. ■

The preceding example motivates the following important definitions.

### Definition 2.3

#### Redundant vectors<sup>4</sup>; linear independence; basis

Consider vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ .

- a. We say that a vector  $\vec{v}_i$  in the list  $\vec{v}_1, \dots, \vec{v}_m$  is *redundant* if  $\vec{v}_i$  is a linear combination of the preceding vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}$ .<sup>5</sup>
- b. The vectors  $\vec{v}_1, \dots, \vec{v}_m$  are called *linearly independent* if none of them is redundant. Otherwise, the vectors are called *linearly dependent* (meaning that at least one of them is redundant).<sup>6</sup>
- c. We say that the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in a subspace  $V$  of  $\mathbb{R}^n$  form a *basis* of  $V$  if they span  $V$  and are linearly independent.<sup>7</sup>

Let's take another look at Example 4: In the list

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

---

<sup>4</sup>The notion of a redundant vector is not part of the established vocabulary of linear algebra. However, we will find this concept quite useful in discussing linear independence.

<sup>5</sup>We call the first vector,  $\vec{v}_1$ , redundant if it is the zero vector. This agrees with the convention that the *empty linear combination* of vectors is the zero vector.

<sup>6</sup>A list of alternative characterizations of linear independence will be presented in Summary 2.9. In many texts, characterization (iv) of that list is used to define linear independence.

<sup>7</sup>An alternative characterization of a basis will be presented in Theorem 2.10.

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

of column vectors of  $A$ , the vectors  $\vec{v}_2$  and  $\vec{v}_4$  are redundant, since  $\vec{v}_2 = 2\vec{v}_1$  and  $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$ . If we omit the redundant vectors from the list, then the remaining vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

are linearly independent; they form a basis of  $V = \text{image}(A)$ .

We can generalize the result of Example 4.

#### Theorem 2.4

##### Basis of the image

To construct a basis of the image of a matrix  $A$ , list all the column vectors of  $A$ , and omit the redundant vectors from this list.

But how can we identify the redundant column vectors? In simple cases, this can often be done by inspection (as in Example 4); in the next section we will develop a general algorithm, based on Gaussian elimination.

#### EXAMPLE 5

Are the following vectors in  $\mathbb{R}^7$  linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

##### Solution

Let's look for redundant vectors in this list. Vectors  $\vec{v}_1$  and  $\vec{v}_2$  are clearly nonredundant, since  $\vec{v}_1$  is nonzero and  $\vec{v}_2$  fails to be a scalar multiple of  $\vec{v}_1$  (look at the fourth components). Looking at the last components, we realize that  $\vec{v}_3$  cannot be a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ , since any linear combination of  $\vec{v}_1$  and  $\vec{v}_2$  will have a 0 in the last component, while the last component of  $\vec{v}_3$  is 7. Looking at the second components, we can see that  $\vec{v}_4$  isn't a linear combination of  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . Thus, the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are linearly independent. ■

We will frequently use the approach of Example 5 to show linear independence.

#### Theorem 2.5

##### Linear independence and zero components

Consider vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . If  $\vec{v}_1$  is nonzero, and if each of the vectors  $\vec{v}_i$  (for  $i \geq 2$ ) has a nonzero entry in a component where all the preceding vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}$  have a 0, then the vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent.

To understand what we are trying to say in Theorem 2.5, take another look at the vectors in Example 5.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

$$\begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

**EXAMPLE 6** Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

linearly independent?

### Solution

Theorem 2.5 doesn't help here, since these vectors don't have any zero components. The vectors  $\vec{v}_1$  and  $\vec{v}_2$  are clearly nonredundant, as  $\vec{v}_1$  is nonzero and  $\vec{v}_2$  fails to be a scalar multiple of  $\vec{v}_1$ . To see whether vector  $\vec{v}_3$  is redundant, we need to examine whether  $\vec{v}_3$  can be written as  $\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$ . Considering the augmented matrix

$$M = \left[ \begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array} \right], \quad \text{with rref}(M) = \left[ \begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right],$$

we find the unique solution  $c_1 = -1$ ,  $c_2 = 2$ , so that

$$\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2.$$

It turns out that vector  $\vec{v}_3$  is redundant, making vectors  $\vec{v}_1$ ,  $\vec{v}_2$ ,  $\vec{v}_3$  linearly dependent. ■

For good reasons, mathematicians like to write their equations in the form

$$(\text{Something}) = 0.$$

Applying this principle,<sup>8</sup> we can write the equation  $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$  from Example 6 as

$$\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

This equation is called a *linear relation* among the vectors  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$ .

### Definition 2.6

#### Linear Relations

Consider the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . An equation of the form

$$c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$$

is called a (linear) *relation* among the vectors  $\vec{v}_1, \dots, \vec{v}_m$ . There is always the *trivial relation*, with  $c_1 = \dots = c_m = 0$ . *Nontrivial relations* (where at least one coefficient  $c_i$  is nonzero) may or may not exist among the vectors  $\vec{v}_1, \dots, \vec{v}_m$ .

---

<sup>8</sup>This method was popularized by Descartes, and is often credited to him, but it was used earlier by the English geographer Thomas Harriot (1560–1621). For more on “Harriot's Principle,” see W. P. Berlinghoff and F. Q. Gouvêa, *Math Through the Ages*, Oxton House Publishers and MAA, 2004.

Example 6 suggests the following result.

**Theorem 2.7**

**Relations and linear dependence**

The vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$  are linearly dependent if (and only if) there are nontrivial relations among them.

**Proof**

- Suppose vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly dependent, and  $\vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$  is a redundant vector in this list. Then we can generate a nontrivial relation by subtracting  $\vec{v}_i$  from both sides:  $c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i = \vec{0}$ .
- Conversely, if there is a nontrivial relation  $c_1\vec{v}_1 + \dots + c_i\vec{v}_i + \dots + c_m\vec{v}_m = \vec{0}$ , where  $i$  is the highest index such that  $c_i \neq 0$ , then we can solve for  $\vec{v}_i$  and thus express  $\vec{v}_i$  as a linear combination of the preceding vectors:

$$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \dots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}.$$

This shows that vector  $\vec{v}_i$  is redundant, so that vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly dependent, as claimed. ■

**EXAMPLE 7**

Suppose the column vectors of an  $n \times m$  matrix  $A$  are linearly independent. Find the kernel of matrix  $A$ .

**Solution**

We need to solve the equation

$$A\vec{x} = \vec{0} \quad \text{or} \quad \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0} \quad \text{or} \quad x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}.$$

We see that finding the kernel of  $A$  amounts to finding the relations among the column vectors of  $A$ . By Theorem 2.7, there is only the trivial relation, with  $x_1 = \dots = x_m = 0$ , so that  $\ker(A) = \{\vec{0}\}$ . ■

Let us summarize the findings of Example 7.

**Theorem 2.8**

**Kernel and relations**

The vectors in the kernel of an  $n \times m$  matrix  $A$  correspond to the linear relations among the column vectors  $\vec{v}_1, \dots, \vec{v}_m$  of  $A$ : The equation

$$A\vec{x} = \vec{0} \quad \text{means that} \quad x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}.$$

In particular, the column vectors of  $A$  are linearly independent if (and only if)  $\ker(A) = \{\vec{0}\}$ , or, equivalently, if  $\text{rank}(A) = m$ . This condition implies that  $m \leq n$ .

Thus, we can find at most  $n$  linearly independent vectors in  $\mathbb{R}^n$ .

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

**EXAMPLE 8** Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

to illustrate the connection between redundant column vectors, relations among the column vectors, and the kernel. See Example 6.

$$\begin{array}{c} \text{Redundant column vector: } \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\ \uparrow \\ \text{Relation among column vectors: } 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \vec{0} \\ \uparrow \\ \text{Vector } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ is in } \ker(A), \text{ since } \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \blacksquare \end{array}$$

In the following summary we list the various characterizations of linear independence discussed thus far (in Definition 2.3b, Theorem 2.7, and Theorem 2.8). We include one new characterization, (iii). The proof of the equivalence of statements (iii) and (iv) is left to the reader as Exercise 35; it is analogous to the proof of Theorem 2.7.

### SUMMARY 2.9 | Various characterizations of linear independence

For a list  $\vec{v}_1, \dots, \vec{v}_m$  of vectors in  $\mathbb{R}^n$ , the following statements are equivalent:

- i. Vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent.
- ii. None of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  is redundant, meaning that none of them is a linear combination of preceding vectors.
- iii. None of the vectors  $\vec{v}_i$  is a linear combination of the other vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$  in the list.
- iv. There is only the trivial relation among the vectors  $\vec{v}_1, \dots, \vec{v}_m$ , meaning that the equation  $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$  has only the solution  $c_1 = \dots = c_m = 0$ .

v.  $\ker \begin{bmatrix} | & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & | \end{bmatrix} = \{\vec{0}\}.$

vi.  $\text{rank} \begin{bmatrix} | & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & | \end{bmatrix} = m.$

We conclude this section with an important alternative characterization of a basis. See Definition 2.3c.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

**EXAMPLE 9** If  $\vec{v}_1, \dots, \vec{v}_m$  is a basis of a subspace  $V$  of  $\mathbb{R}^n$ , and if  $\vec{v}$  is a vector in  $V$ , how many solutions  $c_1, \dots, c_m$  does the equation

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$$

have?

### Solution

There is at least one solution, since the vectors  $\vec{v}_1, \dots, \vec{v}_m$  span  $V$  (that's part of the definition of a basis). Suppose we have two representations

$$\begin{aligned}\vec{v} &= c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m \\ &= d_1 \vec{v}_1 + \cdots + d_m \vec{v}_m.\end{aligned}$$

By subtraction, we find

$$(c_1 - d_1) \vec{v}_1 + \cdots + (c_m - d_m) \vec{v}_m = \vec{0},$$

a relation among the vectors  $\vec{v}_1, \dots, \vec{v}_m$ . Since the vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent, this must be the trivial relation, and we have  $c_1 - d_1 = 0, \dots, c_m - d_m = 0$ , or  $c_1 = d_1, \dots, c_m = d_m$ . It turns out that the two representations  $\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$  and  $\vec{v} = d_1 \vec{v}_1 + \cdots + d_m \vec{v}_m$  are identical. We have shown that there is one and only one way to write  $\vec{v}$  as a linear combination of the basis vectors  $\vec{v}_1, \dots, \vec{v}_m$ . ■

Let us summarize.

### Theorem 2.10

#### Basis and unique representation

Consider the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in a subspace  $V$  of  $\mathbb{R}^n$ .

The vectors  $\vec{v}_1, \dots, \vec{v}_m$  form a basis of  $V$  if (and only if) every vector  $\vec{v}$  in  $V$  can be expressed *uniquely* as a linear combination

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m.$$

(In Section 4, we will call the coefficients  $c_1, \dots, c_m$  the *coordinates* of  $\vec{v}$  with respect to the basis  $\vec{v}_1, \dots, \vec{v}_m$ .)

#### Proof

In Example 9 we have shown only one part of Theorem 2.10; we still need to verify that the uniqueness of the representation  $\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$  (for every  $\vec{v}$  in  $V$ ) implies that  $\vec{v}_1, \dots, \vec{v}_m$  is a basis of  $V$ . Clearly, the vectors  $\vec{v}_1, \dots, \vec{v}_m$  span  $V$ , since every  $\vec{v}$  in  $V$  can be written as a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ .

To show the linear independence of vectors  $\vec{v}_1, \dots, \vec{v}_m$ , consider a relation  $c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = \vec{0}$ . This relation is a representation of the zero vector as a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ . But this representation is unique, with  $c_1 = \cdots = c_m = 0$ , so that  $c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = \vec{0}$  must be the trivial relation. We have shown that vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent. ■

Consider the plane  $V = \text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$  introduced in Example 4. (Take another look at Figure 4.)

We can write

$$\begin{aligned}\vec{v}_4 &= 1\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3 + 0\vec{v}_4 \\ &= 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 1\vec{v}_4,\end{aligned}$$

illustrating the fact that the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  do not form a basis of  $V$ . However, every vector  $\vec{v}$  in  $V$  can be expressed *uniquely* as a linear combination of  $\vec{v}_1$  and  $\vec{v}_3$  alone, meaning that the vectors  $\vec{v}_1, \vec{v}_3$  do form a basis of  $V$ .

## EXERCISES 2

**GOAL** Check whether or not a subset of  $\mathbb{R}^n$  is a subspace. Apply the concept of linear independence (in terms of Definition 2.3, Theorem 2.7, and Theorem 2.8). Apply the concept of a basis, both in terms of Definition 2.3 and in terms of Theorem 2.10.

Which of the sets  $W$  in Exercises 1 through 3 are subspaces of  $\mathbb{R}^3$ ?

1.  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

2.  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$

3.  $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

4. Consider the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . Is  $\text{span}(\vec{v}_1, \dots, \vec{v}_m)$  necessarily a subspace of  $\mathbb{R}^n$ ? Justify your answer.

5. Give a geometrical description of all subspaces of  $\mathbb{R}^3$ . Justify your answer.

6. Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ .

a. Is the intersection  $V \cap W$  necessarily a subspace of  $\mathbb{R}^n$ ?

b. Is the union  $V \cup W$  necessarily a subspace of  $\mathbb{R}^n$ ?

7. Consider a nonempty subset  $W$  of  $\mathbb{R}^n$  that is closed under addition and under scalar multiplication. Is  $W$  necessarily a subspace of  $\mathbb{R}^n$ ? Explain.

8. Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

9. Consider the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ , with  $\vec{v}_m = \vec{0}$ . Are these vectors linearly independent?

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

10.  $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

11.  $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

12.  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

13.  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14.  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

15.  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

16.  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

17.  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

18.  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

19.  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

20.  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$

In Exercises 21 through 26, find a redundant column vector of the given matrix  $A$ , and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of  $A$ . (This procedure is illustrated in Example 8.)

21.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

22.  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

23.  $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

24.  $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25.  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

26.  $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27.  $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

28.  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

29.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

30.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

31.  $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$

32.  $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

33.  $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

34. Consider the  $5 \times 4$  matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

We are told that the vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$  is in the kernel of  $A$ .

Write  $\vec{v}_4$  as a linear combination of  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .

35. Show that there is a nontrivial relation among the vectors  $\vec{v}_1, \dots, \vec{v}_m$  if (and only if) at least one of the vectors  $\vec{v}_i$  is a linear combination of the other vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$ .
36. Consider a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^p$  and some linearly dependent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . Are the vectors  $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$  necessarily linearly dependent? How can you tell?

37. Consider a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^p$  and some linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . Are the vectors  $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$  necessarily linearly independent? How can you tell?

38. a. Let  $V$  be a subspace of  $\mathbb{R}^n$ . Let  $m$  be the largest number of linearly independent vectors we can find in  $V$ . (Note that  $m \leq n$ , by Theorem 2.8.) Choose linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $V$ . Show that the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  span  $V$  and are therefore a basis of  $V$ . This exercise shows that any subspace of  $\mathbb{R}^n$  has a basis.

If you are puzzled, think first about the special case when  $V$  is a plane in  $\mathbb{R}^3$ . What is  $m$  in this case?

- b. Show that any subspace  $V$  of  $\mathbb{R}^n$  can be represented as the image of a matrix.

39. Consider some linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$  and a vector  $\vec{v}$  in  $\mathbb{R}^n$  that is not contained in the span of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ . Are the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}$  necessarily linearly independent? Justify your answer.

40. Consider an  $n \times p$  matrix  $A$  and a  $p \times m$  matrix  $B$ . We are told that the columns of  $A$  and the columns of  $B$  are linearly independent. Are the columns of the product  $AB$  linearly independent as well? Hint: Exercise 1.51 is useful.

41. Consider an  $m \times n$  matrix  $A$  and an  $n \times m$  matrix  $B$  (with  $n \neq m$ ) such that  $AB = I_m$ . (We say that  $A$  is a *left inverse* of  $B$ .) Are the columns of  $B$  linearly independent? What about the columns of  $A$ ?

42. Consider some perpendicular unit vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . Show that these vectors are necessarily linearly independent. Hint: Form the dot product of  $\vec{v}_i$  and both sides of the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}.$$

43. Consider three linearly independent vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^n$ . Are the vectors  $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$  linearly independent as well? How can you tell?

44. Consider linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ , and let  $A$  be an invertible  $m \times m$  matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix} A$$

45. Are the columns of an invertible matrix linearly independent?

46. Find a basis of the kernel of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}.$$

Justify your answer carefully; that is, explain how you know that the vectors you found are linearly independent and span the kernel.

47. Consider three linearly independent vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^4$ . Find

$$\text{rref } \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}.$$

48. Express the plane  $V$  in  $\mathbb{R}^3$  with equation  $3x_1 + 4x_2 + 5x_3 = 0$  as the kernel of a matrix  $A$  and as the image of a matrix  $B$ .

49. Express the line  $L$  in  $\mathbb{R}^3$  spanned by the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  as the image of a matrix  $A$  and as the kernel of a matrix  $B$ .

50. Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ . Let  $V + W$  be the set of all vectors in  $\mathbb{R}^n$  of the form  $\vec{v} + \vec{w}$ , where  $\vec{v}$  is in  $V$  and  $\vec{w}$  is in  $W$ . Is  $V + W$  necessarily a subspace of  $\mathbb{R}^n$ ?

If  $V$  and  $W$  are two distinct lines in  $\mathbb{R}^3$ , what is  $V + W$ ? Draw a sketch.

51. Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$  whose intersection consists only of the vector  $\vec{0}$ .

- a. Consider linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  in  $V$  and  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  in  $W$ . Explain why the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  are linearly independent.

- b. Consider a basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  of  $V$  and a basis  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  of  $W$ . Explain why  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  is a basis of  $V + W$ . See Exercise 50.

52. For which values of the constants  $a, b, c, d, e$ , and  $f$  are the following vectors linearly independent? Justify your answer.

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}$$

53. Consider a subspace  $V$  of  $\mathbb{R}^n$ . We define the *orthogonal complement*  $V^\perp$  of  $V$  as the set of those vectors  $\vec{w}$  in  $\mathbb{R}^n$  that are perpendicular to all vectors in  $V$ ; that is,  $\vec{w} \cdot \vec{v} = 0$ , for all  $\vec{v}$  in  $V$ . Show that  $V^\perp$  is a subspace of  $\mathbb{R}^n$ .

54. Consider the line  $L$  spanned by  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  in  $\mathbb{R}^3$ . Find a basis of  $L^\perp$ . See Exercise 53.

55. Consider the subspace  $L$  of  $\mathbb{R}^5$  spanned by the given vector. Find a basis of  $L^\perp$ . See Exercise 53.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

56. For which values of the constants  $a, b, \dots, m$  are the given vectors linearly independent?

$$\begin{bmatrix} a \\ b \\ c \\ d \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} e \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} f \\ g \\ h \\ i \\ j \\ 1 \end{bmatrix}, \quad \begin{bmatrix} k \\ m \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

57. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

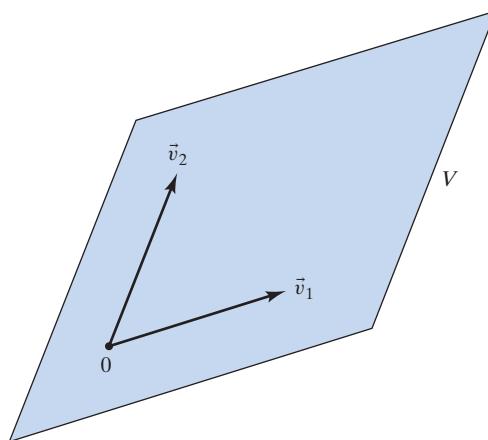
Note that matrix  $A$  is in reduced row-echelon form.

For which positive integers  $j = 1, \dots, 7$  does there exist a vector  $\vec{x}$  in the kernel of  $A$  such that the  $j$ th component  $x_j$  of  $\vec{x}$  is nonzero, while all the components  $x_{j+1}, \dots, x_7$  are zero?

58. Consider an  $n \times m$  matrix  $A$ . For which positive integers  $j = 1, \dots, m$  does there exist a vector  $\vec{x}$  in the kernel of  $A$  such that the  $j$ th component  $x_j$  of  $\vec{x}$  is nonzero, while all the components  $x_{j+1}, \dots, x_m$  are zero? Use Exercise 57 as a guide. Give your answer in terms of the redundant column vectors of  $A$ .

## 3 The Dimension of a Subspace of $\mathbb{R}^n$

Consider a plane  $V$  in  $\mathbb{R}^3$ . Using our geometric intuition, we observe that all bases of  $V$  consist of two vectors. (Any two nonparallel vectors in  $V$  will do; see Figure 1.) One vector is not enough to span  $V$ , and three or more vectors are linearly dependent. It turns out that, more generally, all bases of a subspace  $V$  of  $\mathbb{R}^n$  consist of the same number of vectors. In order to prove this important fact, we need an auxiliary result.



**Figure 1** The vectors  $\vec{v}_1, \vec{v}_2$  form a basis of  $V$ .

**Theorem 3.1**

Consider vectors  $\vec{v}_1, \dots, \vec{v}_p$  and  $\vec{w}_1, \dots, \vec{w}_q$  in a subspace  $V$  of  $\mathbb{R}^n$ . If the vectors  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent, and the vectors  $\vec{w}_1, \dots, \vec{w}_q$  span  $V$ , then  $q \geq p$ .

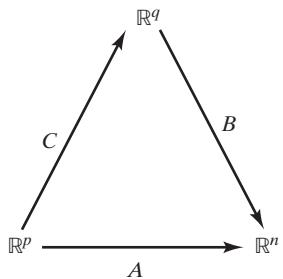
For example, let  $V$  be a plane in  $\mathbb{R}^3$ . Our geometric intuition tells us that we can find *at most* two linearly independent vectors in  $V$ , so that  $2 \geq p$ , and we need *at least* two vectors to span  $V$ , so that  $q \geq 2$ . Therefore, the inequality  $q \geq p$  does indeed hold in this case.

**Proof**

This proof is rather technical and not very illuminating. In the next section, when we study coordinate systems, we will gain a more conceptual understanding of this matter.

Consider the matrices

$$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & | \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} | & | \\ \vec{w}_1 & \cdots & \vec{w}_q \\ | & | \end{bmatrix}.$$



**Figure 2**

Note that  $\text{im}(B) = V$ , since the vectors  $\vec{w}_1, \dots, \vec{w}_q$  span  $V$ . The vectors  $\vec{v}_1, \dots, \vec{v}_p$  are in the image of  $B$ , so that we can write

$$\vec{v}_1 = B\vec{u}_1, \quad \dots, \quad \vec{v}_p = B\vec{u}_p$$

for some vectors  $\vec{u}_1, \dots, \vec{u}_p$  in  $\mathbb{R}^q$ . We can combine these equations and write

$$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & | \end{bmatrix} = B \underbrace{\begin{bmatrix} | & | \\ \vec{u}_1 & \cdots & \vec{u}_p \\ | & | \end{bmatrix}}_C, \quad \text{or} \quad A = BC.$$

See Figure 2.

The kernel of  $C$  is a subset of the kernel of  $A$  (if  $C\vec{x} = \vec{0}$ , then  $A\vec{x} = BC\vec{x} = \vec{0}$ ). But the kernel of  $A$  is  $\{\vec{0}\}$ , since the vectors  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent. Therefore, the kernel of  $C$  is  $\{\vec{0}\}$  as well. Theorem 1.7b now tells us that the  $q \times p$  matrix  $C$  has at least as many rows as it has columns, that is,  $q \geq p$ , as claimed. ■

**Theorem 3.2**
**Number of vectors in a basis**

All bases of a subspace  $V$  of  $\mathbb{R}^n$  consist of the same number of vectors.

**Proof**

Consider two bases  $\vec{v}_1, \dots, \vec{v}_p$  and  $\vec{w}_1, \dots, \vec{w}_q$  of  $V$ . Since the vectors  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent and the vectors  $\vec{w}_1, \dots, \vec{w}_q$  span  $V$ , we have  $q \geq p$ , by Theorem 3.1. Likewise, since the vectors  $\vec{w}_1, \dots, \vec{w}_q$  are linearly independent and the vectors  $\vec{v}_1, \dots, \vec{v}_p$  span  $V$ , we have  $p \geq q$ . Therefore,  $p = q$ . ■

Consider a line  $L$  and a plane  $V$  in  $\mathbb{R}^3$ . A basis of  $L$  consists of just one vector (any nonzero vector in  $L$  will do), while all bases of  $V$  consist of two vectors. A basis of  $\mathbb{R}^3$  consists of three vectors. (The standard vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  are one possible choice.) In each case, the number of vectors in a basis corresponds to what we intuitively sense to be the dimension of the subspace.

**Definition 3.3**

**Dimension**

Consider a subspace  $V$  of  $\mathbb{R}^n$ . The number of vectors in a basis of  $V$  is called the *dimension* of  $V$ , denoted by  $\dim(V)$ .<sup>9</sup>

This algebraic definition of dimension represents a major advance in the development of linear algebra, and indeed of mathematics as a whole: It allows us to conceive of spaces with more than three dimensions. This idea is often poorly understood in popular culture, where some mysticism still surrounds higher-dimensional spaces. The German mathematician Hermann Weyl (1855–1955) put it this way: “We are by no means obliged to seek illumination from the mystic doctrines of spiritists to obtain a clearer vision of multidimensional geometry” (*Raum, Zeit, Materie*, 1918).

The first mathematician who thought about dimension from an algebraic point of view may have been the Frenchman Jean Le Rond d’Alembert (1717–1783). In the article on dimension in the *Encyclopédie*, he wrote the following:

The way of considering quantities having more than three dimensions is just as right as the other, because letters can always be viewed as representing numbers, whether rational or not. I said above that it was not possible to conceive more than three dimensions. A thoughtful gentleman [*un homme d'esprit*] with whom I am acquainted believes that nevertheless one could view duration as a fourth dimension. . . . This idea may be challenged, but it has, it seems to me, some merit, were it only that of being new [*cette idée peut être contestée, mais elle a, ce me semble, quelque mérite, quand ce ne seroit que celui de la nouveauté*]. (*Encyclopédie*, vol. 4, 1754)

This *homme d'esprit* was no doubt d’Alembert himself, afraid of being attacked for what appeared as a risky idea at that time.

The idea of dimension was later studied much more systematically by the German mathematician Hermann Günther Grassmann (1809–1877), who introduced the concept of a subspace of  $\mathbb{R}^n$ . In fact, most of the concepts discussed in this chapter can be traced back to Grassmann’s work. Grassmann presented his ideas in 1844 in the book *Die lineare Ausdehnungslehre, ein neuer Zweig der Mathematik* (*The Theory of Linear Extension, a New Branch of Mathematics*). Grassmann’s methods were only slowly adopted, partly because of his obscure writing. He used unfamiliar authentic German terms, rather than the customary Latin, for mathematical concepts; he writes about “Schatten,” shadows, for example, rather than projections. While his ideas have survived, most of his terminology has not.

Similar work was done by the Swiss mathematician Ludwig Schläfli (1814–1895), a contemporary of Grassmann.

Today, dimension is a standard and central tool in mathematics, as well as in physics and statistics. The concept can be applied to certain nonlinear subsets of  $\mathbb{R}^n$ , called manifolds, generalizing the idea of curves and surfaces in  $\mathbb{R}^3$ .

After this brief historical digression, let us return to the more mundane: What is the dimension of  $\mathbb{R}^n$ ? We expect this dimension to be  $n$ , of course. This is indeed the case: The vectors  $\vec{e}_1, \dots, \vec{e}_n$  form a basis, called the *standard basis* of  $\mathbb{R}^n$ .

A plane  $V$  in  $\mathbb{R}^3$  is two dimensional. Earlier, we mentioned that we cannot find more than two linearly independent vectors in  $V$  and that we need at least two

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<sup>9</sup>For this definition to make sense, we have to be sure that any subspace of  $\mathbb{R}^n$  has a basis. This verification is left as Exercise 2.38a.

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

vectors to span  $V$ . If two vectors in  $V$  are linearly independent, then they form a basis of  $V$ . Likewise, if two vectors span  $V$ , then they form a basis of  $V$ .

We can generalize these observations as follows:

#### Theorem 3.4

#### Independent vectors and spanning vectors in a subspace of $\mathbb{R}^n$

Consider a subspace  $V$  of  $\mathbb{R}^n$  with  $\dim(V) = m$ .

- a. We can find *at most*  $m$  linearly independent vectors in  $V$ .
- b. We need *at least*  $m$  vectors to span  $V$ .
- c. If  $m$  vectors in  $V$  are linearly independent, then they form a basis of  $V$ .
- d. If  $m$  vectors in  $V$  span  $V$ , then they form a basis of  $V$ .

Part (a) allows us to *define* the dimension of  $V$  alternatively as the maximal number of linearly independent vectors in  $V$ . Likewise, part (b) tells us that the dimension of  $V$  is the minimal number of vectors needed to span  $V$ .

In parts (c) and (d) we make the following point: By Definition 2.3, some vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $V$  form a basis of  $V$  if they are linearly independent *and* span  $V$ . However, if we are dealing with “the right number” of vectors (namely,  $m$ , the dimension of  $V$ ), then it suffices to check only one of the two properties; the other will then follow “automatically.”

**Proof** We prove Theorem 3.4, parts (a) and (c). We leave the proofs of parts (b) and (d) as Exercises 80 and 81.

- a. Consider linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  in  $V$ , and let  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$  be a basis of  $V$ . Since the vectors  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$  span  $V$ , we have  $p \leq m$ , by Theorem 3.1, as claimed.
- c. Consider linearly independent vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $V$ . We have to show that the vectors  $\vec{v}_1, \dots, \vec{v}_m$  span  $V$ . If  $\vec{v}$  is any vector in  $V$ , then the  $m+1$  vectors  $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$  will be linearly dependent, by part (a). Since vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent and therefore nonredundant, vector  $\vec{v}$  must be redundant in the list  $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$ , meaning that  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ . Since  $\vec{v}$  is an arbitrary vector in  $V$ , we have shown that vectors  $\vec{v}_1, \dots, \vec{v}_m$  span  $V$ , as claimed. ■

In Section 2, we saw that the kernel and image of a linear transformation are subspaces of the domain and the target space of the transformation, respectively. We will now examine how we can find bases of the image and kernel and thus determine their dimensions.

#### Finding Bases of Kernel and Image

#### EXAMPLE 1

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}.$$

- a. Find a basis of the kernel of  $A$ , and thus determine the dimension of the kernel.
- b. Find a basis of the image of  $A$ , and thus determine the dimension of the image.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

### Solution

- a. We will solve the linear system  $A\vec{x} = \vec{0}$ , by Gaussian elimination. From Example 1.11 we know that

$$B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that  $\ker(A) = \ker(B)$ , by the definition of the reduced row-echelon form. Equation  $A\vec{x} = \vec{0}$  can be solved by solving the simpler equation  $B\vec{x} = \vec{0}$  instead. In Example 1.11 we saw that the vectors in  $\ker(A) = \ker(B)$  are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s & -3t & +4r \\ s & 4t & -5r \\ & t & r \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_1} + t \underbrace{\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_2} + r \underbrace{\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_3},$$

where  $s, t$ , and  $r$  are arbitrary constants.

We claim that the three vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$  form a basis of the kernel of  $A$ . The preceding equation,  $\vec{x} = s\vec{w}_1 + t\vec{w}_2 + r\vec{w}_3$ , shows that the vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$  span the kernel.

Theorem 2.5 tells us that the vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$  are linearly independent, since each has a 1 in a component where the other two vectors have a 0; these components correspond to the free variables  $x_2, x_4$ , and  $x_5$ .

Thus, a basis of the kernel of  $A$  is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix},$$

and  $\dim(\ker A) = 3$ .

- b. To construct a basis of the image of  $A$  by means of Theorem 2.4, we need to find the redundant columns of  $A$ . Let's see how we can use  $B = \text{rref}(A)$  to carry out this task. To keep track of the columns of  $A$  and  $B$ , we will denote the columns of  $A$  by  $\vec{a}_1, \dots, \vec{a}_5$  and those of  $B$  by  $\vec{b}_1, \dots, \vec{b}_5$ .

The redundant columns of  $B = \text{rref}(A)$  are easy to spot. They are the columns that do not contain a leading 1, namely,  $\vec{b}_2 = 2\vec{b}_1, \vec{b}_4 = 3\vec{b}_1 - 4\vec{b}_3$ , and  $\vec{b}_5 = -4\vec{b}_1 + 5\vec{b}_3$ .

And here comes the key observation: The redundant columns of  $A$  correspond to those of  $B$ , meaning that  $\vec{a}_i$  is redundant if and only if  $\vec{b}_i$  is redundant. We will illustrate this fact by means of an example. We know that  $\vec{b}_5$  is redundant, with  $\vec{b}_5 = -4\vec{b}_1 + 5\vec{b}_3$ . This induces the relation  $4\vec{b}_1 - 5\vec{b}_3 + \vec{b}_5 = \vec{0}$ , meaning that the vector

$$\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

is in  $\ker(B) = \ker(A)$ ; see part (a) of this example. But this in turn induces the relation  $4\vec{a}_1 - 5\vec{a}_3 + \vec{a}_5 = \vec{0}$  among the columns of  $A$ , showing that  $\vec{a}_5$  is redundant, with  $\vec{a}_5 = -4\vec{a}_1 + 5\vec{a}_3$ .<sup>10</sup>

Thus, the redundant columns of  $A$  are  $\vec{a}_2 = 2\vec{a}_1$ ,  $\vec{a}_4 = 3\vec{a}_1 - 4\vec{a}_3$  and  $\vec{a}_5 = -4\vec{a}_1 + 5\vec{a}_3$ . By Theorem 2.4, the nonredundant columns  $\vec{a}_1$  and  $\vec{a}_3$  form a basis of the image of  $A$ .

Thus, a basis of the image of  $A$  is

$$\begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix}$$

and  $\dim(\text{im } A) = 2$ . ■

Using Example 1b as a guide, we can establish the following general rule for finding a basis of the image of a matrix.

#### Theorem 3.5

##### Using rref to construct a basis of the image

To construct a basis of the image of  $A$ , pick the column vectors of  $A$  that correspond to the columns of  $\text{rref}(A)$  containing the leading 1's.

Again, here are the three main points that make this procedure work:

- The nonredundant column vectors of  $A$  form a basis of the image of  $A$  (Theorem 2.4).
- The redundant columns of  $A$  correspond to those of  $\text{rref}(A)$ .
- The nonredundant column vectors of  $\text{rref}(A)$  are those containing the leading 1's.

Note that in Theorem 3.5 you need to pick columns of matrix  $A$ , not of  $\text{rref}(A)$ , because the matrices  $A$  and  $\text{rref}(A)$  need not have the same image. See Exercise 1.44b.

In Theorem 3.5, we are constructing a basis of  $\text{im}(A)$  that contains as many vectors as there are leading 1's in  $\text{rref}(A)$ . By the definition of the rank of a matrix, this number is the rank of  $A$ .

#### Theorem 3.6

##### Dimension of the image

For any matrix  $A$ ,

$$\dim(\text{im } A) = \text{rank}(A).$$

Let's get back to the kernel. In Example 1a we are constructing a basis of the kernel of an  $n \times m$  matrix  $A$  that contains as many vectors as there are free variables. Thus,

$$\begin{aligned} \dim(\ker A) &= \binom{\text{number of free variables}}{} = \binom{\text{total number of variables}}{} - \binom{\text{number of leading variables}}{} \\ &= m - \text{rank}(A). \end{aligned}$$

Adding up the equations  $\dim(\ker A) = m - \text{rank}(A)$  and  $\dim(\text{im } A) = \text{rank}(A)$ , we find the remarkable equation  $\dim(\ker A) + \dim(\text{im } A) = m$  for any  $n \times m$  matrix  $A$ .

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<sup>10</sup>A general proof of the claim that the redundant columns of  $A$  correspond to those of  $B$  goes along similar lines. Suppose  $\vec{b}_i$  is redundant, with  $\vec{b}_i = c_1\vec{b}_1 + \cdots + c_{i-1}\vec{b}_{i-1}$ . This induces a relation  $-c_1\vec{b}_1 - \cdots - c_{i-1}\vec{b}_{i-1} + \vec{b}_i = \vec{0}$ , and so forth, as above.

**Theorem 3.7**
**Rank-nullity theorem**

For any  $n \times m$  matrix  $A$ , the equation

$$\dim(\ker A) + \dim(\text{im}A) = m$$

holds. The dimension of  $\ker(A)$  is called the *nullity* of  $A$ , and in Theorem 3.6 we observed that  $\dim(\text{im}A) = \text{rank}(A)$ . Thus, we can write the preceding equation alternatively as

$$(\text{nullity of } A) + (\text{rank of } A) = m.$$

Some authors go so far as to call this the *fundamental theorem of linear algebra*.

We can write the rank-nullity theorem as

$$m - \dim(\ker A) = \dim(\text{im}A);$$

we can interpret this formula geometrically as follows.

Consider the linear transformation

$$T(\vec{x}) = A\vec{x} \quad \text{from } \mathbb{R}^m \text{ to } \mathbb{R}^n.$$

Note that  $m$  is the dimension of the domain of transformation  $T$ . The quantity  $\text{nullity}(A) = \dim(\ker A)$  counts the dimensions that “collapse” as we perform transformation  $T$ , and  $\text{rank}(A) = \dim(\text{im}A)$  counts the dimensions that “survive” transformation  $T$ .

**EXAMPLE 2** Consider the orthogonal projection  $T$  onto a plane  $V$  in  $\mathbb{R}^3$ . See Figure 3. Here, the dimension of the domain is  $m = 3$ , one dimension collapses (the kernel of  $T$  is the

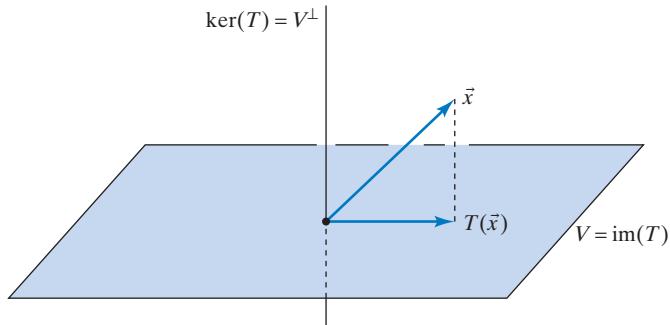


Figure 3

line  $V^\perp$  orthogonal to  $V$ ), and we are left with the two-dimensional  $\text{im}(T) = V$ . See Examples 1.5 and 1.9.

$$\begin{array}{rcl} m - \dim(\ker T) & = & \dim(\text{image } T) \\ \uparrow & - & \uparrow \\ 3 & - & 1 = & 2 \end{array}$$

■

If we can find the redundant column vectors of a matrix  $A$  by inspection, then we can construct bases for the image and kernel of  $A$  without computing the reduced row-echelon form of  $A$ . This shortcut is illustrated in the following example.

**EXAMPLE 3** Find bases of the image and kernel of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}.$$

### Solution

We can spot the redundant columns,  $\vec{v}_2 = 2\vec{v}_1$ ,  $\vec{v}_3 = \vec{0}$ , and  $\vec{v}_5 = \vec{v}_1 + \vec{v}_4$ . Proposition 2.4 tells us that the nonredundant columns

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

form a basis of the image of  $A$ . Thus,  $\dim(\text{im } A) = 2$ .

Applying the method outlined in Example 2.8 to the redundant vectors  $\vec{v}_2$ ,  $\vec{v}_3$ , and  $\vec{v}_5$ , we can generate three vectors in the kernel of  $A$ . We will organize our work in a table.

Redundant Vector	Relation	Vector in Kernel of $A$
$\vec{v}_2 = 2\vec{v}_1$	$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$	$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
$\vec{v}_3 = \vec{0}$	$\vec{v}_3 = \vec{0}$	$\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$\vec{v}_5 = \vec{v}_1 + \vec{v}_4$	$-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$	$\vec{w}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

To facilitate the transition from the relation to the vector in the kernel, it can be useful to write the coefficients of a relation above the corresponding columns of the matrix,<sup>11</sup> as follows (for the last relation):

$$\begin{array}{ccccc} -1 & 0 & 0 & -1 & 1 \\ \left[ \begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{array} \right] \end{array}.$$

We claim that the three vectors  $\vec{w}_2$ ,  $\vec{w}_3$ ,  $\vec{w}_5$  constructed above form a *basis of the kernel of  $A$* . Theorem 2.5 tells us that these vectors are linearly independent, since

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<sup>11</sup>We will refer to these numbers above the matrix as the *Kyle numbers*, after Kyle Burke (Colby 2003), who introduced them.

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

vector  $\vec{w}_i$  has a 1 in the  $i$ th component, while the preceding vectors have a 0 in that component.

From Theorem 3.7, we know that  $\dim(\ker A) = 5 - \dim(\text{im } A) = 3$ . Because  $\vec{w}_2, \vec{w}_3, \vec{w}_5$  are three linearly independent vectors in the three-dimensional space  $\ker(A)$ , they form a basis of  $\ker(A)$ , by Theorem 3.4c. ■

More generally, if  $A$  is an  $n \times m$  matrix, then this procedure generates as many linearly independent vectors in  $\ker(A)$  as there are redundant columns in  $A$ . But this number is

$$m - \left( \begin{array}{c} \text{number of} \\ \text{nonredundant columns} \end{array} \right) = m - \dim(\text{im } A) = \dim(\ker A),$$

by Theorem 3.7, showing that we have enough vectors to form a basis of the kernel (again, we are invoking Theorem 3.4c).

#### Theorem 3.8

#### Finding bases of the kernel and image by inspection

Suppose you are able to spot the redundant columns of a matrix  $A$ .

Express each redundant column as a linear combination of the preceding columns,  $\vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$ ; write a corresponding relation,  $-c_1\vec{v}_1 - \dots - c_{i-1}\vec{v}_{i-1} + \vec{v}_i = \vec{0}$ ; and generate the vector

$$\begin{bmatrix} -c_1 \\ \vdots \\ -c_{i-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in the kernel of  $A$ . The vectors so constructed form a basis of the kernel of  $A$ .

The nonredundant columns form a basis of the image of  $A$ .

The use of Kyle numbers can facilitate this procedure. See Example 3.

#### Bases of $\mathbb{R}^n$

We know that any basis of  $\mathbb{R}^n$  consists of  $n$  vectors, since we have the standard basis  $\vec{e}_1, \dots, \vec{e}_n$  (recall Theorem 3.2). Conversely, how can we tell whether  $n$  given vectors  $\vec{v}_1, \dots, \vec{v}_n$  in  $\mathbb{R}^n$  form a basis?

By Theorem 2.10, the vectors  $\vec{v}_1, \dots, \vec{v}_n$  form a basis of  $\mathbb{R}^n$  if (and only if) every vector  $\vec{b}$  in  $\mathbb{R}^n$  can be written uniquely as a linear combination of the vectors  $\vec{v}_1, \dots, \vec{v}_n$ :

$$\vec{b} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

By the definition of invertibility, the linear system

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$$

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has a unique solution for all  $\vec{b}$  if (and only if) the  $n \times n$  matrix

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

is invertible. We have shown the following result:

#### Theorem 3.9

##### Bases of $\mathbb{R}^n$

The vectors  $\vec{v}_1, \dots, \vec{v}_n$  in  $\mathbb{R}^n$  form a basis of  $\mathbb{R}^n$  if (and only if) the matrix

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

is invertible.

**EXAMPLE 4** For which values of the constant  $k$  do the following vectors form a basis of  $\mathbb{R}^3$ ?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

#### Solution

We need to examine when the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{bmatrix}$$

is invertible. This matrix reduces to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & (1-k)/2 \\ 0 & 0 & k^2 - 1 \end{bmatrix}.$$

We can reduce this matrix all the way to  $I_3$  if (and only if)  $k^2 - 1 \neq 0$ , that is, if  $k$  is neither 1 nor  $-1$ .

Thus, the three given vectors form a basis of  $\mathbb{R}^3$  if (and only if)  $k$  is neither 1 nor  $-1$ . ■

Theorem 3.4, parts (c) and (d), applied to  $V = \mathbb{R}^n$ , and Theorem 3.9 provide us with three new characterizations of invertible matrices.

#### SUMMARY 3.10

##### Various characterizations of invertible matrices

For an  $n \times n$  matrix  $A$ , the following statements are equivalent.

- i.  $A$  is invertible.
- ii. The linear system  $A\vec{x} = \vec{b}$  has a unique solution  $\vec{x}$ , for all  $\vec{b}$  in  $\mathbb{R}^n$ .
- iii.  $\text{rref}(A) = I_n$ .
- iv.  $\text{rank}(A) = n$ .
- v.  $\text{im}(A) = \mathbb{R}^n$ .
- vi.  $\ker(A) = \{\vec{0}\}$ .
- vii. The column vectors of  $A$  form a basis of  $\mathbb{R}^n$ .
- viii. The column vectors of  $A$  span  $\mathbb{R}^n$ .
- ix. The column vectors of  $A$  are linearly independent.

## EXERCISES 3

**GOAL** Use the concept of dimension. Find a basis of the kernel and of the image of a linear transformation.

In Exercises 1 through 20, find the redundant column vectors of the given matrix  $A$  “by inspection.” Then find a basis of the image of  $A$  and a basis of the kernel of  $A$ .

1. 
$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

9. 
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

10. 
$$\begin{bmatrix} 1 & -3 \\ 2 & -6 \\ 3 & -9 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

12. 
$$\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

13. 
$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

14. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

15. 
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

16. 
$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

17. 
$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

18. 
$$\begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

19. 
$$\begin{bmatrix} 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

20. 
$$\begin{bmatrix} 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In Exercises 21 through 25, find the reduced row-echelon form of the given matrix  $A$ . Then find a basis of the image of  $A$  and a basis of the kernel of  $A$ .

21. 
$$\begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix}$$

22. 
$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$$

23. 
$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

24. 
$$\begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$$

25. 
$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$$

26. Consider the matrices

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- a. Which of the matrices in this list have the same kernel as matrix  $C$ ?
- b. Which of the matrices in this list have the same image as matrix  $C$ ?
- c. Which of these matrices has an image that is different from the images of all the other matrices in the list?

27. Determine whether the following vectors form a basis of  $\mathbb{R}^4$ :

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

28. For which value(s) of the constant  $k$  do the vectors below form a basis of  $\mathbb{R}^4$ ?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ k \end{bmatrix}$$

29. Find a basis of the subspace of  $\mathbb{R}^3$  defined by the equation

$$2x_1 + 3x_2 + x_3 = 0.$$

30. Find a basis of the subspace of  $\mathbb{R}^4$  defined by the equation

$$2x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

31. Let  $V$  be the subspace of  $\mathbb{R}^4$  defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Find a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^4$  such that  $\ker(T) = \{0\}$  and  $\text{im}(T) = V$ . Describe  $T$  by its matrix  $A$ .

32. Find a basis of the subspace of  $\mathbb{R}^4$  that consists of all vectors perpendicular to both

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

33. A subspace  $V$  of  $\mathbb{R}^n$  is called a *hyperplane* if  $V$  is defined by a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0,$$

where at least one of the coefficients  $c_i$  is nonzero. What is the dimension of a hyperplane in  $\mathbb{R}^n$ ? Justify your answer carefully. What is a hyperplane in  $\mathbb{R}^3$ ? What is it in  $\mathbb{R}^2$ ?

34. Consider a subspace  $V$  in  $\mathbb{R}^m$  that is defined by  $n$  homogeneous linear equations:

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m & = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m & = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m & = 0 \end{vmatrix}.$$

What is the relationship between the dimension of  $V$  and the quantity  $m - n$ ? State your answer as an inequality. Explain carefully.

35. Consider a nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$ . What is the dimension of the space of all vectors in  $\mathbb{R}^n$  that are perpendicular to  $\vec{v}$ ?

36. Can you find a  $3 \times 3$  matrix  $A$  such that  $\text{im}(A) = \ker(A)$ ? Explain.

37. Give an example of a  $4 \times 5$  matrix  $A$  with  $\dim(\ker A) = 3$ .

38. a. Consider a linear transformation  $T$  from  $\mathbb{R}^5$  to  $\mathbb{R}^3$ . What are the possible values of  $\dim(\ker T)$ ? Explain.

- b. Consider a linear transformation  $T$  from  $\mathbb{R}^4$  to  $\mathbb{R}^7$ . What are the possible values of  $\dim(\text{im } T)$ ? Explain.

39. We are told that a certain  $5 \times 5$  matrix  $A$  can be written as

$$A = BC,$$

where  $B$  is a  $5 \times 4$  matrix and  $C$  is  $4 \times 5$ . Explain how you know that  $A$  is not invertible.

*In Exercises 40 through 43, consider the problem of fitting a conic through  $m$  given points  $P_1(x_1, y_1), \dots, P_m(x_m, y_m)$  in the plane. Note that a conic is a curve in  $\mathbb{R}^2$  that can be described by an equation of the form  $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$ , where at least one of the coefficients  $c_i$  is nonzero.*

40. Explain why fitting a conic through the points  $P_1(x_1, y_1), \dots, P_m(x_m, y_m)$  amounts to finding the kernel of an  $m \times 6$  matrix  $A$ . Give the entries of the  $i$ th row of  $A$ .

Note that a one-dimensional subspace of the kernel of  $A$  defines a unique conic, since the equations  $f(x, y) = 0$  and  $kf(x, y) = 0$  describe the same conic.

41. How many conics can you fit through four distinct points  $P_1(x_1, y_1), \dots, P_4(x_4, y_4)$ ?

42. How many conics can you fit through five distinct points  $P_1(x_1, y_1), \dots, P_5(x_5, y_5)$ ? Describe all possible scenarios, and give an example in each case.

43. How many conics can you fit through six distinct points  $P_1(x_1, y_1), \dots, P_6(x_6, y_6)$ ? Describe all possible scenarios, and give an example in each case.

*For Exercises 44 through 61, consider the problem of fitting a cubic through  $m$  given points  $P_1(x_1, y_1), \dots, P_m(x_m, y_m)$  in the plane. A cubic is a curve in  $\mathbb{R}^2$  that can be described by an equation of the form  $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 = 0$ , where at least one of the coefficients  $c_i$  is nonzero. If  $k$  is any nonzero constant, then the equations  $f(x, y) = 0$  and  $kf(x, y) = 0$  define the same cubic.*

44. Show that the cubics through the points  $(0,0)$ ,  $(1,0)$ ,  $(2,0)$ ,  $(0,1)$ , and  $(0,2)$  can be described by equations of the form  $c_5xy + c_7(x^3 - 3x^2 + 2x) + c_8x^2y + c_9xy^2 + c_{10}(y^3 - 3y^2 + 2y) = 0$ , where at least one of the coefficients  $c_i$  is nonzero. Alternatively, this equation can be written as  $c_7x(x-1)(x-2) + c_{10}y(y-1)(y-2) + xy(c_5 + c_8x + c_9y) = 0$ .

45. Show that the cubics through the points  $(0,0)$ ,  $(1,0)$ ,  $(2,0)$ ,  $(3,0)$ ,  $(0,1)$ ,  $(0,2)$ , and  $(0,3)$  can be described by equations of the form  $c_5xy + c_8x^2y + c_9xy^2 = 0$ , where at least one of the coefficients  $c_5$ ,  $c_8$ , and  $c_9$  is nonzero. Alternatively, this equation can be written as  $xy(c_5 + c_8x + c_9y) = 0$ . Describe these cubics geometrically.

*In Problems 46 through 55, find all the cubics through the given points. You may use the results from Exercises 44 and 45 throughout. If there is a unique cubic, make a rough sketch of it. If there are infinitely many cubics, sketch two of them.*

46.  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1)$

47.  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1), (2, 2)$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

- 48.**  $(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (0, 1), (0, 2), (0, 3), (1, 1)$
- 49.**  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1), (2, 2), (2, 1)$
- 50.**  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1), (2, 2), (3, 3)$
- 51.**  $(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 1)$
- 52.**  $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2)$
- 53.**  $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (3, 2)$
- 54.**  $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)$
- 55.**  $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2), (3, 3)$
- 56.** Explain why fitting a cubic through the  $m$  points  $P_1(x_1, y_1), \dots, P_m(x_m, y_m)$  amounts to finding the kernel of an  $m \times 10$  matrix  $A$ . Give the entries of the  $i$ th row of  $A$ .
- 57.** How many cubics can you fit through eight distinct points  $P_1(x_1, y_1), \dots, P_8(x_8, y_8)$ ?
- 58.** How many cubics can you fit through nine distinct points  $P_1(x_1, y_1), \dots, P_9(x_9, y_9)$ ? Describe all possible scenarios, and give an example in each case.
- 59.** How many cubics can you fit through 10 distinct points  $P_1(x_1, y_1), \dots, P_{10}(x_{10}, y_{10})$ ? Describe all possible scenarios, and give an example in each case.
- 60.** On September 30, 1744, the Swiss mathematician Gabriel Cramer (1704–1752) wrote a remarkable letter to his countryman Leonhard Euler, concerning the issue of fitting a cubic to given points in the plane. He states two “facts” about cubics: (1) Any nine distinct points determine a unique cubic. (2) Two cubics can intersect in nine points. Cramer points out that these two statements are incompatible. If we consider two specific cubics that intersect in nine points (such as  $x^3 - x = 0$  and  $y^3 - y = 0$ ), then there is more than one cubic through these nine points, contradicting the first “fact.” Something is terribly wrong here, and Cramer asks Euler, the greatest mathematician of that time, to resolve this apparent contradiction. (This issue is now known as the Cramer–Euler paradox.)
- Euler worked on the problem for a while and put his answer into an article he submitted in 1747, “Sur une contradiction apparente dans la doctrine des lignes courbes” [*Mémoires de l'Académie des Sciences de Berlin*, 4 (1750): 219–233].
- Using Exercises 46 through 59 as a guide, explain which of the so-called facts stated by Cramer is wrong, thus resolving the paradox.
- 61.** Find all points  $P$  in the plane such that you can fit infinitely many cubics through the points  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 1), P$ .
- 62.** Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ , where  $V$  is contained in  $W$ . Explain why  $\dim(V) \leq \dim(W)$ . (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with  $\mathbb{R}^n$ .)
- 63.** Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ , where  $V$  is contained in  $W$ . In Exercise 62 we learned that  $\dim(V) \leq \dim(W)$ . Show that if  $\dim(V) = \dim(W)$ , then  $V = W$ .
- 64.** Consider a subspace  $V$  of  $\mathbb{R}^n$  with  $\dim(V) = n$ . Explain why  $V = \mathbb{R}^n$ .
- 65.** Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ , with  $V \cap W = \{0\}$ . What is the relationship among  $\dim(V)$ ,  $\dim(W)$ , and  $\dim(V + W)$ ? (For the definition of  $V + W$ , see Exercise 2.50; Exercise 2.51 is helpful.)
- 66.** Two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$  are called *complements* if any vector  $\vec{x}$  in  $\mathbb{R}^n$  can be expressed uniquely as  $\vec{x} = \vec{v} + \vec{w}$ , where  $\vec{v}$  is in  $V$  and  $\vec{w}$  is in  $W$ . Show that  $V$  and  $W$  are complements if (and only if)  $V \cap W = \{0\}$  and  $\dim(V) + \dim(W) = n$ .
- 67.** Consider linearly independent vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  in a subspace  $V$  of  $\mathbb{R}^n$  and vectors  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  that span  $V$ . Show that there is a basis of  $V$  that consists of *all* the  $\vec{v}_i$  and *some* of the  $\vec{w}_j$ . Hint: Find a basis of the image of the matrix
- $$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \cdots & \vec{v}_p & \vec{w}_1 & \cdots & \vec{w}_q \\ | & | & | & | \end{bmatrix}.$$

- 68.** Use Exercise 67 to construct a basis of  $\mathbb{R}^4$  that consists of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix},$$

and some of the vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ , and  $\vec{e}_4$  in  $\mathbb{R}^4$ .

- 69.** Consider two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$ . Show that

$$\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V + W).$$

For the definition of  $V + W$ , see Exercise 2.50.

Hint: Pick a basis  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$  of  $V \cap W$ . Using Exercise 67, construct bases  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  of  $V$  and  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  of  $W$ . Show that  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$  is a basis of  $V + W$ . Demonstrating linear independence is somewhat challenging.

- 70.** Use Exercise 69 to answer the following question: If  $V$  and  $W$  are subspaces of  $\mathbb{R}^{10}$ , with  $\dim(V) = 6$  and  $\dim(W) = 7$ , what are the possible dimensions of  $V \cap W$ ?

In Exercises 71 through 74, we will study the row space of a matrix. The row space of an  $n \times m$  matrix  $A$  is defined as the span of the row vectors of  $A$  (i.e., the set of their linear combinations). For example, the row space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

is the set of all row vectors of the form

$$a[1 \ 2 \ 3 \ 4] + b[1 \ 1 \ 1 \ 1] + c[2 \ 2 \ 2 \ 3].$$

71. Find a basis of the row space of the matrix

$$E = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

72. Consider an  $n \times m$  matrix  $E$  in reduced row-echelon form. Using your work in Exercise 71 as a guide, explain how you can find a basis of the row space of  $E$ . What is the relationship between the dimension of the row space and the rank of  $E$ ?

73. Consider an arbitrary  $n \times m$  matrix  $A$ .

- a. What is the relationship between the row spaces of  $A$  and  $E = \text{rref}(A)$ ? Hint: Examine how the row space is affected by elementary row operations.
- b. What is the relationship between the dimension of the row space of  $A$  and the rank of  $A$ ?

74. Find a basis of the row space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix}.$$

75. Consider an  $n \times n$  matrix  $A$ . Show that there exist scalars  $c_0, c_1, \dots, c_n$  (not all zero) such that the matrix  $c_0I_n + c_1A + c_2A^2 + \dots + c_nA^n$  is noninvertible. Hint: Pick an arbitrary nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$ . Then the  $n+1$  vectors  $\vec{v}, A\vec{v}, A^2\vec{v}, \dots, A^n\vec{v}$  will be linearly dependent. (Much more is true: There are scalars  $c_0, c_1, \dots, c_n$ , not all zero, such that  $c_0I_n + c_1A + c_2A^2 + \dots + c_nA^n = 0$ . You are not asked to demonstrate this fact here.)

76. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Find scalars  $c_0, c_1, c_2$  (not all zero) such that the matrix  $c_0I_2 + c_1A + c_2A^2$  is noninvertible. See Exercise 75.

77. Consider an  $n \times m$  matrix  $A$ . Show that the rank of  $A$  is  $n$  if (and only if)  $A$  has an invertible  $n \times n$  submatrix (i.e., a matrix obtained by deleting  $m-n$  columns of  $A$ ).

78. An  $n \times n$  matrix  $A$  is called *nilpotent* if  $A^m = 0$  for some positive integer  $m$ . Examples are triangular

matrices whose entries on the diagonal are all 0. Consider a nilpotent  $n \times n$  matrix  $A$ , and choose the smallest number  $m$  such that  $A^m = 0$ . Pick a vector  $\vec{v}$  in  $\mathbb{R}^n$  such that  $A^{m-1}\vec{v} \neq \vec{0}$ . Show that the vectors  $\vec{v}, A\vec{v}, A^2\vec{v}, \dots, A^{m-1}\vec{v}$  are linearly independent. Hint: Consider a relation  $c_0\vec{v} + c_1A\vec{v} + c_2A^2\vec{v} + \dots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ . Multiply both sides of the equation with  $A^{m-1}$  to show that  $c_0 = 0$ . Next, show that  $c_1 = 0$ , and so on.

79. Consider a nilpotent  $n \times n$  matrix  $A$ . Use the result demonstrated in Exercise 78 to show that  $A^n = 0$ .

80. Explain why you need at least  $m$  vectors to span a space of dimension  $m$ . See Theorem 3.4b.

81. Prove Theorem 3.4d: If  $m$  vectors span an  $m$ -dimensional space, they form a basis of the space.

82. If a  $3 \times 3$  matrix  $A$  represents the projection onto a plane in  $\mathbb{R}^3$ , what is  $\text{rank}(A)$ ?

83. Consider a  $4 \times 2$  matrix  $A$  and a  $2 \times 5$  matrix  $B$ .

- a. What are the possible dimensions of the *kernel* of  $AB$ ?
- b. What are the possible dimensions of the *image* of  $AB$ ?

84. Consider two  $n \times m$  matrices  $A$  and  $B$ . What can you say about the relationship among the quantities  $\text{rank}(A)$ ,  $\text{rank}(B)$ , and  $\text{rank}(A+B)$ ?

85. Consider an  $n \times p$  matrix  $A$  and a  $p \times m$  matrix  $B$ .

- a. What can you say about the relationship between  $\text{rank}(A)$  and  $\text{rank}(AB)$ ?
- b. What can you say about the relationship between  $\text{rank}(B)$  and  $\text{rank}(AB)$ ?

86. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Show that the kernels of matrices  $A$  and  $B$  are different.

Hint: Think about ways to write the fifth column as a linear combination of the preceding columns.

87. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Show that the kernels of matrices  $A$  and  $B$  are different.  
*Hint:* Think about ways to write the fifth column as a linear combination of the preceding columns.

88. Let  $A$  and  $B$  be two matrices of the same size, with  $A \neq B$ , both in reduced row-echelon form. Show that  $\ker(A) \neq \ker(B)$ . *Hint:* Focus on the first column in which the two matrices differ, say, the  $k$ th columns  $\vec{a}_k$  and  $\vec{b}_k$  of  $A$  and  $B$ , respectively. Explain why at least one of the columns  $\vec{a}_k$  and  $\vec{b}_k$  fails to contain a leading 1. Thus, reversing the roles of matrices  $A$  and  $B$  if necessary, we can assume that  $\vec{a}_k$  does not contain a

leading 1. We can write  $\vec{a}_k$  as a linear combination of preceding columns and use this representation to construct a vector in the kernel of  $A$ . Show that this vector fails to be in the kernel of  $B$ . Use Exercises 86 and 87 as a guide.

89. Suppose a matrix  $A$  in reduced row-echelon form can be obtained from a matrix  $M$  by a sequence of elementary row operations. Show that  $A = \text{rref}(M)$ . *Hint:* Both  $A$  and  $\text{rref}(M)$  are in reduced row-echelon form, and they have the same kernel. Exercise 88 is helpful.
90. Consider a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$ . Using a geometric argument, describe the image and the kernel of the linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  given by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

## 4 Coordinates

Coordinates are one of the “great ideas” of mathematics. René Descartes (1596–1650) is credited with having introduced them, in an appendix to his treatise *Discours de la Méthode* (Leyden, 1637). Myth has it that the idea came to him as he was laying on his back in bed one lazy Sunday morning, watching a fly on the ceiling above him. It occurred to him that he could describe the position of the fly by giving its distance from two walls.

Descartes’s countryman Pierre de Fermat (1601–1665) independently developed the basic principles of analytic geometry, at about the same time, but he did not publish his work.

In this section, we will discuss coordinates systematically.

**EXAMPLE I** Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in  $\mathbb{R}^3$ , and define the plane  $V = \text{span}(\vec{v}_1, \vec{v}_2)$  in  $\mathbb{R}^3$ . Is the vector

$$\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

on the plane  $V$ ? Visualize your answer.

### Solution

We have to examine whether there exist scalars  $c_1$  and  $c_2$  such that  $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ . This problem amounts to solving the linear system with augmented matrix

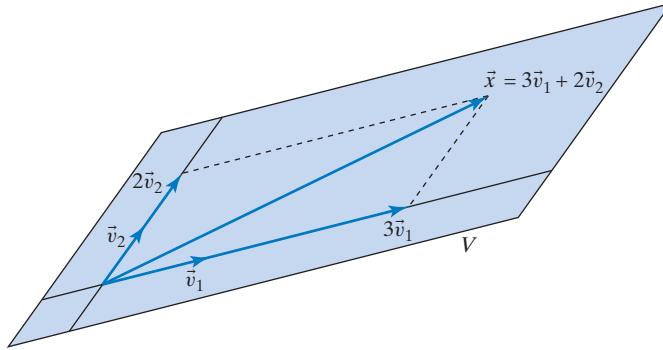
$$M = \left[ \begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array} \right], \quad \text{and} \quad \text{rref}(M) = \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right].$$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

This system is consistent, with the unique solution  $c_1 = 3$  and  $c_2 = 2$ , so that

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = 3\vec{v}_1 + 2\vec{v}_2.$$

In Figure 1, we represent this solution geometrically. It turns out that the vector is indeed on the plane  $V$ .

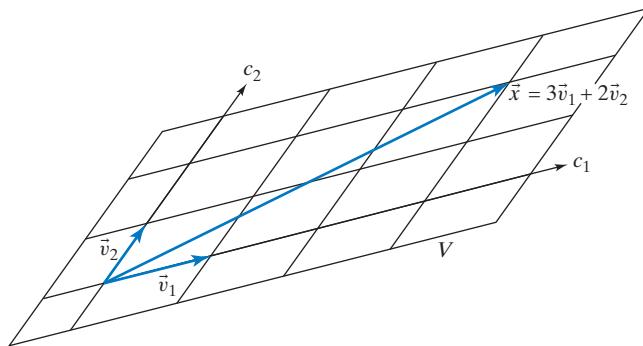


**Figure 1**

To visualize the coefficients 3 and 2 in the linear combination  $\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$ , it is suggestive to introduce a *coordinate grid* on the plane  $V$ , with the axes pointing in the directions of the vectors  $\vec{v}_1$  and  $\vec{v}_2$ , as in Figure 2, where we label the axes  $c_1$  and  $c_2$ . In this grid, our vector  $\vec{x}$  has the coordinates  $c_1 = 3$  and  $c_2 = 2$ . The *coordinate vector* of  $\vec{v} = 3\vec{v}_1 + 2\vec{v}_2$  in this coordinate system is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can think of  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$  as the *address* of  $\vec{x}$  in the  $c_1$ - $c_2$  coordinate system. By introducing  $c_1$ - $c_2$  coordinates in  $V$ , we transform the plane  $V$  into  $\mathbb{R}^2$ .



**Figure 2**

Don't be alarmed by the fact that the axes aren't perpendicular; Cartesian coordinates work just as well with oblique axes.

The following notation can be helpful when discussing coordinates, although it is a bit heavy. Let's denote the basis  $\vec{v}_1, \vec{v}_2$  of  $V$  by  $\mathfrak{B}$ . Then the coordinate vector of  $\vec{x}$  with respect to  $\mathfrak{B}$  is denoted by  $[\vec{x}]_{\mathfrak{B}}$ .

It is customary to denote bases with capital letters in the Fraktur typeface:

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

$$\text{If } \vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = 3\vec{v}_1 + 2\vec{v}_2, \text{ then } [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \quad \blacksquare$$

Let's generalize the ideas introduced in Example 1.

#### Definition 4.1

#### Coordinates in a subspace of $\mathbb{R}^n$

Consider a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$  of a subspace  $V$  of  $\mathbb{R}^n$ . By Theorem 2.10, any vector  $\vec{x}$  in  $V$  can be written uniquely as

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m.$$

The scalars  $c_1, c_2, \dots, c_m$  are called the  $\mathfrak{B}$ -coordinates of  $\vec{x}$ , and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

is the  $\mathfrak{B}$ -coordinate vector of  $\vec{x}$ , denoted by  $[\vec{x}]_{\mathfrak{B}}$ . Thus,

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{means that } \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m.$$

Note that

$$\vec{x} = S [\vec{x}]_{\mathfrak{B}}, \quad \text{where } S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}, \text{ an } n \times m \text{ matrix.}$$

The last equation,  $\vec{x} = S [\vec{x}]_{\mathfrak{B}}$ , follows directly from the definition of coordinates:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = S [\vec{x}]_{\mathfrak{B}}.$$

In Example 1 we considered the case where

$$\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}, \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

You can verify that

$$\vec{x} = S [\vec{x}]_{\mathfrak{B}}, \quad \text{or} \quad \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

It turns out that coordinates have some important linearity properties:

### Theorem 4.2

#### Linearity of Coordinates

If  $\mathfrak{B}$  is a basis of a subspace  $V$  of  $\mathbb{R}^n$ , then

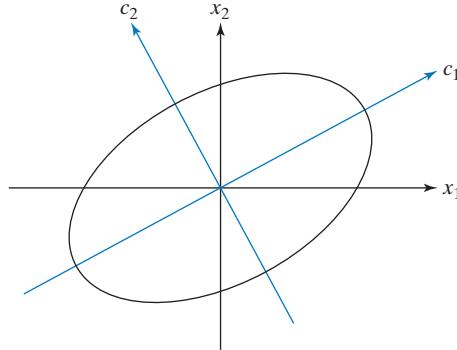
- a.  $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}}$ , for all vectors  $\vec{x}$  and  $\vec{y}$  in  $V$ , and
- b.  $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}}$ , for all  $\vec{x}$  in  $V$  and for all scalars  $k$ .

**Proof** We will prove property (b) and leave part (a) as Exercise 51. Let  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ . If  $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$ , then  $k\vec{x} = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \dots + kc_m\vec{v}_m$ , so that

$$[k\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} kc_1 \\ kc_2 \\ \vdots \\ kc_m \end{bmatrix} = k \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = k[\vec{x}]_{\mathfrak{B}},$$

as claimed. ■

As an important special case of Definition 4.1, consider the case when  $V$  is all of  $\mathbb{R}^n$ . It is often useful to work with bases of  $\mathbb{R}^n$  other than the standard basis,  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ . When dealing with the ellipse in Figure 3, for example, the  $c_1$ - $c_2$ -axes aligned with the principal axes may be preferable to the standard  $x_1$ - $x_2$ -axes.



**Figure 3**

**EXAMPLE 2** Consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of vectors  $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ .

- a. If  $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ , find  $[\vec{x}]_{\mathfrak{B}}$ .
- b. If  $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ , find  $\vec{y}$ .

#### Solution

- a. To find the  $\mathfrak{B}$ -coordinates of vector  $\vec{x}$ , we write  $\vec{x}$  as a linear combination of the basis vectors:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \quad \text{or} \quad \begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

The solution is  $c_1 = 4$ ,  $c_2 = 2$ , so that  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ .

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

Alternatively, we can solve the equation  $\vec{x} = S [\vec{x}]_{\mathfrak{B}}$  for  $[\vec{x}]_{\mathfrak{B}}$ :

$$[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

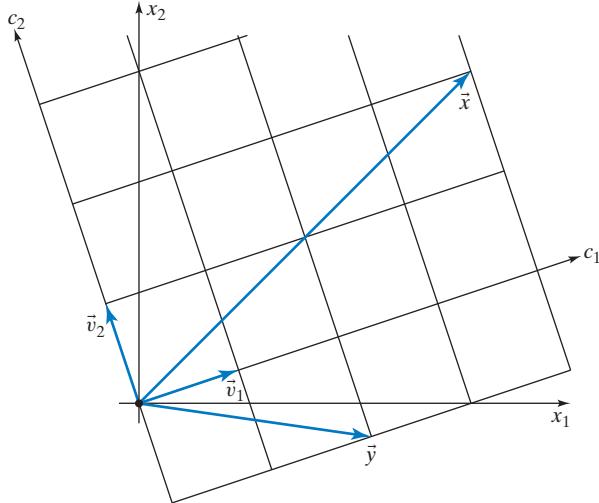
b. By definition of coordinates,  $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  means that

$$\vec{y} = 2\vec{v}_1 + (-1)\vec{v}_2 = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

Alternatively, use the formula

$$\vec{y} = S [\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

These results are illustrated in Figure 4. ■



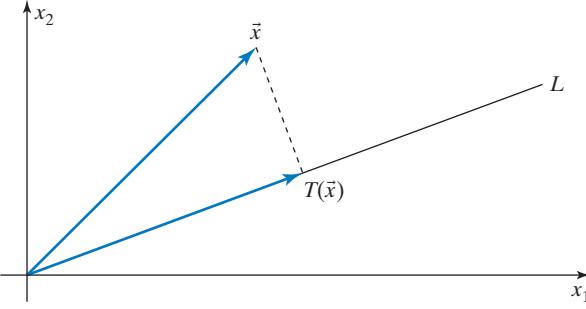
**Figure 4**

We will now go a step further and see how we can express a linear transformation in coordinates.

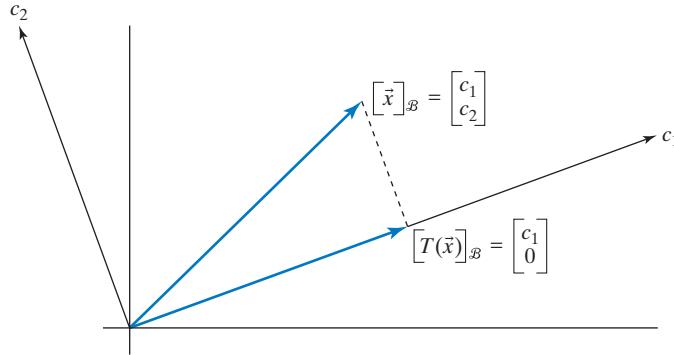
**EXAMPLE 3** Let  $L$  be the line in  $\mathbb{R}^2$  spanned by vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Let  $T$  be the linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that projects any vector  $\vec{x}$  orthogonally onto line  $L$ , as shown in Figure 5. We can facilitate the study of  $T$  by introducing a coordinate system where  $L$  is one of the axes (say, the  $c_1$ -axis), with the  $c_2$ -axis perpendicular to  $L$ , as illustrated in Figure 6. If we use this coordinate system, then  $T$  transforms  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  into  $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ . In  $c_1-c_2$  coordinates,  $T$  is represented by the matrix  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , since

$$\begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions



**Figure 5**



**Figure 6**

Let's make these ideas more precise. We start by introducing a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  of  $\mathbb{R}^2$  with vector  $\vec{v}_1$  on line  $L$  and vector  $\vec{v}_2$  perpendicular to  $L$ , for example,

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

If  $\vec{x} = \underbrace{c_1 \vec{v}_1}_{\text{in } L} + \underbrace{c_2 \vec{v}_2}_{\text{in } L^\perp}$ , then  $T(\vec{x}) = \text{proj}_L(\vec{x}) = c_1 \vec{v}_1$ . Equivalently, if

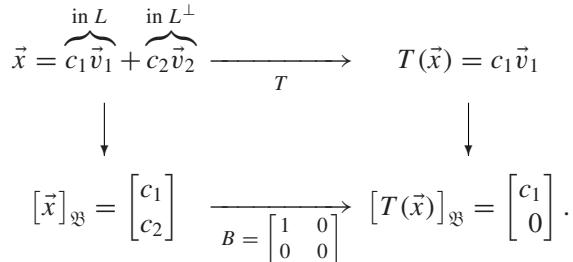
$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \text{then} \quad [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix};$$

see Figure 6.

The matrix  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  that transforms  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  into  $[T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$  is called the  $\mathfrak{B}$ -matrix of  $T$ :

$$[T(\vec{x})]_{\mathfrak{B}} = B [\vec{x}]_{\mathfrak{B}}.$$

We can organize our work in a diagram as follows:



## Subspaces of $\mathbb{R}^n$ and Their Dimensions

When setting up such a diagram, we begin in the top left by writing an arbitrary input vector  $\vec{x}$  as a linear combination of the vectors in the given basis  $\mathfrak{B}$ . In the top right we have  $T(\vec{x})$ , again written as a linear combination of the vectors of basis  $\mathfrak{B}$ . The corresponding entries below are the coordinate vectors  $[\vec{x}]_{\mathfrak{B}}$  and  $[T(\vec{x})]_{\mathfrak{B}}$ . Finding those is a routine step that requires no computational work, since  $\vec{x}$  and  $T(\vec{x})$  have been written as linear combinations of the basis vectors already. Finally, we find the matrix  $B$  that transforms  $[\vec{x}]_{\mathfrak{B}}$  into  $[T(\vec{x})]_{\mathfrak{B}}$ ; this is again a routine step. ■

Let's generalize the ideas of Example 3.

### Theorem 4.3

#### The matrix of a linear transformation

Consider a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  and a basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$  of  $\mathbb{R}^n$ . Then there exists a unique  $n \times n$  matrix  $B$  that transforms  $[\vec{x}]_{\mathfrak{B}}$  into  $[T(\vec{x})]_{\mathfrak{B}}$ :

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ . This matrix  $B$  is called the  $\mathfrak{B}$ -matrix of  $T$ . We can construct  $B$  column by column, as follows:

$$B = \begin{bmatrix} | & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & | \end{bmatrix}.$$

**Proof** If we write  $\vec{x}$  in  $\mathfrak{B}$ -coordinates,  $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ , then we have

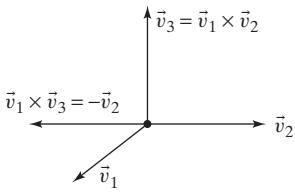
$$\begin{aligned} [T(\vec{x})]_{\mathfrak{B}} &= [T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)]_{\mathfrak{B}} \\ &\stackrel{\text{step 2}}{=} [\underbrace{c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)}_{\text{step 3}}]_{\mathfrak{B}} \\ &\stackrel{\text{step 3}}{=} c_1[T(\vec{v}_1)]_{\mathfrak{B}} + \dots + c_n[T(\vec{v}_n)]_{\mathfrak{B}} \\ &= \begin{bmatrix} | & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} | & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & | \end{bmatrix}}_B [\vec{x}]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}, \end{aligned}$$

as claimed. In step 2 we are using the linearity of  $T$ , and step 3 follows from the linearity of coordinates, Theorem 4.2. ■

### EXAMPLE 4

Consider two perpendicular unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^3$ . Form the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$  of  $\mathbb{R}^3$ , where  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ . Note that  $\vec{v}_3$  is perpendicular to both  $\vec{v}_1$  and  $\vec{v}_2$ , and  $\vec{v}_3$  is a unit vector, since  $\|\vec{v}_3\| = \|\vec{v}_1 \times \vec{v}_2\| = \|\vec{v}_1\| \|\vec{v}_2\| \sin(\pi/2) = 1 \cdot 1 \cdot 1 = 1$ .

- a. Draw a sketch to find  $\vec{v}_1 \times \vec{v}_3$ .
- b. Find the  $\mathfrak{B}$ -matrix  $B$  of the linear transformation  $T(\vec{x}) = \vec{v}_1 \times \vec{x}$ .

**Solution**

**Figure 7**

a. Note first that  $\vec{v}_1 \times \vec{v}_3$  is a unit vector.

Figure 7 illustrates that  $\vec{v}_1 \times \vec{v}_3 = -\vec{v}_2$ .

b. We will organize our work in a diagram, as in Example 3.

$$\begin{array}{ccc}
 \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 & \xrightarrow{T} & T(\vec{x}) = \vec{v}_1 \times (c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3) \\
 \downarrow & & = c_1(\vec{v}_1 \times \vec{v}_1) + c_2(\vec{v}_1 \times \vec{v}_2) + c_3(\vec{v}_1 \times \vec{v}_3) \\
 \left[ \begin{matrix} \vec{x} \\ \end{matrix} \right]_{\mathfrak{B}} = \left[ \begin{matrix} c_1 \\ c_2 \\ c_3 \\ \end{matrix} \right] & \xrightarrow{B} & \left[ \begin{matrix} T(\vec{x}) \\ \end{matrix} \right]_{\mathfrak{B}} = \left[ \begin{matrix} 0 \\ -c_3 \\ c_2 \\ \end{matrix} \right] \\
 B = \left[ \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ \end{matrix} \right]
 \end{array}$$

Alternatively, we can construct  $B$  column by column,

$$B = \left[ \begin{matrix} | & | & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & [T(\vec{v}_3)]_{\mathfrak{B}} \\ | & | & | \end{matrix} \right].$$

We have  $T(\vec{v}_1) = \vec{v}_1 \times \vec{v}_1 = \vec{0}$ ,  $T(\vec{v}_2) = \vec{v}_1 \times \vec{v}_2 = \vec{v}_3$ , and  $T(\vec{v}_3) = \vec{v}_1 \times \vec{v}_3 = -\vec{v}_2$ , so that

$$\begin{array}{ccc}
 [T(\vec{v}_1)]_{\mathfrak{B}} = \left[ \begin{matrix} 0 \\ 0 \\ 0 \\ \end{matrix} \right], & [T(\vec{v}_2)]_{\mathfrak{B}} = \left[ \begin{matrix} 0 \\ 0 \\ 1 \\ \end{matrix} \right], & [T(\vec{v}_3)]_{\mathfrak{B}} = \left[ \begin{matrix} 0 \\ -1 \\ 0 \\ \end{matrix} \right] \\
 & \searrow & \downarrow \\
 B = \left[ \begin{matrix} | & | & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & [T(\vec{v}_3)]_{\mathfrak{B}} \\ | & | & | \end{matrix} \right] = \left[ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ \end{matrix} \right]
 \end{array}$$

Note the block  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  in matrix  $B$ , representing a rotation through  $\frac{\pi}{2}$ .

Writing  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , we can interpret transformation  $T$  geometrically. It is the orthogonal projection onto the  $c_2$ - $c_3$ -plane followed by a rotation through  $\frac{\pi}{2}$  about the  $c_1$ -axis, counter-clockwise as viewed from the positive  $c_1$ -axis. ■

In Example 4, when finding the matrix  $B$  of the linear transformation  $T$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ , it helps to write the basis vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  next to the rows and the values  $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$  above the columns:

$$B = \begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) & T(\vec{v}_3) \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{matrix}.$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

This reminds us that the components of the  $j$ th column vector of  $B$  are the coordinates of  $T(\vec{v}_j)$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ , meaning that  $T(\vec{v}_1) = \vec{0}$ ,  $T(\vec{v}_2) = 1\vec{v}_3$ , and  $T(\vec{v}_3) = (-1)\vec{v}_2$ .

#### EXAMPLE 5

As in Example 3, let  $T$  be the linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that projects any vector orthogonally onto the line  $L$  spanned by the vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . In Example 3, we found that the matrix of  $B$  of  $T$  with respect to the basis  $\mathfrak{B} = \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$  is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

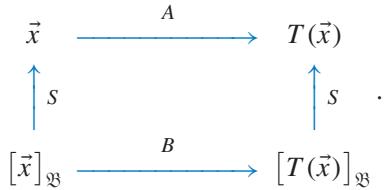
What is the relationship between  $B$  and the *standard matrix*  $A$  of  $T$  [such that  $T(\vec{x}) = A\vec{x}$ ]? We can think of  $A$  as the matrix of  $T$  with respect to the standard basis  $\mathfrak{A} = (\vec{e}_1, \vec{e}_2)$  of  $\mathbb{R}^2$ , in the sense of Definition 4.3. (Think about it!)

#### Solution

Recall from Definition 4.1 that

$$\vec{x} = S \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}, \quad \text{where } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix},$$

and consider the following diagram:



Note that  $T(\vec{x}) = AS \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = SB \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$  for all  $\vec{x}$  in  $\mathbb{R}^2$ , so that

$$AS = SB, \quad B = S^{-1}AS, \quad \text{and} \quad A = SBS^{-1}.$$

We can use the last formula to find the standard matrix  $A$  of  $T$ :

$$A = SBS^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left( \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \right) = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}.$$

■

#### Theorem 4.4

#### Standard matrix versus $\mathfrak{B}$ -matrix

Consider a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  and a basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$  of  $\mathbb{R}^n$ . Let  $B$  be the  $\mathfrak{B}$ -matrix of  $T$ , and let  $A$  be the standard matrix of  $T$  [such that  $T(\vec{x}) = A\vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^n$ ]. Then

$$AS = SB, \quad B = S^{-1}AS, \quad \text{and} \quad A = SBS^{-1}, \quad \text{where} \quad S = \begin{bmatrix} | & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & | \end{bmatrix}.$$

The formulas in Theorem 4.4 motivate the following definition.

**Definition 4.5**
**Similar matrices**

Consider two  $n \times n$  matrices  $A$  and  $B$ . We say that  $A$  is similar to  $B$  if there exists an invertible matrix  $S$  such that

$$AS = SB, \quad \text{or} \quad B = S^{-1}AS.$$

Thus, two matrices are similar if they represent the same linear transformation with respect to different bases.

**EXAMPLE 6** Is matrix  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$  similar to  $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ ?

**Solution**

At this early stage of the course, we have to tackle this problem with “brute force,” using Definition 4.5.

We are looking for a matrix  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $AS = SB$ , or

$$\begin{bmatrix} x + 2z & y + 2t \\ 4x + 3z & 4y + 3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}.$$

These equations simplify to

$$z = 2x, \quad t = -y,$$

so that any invertible matrix of the form

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

does the job. Note that  $\det(S) = -3xy$ . By the theorem of inverse and determinant of a  $2 \times 2$  matrix  $S$  is invertible if  $\det(S) = -3xy \neq 0$ , meaning that neither  $x$  nor  $y$  is zero. For example, we can let  $x = y = 1$ , so that  $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ .

Matrix  $A$  turns out to be similar to  $B$ . ■

**EXAMPLE 7**

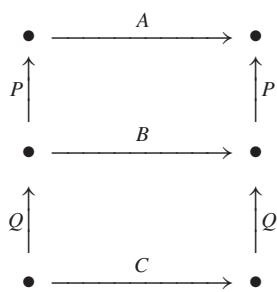
Show that if matrix  $A$  is similar to  $B$ , then its power  $A^t$  is similar to  $B^t$ , for all positive integers  $t$ . (That is,  $A^2$  is similar to  $B^2$ ,  $A^3$  is similar to  $B^3$ , and so on.)

**Solution**

We know that  $B = S^{-1}AS$  for some invertible matrix  $S$ . Now

$$B^t = \underbrace{(S^{-1}AS)(S^{-1}AS) \cdots (S^{-1}AS)(S^{-1}AS)}_{t \text{ times}} = S^{-1}A^tS,$$

proving our claim. Note the cancellation of many terms of the form  $SS^{-1}$ . ■



**Figure 8**

Here are some noteworthy facts about similar matrices.

**Theorem 4.6**

**Similarity is an equivalence relation**

- a. An  $n \times n$  matrix  $A$  is similar to  $A$  itself (*reflexivity*).
- b. If  $A$  is similar to  $B$ , then  $B$  is similar to  $A$  (*symmetry*).
- c. If  $A$  is similar to  $B$  and  $B$  is similar to  $C$ , then  $A$  is similar to  $C$  (*transitivity*).

**Proof** We will prove transitivity, leaving reflexivity and symmetry as Exercise 65.

The assumptions of part (c) mean that there exist invertible matrices  $P$  and  $Q$  such that  $AP = PB$  and  $BQ = QC$ . Using Figure 8 as a guide, we find that  $APQ = PBQ = PQC$ . We see that  $AS = SC$ , where  $S = PQ$  is invertible, proving that  $A$  is similar to  $C$ . ■

Problems in geometry, physics, or engineering can often be simplified by choosing a coordinate system that is “well adjusted” to the situation at hand. Take another look at Example 3.

**EXAMPLE 8**

Given a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , we will often be interested in finding a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T$  turns out to be “simple” in some sense. Ideally, we might want  $B$  to be a *diagonal* matrix  $B = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ , as in Example 3, where we found  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . What is required of  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  to make the  $\mathfrak{B}$ -matrix  $B$  of  $T$  diagonal? Let’s write auxiliary vectors as discussed after Example 4:

$$B = \begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) \\ c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}.$$

We see that the matrix  $B$  is diagonal,  $B = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ , if and only if  $T(\vec{v}_1) = c_1 \vec{v}_1$  and  $T(\vec{v}_2) = c_2 \vec{v}_2$ , meaning that  $T(\vec{v}_1)$  is a *scalar multiple* of  $\vec{v}_1$ , and  $T(\vec{v}_2)$  is a *scalar multiple* of  $\vec{v}_2$ . From a geometrical point of view, this means that  $T(\vec{v}_j)$  is *parallel* to  $\vec{v}_j$ , for  $j = 1$  and 2. ■

We can generalize.

**Theorem 4.7**

**When is the  $\mathfrak{B}$ -matrix of  $T$  diagonal?**

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Let  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$  be a basis of  $\mathbb{R}^n$ .

Then the  $\mathfrak{B}$ -matrix  $B$  of  $T$  is *diagonal* if and only if  $T(\vec{v}_1) = c_1 \vec{v}_1, \dots, T(\vec{v}_n) = c_n \vec{v}_n$  for some scalars  $c_1, \dots, c_n$ .

From a geometrical point of view, this means that  $T(\vec{v}_j)$  is *parallel* to  $\vec{v}_j$  for all  $j = 1, \dots, n$ .

The following diagram, generalizing Example 8, illustrates this result:

$$B = \begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) & \dots & T(\vec{v}_n) \\ c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}.$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

A basis  $\vec{v}_1, \dots, \vec{v}_n$  such that  $T(\vec{v}_j)$  is parallel to  $\vec{v}_j$  can often be found geometrically (see Example 9 and Exercises 37 through 42) or algebraically. See Exercise 82.

**EXAMPLE 9** Let the linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  be the reflection about a line  $L$  through the origin. See Figure 9. Find a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T$  is diagonal.

#### Solution

According to Example 8 or Theorem 4.7, we need to find a basis  $\vec{v}_1, \vec{v}_2$  of  $\mathbb{R}^2$  such that  $T(\vec{v}_1) = c_1\vec{v}_1$  and  $T(\vec{v}_2) = c_2\vec{v}_2$  for some scalars  $c_1$  and  $c_2$ . Thinking about this problem from a geometrical point of view, we realize that we can choose a nonzero vector  $\vec{v}_1$  parallel to  $L$ , with  $T(\vec{v}_1) = \vec{v}_1 = 1\vec{v}_1$ , and a nonzero vector  $\vec{v}_2$  perpendicular to  $L$ , with  $T(\vec{v}_2) = -\vec{v}_2 = (-1)\vec{v}_2$ . See Figure 10. Then the matrix of  $T$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  will be

$$B = \begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{matrix} \vec{v}_1 \\ \vec{v}_2 \end{matrix},$$

a diagonal matrix as required. ■

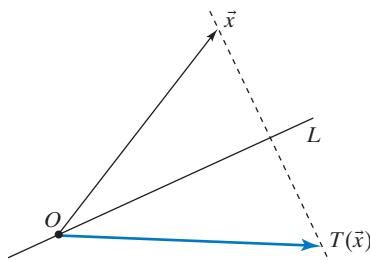


Figure 9

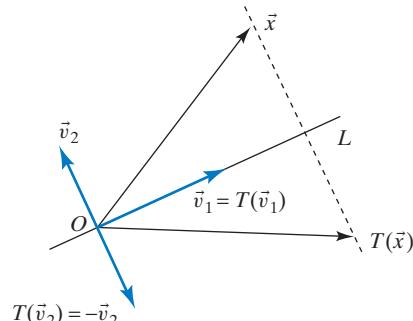


Figure 10

**EXAMPLE 10** Consider the matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , representing the rotation through an angle of  $\theta = \pi/2$  in  $\mathbb{R}^2$ . See Figure 11. Find a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix  $B$  of the transformation  $T(\vec{x}) = A\vec{x}$  is diagonal, or explain why no such basis exists.

#### Solution

According to Example 8 or Theorem 4.7, we need to make an attempt to find a basis  $\vec{v}_1, \vec{v}_2$  of  $\mathbb{R}^2$  such that  $T(\vec{v}_1) = c_1\vec{v}_1$  and  $T(\vec{v}_2) = c_2\vec{v}_2$  for some scalars  $c_1$  and  $c_2$ . If  $\vec{x}$  is any nonzero vector, then  $T(\vec{x})$  is perpendicular to  $\vec{x}$ , so that  $T(\vec{x})$  fails to be parallel to  $\vec{x}$ . Thus, it is impossible to find a basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  such that  $T(\vec{v}_j)$  is a scalar multiple of  $\vec{v}_j$  for  $j = 1$  and 2. A basis  $\mathfrak{B}$  with the required property fails to exist. ■

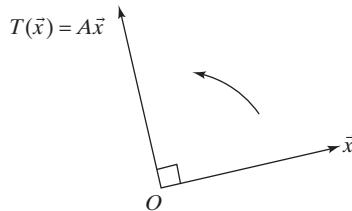


Figure 11

## EXERCISES 4

**GOAL** Use the concept of coordinates. Apply the definition of the matrix of a linear transformation with respect to a basis. Relate this matrix to the standard matrix of the transformation. Find the matrix of a linear transformation (with respect to any basis) column by column. Use the concept of similarity.

In Exercises 1 through 18, determine whether the vector  $\vec{x}$  is in the span  $V$  of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  (proceed “by inspection” if possible, and use the reduced row-echelon form if necessary). If  $\vec{x}$  is in  $V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ .

$$1. \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$2. \vec{x} = \begin{bmatrix} 23 \\ 29 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 46 \\ 58 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 61 \\ 67 \end{bmatrix}$$

$$3. \vec{x} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 23 \\ 29 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 31 \\ 37 \end{bmatrix}$$

$$4. \vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$5. \vec{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$6. \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$7. \vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$8. \vec{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$9. \vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$10. \vec{x} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$11. \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

$$12. \vec{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$13. \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$14. \vec{x} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$15. \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$16. \vec{x} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$17. \vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$18. \vec{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

In Exercises 19 through 24, find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ . For practice, solve each problem in three ways: (a) Use the formula  $B = S^{-1}AS$ , (b) use a commutative diagram (as in Examples 3 and 4), and (c) construct  $B$  “column by column.”

$$19. A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$20. A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$21. A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$22. A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$23. A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$24. A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

In Exercises 25 through 30, find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ .

25.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

26.  $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

27.  $A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix};$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

28.  $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix};$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

29.  $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix};$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

30.  $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix};$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Let  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$  be any basis of  $\mathbb{R}^3$  consisting of perpendicular unit vectors, such that  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ . In Exercises 31 through 36, find the  $\mathfrak{B}$ -matrix  $B$  of the given linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ . Interpret  $T$  geometrically.

31.  $T(\vec{x}) = \vec{v}_2 \times \vec{x}$

32.  $T(\vec{x}) = \vec{x} \times \vec{v}_3$

33.  $T(\vec{x}) = (\vec{v}_2 \cdot \vec{x})\vec{v}_2$

34.  $T(\vec{x}) = \vec{x} - 2(\vec{v}_3 \cdot \vec{x})\vec{v}_3$

35.  $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_2$

36.  $T(\vec{x}) = \vec{v}_1 \times \vec{x} + (\vec{v}_1 \cdot \vec{x})\vec{v}_1$

In Exercises 37 through 42, find a basis  $\mathfrak{B}$  of  $\mathbb{R}^n$  such that the  $\mathfrak{B}$ -matrix  $B$  of the given linear transformation  $T$  is diagonal.

37. Orthogonal projection  $T$  onto the line in  $\mathbb{R}^2$  spanned by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

38. Reflection  $T$  about the line in  $\mathbb{R}^2$  spanned by

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

39. Reflection  $T$  about the line in  $\mathbb{R}^3$  spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

40. Orthogonal projection  $T$  onto the line in  $\mathbb{R}^3$  spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

41. Orthogonal projection  $T$  onto the plane  $3x_1 + x_2 + 2x_3 = 0$  in  $\mathbb{R}^3$

42. Reflection  $T$  about the plane  $x_1 - 2x_2 + 2x_3 = 0$  in  $\mathbb{R}^3$

43. Consider the plane  $x_1 + 2x_2 + x_3 = 0$  with basis  $\mathfrak{B}$  consisting of vectors

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \text{ If } [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \text{ find } \vec{x}.$$

44. Consider the plane  $2x_1 - 3x_2 + 4x_3 = 0$  with basis  $\mathfrak{B}$  consisting of vectors

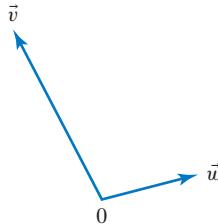
$$\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}. \text{ If } [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{ find } \vec{x}.$$

45. Consider the plane  $2x_1 - 3x_2 + 4x_3 = 0$ . Find a basis  $\mathfrak{B}$  of this plane such that  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  for  $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ .

46. Consider the plane  $x_1 + 2x_2 + x_3 = 0$ . Find a basis  $\mathfrak{B}$  of this plane such that  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  for  $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ .

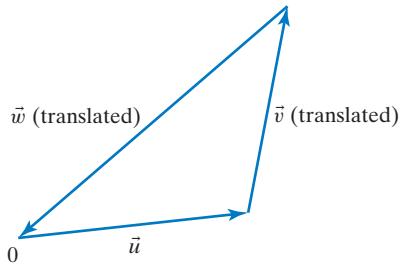
47. Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . We are told that the matrix of  $T$  with respect to the basis  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Find the standard matrix of  $T$  in terms of  $a, b, c$ , and  $d$ .

48. In the accompanying figure, sketch the vector  $\vec{x}$  with  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , where  $\mathfrak{B}$  is the basis of  $\mathbb{R}^2$  consisting of the vectors  $\vec{v}, \vec{w}$ .

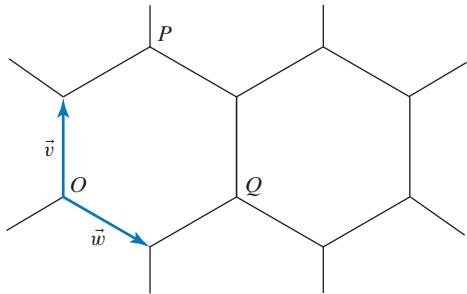


## Subspaces of $\mathbb{R}^n$ and Their Dimensions

49. Consider the vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  sketched in the accompanying figure. Find the coordinate vector of  $\vec{w}$  with respect to the basis  $\vec{u}$ ,  $\vec{v}$ .



50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of the vectors  $\vec{v}$ ,  $\vec{w}$  in the following sketch:



- a. Find the coordinate vectors  $[\overrightarrow{OP}]_{\mathfrak{B}}$  and  $[\overrightarrow{OQ}]_{\mathfrak{B}}$ .  
*Hint:* Sketch the coordinate grid defined by the basis  $\mathfrak{B} = (\vec{v}, \vec{w})$ .
- b. We are told that  $[\overrightarrow{OR}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ . Sketch the point R. Is R a vertex or a center of a tile?
- c. We are told that  $[\overrightarrow{OS}]_{\mathfrak{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$ . Is S a center or a vertex of a tile?
51. Prove part (a) of Theorem 4.2.

52. If  $\mathfrak{B}$  is a basis of  $\mathbb{R}^n$ , is the transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  given by

$$T(\vec{x}) = [\vec{x}]_{\mathfrak{B}}$$

linear? Justify your answer.

53. Consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of the vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . We are told that  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$  for a certain vector  $\vec{x}$  in  $\mathbb{R}^2$ . Find  $\vec{x}$ .
54. Let  $\mathfrak{B}$  be the basis of  $\mathbb{R}^n$  consisting of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ , and let  $\mathfrak{T}$  be some other basis of  $\mathbb{R}^n$ . Is

$$[\vec{v}_1]_{\mathfrak{T}}, [\vec{v}_2]_{\mathfrak{T}}, \dots, [\vec{v}_n]_{\mathfrak{T}}$$

a basis of  $\mathbb{R}^n$  as well? Explain.

55. Consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of the vectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and let  $\mathfrak{N}$  be the basis consisting of  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . Find a matrix  $P$  such that

$$[\vec{x}]_{\mathfrak{N}} = P [\vec{x}]_{\mathfrak{B}},$$

for all  $\vec{x}$  in  $\mathbb{R}^2$ .

56. Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

57. Show that if a  $3 \times 3$  matrix  $A$  represents the reflection about a plane, then  $A$  is similar to the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

58. Consider a  $3 \times 3$  matrix  $A$  and a vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $A^3\vec{v} = \vec{0}$ , but  $A^2\vec{v} \neq \vec{0}$ .

- a. Show that the vectors  $A^2\vec{v}$ ,  $A\vec{v}$ ,  $\vec{v}$  form a basis of  $\mathbb{R}^3$ . *Hint:* It suffices to show linear independence. Consider a relation  $c_1A^2\vec{v} + c_2A\vec{v} + c_3\vec{v} = \vec{0}$  and multiply by  $A^2$  to show that  $c_3 = 0$ .
- b. Find the matrix of the transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $A^2\vec{v}$ ,  $A\vec{v}$ ,  $\vec{v}$ .

59. Is matrix  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  similar to matrix  $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ ?

60. Is matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  similar to matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ?

61. Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{x} \quad \text{is} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

62. Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x} \quad \text{is} \quad B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

63. Is matrix  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$  similar to matrix  $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$  for all  $p$  and  $q$ ?

64. Is matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  similar to matrix  $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$  for all  $a, b, c, d$ ?

65. Prove parts (a) and (b) of Theorem 4.6.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

66. Consider a matrix  $A$  of the form  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where

$a^2 + b^2 = 1$  and  $a \neq 1$ . Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}$ . Interpret the answer geometrically.

67. If  $c \neq 0$ , find the matrix of the linear transformation  $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$  with respect to basis  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}$ .

68. Find an invertible  $2 \times 2$  matrix  $S$  such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form  $\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$ . See Exercise 67.

69. If  $A$  is a  $2 \times 2$  matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that  $A$  is similar to a diagonal matrix  $D$ . Find an invertible  $S$  such that  $S^{-1}AS = D$ .

70. Is there a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that  $\mathfrak{B}$ -matrix  $B$  of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular? Hint: Think about the first column of  $B$ .

71. Suppose that matrix  $A$  is similar to  $B$ , with  $B = S^{-1}AS$ .

- a. Show that if  $\vec{x}$  is in  $\ker(B)$ , then  $S\vec{x}$  is in  $\ker(A)$ .
- b. Show that  $\text{nullity}(A) = \text{nullity}(B)$ . Hint: If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  is a basis of  $\ker(B)$ , then the vectors  $S\vec{v}_1, S\vec{v}_2, \dots, S\vec{v}_p$  in  $\ker(A)$  are linearly independent. Now reverse the roles of  $A$  and  $B$ .

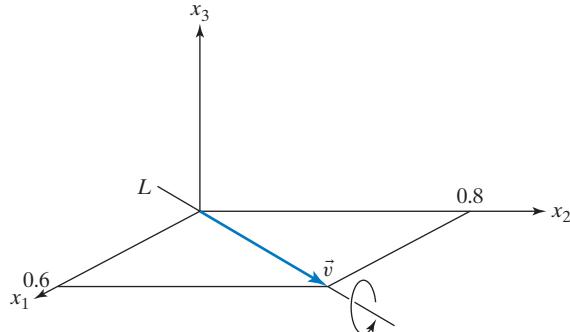
72. If  $A$  is similar to  $B$ , what is the relationship between  $\text{rank}(A)$  and  $\text{rank}(B)$ ? See Exercise 71.

73. Let  $L$  be the line in  $\mathbb{R}^3$  spanned by the vector

$$\vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}.$$

Let  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  be the rotation about this line through an angle of  $\pi/2$ , in the direction indicated in

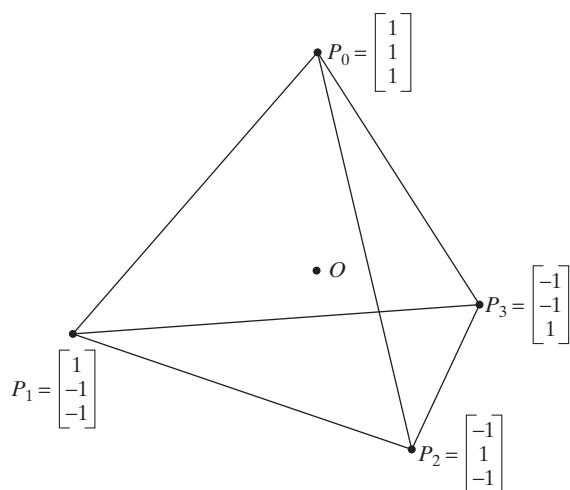
the accompanying sketch. Find the matrix  $A$  such that  $T(\vec{x}) = A\vec{x}$ .



74. Consider the regular tetrahedron in the accompanying sketch whose center is at the origin. Let  $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$  be the position vectors of the four vertices of the tetrahedron:

$$\vec{v}_0 = \overrightarrow{OP_0}, \dots, \vec{v}_3 = \overrightarrow{OP_3}.$$

- a. Find the sum  $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ .
- b. Find the coordinate vector of  $\vec{v}_0$  with respect to the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .
- c. Let  $T$  be the linear transformation with  $T(\vec{v}_0) = \vec{v}_3$ ,  $T(\vec{v}_3) = \vec{v}_1$ , and  $T(\vec{v}_1) = \vec{v}_0$ . What is  $T(\vec{v}_2)$ ? Describe the transformation  $T$  geometrically (as a reflection, rotation, projection, or whatever). Find the matrix  $B$  of  $T$  with respect to the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . What is  $B^3$ ? Explain.



75. Find the matrix  $B$  of the rotation  $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$  with respect to the basis  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ . Interpret your answer geometrically.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

76. If  $t$  is any real number, what is the matrix  $B$  of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \vec{x}$$

with respect to basis  $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ ? Interpret your answer geometrically.

77. Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Let  $B$  be the matrix of  $T$  with respect to the basis  $\vec{e}_n, \vec{e}_{n-1}, \dots, \vec{e}_2, \vec{e}_1$  of  $\mathbb{R}^n$ . Describe the entries of  $B$  in terms of the entries of  $A$ .

78. This problem refers to Leontief's input-output model. Consider three industries  $I_1, I_2, I_3$ , each of which produces only one good, with unit prices  $p_1 = 2, p_2 = 5, p_3 = 10$  (in U.S. dollars), respectively. Let the three products be labeled good 1, good 2, and good 3. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}$$

be the matrix that lists the interindustry demand in terms of dollar amounts. The entry  $a_{ij}$  tells us how many dollars' worth of good  $i$  are required to produce one dollar's worth of good  $j$ . Alternatively, the interindustry demand can be measured in units of goods by means of the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

where  $b_{ij}$  tells us how many units of good  $i$  are required to produce one unit of good  $j$ . Find the matrix  $B$  for the economy discussed here. Also, write an equation relating the three matrices  $A, B$ , and  $S$ , where

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

is the diagonal matrix listing the unit prices on the diagonal. Justify your answer carefully.

79. Consider the matrix  $A = \begin{bmatrix} 11 & -30 \\ 4 & -11 \end{bmatrix}$ . Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T(\vec{x}) = A\vec{x}$  is  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

80. Consider the matrix  $A = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$ . Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T(\vec{x}) = A\vec{x}$  is  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

81. Consider the linear transformation  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_2 + x_3 \end{bmatrix}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

- a. Find all vectors of the form  $\vec{x} = \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix}$  such that  $T(\vec{x})$  is a scalar multiple of  $\vec{x}$ . Be prepared to deal with irrational numbers.  
 b. Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^3$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T$  is diagonal.

82. Consider the linear transformation  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 3x_3 - 2x_2 \end{bmatrix}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

- a. Find all vectors of the form  $\vec{x} = \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix}$  such that  $T(\vec{x})$  is a scalar multiple of  $\vec{x}$ .  
 b. Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^3$  such that the  $\mathfrak{B}$ -matrix  $B$  of  $T$  is diagonal.

## Exercises

### TRUE OR FALSE?

1. If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  and  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$  are any two bases of a subspace  $V$  of  $\mathbb{R}^{10}$ , then  $n$  must equal  $m$ .
2. If  $A$  is a  $5 \times 6$  matrix of rank 4, then the nullity of  $A$  is 1.
3. The image of a  $3 \times 4$  matrix is a subspace of  $\mathbb{R}^4$ .
4. The span of vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  consists of all linear combinations of vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ .

5. If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are linearly independent vectors in  $\mathbb{R}^n$ , then they must form a basis of  $\mathbb{R}^n$ .
6. There exists a  $5 \times 4$  matrix whose image consists of all of  $\mathbb{R}^5$ .
7. The kernel of any invertible matrix consists of the zero vector only.
8. The identity matrix  $I_n$  is similar to all invertible  $n \times n$  matrices.

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

9. If  $2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w}$ , then vectors  $\vec{u}, \vec{v}, \vec{w}$  must be linearly dependent.
10. The column vectors of a  $5 \times 4$  matrix must be linearly dependent.
11. If matrix  $A$  is similar to matrix  $B$ , and  $B$  is similar to  $C$ , then  $C$  must be similar to  $A$ .
12. If a subspace  $V$  of  $\mathbb{R}^n$  contains none of the standard vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ , then  $V$  consists of the zero vector only.
13. If vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are linearly independent, then vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  must be linearly independent as well.
14. The vectors of the form  $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$  (where  $a$  and  $b$  are arbitrary real numbers) form a subspace of  $\mathbb{R}^4$ .
15. Matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .
16. Vectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$  form a basis of  $\mathbb{R}^3$ .
17. If the kernel of a matrix  $A$  consists of the zero vector only, then the column vectors of  $A$  must be linearly independent.
18. If the image of an  $n \times n$  matrix  $A$  is all of  $\mathbb{R}^n$ , then  $A$  must be invertible.
19. If vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  span  $\mathbb{R}^4$ , then  $n$  must be equal to 4.
20. If vectors  $\vec{u}, \vec{v}$ , and  $\vec{w}$  are in a subspace  $V$  of  $\mathbb{R}^n$ , then vector  $2\vec{u} - 3\vec{v} + 4\vec{w}$  must be in  $V$  as well.
21. If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $AB$  must be similar to  $BA$ .
22. If  $A$  is an invertible  $n \times n$  matrix, then the kernels of  $A$  and  $A^{-1}$  must be equal.
23. Matrix  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .
24. Vectors  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$  are linearly independent.
25. If a subspace  $V$  of  $\mathbb{R}^3$  contains the standard vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ , then  $V$  must be  $\mathbb{R}^3$ .
26. If a  $2 \times 2$  matrix  $P$  represents the orthogonal projection onto a line in  $\mathbb{R}^2$ , then  $P$  must be similar to matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .
27. If  $A$  and  $B$  are  $n \times n$  matrices, and vector  $\vec{v}$  is in the kernel of both  $A$  and  $B$ , then  $\vec{v}$  must be in the kernel of matrix  $AB$  as well.
28. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.
29. If  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are any three distinct vectors in  $\mathbb{R}^3$ , then there must be a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  such that  $T(\vec{v}_1) = \vec{e}_1$ ,  $T(\vec{v}_2) = \vec{e}_2$ , and  $T(\vec{v}_3) = \vec{e}_3$ .
30. If vectors  $\vec{u}, \vec{v}, \vec{w}$  are linearly dependent, then vector  $\vec{w}$  must be a linear combination of  $\vec{u}$  and  $\vec{v}$ .
31.  $\mathbb{R}^2$  is a subspace of  $\mathbb{R}^3$ .
32. If an  $n \times n$  matrix  $A$  is similar to matrix  $B$ , then  $A + 7I_n$  must be similar to  $B + 7I_n$ .
33. If  $V$  is any three-dimensional subspace of  $\mathbb{R}^5$ , then  $V$  has infinitely many bases.
34. Matrix  $I_n$  is similar to  $2I_n$ .
35. If  $AB = 0$  for two  $2 \times 2$  matrices  $A$  and  $B$ , then  $BA$  must be the zero matrix as well.
36. If  $A$  and  $B$  are  $n \times n$  matrices, and vector  $\vec{v}$  is in the image of both  $A$  and  $B$ , then  $\vec{v}$  must be in the image of matrix  $A + B$  as well.
37. If  $V$  and  $W$  are subspaces of  $\mathbb{R}^n$ , then their union  $V \cup W$  must be a subspace of  $\mathbb{R}^n$  as well.
38. If the kernel of a  $5 \times 4$  matrix  $A$  consists of the zero vector only and if  $A\vec{v} = A\vec{w}$  for two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^4$ , then vectors  $\vec{v}$  and  $\vec{w}$  must be equal.
39. If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  and  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$  are two bases of  $\mathbb{R}^n$ , then there exists a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that  $T(\vec{v}_1) = \vec{w}_1$ ,  $T(\vec{v}_2) = \vec{w}_2, \dots, T(\vec{v}_n) = \vec{w}_n$ .
40. If matrix  $A$  represents a rotation through  $\pi/2$  and matrix  $B$  a rotation through  $\pi/4$ , then  $A$  is similar to  $B$ .
41. There exists a  $2 \times 2$  matrix  $A$  such that  $\text{im}(A) = \ker(A)$ .
42. If two  $n \times n$  matrices  $A$  and  $B$  have the same rank, then they must be similar.
43. If  $A$  is similar to  $B$ , and  $A$  is invertible, then  $B$  must be invertible as well.
44. If  $A^2 = 0$  for a  $10 \times 10$  matrix  $A$ , then the inequality  $\text{rank}(A) \leq 5$  must hold.
45. For every subspace  $V$  of  $\mathbb{R}^3$ , there exists a  $3 \times 3$  matrix  $A$  such that  $V = \text{im}(A)$ .
46. There exists a nonzero  $2 \times 2$  matrix  $A$  that is similar to  $2A$ .
47. If the  $2 \times 2$  matrix  $R$  represents the reflection about a line in  $\mathbb{R}^2$ , then  $R$  must be similar to matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

## Subspaces of $\mathbb{R}^n$ and Their Dimensions

- 48.** If  $A$  is similar to  $B$ , then there exists one *and only one* invertible matrix  $S$  such that  $S^{-1}AS = B$ .
- 49.** If the kernel of a  $5 \times 4$  matrix  $A$  consists of the zero vector alone, and if  $AB = AC$  for two  $4 \times 5$  matrices  $B$  and  $C$ , then matrices  $B$  and  $C$  must be equal.
- 50.** If  $A$  is any  $n \times n$  matrix such that  $A^2 = A$ , then the image of  $A$  and the kernel of  $A$  have only the zero vector in common.
- 51.** There exists a  $2 \times 2$  matrix  $A$  such that  $A^2 \neq 0$  and  $A^3 = 0$ .
- 52.** If  $A$  and  $B$  are  $n \times m$  matrices such that the image of  $A$  is a subset of the image of  $B$ , then there must exist an  $m \times m$  matrix  $C$  such that  $A = BC$ .
- 53.** Among the  $3 \times 3$  matrices whose entries are all 0's and 1's, most are invertible.

### Answers to Odd-Numbered Exercises

**1** Answers to more theoretical questions are omitted.

1.  $\ker A = \{\vec{0}\}$

3.  $\vec{e}_1, \vec{e}_2$

5.  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

7.  $\ker A = \{\vec{0}\}$

9.  $\ker A = \{\vec{0}\}$

11.  $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

13.  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

15.  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

17. All of  $\mathbb{R}^2$

19. The line spanned by  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

21. All of  $\mathbb{R}^3$

23. kernel is  $\{\vec{0}\}$ , image is all of  $\mathbb{R}^2$ .

25. Same as Exercise 23

27.  $f(x) = x^3 - x$

29.  $f \begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{bmatrix}$

(compare with spherical coordinates)

31.  $A = \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

33.  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$

35.  $\ker T$  is the plane with normal vector  $\vec{v}$ ;  $\text{im } T = \mathbb{R}$ .

37.  $\text{im } A = \text{span}(\vec{e}_1, \vec{e}_2); \ker A = \text{span}(\vec{e}_1); \text{im}(A^2) = \text{span}(\vec{e}_1); \ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2); A^3 = 0 \text{ so } \ker(A^3) = \mathbb{R}^3 \text{ and } \text{im}(A^3) = \{0\}$

39. a.  $\ker B$  is contained in  $\ker(AB)$ , but they need not be equal.

b.  $\text{im}(AB)$  is contained in  $\text{im } A$ , but they need not be equal.

41. a.  $\text{im } A$  is the line spanned by  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ , and  $\ker A$  is the perpendicular line, spanned by  $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

b.  $A^2 = A$ ; if  $\vec{v}$  is in  $\text{im } A$ , then  $A\vec{v} = \vec{v}$ .

c. Orthogonal projection onto the line spanned by  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

43. Suppose  $A$  is an  $n \times m$  matrix of rank  $r$ . Let  $B$  be the matrix you get when you omit the first  $r$  rows and the first  $m$  columns of rref  $[A \mid I_n]$ . (What can you do when  $r = n$ ?)

45. There are  $m-r$  nonleading variables, which can be chosen freely. The general vector in the kernel can be written as a linear combination of  $m-r$  vectors, with the nonleading variables as coefficients.

47.  $\text{im } T = L_2$  and  $\ker T = L_1$

51.  $\ker(AB) = \{\vec{0}\}$

53. a.  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

b.  $\ker H = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ , by part a, and  $\text{im } M = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ , by Theorem 1.3. Thus  $\ker H = \text{im}(M)$ .  $H(M\vec{x}) = \vec{0}$ , since  $M\vec{x}$  is in  $\text{im } M = \ker H$ .

**2** Answers to more theoretical questions are omitted.

1. Not a subspace

3.  $W$  is a subspace.

7. Yes

9. Dependent

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

11. Independent

15. Dependent

19.  $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$  are redundant.

21.  $\vec{v}_2 = \vec{v}_1$ , or,  $\vec{v}_1 - \vec{v}_2 = \vec{0}$ , so that  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is in the kernel.

23.  $\vec{v}_1 = \vec{0}$ , so that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is in the kernel.

25.  $\vec{v}_3 = \vec{v}_1$ , or,  $\vec{v}_1 - \vec{v}_3 = \vec{0}$ , so that  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  is in the kernel.

27.  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

29.  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

31.  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

33.  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

35. Suppose there is a nontrivial relation  $c_1\vec{v}_1 + \dots + c_i\vec{v}_i + \dots + c_m\vec{v}_m = \vec{0}$ , with  $c_i \neq 0$ . We can solve this equation for  $\vec{v}_i$  and thus express  $\vec{v}_i$  as a linear combination of the other vectors in the list.

Conversely, if  $\vec{v}_i$  is a linear combination of the other vectors,  $\vec{v}_i = \dots$ , then we can subtract  $\vec{v}_i$  from both sides of the equation to obtain a nontrivial relation (the coefficient of  $\vec{v}_i$  will be  $-1$ ).

37. The vectors  $T(\vec{v}_1), \dots, T(\vec{v}_m)$  are not necessarily independent.

39. The vectors  $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$  are linearly independent.

41. The columns of  $B$  are linearly independent, while the columns of  $A$  are dependent.

43. The vectors are linearly independent.

45. Yes

47.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

49.  $L = \text{im} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

51. a. Consider a relation  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p + d_1\vec{w}_1 + \dots + d_q\vec{w}_q = \vec{0}$ . Then  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = -d_1\vec{w}_1 - \dots - d_q\vec{w}_q$  is  $\vec{0}$ , because this vector is both in  $V$  and in  $W$ . The claim follows.

b. From part a we know that the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  are linearly independent. Consider a vector  $\vec{x}$  in  $V + W$ . By the

13. Dependent

17. Independent

definition of  $V + W$ , we can write  $\vec{x} = \vec{v} + \vec{w}$  for a  $\vec{v}$  in  $V$  and a  $\vec{w}$  in  $W$ . The  $\vec{v}$  is a linear combination of the  $\vec{v}_i$ , and  $\vec{w}$  is a linear combination of the  $\vec{w}_j$ . This shows that the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  span  $V + W$ .

55.  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

57. For  $j = 1, 3, 6$ , and  $7$ , corresponding to the columns that do not contain leading 1's

3 Answers to more theoretical questions are omitted.

1.  $\vec{v}_2 = 3\vec{v}_1$ ; basis of image:  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

3. No redundant vectors; basis of image:  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ;  
basis of kernel:  $\emptyset$

5.  $\vec{v}_3 = 3\vec{v}_1$ ; basis of image:  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ ;  
basis of kernel:  $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

7.  $\vec{v}_2 = 2\vec{v}_1$ ; basis of image:  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

9.  $\vec{v}_2 = 2\vec{v}_1$ ; basis of image:  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ;  
basis of kernel:  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

11.  $\vec{v}_3 = \vec{v}_1$ ; basis of image:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

13.  $\vec{v}_2 = 2\vec{v}_1, \vec{v}_3 = 3\vec{v}_1$ ; basis of image:  $[1]$ ;

basis of kernel:  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

15.  $\vec{v}_3 = 2\vec{v}_1 + 2\vec{v}_2, \vec{v}_4 = \vec{0}$ ;

basis of image:  $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

17.  $\vec{v}_1 = \vec{0}, \vec{v}_3 = 2\vec{v}_2, \vec{v}_5 = 3\vec{v}_2 + 4\vec{v}_4$ ;

basis of image:  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

19.  $\vec{v}_3 = 5\vec{v}_1 + 4\vec{v}_2, \vec{v}_4 = 3\vec{v}_1 + 2\vec{v}_2$ ;

basis of image:  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -5 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

21. rref  $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ ;

basis of image:  $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$

23. rref  $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ;

basis of image:  $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

25. rref  $A = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ;

basis of image:  $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$ ;

basis of kernel:  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

27. They do.

29.  $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

31.  $A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

33. The dimension of a hyperplane in  $\mathbb{R}^n$  is  $n - 1$ .

35. The dimension is  $n - 1$ .

37.  $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

39.  $\ker C$  is at least one-dimensional, and  $\ker C$  is contained in  $\ker A$ .

41. To fit a conic through a given point  $P_j(x_j, y_j)$ , we need to solve the equation  $c_1 + c_2x_j + c_3y_j + c_4x_j^2 + c_5x_jy_j + c_6y_j^2 = 0$ , a homogeneous linear equation in the six unknowns  $c_1, \dots, c_6$ . Thus, fitting a conic to four given points  $P_1(x_1, y_1), \dots, P_4(x_4, y_4)$  amounts to solving a system of four homogeneous linear equations with six unknowns. This in turn amounts to finding the kernel of a  $4 \times 6$  matrix  $A$ . This kernel is at least two-dimensional. Since every one-dimensional subspace of  $\ker A$  defines a unique conic (see Exercise 40), there will be infinitely many such conics.

43. Building on our work in Exercise 41, we observe that fitting a conic through six points amounts to finding the kernel of a  $6 \times 6$  matrix  $A$ . There will be no such conic if  $\ker A = \{\vec{0}\}$ , one conic if  $\dim(\ker A) = 1$ , and infinitely many conics if  $\dim(\ker A) > 1$ . To give an example for each case, we note that the unique conic  $xy = 0$  runs through the points  $(0, 0), (1, 0), (2, 0), (0, 1), (0, 2)$ . Thus, there is no conic through the points  $(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)$ , whereas the only conic through  $(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (0, 3)$  is

$xy = 0$ . There are infinitely many conics through  $(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)$ .

45. A cubic runs through  $(0, 0)$  if  $c_1 = 0$ . Now the cubic runs through the points  $(1, 0)$ ,  $(2, 0)$ , and  $(3, 0)$  as well if

$$\begin{aligned} c_2 + c_4 + c_7 &= 0 \\ 2c_2 + 4c_4 + 8c_7 &= 0 \\ 3c_3 + 9c_4 + 27c_7 &= 0. \end{aligned}$$

Gaussian elimination shows that  $c_2 = c_4 = c_7 = 0$ . Analogously, the cubic runs through the points  $(0, 1)$ ,  $(0, 2)$ , and  $(0, 3)$  if  $c_3 = c_6 = c_{10} = 0$ . The claim follows.

Each such cubic is the union of the  $x$ -axis, the  $y$ -axis, and an arbitrary line in  $\mathbb{R}^2$ .

47. Plugging the points  $(1, 1)$  and  $(2, 2)$  into the solution  $xy(c_5 + c_8x + c_9y) = 0$  of Problem 45, we find  $c_5 = 0$  and  $c_9 = -c_8$ , so that the solutions are of the form  $xy(c_8x - c_8y) = 0$ , where  $c_8 \neq 0$ . Division by  $c_8$  produces the unique solution  $xy(x - y) = 0$ , the union of the  $y$ -axis, the  $x$ -axis, and the diagonal  $y = x$ .

49. No such cubic exists, since the unique cubic through the first 9 points does not pass through  $(2, 1)$ . See Exercise 47.

51. Plugging the points  $(4, 0)$ ,  $(0, 4)$  and  $(1, 1)$  into the solution  $xy(c_5 + c_8x + c_9y) = 0$  of Problem 45, we find that  $c_5 + c_8 + c_9 = 0$ . Define  $a = c_8$  and  $b = c_9$ . There are infinitely many cubics running through the given points, of the form  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Each such cubic is the union of the  $y$ -axis, the  $x$ -axis, and any line through the point  $(1, 1)$ .

53. Plugging the points  $(1, 1)$ ,  $(2, 1)$ ,  $(1, 2)$ , and  $(3, 2)$  into the solutions of Problem 44, we find that  $c_5 = c_7 = c_8 = c_9 = 0$ , so that the solutions are of the form  $c_{10}y(y - 1)(y - 2) = 0$ , where  $c_{10} \neq 0$ . Division by  $c_{10}$  produces the unique solution  $y(y - 1)(y - 2) = 0$ , the union of the three horizontal lines  $y = 0$ ,  $y = 1$ , and  $y = 2$ .

55. Plugging the points  $(1, 1)$ ,  $(2, 1)$ ,  $(1, 2)$ ,  $(2, 2)$ , and  $(3, 3)$  into the solutions of Problem 44, we find that  $c_5 = c_8 = c_9 = 0$  and  $c_7 + c_{10} = 0$ , so that the solutions are of the form  $c_7x(x - 1)(x - 2) - c_7y(y - 1)(y - 2) = 0$ , where  $c_7 \neq 0$ . Division by  $c_7$  produces the unique solution  $x(x - 1)(x - 2) = y(y - 1)(y - 2)$ , the union of the diagonal  $y = x$  with an ellipse.

57. See Exercises 41 and 56. Since the kernel of the  $8 \times 10$  matrix  $A$  is at least two-dimensional, and because every one-dimensional subspace of  $\ker A$  defines a unique cubic (compare with Exercise 40), there will be infinitely many such cubics.

59. There may be no such cubic [as in Exercise 49], exactly one [take the 9 points in Exercise 47 and add  $(-1, -1)$ ], or infinitely many [as in Exercise 51].

63. A basis of  $V$  is also a basis of  $W$ , by Theorem 3.4c.

65.  $\dim(V+W) = \dim V + \dim W$ , by Exercise 2.51.

67. The first  $p$  columns of rref  $A$  contain leading 1's because the  $\vec{v}_i$  are linearly independent. Now apply Theorem 3.5.

- $$71. \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

73. a.  $A$  and  $E$  have the same row space, since elementary row operations leave the row space unchanged.

- b.  $\text{rank } A = \dim(\text{rowspace}(A))$ , by part a and Exercise 72.

77. Suppose  $\text{rank } A = n$ . The submatrix of  $A$  consisting of the  $n$  pivot columns of  $A$  is invertible, since the pivot columns are linearly independent.

Conversely, if  $A$  has an invertible  $n \times n$  submatrix, then the columns of that submatrix span  $\mathbb{R}^n$ , so  $\text{im } A = \mathbb{R}^n$  and  $\text{rank } A = n$ .

79. Let  $m$  be the smallest number such that  $A^m = 0$ . By Exercise 78, there are  $m$  linearly independent vectors in  $\mathbb{R}^n$ ; therefore,  $m \leq n$ , and  $A^n = A^m A^{n-m} = 0$ .



85. a.  $\text{rank}(AB) \leq \text{rank } A$     b.  $\text{rank}(AB) \leq \text{rank } B$

**4** Answers to more theoretical questions are omitted.

$$1. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad 3. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$5. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad 7. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

9.  $\vec{x}$  isn't in  $V$ .      11.  $[\vec{x}]_{\mathfrak{V}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$$13. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad 15. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}$$

$$17. \quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad 19. \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$21. B = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} \quad 23. B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$25. B = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix} \quad 27. B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### Subspaces of $\mathbb{R}^n$ and Their Dimensions

29.  $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

31.  $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

33.  $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

35.  $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

37.  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , for example

39.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ , for example

41.  $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ , for example

43.  $\vec{x} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$

45. If  $\vec{v}$  is any vector in the plane that is not parallel to  $\vec{x}$ , then  $\vec{v}, \frac{1}{3}(\vec{x} - 2\vec{v})$  is a basis with the desired

property. For example,  $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$  gives the basis

$$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} -4 \\ -4 \\ -1 \end{bmatrix}.$$

47.  $A = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$

49.  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

53.  $\vec{x} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$

55.  $\frac{1}{2} \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

57. Consider a basis with two vectors parallel to the plane, and one vector perpendicular.

59. Yes

61.  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , for example

63. Yes

67.  $B = \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix}$

69. If  $S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ , then  $S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ .

71. a. If  $B\vec{x} = S^{-1}AS\vec{x} = \vec{0}$ , then  $A(S\vec{x}) = \vec{0}$ .

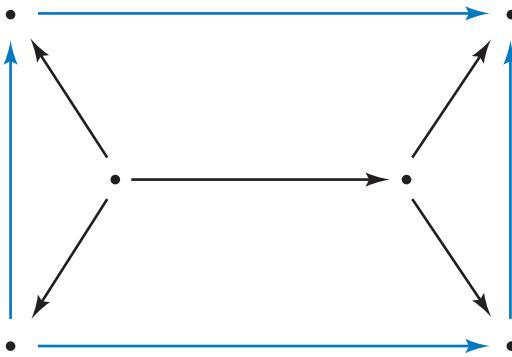
b. Since we have the  $p$  linearly independent vectors  $S\vec{v}_1, S\vec{v}_2, \dots, S\vec{v}_p$  in  $\ker A$ , we know that  $\dim(\ker B) = p \leq \dim(\ker A)$ , by Theorem 3.4a. Reversing the roles of  $A$  and  $B$ , we find that  $\dim(\ker A) \leq \dim(\ker B)$ . Thus,  $\text{nullity } A = \dim(\ker A) = \dim(\ker B) = \text{nullity } B$ .

73.  $\begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}$

75.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

77.  $b_{ij} = a_{n+1-i, n+1-j}$

79. By Theorem 4.7, we seek a basis  $\vec{v}_1, \vec{v}_2$  such that  $A\vec{v}_1 = \vec{v}_1$  and  $A\vec{v}_2 = -\vec{v}_2$ . Solving the linear systems  $A\vec{x} = \vec{x}$  and  $A\vec{x} = -\vec{x}$  [or  $(A - I_2)\vec{x} = \vec{0}$  and  $(A + I_2)\vec{x} = \vec{0}$ ], we find  $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ , for example.



# Linear Spaces

## I Introduction to Linear Spaces

Some of the key words of the language of linear algebra as applied to vectors in  $\mathbb{R}^n$  are linear combination, linear transformation, kernel, image, subspace, span, linear independence, basis, dimension, and coordinates. Note that all these concepts can be defined in terms of sums and scalar multiples of vectors. In this chapter, we will see that it can be both natural and useful to apply this language to other mathematical objects, such as functions, matrices, equations, or infinite sequences. Indeed, linear algebra provides a unifying language used throughout modern mathematics and physics.

Here is an introductory example:

**EXAMPLE I** Consider the differential equation<sup>1</sup> (DE)

$$f''(x) + f(x) = 0, \quad \text{or} \quad f''(x) = -f(x).$$

We are asked to find all twice-differentiable functions  $f(x)$  whose second derivative is the negative of the function itself. Recalling the derivative rules from your introductory calculus class, you will (hopefully) note that

$$\sin(x) \quad \text{and} \quad \cos(x)$$

are solutions of this DE.

Can you find any other solutions?

Note that the solution set of this DE is closed under addition and under scalar multiplication. If  $f_1(x)$  and  $f_2(x)$  are solutions, then so is  $f(x) = f_1(x) + f_2(x)$ , since

$$f''(x) = f_1''(x) + f_2''(x) = -f_1(x) - f_2(x) = -f(x).$$

Likewise, if  $f_1(x)$  is a solution and  $k$  is any scalar, then  $f(x) = kf_1(x)$  is a solution of the DE as well. (Verify this!)

---

<sup>1</sup>A differential equation is an equation involving derivatives of an unknown function. No previous knowledge of DEs is expected here.

## Linear Spaces

It follows that all “linear combinations”<sup>2</sup>

$$f(x) = c_1 \sin(x) + c_2 \cos(x)$$

are solutions of this DE. It can be shown that all solutions are of this form; we leave the proof as Exercise 58.

Let  $F(\mathbb{R}, \mathbb{R})$  be the set of all functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Since the solution set  $V$  of our DE is closed under addition and scalar multiplication, we can say that  $V$  is a “subspace” of  $F(\mathbb{R}, \mathbb{R})$ .

How many solutions does this differential equation have? There are infinitely many solutions, of course, but we can use the language of linear algebra to give a more informative answer. The functions  $\sin(x)$  and  $\cos(x)$  form a “basis” of the “solution space”  $V$ , so that the “dimension” of  $V$  is 2.

In summary, the solutions of our DE form a two-dimensional subspace of  $F(\mathbb{R}, \mathbb{R})$ , with basis  $\sin(x)$  and  $\cos(x)$ . ■

We will now make the informal ideas presented in Example 1 more precise.

Note again that all the basic concepts of linear algebra can be defined in terms of sums and scalar multiples. Whenever we are dealing with a set [such as  $F(\mathbb{R}, \mathbb{R})$  in Example 1] whose elements can be added and multiplied by scalars, subject to certain rules, then we can apply the language of linear algebra just as we do for vectors in  $\mathbb{R}^n$ . These “certain rules” are spelled out in Definition 1.1. Compare this definition with the rules of vector algebra.

### Definition 1.1

#### Linear spaces (or vector spaces)

A *linear space*<sup>3</sup>  $V$  is a set endowed with a rule for addition (if  $f$  and  $g$  are in  $V$ , then so is  $f + g$ ) and a rule for scalar multiplication (if  $f$  is in  $V$  and  $k$  in  $\mathbb{R}$ , then  $kf$  is in  $V$ ) such that these operations satisfy the following eight rules<sup>4</sup> (for all  $f, g, h$  in  $V$  and all  $c, k$  in  $\mathbb{R}$ ):

1.  $(f + g) + h = f + (g + h)$ .
2.  $f + g = g + f$ .
3. There exists a *neutral element*  $n$  in  $V$  such that  $f + n = f$ , for all  $f$  in  $V$ . This  $n$  is unique and denoted by 0.
4. For each  $f$  in  $V$ , there exists a  $g$  in  $V$  such that  $f + g = 0$ . This  $g$  is unique and denoted by  $(-f)$ .
5.  $k(f + g) = kf + kg$ .
6.  $(c + k)f = cf + kf$ .
7.  $c(kf) = (ck)f$ .
8.  $1f = f$ .

This definition contains a lot of fine print. In brief, a linear space is a set with two reasonably defined operations, addition and scalar multiplication, that allow us to

<sup>2</sup>We are cautious here and use quotes, since the term “linear combination” has been officially defined for vectors in  $\mathbb{R}^n$  only.

<sup>3</sup>The term “vector space” is more commonly used in English (but it’s *espace linéaire* in French). We prefer the term “linear space” to avoid the confusion that some students experience with the term “vector” in this abstract sense.

<sup>4</sup>These axioms were established by the Italian mathematician Giuseppe Peano (1858–1932) in his *Calcolo Geometrico* of 1888. Peano calls  $V$  a “linear system.”

## Linear Spaces

form linear combinations. All the other basic concepts of linear algebra in turn rest on the concept of a linear combination.

**EXAMPLE 2** In  $\mathbb{R}^n$ , the prototype linear space, the neutral element is the zero vector,  $\vec{0}$ . ■

Probably the most important examples of linear spaces, besides  $\mathbb{R}^n$ , are *spaces of functions*.

**EXAMPLE 3** Let  $F(\mathbb{R}, \mathbb{R})$  be the set of all functions from  $\mathbb{R}$  to  $\mathbb{R}$  (see Example 1), with the operations

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ \text{and} \\ (kf)(x) &= kf(x).\end{aligned}$$

Then  $F(\mathbb{R}, \mathbb{R})$  is a linear space. The neutral element is the zero function,  $f(x) = 0$  for all  $x$ . ■

**EXAMPLE 4** If addition and scalar multiplication are given, then  $\mathbb{R}^{n \times m}$ , the set of all  $n \times m$  matrices, is a linear space. The neutral element is the zero matrix, whose entries are all zero. ■

**EXAMPLE 5** The set of all infinite sequences of real numbers is a linear space, where addition and scalar multiplication are defined term by term:

$$\begin{aligned}(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) &= (x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots) \\ k(x_0, x_1, x_2, \dots) &= (kx_0, kx_1, kx_2, \dots).\end{aligned}$$

The neutral element is the sequence

$$(0, 0, 0, \dots).$$
 ■

**EXAMPLE 6** The linear equations in three unknowns,

$$ax + by + cz = d,$$

where  $a, b, c$ , and  $d$  are constants, form a linear space.

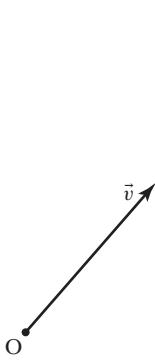
The operations (addition and scalar multiplication) are familiar from the process of Gaussian elimination. The neutral element is the equation  $0 = 0$  (with  $a = b = c = d = 0$ ). ■

**EXAMPLE 7** Consider the plane  $\mathcal{P}$  with a point designated as the origin,  $O$ , but without a coordinate system (the coordinate-free plane). A *geometric vector*  $\vec{v}$  in this plane is an arrow (a directed line segment) with its tail at the origin, as shown in Figure 1. The sum  $\vec{v} + \vec{w}$  of two vectors  $\vec{v}$  and  $\vec{w}$  is defined by means of a parallelogram, as illustrated in Figure 2. If  $k$  is a positive scalar, then vector  $k\vec{v}$  points in the same direction as  $\vec{v}$ , but  $k\vec{v}$  is  $k$  times as long as  $\vec{v}$ ; see Figure 3. If  $k$  is negative, then  $k\vec{v}$  points in the opposite direction, and it is  $|k|$  times as long as  $\vec{v}$ ; see Figure 4. The geometric vectors in the plane with these operations form a linear space,  $\tilde{\mathcal{P}}$ . The neutral element of  $\tilde{\mathcal{P}}$  is the zero vector  $\vec{0}$ , with tail and head at the origin.

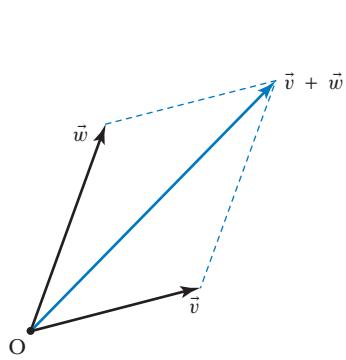
By introducing a *coordinate system*, we can identify the space  $\tilde{\mathcal{P}}$  of geometric vectors with  $\mathbb{R}^2$ ; this was the great idea of Descartes's *Analytic Geometry*. In Section 3, we will study this idea more systematically. ■

**EXAMPLE 8** Let  $\mathbb{C}$  be the set of the *complex numbers*. We trust that you have at least a fleeting acquaintance with complex numbers. Without attempting a definition, we recall that

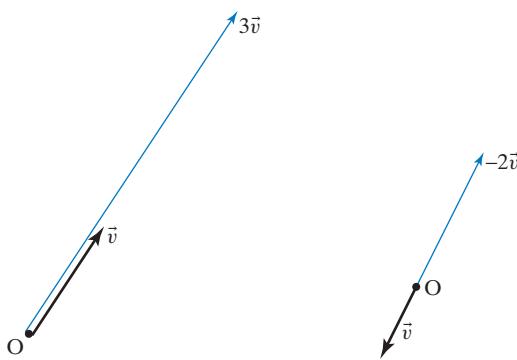
## Linear Spaces



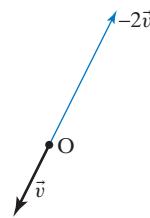
**Figure 1**



**Figure 2**



**Figure 3**



**Figure 4**

a complex number can be expressed as  $z = a + bi$ , where  $a$  and  $b$  are real numbers. Addition of complex numbers is defined in a natural way, by the rule

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

If  $k$  is a real scalar, we define

$$k(a + ib) = ka + i(kb).$$

There is also a (less natural) rule for the multiplication of complex numbers, but we are not concerned with this operation here.

The complex numbers  $\mathbb{C}$  with the two operations just given form a linear space; the neutral element is the complex number  $0 = 0 + 0i$ . ■

We say that an element  $f$  of a linear space is a *linear combination* of the elements  $f_1, f_2, \dots, f_n$  if

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

for some scalars  $c_1, c_2, \dots, c_n$ .

### EXAMPLE 9

Let  $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ . Show that  $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$  is a linear combination of  $A$  and  $I_2$ .

### Solution

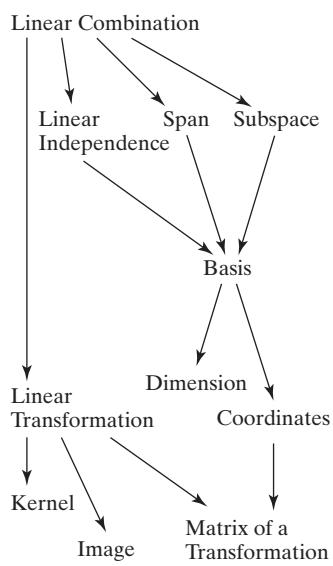
We have to find scalars  $c_1$  and  $c_2$  such that

$$A^2 = c_1 A + c_2 I_2,$$

or

$$\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this simple example, we can see by inspection that  $c_1 = 3$  and  $c_2 = 2$ . We could do this problem more systematically and solve a system of four linear equations in two unknowns. ■



**Figure 5**

Since the basic notions of linear algebra (initially introduced for  $\mathbb{R}^n$ ) are defined in terms of linear combinations, we can now generalize these notions without modifications. A short version of the rest of this chapter would say that the concepts of linear transformation, kernel, image, linear independence, span, subspace, basis, dimension, and coordinates can be defined for a linear space in just the same way as for  $\mathbb{R}^n$ . Figure 5 illustrates the logical dependencies between the key concepts of linear algebra introduced thus far.

## Linear Spaces

What follows is the long version, with many examples.

### Definition 1.2

#### Subspaces

A subset  $W$  of a linear space  $V$  is called a *subspace* of  $V$  if

- a.  $W$  contains the neutral element  $0$  of  $V$ .
- b.  $W$  is closed under addition (if  $f$  and  $g$  are in  $W$ , then so is  $f + g$ ).
- c.  $W$  is closed under scalar multiplication (if  $f$  is in  $W$  and  $k$  is a scalar, then  $kf$  is in  $W$ ).

We can summarize parts b and c by saying that  $W$  is closed under linear combinations.

Note that a subspace  $W$  of a linear space  $V$  is a linear space in its own right. (Why do the eight rules listed in Definition 1.1 hold for  $W$ ?)

### EXAMPLE 10

Show that the polynomials of degree  $\leq 2$ , of the form  $f(x) = a + bx + cx^2$ , are a subspace  $W$  of the space  $F(\mathbb{R}, \mathbb{R})$  of all functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

#### Solution

- a.  $W$  contains the neutral element of  $F(\mathbb{R}, \mathbb{R})$ , the zero function  $f(x) = 0$ . Indeed, we can write  $f(x) = 0 + 0x + 0x^2$ .
- b.  $W$  is closed under addition: If two polynomials  $f(x) = a + bx + cx^2$  and  $g(x) = p + qx + rx^2$  are in  $W$ , then their sum  $f(x) + g(x) = (a + p) + (b + q)x + (c + r)x^2$  is in  $W$  as well, since  $f(x) + g(x)$  is a polynomial of degree  $\leq 2$ .
- c.  $W$  is closed under scalar multiplication: If  $f(x) = a + bx + cx^2$  is a polynomial in  $W$  and  $k$  is a constant, then  $kf(x) = ka + (kb)x + (kc)x^2$  is in  $W$  as well. ■

### EXAMPLE 11

Show that the differentiable functions form a subspace  $W$  of  $F(\mathbb{R}, \mathbb{R})$ .

#### Solution

- a. The zero function  $f(x) = 0$  is differentiable, with  $f'(x) = 0$ .
- b.  $W$  is closed under addition: You learned in your introductory calculus class that the sum of two differentiable functions  $f(x)$  and  $g(x)$  is differentiable, with  $(f(x) + g(x))' = f'(x) + g'(x)$ .
- c.  $W$  is closed under scalar multiplication, since any scalar multiple of a differentiable function is differentiable as well. ■

In the next example, we will build on Examples 10 and 11.

### EXAMPLE 12

Here are more subspaces of  $F(\mathbb{R}, \mathbb{R})$ :

- a.  $C^\infty$ , the smooth functions, that is, functions  $f(x)$  from  $\mathbb{R}$  to  $\mathbb{R}$  that have derivatives of all orders,  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$ , and so on. This subspace contains all polynomials, exponential functions,  $\sin(x)$ , and  $\cos(x)$ , for example. However,  $C^\infty$  fails to contain functions such as  $\frac{1}{x}$ ,  $\tan x$ ,  $|x|$ , or  $x^5|x|$ .
- b.  $P$ , the set of all polynomials.
- c.  $P_n$ , the set of all polynomials of degree  $\leq n$ . ■

**EXAMPLE 13** Show that the matrices  $B$  that commute with  $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$  form a subspace of  $\mathbb{R}^{2 \times 2}$ .

**Solution**

- a. The zero matrix  $0$  commutes with  $A$ , since  $A0 = 0A = 0$ .
- b. If matrices  $B_1$  and  $B_2$  commute with  $A$ , then so does matrix  $B = B_1 + B_2$ , since

$$BA = (B_1 + B_2)A = B_1A + B_2A = AB_1 + AB_2 = A(B_1 + B_2) = AB.$$

- c. If  $B$  commutes with  $A$ , then so does  $kB$ , since

$$(kB)A = k(BA) = k(AB) = A(kB).$$

Note that we have not used the special form of  $A$ . We have indeed shown that the  $n \times n$  matrices  $B$  that commute with any given  $n \times n$  matrix  $A$  form a subspace of  $\mathbb{R}^{n \times n}$ . ■

**EXAMPLE 14** Consider the set  $W$  of all noninvertible  $2 \times 2$  matrices. Is  $W$  a subspace of  $\mathbb{R}^{2 \times 2}$ ?

**Solution**

The following example shows that  $W$  isn't closed under addition:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

↑  
in  $W$ 
↑  
not in  $W$

Therefore,  $W$  fails to be a subspace of  $\mathbb{R}^{2 \times 2}$ . ■

Next, we will generalize the notions of span, linear independence, basis, coordinates, and dimension.

**Definition 1.3**

**Span, linear independence, basis, coordinates**

Consider the elements  $f_1, \dots, f_n$  in a linear space  $V$ .

- a. We say that  $f_1, \dots, f_n$  span  $V$  if every  $f$  in  $V$  can be expressed as a linear combination of  $f_1, \dots, f_n$ .
- b. We say that  $f_i$  is redundant if it is a linear combination of  $f_1, \dots, f_{i-1}$ . The elements  $f_1, \dots, f_n$  are called linearly independent if none of them is redundant. This is the case if the equation

$$c_1f_1 + \cdots + c_nf_n = 0$$

has only the trivial solution

$$c_1 = \cdots = c_n = 0.$$

## Linear Spaces

### Definition 1.3

#### Span, linear independence, basis, coordinates (*Continued*)

- c. We say that elements  $f_1, \dots, f_n$  are a *basis* of  $V$  if they span  $V$  and are linearly independent. This means that every  $f$  in  $V$  can be written uniquely as a linear combination  $f = c_1 f_1 + \dots + c_n f_n$ . The coefficients  $c_1, \dots, c_n$  are called the *coordinates* of  $f$  with respect to the basis  $\mathfrak{B} = (f_1, \dots, f_n)$ . The vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

in  $\mathbb{R}^n$  is called the  $\mathfrak{B}$ -*coordinate vector* of  $f$ , denoted by  $[f]_{\mathfrak{B}}$ .

The transformation

$$L(f) = [f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{from } V \text{ to } \mathbb{R}^n$$

is called the  $\mathfrak{B}$ -*coordinate transformation*, sometimes denoted by  $L_{\mathfrak{B}}$ .

The  $\mathfrak{B}$ -coordinate transformation is invertible, with inverse

$$L^{-1} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 f_1 + \dots + c_n f_n.$$

Note in particular that  $L^{-1}(\vec{e}_i) = f_i$ .

We can represent the coordinate transformation and its inverse in the following diagram:

$$f = c_1 f_1 + \dots + c_n f_n \quad \text{in } V \quad \xrightleftharpoons[L_{\mathfrak{B}}]{L_{\mathfrak{B}}^{-1}} \quad [f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{in } \mathbb{R}^n.$$

As in the case of  $\mathbb{R}^n$ , coordinates have important linearity properties.

### Theorem 1.4

#### Linearity of the coordinate transformation $L_{\mathfrak{B}}$

If  $\mathfrak{B}$  is a basis of a linear space  $V$ , then

- a.  $[f+g]_{\mathfrak{B}} = [f]_{\mathfrak{B}} + [g]_{\mathfrak{B}}$ , for all elements  $f$  and  $g$  of  $V$ , and
- b.  $[kf]_{\mathfrak{B}} = k[f]_{\mathfrak{B}}$ , for all  $f$  in  $V$  and for all scalars  $k$ .

The proof is analogous to that of the theorem of linearity of Coordinates.

Now we are ready to introduce the key concept of the dimension of a linear space.

### Theorem 1.5

#### Dimension

If a linear space  $V$  has a basis with  $n$  elements, then all other bases of  $V$  consist of  $n$  elements as well. We say that  $n$  is the *dimension* of  $V$ :

$$\dim(V) = n.$$

To prove this important theorem, consider two bases  $\mathfrak{B} = (f_1, \dots, f_n)$  and  $\mathfrak{C} = (g_1, \dots, g_m)$  of  $V$ ; we have to show that  $n = m$ .

## Linear Spaces

We will show first that the  $m$  vectors  $[g_1]_{\mathfrak{B}}, \dots, [g_m]_{\mathfrak{B}}$  in  $\mathbb{R}^n$  are linearly independent, which implies that  $m \leq n$ . Consider a relation

$$c_1 [g_1]_{\mathfrak{B}} + \cdots + c_m [g_m]_{\mathfrak{B}} = \vec{0}.$$

By Theorem 1.4, we have

$$[c_1 g_1 + \cdots + c_m g_m]_{\mathfrak{B}} = \vec{0}, \quad \text{so that } c_1 g_1 + \cdots + c_m g_m = 0.$$

Since the elements  $g_1, \dots, g_m$  are linearly independent, it follows that  $c_1 = \cdots = c_m = 0$ , meaning that  $c_1 [g_1]_{\mathfrak{B}} + \cdots + c_m [g_m]_{\mathfrak{B}} = \vec{0}$  is the trivial relation, as claimed.

Reversing the roles of the two bases, we see that the  $n$  vectors  $[f_1]_{\mathfrak{C}}, \dots, [f_n]_{\mathfrak{C}}$  in  $\mathbb{R}^m$  are linearly independent, so that  $n \leq m$ .

We can conclude that  $n = m$ , as claimed. ■

**EXAMPLE 15** Find a basis of  $\mathbb{R}^{2 \times 2}$ , the space of all  $2 \times 2$  matrices, and thus determine the dimension of  $\mathbb{R}^{2 \times 2}$ .

### Solution

We can write any  $2 \times 2$  matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span  $\mathbb{R}^{2 \times 2}$ . The four matrices are also linearly independent: None of them is a linear combination of the others, since each has a 1 in a position where the three others have a 0. This shows that

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

is a basis (called the *standard basis* of  $\mathbb{R}^{2 \times 2}$ ), so that  $\dim(\mathbb{R}^{2 \times 2}) = 4$ .

The  $\mathfrak{B}$ -coordinate transformation  $L_{\mathfrak{B}}$  is represented in the following diagram:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{in } \mathbb{R}^{2 \times 2} \quad \xrightarrow{L_{\mathfrak{B}}} \quad [A]_{\mathfrak{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \text{in } \mathbb{R}^4. \quad ■$$

**EXAMPLE 16** Find a basis of  $P_2$ , the space of all polynomials of degree  $\leq 2$ , and thus determine the dimension of  $P_2$ .

### Solution

We can write any polynomial  $f(x)$  of degree  $\leq 2$  as

$$f(x) = a + bx + cx^2 = a \cdot 1 + b \cdot x + c \cdot x^2,$$

showing that the monomials  $1, x, x^2$  span  $P_2$ . We leave it as an exercise for the reader to verify the linear independence of these monomials. Thus,  $\mathfrak{B} = (1, x, x^2)$  is a basis (called the *standard basis* of  $P_2$ ), so that  $\dim(P_2) = 3$ .

The  $\mathfrak{B}$ -coordinate transformation  $L_{\mathfrak{B}}$  is represented in the following diagram:

$$f(x) = a + bx + cx^2 \quad \text{in } P_2 \quad \xrightarrow{L_{\mathfrak{B}}} \quad [f(x)]_{\mathfrak{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{in } \mathbb{R}^3. \quad ■$$

## Linear Spaces

Using Examples 15 and 16 as a guide, we can present the following strategy for finding a basis of a linear space.

### SUMMARY 1.6 | Finding a basis of a linear space $V$

- Write down a typical element of  $V$ , in terms of some arbitrary constants.
- Using the arbitrary constants as coefficients, express your typical element as a linear combination of some elements of  $V$ .
- Verify that the elements of  $V$  in this linear combination are linearly independent; then they will form a basis of  $V$ .

**EXAMPLE 17** Find a basis of the space  $V$  of all matrices  $B$  that commute with  $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ . See Example 13.

#### Solution

We need to find all matrices  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The entries of  $B$  must satisfy the linear equations

$$2b = c, \quad a + 3b = d, \quad 2d = 2a + 3c, \quad c + 3d = 2b + 3d.$$

The last two equations are redundant, so that the matrices  $B$  in  $V$  are of the form

$$B = \begin{bmatrix} a & b \\ 2b & a + 3b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = aI_2 + bA.$$

Since the matrices  $I_2$  and  $A$  are linearly independent, a basis of  $V$  is

$$(I_2, A) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \right).$$
 ■

In the introductory example of this section, we found that the solutions of the differential equation

$$f''(x) + f(x) = 0$$

form a two-dimensional subspace of  $C^\infty$ , with basis  $(\cos x, \sin x)$ .

We can generalize this result as follows:

### Theorem 1.7

#### Linear differential equations

The solutions of the differential equation

$$f''(x) + af'(x) + bf(x) = 0 \quad (\text{where } a \text{ and } b \text{ are constants})$$

form a two-dimensional subspace of the space  $C^\infty$  of smooth functions.

More generally, the solutions of the differential equation

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_1f'(x) + a_0f(x) = 0$$

**Theorem 1.7****Linear differential equations (*Continued*)**

(where  $a_0, \dots, a_{n-1}$  are constants) form an  $n$ -dimensional subspace of  $C^\infty$ . A differential equation of this form is called an  *$n$ th-order linear differential equation with constant coefficients*.

Second-order linear DEs are frequently used to model oscillatory phenomena in physics. Important examples are damped harmonic motion and *LC* circuits.

Consider how cumbersome it would be to state the second part of Theorem 1.7 without using the language of linear algebra. (Try it!) This may convince you that it can be both natural and useful to apply the language of linear algebra to functions.

**EXAMPLE 18**

Find all solutions of the DE

$$f''(x) + f'(x) - 6f(x) = 0.$$

*Hint:* Find all exponential functions  $f(x) = e^{kx}$  that solve the DE.

**Solution**

An exponential function  $f(x) = e^{kx}$  solves the DE if

$$\begin{aligned} f''(x) + f'(x) - 6f(x) &= k^2 e^{kx} + k e^{kx} - 6e^{kx} \\ &= (k^2 + k - 6)e^{kx} = (k+3)(k-2)e^{kx} = 0. \end{aligned}$$

The solutions are  $k = 2$  and  $k = -3$ . Thus,  $e^{2x}$  and  $e^{-3x}$  are solutions of the DE. (Check this!) Theorem 1.7 tells us that the solution space is two-dimensional. Thus, the linearly independent functions  $e^{2x}, e^{-3x}$  form a basis of  $V$ , and all solutions are of the form

$$f(x) = c_1 e^{2x} + c_2 e^{-3x}. \quad \blacksquare$$

**EXAMPLE 19**

Let  $f_1, \dots, f_n$  be polynomials. Explain why these polynomials will not span the space  $P$  of all polynomials.

**Solution**

Let  $N$  be the maximum of the degrees of the polynomials  $f_1, \dots, f_n$ . Then all linear combinations of  $f_1, \dots, f_n$  are in  $P_N$ , the space of polynomials of degree  $\leq N$ . Any polynomial of higher degree, such as  $f(x) = x^{N+1}$ , will not be in the span of  $f_1, \dots, f_n$ , proving our claim. ■

Example 19 implies that the space  $P$  of all polynomials does not have a finite basis  $f_1, \dots, f_n$ .

**Definition 1.8****Finite dimensional linear spaces**

A linear space  $V$  is called *finite dimensional* if it has a (finite) basis  $f_1, \dots, f_n$ , so that we can define its dimension  $\dim(V) = n$ . See Definition 1.5. Otherwise, the space is called *infinite dimensional*.<sup>5</sup>

As we have just seen, the space  $P$  of all polynomials is infinite dimensional (as was known to Peano in 1888).

---

<sup>5</sup>More advanced texts introduce the concept of an *infinite basis*.

## Linear Spaces

Take another look at the linear spaces introduced in Examples 1 through 8 of this section and see which of them are finite dimensional.

The basic theory of infinite dimensional spaces of functions was established by David Hilbert (1862–1943) and his student Erhard Schmidt (1876–1959), in the first decade of the twentieth century, based on their work on integral equations. A more general and axiomatic approach was presented by Stefan Banach (1892–1945) in his 1920 doctoral thesis. These topics (Hilbert spaces, Banach spaces) would be discussed in a course on *functional analysis* rather than linear algebra.

### EXERCISES I

**GOAL** Find a basis of a linear space and thus determine its dimension. Examine whether a subset of a linear space is a subspace.

Which of the subsets of  $P_2$  given in Exercises 1 through 5 are subspaces of  $P_2$  (see Example 16)? Find a basis for those that are subspaces.

1.  $\{p(t): p(0) = 2\}$
2.  $\{p(t): p(2) = 0\}$
3.  $\{p(t): p'(1) = p(2)\}$  ( $p'$  is the derivative.)
4.  $\{p(t): \int_0^1 p(t) dt = 0\}$
5.  $\{p(t): p(-t) = -p(t), \text{ for all } t\}$

Which of the subsets  $V$  of  $\mathbb{R}^{3 \times 3}$  given in Exercises 6 through 11 are subspaces of  $\mathbb{R}^{3 \times 3}$ ?

6. The invertible  $3 \times 3$  matrices
7. The diagonal  $3 \times 3$  matrices
8. The upper triangular  $3 \times 3$  matrices
9. The  $3 \times 3$  matrices whose entries are all greater than or equal to zero
10. The  $3 \times 3$  matrices  $A$  such that vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is in the kernel of  $A$
11. The  $3 \times 3$  matrices in reduced row-echelon form

Let  $V$  be the space of all infinite sequences of real numbers. See Example 5. Which of the subsets of  $V$  given in Exercises 12 through 15 are subspaces of  $V$ ?

12. The arithmetic sequences [i.e., sequences of the form  $(a, a+k, a+2k, a+3k, \dots)$ , for some constants  $a$  and  $k$ ]
13. The geometric sequences [i.e., sequences of the form  $(a, ar, ar^2, ar^3, \dots)$ , for some constants  $a$  and  $r$ ]
14. The sequences  $(x_0, x_1, \dots)$  that converge to zero (i.e.,  $\lim_{n \rightarrow \infty} x_n = 0$ )
15. The square-summable sequences  $(x_0, x_1, \dots)$  (i.e., those for which  $\sum_{i=0}^{\infty} x_i^2$  converges)

Find a basis for each of the spaces  $V$  in Exercises 16 through 36, and determine its dimension.

16.  $\mathbb{R}^{3 \times 2}$
17.  $\mathbb{R}^{n \times m}$
18.  $P_n$
19. The real linear space  $\mathbb{C}^2$
20. The space of all matrices  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  in  $\mathbb{R}^{2 \times 2}$  such that  $a + d = 0$
21. The space of all diagonal  $2 \times 2$  matrices
22. The space of all diagonal  $n \times n$  matrices
23. The space of all lower triangular  $2 \times 2$  matrices
24. The space of all upper triangular  $3 \times 3$  matrices
25. The space of all polynomials  $f(t)$  in  $P_2$  such that  $f(1) = 0$
26. The space of all polynomials  $f(t)$  in  $P_3$  such that  $f(1) = 0$  and  $\int_{-1}^1 f(t) dt = 0$
27. The space of all  $2 \times 2$  matrices  $A$  that commute with  $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
28. The space of all  $2 \times 2$  matrices  $A$  that commute with  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
29. The space of all  $2 \times 2$  matrices  $A$  such that  $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
30. The space of all  $2 \times 2$  matrices  $A$  such that  $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
31. The space of all  $2 \times 2$  matrices  $S$  such that  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
32. The space of all  $2 \times 2$  matrices  $S$  such that  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S = S \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

## Linear Spaces

33. The space of all  $2 \times 2$  matrices  $S$  such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S = S$$

34. The space of all  $2 \times 2$  matrices  $S$  such that

$$\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} S = S$$

35. The space of all  $3 \times 3$  matrices  $A$  that commute with

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

36. The space of all  $3 \times 3$  matrices  $A$  that commute with

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

37. If  $B$  is a diagonal  $3 \times 3$  matrix, what are the possible dimensions of the space  $V$  of all  $3 \times 3$  matrices  $A$  that commute with  $B$ ? Use Exercises 35 and 36 as a guide.

38. If  $B$  is a diagonal  $4 \times 4$  matrix, what are the possible dimensions of the space  $V$  of all  $4 \times 4$  matrices  $A$  that commute with  $B$ ?

39. What is the dimension of the space of all upper triangular  $n \times n$  matrices?

40. If  $\vec{c}$  is any vector in  $\mathbb{R}^n$ , what are the possible dimensions of the space  $V$  of all  $n \times n$  matrices  $A$  such that  $A\vec{c} = \vec{0}$ ?

41. If  $B$  is any  $3 \times 3$  matrix, what are the possible dimensions of the space  $V$  of all  $3 \times 3$  matrices  $A$  such that  $BA = 0$ ?

42. If  $B$  is any  $n \times n$  matrix, what are the possible dimensions of the space  $V$  of all  $n \times n$  matrices  $A$  such that  $BA = 0$ ?

43. If matrix  $A$  represents the reflection about a line  $L$  in  $\mathbb{R}^2$ , what is the dimension of the space  $V$  of all matrices  $S$  such that

$$AS = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ?$$

*Hint:* Write  $S = [\vec{v} \ \vec{w}]$ , and show that  $\vec{v}$  must be parallel to  $L$ , while  $\vec{w}$  must be perpendicular to  $L$ .

44. If matrix  $A$  represents the orthogonal projection onto a plane  $V$  in  $\mathbb{R}^3$ , what is the dimension of the space  $V$  of all matrices  $S$  such that

$$AS = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

See Exercise 43.

45. Find a basis of the space  $V$  of all  $3 \times 3$  matrices  $A$  that commute with

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and thus determine the dimension of  $V$ .

46. In the linear space of infinite sequences, consider the subspace  $W$  of arithmetic sequences. See Exercise 12. Find a basis for  $W$ , and thus determine the dimension of  $W$ .

47. A function  $f(t)$  from  $\mathbb{R}$  to  $\mathbb{R}$  is called even if  $f(-t) = f(t)$ , for all  $t$  in  $\mathbb{R}$ , and odd if  $f(-t) = -f(t)$ , for all  $t$ . Are the even functions a subspace of  $F(\mathbb{R}, \mathbb{R})$ , the space of all functions from  $\mathbb{R}$  to  $\mathbb{R}$ ? What about the odd functions? Justify your answers carefully.

48. Find a basis of each of the following linear spaces, and thus determine their dimensions. See Exercise 47.

- a.  $\{f \in P_4 : f \text{ is even}\}$   
 b.  $\{f \in P_4 : f \text{ is odd}\}$

49. Let  $L(\mathbb{R}^m, \mathbb{R}^n)$  be the set of all linear transformations from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Is  $L(\mathbb{R}^m, \mathbb{R}^n)$  a subspace of  $F(\mathbb{R}^m, \mathbb{R}^n)$ , the space of all functions from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ? Justify your answer carefully.

50. Find all the solutions of the differential equation  $f''(x) + 8f'(x) - 20f(x) = 0$ .

51. Find all the solutions of the differential equation  $f''(x) - 7f'(x) + 12f(x) = 0$ .

52. Make up a second-order linear DE whose solution space is spanned by the functions  $e^{-x}$  and  $e^{-5x}$ .

53. Show that in an  $n$ -dimensional linear space we can find at most  $n$  linearly independent elements. *Hint:* Consider the proof of Theorem 1.5.

54. Show that if  $W$  is a subspace of an  $n$ -dimensional linear space  $V$ , then  $W$  is finite dimensional as well, and  $\dim(W) \leq n$ .

55. Show that the space  $F(\mathbb{R}, \mathbb{R})$  of all functions from  $\mathbb{R}$  to  $\mathbb{R}$  is infinite dimensional.

56. Show that the space of infinite sequences of real numbers is infinite dimensional.

57. We say that a linear space  $V$  is *finitely generated* if it can be spanned by finitely many elements. Show that a finitely generated space is in fact finite dimensional (and vice versa, of course). Furthermore, if the elements  $g_1, \dots, g_m$  span  $V$ , then  $\dim(V) \leq m$ .

58. In this exercise we will show that the functions  $\cos(x)$  and  $\sin(x)$  span the solution space  $V$  of the differential equation  $f''(x) = -f(x)$ . See Example 1 of this section.

## Linear Spaces

- a. Show that if  $g(x)$  is in  $V$ , then the function  $(g(x))^2 + (g'(x))^2$  is constant. *Hint:* Consider the derivative.
- b. Show that if  $g(x)$  is in  $V$ , with  $g(0) = g'(0) = 0$ , then  $g(x) = 0$  for all  $x$ .
- c. If  $f(x)$  is in  $V$ , then  $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$  is in  $V$  as well (why?). Verify that  $g(0) = 0$  and  $g'(0) = 0$ . We can conclude that  $g(x) = 0$  for all  $x$ , so that  $f(x) = f(0)\cos(x) + f'(0)\sin(x)$ . It follows that the functions  $\cos(x)$  and  $\sin(x)$  span  $V$ , as claimed.
59. Show that if 0 is the neutral element of a linear space  $V$ , then  $k0 = 0$ , for all scalars  $k$ .
60. Consider the sequence  $(f_0, f_1, f_2, \dots)$  recursively defined by  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_n = f_{n-2} + f_{n-1}$  for all  $n = 2, 3, 4, \dots$ . This is known as the *Fibonacci sequence*.
- In this exercise you are invited to derive a *closed formula* for  $f_n$ , expressing  $f_n$  in terms of  $n$ , rather than recursively in terms of  $f_{n-1}$  and  $f_{n-2}$ .
- a. Find the terms  $f_0, f_1, \dots, f_9, f_{10}$  of the Fibonacci sequence.
- b. In the space  $V$  of all infinite sequences of real numbers (see Example 5), consider the subset  $W$  of all sequences  $(x_0, x_1, x_2, \dots)$  that satisfy the recursive equation  $x_n = x_{n-2} + x_{n-1}$  for all  $n = 2, 3, 4, \dots$ . Note that the Fibonacci sequence belongs to  $W$ . Show that  $W$  is a subspace of  $V$ , and find a basis of  $W$  (write the first five terms  $x_0, \dots, x_4$  of each sequence in your basis). Determine the dimension of  $W$ .
- c. Find all *geometric sequences* of the form  $(1, r, r^2, \dots)$  in  $W$ . Can you form a basis of  $W$  consisting of such sequences? (Be prepared to deal with irrational numbers.)
- d. Write the Fibonacci sequence as a linear combination of geometric sequences. Use your answer to find a closed formula for  $f_n$ .<sup>6</sup>
- e. Explain why  $f_n$  is the integer closest to  $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$ , for all  $n = 0, 1, 2, \dots$ . Use technology to find  $f_{50}$ .<sup>7</sup>
- f. Find  $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n}$ .

## 2 Linear Transformations and Isomorphisms

In this section, we will define the concepts of a linear transformation, image, kernel, rank, and nullity in the context of linear spaces.

### Definition 2.1

#### Linear transformations, image, kernel, rank, nullity

Consider two linear spaces  $V$  and  $W$ . A function  $T$  from  $V$  to  $W$  is called a *linear transformation* if

$$T(f + g) = T(f) + T(g) \quad \text{and} \quad T(kf) = kT(f)$$

for all elements  $f$  and  $g$  of  $V$  and for all scalars  $k$ . These two rules are referred to as the *sum rule* and the *constant-multiple rule*, respectively.

For a linear transformation  $T$  from  $V$  to  $W$ , we let

$$\text{im}(T) = \{T(f) : f \text{ in } V\}$$

and

$$\ker(T) = \{f \text{ in } V : T(f) = 0\}.$$

Note that  $\text{im}(T)$  is a subspace of target space  $W$  and that  $\ker(T)$  is a subspace of domain  $V$ .

---

<sup>6</sup>This is known as *Binet's formula*, named after the French mathematician Jacques Binet (1786–1856), although the result was known to Leonhard Euler and Daniel Bernoulli more than a century earlier.

<sup>7</sup>The number  $\frac{1+\sqrt{5}}{2} \approx 1.618$ , known as the *golden ratio* or *golden section*, is of interest not just to mathematicians, but to painters, musicians, architects, and biologists as well.

**Definition 2.1**
**Linear transformations, image, kernel, rank, nullity (Continued)**

If the image of  $T$  is finite dimensional, then  $\dim(\text{im } T)$  is called the *rank* of  $T$ , and if the kernel of  $T$  is finite dimensional, then  $\dim(\ker T)$  is the *nullity* of  $T$ .

If  $V$  is finite dimensional, then the rank-nullity theorem holds. See the rank-nullity theorem:

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) = \dim(\text{im } T) + \dim(\ker T).$$

A proof of the rank-nullity theorem is outlined in Exercise 81.

**EXAMPLE 1**

Consider the transformation  $D(f) = f'$  from  $C^\infty$  to  $C^\infty$ . It follows from the rules of calculus that  $D$  is a linear transformation:

$$\begin{aligned} D(f+g) &= (f+g)' = f' + g' \quad \text{equals} \quad D(f) + D(g) = f' + g' \quad \text{and} \\ D(kf) &= (kf)' = kf' \quad \text{equals} \quad kD(f) = kf'. \end{aligned}$$

Here  $f$  and  $g$  are smooth functions, and  $k$  is a constant.

What is the *kernel* of  $D$ ? This kernel consists of all smooth functions  $f$  such that  $D(f) = f' = 0$ . As you may recall from calculus, these are the constant functions  $f(x) = k$ . Therefore, the kernel of  $D$  is one-dimensional; the function  $f(x) = 1$  is a basis. The nullity of  $D$  is 1.

What about the *image* of  $D$ ? The image consists of all smooth functions  $g$  such that  $g = D(f) = f'$  for some function  $f$  in  $C^\infty$  (i.e., all smooth functions  $g$  that have a smooth antiderivative  $f$ ). The fundamental theorem of calculus implies that all smooth functions (in fact, all continuous functions) have an antiderivative. We can conclude that

$$\text{im}(D) = C^\infty.$$

**EXAMPLE 2**

Let  $C[0, 1]$  be the linear space of all continuous functions from the closed interval  $[0, 1]$  to  $\mathbb{R}$ . We define the transformation

$$I(f) = \int_0^1 f(x) dx \quad \text{from } C[0, 1] \text{ to } \mathbb{R}.$$

We adopt the simplified notation  $I(f) = \int_0^1 f$ . To check that  $I$  is linear, we apply basic rules of integration:

$$I(f+g) = \int_0^1 (f+g) = \int_0^1 f + \int_0^1 g \quad \text{equals} \quad I(f) + I(g) = \int_0^1 f + \int_0^1 g$$

and

$$I(kf) = \int_0^1 (kf) = k \int_0^1 f \quad \text{equals} \quad kI(f) = k \int_0^1 f.$$

What is the image of  $I$ ? The image of  $I$  consists of all real numbers  $b$  such that

$$b = I(f) = \int_0^1 f,$$

for some continuous function  $f$ . One of many possible choices for  $f$  is the constant function  $f(x) = b$ . Therefore,

$$\text{im}(I) = \mathbb{R}, \quad \text{and} \quad \text{rank}(I) = 1.$$

We leave it to the reader to think about the kernel of  $I$ .

## Linear Spaces

**EXAMPLE 3** Let  $V$  be the space of all infinite sequences of real numbers. Consider the transformation

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$$

from  $V$  to  $V$ . (We drop the first term,  $x_0$ , of the sequence.)

- a. Show that  $T$  is a linear transformation.
- b. Find the kernel of  $T$ .
- c. Is the sequence  $(1, 2, 3, \dots)$  in the image of  $T$ ?
- d. Find the image of  $T$ .

### Solution

a.  $T((x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots)) = T(x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots)$   
 $= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots)$

equals

$$T(x_0, x_1, x_2, \dots) + T(y_0, y_1, y_2, \dots) = (x_1, x_2, x_3, \dots) + (y_1, y_2, y_3, \dots)$$

$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots).$$

We leave it to the reader to verify the constant-multiple rule.

- b. The kernel consists of everything that is transformed to zero, that is, all sequences  $(x_0, x_1, x_2, \dots)$  such that

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots) = (0, 0, 0, \dots).$$

This means that entries  $x_1, x_2, x_3, \dots$  all have to be zero, while  $x_0$  is arbitrary. Thus,  $\ker(T)$  consists of all sequences of the form  $(x_0, 0, 0, \dots)$ , where  $x_0$  is arbitrary. The kernel of  $T$  is one-dimensional, with basis  $(1, 0, 0, 0, \dots)$ . The nullity of  $T$  is 1.

- c. We need to find a sequence  $(x_0, x_1, x_2, \dots)$  such that

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots) = (1, 2, 3, \dots).$$

It is required that  $x_1 = 1, x_2 = 2, x_3 = 3, \dots$ , and we can choose any value for  $x_0$ , for example,  $x_0 = 0$ . Thus,

$$(1, 2, 3, \dots) = T(0, 1, 2, 3, \dots)$$

is indeed in the image of  $T$ .

- d. Mimicking our solution in part (c), we can write any sequence  $(b_0, b_1, b_2, \dots)$  as

$$(b_0, b_1, b_2, \dots) = T(0, b_0, b_1, b_2, \dots),$$

so that  $\text{im}(T) = V$ . ■

**EXAMPLE 4** Consider the transformation

$$L \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \text{from } \mathbb{R}^{2 \times 2} \text{ to } \mathbb{R}^4.$$

Note that  $L$  is the coordinate transformation  $L_{\mathfrak{B}}$  with respect to the standard basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of  $\mathbb{R}^{2 \times 2}$ ; see Example 1.15. Being a coordinate transformation,  $L$  is both linear and invertible; see Theorem 1.4.

## Linear Spaces

Note that the elements of both  $\mathbb{R}^{2 \times 2}$  and  $\mathbb{R}^4$  are described by a list of four scalars  $a, b, c$ , and  $d$ . The linear transformation  $L$  merely “rearranges” these scalars, and  $L^{-1}$  puts them back into their original places in  $\mathbb{R}^{2 \times 2}$ .

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \xrightarrow{L} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\quad \xleftarrow{L^{-1}}$$

The linear spaces  $\mathbb{R}^{2 \times 2}$  and  $\mathbb{R}^4$  have essentially the *same structure*. We say that the linear spaces  $\mathbb{R}^{2 \times 2}$  and  $\mathbb{R}^4$  are *isomorphic*, from Greek *ἴσος* (isos), same, and *μορφή* (morphe), structure. The invertible linear transformation  $L$  is called an *isomorphism*. ■

### Definition 2.2

#### Isomorphisms and isomorphic spaces

An invertible linear transformation  $T$  is called an *isomorphism*. We say that the linear space  $V$  is isomorphic to the linear space  $W$  if there exists an isomorphism  $T$  from  $V$  to  $W$ .

We can generalize our findings in Example 4.

### Theorem 2.3

#### Coordinate transformations are isomorphisms

If  $\mathfrak{B} = (f_1, f_2, \dots, f_n)$  is a basis of a linear space  $V$ , then the *coordinate transformation*  $L_{\mathfrak{B}}(f) = [f]_{\mathfrak{B}}$  from  $V$  to  $\mathbb{R}^n$  is an isomorphism. Thus,  $V$  is isomorphic to  $\mathbb{R}^n$ ; the linear spaces  $V$  and  $\mathbb{R}^n$  have the same structure.

$$f = c_1 f_1 + \cdots + c_n f_n \quad \text{in } V \quad \xrightarrow{(L_{\mathfrak{B}})^{-1}} \quad [f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{in } \mathbb{R}^n$$

Let’s reiterate the main point: *Any n-dimensional linear space V is isomorphic to  $\mathbb{R}^n$* . This means that we don’t need a new theory for finite dimensional spaces. By introducing coordinates, we can transform any  $n$ -dimensional linear space into  $\mathbb{R}^n$ . Infinite dimensional linear spaces, on the other hand, are largely beyond the reach of the methods of elementary linear algebra.

### EXAMPLE 5

Show that the transformation

$$T(A) = S^{-1}AS \quad \text{from } \mathbb{R}^{2 \times 2} \text{ to } \mathbb{R}^{2 \times 2}$$

is an isomorphism, where  $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ .

#### Solution

We need to show that  $T$  is a linear transformation, and that  $T$  is invertible.

Let’s check the linearity first:

$$\begin{aligned} T(A_1 + A_2) &= S^{-1}(A_1 + A_2)S = S^{-1}(A_1S + A_2S) = S^{-1}A_1S + S^{-1}A_2S \\ &\quad \text{equals} \\ T(A_1) + T(A_2) &= S^{-1}A_1S + S^{-1}A_2S, \end{aligned}$$

and

$$T(kA) = S^{-1}(kA)S = k(S^{-1}AS) \quad \text{equals} \quad kT(A) = k(S^{-1}AS).$$

The most direct way to show that a function is invertible is to exhibit the inverse. Here we need to solve the equation  $B = S^{-1}AS$  for input  $A$ . We find that  $A = SBS^{-1}$ , so that  $T$  is indeed invertible. The inverse transformation is

$$T^{-1}(B) = SBS^{-1}. \quad \blacksquare$$

**Theorem 2.4**

**Properties of isomorphisms**

- a. A linear transformation  $T$  from  $V$  to  $W$  is an isomorphism if (and only if)  $\ker(T) = \{0\}$  and  $\text{im}(T) = W$ .

*In parts (b) through (d), the linear spaces  $V$  and  $W$  are assumed to be finite dimensional.*

- b. The linear space  $V$  is isomorphic to  $W$  if (and only if)  $\dim(V) = \dim(W)$ .
- c. Suppose  $T$  is a linear transformation from  $V$  to  $W$  with  $\ker(T) = \{0\}$ . If  $\dim(V) = \dim(W)$ , then  $T$  is an isomorphism.
- d. Suppose  $T$  is a linear transformation from  $V$  to  $W$  with  $\text{im}(T) = W$ . If  $\dim(V) = \dim(W)$ , then  $T$  is an isomorphism.

**Proof**

- a. Suppose first that  $T$  is an isomorphism. To find the kernel of  $T$ , we have to solve the equation  $T(f) = 0$ . Applying  $T^{-1}$  on both sides, we find that  $f = T^{-1}(0) = 0$ , so that  $\ker(T) = \{0\}$ , as claimed. See Exercise 75. To see that  $\text{im}(T) = W$ , note that any  $g$  in  $W$  can be written as  $g = T(T^{-1}(g))$ .

Conversely, suppose that  $\ker(T) = \{0\}$  and  $\text{im}(T) = W$ . We have to show that  $T$  is invertible; that is, the equation  $T(f) = g$  has a unique solution  $f$  for every  $g$  in  $W$  (by the definition of invertible functions). There is at least one solution  $f$ , since  $\text{im}(T) = W$ . Consider two solutions  $f_1$  and  $f_2$ , so that  $T(f_1) = T(f_2) = g$ . Then

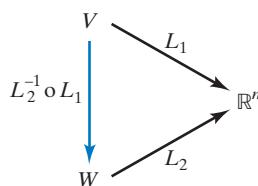
$$0 = T(f_1) - T(f_2) = T(f_1 - f_2),$$

so that  $f_1 - f_2$  is in the kernel of  $T$ . Since the kernel of  $T$  is  $\{0\}$ , we must have  $f_1 - f_2 = 0$  and  $f_1 = f_2$ , as claimed.

- b. First assume that  $V$  is isomorphic to  $W$ , and let  $T$  be an isomorphism from  $V$  to  $W$ . Now apply the rank-nullity theorem (Definition 2.1) to  $T$ :

$$\dim(V) = \dim(\ker T) + \dim(\text{im } T) = 0 + \dim(W) = \dim(W), \text{ by part (a).}$$

Conversely, suppose that  $\dim(V) = \dim(W) = n$ . Then, by Theorem 2.3, both  $V$  and  $W$  are isomorphic to  $\mathbb{R}^n$ . If  $L_1$  is an isomorphism from  $V$  to  $\mathbb{R}^n$  and  $L_2$  is an isomorphism from  $W$  to  $\mathbb{R}^n$ , then  $L_2^{-1} \circ L_1$  will be an isomorphism from  $V$  to  $W$ , by Exercises 76 and 77. Thus,  $V$  is isomorphic to  $W$ , as claimed.



## Linear Spaces

- c. By part (a), it suffices to show that  $\text{im}(T) = W$ , or, equivalently, that  $\dim(\text{im } T) = \dim(W)$ . But this claim follows from the rank-nullity theorem:

$$\dim(W) = \dim(V) = \dim(\ker T) + \dim(\text{im } T) = \dim(\text{im } T).$$

- d. By part (a), it suffices to show that  $\ker(T) = \{0\}$ . The proof is analogous to part (c). ■

### EXAMPLE 6

- a. Is the linear transformation

$$L(f(x)) = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} \quad \text{from } P_3 \text{ to } \mathbb{R}^3 \text{ an isomorphism?}$$

- b. Is the linear transformation

$$T(f(x)) = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} \quad \text{from } P_2 \text{ to } \mathbb{R}^3 \text{ an isomorphism?}$$

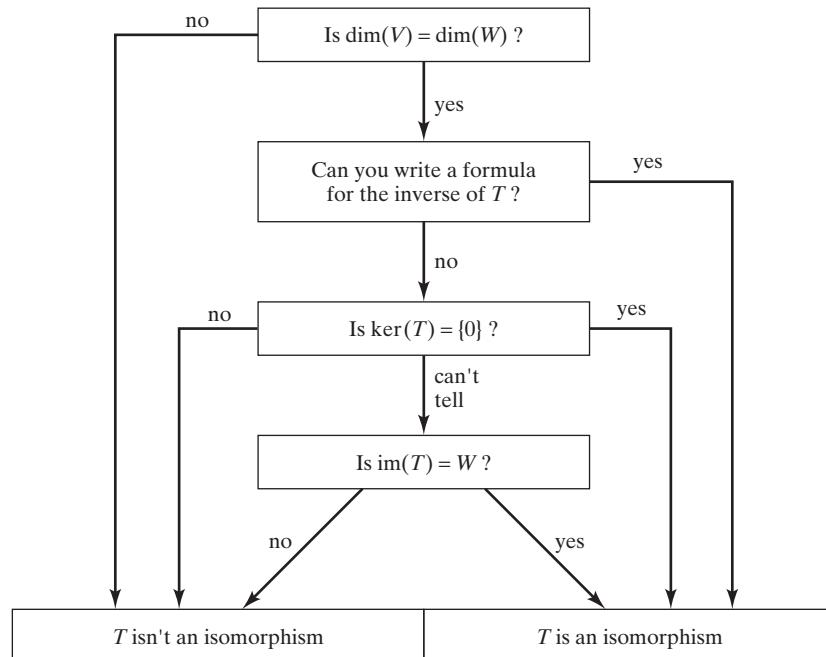
### Solution

- a. Consider Theorem 2.4b. Since  $\dim(P_3) = 4$  and  $\dim(\mathbb{R}^3) = 3$ , the spaces  $P_3$  and  $\mathbb{R}^3$  fail to be isomorphic, so that  $L$  fails to be an isomorphism.

- b. In this case, the domain and target space have the same dimension,

$$\dim(P_2) = \dim(\mathbb{R}^3) = 3.$$

This fact alone does not guarantee that  $T$  is an isomorphism, however. Let's find the kernel of  $T$  and use Theorem 2.4c. The kernel of  $T$  consists of all polynomials  $f(x)$  in  $P_2$  such that



**Figure 1** Is the linear transformation  $T$  from  $V$  to  $W$  an isomorphism? ( $V$  and  $W$  are finite dimensional linear spaces.)

$$T(f(x)) = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

that is,  $f(1) = 0$ ,  $f(2) = 0$ , and  $f(3) = 0$ . Since a *nonzero* polynomial in  $P_2$  has at most two zeros, the zero polynomial is the only solution, so that  $\ker(T) = \{0\}$ . Thus,  $T$  is indeed an isomorphism. ■

The diagram in Figure 1 can be a useful guide in determining whether a given linear transformation  $T$  from  $V$  to  $W$  is an isomorphism. Here we assume that both  $V$  and  $W$  are *finite dimensional*. (We leave it as an exercise for the reader to adapt the diagram to the case of infinite dimensional spaces.)

## EXERCISES 2

**GOAL** Examine whether a transformation is linear. Find the image and kernel of a linear transformation. Examine whether a linear transformation is an isomorphism.

Find out which of the transformations in Exercises 1 through 50 are linear. For those that are linear, determine whether they are isomorphisms.

1.  $T(M) = M + I_2$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
2.  $T(M) = 7M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
3.  $T(M) =$  (sum of the diagonal entries of  $M$ ) from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$
4.  $T(M) = \det(M)$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$
5.  $T(M) = M^2$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
6.  $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
7.  $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
8.  $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
9.  $T(M) = S^{-1}MS$ , where  $S = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ , from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
10.  $T(M) = PMP^{-1}$ , where  $P = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ , from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
11.  $T(M) = PMQ$ , where  $P = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$  and  $Q = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$ , from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
12.  $T(c) = c \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$  from  $\mathbb{R}$  to  $\mathbb{R}^{2 \times 2}$
13.  $T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

14.  $T(M) = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} M - M \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
  15.  $T(M) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} M - M \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
  16.  $T(M) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$
  17.  $T(x + iy) = x$  from  $\mathbb{C}$  to  $\mathbb{C}$
  18.  $T(x + iy) = x^2 + y^2$  from  $\mathbb{C}$  to  $\mathbb{C}$
  19.  $T(x + iy) = i(x + iy)$  from  $\mathbb{C}$  to  $\mathbb{C}$
  20.  $T(x + iy) = x - iy$  from  $\mathbb{C}$  to  $\mathbb{C}$
  21.  $T(x + iy) = y + ix$  from  $\mathbb{C}$  to  $\mathbb{C}$
  22.  $T(f(t)) = \int_{-2}^3 f(t) dt$  from  $P_2$  to  $\mathbb{R}$
  23.  $T(f(t)) = f(7)$  from  $P_2$  to  $\mathbb{R}$
  24.  $T(f(t)) = f''(t)f(t)$  from  $P_2$  to  $P_2$
  25.  $T(f(t)) = f''(t) + 4f'(t)$  from  $P_2$  to  $P_2$
  26.  $T(f(t)) = f(-t)$  from  $P_2$  to  $P_2$ , that is,  $T(a + bt + ct^2) = a - bt + ct^2$
  27.  $T(f(t)) = f(2t)$  from  $P_2$  to  $P_2$ , that is,  $T(a + bt + ct^2) = a + 2bt + 4ct^2$
  28.  $T(f(t)) = f(2t) - f(t)$  from  $P_2$  to  $P_2$
  29.  $T(f(t)) = f'(t)$  from  $P_2$  to  $P_2$
  30.  $T(f(t)) = t(f'(t))$  from  $P_2$  to  $P_2$
  31.  $T(f(t)) = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$  from  $P_2$  to  $\mathbb{R}^{2 \times 2}$
  32.  $T(f(t)) = f'(t) + t^2$  from  $P_2$  to  $P_2$
- In Exercises 33 through 36,  $V$  denotes the space of infinite sequences of real numbers.
33.  $T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots)$  from  $V$  to  $V$  (we are dropping every other term)
  34.  $T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)$  from  $V$  to  $V$

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- 35.**  $T(f(t)) = (f(0), f'(0), f''(0), f'''(0), \dots)$  from  $P$  to  $V$ , where  $P$  denotes the space of all polynomials
- 36.**  $T(f(t)) = (f(0), f(1), f(2), f(3), \dots)$  from  $P$  to  $V$ , where  $P$  denotes the space of all polynomials
- 37.**  $T(f) = f + f'$  from  $C^\infty$  to  $C^\infty$
- 38.**  $T(f) = f + f''$  from  $C^\infty$  to  $C^\infty$
- 39.**  $T(f) = f'' - 5f' + 6f$  from  $C^\infty$  to  $C^\infty$
- 40.**  $T(f) = f'' + 2f' + f$  from  $C^\infty$  to  $C^\infty$
- 41.**  $T(f(t)) = f(t) + f''(t) + \sin(t)$  from  $C^\infty$  to  $C^\infty$
- 42.**  $T(f(t)) = \begin{bmatrix} f(7) \\ f(11) \end{bmatrix}$  from  $P_2$  to  $\mathbb{R}^2$
- 43.**  $T(f(t)) = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix}$  from  $P_2$  to  $\mathbb{R}^3$
- 44.**  $T(f(t)) = \begin{bmatrix} f(1) \\ f'(2) \\ f(3) \end{bmatrix}$  from  $P_2$  to  $\mathbb{R}^3$
- 45.**  $T(f(t)) = t(f(t))$  from  $P$  to  $P$
- 46.**  $T(f(t)) = (t-1)f(t)$  from  $P$  to  $P$
- 47.**  $T(f(t)) = \int_0^t f(x) dx$  from  $P$  to  $P$
- 48.**  $T(f(t)) = f'(t)$  from  $P$  to  $P$
- 49.**  $T(f(t)) = f(t^2)$  from  $P$  to  $P$
- 50.**  $T(f(t)) = \frac{f(t+2) - f(t)}{2}$  from  $P$  to  $P$
- 51.** Find the kernel and nullity of the transformation  $T$  in Exercise 13.
- 52.** Find the kernel and nullity of the transformation  $T$  in Exercise 6.
- 53.** Find the image, rank, kernel, and nullity of the transformation  $T$  in Exercise 25.
- 54.** Find the image, rank, kernel, and nullity of the transformation  $T$  in Exercise 22.
- 55.** Find the image and kernel of the transformation  $T$  in Exercise 33.
- 56.** Find the image, rank, kernel, and nullity of the transformation  $T$  in Exercise 30.
- 57.** Find the kernel and nullity of the transformation  $T$  in Exercise 39.
- 58.** Find the image and kernel of the transformation  $T$  in Exercise 34.
- 59.** For the transformation  $T$  in Exercise 23, find the kernel, nullity, rank, and image, in this order.
- 60.** For the transformation  $T$  in Exercise 42, find the kernel, nullity, rank, and image, in this order.
- 61.** Find the image and kernel of the transformation  $T$  in Exercise 45.
- 62.** Find the image and kernel of the transformation  $T$  in Exercise 48.
- 63.** Define an isomorphism from  $P_3$  to  $\mathbb{R}^3$ , if you can.
- 64.** Define an isomorphism from  $P_3$  to  $\mathbb{R}^{2 \times 2}$ , if you can.
- 65.** We will define a transformation  $T$  from  $\mathbb{R}^{n \times m}$  to  $F(\mathbb{R}^m, \mathbb{R}^n)$ ; recall that  $F(\mathbb{R}^m, \mathbb{R}^n)$  is the space of all functions from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . For a matrix  $A$  in  $\mathbb{R}^{n \times m}$ , the value  $T(A)$  will be a function from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ; thus, we need to define  $(T(A))(\vec{v})$  for a vector  $\vec{v}$  in  $\mathbb{R}^m$ . We let
- $$(T(A))(\vec{v}) = A\vec{v}.$$
- a.** Show that  $T$  is a linear transformation.
- b.** Find the kernel of  $T$ .
- c.** Show that the image of  $T$  is the space  $L(\mathbb{R}^m, \mathbb{R}^n)$  of all linear transformations from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . See Exercise 1.19.
- d.** Find the dimension of  $L(\mathbb{R}^m, \mathbb{R}^n)$ .
- 66.** Find the kernel and nullity of the linear transformation  $T(f) = f - f'$  from  $C^\infty$  to  $C^\infty$ .
- 67.** For which constants  $k$  is the linear transformation
- $$T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$$
- an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ ?
- 68.** For which constants  $k$  is the linear transformation
- $$T(M) = M \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix} M$$
- an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ ?
- 69.** If matrix  $A$  is similar to  $B$ , is  $T(M) = AM - MB$  an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ ?
- 70.** For which real numbers  $c_0, c_1, \dots, c_n$  is the linear transformation
- $$T(f(t)) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}$$
- an isomorphism from  $P_n$  to  $\mathbb{R}^{n+1}$ ?

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- 71.** Does there exist a polynomial  $f(t)$  of degree  $\leq 4$  such that  $f(2) = 3$ ,  $f(3) = 5$ ,  $f(5) = 7$ ,  $f(7) = 11$ , and  $f(11) = 2$ ? If so, how many such polynomials are there? Hint: Use Exercise 70.

In Exercises 72 through 74, let  $Z_n$  be the set of all polynomials of degree  $\leq n$  such that  $f(\mathbf{0}) = \mathbf{0}$ .

- 72.** Show that  $Z_n$  is a subspace of  $P_n$ , and find the dimension of  $Z_n$ .
- 73.** Is the linear transformation  $T(f(t)) = \int_0^t f(x) dx$  an isomorphism from  $P_{n-1}$  to  $Z_n$ ?
- 74.** Define an isomorphism from  $Z_n$  to  $P_{n-1}$  (think calculus!).
- 75.** Show that if  $T$  is a linear transformation from  $V$  to  $W$ , then  $T(0_V) = 0_W$ , where  $0_V$  and  $0_W$  are the neutral elements of  $V$  and  $W$ , respectively. If  $T$  is an isomorphism, show that  $T^{-1}(0_W) = 0_V$ .
- 76.** Prove that if  $T$  is an isomorphism from  $V$  to  $W$ , then  $T^{-1}$  is an isomorphism from  $W$  to  $V$ . Hint:  $T^{-1}(f + g) = T^{-1}(T(T^{-1}(f)) + T(T^{-1}(g)))$ .

- 77.** If  $T$  is a linear transformation from  $V$  to  $W$  and  $L$  is a linear transformation from  $W$  to  $U$ , is the composite transformation  $L \circ T$  from  $V$  to  $U$  linear? How can you tell? If  $T$  and  $L$  are isomorphisms, is  $L \circ T$  an isomorphism as well?

- 78.** Let  $\mathbb{R}^+$  be the set of positive real numbers. On  $\mathbb{R}^+$  we define the “exotic” operations

$$x \oplus y = xy \quad (\text{usual multiplication})$$

and

$$k \odot x = x^k.$$

- a.** Show that  $\mathbb{R}^+$  with these operations is a linear space; find a basis of this space.
- b.** Show that  $T(x) = \ln(x)$  is a linear transformation from  $\mathbb{R}^+$  to  $\mathbb{R}$ , where  $\mathbb{R}$  is endowed with the ordinary operations. Is  $T$  an isomorphism?
- 79.** Is it possible to define “exotic” operations on  $\mathbb{R}^2$ , so that  $\dim(\mathbb{R}^2) = 1$ ?
- 80.** Let  $X$  be the set of all students in your linear algebra class. Can you define operations on  $X$  that make  $X$  into a real linear space? Explain.

- 81.** In this exercise, we will outline a proof of the rank-nullity theorem: If  $T$  is a linear transformation from  $V$  to  $W$ , where  $V$  is finite dimensional, then

$$\begin{aligned}\dim(V) &= \dim(\text{im } T) + \dim(\ker T) \\ &= \text{rank}(T) + \text{nullity}(T).\end{aligned}$$

- a.** Explain why  $\ker(T)$  and  $\text{image}(T)$  are finite dimensional. Hint: Use Exercises 1.54 and 1.57.

Now consider a basis  $v_1, \dots, v_n$  of  $\ker(T)$ , where  $n = \text{nullity}(T)$ , and a basis  $w_1, \dots, w_r$  of  $\text{im}(T)$ , where  $r = \text{rank}(T)$ . Consider elements  $u_1, \dots, u_r$  in  $V$  such that  $T(u_i) = w_i$  for  $i = 1, \dots, r$ . Our goal is to show that the  $r+n$  elements  $u_1, \dots, u_r, v_1, \dots, v_n$  form a basis of  $V$ ; this will prove our claim.

- b.** Show that the elements  $u_1, \dots, u_r, v_1, \dots, v_n$  are linearly independent. Hint: Consider a relation  $c_1u_1 + \dots + c_ru_r + d_1v_1 + \dots + d_nv_n = 0$ , apply transformation  $T$  to both sides, and take it from there.
- c.** Show that the elements  $u_1, \dots, u_r, v_1, \dots, v_n$  span  $V$ . Hint: Consider an arbitrary element  $v$  in  $V$ , and write  $T(v) = d_1w_1 + \dots + d_rw_r$ . Now show that the element  $v - d_1u_1 - \dots - d_ru_r$  is in the kernel of  $T$ , so that  $v - d_1u_1 - \dots - d_ru_r$  can be written as a linear combination of  $v_1, \dots, v_n$ .

- 82.** Prove the following variant of the rank-nullity theorem: If  $T$  is a linear transformation from  $V$  to  $W$ , and if  $\ker T$  and  $\text{im } T$  are both finite dimensional, then  $V$  is finite dimensional as well, and  $\dim V = \dim(\ker T) + \dim(\text{im } T)$ .

- 83.** Consider linear transformations  $T$  from  $V$  to  $W$  and  $L$  from  $W$  to  $U$ . If  $\ker T$  and  $\ker L$  are both finite dimensional, show that  $\ker(L \circ T)$  is finite dimensional as well, and  $\dim(\ker(L \circ T)) \leq \dim(\ker T) + \dim(\ker L)$ . Hint: Restrict  $T$  to  $\ker(L \circ T)$  and apply the rank-nullity theorem, as presented in Exercise 82.

- 84.** Consider linear transformations  $T$  from  $V$  to  $W$  and  $L$  from  $W$  to  $U$ . If  $\ker T$  and  $\ker L$  are both finite dimensional, and if  $\text{im } T = W$ , show that  $\ker(L \circ T)$  is finite dimensional as well and that  $\dim(\ker(L \circ T)) = \dim(\ker T) + \dim(\ker L)$ .

## 3 The Matrix of a Linear Transformation

Next we will examine how we can express a linear transformation in coordinates.

**EXAMPLE I** Consider the linear transformation

$$T(f) = f' + f'' \quad \text{from } P_2 \text{ to } P_2,$$

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or, written more explicitly,

$$T(f(x)) = f'(x) + f''(x).$$

Since  $P_2$  is isomorphic to  $\mathbb{R}^3$ , this is essentially a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , given by a  $3 \times 3$  matrix  $B$ . Let's see how we can find this matrix.

If we let  $f(x) = a + bx + cx^2$ , then we can write transformation  $T$  as

$$\begin{aligned} T(a + bx + cx^2) &= (a + bx + cx^2)' + (a + bx + cx^2)'' \\ &= (b + 2cx) + 2c = (b + 2c) + 2cx. \end{aligned}$$

Next let's write the input  $f(x) = a + bx + cx^2$  and the output  $T(f(x)) = (b + 2c) + 2cx$  in coordinates with respect to the standard basis  $\mathfrak{B} = (1, x, x^2)$  of  $P_2$ :

$$\begin{array}{ccc} f(x) = a + bx + cx^2 & \xrightarrow{T} & T(f(x)) = (b + 2c) + 2cx \\ \downarrow L_{\mathfrak{B}} & & \downarrow L_{\mathfrak{B}} \\ [f(x)]_{\mathfrak{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \longrightarrow & [T(f(x))]_{\mathfrak{B}} = \begin{bmatrix} b+2c \\ 2c \\ 0 \end{bmatrix}. \end{array}$$

Written in  $\mathfrak{B}$ -coordinates, transformation  $T$  takes  $[f(x)]_{\mathfrak{B}}$  to

$$[T(f(x))]_{\mathfrak{B}} = \begin{bmatrix} b+2c \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} [f(x)]_{\mathfrak{B}}.$$

The matrix

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

is called the  $\mathfrak{B}$ -matrix of transformation  $T$ . It describes the transformation  $T$  if the input and output are written in  $\mathfrak{B}$ -coordinates. Let us summarize our work in two diagrams:

$$\begin{array}{ccc} P_2 & \xrightarrow{T} & P_2 \\ \downarrow L_{\mathfrak{B}} & & \downarrow L_{\mathfrak{B}} \\ \mathbb{R}^3 & \xrightarrow{B} & \mathbb{R}^3, \end{array} \quad \begin{array}{ccc} f & \xrightarrow{T} & T(f) \\ \downarrow L_{\mathfrak{B}} & & \downarrow L_{\mathfrak{B}} \\ [f]_{\mathfrak{B}} & \xrightarrow{B} & [T(f)]_{\mathfrak{B}}. \end{array}$$



### Definition 3.1

#### The $\mathfrak{B}$ -matrix of a linear transformation

Consider a linear transformation  $T$  from  $V$  to  $V$ , where  $V$  is an  $n$ -dimensional linear space. Let  $\mathfrak{B}$  be a basis of  $V$ . Consider the linear transformation  $L_{\mathfrak{B}} \circ T \circ L_{\mathfrak{B}}^{-1}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , with standard matrix  $B$ , meaning that  $B\vec{x} = L_{\mathfrak{B}}(T(L_{\mathfrak{B}}^{-1}(\vec{x})))$  for all  $\vec{x}$  in  $\mathbb{R}^n$ . This matrix  $B$  is called the  $\mathfrak{B}$ -matrix of transformation  $T$ . See the accompanying diagrams. Letting  $f = L_{\mathfrak{B}}^{-1}(\vec{x})$  and  $\vec{x} = [f]_{\mathfrak{B}}$ , we find that

$$[T(f)]_{\mathfrak{B}} = B[f]_{\mathfrak{B}}, \quad \text{for all } f \text{ in } V.$$

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Consider the following diagrams.

$$\begin{array}{ccc}
 V & \xrightarrow{T} & V \\
 L_{\mathfrak{B}} \downarrow & & \downarrow L_{\mathfrak{B}} \\
 \mathbb{R}^n & \xrightarrow{B} & \mathbb{R}^n
 \end{array}
 \quad
 \begin{array}{ccc}
 f & \xrightarrow{T} & T(f) \\
 L_{\mathfrak{B}} \downarrow & & \downarrow L_{\mathfrak{B}} \\
 [f]_{\mathfrak{B}} & \xrightarrow{B} & [T(f)]_{\mathfrak{B}}
 \end{array}$$

We can write  $B$  in terms of its columns. Suppose that  $\mathfrak{B} = (f_1, \dots, f_i, \dots, f_n)$ . Then

$$[T(f_i)]_{\mathfrak{B}} = B [f_i]_{\mathfrak{B}} = B \vec{e}_i = (i\text{th column of } B).$$

### Theorem 3.2

#### The columns of the $\mathfrak{B}$ -matrix of a linear transformation

Consider a linear transformation  $T$  from  $V$  to  $V$ , and let  $B$  be the matrix of  $T$  with respect to a basis  $\mathfrak{B} = (f_1, \dots, f_n)$  of  $V$ . Then

$$B = \left[ \begin{array}{ccc|c} & & & \\ [T(f_1)]_{\mathfrak{B}} & \cdots & [T(f_n)]_{\mathfrak{B}} & \\ & & & \end{array} \right].$$

The columns of  $B$  are the  $\mathfrak{B}$ -coordinate vectors of the transforms of the basis elements  $f_1, \dots, f_n$  of  $V$ .

**EXAMPLE 2** Use Theorem 3.2 to find the matrix  $B$  of the linear transformation  $T(f) = f' + f''$  from  $P_2$  to  $P_2$  with respect to the standard basis  $\mathfrak{B} = (1, x, x^2)$ ; see Example 1.

#### Solution

By Theorem 3.2, we have

$$B = \left[ \begin{array}{ccc} [T(1)]_{\mathfrak{B}} & [T(x)]_{\mathfrak{B}} & [T(x^2)]_{\mathfrak{B}} \end{array} \right].$$

Now

$$\begin{aligned}
 T(1) &= 1' + 1'' & T(x) &= x' + x'' & T(x^2) &= (x^2)' + (x^2)'' \\
 &= 0 & &= 1 & &= 2 + 2x
 \end{aligned}$$

$$\begin{aligned}
 [T(1)]_{\mathfrak{B}} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & [T(x)]_{\mathfrak{B}} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & [T(x^2)]_{\mathfrak{B}} &= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \\
 B &= \left[ \begin{array}{ccc} [T(1)]_{\mathfrak{B}} & [T(x)]_{\mathfrak{B}} & [T(x^2)]_{\mathfrak{B}} \end{array} \right] & & & & \\
 &= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

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It is suggestive to write the basis elements 1,  $x$ , and  $x^2$  next to the rows of matrix  $B$ , while the values  $T(1) = 0$ ,  $T(x) = 1$ , and  $T(x^2) = 2 + 2x$  are written above the columns:

$$B = \begin{bmatrix} T(1) & T(x) & T(x^2) \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

The last column of  $B$ , for example, indicates that  $T(x^2) = 2 \cdot 1 + 2 \cdot x = 2 + 2x$ . ■

A problem concerning a linear transformation  $T$  can often be done by solving the corresponding problem for the matrix  $B$  of  $T$  with respect to some basis  $\mathfrak{B}$ . We can use this technique to find the image and kernel of  $T$ , to determine whether  $T$  is an isomorphism (this is the case if  $B$  is invertible), or to solve an equation  $T(f) = g$  for  $f$  if  $g$  is given (this amounts to solving the linear system  $B [f]_{\mathfrak{B}} = [g]_{\mathfrak{B}}$ ).

**EXAMPLE 3** Let  $V$  be the span of  $\cos(x)$  and  $\sin(x)$  in  $C^\infty$ ; note that  $V$  consists of all trigonometric functions of the form  $f(x) = a \cos(x) + b \sin(x)$ . Consider the transformation

$$T(f) = 3f + 2f' - f'' \quad \text{from } V \text{ to } V.$$

We are told that  $T$  is a linear transformation.

- a. Using Theorem 3.2, find the matrix  $B$  of  $T$  with respect to the basis  $\mathfrak{B} = (\cos(x), \sin(x))$ .
- b. Is  $T$  an isomorphism?

### Solution

- a. Here

$$B = \begin{bmatrix} [T(\cos x)]_{\mathfrak{B}} & [T(\sin x)]_{\mathfrak{B}} \end{bmatrix}.$$

Now

$$\begin{aligned} T(\cos x) &= 3 \cos x - 2 \sin x + \cos x \\ &= 4 \cos x - 2 \sin x \\ T(\sin x) &= 3 \sin x + 2 \cos x + \sin x \\ &= 2 \cos x + 4 \sin x. \end{aligned}$$

Then

$$B = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}.$$

Matrix  $B$  represents a rotation combined with a scaling.

- b. Matrix  $B$  is invertible, since  $\det(B) = ad - bc = 20 \neq 0$ . Thus transformation  $T$  is invertible as well, so that  $T$  is an isomorphism (we were told that  $T$  is linear). ■

**EXAMPLE 4** Consider the linear transformation

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{from } \mathbb{R}^{2 \times 2} \text{ to } \mathbb{R}^{2 \times 2}.$$

- a. Find the matrix  $B$  of  $T$  with respect to the standard basis  $\mathfrak{B}$  of  $\mathbb{R}^{2 \times 2}$ .
- b. Find bases of the image and kernel of  $B$ .

## Linear Spaces

- c. Find bases of the image and kernel of  $T$ , and thus determine the rank and nullity of transformation  $T$ .

### Solution

- a. For the sake of variety, let us find  $B$  by means of a diagram.

$$\begin{array}{ccc}
 \left[ \begin{matrix} a & b \\ c & d \end{matrix} \right] & \xrightarrow{T} & \left[ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right] \left[ \begin{matrix} a & b \\ c & d \end{matrix} \right] - \left[ \begin{matrix} a & b \\ c & d \end{matrix} \right] \left[ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right] \\
 & & = \left[ \begin{matrix} c & d \\ 0 & 0 \end{matrix} \right] - \left[ \begin{matrix} 0 & a \\ 0 & c \end{matrix} \right] = \left[ \begin{matrix} c & d-a \\ 0 & -c \end{matrix} \right]
 \end{array}$$

$$\begin{array}{ccc}
 L_{\mathfrak{B}} \downarrow & & \downarrow L_{\mathfrak{B}} \\
 \left[ \begin{matrix} a \\ b \\ c \\ d \end{matrix} \right] & \xrightarrow{B} & \left[ \begin{matrix} c \\ d-a \\ 0 \\ -c \end{matrix} \right]
 \end{array}$$

We see that

$$B = \left[ \begin{matrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{matrix} \right].$$

- b. Note that columns  $\vec{v}_2$  and  $\vec{v}_4$  of  $B$  are redundant, with  $\vec{v}_2 = \vec{0}$  and  $\vec{v}_4 = -\vec{v}_1$ , or  $\vec{v}_1 + \vec{v}_4 = \vec{0}$ . Thus, the nonredundant columns

$$\vec{v}_1 = \left[ \begin{matrix} 0 \\ -1 \\ 0 \\ 0 \end{matrix} \right], \quad \vec{v}_3 = \left[ \begin{matrix} 1 \\ 0 \\ 0 \\ -1 \end{matrix} \right] \quad \text{form a basis of } \text{im}(B),$$

and

$$\left[ \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right], \quad \left[ \begin{matrix} 1 \\ 0 \\ 0 \\ 1 \end{matrix} \right] \quad \text{form a basis of } \ker(B).$$

- c. We apply  $L_{\mathfrak{B}}^{-1}$  to transform the vectors we found in part (b) back into  $\mathbb{R}^{2 \times 2}$ , the domain and target space of transformation  $T$ :

$$\left[ \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} \right], \quad \left[ \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right] \quad \text{is a basis of } \text{im}(T),$$

and

$$\left[ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right], \quad I_2 = \left[ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right] \quad \text{is a basis of } \ker(T).$$

Thus,  $\text{rank}(T) = \dim(\text{im } T) = 2$  and  $\text{nullity}(T) = \dim(\ker T) = 2$ . ■

### Change of Basis

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are two bases of a linear space  $V$ , what is the relationship between the coordinate vectors  $[f]_{\mathfrak{A}}$  and  $[f]_{\mathfrak{B}}$ , for an element  $f$  of  $V$ ?

**Definition 3.3**
**Change of basis matrix**

Consider two bases  $\mathfrak{A}$  and  $\mathfrak{B}$  of an  $n$ -dimensional linear space  $V$ . Consider the linear transformation  $L_{\mathfrak{A}} \circ L_{\mathfrak{B}}^{-1}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , with standard matrix  $S$ , meaning that  $S\vec{x} = L_{\mathfrak{A}}(L_{\mathfrak{B}}^{-1}(\vec{x}))$  for all  $\vec{x}$  in  $\mathbb{R}^n$ . This invertible matrix  $S$  is called the *change of basis matrix* from  $\mathfrak{B}$  to  $\mathfrak{A}$ , sometimes denoted by  $S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ . See the accompanying diagrams. Letting  $f = L_{\mathfrak{B}}^{-1}(\vec{x})$  and  $\vec{x} = [f]_{\mathfrak{B}}$ , we find that

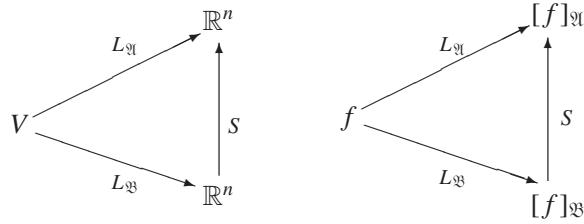
$$[f]_{\mathfrak{A}} = S[f]_{\mathfrak{B}}, \quad \text{for all } f \text{ in } V.$$

If  $\mathfrak{B} = (b_1, \dots, b_i, \dots, b_n)$ , then

$$[b_i]_{\mathfrak{A}} = S[b_i]_{\mathfrak{B}} = S\vec{e}_i = (\text{i-th column of } S),$$

so that

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} & & \\ [b_1]_{\mathfrak{A}} & \cdots & [b_n]_{\mathfrak{A}} \\ & & \end{bmatrix}.$$



What is the relationship between the change of basis matrices  $S_{\mathfrak{B} \rightarrow \mathfrak{A}}$  and  $S_{\mathfrak{A} \rightarrow \mathfrak{B}}$ ? Solving the equation  $[f]_{\mathfrak{A}} = (S_{\mathfrak{B} \rightarrow \mathfrak{A}})[f]_{\mathfrak{B}}$  for  $[f]_{\mathfrak{B}}$ , we find that  $[f]_{\mathfrak{B}} = (S_{\mathfrak{B} \rightarrow \mathfrak{A}})^{-1}[f]_{\mathfrak{A}}$ , so that

$$S_{\mathfrak{A} \rightarrow \mathfrak{B}} = (S_{\mathfrak{B} \rightarrow \mathfrak{A}})^{-1}.$$

**EXAMPLE 5** Let  $V$  be the subspace of  $C^\infty$  spanned by the functions  $e^x$  and  $e^{-x}$ , with the bases  $\mathfrak{A} = (e^x, e^{-x})$  and  $\mathfrak{B} = (e^x + e^{-x}, e^x - e^{-x})$ . Find the change of basis matrix  $S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

**Solution**

By Definition 3.3,

$$S = \begin{bmatrix} [e^x + e^{-x}]_{\mathfrak{A}} & [e^x - e^{-x}]_{\mathfrak{A}} \end{bmatrix}.$$

Now

$$e^x + e^{-x} = 1 \cdot e^x + 1 \cdot e^{-x} \quad e^x - e^{-x} = 1 \cdot e^x + (-1) \cdot e^{-x}$$

$$[e^x + e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [e^x - e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

## Linear Spaces

It is suggestive to write the functions  $e^x$  and  $e^{-x}$  of basis  $\mathfrak{A}$  next to the rows of matrix  $S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ , while the functions  $e^x + e^{-x}$  and  $e^x - e^{-x}$  of basis  $\mathfrak{B}$  are written above the columns:

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} e^x + e^{-x} & e^x - e^{-x} \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^x \\ e^{-x} \end{bmatrix}.$$

The second column of matrix  $S$  indicates that  $e^x - e^{-1} = 1 \cdot e^x + (-1) \cdot e^{-x}$ . ■

**EXAMPLE 6** Let  $\mathfrak{A} = (\vec{e}_1, \dots, \vec{e}_n)$  be the standard basis of  $\mathbb{R}^n$ , and let  $\mathfrak{B} = (\vec{b}_1, \dots, \vec{b}_n)$  be an arbitrary basis of  $\mathbb{R}^n$ . Find the change of basis matrix  $S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

### Solution

By Definition 3.3,

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} [\vec{b}_1]_{\mathfrak{A}} & \cdots & [\vec{b}_n]_{\mathfrak{A}} \end{bmatrix}.$$

But note that

$$[\vec{x}]_{\mathfrak{A}} = \vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n,$$

since

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n;$$

the components of  $\vec{x}$  are its coordinates with respect to the standard basis. Thus,

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}.$$

Compare this with the definition of coordinates in a subspace of  $\mathbb{R}^n$ . ■

**EXAMPLE 7** The equation  $x_1 + x_2 + x_3 = 0$  defines a plane  $V$  in  $\mathbb{R}^3$ . In this plane, consider the two bases

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \quad \text{and} \quad \mathfrak{B} = (\vec{b}_1, \vec{b}_2) = \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right).$$

- a. Find the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ .
- b. Verify that the matrix  $S$  in part (a) satisfies the equation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} S.$$

### Solution

- a. By inspection, we find that  $\vec{b}_1 = 2\vec{a}_1 + \vec{a}_2$  and  $\vec{b}_2 = -\vec{a}_1 + 4\vec{a}_2$ , so that

$$S_{\mathfrak{B} \rightarrow \mathfrak{A}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \\ 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix}.$$

## Linear Spaces

**b.** We can verify that

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} S.$$

This equation reflects the fact that the two columns of  $S$  are the coordinate vectors of  $\vec{b}_1$  and  $\vec{b}_2$  with respect to the basis  $\mathfrak{A} = (\vec{a}_1, \vec{a}_2)$ . We can illustrate this equation with a commutative diagram, where  $\vec{x}$  represents a vector in  $V$ :

$$\begin{array}{ccc} [\vec{x}]_{\mathfrak{A}} & & [\vec{a}_1 \quad \vec{a}_2] \\ \uparrow S & \searrow & \rightarrow \vec{x} \\ [\vec{x}]_{\mathfrak{B}} & & [\vec{b}_1 \quad \vec{b}_2] \end{array}$$

Let us remind ourselves where the equation  $\vec{x} = [\vec{b}_1 \quad \vec{b}_2] [\vec{x}]_{\mathfrak{B}}$  comes from: If  $c_1, c_2$  are the coordinates of  $\vec{x}$  with respect to  $\mathfrak{B}$ , then

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} [\vec{x}]_{\mathfrak{B}}.$$

■

We can generalize.

### Theorem 3.4

#### Change of basis in a subspace of $\mathbb{R}^n$

Consider a subspace  $V$  of  $\mathbb{R}^n$  with two bases  $\mathfrak{A} = (\vec{a}_1, \dots, \vec{a}_m)$  and  $\mathfrak{B} = (\vec{b}_1, \dots, \vec{b}_m)$ . Let  $S$  be the change of basis matrix from  $\mathfrak{B}$  to  $\mathfrak{A}$ . Then the following equation holds:

$$\begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{bmatrix} S.$$

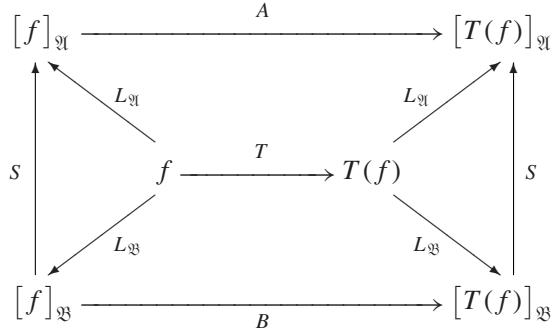
As in Example 7, this equation can be justified by means of a commutative diagram:

$$\begin{array}{ccc} [\vec{x}]_{\mathfrak{A}} & & [\vec{a}_1 \quad \cdots \quad \vec{a}_m] \\ \uparrow S & \searrow & \rightarrow \vec{x} \\ [\vec{x}]_{\mathfrak{B}} & & [\vec{b}_1 \quad \cdots \quad \vec{b}_m] \end{array}$$

Now consider a linear transformation  $T$  from  $V$  to  $V$ , where  $V$  is a finite dimensional linear space. Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two bases of  $V$ , and let  $A$  and  $B$  be the  $\mathfrak{A}$ - and the  $\mathfrak{B}$ -matrix of  $T$ , respectively. What is the relationship among  $A$ ,  $B$ , and the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ ?

## Linear Spaces

Consider the following diagram.



We see that  $[T(f)]_A = AS [f]_B = SB [f]_B$  for all  $f$  in  $V$ , so that  $AS = SB$ .

### Theorem 3.5

#### Change of basis for the matrix of a linear transformation

Let  $V$  be a linear space with two given bases  $\mathfrak{A}$  and  $\mathfrak{B}$ . Consider a linear transformation  $T$  from  $V$  to  $V$ , and let  $A$  and  $B$  be the  $\mathfrak{A}$ -matrix and  $\mathfrak{B}$ -matrix of  $T$ , respectively. Let  $S$  be the change of basis matrix from  $\mathfrak{B}$  to  $\mathfrak{A}$ . Then  $A$  is similar to  $B$ , and

$$AS = SB \quad \text{or} \quad A = SBS^{-1} \quad \text{or} \quad B = S^{-1}AS.$$

**EXAMPLE 8** As in Example 5, let  $V$  be the linear space spanned by the functions  $e^x$  and  $e^{-x}$ , with the bases  $\mathfrak{A} = (e^x, e^{-x})$  and  $\mathfrak{B} = (e^x + e^{-x}, e^x - e^{-x})$ . Consider the linear transformation  $D(f) = f'$  from  $V$  to  $V$ .

- a. Find the  $\mathfrak{A}$ -matrix  $A$  of  $D$ .
- b. Use part (a), Theorem 3.5, and Example 5 to find the  $\mathfrak{B}$ -matrix  $B$  of  $D$ .
- c. Use Theorem 3.2 to find the  $\mathfrak{B}$ -matrix  $B$  of  $D$  in terms of its columns.

#### Solution

- a. Let's use a diagram. Recall that  $(e^{-x})' = -e^{-x}$ , by the chain rule.

$$\begin{array}{ccc} ae^x + be^{-x} & \xrightarrow{D} & ae^x - be^{-x} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \end{bmatrix} & \xrightarrow{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} & \begin{bmatrix} a \\ -b \end{bmatrix} \end{array}$$

- b. In Example 5 we found the change of basis matrix  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ . Now

$$B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

- c. Note that  $D(e^x + e^{-x}) = e^x - e^{-x}$  and  $D(e^x - e^{-x}) = e^x + e^{-x}$ . Thus,

$$B = \begin{bmatrix} D(e^x + e^{-x}) & D(e^x - e^{-x}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^x + e^{-x} & e^x - e^{-x} \\ e^x - e^{-x} & e^x + e^{-x} \end{bmatrix}.$$
■

## EXERCISES 3

**GOAL** Use the concept of coordinates. Find the matrix of a linear transformation. Use this matrix to find the image and kernel of a transformation.

1. Are the polynomials  $f(t) = 7 + 3t + t^2$ ,  $g(t) = 9 + 9t + 4t^2$ , and  $h(t) = 3 + 2t + t^2$  linearly independent?

2. Are the matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}$$

linearly independent?

3. Do the polynomials  $f(t) = 1 + 2t + 9t^2 + t^3$ ,  $g(t) = 1 + 7t + 7t^3$ ,  $h(t) = 1 + 8t + t^2 + 5t^3$ , and  $k(t) = 1 + 8t + 4t^2 + 8t^3$  form a basis of  $P_3$ ?

4. Consider the polynomials  $f(t) = t + 1$  and  $g(t) = (t+2)(t+k)$ , where  $k$  is an arbitrary constant. For which values of the constant  $k$  are the three polynomials  $f(t)$ ,  $tf(t)$ , and  $g(t)$  a basis of  $P_2$ ?

In Exercises 5 through 40, find the matrix of the given linear transformation  $T$  with respect to the given basis. If no basis is specified, use the standard basis:  $\mathfrak{A} = (1, t, t^2)$  for  $P_2$ ,

$$\mathfrak{A} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

for  $\mathbb{R}^{2 \times 2}$ , and  $\mathfrak{A} = (1, i)$  for  $\mathbb{C}$ . For the space  $U^{2 \times 2}$  of upper triangular  $2 \times 2$  matrices, use the basis

$$\mathfrak{A} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

unless another basis is given. In each case, determine whether  $T$  is an isomorphism. If  $T$  isn't an isomorphism, find bases of the kernel and image of  $T$ , and thus determine the rank of  $T$ .

5.  $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$

6.  $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$ , with respect to the basis  $\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$

7.  $T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$ , with respect to the basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

8.  $T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$

9.  $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$

10.  $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$

11.  $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$  from  $U^{2 \times 2}$  to  $U^{2 \times 2}$ , with respect to the basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

12.  $T(M) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

13.  $T(M) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

14.  $T(M) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ , with respect to the basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right)$$

15.  $T(x + iy) = x - iy$  from  $\mathbb{C}$  to  $\mathbb{C}$

16.  $T(x + iy) = x - iy$  from  $\mathbb{C}$  to  $\mathbb{C}$ , with respect to the basis  $\mathfrak{B} = (1+i, 1-i)$

17.  $T(z) = iz$  from  $\mathbb{C}$  to  $\mathbb{C}$

18.  $T(z) = (2+3i)z$  from  $\mathbb{C}$  to  $\mathbb{C}$

19.  $T(z) = (p+iq)z$  from  $\mathbb{C}$  to  $\mathbb{C}$ , where  $p$  and  $q$  are arbitrary real numbers

20.  $T(f) = f'$  from  $P_2$  to  $P_2$

21.  $T(f) = f' - 3f$  from  $P_2$  to  $P_2$

22.  $T(f) = f'' + 4f'$  from  $P_2$  to  $P_2$

23.  $T(f(t)) = f(3)$  from  $P_2$  to  $P_2$

24.  $T(f(t)) = f(3)$  from  $P_2$  to  $P_2$ , with respect to the basis  $\mathfrak{B} = (1, t-3, (t-3)^2)$

25.  $T(f(t)) = f(-t)$  from  $P_2$  to  $P_2$

26.  $T(f(t)) = f(2t)$  from  $P_2$  to  $P_2$

27.  $T(f(t)) = f(2t-1)$  from  $P_2$  to  $P_2$

28.  $T(f(t)) = f(2t-1)$  from  $P_2$  to  $P_2$ , with respect to the basis  $\mathfrak{B} = (1, t-1, (t-1)^2)$

29.  $T(f(t)) = \int_0^2 f(t) dt$  from  $P_2$  to  $P_2$

## Linear Spaces

30.  $T(f(t)) = \frac{f(t+h) - f(t)}{h}$  from  $P_2$  to  $P_2$ , where  $h$  is a nonzero constant. Interpret transformation  $T$  geometrically.

31.  $T(f(t)) = \frac{f(t+h) - f(t-h)}{2h}$  from  $P_2$  to  $P_2$ , where  $h$  is a nonzero constant. Interpret transformation  $T$  geometrically.

32.  $T(f(t)) = f(1) + f'(1)(t-1)$  from  $P_2$  to  $P_2$ . Interpret transformation  $T$  geometrically.

33.  $T(f(t)) = f(1) + f'(1)(t-1)$  from  $P_2$  to  $P_2$ , with respect to the basis  $\mathfrak{B} = (1, t-1, (t-1)^2)$

34.  $T(M) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} M - M \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

35.  $T(M) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} M - M \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

36.  $T(M) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} M - M \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

37.  $T(M) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} M - M \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ , with respect to the basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

38.  $T(M) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M - M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ , with respect to the basis

$$\mathfrak{B} = \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right)$$

39.  $T(M) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M - M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

40.  $T(M) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} M - M \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

41. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 6 to the standard basis  $\mathfrak{A}$  of  $U^{2 \times 2}$  considered in Exercise 5.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 6 and 5, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

42. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 7 to the standard basis  $\mathfrak{A}$  of  $U^{2 \times 2}$  considered in Exercise 8.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 7 and 8, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

43. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 11 to the standard basis  $\mathfrak{A}$  of  $U^{2 \times 2}$  considered in Exercise 10.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 11 and 10, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

44. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 14 to the standard basis  $\mathfrak{A}$  of  $\mathbb{R}^{2 \times 2}$  considered in Exercise 13.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 14 and 13, respectively.

45. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 16 to the standard basis  $\mathfrak{A} = (1, i)$  of  $\mathbb{C}$  considered in Exercise 15.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 16 and 15, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

46. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 24 to the standard basis  $\mathfrak{A} = (1, t, t^2)$  of  $P_2$  considered in Exercise 23.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 24 and 23, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

47. a. Find the change of basis matrix  $S$  from the basis  $\mathfrak{B}$  considered in Exercise 28 to the standard basis  $\mathfrak{A} = (1, t, t^2)$  of  $P_2$  considered in Exercise 27.

b. Verify the formula  $SB = AS$  for the matrices  $B$  and  $A$  you found in Exercises 28 and 27, respectively.

c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

*In Exercises 48 through 53, let  $V$  be the space spanned by the two functions  $\cos(t)$  and  $\sin(t)$ . In each exercise, find the matrix of the given transformation  $T$  with respect to the basis  $\cos(t), \sin(t)$ , and determine whether  $T$  is an isomorphism.*

48.  $T(f) = f'$

49.  $T(f) = f'' + 2f' + 3f$

50.  $T(f) = f'' + af' + bf$ , where  $a$  and  $b$  are arbitrary real numbers. Find all the values of  $a$  and  $b$  such that  $T$  is an isomorphism.

51.  $T(f(t)) = f(t - \pi/2)$

52.  $T(f(t)) = f(t - \pi/4)$

53.  $T(f(t)) = f(t - \theta)$ , where  $\theta$  is an arbitrary real number. Hint: Use the addition theorems for sine and cosine.

*In Exercises 54 through 58, let  $V$  be the plane with equation  $x_1 + 2x_2 + 3x_3 = 0$  in  $\mathbb{R}^3$ . In each exercise, find the matrix  $B$  of the given transformation  $T$  from  $V$  to  $V$ , with*

## Linear Spaces

respect to the basis  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ . Note that the domain

and target space of  $T$  are restricted to the plane  $V$ , so that  $B$  will be a  $2 \times 2$  matrix.

54. The orthogonal projection onto the line spanned by vector  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

55. The orthogonal projection onto the line spanned by vector  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

56.  $T(\vec{x}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{x}$ .

57.  $T(\vec{x}) = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$

58.  $T(\vec{x}) = \left( \vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

59. Consider a linear transformation  $T$  from  $V$  to  $V$  with  $\ker(T) = \{0\}$ . If  $V$  is finite dimensional, then  $T$  is an isomorphism, since the matrix of  $T$  will be invertible. Show that this is not necessarily the case if  $V$  is infinite dimensional: Give an example of a linear transformation  $T$  from  $P$  to  $P$  with  $\ker(T) = \{0\}$  that is not an isomorphism. (Recall that  $P$  is the space of all polynomials.)

60. In the plane  $V$  defined by the equation  $2x_1 + x_2 - 2x_3 = 0$ , consider the bases

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left( \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right)$$

and

$$\mathfrak{B} = (\vec{b}_1, \vec{b}_2) = \left( \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right).$$

- a. Find the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ .  
 b. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .  
 c. Write an equation relating the matrices  $[\vec{a}_1 \quad \vec{a}_2]$ ,  $[\vec{b}_1 \quad \vec{b}_2]$ , and  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

61. In  $\mathbb{R}^2$ , consider the bases

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right)$$

and

$$\mathfrak{B} = (\vec{b}_1, \vec{b}_2) = \left( \begin{bmatrix} 5 \\ -10 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \end{bmatrix} \right).$$

- a. Find the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ . Interpret the transformation defined by  $S$  geometrically.  
 b. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .  
 c. Write an equation relating the matrices  $[\vec{a}_1 \quad \vec{a}_2]$ ,  $[\vec{b}_1 \quad \vec{b}_2]$ , and  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

62. In the plane  $V$  defined by the equation  $x_1 - 2x_2 + 2x_3 = 0$ , consider the basis

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left( \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right).$$

- a. Construct another basis  $\mathfrak{B} = (\vec{b}_1, \vec{b}_2)$  of  $V$ , such that neither  $\vec{b}_1$  nor  $\vec{b}_2$  has any zero components.  
 b. Find the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ .  
 c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .  
 d. Write an equation relating the matrices  $[\vec{a}_1 \quad \vec{a}_2]$ ,  $[\vec{b}_1 \quad \vec{b}_2]$ , and  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

63. In the plane  $V$  defined by the equation  $x_1 + 3x_2 - 2x_3 = 0$ , consider the basis

$$\mathfrak{A} = (\vec{a}_1, \vec{a}_2) = \left( \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right).$$

- a. Construct another basis  $\mathfrak{B} = (\vec{b}_1, \vec{b}_2)$  of  $V$ , such that neither  $\vec{b}_1$  nor  $\vec{b}_2$  has any negative components.  
 b. Find the change of basis matrix  $S$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ .  
 c. Find the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$ .  
 d. Write an equation relating the matrices  $[\vec{a}_1 \quad \vec{a}_2]$ ,  $[\vec{b}_1 \quad \vec{b}_2]$ , and  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .

64. Let  $V$  be the space of all upper triangular  $2 \times 2$  matrices. Consider the linear transformation

$$T \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aI_2 + bQ + cQ^2$$

from  $V$  to  $V$ , where  $Q = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ .

- a. Find the matrix  $A$  of  $T$  with respect to the basis

$$\mathfrak{A} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

- b. Find bases of the image and kernel of  $T$ , and thus determine the rank of  $T$ .

## Linear Spaces

65. Let  $V$  be the subspace of  $\mathbb{R}^{2 \times 2}$  spanned by the matrices

$$I_2 \text{ and } Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } b \neq 0.$$

- a. Compute  $Q^2$  and find the coordinate vector  $[Q^2]_{\mathfrak{B}}$ , where  $\mathfrak{B} = (I_2, Q)$ .
- b. Consider the linear transformation  $T(M) = MQ$  from  $V$  to  $V$ . Find the  $\mathfrak{B}$ -matrix  $B$  of  $T$ . For which matrices  $Q$  is  $T$  an isomorphism?
- c. If  $T$  fails to be an isomorphism, find the image and kernel of  $T$ . What is the rank of  $T$  in that case?

66. Let  $V$  be the linear space of all functions in two variables of the form  $q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ . Consider the linear transformation

$$T(f) = x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2}$$

from  $V$  to  $V$ .

- a. Find the matrix  $\mathfrak{B}$  of  $T$  with respect to the basis  $x_1^2, x_1x_2, x_2^2$  of  $V$ .
- b. Find bases of the kernel and image of  $T$ .

67. Let  $V$  be the linear space of all functions of the form

$$f(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t).$$

Consider the linear transformation  $T$  from  $V$  to  $V$  given by

$$T(f) = f'' + f.$$

- a. Find the matrix of  $T$  with respect to the basis  $\cos(t), \sin(t), t \cos(t), t \sin(t)$  of  $V$ .
- b. Find all solutions  $f$  in  $V$  of the differential equation

$$T(f) = f'' + f = \cos(t).$$

Graph your solution(s). [The differential equation  $f'' + f = \cos(t)$  describes a forced undamped oscillator. In this example, we observe the phenomenon of *resonance*.]

68. Consider the linear space  $V$  of all infinite sequences of real numbers. We define the subset  $W$  of  $V$  consisting of all sequences  $(x_0, x_1, x_2, \dots)$  such that  $x_{n+2} = x_{n+1} + 6x_n$  for all  $n \geq 0$ .

- a. Show that  $W$  is a *subspace* of  $V$ .
- b. Determine the dimension of  $W$ .
- c. Does  $W$  contain any geometric sequences of the form  $(1, c, c^2, c^3, \dots)$ , for some constant  $c$ ? Find all such sequences in  $W$ .
- d. Can you find a basis of  $W$  consisting of geometric sequences?
- e. Consider the sequence in  $W$  whose first two terms are  $x_0 = 0, x_1 = 1$ . Find  $x_2, x_3, x_4$ . Find a closed formula for the  $n$ th term  $x_n$  of this sequence. Hint: Write this sequence as a linear combination of the sequences you found in part (d).

69. Consider a basis  $f_1, \dots, f_n$  of  $P_{n-1}$ . Let  $a_1, \dots, a_n$  be distinct real numbers. Consider the  $n \times n$  matrix  $M$  whose  $ij$ th entry is  $f_j(a_i)$ . Show that the matrix  $M$  is invertible. Hint: If the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is in the kernel of  $M$ , then the polynomial  $f = c_1 f_1 + \dots + c_n f_n$  in  $P_{n-1}$  vanishes at  $a_1, \dots, a_n$ ; therefore,  $f = 0$ .

70. Let  $a_1, \dots, a_n$  be distinct real numbers. Show that there exist “weights”  $w_1, \dots, w_n$  such that

$$\int_{-1}^1 f(t) dt = \sum_{i=1}^n w_i f(a_i),$$

for all polynomials  $f(t)$  in  $P_{n-1}$ . Hint: It suffices to prove the claim for a basis  $f_1, \dots, f_n$  of  $P_{n-1}$ . Exercise 69 is helpful.

71. Find the weights  $w_1, w_2, w_3$  in Exercise 70 for  $a_1 = -1, a_2 = 0, a_3 = 1$ . Compare this with Simpson’s rule in calculus.

72. In all parts of this problem, let  $V$  be the set of all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that  $x_3 = x_1 + x_2$  and  $x_4 = x_2 + x_3$ .

- a. Represent  $V$  as the kernel of a matrix  $M$ . Find the rank of  $M$  and the dimension of  $V$ . Show that

$$\mathfrak{A} = \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right) \text{ is a basis of } V.$$

- b. Find all vectors of the form  $\begin{bmatrix} 1 \\ r \\ r^2 \\ r^3 \end{bmatrix}$  that are contained in  $V$ . (Be prepared to deal with irrational numbers.) Can you form a basis  $\mathfrak{B}$  of  $V$  consisting of such vectors?

- c. Consider the linear transformation  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_3 + x_4 \end{bmatrix}$  from  $\mathbb{R}^4$  to  $\mathbb{R}^4$ . If  $\vec{x}$  is a vector in  $V$ , show that  $T(\vec{x})$  is in  $V$  as well. Thus,  $T$  induces a linear transformation from  $V$  to  $V$ , which we will denote by  $F$ .

- d. Find the matrix  $A$  of  $F$  with respect to the basis  $\mathfrak{A}$  from part (a). [Note that  $A$  will be a  $2 \times 2$  matrix, since  $\dim(V) = 2$ .]

- e. Find the matrix  $B$  of  $F$  with respect to your basis  $\mathfrak{B}$  from part (b).
- f. Find the change of basis matrix  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .
- g. Write an equation relating the matrices  $A$ ,  $B$ , and  $S$ , and check that this equation holds for the matrices you found in parts (d), (e), and (f).
73. As in Problem 72, let  $V$  be the set of all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that  $x_3 = x_1 + x_2$  and  $x_4 = x_2 + x_3$ . In Problem 72 we see that  $V$  is a subspace of  $\mathbb{R}^4$  with  $\dim(V) = 2$ .
- a. Consider the linear transformation  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{bmatrix}$  from  $\mathbb{R}^4$  to  $\mathbb{R}^4$ . Show that  $T(\vec{x})$  is orthogonal to  $\vec{x}$ , for all  $\vec{x}$  in  $\mathbb{R}^4$ . If  $\vec{x}$  is a vector in  $V$ , show

that  $T(\vec{x})$  is in  $V$  as well. Thus,  $T$  induces a linear transformation from  $V$  to  $V$ , which we will denote by  $F$ .

- b. Find the matrix  $A$  of  $F$  with respect to the basis  $\mathfrak{A} = \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right)$ .
- c. Find the matrix  $B$  of  $F$  with respect to the basis  $\mathfrak{B} = \left( \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$ .
- d. Find the change of basis matrix  $S = S_{\mathfrak{B} \rightarrow \mathfrak{A}}$ .
- e. Write an equation relating the matrices  $A$ ,  $B$ , and  $S$ , and check that this equation holds for the matrices you found in parts (b), (c), and (d).
- f. Does there exist a basis  $\mathfrak{C}$  of  $V$  such that the  $\mathfrak{C}$ -matrix  $C$  of  $F$  is diagonal?

## Exercises

### TRUE OR FALSE?

1. The polynomials of degree less than 7 form a seven-dimensional subspace of the linear space of all polynomials.
2. The function  $T(f) = 3f - 4f'$  from  $C^\infty$  to  $C^\infty$  is a linear transformation.
3. The lower triangular  $2 \times 2$  matrices form a subspace of the space of all  $2 \times 2$  matrices.
4. The kernel of a linear transformation is a subspace of the domain.
5. The space  $\mathbb{R}^{2 \times 3}$  is five-dimensional.
6. If  $f_1, \dots, f_n$  is a basis of a linear space  $V$ , then any element of  $V$  can be written as a linear combination of  $f_1, \dots, f_n$ .
7. The space  $P_1$  is isomorphic to  $\mathbb{C}$ .
8. If the kernel of a linear transformation  $T$  from  $P_4$  to  $P_4$  is  $\{0\}$ , then  $T$  must be an isomorphism.
9. If  $W_1$  and  $W_2$  are subspaces of a linear space  $V$ , then the intersection  $W_1 \cap W_2$  must be a subspace of  $V$  as well.
10. If  $T$  is a linear transformation from  $P_6$  to  $\mathbb{R}^{2 \times 2}$ , then the kernel of  $T$  must be three-dimensional.
11. All bases of  $P_3$  contain at least one polynomial of degree  $\leq 2$ .
12. If  $T$  is an isomorphism, then  $T^{-1}$  must be an isomorphism as well.

13. The linear transformation  $T(f) = f + f''$  from  $C^\infty$  to  $C^\infty$  is an isomorphism.
14. All linear transformations from  $P_3$  to  $\mathbb{R}^{2 \times 2}$  are isomorphisms.
15. If  $T$  is a linear transformation from  $V$  to  $V$ , then the intersection of  $\text{im}(T)$  and  $\ker(T)$  must be  $\{0\}$ .
16. The space of all upper triangular  $4 \times 4$  matrices is isomorphic to the space of all lower triangular  $4 \times 4$  matrices.
17. Every polynomial of degree 3 can be expressed as a linear combination of the polynomial  $(t - 3)$ ,  $(t - 3)^2$ , and  $(t - 3)^3$ .
18. If a linear space  $V$  can be spanned by 10 elements, then the dimension of  $V$  must be  $\leq 10$ .
19. The function  $T(M) = \det(M)$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$  is a linear transformation.
20. There exists a  $2 \times 2$  matrix  $A$  such that the space  $V$  of all matrices commuting with  $A$  is one-dimensional.
21. The linear transformation  $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} M$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$  has rank 1.
22. If the matrix of a linear transformation  $T$  (with respect to some basis) is  $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$ , then there must exist a nonzero element  $f$  in the domain of  $T$  such that  $T(f) = 3f$ .

## Linear Spaces

23. The kernel of the linear transformation  $T(f(t)) = f(t^2)$  from  $P$  to  $P$  is  $\{0\}$ .
24. If  $S$  is any invertible  $2 \times 2$  matrix, then the linear transformation  $T(M) = SMS$  is an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ .
25. There exists a  $2 \times 2$  matrix  $A$  such that the space of all matrices commuting with  $A$  is two-dimensional.
26. There exists a basis of  $\mathbb{R}^{2 \times 2}$  that consists of four invertible matrices.
27. If the image of a linear transformation  $T$  from  $P$  to  $P$  is all of  $P$ , then  $T$  must be an isomorphism.
28. If  $f_1, f_2, f_3$  is a basis of a linear space  $V$ , then  $f_1, f_1 + f_2, f_1 + f_2 + f_3$  must be a basis of  $V$  as well.
29. If  $a, b$ , and  $c$  are distinct real numbers, then the polynomials  $(x - b)(x - c)$ ,  $(x - a)(x - c)$ , and  $(x - a)(x - b)$  must be linearly independent.
30. The linear transformation  $T(f(t)) = f(4t - 3)$  from  $P$  to  $P$  is an isomorphism.
31. If  $W$  is a subspace of  $V$ , and if  $W$  is finite dimensional, then  $V$  must be finite dimensional as well.
32. There exists a linear transformation from  $\mathbb{R}^{3 \times 3}$  to  $\mathbb{R}^{2 \times 2}$  whose kernel consists of all lower triangular  $3 \times 3$  matrices, while the image consists of all upper triangular  $2 \times 2$  matrices.
33. Every two-dimensional subspace of  $\mathbb{R}^{2 \times 2}$  contains at least one invertible matrix.
34. If  $\mathfrak{A} = (f, g)$  and  $\mathfrak{B} = (f, f + g)$  are two bases of a linear space  $V$ , then the change of basis matrix from  $\mathfrak{A}$  to  $\mathfrak{B}$  is  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .
35. If the matrix of a linear transformation  $T$  with respect to a basis  $(f, g)$  is  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , then the matrix of  $T$  with respect to the basis  $(g, f)$  is  $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ .
36. The linear transformation  $T(f) = f'$  from  $P_n$  to  $P_n$  has rank  $n$ , for all positive integers  $n$ .
37. If the matrix of a linear transformation  $T$  (with respect to some basis) is  $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ , then  $T$  must be an isomorphism.
38. There exists a subspace of  $\mathbb{R}^{3 \times 4}$  that is isomorphic to  $P_9$ .
39. There exist two distinct subspaces  $W_1$  and  $W_2$  of  $\mathbb{R}^{2 \times 2}$  whose union  $W_1 \cup W_2$  is a subspace of  $\mathbb{R}^{2 \times 2}$  as well.
40. There exists a linear transformation from  $P$  to  $P_5$  whose image is all of  $P_5$ .
41. If  $f_1, \dots, f_n$  are polynomials such that the degree of  $f_k$  is  $k$  (for  $k = 1, \dots, n$ ), then  $f_1, \dots, f_n$  must be linearly independent.
42. The transformation  $D(f) = f'$  from  $C^\infty$  to  $C^\infty$  is an isomorphism.
43. If  $T$  is a linear transformation from  $P_4$  to  $W$  with  $\text{im}(T) = W$ , then the inequality  $\dim(W) \leq 5$  must hold.
44. The kernel of the linear transformation
- $$T(f(t)) = \int_0^1 f(t) dt$$
- from  $P$  to  $\mathbb{R}$  is finite dimensional.
45. If  $T$  is a linear transformation from  $V$  to  $V$ , then  $\{f \text{ in } V : T(f) = f\}$  must be a subspace of  $V$ .
46. If  $T$  is a linear transformation from  $P_6$  to  $P_6$  that transforms  $t^k$  into a polynomial of degree  $k$  (for  $k = 0, 1, \dots, 6$ ), then  $T$  must be an isomorphism.
47. There exist invertible  $2 \times 2$  matrices  $P$  and  $Q$  such that the linear transformation  $T(M) = PM - MQ$  is an isomorphism.
48. There exists a linear transformation from  $P_6$  to  $\mathbb{C}$  whose kernel is isomorphic to  $\mathbb{R}^{2 \times 2}$ .
49. If  $f_1, f_2, f_3$  is a basis of a linear space  $V$ , and if  $f$  is any element of  $V$ , then the elements  $f_1 + f, f_2 + f, f_3 + f$  must form a basis of  $V$  as well.
50. There exists a two-dimensional subspace of  $\mathbb{R}^{2 \times 2}$  whose nonzero elements are all invertible.
51. The space  $P_{11}$  is isomorphic to  $\mathbb{R}^{3 \times 4}$ .
52. If  $T$  is a linear transformation from  $V$  to  $W$ , and if both  $\text{im}(T)$  and  $\ker(T)$  are finite dimensional, then  $W$  must be finite dimensional.
53. If  $T$  is a linear transformation from  $V$  to  $\mathbb{R}^{2 \times 2}$  with  $\ker(T) = \{0\}$ , then the inequality  $\dim(V) \leq 4$  must hold.
54. The function
- $$T(f(t)) = \frac{d}{dt} \int_2^{3t+4} f(x) dx$$
- from  $P_5$  to  $P_5$  is an isomorphism.
55. Any four-dimensional linear space has infinitely many three-dimensional subspaces.
56. If the matrix of a linear transformation  $T$  (with respect to some basis) is  $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$ , then there must exist a nonzero element  $f$  in the domain of  $T$  such that  $T(f) = 4f$ .

## Linear Spaces

- 57.** If the image of a linear transformation  $T$  is infinite dimensional, then the domain of  $T$  must be infinite dimensional.
- 58.** There exists a  $2 \times 2$  matrix  $A$  such that the space of all matrices commuting with  $A$  is three-dimensional.
- 59.** If  $A$ ,  $B$ ,  $C$ , and  $D$  are noninvertible  $2 \times 2$  matrices, then the matrices  $AB$ ,  $AC$ , and  $AD$  must be linearly dependent.
- 60.** There exist two distinct three-dimensional subspaces  $W_1$  and  $W_2$  of  $P_4$  such that the union  $W_1 \cup W_2$  is a subspace of  $P_4$  as well.
- 61.** If the elements  $f_1, \dots, f_n$  (where  $f_1 \neq 0$ ) are linearly dependent, then one element  $f_k$  can be expressed *uniquely* as a linear combination of the preceding elements  $f_1, \dots, f_{k-1}$ .
- 62.** There exists a  $3 \times 3$  matrix  $P$  such that the linear transformation  $T(M) = MP - PM$  from  $\mathbb{R}^{3 \times 3}$  to  $\mathbb{R}^{3 \times 3}$  is an isomorphism.
- 63.** If  $f_1, f_2, f_3, f_4, f_5$  are elements of a linear space  $V$ , and if there are exactly two redundant elements in the list  $f_1, f_2, f_3, f_4, f_5$ , then there must be exactly two redundant elements in the list  $f_2, f_4, f_5, f_1, f_3$  as well.
- 64.** There exists a linear transformation  $T$  from  $P_6$  to  $P_6$  such that the kernel of  $T$  is isomorphic to the image of  $T$ .
- 65.** If  $T$  is a linear transformation from  $V$  to  $W$ , and if both  $\text{im}(T)$  and  $\ker(T)$  are finite dimensional, then  $V$  must be finite dimensional.
- 66.** If the matrix of a linear transformation  $T$  (with respect to some basis) is  $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$ , then there must exist a nonzero element  $f$  in the domain of  $T$  such that  $T(f) = 5f$ .
- 67.** Every three-dimensional subspace of  $\mathbb{R}^{2 \times 2}$  contains at least one invertible matrix.

### Answers to Odd-Numbered Exercises

**1** Answers to more theoretical questions are omitted.

1. Not a subspace
3. Subspace with basis  $1 - t, 2 - t^2$
5. Subspace with basis  $t$
7. Subspace
9. Not a subspace
11. Not a subspace
13. Not a subspace
15. Subspace
17. Matrices with one entry equal to 1 and all other entries equal to 0. The dimension is  $mn$ .
19. A basis is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}$ , so that the dimension is 4.
21. A basis is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , so that the dimension is 2.
23. A basis is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , so that the dimension is 3.
25. A basis is  $1 - t, 1 - t^2$ , so that the dimension is 2.
27. A basis is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , so that the dimension is 2.
29. A basis is  $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ , so that the dimension is 2.
31. A basis is  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ , so that the dimension is 2.

**33.** Only the zero matrix has this property, so that the basis is  $\emptyset$ , and the dimension is 0.

- 35.** A basis is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , and the dimension is 3.

37. 3, 5, or 9
39.  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
41. 0, 3, 6, or 9
43. 2
45.  $\dim(V) = 3$
47. Yes and yes
49. Yes
51.  $f(x) = ae^{3x} + be^{4x}$

**2** Answers to more theoretical questions are omitted.

1. Nonlinear
3. Linear, not an isomorphism
5. Nonlinear
7. Isomorphism
9. Isomorphism
11. Isomorphism
13. Linear, not an isomorphism
15. Isomorphism
17. Linear, not an isomorphism
19. Isomorphism
21. Isomorphism
23. Linear, not an isomorphism
25. Linear, not an isomorphism

## Linear Spaces

27. Isomorphism  
 29. Linear, not an isomorphism  
 31. Linear, not an isomorphism  
 33. Linear, not an isomorphism  
 35. Linear, not an isomorphism  
 37. Linear, not an isomorphism  
 39. Linear, not an isomorphism  
 41. Nonlinear      43. Isomorphism  
 45. Linear, not an isomorphism  
 47. Linear, not an isomorphism  
 49. Linear, not an isomorphism  
 51.  $\ker T$  consists of all matrices of the form  $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ , so that the nullity is 2.  
 53. The image consists of all linear functions, of the form  $mt+b$ , so that the rank is 2. The kernel consists of the constant functions, so that the nullity is 1.  
 55. The image consists of all infinite sequences, and the kernel consists of all sequences of the form  $(0, x_1, 0, x_3, 0, x_5, \dots)$ .  
 57. The kernel consists of all functions of the form  $ae^{2t} + be^{3t}$ , so that the nullity is 2.  
 59. The kernel has the basis  $t - 7, (t - 7)^2$ , so that the nullity is 2. The image is all of  $\mathbb{R}$ , so that the rank is 1.  
 61. The kernel consists of the zero function alone, and the image consists of all polynomials  $g(t)$  whose constant term is zero [that is,  $g(0) = 0$ ].  
 63. Impossible, since  $\dim(P_3) \neq \dim(\mathbb{R}^3)$ .  
 65. b.  $\ker T$  consists of the zero matrix alone.  
 d. This dimension is  $mn$ .  
 67. For all  $k$  except  $k = 2$  and  $k = 4$   
 69. No; if  $B = S^{-1}AS$ , then  $T(S) = 0$ .  
 71. Yes, there is exactly one such polynomial.  
 73. Yes      77. Yes and yes  
 79. Surprisingly, yes  
 83. The transformation  $T$  induces a transformation  $\tilde{T}$  from  $\ker(L \circ T)$  to  $\ker L$ , with  $\ker \tilde{T} = \ker T$ . Applying the rank-nullity theorem as stated in Exercise 82 to  $\tilde{T}$ , we find that  $\dim(\ker(L \circ T)) = \dim(\ker \tilde{T}) + \dim(\text{im } \tilde{T}) \leq \dim(\ker T) + \dim(\ker L)$ , since  $\text{im } \tilde{T}$  is a subspace of the kernel of  $L$ .

**3** Answers to more theoretical questions are omitted.

1. Yes      3. Yes  
 5.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$       7.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$   
 9.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       11.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

13.  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$       15.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$   
 17.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$       19.  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$   
 21.  $\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$       23.  $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$   
 25.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       27.  $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$   
 29.  $\begin{bmatrix} 2 & 2 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$       31.  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$   
 33.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$       35.  $\begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$   
 37.  $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$       39.  $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$   
 41. a.  $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$   
 c.  $S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$   
 43. a.  $S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$   
 c.  $S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$   
 45. a.  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$       c.  $S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$   
 47. a.  $S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$   
 c.  $S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$   
 49.  $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$       51.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$   
 53.  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$       55.  $\begin{bmatrix} 2/9 & -14/9 \\ -1/9 & 7/9 \end{bmatrix}$

## Linear Spaces

57. 
$$\begin{bmatrix} -1 & 3 \\ -1 & 0 \end{bmatrix}$$

59.  $T(f(t)) = t \cdot f(t)$  from  $P$  to  $P$ , for example

61. a.  $S = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$       b.  $S^{-1} = \frac{1}{25} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

63. a.  $\vec{b}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ , for example

b.  $S = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$       c.  $S^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$

65. a.  $P^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}$  and

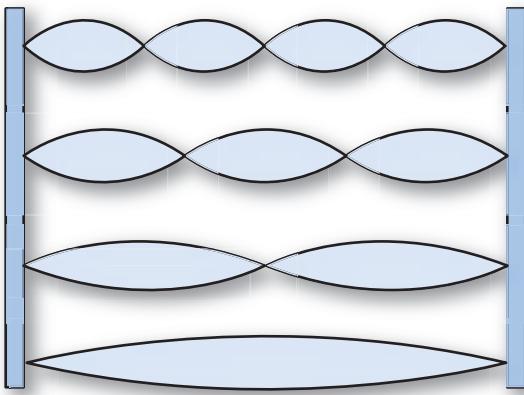
$$[P^2]_{\mathfrak{B}} = \begin{bmatrix} bc - ad \\ a + d \end{bmatrix}$$

b.  $B = \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix}$ .  $T$  is an isomorphism if  $B$  is invertible.

c. In this case,  $\text{im } T$  is spanned by  $P$  and  $\ker T$  is spanned by  $(a+d)I_2 - P = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

67. a.  $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$       b.  $f(t) = \frac{1}{2}t \sin(t)$

73. b.  $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$       c.  $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$   
 d.  $S = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$       e.  $AS = SA$       f. No



# Orthogonality and Least Squares

## I Orthogonal Projections and Orthonormal Bases

In this chapter, we will generalize the ideas on some linear transformations that are important in geometry: orthogonal projections, reflections, and rotations in particular. In Sections 1 and 2, we will discuss the orthogonal projection onto a subspace  $V$  of  $\mathbb{R}^n$ . In Section 3, we will study linear transformations that preserve length, such as reflections and rotations. In Section 4, we will present an important application of orthogonal projections: the method of least squares in statistics. Finally, in Section 5, we will go a step further and generalize all these ideas from  $\mathbb{R}^n$  to linear spaces.

First, we will discuss some basic concepts of geometry.

### Definition 1.1

#### Orthogonality, length, unit vectors

- a. Two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are called *perpendicular* or *orthogonal*<sup>1</sup> if  $\vec{v} \cdot \vec{w} = 0$ .
- b. The *length* (or magnitude or norm) of a vector  $\vec{v}$  in  $\mathbb{R}^n$  is  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ .
- c. A vector  $\vec{u}$  in  $\mathbb{R}^n$  is called a *unit vector* if its length is 1 (i.e.,  $\|\vec{u}\| = 1$ , or  $\vec{u} \cdot \vec{u} = 1$ ).

If  $\vec{v}$  is a nonzero vector in  $\mathbb{R}^n$ , then

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$$

is a unit vector. See Exercise 25b.

A vector  $\vec{x}$  in  $\mathbb{R}^n$  is said to be *orthogonal* to a subspace  $V$  of  $\mathbb{R}^n$  if  $\vec{x}$  is orthogonal to all the vectors  $\vec{v}$  in  $V$ , meaning that  $\vec{x} \cdot \vec{v} = 0$  for all vectors  $\vec{v}$  in  $V$ .

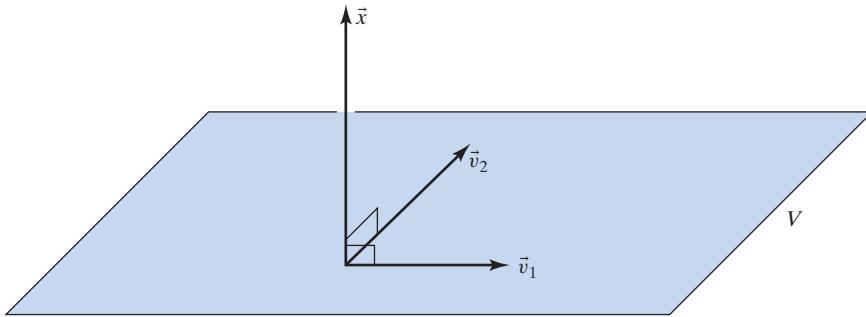
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<sup>1</sup>The two terms are synonymous: *Perpendicular* comes from Latin and *orthogonal* from Greek.

## Orthogonality and Least Squares

If we are given a basis  $\vec{v}_1, \dots, \vec{v}_m$  of  $V$ , then  $\vec{x}$  is orthogonal to  $V$  if (and only if)  $\vec{x}$  is orthogonal to all the vectors  $\vec{v}_1, \dots, \vec{v}_m$ . See Exercise 22.

For example, a vector  $\vec{x}$  in  $\mathbb{R}^3$  is orthogonal to a plane  $V$  in  $\mathbb{R}^3$  if (and only if)  $\vec{x}$  is orthogonal to two vectors  $\vec{v}_1, \vec{v}_2$  that form a basis of  $V$ . See Figure 1.



**Figure 1**

### Definition 1.2

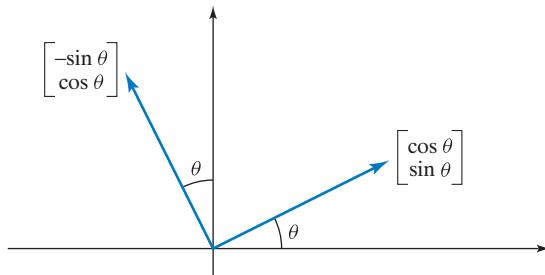
#### Orthonormal vectors

The vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$  in  $\mathbb{R}^n$  are called *orthonormal* if they are all unit vectors and orthogonal to one another:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

**EXAMPLE 1** The vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$  in  $\mathbb{R}^n$  are orthonormal. ■

**EXAMPLE 2** For any scalar  $\theta$ , the vectors  $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  and  $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$  are orthonormal. See Figure 2. ■



**Figure 2**

**EXAMPLE 3** The vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in  $\mathbb{R}^4$  are orthonormal. (Verify this.) Can you find a vector  $\vec{u}_4$  in  $\mathbb{R}^4$  such that all the vectors  $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$  are orthonormal? See Exercise 16. ■

## Orthogonality and Least Squares

The following properties of orthonormal vectors are often useful:

### Theorem 1.3

#### Properties of orthonormal vectors

- a. Orthonormal vectors are linearly independent.
- b. Orthonormal vectors  $\vec{u}_1, \dots, \vec{u}_n$  in  $\mathbb{R}^n$  form a basis of  $\mathbb{R}^n$ .

#### *Proof*

- a. Consider a relation

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_i\vec{u}_i + \dots + c_m\vec{u}_m = \vec{0}$$

among the orthonormal vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$  in  $\mathbb{R}^n$ . Let us form the dot product of each side of this equation with  $\vec{u}_i$ :

$$(c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_i\vec{u}_i + \dots + c_m\vec{u}_m) \cdot \vec{u}_i = \vec{0} \cdot \vec{u}_i = 0.$$

Because the dot product is distributive,

$$c_1(\vec{u}_1 \cdot \vec{u}_i) + c_2(\vec{u}_2 \cdot \vec{u}_i) + \dots + c_i(\vec{u}_i \cdot \vec{u}_i) + \dots + c_m(\vec{u}_m \cdot \vec{u}_i) = 0.$$

We know that  $\vec{u}_i \cdot \vec{u}_i = 1$ , and all other dot products  $\vec{u}_j \cdot \vec{u}_i$  are zero. Therefore,  $c_i = 0$ . Since this holds for all  $i = 1, \dots, m$ , it follows that the vectors  $\vec{u}_1, \dots, \vec{u}_m$  are linearly independent.

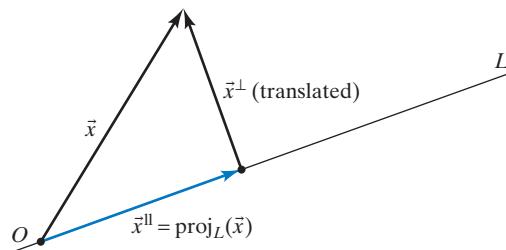
- b. This follows from part (a) (Any  $n$  linearly independent vectors in  $\mathbb{R}^n$  form a basis of  $\mathbb{R}^n$ ). ■

### Orthogonal Projections

If  $\vec{x}$  is a vector in  $\mathbb{R}^2$  and  $L$  is a line in  $\mathbb{R}^2$ , then we can resolve the vector  $\vec{x}$  into a component  $\vec{x}^{\parallel}$  parallel to  $L$  and a component  $\vec{x}^{\perp}$  perpendicular to  $L$ ,

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

and this decomposition is unique. The vector  $\vec{x}^{\parallel}$  is called the orthogonal projection of  $\vec{x}$  onto  $L$ . See Figure 3.



**Figure 3**

Let's see how we can generalize this idea to any subspace  $V$  of  $\mathbb{R}^n$ .

### Theorem 1.4

#### Orthogonal projection

Consider a vector  $\vec{x}$  in  $\mathbb{R}^n$  and a subspace  $V$  of  $\mathbb{R}^n$ . Then we can write

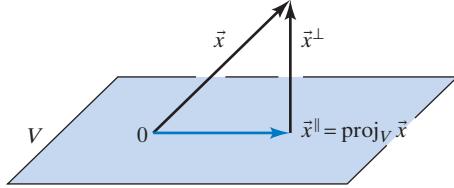
$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where  $\vec{x}^{\parallel}$  is in  $V$  and  $\vec{x}^{\perp}$  is perpendicular to  $V$ , and this representation is unique.

**Theorem 1.4**
**Orthogonal projection (*Continued*)**

The vector  $\vec{x}^{\parallel}$  is called the *orthogonal projection* of  $\vec{x}$  onto  $V$ , denoted by  $\text{proj}_V \vec{x}$ . See Figure 4.

The transformation  $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\parallel}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  is linear.



**Figure 4**

**Proof** Consider an *orthonormal* basis  $\vec{u}_1, \dots, \vec{u}_m$  of  $V$  (see Definition 1.2).<sup>2</sup> If a decomposition  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$  (with  $\vec{x}^{\parallel}$  in  $V$  and  $\vec{x}^{\perp}$  orthogonal to  $V$ ) does exist, then we can write

$$\vec{x}^{\parallel} = c_1 \vec{u}_1 + \dots + c_i \vec{u}_i + \dots + c_m \vec{u}_m,$$

for some coefficients  $c_1, \dots, c_i, \dots, c_m$  yet to be determined (since  $\vec{x}^{\parallel}$  is in  $V$ ). We know that

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - c_1 \vec{u}_1 - \dots - c_i \vec{u}_i - \dots - c_m \vec{u}_m$$

is orthogonal to  $V$ , meaning that  $\vec{x} - c_1 \vec{u}_1 - \dots - c_i \vec{u}_i - \dots - c_m \vec{u}_m$  is orthogonal to all the vectors  $\vec{u}_i$  in  $V$ :

$$\begin{aligned} 0 &= \vec{u}_i \cdot (\vec{x} - c_1 \vec{u}_1 - \dots - c_i \vec{u}_i - \dots - c_m \vec{u}_m) \\ &= \vec{u}_i \cdot \vec{x} - c_1 \underbrace{(\vec{u}_i \cdot \vec{u}_1)}_0 - \dots - c_i \underbrace{(\vec{u}_i \cdot \vec{u}_i)}_1 - \dots - c_m \underbrace{(\vec{u}_i \cdot \vec{u}_m)}_0 = \vec{u}_i \cdot \vec{x} - c_i \end{aligned}$$

See Definition 1.2. It follows that  $c_i = \vec{u}_i \cdot \vec{x}$ , so that

$$\vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 + \dots + (\vec{u}_i \cdot \vec{x}) \vec{u}_i + \dots + (\vec{u}_m \cdot \vec{x}) \vec{u}_m$$

and

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 - \dots - (\vec{u}_i \cdot \vec{x}) \vec{u}_i - \dots - (\vec{u}_m \cdot \vec{x}) \vec{u}_m.$$

Note that  $\vec{u}_i \cdot \vec{x}^{\perp} = 0$ , by construction, so that  $\vec{x}^{\perp}$  is orthogonal to  $V$ , as required. (Recall the remarks preceding Figure 1.)

We leave the verification of the linearity of the transformation  $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\parallel}$  as Exercise 24. ■

<sup>2</sup>In the next section we will introduce an algorithm for constructing an orthonormal basis of any subspace  $V$  of  $\mathbb{R}^n$ . Here we need to convince ourselves merely that such a basis of  $V$  does indeed exist. We will present a proof by induction on  $m = \dim(V)$ , for a fixed  $n$ . If  $\dim(V) = 1$ , then a unit vector  $\vec{u}$  in  $V$  will form an orthonormal basis of  $V$ . For the induction step from  $m - 1$  to  $m$ , consider a subspace  $V$  of  $\mathbb{R}^n$  with  $\dim(V) = m$ , and let  $\vec{u}$  be a unit vector in  $V$ . Consider the linear transformation  $T(\vec{x}) = \vec{x} \cdot \vec{u}$  from  $V$  to  $\mathbb{R}$ . By the rank-nullity theorem, the kernel of  $T$  will be an  $(m - 1)$ -dimensional subspace  $W$  of  $V$ , consisting of all vectors  $\vec{x}$  in  $V$  that are orthogonal to  $\vec{u}$  [since  $T(\vec{x}) = \vec{x} \cdot \vec{u} = 0$ ]. By the induction hypothesis, there exists an orthonormal basis  $(\vec{u}_1, \dots, \vec{u}_{m-1})$  of  $W$ , and  $(\vec{u}_1, \dots, \vec{u}_{m-1}, \vec{u})$  will be an orthonormal basis of  $V$ .

**Theorem 1.5**
**Formula for the orthogonal projection**

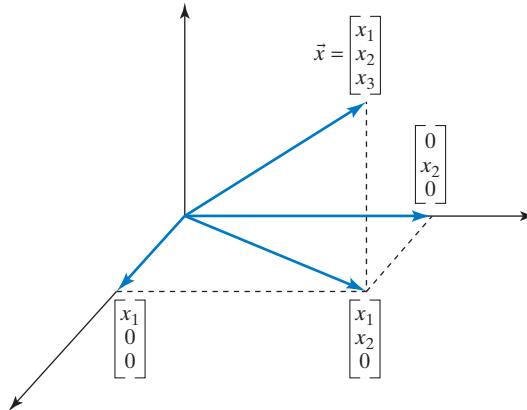
If  $V$  is a subspace of  $\mathbb{R}^n$  with an orthonormal basis  $\vec{u}_1, \dots, \vec{u}_m$ , then

$$\text{proj}_V \vec{x} = \vec{x}^\parallel = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m.$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ .

Note that  $\text{proj}_V \vec{x}$  is the sum of all vectors  $(\vec{u}_i \cdot \vec{x})\vec{u}_i$ , for  $i = 1, \dots, m$ , representing the orthogonal projections of  $\vec{x}$  onto the lines spanned by the basis vectors  $\vec{u}_1, \dots, \vec{u}_m$  of  $V$ .

For example, projecting a vector in  $\mathbb{R}^3$  orthogonally onto the  $x_1$ - $x_2$ -plane amounts to the same as projecting it onto the  $x_1$ -axis, then onto the  $x_2$ -axis, and then adding the resultant vectors. See Figure 5.



**Figure 5**

**EXAMPLE 4** Consider the subspace  $V = \text{im}(A)$  of  $\mathbb{R}^4$ , where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Find  $\text{proj}_V \vec{x}$ , for

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}.$$

**Solution**

Since the two column vectors of  $A$  happen to be linearly independent, they form a basis of  $V$ . Since they happen to be orthogonal, we can construct an orthonormal basis of  $V$  merely by dividing each of these two vectors by its length (2 for both vectors):

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

## Orthogonality and Least Squares

Then

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 = 6\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

To check this answer, verify that  $\vec{x} - \text{proj}_V \vec{x}$  is perpendicular to both  $\vec{u}_1$  and  $\vec{u}_2$ . ■

What happens when we apply Theorem 1.5 to the subspace  $V = \mathbb{R}^n$  of  $\mathbb{R}^n$  with orthonormal basis  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ ? Clearly,  $\text{proj}_V \vec{x} = \vec{x}$ , for all  $\vec{x}$  in  $\mathbb{R}^n$  by Theorem 1.4. Therefore,

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n,$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ .

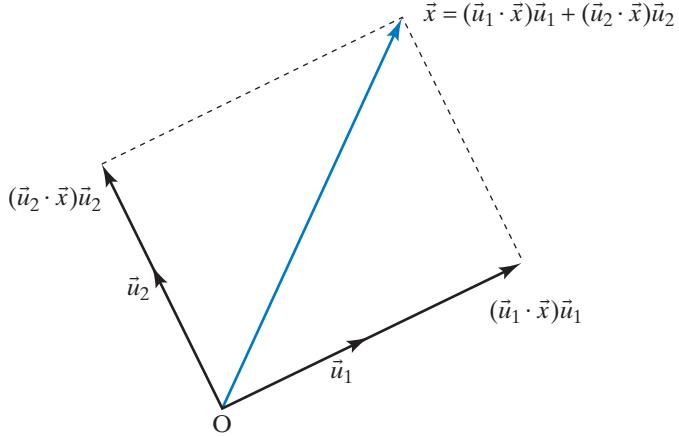
### Theorem 1.6

Consider an orthonormal basis  $\vec{u}_1, \dots, \vec{u}_n$  of  $\mathbb{R}^n$ . Then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n,$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ .

This means that if you project  $\vec{x}$  onto all the lines spanned by the basis vectors  $\vec{u}_i$  and add the resultant vectors, you get the vector  $\vec{x}$  back. Figure 6 illustrates this in the case  $n = 2$ .



**Figure 6**

What is the practical significance of Theorem 1.6? Whenever we have a basis  $\vec{v}_1, \dots, \vec{v}_n$  of  $\mathbb{R}^n$ , any vector  $\vec{x}$  in  $\mathbb{R}^n$  can be expressed uniquely as a linear combination of the  $\vec{v}_i$ :

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n.$$

To find the coordinates  $c_i$ , we generally need to solve a linear system, which may involve a fair amount of computation. However, if we are dealing with an *orthonormal* basis  $\vec{u}_1, \dots, \vec{u}_n$ , then we can find the  $c_i$  much more easily, using the formula

$$c_i = \vec{u}_i \cdot \vec{x}.$$

### EXAMPLE 5

Consider the orthogonal projection  $T(\vec{x}) = \text{proj}_V \vec{x}$  onto a subspace  $V$  of  $\mathbb{R}^n$ . Describe the image and kernel of  $T$ .

## Orthogonality and Least Squares

### Solution

By definition of an orthogonal projection, the image of  $T$  is the subspace  $V$ , while the kernel of  $T$  consists of all those vectors  $\vec{x}$  in  $\mathbb{R}^n$  such that  $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^\parallel = \vec{0}$ , meaning that  $\vec{x} = \vec{x}^\perp$ . In other words, the kernel consists of those vectors  $\vec{x}$  in  $\mathbb{R}^n$  that are orthogonal to  $V$ . This space deserves a name. ■

### Definition 1.7

#### Orthogonal complement

Consider a subspace  $V$  of  $\mathbb{R}^n$ . The *orthogonal complement*  $V^\perp$  of  $V$  is the set of those vectors  $\vec{x}$  in  $\mathbb{R}^n$  that are orthogonal to all vectors in  $V$ :

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\}.$$

Note that  $V^\perp$  is the kernel of the orthogonal projection onto  $V$ .

Take another look at Figures 3 and 4, and identify the kernel of the projection in each case.

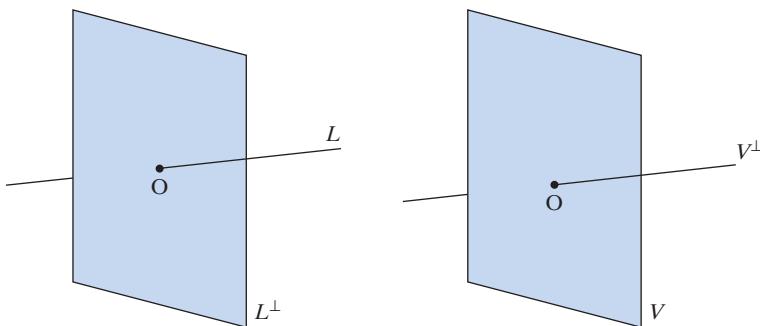
In Figure 7, we sketch the orthogonal complements of a line  $L$  and of a plane  $V$  in  $\mathbb{R}^3$ . Note that both  $L^\perp$  and  $V^\perp$  are subspaces of  $\mathbb{R}^3$ . Furthermore,

$$\dim(L) + \dim(L^\perp) = 1 + 2 = 3 = \dim(\mathbb{R}^3),$$

and

$$\dim(V) + \dim(V^\perp) = 2 + 1 = 3 = \dim(\mathbb{R}^3).$$

We can generalize these observations.



**Figure 7**

### Theorem 1.8

#### Properties of the orthogonal complement

Consider a subspace  $V$  of  $\mathbb{R}^n$ .

- a. The orthogonal complement  $V^\perp$  of  $V$  is a subspace of  $\mathbb{R}^n$ .
- b. The intersection of  $V$  and  $V^\perp$  consists of the zero vector:  $V \cap V^\perp = \{\vec{0}\}$ .
- c.  $\dim(V) + \dim(V^\perp) = n$ .
- d.  $(V^\perp)^\perp = V$ .

### Proof

- a. If  $T(\vec{x}) = \text{proj}_V \vec{x}$ , then  $V^\perp = \ker(T)$ , a subspace of  $\mathbb{R}^n$ .
- b. If a vector  $\vec{x}$  is in  $V$  as well as in  $V^\perp$ , then  $\vec{x}$  is orthogonal to itself:  $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2 = 0$ , so that  $\vec{x}$  must equal  $\vec{0}$ , as claimed.

## Orthogonality and Least Squares

- c. We can apply the rank-nullity theorem to the linear transformation  $T(\vec{x}) = \text{proj}_V \vec{x}$ :

$$n = \dim(\text{im } T) + \dim(\ker T) = \dim(V) + \dim(V^\perp).$$

- d. We leave this proof as Exercise 23. ■

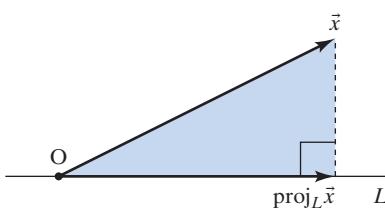
### From Pythagoras to Cauchy

#### EXAMPLE 6

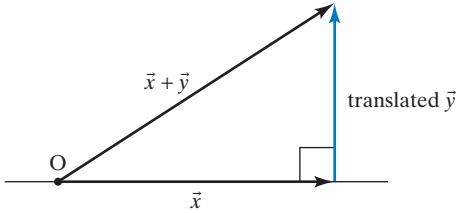
Consider a line  $L$  in  $\mathbb{R}^3$  and a vector  $\vec{x}$  in  $\mathbb{R}^3$ . What can you say about the relationship between the lengths of the vectors  $\vec{x}$  and  $\text{proj}_L \vec{x}$ ?

#### Solution

Applying the Pythagorean theorem to the shaded right triangle in Figure 8, we find that  $\|\text{proj}_L \vec{x}\| \leq \|\vec{x}\|$ . The statement is an equality if (and only if)  $\vec{x}$  is on  $L$ . ■



**Figure 8**



**Figure 9**

Does this inequality hold in higher-dimensional cases? We have to examine whether the Pythagorean theorem holds in  $\mathbb{R}^n$ .

#### Theorem 1.9

##### Pythagorean theorem

Consider two vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ . The equation

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

holds if (and only if)  $\vec{x}$  and  $\vec{y}$  are orthogonal. See Figure 9.

#### Proof

The verification is straightforward:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + 2(\vec{x} \cdot \vec{y}) + \vec{y} \cdot \vec{y} = \|\vec{x}\|^2 + 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2 \\ &= \|\vec{x}\|^2 + \|\vec{y}\|^2 \quad \text{if (and only if) } \vec{x} \cdot \vec{y} = 0. \end{aligned} \quad ■$$

Now we can generalize Example 6.

#### Theorem 1.10

##### An inequality for the magnitude of $\text{proj}_V \vec{x}$

Consider a subspace  $V$  of  $\mathbb{R}^n$  and a vector  $\vec{x}$  in  $\mathbb{R}^n$ . Then

$$\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|.$$

The statement is an equality if (and only if)  $\vec{x}$  is in  $V$ .

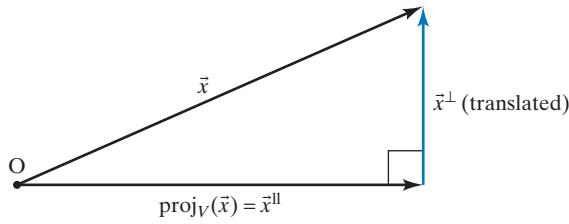
#### Proof

We apply the Pythagorean theorem (see Figure 10):

$$\|\vec{x}\|^2 = \|\text{proj}_V \vec{x}\|^2 + \|\vec{x}^\perp\|^2.$$

It follows that  $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$ , as claimed. ■

## Orthogonality and Least Squares

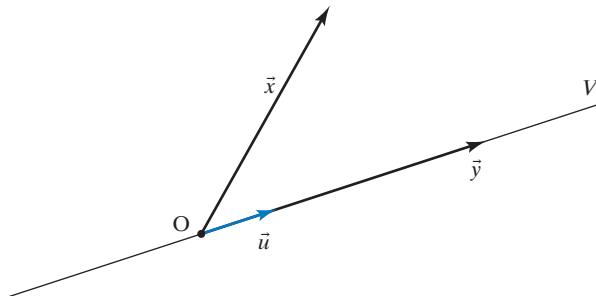


**Figure 10**

For example, let  $V$  be a one-dimensional subspace of  $\mathbb{R}^n$  spanned by a (nonzero) vector  $\vec{y}$ . We introduce the unit vector

$$\vec{u} = \frac{1}{\|\vec{y}\|} \vec{y}$$

in  $V$ . See Figure 11.



**Figure 11**

We know that

$$\text{proj}_V \vec{x} = (\vec{x} \cdot \vec{u}) \vec{u},$$

for any  $\vec{x}$  in  $\mathbb{R}^n$ . Theorem 1.10 tells us that

$$\|\vec{x}\| \geq \|\text{proj}_V \vec{x}\| = \|(\vec{x} \cdot \vec{u}) \vec{u}\| \stackrel{\text{step 3}}{\leq} |\vec{x} \cdot \vec{u}| = \left| \vec{x} \cdot \left( \frac{1}{\|\vec{y}\|} \vec{y} \right) \right| = \frac{1}{\|\vec{y}\|} |\vec{x} \cdot \vec{y}|.$$

To justify step 3, note that  $\|k\vec{v}\| = |k|\|\vec{v}\|$ , for all vectors  $\vec{v}$  in  $\mathbb{R}^n$  and all scalars  $k$ . See Exercise 25a. We conclude that

$$\frac{|\vec{x} \cdot \vec{y}|}{\|\vec{y}\|} \leq \|\vec{x}\|.$$

Multiplying both sides of this inequality by  $\|\vec{y}\|$ , we find the following useful result:

### Theorem 1.11

#### Cauchy–Schwarz inequality<sup>3</sup>

If  $\vec{x}$  and  $\vec{y}$  are vectors in  $\mathbb{R}^n$ , then

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|.$$

This statement is an equality if (and only if)  $\vec{x}$  and  $\vec{y}$  are parallel.

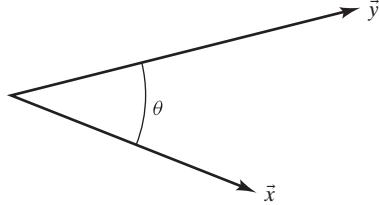
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<sup>3</sup>Named after the French mathematician Augustin-Louis Cauchy (1789–1857) and the German mathematician Hermann Amandus Schwarz (1843–1921).

## Orthogonality and Least Squares

Consider two nonzero vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^3$ . You may know an expression for the dot product  $\vec{x} \cdot \vec{y}$  in terms of the angle  $\theta$  between the two vectors (see Figure 12):

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \theta.$$



**Figure 12**

This formula allows us to find the angle between two nonzero vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^3$ :

$$\cos \theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \quad \text{or} \quad \theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}.$$

In  $\mathbb{R}^n$ , where we have no intuitive notion of an angle between two vectors, we can use this formula to *define* the angle:

### Definition 1.12

#### Angle between two vectors

Consider two nonzero vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ . The angle  $\theta$  between these vectors is defined as

$$\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}.$$

Note that  $\theta$  is between 0 and  $\pi$ , by definition of the inverse cosine function.

We have to make sure that

$$\arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$$

is defined; that is,

$$\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$$

is between  $-1$  and  $1$ , or, equivalently,

$$\left| \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \right| = \frac{|\vec{x} \cdot \vec{y}|}{\|\vec{x}\| \|\vec{y}\|} \leq 1.$$

But this follows from the Cauchy–Schwarz inequality,  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ .

### EXAMPLE 7

Find the angle between the vectors

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

#### Solution

$$\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \arccos \frac{1}{1 \cdot 2} = \frac{\pi}{3}$$

■

Here is an application to statistics of some concepts introduced in this section.

## Orthogonality and Least Squares

### Correlation (Optional)

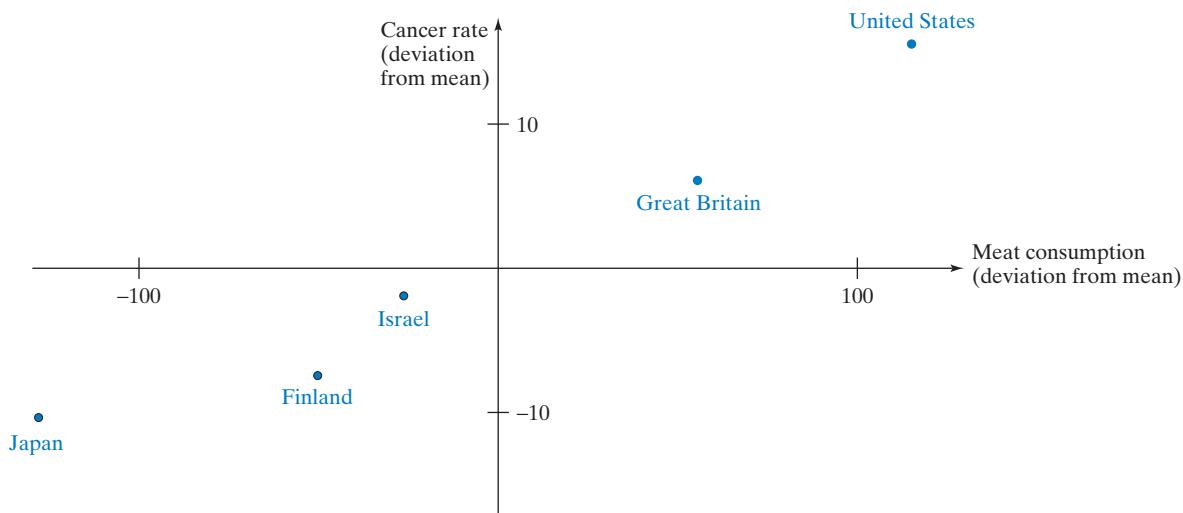
Consider the meat consumption (in grams per day per person) and incidence of colon cancer (per 100,000 women per year) in various industrialized countries.

Country	Meat Consumption	Cancer Rate
Japan	26	7.5
Finland	101	9.8
Israel	124	16.4
Great Britain	205	23.3
United States	284	34.0
Mean	148	18.2

Can we detect a positive or negative *correlation*<sup>4</sup> between meat consumption and cancer rate? Does a country with high meat consumption have high cancer rates, and vice versa? By *high*, we mean “above average,” of course. A quick look at the data shows such a *positive correlation*: In Great Britain and the United States, both meat consumption and cancer rate are above average. In the three other countries, they are below average. This positive correlation becomes more apparent when we list the preceding data as *deviations from the mean* (above or below the average).

Country	Meat Consumption (Deviation from Mean)	Cancer Rate (Deviation from Mean)
Japan	-122	-10.7
Finland	-47	-8.4
Israel	-24	-1.8
Great Britain	57	5.1
United States	136	15.8

Perhaps even more informative is a *scatter plot* of the deviation data. See Figure 13.



**Figure 13**

---

<sup>4</sup>We are using the term “correlation” in a colloquial, *qualitative* sense. Our goal is to *quantify* this term.

## Orthogonality and Least Squares

A positive correlation is indicated when most of the data points (in our case, all of them) are located in the first and third quadrant.

To process these data numerically, it is convenient to represent the deviation for both characteristics (meat consumption and cancer rate) as vectors in  $\mathbb{R}^5$ :

$$\vec{x} = \begin{bmatrix} -122 \\ -47 \\ -24 \\ 57 \\ 136 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -10.7 \\ -8.4 \\ -1.8 \\ 5.1 \\ 15.8 \end{bmatrix}.$$

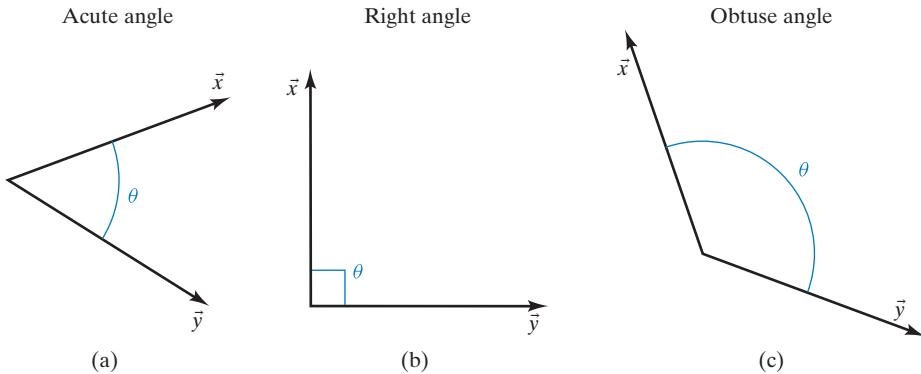
We will call these two vectors the *deviation vectors* of the two characteristics.

In the case of a positive correlation, most of the corresponding entries  $x_i, y_i$  of the deviation vectors have the same sign (both positive or both negative). In our example, this is the case for all entries. This means that the product  $x_i y_i$  will be positive most of the time; hence, the sum of all these products will be positive. But this sum is simply the dot product of the two deviation vectors.

Still using the term “correlation” in a colloquial sense, we conclude the following:

Consider two characteristics of a population, with deviation vectors  $\vec{x}$  and  $\vec{y}$ . There is a *positive correlation* between the two characteristics if (and only if)  $\vec{x} \cdot \vec{y} > 0$ .

A positive correlation between the characteristics means that the angle  $\theta$  between the deviation vectors is less than  $90^\circ$ . See Figure 14.



**Figure 14** (a) Positive correlation:  $\vec{x} \cdot \vec{y} > 0$ . (b) No correlation:  $\vec{x} \cdot \vec{y} = 0$ . (c) Negative correlation:  $\vec{x} \cdot \vec{y} < 0$ .

We can use the cosine of the angle  $\theta$  between  $\vec{x}$  and  $\vec{y}$  as a quantitative measure for the correlation between the two characteristics.

### Definition 1.13

#### Correlation coefficient

The *correlation coefficient*  $r$  between two characteristics of a population is the cosine of the angle  $\theta$  between the deviation vectors  $\vec{x}$  and  $\vec{y}$  for the two characteristics:

$$r = \cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}.$$

## Orthogonality and Least Squares

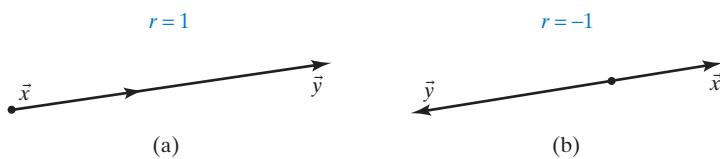
In the case of meat consumption and cancer, we find that

$$r \approx \frac{4182.9}{198.53 \cdot 21.539} \approx 0.9782.$$

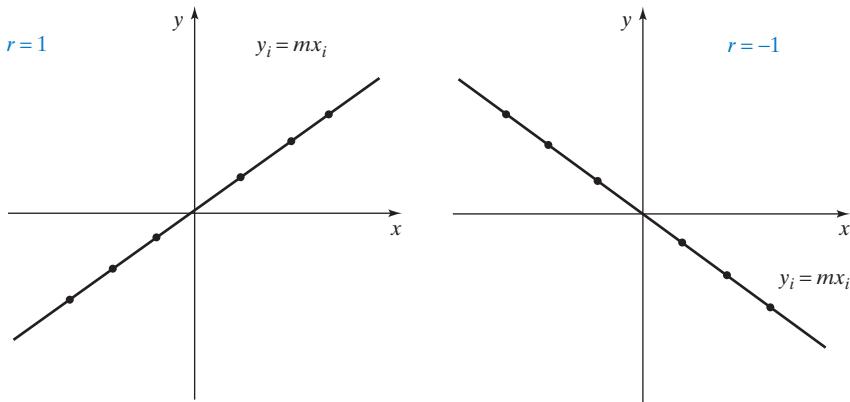
The angle between the two deviation vectors is  $\arccos(r) \approx 0.21$  (radians)  $\approx 12^\circ$ .

Note that the length of the deviation vectors is irrelevant for the correlation: If we had measured the cancer rate per 1,000,000 women (instead of 100,000), the vector  $\vec{y}$  would be 10 times longer, but the correlation would be the same.

The correlation coefficient  $r$  is always between  $-1$  and  $1$ ; the cases when  $r = 1$  (representing a perfect positive correlation) and  $r = -1$  (perfect negative correlation) are of particular interest. See Figure 15. In both cases, the data points  $(x_i, y_i)$  will be on the straight line  $y = mx$ . See Figure 16.



**Figure 15** (a)  $\vec{y} = m\vec{x}$ , for positive  $m$ . (b)  $\vec{y} = m\vec{x}$ , for negative  $m$ .



**Figure 16**

Note that even a strong positive correlation (an  $r$  close to 1) does not necessarily imply a causal relationship. Based only on the work we did above, we cannot conclude that high meat consumption causes colon cancer. Take a statistics course to learn more about these important issues!

## EXERCISES I

**GOAL** Apply the basic concepts of geometry in  $\mathbb{R}^n$ : length, angles, orthogonality. Use the idea of an orthogonal projection onto a subspace. Find this projection if an orthonormal basis of the subspace is given.

Find the length of each of the vectors  $\vec{v}$  in Exercises 1 through 3.

$$\begin{array}{lll} 1. \vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix} & 2. \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} & 3. \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \end{array}$$

## Orthogonality and Least Squares

Find the angle  $\theta$  between each of the pairs of vectors  $\vec{u}$  and  $\vec{v}$  in Exercises 4 through 6.

4.  $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

5.  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

6.  $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

For each pair of vectors  $\vec{u}, \vec{v}$  listed in Exercises 7 through 9, determine whether the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$  is acute, obtuse, or right.

7.  $\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

8.  $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -8 \\ 5 \end{bmatrix}$

9.  $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}$

10. For which value(s) of the constant  $k$  are the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

perpendicular?

11. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{in } \mathbb{R}^n.$$

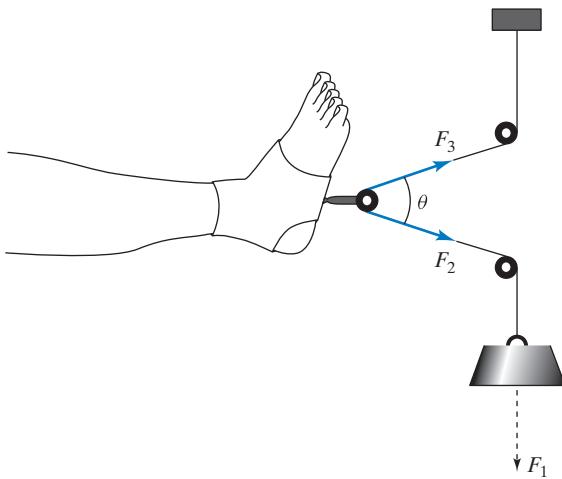
- a. For  $n = 2, 3, 4$ , find the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$ . For  $n = 2$  and 3, represent the vectors graphically.  
b. Find the limit of  $\theta$  as  $n$  approaches infinity.

12. Give an algebraic proof for the triangle inequality

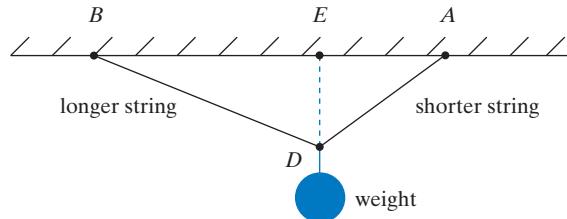
$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

Draw a sketch. Hint: Expand  $\|\vec{v} + \vec{w}\|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})$ . Then use the Cauchy-Schwarz inequality.

13. *Leg traction.* The accompanying figure shows how a leg may be stretched by a pulley line for therapeutic purposes. We denote by  $\vec{F}_1$  the vertical force of the weight. The string of the pulley line has the same tension everywhere; hence, the forces  $\vec{F}_2$  and  $\vec{F}_3$  have the same magnitude as  $\vec{F}_1$ . Assume that the magnitude of each force is 10 pounds. Find the angle  $\theta$  so that the magnitude of the force exerted on the leg is 16 pounds. Round your answer to the nearest degree. (Adapted from E. Batschelet, *Introduction to Mathematics for Life Scientists*, Springer, 1979.)

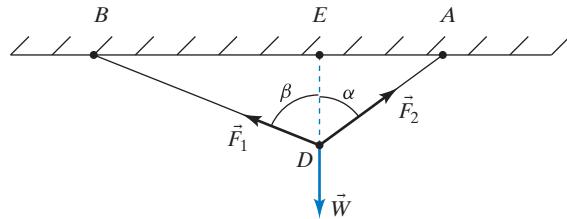


14. *Leonardo da Vinci and the resolution of forces.* Leonardo (1452–1519) asked himself how the weight of a body, supported by two strings of different length, is apportioned between the two strings.



Three forces are acting at the point  $D$ : the tensions  $\vec{F}_1$  and  $\vec{F}_2$  in the strings and the weight  $\vec{W}$ . Leonardo believed that

$$\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\overline{EA}}{\overline{EB}}.$$



Was he right? (Source: *Les Manuscrits de Léonard de Vinci*, published by Ravaïsson-Mollien, Paris, 1890.) Hint: Resolve  $\vec{F}_1$  into a horizontal and a vertical component; do the same for  $\vec{F}_2$ . Since the system is at rest, the equation  $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$  holds. Express the ratios

$$\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} \quad \text{and} \quad \frac{\overline{EA}}{\overline{EB}}$$

in terms of  $\alpha$  and  $\beta$ , using trigonometric functions, and compare the results.

## Orthogonality and Least Squares

15. Consider the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

Find a basis of the subspace of  $\mathbb{R}^4$  consisting of all vectors perpendicular to  $\vec{v}$ .

16. Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in  $\mathbb{R}^4$ . Can you find a vector  $\vec{u}_4$  in  $\mathbb{R}^4$  such that the vectors  $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$  are orthonormal? If so, how many such vectors are there?

17. Find a basis for  $W^\perp$ , where

$$W = \text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right).$$

18. Here is an infinite-dimensional version of Euclidean space: In the space of all infinite sequences, consider the subspace  $\ell_2$  of square-summable sequences [i.e., those sequences  $(x_1, x_2, \dots)$  for which the infinite series  $x_1^2 + x_2^2 + \dots$  converges]. For  $\vec{x}$  and  $\vec{y}$  in  $\ell_2$ , we define

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots}, \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots.$$

(Why does the series  $x_1 y_1 + x_2 y_2 + \dots$  converge?)

- a. Check that  $\vec{x} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots)$  is in  $\ell_2$ , and find  $\|\vec{x}\|$ . Recall the formula for the geometric series:  $1 + a + a^2 + a^3 + \dots = 1/(1-a)$ , if  $-1 < a < 1$ .
- b. Find the angle between  $(1, 0, 0, \dots)$  and  $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ .
- c. Give an example of a sequence  $(x_1, x_2, \dots)$  that converges to 0 (i.e.,  $\lim_{n \rightarrow \infty} x_n = 0$ ) but does not belong to  $\ell_2$ .
- d. Let  $L$  be the subspace of  $\ell_2$  spanned by  $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ . Find the orthogonal projection of  $(1, 0, 0, \dots)$  onto  $L$ .

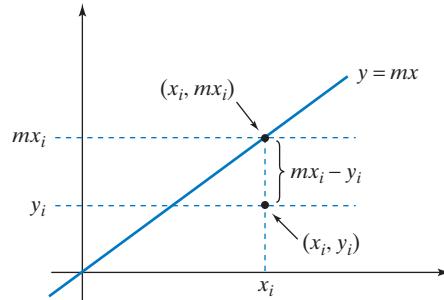
The Hilbert space  $\ell_2$  was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of  $\ell_2$ . Today, this space is used in many other applications, including economics. See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.

19. For a line  $L$  in  $\mathbb{R}^2$ , draw a sketch to interpret the following transformations geometrically:

- a.  $T(\vec{x}) = \vec{x} - \text{proj}_L \vec{x}$
- b.  $T(\vec{x}) = \vec{x} - 2\text{proj}_L \vec{x}$
- c.  $T(\vec{x}) = 2\text{proj}_L \vec{x} - \vec{x}$

20. Refer to Figure 13 of this section. The *least-squares line* for these data is the line  $y = mx$  that fits the data best, in that the sum of the squares of the vertical distances between the line and the data points is minimal. We want to minimize the sum

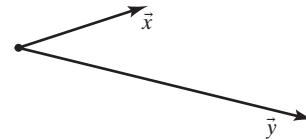
$$(mx_1 - y_1)^2 + (mx_2 - y_2)^2 + \dots + (mx_5 - y_5)^2.$$



In vector notation, to minimize the sum means to find the scalar  $m$  such that

$$\|m\vec{x} - \vec{y}\|^2$$

is minimal. Arguing geometrically, explain how you can find  $m$ . Use the accompanying sketch, which is not drawn to scale.



Find  $m$  numerically, and explain the relationship between  $m$  and the correlation coefficient  $r$ . You may find the following information helpful:

$$\vec{x} \cdot \vec{y} = 4182.9, \quad \|\vec{x}\| \approx 198.53, \quad \|\vec{y}\| \approx 21.539.$$

To check whether your solution  $m$  is reasonable, draw the line  $y = mx$  in Figure 13. (A more thorough discussion of least-squares approximations will follow in Section 4.)

21. Find scalars  $a, b, c, d, e, f, g$  such that the vectors

$$\begin{bmatrix} a \\ d \\ f \end{bmatrix}, \quad \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \quad \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$$

are orthonormal.

22. Consider a basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  of a subspace  $V$  of  $\mathbb{R}^n$ . Show that a vector  $\vec{x}$  in  $\mathbb{R}^n$  is orthogonal to  $V$  if (and only if)  $\vec{x}$  is orthogonal to all the vectors  $\vec{v}_1, \dots, \vec{v}_m$ .
23. Prove Theorem 1.8d.  $(V^\perp)^\perp = V$  for any subspace  $V$  of  $\mathbb{R}^n$ . Hint: Show that  $V \subseteq (V^\perp)^\perp$ , by the definition of  $V^\perp$ ; then show that  $\dim(V) = \dim((V^\perp)^\perp)$ , by Theorem 1.8c.
24. Complete the proof of Theorem 1.4: Orthogonal projections are linear transformations.

## Orthogonality and Least Squares

25. a. Consider a vector  $\vec{v}$  in  $\mathbb{R}^n$ , and a scalar  $k$ . Show that

$$\|k\vec{v}\| = |k|\|\vec{v}\|.$$

- b. Show that if  $\vec{v}$  is a nonzero vector in  $\mathbb{R}^n$ , then  $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$  is a unit vector.

26. Find the orthogonal projection of  $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$  onto the subspace of  $\mathbb{R}^3$  spanned by

$$\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

27. Find the orthogonal projection of  $9\vec{e}_1$  onto the subspace of  $\mathbb{R}^4$  spanned by

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

28. Find the orthogonal projection of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

onto the subspace of  $\mathbb{R}^4$  spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

29. Consider the orthonormal vectors  $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$  in  $\mathbb{R}^{10}$ . Find the length of the vector

$$\vec{x} = 7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5.$$

30. Consider a subspace  $V$  of  $\mathbb{R}^n$  and a vector  $\vec{x}$  in  $\mathbb{R}^n$ . Let  $\vec{y} = \text{proj}_V \vec{x}$ . What is the relationship between the following quantities?

$$\|\vec{y}\|^2 \quad \text{and} \quad \vec{y} \cdot \vec{x}$$

31. Consider the orthonormal vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$  in  $\mathbb{R}^n$ , and an arbitrary vector  $\vec{x}$  in  $\mathbb{R}^n$ . What is the relationship between the following two quantities?

$$p = (\vec{u}_1 \cdot \vec{x})^2 + (\vec{u}_2 \cdot \vec{x})^2 + \dots + (\vec{u}_m \cdot \vec{x})^2 \quad \text{and} \quad \|\vec{x}\|^2$$

When are the two quantities equal?

32. Consider two vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^n$ . Form the matrix

$$G = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{bmatrix}.$$

For which choices of  $\vec{v}_1$  and  $\vec{v}_2$  is the matrix  $G$  invertible?

33. Among all the vectors in  $\mathbb{R}^n$  whose components add up to 1, find the vector of minimal length. In the case  $n = 2$ , explain your solution geometrically.

34. Among all the unit vectors in  $\mathbb{R}^n$ , find the one for which the sum of the components is maximal. In the case  $n = 2$ , explain your answer geometrically, in terms of the unit circle and the level curves of the function  $x_1 + x_2$ .

35. Among all the unit vectors  $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  in  $\mathbb{R}^3$ , find the one for which the sum  $x + 2y + 3z$  is minimal.

36. There are three exams in your linear algebra class, and you theorize that your score in each exam (out of 100) will be numerically equal to the number of hours you study for that exam. The three exams count 20%, 30%, and 50%, respectively, toward the final grade. If your (modest) goal is to score 76% in the course, how many hours  $a$ ,  $b$ , and  $c$  should you study for each of the three exams to minimize quantity  $a^2 + b^2 + c^2$ ? This quadratic model reflects the fact that it may be four times as painful to study for 10 hours than for just 5 hours.

37. Consider a plane  $V$  in  $\mathbb{R}^3$  with orthonormal basis  $\vec{u}_1, \vec{u}_2$ . Let  $\vec{x}$  be a vector in  $\mathbb{R}^3$ . Find a formula for the reflection  $R(\vec{x})$  of  $\vec{x}$  about the plane  $V$ .

38. Consider three unit vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$  in  $\mathbb{R}^n$ . We are told that  $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = 1/2$ . What are the possible values of  $\vec{v}_2 \cdot \vec{v}_3$ ? What could the angle between the vectors  $\vec{v}_2$  and  $\vec{v}_3$  be? Give examples; draw sketches for the cases  $n = 2$  and  $n = 3$ .

39. Can you find a line  $L$  in  $\mathbb{R}^n$  and a vector  $\vec{x}$  in  $\mathbb{R}^n$  such that

$$\vec{x} \cdot \text{proj}_L \vec{x}$$

is negative? Explain, arguing algebraically.

*In Exercises 40 through 46, consider vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^4$ ; we are told that  $\vec{v}_i \cdot \vec{v}_j$  is the entry  $a_{ij}$  of matrix A.*

$$A = \begin{bmatrix} 3 & 5 & 11 \\ 5 & 9 & 20 \\ 11 & 20 & 49 \end{bmatrix}$$

40. Find  $\|\vec{v}_2\|$ .

41. Find the angle enclosed by vectors  $\vec{v}_2$  and  $\vec{v}_3$ .

42. Find  $\|\vec{v}_1 + \vec{v}_2\|$ .

43. Find  $\text{proj}_{\vec{v}_2}(\vec{v}_1)$ , expressed as a scalar multiple of  $\vec{v}_2$ .

44. Find a nonzero vector  $\vec{v}$  in  $\text{span}(\vec{v}_2, \vec{v}_3)$  such that  $\vec{v}$  is orthogonal to  $\vec{v}_3$ . Express  $\vec{v}$  as a linear combination of  $\vec{v}_2$  and  $\vec{v}_3$ .

45. Find  $\text{proj}_V(\vec{v}_1)$ , where  $V = \text{span}(\vec{v}_2, \vec{v}_3)$ . Express your answer as a linear combination of  $\vec{v}_2$  and  $\vec{v}_3$ .

46. Find  $\text{proj}_V(\vec{v}_3)$ , where  $V = \text{span}(\vec{v}_1, \vec{v}_2)$ . Express your answer as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ .

## 2 Gram–Schmidt Process and QR Factorization

One of the main themes of this chapter is the study of the orthogonal projection onto a subspace  $V$  of  $\mathbb{R}^n$ . In Theorem 1.5, we gave a formula for this projection,

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m,$$

where  $\vec{u}_1, \dots, \vec{u}_m$  is an orthonormal basis of  $V$ . Now we will show how to construct such an orthonormal basis. We will present an algorithm that allows us to convert any basis  $\vec{v}_1, \dots, \vec{v}_m$  of a subspace  $V$  of  $\mathbb{R}^n$  into an orthonormal basis  $\vec{u}_1, \dots, \vec{u}_m$  of  $V$ .

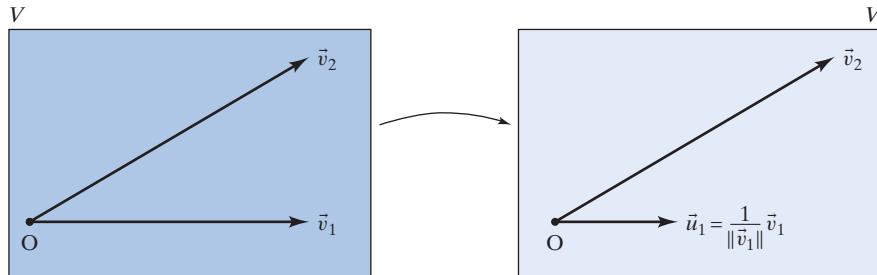
Let us first think about low-dimensional cases. If  $V$  is a line with basis  $\vec{v}_1$ , we can find an orthonormal basis  $\vec{u}_1$  simply by dividing  $\vec{v}_1$  by its length:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1.$$

When  $V$  is a plane with basis  $\vec{v}_1, \vec{v}_2$ , we first construct

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1$$

as before. See Figure 1.



**Figure 1**

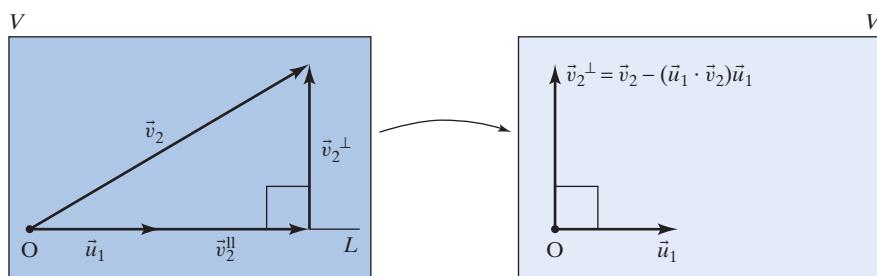
Now comes the crucial step: We have to find a vector in  $V$  orthogonal to  $\vec{u}_1$ . (Initially, we will not insist that this vector be a unit vector.) Let's resolve the vector  $\vec{v}_2$  into its components parallel and perpendicular to the line  $L$  spanned by  $\vec{u}_1$ :

$$\vec{v}_2 = \vec{v}_2^\parallel + \vec{v}_2^\perp$$

See Figure 2. Then the vector

$$\vec{v}_2^\perp = \vec{v}_2 - \vec{v}_2^\parallel = \vec{v}_2 - \text{proj}_L(\vec{v}_2) = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

is orthogonal to  $\vec{u}_1$ .

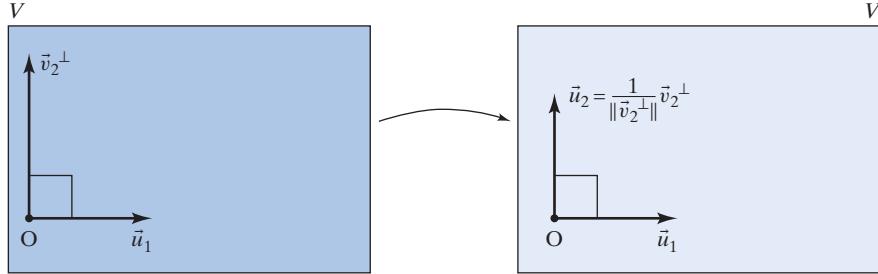


**Figure 2**

### Orthogonality and Least Squares

The last step is straightforward: We divide the vector  $\vec{v}_2^\perp$  by its length to get the second vector  $\vec{u}_2$  of an orthonormal basis (see Figure 3):

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|} \vec{v}_2^\perp.$$



**Figure 3**

#### EXAMPLE I

- a. Find an orthonormal basis  $\vec{u}_1, \vec{u}_2$  of the subspace

$$V = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} \right) \text{ of } \mathbb{R}^4, \text{ with basis } \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}.$$

- b. Find the change of basis matrix  $R$  from the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$  to the basis  $\mathfrak{A} = (\vec{u}_1, \vec{u}_2)$  you constructed in part a.

#### Solution

- a. Following the three steps illustrated in Figures 1, 2, and 3, we will compute first  $\vec{u}_1$ , then  $\vec{v}_2^\perp$ , and finally  $\vec{u}_2$ :

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

$$\vec{v}_2^\perp = \vec{v}_2 - \vec{v}_2^\parallel = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1 = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} - 10 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix},$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|} \vec{v}_2^\perp = \frac{1}{8} \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

We have found an orthonormal basis  $\mathfrak{A}$  of  $V$ :

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

### Orthogonality and Least Squares

- b.** Recall that the columns of the change of basis matrix  $R$  from  $\mathfrak{B}$  to  $\mathfrak{A}$  are the coordinate vectors of  $\vec{v}_1$  and  $\vec{v}_2$  with respect to basis  $\mathfrak{A} = (\vec{u}_1, \vec{u}_2)$ .

A straightforward computation shows that  $\vec{v}_1 = 2\vec{u}_1$  and  $\vec{v}_2 = 10\vec{u}_1 + 8\vec{u}_2$ , so that

$$R = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \\ 2 & 10 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix}.$$

(Later in this section, we will develop more efficient methods for finding the entries of  $R$ .)

To express the relationship between the bases  $\mathfrak{A}$  and  $\mathfrak{B}$  in matrix form, we write

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} R, \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & 9 \\ 1 & 1 \end{bmatrix}}_M = \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 2 & 10 \\ 0 & 8 \end{bmatrix}}_R.$$

In this context, it is customary to denote the matrices on the right-hand side by  $Q$  and  $R$ . Note that we have written the  $4 \times 2$  matrix  $M$  with columns  $\vec{v}_1$  and  $\vec{v}_2$  as the product of the  $4 \times 2$  matrix  $Q$  with orthonormal columns and the upper triangular  $2 \times 2$  matrix  $R$  with positive entries on the diagonal. This is referred to as the *QR factorization* of matrix  $M$ . Matrix  $Q$  stores the orthonormal basis  $\vec{u}_1, \vec{u}_2$  we constructed, and matrix  $R$  gives the relationship between the “old” basis  $\vec{v}_1, \vec{v}_2$ , and the “new” basis  $\vec{u}_1, \vec{u}_2$  of  $V$ . ■

Now that we know how to find an orthonormal basis of a plane, how would we proceed in the case of a three-dimensional subspace  $V$  of  $\mathbb{R}^n$  with basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ ? We can first find an orthonormal basis  $\vec{u}_1, \vec{u}_2$  of the plane  $E = \text{span}(\vec{v}_1, \vec{v}_2)$ , as illustrated in Example 1. Next we resolve the vector  $\vec{v}_3$  into its components parallel and perpendicular to the plane  $E$ :

$$\vec{v}_3 = \vec{v}_3^{\parallel} + \vec{v}_3^{\perp},$$

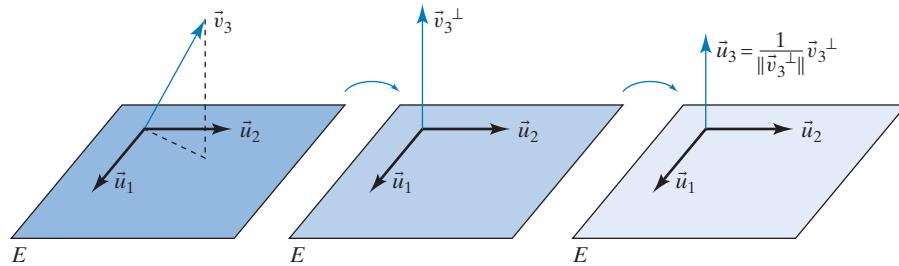
so that

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel} = \vec{v}_3 - \text{proj}_E(\vec{v}_3) = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2.$$

Finally, we let

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|} \vec{v}_3^{\perp}.$$

See Figure 4.



**Figure 4**

## Orthogonality and Least Squares

Generalizing this method, we can construct an orthonormal basis of any subspace  $V$  of  $\mathbb{R}^n$ . Unfortunately, the terminology gets a bit heavy in the general case. Conceptually, the method is pretty straightforward, however: We keep computing perpendicular components of vectors, using the formula for projection, and we keep dividing vectors by their length to generate unit vectors.

### Theorem 2.1

#### The Gram–Schmidt process<sup>5</sup>

Consider a basis  $\vec{v}_1, \dots, \vec{v}_m$  of a subspace  $V$  of  $\mathbb{R}^n$ . For  $j = 2, \dots, m$ , we resolve the vector  $\vec{v}_j$  into its components parallel and perpendicular to the span of the preceding vectors,  $\vec{v}_1, \dots, \vec{v}_{j-1}$ :

$$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}, \quad \text{with respect to } \text{span}(\vec{v}_1, \dots, \vec{v}_{j-1}).$$

Then

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1, \quad \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp}, \dots, \quad \vec{u}_j = \frac{1}{\|\vec{v}_j^{\perp}\|} \vec{v}_j^{\perp}, \dots, \quad \vec{u}_m = \frac{1}{\|\vec{v}_m^{\perp}\|} \vec{v}_m^{\perp}$$

is an orthonormal basis of  $V$ . By Theorem 1.7, we have

$$\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{\parallel} = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j) \vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j) \vec{u}_{j-1}.$$

If you are puzzled by these formulas, go back to the cases where  $V$  is a two- or three-dimensional space; take another good look at Figures 1 through 4.

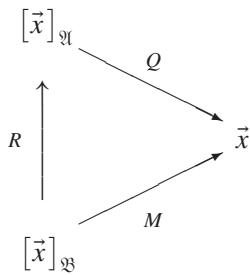
### The QR Factorization

The Gram–Schmidt process represents a change of basis from the “old” basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  to a “new,” orthonormal basis  $\mathfrak{A} = (\vec{u}_1, \dots, \vec{u}_m)$  of  $V$ ; it is most succinctly described in terms of the change of basis matrix  $R$  from  $\mathfrak{B}$  to  $\mathfrak{A}$ , as discussed in Example 1.

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_M = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_Q R.$$

Again, it is customary to denote the matrices on the right-hand side by  $Q$  and  $R$ ; the preceding equation is called the *QR factorization* of  $M$ .

We can represent the relationship among the matrices  $M$ ,  $Q$ , and  $R$  in a commutative diagram, where  $\vec{x}$  is a vector in  $V$ .



<sup>5</sup>Named after the Danish actuary Jørgen Gram (1850–1916) and the German mathematician Erhardt Schmidt (1876–1959).

## Orthogonality and Least Squares

What do the entries of the change of basis matrix  $R$  look like? We know that the entries in the  $j$ th column of  $R$  are the coordinates of  $\vec{v}_j$  with respect to the basis  $\mathfrak{A} = (\vec{u}_1, \dots, \vec{u}_m)$ . Using the equations in Theorem 2.1, we can write

$$\begin{aligned}\vec{v}_j &= \vec{v}_j^{\parallel} + \vec{v}_j^{\perp} \\ &= \underbrace{(\vec{u}_1 \cdot \vec{v}_j) \vec{u}_1 + \cdots + (\vec{u}_i \cdot \vec{v}_j) \vec{u}_i}_{\vec{v}_j^{\parallel}} + \cdots + \underbrace{(\vec{u}_{j-1} \cdot \vec{v}_j) \vec{u}_{j-1}}_{\vec{v}_j^{\perp}} + \underbrace{\|\vec{v}_j^{\perp}\| \vec{u}_j}_{r_{jj}}.\end{aligned}$$

It follows that  $r_{ij} = \vec{u}_i \cdot \vec{v}_j$  if  $i < j$ ;  $r_{jj} = \|\vec{v}_j^{\perp}\|$ ; and  $r_{ij} = 0$  if  $i > j$ . The last equation implies that  $R$  is upper triangular. (The first diagonal entry is  $r_{11} = \|\vec{v}_1\|$ , since  $\vec{v}_1 = \|\vec{v}_1\| \vec{u}_1$ .)

### Theorem 2.2

#### **$QR$ factorization**

Consider an  $n \times m$  matrix  $M$  with linearly independent columns  $\vec{v}_1, \dots, \vec{v}_m$ . Then there exists an  $n \times m$  matrix  $Q$  whose columns  $\vec{u}_1, \dots, \vec{u}_m$  are orthonormal and an upper triangular matrix  $R$  with positive diagonal entries such that

$$M = QR.$$

This representation is unique. Furthermore,  $r_{11} = \|\vec{v}_1\|$ ,  $r_{jj} = \|\vec{v}_j^{\perp}\|$  (for  $j = 2, \dots, m$ ), and  $r_{ij} = \vec{u}_i \cdot \vec{v}_j$  (for  $i < j$ ).

Take another look at Example 1, where  $L = V_1 = \text{span}(\vec{v}_1)$ .

The verification of the uniqueness of the  $QR$  factorization is left as Exercise 3.51. To find the  $QR$  factorization of a matrix  $M$ , we perform the Gram–Schmidt process on the columns of  $M$ , constructing  $R$  and  $Q$  column by column. No extra computations are required: All the information necessary to build  $R$  and  $Q$  is provided by the Gram–Schmidt process.  $QR$  factorization is an effective way to organize and record the work performed in the Gram–Schmidt process; it is useful for many computational and theoretical purposes.

**EXAMPLE 2** Find the  $QR$  factorization of the matrix  $M = \begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & -8 \end{bmatrix}$ .

#### **Solution**

Here

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}.$$

As in Example 1, the  $QR$  factorization of  $M$  will have the form

$$M = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \|\vec{v}_2^{\perp}\| \end{bmatrix}}_R.$$

## Orthogonality and Least Squares

We will compute the entries of  $R$  and the columns of  $Q$  step by step:

$$\begin{aligned} r_{11} &= \|\vec{v}_1\| = 3, \quad \vec{u}_1 = \frac{1}{r_{11}}\vec{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \\ r_{12} &= \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix} = 9, \quad \vec{v}_2^\perp = \vec{v}_2 - r_{12}\vec{u}_1 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}, \\ r_{22} &= \left\| \vec{v}_2^\perp \right\| = \sqrt{36} = 6, \quad \vec{u}_2 = \frac{1}{r_{22}}\vec{v}_2^\perp = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & -8 \end{bmatrix} &= M = QR = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \\ &= \underbrace{\left( \frac{1}{3} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{bmatrix} \right)}_Q \underbrace{\begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}}_R. \end{aligned}$$

Draw pictures analogous to Figures 1 through 3 to illustrate these computations! ■

Let us outline the algorithm we used in Example 2.

### Theorem 2.3

**QR factorization**

Consider an  $n \times m$  matrix  $M$  with linearly independent columns  $\vec{v}_1, \dots, \vec{v}_m$ . Then the columns  $\vec{u}_1, \dots, \vec{u}_m$  of  $Q$  and the entries  $r_{ij}$  of  $R$  can be computed in the following order:

- first column of  $R$ , first column of  $Q$ ;
- second column of  $R$ , second column of  $Q$ ;
- third column of  $R$ , third column of  $Q$ ;
- and so on.

More specifically,

$$\begin{aligned} r_{11} &= \|\vec{v}_1\|, \quad \vec{u}_1 = \frac{1}{r_{11}}\vec{v}_1; \\ r_{12} &= \vec{u}_1 \cdot \vec{v}_2, \quad \vec{v}_2^\perp = \vec{v}_2 - r_{12}\vec{u}_1, \quad r_{22} = \left\| \vec{v}_2^\perp \right\|, \quad \vec{u}_2 = \frac{1}{r_{22}}\vec{v}_2^\perp; \\ r_{13} &= \vec{u}_1 \cdot \vec{v}_3, \quad r_{23} = \vec{u}_2 \cdot \vec{v}_3, \quad \vec{v}_3^\perp = \begin{cases} \vec{v}_3 - r_{13}\vec{u}_1 \\ -r_{23}\vec{u}_2 \end{cases}, \\ r_{33} &= \left\| \vec{v}_3^\perp \right\|, \quad \vec{u}_3 = \frac{1}{r_{33}}\vec{v}_3^\perp; \end{aligned}$$

and so on.

For matrices  $M$  with more than three columns, the computation of the  $QR$  factorization is tedious, and may best be left to a machine (unless  $M$  is of a particularly simple form).

## EXERCISES 2

**GOAL** Perform the Gram–Schmidt process, and thus find the QR factorization of a matrix.

Using paper and pencil, perform the Gram–Schmidt process on the sequences of vectors given in Exercises 1 through 14.

1.  $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

2.  $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

3.  $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$

5.  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

7.  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$

9.  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}$

11.  $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 14 \\ 10 \end{bmatrix}$

13.  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$

15.  $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

17.  $\begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix}$

19.  $\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$

2.  $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

4.  $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

6.  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

8.  $\begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix}$

10.  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix}$

12.  $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 2 \\ 13 \end{bmatrix}$

14.  $\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix}$

Using paper and pencil, find the QR factorizations of the matrices in Exercises 15 through 28. Compare with Exercises 1 through 14.

16.  $\begin{bmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{bmatrix}$

18.  $\begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$

20.  $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

21.  $\begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

22.  $\begin{bmatrix} 5 & 3 \\ 4 & 6 \\ 2 & 7 \\ 2 & -2 \end{bmatrix}$

23.  $\begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & -5 \\ 1 & 3 \end{bmatrix}$

24.  $\begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 6 \\ 1 & 4 \end{bmatrix}$

25.  $\begin{bmatrix} 4 & 5 \\ 0 & 2 \\ 0 & 14 \\ 3 & 10 \end{bmatrix}$

26.  $\begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 0 & 2 \\ 6 & 13 \end{bmatrix}$

27.  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

28.  $\begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$

29. Perform the Gram–Schmidt process on the following basis of  $\mathbb{R}^2$ :

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Illustrate your work with sketches, as in Figures 1 through 3 of this section.

30. Consider two linearly independent vectors  $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$  in  $\mathbb{R}^2$ . Draw sketches (as in Figures 1 through 3 of this section) to illustrate the Gram–Schmidt process for  $\vec{v}_1, \vec{v}_2$ . You need not perform the process algebraically.

31. Perform the Gram–Schmidt process on the following basis of  $\mathbb{R}^3$ :

$$\vec{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Here,  $a, c$ , and  $f$  are positive constants, and the other constants are arbitrary. Illustrate your work with a sketch, as in Figure 4.

32. Find an orthonormal basis of the plane

$$x_1 + x_2 + x_3 = 0.$$

33. Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

34. Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

## Orthogonality and Least Squares

35. Find an orthonormal basis of the image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

36. Consider the matrix

$$M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Find the  $QR$  factorization of  $M$ .

37. Consider the matrix

$$M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find the  $QR$  factorization of  $M$ .

38. Find the  $QR$  factorization of

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

39. Find an orthonormal basis  $\vec{u}_1, \vec{u}_2, \vec{u}_3$  of  $\mathbb{R}^3$  such that

$$\text{span}(\vec{u}_1) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

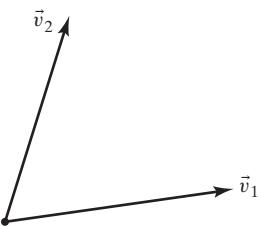
and

$$\text{span}(\vec{u}_1, \vec{u}_2) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right).$$

40. Consider an invertible  $n \times n$  matrix  $A$  whose columns are orthogonal, but not necessarily orthonormal. What does the  $QR$  factorization of  $A$  look like?

41. Consider an invertible upper triangular  $n \times n$  matrix  $A$ . What does the  $QR$  factorization of  $A$  look like?

42. The two column vectors  $\vec{v}_1$  and  $\vec{v}_2$  of a  $2 \times 2$  matrix  $A$  are shown in the accompanying figure. Let  $A = QR$  be the  $QR$  factorization of  $A$ . Represent the diagonal entries  $r_{11}$  and  $r_{22}$  of  $R$  as lengths in the figure. Interpret the product  $r_{11}r_{22}$  as an area.



43. Consider a block matrix

$$A = [A_1 \ A_2]$$

with linearly independent columns. ( $A_1$  is an  $n \times m_1$  matrix, and  $A_2$  is  $n \times m_2$ .) Suppose you know the  $QR$  factorization of  $A$ . Explain how this allows you to find the  $QR$  factorization of  $A_1$ .

44. Consider an  $n \times m$  matrix  $A$  with  $\text{rank}(A) < m$ . Is it always possible to write

$$A = QR,$$

where  $Q$  is an  $n \times m$  matrix with orthonormal columns and  $R$  is upper triangular? Explain.

45. Consider an  $n \times m$  matrix  $A$  with  $\text{rank}(A) = m$ . Is it always possible to write  $A$  as

$$A = QL,$$

where  $Q$  is an  $n \times m$  matrix with orthonormal columns and  $L$  is a lower triangular  $m \times m$  matrix with positive diagonal entries? Explain.

## 3 Orthogonal Transformations and Orthogonal Matrices

In geometry, we are particularly interested in those linear transformations that preserve the length of vectors.

### Definition 3.1

### Orthogonal transformations and orthogonal matrices

A linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  is called *orthogonal* if it preserves the length of vectors:

$$\|T(\vec{x})\| = \|\vec{x}\|, \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

If  $T(\vec{x}) = A\vec{x}$  is an orthogonal transformation, we say that  $A$  is an *orthogonal matrix*.<sup>6</sup>

<sup>6</sup>A list of alternative characterizations of an orthogonal matrix will be presented in Summary 3.8.

## Orthogonality and Least Squares

**EXAMPLE I** The rotation

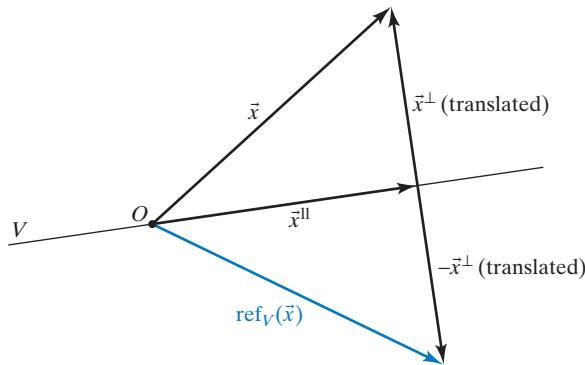
$$T(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}$$

is an orthogonal transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , and

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

is an orthogonal matrix, for all angles  $\theta$ . ■

**EXAMPLE 2** Consider a subspace  $V$  of  $\mathbb{R}^n$ . For a vector  $\vec{x}$  in  $\mathbb{R}^n$ , the vector  $\text{ref}_V(\vec{x}) = \vec{x}^\parallel - \vec{x}^\perp$  is called the reflection of  $\vec{x}$  about  $V$ . See Figure 1. Show that reflections are orthogonal transformations.



**Figure 1**

### Solution

By the Pythagorean theorem, we have

$$\|\text{ref}_V(\vec{x})\|^2 = \|\vec{x}^\parallel\|^2 + \|-\vec{x}^\perp\|^2 = \|\vec{x}^\parallel\|^2 + \|\vec{x}^\perp\|^2 = \|\vec{x}\|^2. \quad ■$$

As the name suggests, orthogonal transformations preserve right angles. In fact, orthogonal transformations preserve all angles. See Exercise 29.

### Theorem 3.2

#### Orthogonal transformations preserve orthogonality

Consider an orthogonal transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . If the vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are orthogonal, then so are  $T(\vec{v})$  and  $T(\vec{w})$ .

**Proof** By the theorem of Pythagoras, we have to show that

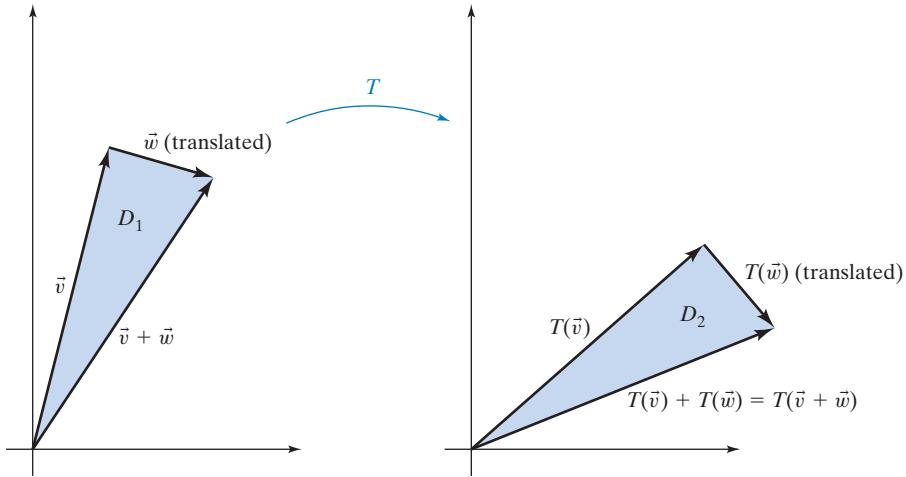
$$\|T(\vec{v}) + T(\vec{w})\|^2 = \|T(\vec{v})\|^2 + \|T(\vec{w})\|^2.$$

Let's see:

$$\begin{aligned} \|T(\vec{v}) + T(\vec{w})\|^2 &= \|T(\vec{v} + \vec{w})\|^2 && (T \text{ is linear}) \\ &= \|\vec{v} + \vec{w}\|^2 && (T \text{ is orthogonal}) \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 && (\vec{v} \text{ and } \vec{w} \text{ are orthogonal}) \\ &= \|T(\vec{v})\|^2 + \|T(\vec{w})\|^2 && (T \text{ is orthogonal}). \end{aligned} \quad ■$$

## Orthogonality and Least Squares

Theorem 3.2 is perhaps better explained with a sketch. See Figure 2.



**Figure 2**

The two shaded triangles are congruent, because corresponding sides are the same length (since  $T$  preserves length). Since  $D_1$  is a right triangle, so is  $D_2$ .

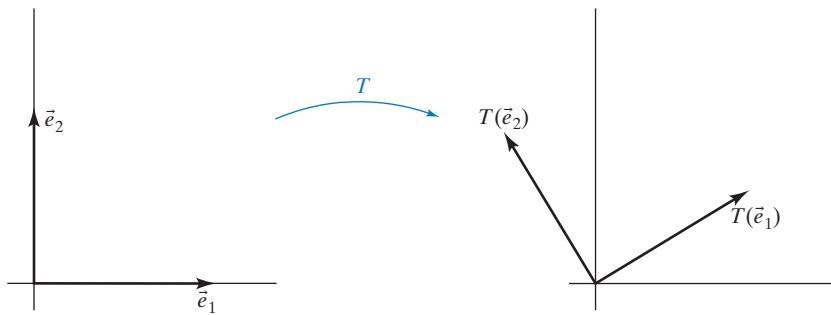
Here is an alternative characterization of orthogonal transformations:

### Theorem 3.3

#### Orthogonal transformations and orthonormal bases

- a. A linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  is orthogonal if (and only if) the vectors  $T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n)$  form an orthonormal basis of  $\mathbb{R}^n$ .
- b. An  $n \times n$  matrix  $A$  is orthogonal if (and only if) its columns form an orthonormal basis of  $\mathbb{R}^n$ .

Figure 3 illustrates part (a) for a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .



**Figure 3**

**Proof** We prove part (a). If  $T$  is an orthogonal transformation, then, by definition, the  $T(\vec{e}_i)$  are unit vectors and, by Theorem 3.2, they are orthogonal. Conversely, suppose the  $T(\vec{e}_i)$  form an orthonormal basis. Consider a vector  $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$  in  $\mathbb{R}^n$ . Then

## Orthogonality and Least Squares

$$\begin{aligned}
 \|T(\vec{x})\|^2 &= \|x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)\|^2 \\
 &= \|x_1 T(\vec{e}_1)\|^2 + \|x_2 T(\vec{e}_2)\|^2 + \cdots + \|x_n T(\vec{e}_n)\|^2 \quad (\text{by Pythagoras}) \\
 &= x_1^2 + x_2^2 + \cdots + x_n^2 \\
 &= \|\vec{x}\|^2.
 \end{aligned}$$

*Note:* A matrix with orthogonal columns need not be an orthogonal matrix. As an example, consider the matrix  $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ .

**EXAMPLE 3** Show that the matrix  $A$  is orthogonal:

$$A = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

### Solution

Check that the columns of  $A$  form an orthonormal basis of  $\mathbb{R}^4$ .

Here are some algebraic properties of orthogonal matrices.

### Theorem 3.4

#### Products and inverses of orthogonal matrices

- a. The product  $AB$  of two orthogonal  $n \times n$  matrices  $A$  and  $B$  is orthogonal.
- b. The inverse  $A^{-1}$  of an orthogonal  $n \times n$  matrix  $A$  is orthogonal.

**Proof** In part (a), the linear transformation  $T(\vec{x}) = AB\vec{x}$  preserves length, because  $\|T(\vec{x})\| = \|A(B\vec{x})\| = \|B\vec{x}\| = \|\vec{x}\|$ . In part (b), the linear transformation  $T(\vec{x}) = A^{-1}\vec{x}$  preserves length, because  $\|A^{-1}\vec{x}\| = \|A(A^{-1}\vec{x})\| = \|\vec{x}\|$ . Figure 4 illustrates property (a).

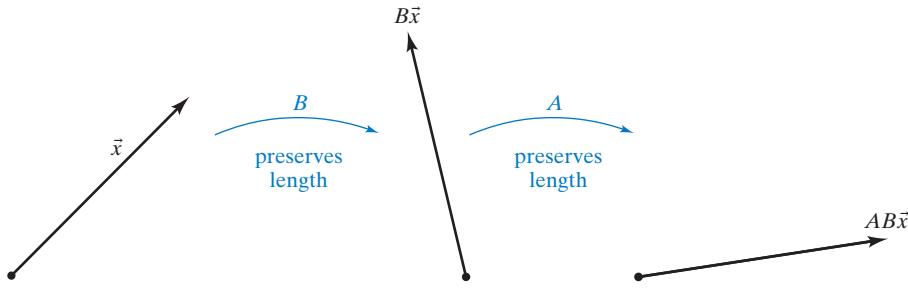


Figure 4

### The Transpose of a Matrix

**EXAMPLE 4** Consider the orthogonal matrix

$$A = \frac{1}{7} \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}.$$

## Orthogonality and Least Squares

Form another  $3 \times 3$  matrix  $B$  whose  $ij$ th entry is the  $j$ ith entry of  $A$ :

$$B = \frac{1}{7} \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}.$$

Note that the rows of  $B$  correspond to the columns of  $A$ .

Compute  $BA$ , and explain the result.

### Solution

$$BA = \frac{1}{49} \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix} \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = I_3$$

This result is no coincidence: The  $ij$ th entry of  $BA$  is the dot product of the  $i$ th row of  $B$  and the  $j$ th column of  $A$ . By definition of  $B$ , this is just the dot product of the  $i$ th column of  $A$  and the  $j$ th column of  $A$ . Since  $A$  is orthogonal, this product is 1 if  $i = j$  and 0 otherwise. ■

Before we can generalize the findings of Example 4, we introduce some new terminology.

### Definition 3.5

#### The transpose of a matrix; symmetric and skew-symmetric matrices

Consider an  $m \times n$  matrix  $A$ .

The transpose  $A^T$  of  $A$  is the  $n \times m$  matrix whose  $ij$ th entry is the  $j$ ith entry of  $A$ : The roles of rows and columns are reversed.

We say that a square matrix  $A$  is *symmetric* if  $A^T = A$ , and  $A$  is called *skew-symmetric* if  $A^T = -A$ .

**EXAMPLE 5** If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 5 \end{bmatrix}$ , then  $A^T = \begin{bmatrix} 1 & 9 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$ . ■

**EXAMPLE 6** The *symmetric*  $2 \times 2$  matrices are those of the form  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ , for example,  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ . The symmetric  $2 \times 2$  matrices form a three-dimensional subspace of  $\mathbb{R}^{2 \times 2}$ , with basis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

The *skew-symmetric*  $2 \times 2$  matrices are those of the form  $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ , for example,  $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ . These form a one-dimensional space with basis  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . ■

Note that the transpose of a (column) vector  $\vec{v}$  is a row vector: If

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{then} \quad \vec{v}^T = [1 \ 2 \ 3].$$

The transpose gives us a convenient way to express the dot product of two (column) vectors as a matrix product.

## Orthogonality and Least Squares

**Theorem 3.6**

If  $\vec{v}$  and  $\vec{w}$  are two (column) vectors in  $\mathbb{R}^n$ , then

$$\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}. \quad \begin{array}{c} \uparrow \\ \text{Dot product} \end{array} \quad \begin{array}{c} \uparrow \\ \text{Matrix product} \end{array}$$

Here we are identifying the  $1 \times 1$  matrix  $\vec{v}^T \vec{w}$  with its sole entry, the scalar  $\vec{v} \cdot \vec{w}$ . Purists may prefer to write  $\vec{v}^T \vec{w} = [\vec{v} \cdot \vec{w}]$ .

For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2.$$

Now we can succinctly state the observation made in Example 4.

**Theorem 3.7**

Consider an  $n \times n$  matrix  $A$ . The matrix  $A$  is orthogonal if (and only if)  $A^T A = I_n$  or, equivalently, if  $A^{-1} = A^T$ .

**Proof** To justify this fact, write  $A$  in terms of its columns:

$$A = \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix}.$$

Then

$$A^T A = \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ \vdots & & - \\ - & \vec{v}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \dots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \dots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \dots & \vec{v}_n \cdot \vec{v}_n \end{bmatrix}.$$

By Theorem 3.3b, this product is  $I_n$  if (and only if)  $A$  is orthogonal. ■

Later in this text, we will frequently work with matrices of the form  $A^T A$ . It is helpful to think of  $A^T A$  as a table displaying the dot products  $\vec{v}_i \cdot \vec{v}_j$  among the columns of  $A$ , as shown above.

We summarize the various characterizations we have found of orthogonal matrices, and we will add one more characterization, (vi), below.

**SUMMARY 3.8**

**Orthogonal matrices**

Consider an  $n \times n$  matrix  $A$ . Then the following statements are equivalent:

- i.  $A$  is an orthogonal matrix.
- ii. The transformation  $L(\vec{x}) = A\vec{x}$  preserves length; that is,  $\|A\vec{x}\| = \|\vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^n$ .
- iii. The columns of  $A$  form an orthonormal basis of  $\mathbb{R}^n$ .
- iv.  $A^T A = I_n$ .
- v.  $A^{-1} = A^T$ .
- vi.  $A$  preserves the dot product, meaning that  $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$  for all  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ .

## Orthogonality and Least Squares

In Exercise 28, we invite the reader to prove that (vi) is equivalent to the characterizations (i) through (v).

Here are some algebraic properties of the transpose.

### Theorem 3.9

#### Properties of the transpose

- a.  $(A + B)^T = A^T + B^T$  for all  $m \times n$  matrices  $A$  and  $B$ .
- b.  $(kA)^T = kA^T$  for all  $m \times n$  matrices  $A$  and for all scalars  $k$ .
- c.  $(AB)^T = B^T A^T$  for all  $m \times p$  matrices  $A$  and for all  $p \times n$  matrices  $B$ .
- d.  $\text{rank}(A^T) = \text{rank}(A)$  for all matrices  $A$ .
- e.  $(A^T)^{-1} = (A^{-1})^T$  for all invertible  $n \times n$  matrices  $A$ .

#### Proof

We leave the proofs of parts (a) and (b) as exercises to the reader. Note that parts (a) and (b) tell us that the function  $L(A) = A^T$  from  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}^{n \times m}$  is a linear transformation (and, in fact, an isomorphism).

- c. Write  $A$  in terms of its rows and  $B$  in terms of its columns,  $A = \begin{bmatrix} -\vec{w}_1 & - \\ \vdots & \\ -\vec{w}_m & - \end{bmatrix}$  and  $B = \begin{bmatrix} | & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & | & | \end{bmatrix}$ , so that  $B^T = \begin{bmatrix} -\vec{v}_1^T & - \\ \vdots & \\ -\vec{v}_n^T & - \end{bmatrix}$  and  $A^T = \begin{bmatrix} | & | \\ \vec{w}_1^T & \cdots & \vec{w}_m^T \\ | & | & | \end{bmatrix}$ .

Now show that the  $i$ th entry of  $(AB)^T$  equals the  $ij$ th entry of  $B^T A^T$ :

$$\begin{aligned} (\text{$i$th entry of } (AB)^T) &= (\text{$j$th entry of } AB) = \vec{w}_j \cdot \vec{v}_i \\ (\text{$ij$th entry of } B^T A^T) &= \vec{v}_i^T \cdot \vec{w}_j^T = \vec{w}_j \cdot \vec{v}_i. \end{aligned}$$

- d. Consider the row space of  $A$  (i.e., the span of the rows of  $A$ ). It is not hard to show that the dimension of this space is  $\text{rank}(A)$ . Indeed, we have

$$\begin{aligned} \text{rank}(A^T) &= \text{dimension of the span of the columns of } A^T \\ &= \text{dimension of the span of the rows of } A \\ &= \text{rank}(A). \end{aligned}$$

- e. We know that

$$AA^{-1} = I_n.$$

Transposing both sides and using part (c), we find that

$$(AA^{-1})^T = (A^{-1})^T A^T = I_n.$$

$A^T$  is invertible and

$$(A^{-1})^T = (A^T)^{-1}. \quad \blacksquare$$

### The Matrix of an Orthogonal Projection

The transpose allows us to write a formula for the matrix of an orthogonal projection. Consider first the orthogonal projection

$$\text{proj}_L \vec{x} = (\vec{u}_1 \cdot \vec{x}) \vec{u}_1$$

## Orthogonality and Least Squares

onto a line  $L$  in  $\mathbb{R}^n$ , where  $\vec{u}_1$  is a unit vector in  $L$ . If we view the vector  $\vec{u}_1$  as an  $n \times 1$  matrix and the scalar  $\vec{u}_1 \cdot \vec{x}$  as a  $1 \times 1$  matrix, we can write

$$\begin{aligned}\text{proj}_L \vec{x} &= \vec{u}_1(\vec{u}_1 \cdot \vec{x}) \\ &= \vec{u}_1 \vec{u}_1^T \vec{x} \\ &= M\vec{x},\end{aligned}$$

where  $M = \vec{u}_1 \vec{u}_1^T$ . Note that  $\vec{u}_1$  is an  $n \times 1$  matrix and  $\vec{u}_1^T$  is  $1 \times n$ , so that  $M$  is  $n \times n$ , as expected.

More generally, consider the projection

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

onto a subspace  $V$  of  $\mathbb{R}^n$  with orthonormal basis  $\vec{u}_1, \dots, \vec{u}_m$ . We can write

$$\begin{aligned}\text{proj}_V \vec{x} &= \vec{u}_1 \vec{u}_1^T \vec{x} + \cdots + \vec{u}_m \vec{u}_m^T \vec{x} \\ &= (\vec{u}_1 \vec{u}_1^T + \cdots + \vec{u}_m \vec{u}_m^T) \vec{x} \\ &= \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} - & \vec{u}_1^T & - \\ \vdots & & \vdots \\ - & \vec{u}_m^T & - \end{bmatrix} \vec{x}.\end{aligned}$$

We have shown the following result:

### Theorem 3.10

#### The matrix of an orthogonal projection

Consider a subspace  $V$  of  $\mathbb{R}^n$  with orthonormal basis  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ . The matrix  $P$  of the orthogonal projection onto  $V$  is

$$P = Q Q^T, \quad \text{where } Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}.$$

Pay attention to the order of the factors ( $Q Q^T$  as opposed to  $Q^T Q$ ). Note that matrix  $P$  is symmetric, since  $P^T = (Q Q^T)^T = (Q^T)^T Q^T = Q Q^T = P$ .

**EXAMPLE 7** Find the matrix  $P$  of the orthogonal projection onto the subspace of  $\mathbb{R}^4$  spanned by

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

#### Solution

Note that the vectors  $\vec{u}_1$  and  $\vec{u}_2$  are orthonormal. Therefore, the matrix is

$$P = Q Q^T = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \quad \blacksquare$$

## EXERCISES 3

**GOAL** Use the various characterizations of orthogonal transformations and orthogonal matrices. Find the matrix of an orthogonal projection. Use the properties of the transpose.

Which of the matrices in Exercises 1 through 4 are orthogonal?

1.  $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

2.  $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

3.  $\frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

4.  $\frac{1}{7} \begin{bmatrix} 2 & 6 & -3 \\ 6 & -3 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

If the  $n \times n$  matrices  $A$  and  $B$  are orthogonal, which of the matrices in Exercises 5 through 11 must be orthogonal as well?

5.  $3A$

6.  $-B$

7.  $AB$

8.  $A + B$

9.  $B^{-1}$

10.  $B^{-1}AB$

11.  $A^T$

If the  $n \times n$  matrices  $A$  and  $B$  are symmetric and  $B$  is invertible, which of the matrices in Exercises 13 through 20 must be symmetric as well?

13.  $3A$

14.  $-B$

15.  $AB$

16.  $A + B$

17.  $B^{-1}$

18.  $A^{10}$

19.  $2I_n + 3A - 4A^2$

20.  $AB^2A$

If  $A$  and  $B$  are arbitrary  $n \times n$  matrices, which of the matrices in Exercises 21 through 26 must be symmetric?

21.  $A^T A$

22.  $BB^T$

23.  $A - A^T$

24.  $A^T BA$

25.  $A^T B^T BA$

26.  $B(A + A^T)B^T$

27. Consider an  $n \times m$  matrix  $A$ , a vector  $\vec{v}$  in  $\mathbb{R}^m$ , and a vector  $\vec{w}$  in  $\mathbb{R}^n$ . Show that

$$(A\vec{v}) \cdot \vec{w} = \vec{v} \cdot (A^T \vec{w}).$$

28. Consider an  $n \times n$  matrix  $A$ . Show that  $A$  is an orthogonal matrix if (and only if)  $A$  preserves the dot product, meaning that  $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$  for all  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ . Hint: In Summary 3.8, show that statement (iv) implies (vi), and (vi) implies (ii).

29. Show that an orthogonal transformation  $L$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  preserves angles: The angle between two nonzero vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  equals the angle between  $L(\vec{v})$  and  $L(\vec{w})$ . Conversely, is any linear transformation that preserves angles orthogonal?

30. Consider a linear transformation  $L$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  that preserves length. What can you say about the kernel of  $L$ ? What is the dimension of the image? What can you say about the relationship between  $n$  and  $m$ ? If  $A$  is the matrix of  $L$ , what can you say about the columns of  $A$ ? What is  $A^T A$ ? What about  $AA^T$ ? Illustrate your answers with an example where  $m = 2$  and  $n = 3$ .

31. Are the rows of an orthogonal matrix  $A$  necessarily orthonormal?

32. a. Consider an  $n \times m$  matrix  $A$  such that  $A^T A = I_m$ . Is it necessarily true that  $AA^T = I_n$ ? Explain.  
 b. Consider an  $n \times n$  matrix  $A$  such that  $A^T A = I_n$ . Is it necessarily true that  $AA^T = I_n$ ? Explain.

33. Find all orthogonal  $2 \times 2$  matrices.

34. Find all orthogonal  $3 \times 3$  matrices of the form

$$\begin{bmatrix} a & b & 0 \\ c & d & 1 \\ e & f & 0 \end{bmatrix}.$$

35. Find an orthogonal transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  such that

$$T \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

36. Find an orthogonal matrix of the form

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}.$$

37. Is there an orthogonal transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  such that

$$T \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}?$$

38. a. Give an example of a (nonzero) skew-symmetric  $3 \times 3$  matrix  $A$ , and compute  $A^2$ .  
 b. If an  $n \times n$  matrix  $A$  is skew-symmetric, is matrix  $A^2$  necessarily skew-symmetric as well? Or is  $A^2$  necessarily symmetric?

39. Consider a line  $L$  in  $\mathbb{R}^n$ , spanned by a unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Consider the matrix  $A$  of the orthogonal projection onto  $L$ . Describe the  $ij$ th entry of  $A$ , in terms of the components  $u_i$  of  $\vec{u}$ .

## Orthogonality and Least Squares

40. Consider the subspace  $W$  of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}.$$

Find the matrix of the orthogonal projection onto  $W$ .

41. Find the matrix  $A$  of the orthogonal projection onto the line in  $\mathbb{R}^n$  spanned by the vector

$$\left. \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\} \quad \text{all } n \text{ components are 1.}$$

42. Let  $A$  be the matrix of an orthogonal projection. Find  $A^2$  in two ways:

- a. Geometrically. (Consider what happens when you apply an orthogonal projection twice.)
- b. By computation, using the formula given in Theorem 3.10.

43. Consider a unit vector  $\vec{u}$  in  $\mathbb{R}^3$ . We define the matrices

$$A = 2\vec{u}\vec{u}^T - I_3 \quad \text{and} \quad B = I_3 - 2\vec{u}\vec{u}^T.$$

Describe the linear transformations defined by these matrices geometrically.

44. Consider an  $n \times m$  matrix  $A$ . Find

$$\dim(\text{im}(A)) + \dim(\ker(A^T)),$$

in terms of  $m$  and  $n$ .

45. For which  $n \times m$  matrices  $A$  does the equation

$$\dim(\ker(A)) = \dim(\ker(A^T))$$

hold? Explain.

46. Consider a  $QR$  factorization

$$M = QR.$$

Show that

$$R = Q^T M.$$

47. If  $A = QR$  is a  $QR$  factorization, what is the relationship between  $A^T A$  and  $R^T R$ ?

48. Consider an invertible  $n \times n$  matrix  $A$ . Can you write  $A$  as  $A = LQ$ , where  $L$  is a lower triangular matrix and  $Q$  is orthogonal? Hint: Consider the  $QR$  factorization of  $A^T$ .

49. Consider an invertible  $n \times n$  matrix  $A$ . Can you write  $A = RQ$ , where  $R$  is an upper triangular matrix and  $Q$  is orthogonal?

50. a. Find all  $n \times n$  matrices that are both orthogonal and upper triangular, with positive diagonal entries.

- b. Show that the  $QR$  factorization of an invertible  $n \times n$  matrix is unique. Hint: If  $A = Q_1 R_1 = Q_2 R_2$ , then the matrix  $Q_2^{-1} Q_1 = R_2 R_1^{-1}$  is both orthogonal and upper triangular, with positive diagonal entries.

51. a. Consider the matrix product  $Q_1 = Q_2 S$ , where both  $Q_1$  and  $Q_2$  are  $n \times m$  matrices with orthonormal columns. Show that  $S$  is an orthogonal matrix. Hint: Compute  $Q_1^T Q_1 = (Q_2 S)^T Q_2 S$ . Note that  $Q_1^T Q_1 = Q_2^T Q_2 = I_m$ .

- b. Show that the  $QR$  factorization of an  $n \times m$  matrix  $M$  is unique. Hint: If  $M = Q_1 R_1 = Q_2 R_2$ , then  $Q_1 = Q_2 R_2 R_1^{-1}$ . Now use part (a) and Exercise 50a.

52. Find a basis of the space  $V$  of all symmetric  $3 \times 3$  matrices, and thus determine the dimension of  $V$ .

53. Find a basis of the space  $V$  of all skew-symmetric  $3 \times 3$  matrices, and thus determine the dimension of  $V$ .

54. Find the dimension of the space of all skew-symmetric  $n \times n$  matrices.

55. Find the dimension of the space of all symmetric  $n \times n$  matrices.

56. Is the transformation  $L(A) = A^T$  from  $\mathbb{R}^{2 \times 3}$  to  $\mathbb{R}^{3 \times 2}$  linear? Is  $L$  an isomorphism?

57. Is the transformation  $L(A) = A^T$  from  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}^{n \times m}$  linear? Is  $L$  an isomorphism?

58. Find the image and kernel of the linear transformation  $L(A) = \frac{1}{2}(A + A^T)$  from  $\mathbb{R}^{n \times n}$  to  $\mathbb{R}^{n \times n}$ . Hint: Think about symmetric and skew-symmetric matrices.

59. Find the image and kernel of the linear transformation  $L(A) = \frac{1}{2}(A - A^T)$  from  $\mathbb{R}^{n \times n}$  to  $\mathbb{R}^{n \times n}$ . Hint: Think about symmetric and skew-symmetric matrices.

60. Find the matrix of the linear transformation  $L(A) = A^T$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$  with respect to the basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

61. Find the matrix of the linear transformation  $L(A) = A - A^T$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$  with respect to the basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

62. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & -5 \\ 2 & 2 & 0 \end{bmatrix}$$

with  $LDU$ -factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the  $LDU$ -factorization of  $A^T$ .

## Orthogonality and Least Squares

- 63.** Consider a symmetric invertible  $n \times n$  matrix  $A$  which admits an  $LDU$ -factorization  $A = LDU$ . Show that  $U = L^T$ . (This is sometimes called the  $LDL^T$ -factorization of a symmetric matrix  $A$ .)
- 64.** This exercise shows one way to define the *quaternions*, discovered in 1843 by the Irish mathematician Sir W. R. Hamilton (1805–1865). Consider the set  $H$  of all  $4 \times 4$  matrices  $M$  of the form

$$M = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix},$$

where  $p, q, r, s$  are arbitrary real numbers. We can write  $M$  more succinctly in partitioned form as

$$M = \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix},$$

where  $A$  and  $B$  are rotation-scaling matrices.

- a. Show that  $H$  is closed under addition: If  $M$  and  $N$  are in  $H$ , then so is  $M + N$ .
  - b. Show that  $H$  is closed under scalar multiplication: If  $M$  is in  $H$  and  $k$  is an arbitrary scalar, then  $kM$  is in  $H$ .
  - c. Parts (a) and (b) show that  $H$  is a subspace of the linear space  $\mathbb{R}^{4 \times 4}$ . Find a basis of  $H$ , and thus determine the dimension of  $H$ .
  - d. Show that  $H$  is closed under multiplication: If  $M$  and  $N$  are in  $H$ , then so is  $MN$ .
  - e. Show that if  $M$  is in  $H$ , then so is  $M^T$ .
  - f. For a matrix  $M$  in  $H$ , compute  $M^T M$ .
  - g. Which matrices  $M$  in  $H$  are invertible? If a matrix  $M$  in  $H$  is invertible, is  $M^{-1}$  necessarily in  $H$  as well?
  - h. If  $M$  and  $N$  are in  $H$ , does the equation  $MN = NM$  always hold?
- 65.** Find all orthogonal  $2 \times 2$  matrices  $A$  such that all the entries of  $10A$  are integers and such that both entries in the first column are positive.
- 66.** Find an orthogonal  $2 \times 2$  matrix  $A$  such that all the entries of  $100A$  are integers while all the entries of  $10A$  fail to be integers.
- 67.** Consider a subspace  $V$  of  $\mathbb{R}^n$  with a basis  $\vec{v}_1, \dots, \vec{v}_m$ ; suppose we wish to find a formula for the orthogonal projection onto  $V$ . Using the methods we have developed thus far, we can proceed in two steps: We use the Gram–Schmidt process to construct an orthonormal

basis  $\vec{u}_1, \dots, \vec{u}_m$  of  $V$ , and then we use Theorem 3.10: The matrix of the orthogonal projection is  $QQ^T$ , where

$$Q = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}.$$

In this exercise we will see how we can write the matrix of the projection directly in terms of the basis  $\vec{v}_1, \dots, \vec{v}_m$  and the matrix

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}.$$

(This issue will be discussed more thoroughly in Section 4; see Theorem 4.7.)

Since  $\text{proj}_V \vec{x}$  is in  $V$ , we can write

$$\text{proj}_V \vec{x} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$$

for some scalars  $c_1, \dots, c_m$  yet to be determined. Now  $\vec{x} - \text{proj}_V(\vec{x}) = \vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m$  is orthogonal to  $V$ , meaning that  $\vec{v}_i \cdot (\vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m) = 0$  for  $i = 1, \dots, m$ .

- a. Use the equation  $\vec{v}_i \cdot (\vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m) = 0$  to show that  $A^T A \vec{c} = A^T \vec{x}$ , where  $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ .
- b. Conclude that  $\vec{c} = (A^T A)^{-1} A^T \vec{x}$  and  $\text{proj}_V \vec{x} = A \vec{c} = A(A^T A)^{-1} A^T \vec{x}$ .

- 68.** The formula  $A(A^T A)^{-1} A^T$  for the matrix of an orthogonal projection is derived in Exercise 67. Now consider the  $QR$  factorization of  $A$ , and express the matrix  $A(A^T A)^{-1} A^T$  in terms of  $Q$ .

- 69.** In  $\mathbb{R}^4$ , consider the subspace  $W$  spanned by the vectors  $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ . Find the matrix  $P_W$  of the orthogonal projection onto  $W$ .

- 70.** In all parts of this problem, let  $V$  be the subspace of all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that  $x_3 = x_1 + x_2$  and  $x_4 = x_2 + x_3$ .
- a. Find the matrix  $P_V$  of the orthogonal projection onto the subspace  $V$  in  $\mathbb{R}^4$ .
  - b. What is the relationship between the subspaces  $W$  and  $V$  defined in Exercises 69 and 70? Consequently, what is the relationship between the matrices  $P_W$  and  $P_V$  in Exercises 69 and 70?

- 71.** An  $n \times n$  matrix  $A$  is said to be a *Hankel matrix* (named after the German mathematician Hermann Hankel, 1839–1873) if  $a_{ij} = a_{i+1,j-1}$  for all  $i = 1, \dots, n-1$  and all  $j = 2, \dots, n$ , meaning that  $A$  has constant positive sloping diagonals. For example, a  $4 \times 4$  Hankel matrix is of the form

## Orthogonality and Least Squares

$$A = \begin{bmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{bmatrix}.$$

Show that the  $n \times n$  Hankel matrices form a subspace of  $\mathbb{R}^{n \times n}$ . Find the dimension of this space.

72. Consider a vector  $\vec{v}$  in  $\mathbb{R}^n$  of the form

$$\vec{v} = \begin{bmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{bmatrix},$$

where  $a$  is any real number. Let  $P$  be the matrix of the orthogonal projection onto  $\text{span}(\vec{v})$ . Describe the entries of  $P$  in terms of  $a$ , and explain why  $P$  is a Hankel matrix. See Exercise 71. As an example, find  $P$  for

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

73. Let  $n$  be an even positive integer. In both parts of this problem, let  $V$  be the subspace of all vectors  $\vec{x}$  in  $\mathbb{R}^n$  such that  $x_{j+2} = x_j + x_{j+1}$ , for all  $j = 1, \dots, n-2$ . (In Exercise 70 we consider the special case  $n = 4$ .) Consider the basis  $\vec{v}, \vec{w}$  of  $V$  with

$$\vec{a} = \begin{bmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ b \\ b^2 \\ \vdots \\ b^{n-1} \end{bmatrix},$$

$$\text{where } a = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{5}}{2}.$$

- a. Show that  $\vec{a}$  is orthogonal to  $\vec{b}$ .
  - b. Explain why the matrix  $P$  of the orthogonal projection onto  $V$  is a Hankel matrix. See Exercises 71 and 72.
74. For any integer  $m$ , we define the *Fibonacci number*  $f_m$  recursively by  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_{j+2} = f_j + f_{j+1}$  for all integers  $j$ .<sup>7</sup>
- a. Find the Fibonacci numbers  $f_m$  for  $m = -5, \dots, 5$ .

- b. Based upon your answer in part (a), describe the relationship between  $f_{-m}$  and  $f_m$ . (For extra credit, give a formal proof by induction on  $m$ .)

Now let  $n$  be a positive integer, with  $n \geq 2$ . Let  $V$  be the two-dimensional subspace of all vectors  $\vec{x}$  in  $\mathbb{R}^n$  such that  $x_{j+2} = x_j + x_{j+1}$ , for all  $j = 1, \dots, n-2$ . See Exercise 73. Note that, by definition, any  $n$  consecutive Fibonacci numbers form a vector in  $V$ . Consider the basis  $\vec{v}, \vec{w}$  of  $V$  with

$$\vec{v} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix},$$

$$\vec{w} = \begin{bmatrix} f_{-n+1} \\ f_{-n+2} \\ \vdots \\ f_{-1} \\ f_0 \end{bmatrix} = \begin{bmatrix} f_{n-1} \\ -f_{n-2} \\ \vdots \\ 1 \\ 0 \end{bmatrix}.$$

We are told that  $\|\vec{v}\|^2 = \|\vec{w}\|^2 = f_{n-1}f_n$ . (For extra credit, give a proof by induction on  $n$ .)

- c. Find the basis  $\vec{v}, \vec{w}$  in the case  $n = 6$ . Verify the identity  $\|\vec{v}\|^2 = \|\vec{w}\|^2 = f_5f_6$ . Also, show that  $\vec{v}$  is orthogonal to  $\vec{w}$ .
- d. Show that  $\vec{v}$  is orthogonal to  $\vec{w}$  for any even positive integer  $n$ .
- e. For an even positive integer  $n$ , let  $P$  be the matrix of the orthogonal projection onto  $V$ . Show that the first column of  $P$  is  $\frac{1}{f_n}\vec{w}$ , while the last column is  $\frac{1}{f_n}\vec{v}$ . Recall from Exercise 73 that  $P$  is a Hankel matrix, and note that a Hankel matrix is determined by its first and last columns. Conclude that

$$P = \frac{1}{f_n} \begin{bmatrix} f_{-n+1} & f_{-n+2} & \cdots & f_{-1} & f_0 \\ f_{-n+2} & f_{-n+3} & \cdots & f_0 & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{-1} & f_0 & \cdots & f_{n-3} & f_{n-2} \\ f_0 & f_1 & \cdots & f_{n-2} & f_{n-1} \end{bmatrix},$$

meaning that the  $ij$ th entry of  $P$  is  $\frac{f_{i+j-n-1}}{f_n}$ .

- f. Find the matrix  $P$  in the case  $n = 6$ .

## 4 Least Squares and Data Fitting

In this section, we will present an important application of the ideas introduced in this chapter. First, we take another look at orthogonal complements and orthogonal projections.

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<sup>7</sup>Exercises 69 through 74 are based on research by Prof. Ben Mathes, Colby College.

## Orthogonality and Least Squares

### Another Characterization of Orthogonal Complements

Consider a subspace  $V = \text{im}(A)$  of  $\mathbb{R}^n$ , where  $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$ . Then

$$\begin{aligned} V^\perp &= \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\} \\ &= \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v}_i \cdot \vec{x} = 0, \text{ for } i = 1, \dots, m\} \\ &= \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v}_i^T \vec{x} = 0, \text{ for } i = 1, \dots, m\}. \end{aligned}$$

In other words,  $V^\perp = (\text{im}A)^\perp$  is the kernel of the matrix

$$A^T = \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ \vdots & & \\ - & \vec{v}_m^T & - \end{bmatrix}.$$

#### Theorem 4.1

For any matrix  $A$ ,

$$(\text{im}A)^\perp = \ker(A^T).$$

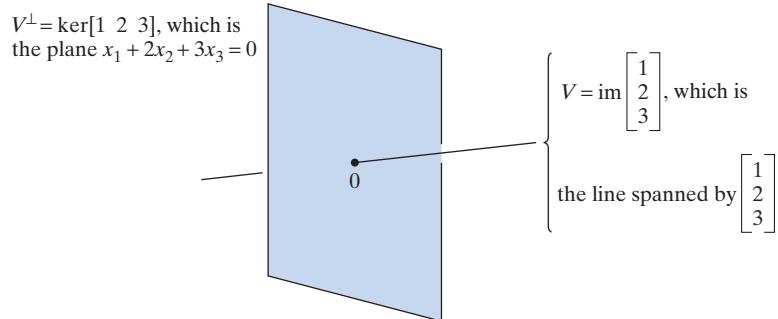
Here is a very simple example: Consider the line

$$V = \text{im} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then

$$V^\perp = \ker [1 \ 2 \ 3]$$

is the plane with equation  $x_1 + 2x_2 + 3x_3 = 0$ . See Figure 1.



**Figure 1**

The following somewhat technical result will be useful later:

#### Theorem 4.2

a. If  $A$  is an  $n \times m$  matrix, then

$$\ker(A) = \ker(A^T A).$$

b. If  $A$  is an  $n \times m$  matrix with  $\ker(A) = \{\vec{0}\}$ , then  $A^T A$  is invertible.

## Orthogonality and Least Squares

### *Proof*

- a. Clearly, the kernel of  $A$  is contained in the kernel of  $A^T A$ . Conversely, consider a vector  $\vec{x}$  in the kernel of  $A^T A$ , so that  $A^T A \vec{x} = \vec{0}$ . Then  $A \vec{x}$  is in the image of  $A$  and in the kernel of  $A^T$ . Since  $\ker(A^T)$  is the orthogonal complement of  $\text{im}(A)$  by Theorem 4.1, the vector  $A \vec{x}$  is  $\vec{0}$  by Theorem 1.8b; that is,  $\vec{x}$  is in the kernel of  $A$ .
- b. Note that  $A^T A$  is an  $m \times m$  matrix. By part (a),  $\ker(A^T A) = \{\vec{0}\}$ , and the square matrix  $A^T A$  is therefore invertible. ■

## An Alternative Characterization of Orthogonal Projections

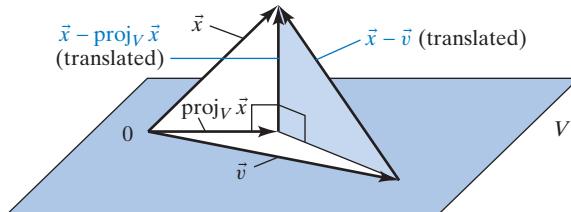
### Theorem 4.3

Consider a vector  $\vec{x}$  in  $\mathbb{R}^n$  and a subspace  $V$  of  $\mathbb{R}^n$ . Then the orthogonal projection  $\text{proj}_V \vec{x}$  is the vector in  $V$  closest to  $\vec{x}$ , in that

$$\|\vec{x} - \text{proj}_V \vec{x}\| < \|\vec{x} - \vec{v}\|,$$

for all  $\vec{v}$  in  $V$  different from  $\text{proj}_V \vec{x}$ .

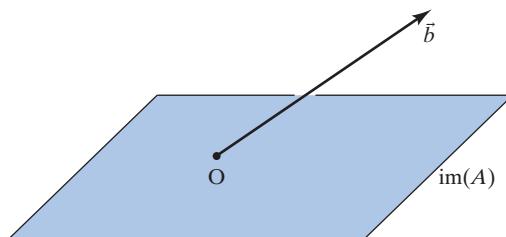
To justify this fact, apply the Pythagorean theorem to the shaded right triangle in Figure 2.



**Figure 2**

## Least-Squares Approximations

Consider an *inconsistent* linear system  $A \vec{x} = \vec{b}$ . The fact that this system is inconsistent means that the vector  $\vec{b}$  is not in the image of  $A$ . See Figure 3.



**Figure 3**

Although this system cannot be solved, we might be interested in finding a good approximate solution. We can try to find a vector  $\vec{x}^*$  such that  $A \vec{x}^*$  is “as close as possible” to  $\vec{b}$ . In other words, we try to minimize the *error*  $\|\vec{b} - A \vec{x}\|$ .

### Definition 4.4

#### Least-squares solution

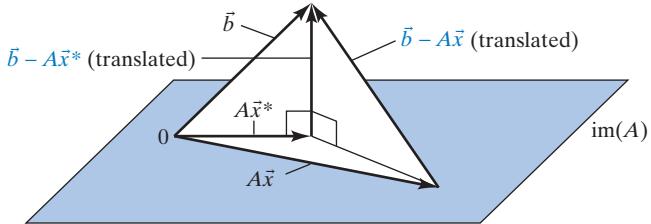
Consider a linear system

$$A \vec{x} = \vec{b},$$

where  $A$  is an  $n \times m$  matrix. A vector  $\vec{x}^*$  in  $\mathbb{R}^m$  is called a *least-squares solution* of this system if  $\|\vec{b} - A \vec{x}^*\| \leq \|\vec{b} - A \vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^m$ .

## Orthogonality and Least Squares

See Figure 4.



**Figure 4**

The term *least-squares solution* reflects the fact that we are minimizing the sum of the squares of the components of the vector  $\vec{b} - A\vec{x}$ .

If the system  $A\vec{x} = \vec{b}$  happens to be consistent, then the least-squares solutions are its exact solutions: The error  $\|\vec{b} - A\vec{x}\|$  is zero.

How can we find the least-squares solutions of a linear system  $A\vec{x} = \vec{b}$ ? Consider the following string of equivalent statements:

$$\begin{aligned}
 & \text{The vector } \vec{x}^* \text{ is a least-squares solution} \\
 & \text{of the system } A\vec{x} = \vec{b}. \\
 & \Updownarrow \text{Definition 4.4} \\
 & \|\vec{b} - A\vec{x}^*\| \leq \|\vec{b} - A\vec{x}\| \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^m. \\
 & \Updownarrow \text{Theorem 4.3} \\
 & A\vec{x}^* = \text{proj}_V \vec{b}, \quad \text{where } V = \text{im}(A). \\
 & \Updownarrow \text{Theorems 1.4 and 4.1} \\
 & \vec{b} - A\vec{x}^* \text{ is in } V^\perp = (\text{im}A)^\perp = \ker(A^T). \\
 & \Updownarrow \\
 & A^T(\vec{b} - A\vec{x}^*) = \vec{0}. \\
 & \Updownarrow \\
 & A^T A\vec{x}^* = A^T \vec{b}.
 \end{aligned}$$

Take another look at Figures 2 and 4.

### Theorem 4.5

#### The normal equation

The least-squares solutions of the system

$$A\vec{x} = \vec{b}$$

are the exact solutions of the (consistent) system

$$A^T A\vec{x} = A^T \vec{b}.$$

The system  $A^T A\vec{x} = A^T \vec{b}$  is called the *normal equation* of  $A\vec{x} = \vec{b}$ .

The case when  $\ker(A) = \{\vec{0}\}$  is of particular importance. Then, the matrix  $A^T A$  is invertible (by Theorem 4.2b), and we can give a closed formula for the least-squares solution.

## Orthogonality and Least Squares

**Theorem 4.6**

If  $\ker(A) = \{\vec{0}\}$ , then the linear system

$$A\vec{x} = \vec{b}$$

has the unique least-squares solution

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}.$$

From a computational point of view, it may be more efficient to solve the normal equation  $A^T A \vec{x} = A^T \vec{b}$  by Gauss–Jordan elimination, rather than by using Theorem 4.6.

**EXAMPLE 1** Use Theorem 4.6 to find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}.$$

What is the geometric relationship between  $A\vec{x}^*$  and  $\vec{b}$ ?

**Solution**

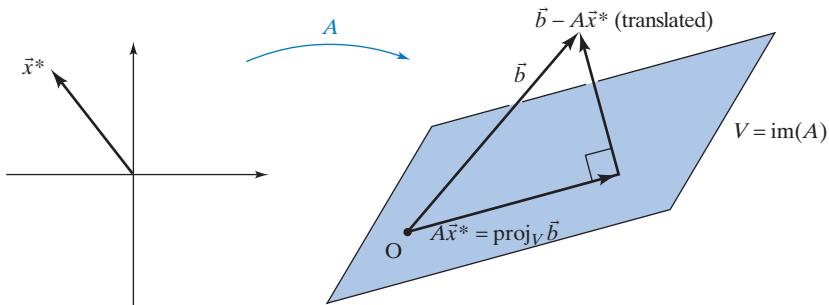
We compute

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad A\vec{x}^* = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$$

Recall that  $A\vec{x}^*$  is the orthogonal projection of  $\vec{b}$  onto the image of  $A$ . Check that

$$\vec{b} - A\vec{x}^* = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is indeed perpendicular to the two column vectors of  $A$ . See Figure 5. ■



**Figure 5**

If  $\vec{x}^*$  is a least-squares solution of the system  $A\vec{x} = \vec{b}$ , then  $A\vec{x}^*$  is the orthogonal projection of  $\vec{b}$  onto  $\text{im}(A)$ . We can use this fact to find a new formula for orthogonal projections. Compare this with Theorems 1.5 and 3.10. Consider a subspace  $V$  of  $\mathbb{R}^n$  and a vector  $\vec{b}$  in  $\mathbb{R}^n$ . Choose a basis  $\vec{v}_1, \dots, \vec{v}_m$  of  $V$ , and form the matrix  $A = [\vec{v}_1 \ \dots \ \vec{v}_m]$ . Note that  $\ker(A) = \{\vec{0}\}$ , since the columns of  $A$  are linearly independent. The least-squares solution of the system  $A\vec{x} = \vec{b}$  is  $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$ . Thus, the orthogonal projection of  $\vec{b}$  onto  $V$  is  $\text{proj}_V \vec{b} = A\vec{x}^* = A(A^T A)^{-1} A^T \vec{b}$ .

**Theorem 4.7**
**The matrix of an orthogonal projection**

Consider a subspace  $V$  of  $\mathbb{R}^n$  with basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ . Let

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}.$$

Then the matrix of the orthogonal projection onto  $V$  is

$$A(A^T A)^{-1} A^T.$$

We are not required to find an *orthonormal* basis of  $V$  here. If the vectors  $\vec{v}_1, \dots, \vec{v}_m$  happen to be orthonormal, then  $A^T A = I_m$  and the formula simplifies to  $AA^T$ . See Theorem 3.10.

**EXAMPLE 2** Find the matrix of the orthogonal projection onto the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

**Solution**

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix},$$

and compute

$$A(A^T A)^{-1} A^T = \frac{1}{10} \begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}.$$
■

**Data Fitting**

Scientists are often interested in fitting a function of a certain type to data they have gathered. The functions considered could be linear, polynomial, rational, trigonometric, or exponential. The equations we have to solve as we fit data are frequently linear.

**EXAMPLE 3** Find a cubic polynomial whose graph passes through the points  $(1, 3)$ ,  $(-1, 13)$ ,  $(2, 1)$ ,  $(-2, 33)$ .

**Solution**

We are looking for a function

$$f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

## Orthogonality and Least Squares

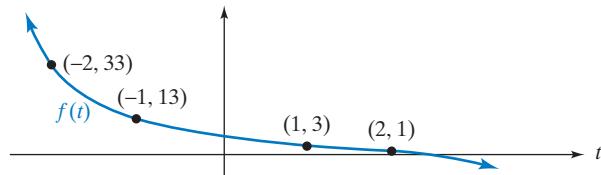
such that  $f(1) = 3$ ,  $f(-1) = 13$ ,  $f(2) = 1$ ,  $f(-2) = 33$ ; that is, we have to solve the linear system

$$\begin{vmatrix} c_0 + c_1 + c_2 + c_3 &= 3 \\ c_0 - c_1 + c_2 - c_3 &= 13 \\ c_0 + 2c_1 + 4c_2 + 8c_3 &= 1 \\ c_0 - 2c_1 + 4c_2 - 8c_3 &= 33 \end{vmatrix}.$$

This linear system has the unique solution

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \\ -1 \end{bmatrix}.$$

Thus, the cubic polynomial whose graph passes through the four given data points is  $f(t) = 5 - 4t + 3t^2 - t^3$ , as shown in Figure 6. ■



**Figure 6**

Frequently, a data-fitting problem leads to a linear system with more equations than variables. (This happens when the number of data points exceeds the number of parameters in the function we seek.) Such a system is usually inconsistent, and we will look for the least-squares solution(s).

**EXAMPLE 4** Fit a quadratic function to the four data points  $(a_1, b_1) = (-1, 8)$ ,  $(a_2, b_2) = (0, 8)$ ,  $(a_3, b_3) = (1, 4)$ , and  $(a_4, b_4) = (2, 16)$ .

### Solution

We are looking for a function  $f(t) = c_0 + c_1t + c_2t^2$  such that

$$\begin{vmatrix} f(a_1) = b_1 \\ f(a_2) = b_2 \\ f(a_3) = b_3 \\ f(a_4) = b_4 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} c_0 - c_1 + c_2 = 8 \\ c_0 = 8 \\ c_0 + c_1 + c_2 = 4 \\ c_0 + 2c_1 + 4c_2 = 16 \end{vmatrix} \quad \text{or} \quad A \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \vec{b},$$

where

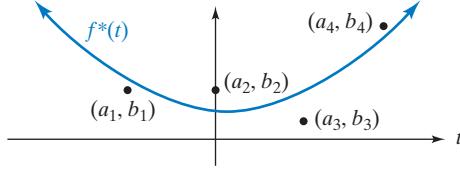
$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}.$$

We have four equations, corresponding to the four data points, but only three unknowns, the three coefficients of a quadratic polynomial. Check that this system is indeed inconsistent. The least-squares solution is

$$\vec{c}^* = \begin{bmatrix} c_0^* \\ c_1^* \\ c_2^* \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}.$$

## Orthogonality and Least Squares

The least-squares approximation is  $f^*(t) = 5 - t + 3t^2$ , as shown in Figure 7.



**Figure 7**

This quadratic function  $f^*(t)$  fits the data points best, in that the vector

$$A\vec{c}^* = \begin{bmatrix} f^*(a_1) \\ f^*(a_2) \\ f^*(a_3) \\ f^*(a_4) \end{bmatrix}$$

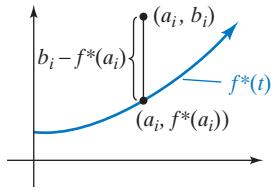
is as close as possible to

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

This means that

$$\|\vec{b} - A\vec{c}^*\|^2 = (b_1 - f^*(a_1))^2 + (b_2 - f^*(a_2))^2 + (b_3 - f^*(a_3))^2 + (b_4 - f^*(a_4))^2$$

is minimal: The sum of the squares of the vertical distances between graph and data points is minimal. See Figure 8. ■



**Figure 8**

### EXAMPLE 5

Find the linear function  $c_0 + c_1 t$  that best fits the data points  $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ , using least squares. Assume that  $a_1 \neq a_2$ .

#### Solution

We attempt to solve the system

$$\begin{vmatrix} c_0 + c_1 a_1 & = b_1 \\ c_0 + c_1 a_2 & = b_2 \\ \vdots & \vdots & \vdots \\ c_0 + c_1 a_n & = b_n \end{vmatrix},$$

or

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

or

$$A \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \vec{b}.$$

## Orthogonality and Least Squares

Note that  $\text{rank}(A) = 2$ , since  $a_1 \neq a_2$ .

The least-squares solution is

$$\begin{aligned} \begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} &= (A^T A)^{-1} A^T \vec{b} = \left( \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ \vdots & & \vdots \\ 1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ \vdots & & \vdots \\ b_1 & & b_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= \begin{bmatrix} n & \sum_i a_i \\ \sum_i a_i & \sum_i a_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i b_i \\ \sum_i a_i b_i \end{bmatrix} \quad (\text{where } \sum_i \text{ refers to the sum for } i = 1, \dots, n) \\ &= \frac{1}{n(\sum_i a_i^2) - (\sum_i a_i)^2} \begin{bmatrix} \sum_i a_i^2 & -\sum_i a_i \\ -\sum_i a_i & n \end{bmatrix} \begin{bmatrix} \sum_i b_i \\ \sum_i a_i b_i \end{bmatrix}. \end{aligned}$$

We have found that

$$\begin{aligned} c_0^* &= \frac{(\sum_i a_i^2)(\sum_i b_i) - (\sum_i a_i)(\sum_i a_i b_i)}{n(\sum_i a_i^2) - (\sum_i a_i)^2} \\ c_1^* &= \frac{n(\sum_i a_i b_i) - (\sum_i a_i)(\sum_i b_i)}{n(\sum_i a_i^2) - (\sum_i a_i)^2}. \end{aligned}$$

These formulas are well known to statisticians. There is no need to memorize them.



We conclude this section with an example for multivariate data fitting.

**EXAMPLE 6** In the accompanying table, we list the scores of five students on the three exams given in a class.

	<b><i>h:</i></b> <b>Hour Exam</b>	<b><i>m:</i></b> <b>Midterm Exam</b>	<b><i>f:</i></b> <b>Final Exam</b>
Gabriel	76	48	43
Kyle	92	92	90
Faruk	68	82	64
Yasmine	86	68	69
Alec	54	70	50

Find the function of the form  $f = c_0 + c_1 h + c_2 m$  that best fits these data, using least squares. What score  $f$  does your formula predict for Marilyn, another student, whose scores in the first two exams were  $h = 92$  and  $m = 72$ ?

### Solution

We attempt to solve the system

$$\begin{cases} c_0 + 76c_1 + 48c_2 = 43 \\ c_0 + 92c_1 + 92c_2 = 90 \\ c_0 + 68c_1 + 82c_2 = 64 \\ c_0 + 86c_1 + 68c_2 = 69 \\ c_0 + 54c_1 + 70c_2 = 50 \end{cases}.$$

## Orthogonality and Least Squares

The least-squares solution is

$$\begin{bmatrix} c_0^* \\ c_1^* \\ c_2^* \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} \approx \begin{bmatrix} -42.4 \\ 0.639 \\ 0.799 \end{bmatrix}.$$

The function that gives the best fit is approximately

$$f = -42.4 + 0.639h + 0.799m.$$

This formula predicts the score

$$f = -42.4 + 0.639 \cdot 92 + 0.799 \cdot 72 \approx 74$$

for Marilyn. ■

## EXERCISES 4

**GOAL** Use the formula  $(\text{im } A)^\perp = \ker(A^T)$ . Apply the characterization of  $\text{proj}_V \vec{x}$  as the vector in  $V$  “closest to  $\vec{x}$ .” Find the least-squares solutions of a linear system  $A\vec{x} = \vec{b}$  using the normal equation  $A^T A\vec{x} = A^T \vec{b}$ .

1. Consider the subspace  $\text{im}(A)$  of  $\mathbb{R}^2$ , where

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Find a basis of  $\ker(A^T)$ , and draw a sketch illustrating the formula

$$(\text{im } A)^\perp = \ker(A^T)$$

in this case.

2. Consider the subspace  $\text{im}(A)$  of  $\mathbb{R}^3$ , where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find a basis of  $\ker(A^T)$ , and draw a sketch illustrating the formula  $(\text{im } A)^\perp = \ker(A^T)$  in this case.

3. Consider a subspace  $V$  of  $\mathbb{R}^n$ . Let  $\vec{v}_1, \dots, \vec{v}_p$  be a basis of  $V$  and  $\vec{w}_1, \dots, \vec{w}_q$  a basis of  $V^\perp$ . Is  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  a basis of  $\mathbb{R}^n$ ? Explain.

4. Let  $A$  be an  $n \times m$  matrix. Is the formula

$$(\ker A)^\perp = \text{im}(A^T)$$

necessarily true? Explain.

5. Let  $V$  be the solution space of the linear system

$$\begin{vmatrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 5x_3 + 4x_4 = 0 \end{vmatrix}.$$

Find a basis of  $V^\perp$ .

6. If  $A$  is an  $n \times m$  matrix, is the formula

$$\text{im}(A) = \text{im}(AA^T)$$

necessarily true? Explain.

7. Consider a symmetric  $n \times n$  matrix  $A$ . What is the relationship between  $\text{im}(A)$  and  $\ker(A)$ ?

8. Consider a linear transformation  $L(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , with  $\ker(L) = \{\vec{0}\}$ . The pseudoinverse  $L^+$  of  $L$  is the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$L^+(\vec{y}) = (\text{the least-squares solution of } L(\vec{x}) = \vec{y}).$$

- a. Show that the transformation  $L^+$  is linear. Find the matrix  $A^+$  of  $L^+$ , in terms of the matrix  $A$  of  $L$ .
- b. If  $L$  is invertible, what is the relationship between  $L^+$  and  $L^{-1}$ ?
- c. What is  $L^+(L(\vec{x}))$ , for  $\vec{x}$  in  $\mathbb{R}^n$ ?
- d. What is  $L(L^+(\vec{y}))$ , for  $\vec{y}$  in  $\mathbb{R}^m$ ?
- e. Find  $L^+$  for the linear transformation

$$L(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}.$$

9. Consider the linear system  $A\vec{x} = \vec{b}$ , where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

- a. Draw a sketch showing the following subsets of  $\mathbb{R}^2$ :
  - the kernel of  $A$ , and  $(\ker A)^\perp$
  - the image of  $A^T$
  - the solution set  $S$  of the system  $A\vec{x} = \vec{b}$
- b. What relationship do you observe between  $\ker(A)$  and  $\text{im}(A^T)$ ? Explain.
- c. What relationship do you observe between  $\ker(A)$  and  $S$ ? Explain.
- d. Find the unique vector  $\vec{x}_0$  in the intersection of  $S$  and  $(\ker A)^\perp$ . Show  $\vec{x}_0$  on your sketch.
- e. What can you say about the length of  $\vec{x}_0$  compared with the length of all other vectors in  $S$ ?

## Orthogonality and Least Squares

10. Consider a consistent system  $A\vec{x} = \vec{b}$ .

- a. Show that this system has a solution  $\vec{x}_0$  in  $(\ker A)^\perp$ .

*Hint:* An arbitrary solution  $\vec{x}$  of the system can be written as  $\vec{x} = \vec{x}_h + \vec{x}_0$ , where  $\vec{x}_h$  is in  $\ker(A)$  and  $\vec{x}_0$  is in  $(\ker A)^\perp$ .

- b. Show that the system  $A\vec{x} = \vec{b}$  has only one solution in  $(\ker A)^\perp$ . *Hint:* If  $\vec{x}_0$  and  $\vec{x}_1$  are two solutions in  $(\ker A)^\perp$ , think about  $\vec{x}_1 - \vec{x}_0$ .

- c. If  $\vec{x}_0$  is the solution in  $(\ker A)^\perp$  and  $\vec{x}_1$  is another solution of the system  $A\vec{x} = \vec{b}$ , show that  $\|\vec{x}_0\| < \|\vec{x}_1\|$ . The vector  $\vec{x}_0$  is called the *minimal solution* of the linear system  $A\vec{x} = \vec{b}$ .

11. Consider a linear transformation  $L(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , where  $\text{rank}(A) = m$ . The *pseudoinverse*  $L^+$  of  $L$  is the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$L^+(\vec{y}) = (\text{the minimal solution of the system } L(\vec{x}) = \vec{y}).$$

See Exercise 10.

- a. Show that the transformation  $L^+$  is linear.  
 b. What is  $L(L^+(\vec{y}))$ , for  $\vec{y}$  in  $\mathbb{R}^m$ ?  
 c. What is  $L^+(L(\vec{x}))$ , for  $\vec{x}$  in  $\mathbb{R}^n$ ?  
 d. Determine the image and kernel of  $L^+$ .  
 e. Find  $L^+$  for the linear transformation

$$L(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}.$$

12. Using Exercise 10 as a guide, define the term *minimal least-squares solution* of a linear system. Explain why the minimal least-squares solution  $\vec{x}^*$  of a linear system  $A\vec{x} = \vec{b}$  is in  $(\ker A)^\perp$ .

13. Consider a linear transformation  $L(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . The *pseudoinverse*  $L^+$  of  $L$  is the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$L^+(\vec{y}) = (\text{the minimal least-squares solution of the system } L(\vec{x}) = \vec{y}).$$

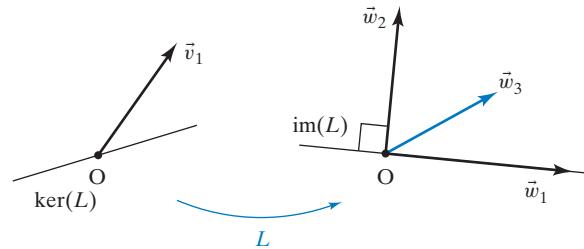
See Exercises 8, 11, and 12 for special cases.

- a. Show that the transformation  $L^+$  is linear.  
 b. What is  $L^+(L(\vec{x}))$ , for  $\vec{x}$  in  $\mathbb{R}^n$ ?  
 c. What is  $L(L^+(\vec{y}))$ , for  $\vec{y}$  in  $\mathbb{R}^m$ ?  
 d. Determine the image and kernel of  $L^+$  [in terms of  $\text{im}(A^T)$  and  $\ker(A^T)$ ].  
 e. Find  $L^+$  for the linear transformation

$$L(\vec{x}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}.$$

14. In the accompanying figure, we show the kernel and the image of a linear transformation  $L$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , together with some vectors  $\vec{v}_1, \vec{w}_1, \vec{w}_2, \vec{w}_3$ . We are told that  $L(\vec{v}_1) = \vec{w}_1$ . For  $i = 1, 2, 3$ , find the vectors  $L^+(\vec{w}_i)$ , where  $L^+$  is the pseudoinverse of  $L$  defined in

Exercise 13. Show your solutions in the figure, and explain how you found them.



15. Consider an  $m \times n$  matrix  $A$  with  $\ker(A) = \{\vec{0}\}$ . Show that there exists an  $n \times m$  matrix  $B$  such that  $BA = I_n$ . *Hint:*  $A^T A$  is invertible.

16. Use the formula  $(\text{im } A)^\perp = \ker(A^T)$  to prove the equation

$$\text{rank}(A) = \text{rank}(A^T).$$

17. Does the equation

$$\text{rank}(A) = \text{rank}(A^T A)$$

hold for all  $n \times m$  matrices  $A$ ? Explain.

18. Does the equation

$$\text{rank}(A^T A) = \text{rank}(AA^T)$$

hold for all  $n \times m$  matrices  $A$ ? Explain. *Hint:* Exercise 17 is useful.

19. Find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Use paper and pencil. Draw a sketch showing the vector  $\vec{b}$ , the image of  $A$ , the vector  $A\vec{x}^*$ , and the vector  $\vec{b} - A\vec{x}^*$ .

20. By using paper and pencil, find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } \vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Verify that the vector  $\vec{b} - A\vec{x}^*$  is perpendicular to the image of  $A$ .

21. Find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 6 & 9 \\ 3 & 8 \\ 2 & 10 \end{bmatrix} \quad \text{and } \vec{b} = \begin{bmatrix} 0 \\ 49 \\ 0 \end{bmatrix}.$$

Determine the error  $\|\vec{b} - A\vec{x}^*\|$ .

## Orthogonality and Least Squares

22. Find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

Determine the error  $\|\vec{b} - A\vec{x}^*\|$ .

23. Find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Explain.

24. Find the least-squares solution  $\vec{x}^*$  of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

Draw a sketch showing the vector  $\vec{b}$ , the image of  $A$ , the vector  $A\vec{x}^*$ , and the vector  $\vec{b} - A\vec{x}^*$ .

25. Find the least-squares solutions  $\vec{x}^*$  of the system  $A\vec{x} = \vec{b}$ , where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Use only paper and pencil. Draw a sketch.

26. Find the least-squares solutions  $\vec{x}^*$  of the system  $A\vec{x} = \vec{b}$ , where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

27. Consider an inconsistent linear system  $A\vec{x} = \vec{b}$ , where  $A$  is a  $3 \times 2$  matrix. We are told that the least-squares solution of this system is  $\vec{x}^* = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ . Consider an orthogonal  $3 \times 3$  matrix  $S$ . Find the least-squares solution(s) of the system  $SA\vec{x} = S\vec{b}$ .

28. Consider an orthonormal basis  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$  in  $\mathbb{R}^n$ . Find the least-squares solution(s) of the system

$$A\vec{x} = \vec{u}_n,$$

where

$$A = \left[ \begin{array}{c|c|c|c} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_{n-1} \\ | & | & & | \end{array} \right].$$

29. Find the least-squares solution of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 10^{-10} & 0 \\ 0 & 10^{-10} \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 10^{-10} \\ 10^{-10} \end{bmatrix}.$$

Describe and explain the difficulties you may encounter if you use technology. Then find the solution using paper and pencil.

30. Fit a linear function of the form  $f(t) = c_0 + c_1 t$  to the data points  $(0, 0), (0, 1), (1, 1)$ , using least squares. Use only paper and pencil. Sketch your solution, and explain why it makes sense.

31. Fit a linear function of the form  $f(t) = c_0 + c_1 t$  to the data points  $(0, 3), (1, 3), (1, 6)$ , using least squares. Sketch the solution.

32. Fit a quadratic polynomial to the data points  $(0, 27), (1, 0), (2, 0), (3, 0)$ , using least squares. Sketch the solution.

33. Find the trigonometric function of the form  $f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$  that best fits the data points  $(0, 0), (1, 1), (2, 2), (3, 3)$ , using least squares. Sketch the solution together with the function  $g(t) = t$ .

34. Find the function of the form

$$f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t) + c_3 \sin(2t) + c_4 \cos(2t)$$

that best fits the data points  $(0, 0), (0.5, 0.5), (1, 1), (1.5, 1.5), (2, 2), (2.5, 2.5), (3, 3)$ , using least squares. Sketch the solution, together with the function  $g(t) = t$ .

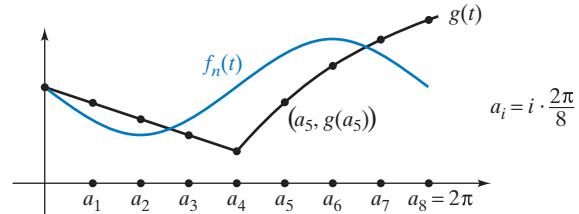
35. Suppose you wish to fit a function of the form

$$f(t) = c + p \sin(t) + q \cos(t)$$

to a given continuous function  $g(t)$  on the closed interval from 0 to  $2\pi$ . One approach is to choose  $n$  equally spaced points  $a_i$  between 0 and  $2\pi$  [ $a_i = i \cdot (2\pi/n)$ , for  $i = 1, \dots, n$ , say]. We can fit a function

$$f_n(t) = c_n + p_n \sin(t) + q_n \cos(t)$$

to the data points  $(a_i, g(a_i))$ , for  $i = 1, \dots, n$ . Now examine what happens to the coefficients  $c_n, p_n, q_n$  of  $f_n(t)$  as  $n$  approaches infinity.



## Orthogonality and Least Squares

To find  $f_n(t)$ , we make an attempt to solve the equations

$$f_n(a_i) = g(a_i), \quad \text{for } i = 1, \dots, n,$$

or

$$\begin{vmatrix} c_n + p_n \sin(a_1) + q_n \cos(a_1) &= g(a_1) \\ c_n + p_n \sin(a_2) + q_n \cos(a_2) &= g(a_2) \\ \vdots &\vdots \\ c_n + p_n \sin(a_n) + q_n \cos(a_n) &= g(a_n) \end{vmatrix},$$

or

$$A_n \begin{bmatrix} c_n \\ p_n \\ q_n \end{bmatrix} = \vec{b}_n,$$

where

$$A_n = \begin{bmatrix} 1 & \sin(a_1) & \cos(a_1) \\ 1 & \sin(a_2) & \cos(a_2) \\ \vdots & \vdots & \vdots \\ 1 & \sin(a_n) & \cos(a_n) \end{bmatrix}, \quad \vec{b}_n = \begin{bmatrix} g(a_1) \\ g(a_2) \\ \vdots \\ g(a_n) \end{bmatrix}.$$

- a. Find the entries of the matrix  $A_n^T A_n$  and the components of the vector  $A_n^T \vec{b}_n$ .

- b. Find

$$\lim_{n \rightarrow \infty} \left( \frac{2\pi}{n} A_n^T A_n \right) \quad \text{and} \quad \lim_{n \rightarrow \infty} \left( \frac{2\pi}{n} A_n^T \vec{b}_n \right).$$

*Hint:* Interpret the entries of the matrix  $(2\pi/n)A_n^T A_n$  and the components of the vector  $(2\pi/n)A_n^T \vec{b}$  as Riemann sums. Then the limits are the corresponding Riemann integrals. Evaluate as many integrals as you can. Note that

$$\lim_{n \rightarrow \infty} \left( \frac{2\pi}{n} A_n^T A_n \right)$$

is a diagonal matrix.

- c. Find

$$\begin{aligned} \lim_{n \rightarrow \infty} \begin{bmatrix} c_n \\ p_n \\ q_n \end{bmatrix} &= \lim_{n \rightarrow \infty} (A_n^T A_n)^{-1} A_n^T \vec{b}_n \\ &= \lim_{n \rightarrow \infty} \left[ \left( \frac{2\pi}{n} A_n^T A_n \right)^{-1} \left( \frac{2\pi}{n} A_n^T \vec{b}_n \right) \right] \\ &= \left[ \lim_{n \rightarrow \infty} \left( \frac{2\pi}{n} A_n^T A_n \right) \right]^{-1} \lim_{n \rightarrow \infty} \left( \frac{2\pi}{n} A_n^T \vec{b}_n \right). \end{aligned}$$

The resulting vector  $\begin{bmatrix} c \\ p \\ q \end{bmatrix}$  gives you the coefficient of the desired function

$$f(t) = \lim_{n \rightarrow \infty} f_n(t).$$

Write  $f(t)$ . The function  $f(t)$  is called the first *Fourier approximation* of  $g(t)$ . The Fourier approximation satisfies a “continuous” least-squares condition, an idea we will make more precise in the next section.

36. Let  $S(t)$  be the number of daylight hours on the  $t$ th day of the year 2012 in Rome, Italy. We are given the following data for  $S(t)$ :

Day	$t$	$S(t)$
February 1	32	10
March 17	77	12
April 30	121	14
May 31	152	15

We wish to fit a trigonometric function of the form

$$f(t) = a + b \sin \left( \frac{2\pi}{366} t \right) + c \cos \left( \frac{2\pi}{366} t \right)$$

to these data. Find the best approximation of this form, using least squares.

How many daylight hours does your model predict for the longest day of the year 2012? (The actual value is 15 hours, 13 minutes, 39 seconds.)

37. The accompanying table lists several commercial airplanes, the year they were introduced, and the number of displays in the cockpit.

Plane	Year $t$	Displays $d$
Douglas DC-3	'35	35
Lockheed Constellation	'46	46
Boeing 707	'59	77
Concorde	'69	133

- a. Fit a linear function of the form  $\log(d) = c_0 + c_1 t$  to the data points  $(t_i, \log(d_i))$ , using least squares.  
b. Use your answer in part (a) to fit an exponential function  $d = ka^t$  to the data points  $(t_i, d_i)$ .  
c. The Airbus A320 was introduced in 1988. Based on your answer in part b, how many displays do you expect in the cockpit of this plane? (There are 93 displays in the cockpit of an Airbus A320. Explain.)  
38. In the accompanying table, we list the height  $h$ , the gender  $g$ , and the weight  $w$  of some young adults.

## Orthogonality and Least Squares

Height $h$ (in Inches above 5 Feet)	Gender $g$ (1 = "Female," 0 = "Male")	Weight $w$ (in Pounds)
2	1	110
12	0	180
5	1	120
11	1	160
6	0	160

Fit a function of the form

$$w = c_0 + c_1 h + c_2 g$$

to these data, using least squares. Before you do the computations, think about the signs of  $c_1$  and  $c_2$ . What signs would you expect if these data were representative of the general population? Why? What is the sign of  $c_0$ ? What is the practical significance of  $c_0$ ?

39. In the accompanying table, we list the estimated number  $g$  of genes and the estimated number  $z$  of cell types for various organisms.

Organism	Number of Genes, $g$	Number of Cell Types, $z$
Humans	600,000	250
Annelid worms	200,000	60
Jellyfish	60,000	25
Sponges	10,000	12
Yeast	2,500	5

- a. Fit a function of the form  $\log(z) = c_0 + c_1 \log(g)$  to the data points  $(\log(g_i), \log(z_i))$ , using least squares.
- b. Use your answer in part (a) to fit a power function  $z = kg^n$  to the data points  $(g_i, z_i)$ .
- c. Using the theory of self-regulatory systems, scientists developed a model that predicts that  $z$  is a

square-root function of  $g$  (i.e.,  $a = k\sqrt{g}$ , for some constant  $k$ ). Is your answer in part (b) reasonably close to this form?

40. Consider the data in the following table.

Planet	Mean Distance from the Sun (in Astronomical Units)	Period of Revolution (in Earth Years)
Mercury	0.387	0.241
Earth	1.000	1.000
Jupiter	5.203	11.86
Uranus	19.19	84.04
Pluto	39.53	248.6

Use the methods discussed in Exercise 39 to fit a power function of the form  $D = ka^n$  to these data. Explain, in terms of Kepler's laws of planetary motion. Explain why the constant  $k$  is close to 1.

41. In the accompanying table, we list the public debt  $D$  of the United States (in billions of dollars), in various years  $t$  (as of September 30).

Year	1975	1985	1995	2005
$D$	533	1,823	4,974	7,933

- a. Letting  $t = 0$  in 1975, fit a linear function of the form  $\log(D) = c_0 + c_1 t$  to the data points  $(t_i, \log(D_i))$ , using least squares. Use the result to fit an exponential function to the data points  $(t_i, D_i)$ .
- b. What debt does your formula in part (a) predict for 2015?
- 42. If  $A$  is any matrix, show that the linear transformation  $L(\vec{x}) = A\vec{x}$  from  $\text{im}(A^T)$  to  $\text{im}(A)$  is an isomorphism. This provides yet another proof of the formula  $\text{rank}(A) = \text{rank}(A^T)$ .

## 5 Inner Product Spaces

In this chapter, a new operation for vectors in  $\mathbb{R}^n$  takes center stage: the *dot product*. In Sections 1 through 4, we studied concepts that are defined in terms of the dot product, the most important of them being the *length* of vectors and *orthogonality* of vectors. In this section, we will see that it can be useful to define a product analogous to the dot product in a linear space other than  $\mathbb{R}^n$ . These generalized dot products are called *inner products*. Once we have an inner product in a linear space, we can define length and orthogonality in that space just as in  $\mathbb{R}^n$ , and we can generalize all the key ideas and theorems of Sections 1 through 4.

**Definition 5.1**
**Inner products and inner product spaces**

An *inner product* in a linear space  $V$  is a rule that assigns a real scalar (denoted by  $\langle f, g \rangle$ ) to any pair  $f, g$  of elements of  $V$ , such that the following properties hold for all  $f, g, h$  in  $V$ , and all  $c$  in  $\mathbb{R}$ :

- a.  $\langle f, g \rangle = \langle g, f \rangle$  (symmetry).
- b.  $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$ .
- c.  $\langle cf, g \rangle = c\langle f, g \rangle$ .
- d.  $\langle f, f \rangle > 0$ , for all nonzero  $f$  in  $V$  (positive definiteness).

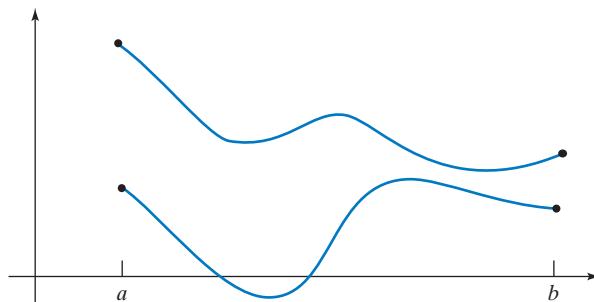
A linear space endowed with an inner product is called an *inner product space*.

Properties (b) and (c) express the fact that  $T(f) = \langle f, g \rangle$  is a linear transformation from  $V$  to  $\mathbb{R}$ , for a fixed  $g$  in  $V$ .

Roughly speaking, an inner product space behaves like  $\mathbb{R}^n$  as far as addition, scalar multiplication, and the dot product are concerned.

**EXAMPLE 1**

Consider the linear space  $C[a, b]$  consisting of all continuous functions whose domain is the closed interval  $[a, b]$ , where  $a < b$ . See Figure 1.


**Figure 1**

For functions  $f$  and  $g$  in  $C[a, b]$ , we define

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt.$$

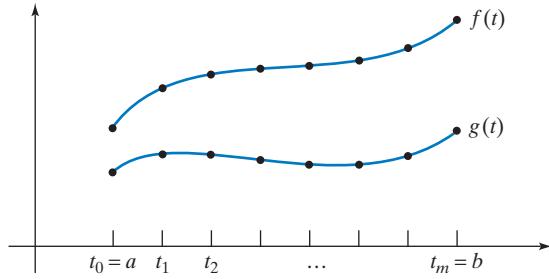
The verification of the first three axioms for an inner product is straightforward. For example,

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt = \int_a^b g(t)f(t) dt = \langle g, f \rangle.$$

The verification of the last axiom requires a bit of calculus. We leave it as Exercise 1.

Recall that the Riemann integral  $\int_a^b f(t)g(t) dt$  is the limit of the Riemann sum  $\sum_{i=1}^m f(t_k)g(t_k)\Delta t$ , where the  $t_k$  can be chosen as equally spaced points on the interval  $[a, b]$ . See Figure 2.

## Orthogonality and Least Squares



**Figure 2**

Then

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt \approx \sum_{k=1}^m f(t_k)g(t_k)\Delta t = \left( \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_m) \end{bmatrix}, \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_m) \end{bmatrix} \right) \Delta t$$

for large  $m$ .

This approximation shows that the inner product  $\langle f, g \rangle = \int_a^b f(t)g(t) dt$  for functions is a continuous version of the dot product: The more subdivisions you choose, the better the dot product on the right will approximate the inner product  $\langle f, g \rangle$ . █

**EXAMPLE 2** Let  $\ell_2$  be the space of all “square-summable” infinite sequences, that is, sequences

$$\vec{x} = (x_0, x_1, x_2, \dots, x_n, \dots)$$

such that  $\sum_{i=0}^{\infty} x_i^2 = x_0^2 + x_1^2 + \dots$  converges. In this space we can define the inner product

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=0}^{\infty} x_i y_i = x_0 y_0 + x_1 y_1 + \dots$$

(Show that this series converges.) The verification of the axioms is straightforward. █

**EXAMPLE 3** The *trace* of a square matrix is the sum of its diagonal entries. For example,

$$\text{trace} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4 = 5.$$

In  $\mathbb{R}^{n \times m}$ , the space of all  $n \times m$  matrices, we can define the inner product

$$\langle A, B \rangle = \text{trace}(A^T B).$$

We will verify the first and fourth axioms.

$$\langle A, B \rangle = \text{trace}(A^T B) = \text{trace}((A^T B)^T) = \text{trace}(B^T A) = \langle B, A \rangle$$

To check that  $\langle A, A \rangle > 0$  for nonzero  $A$ , write  $A$  in terms of its columns:

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}.$$

## Orthogonality and Least Squares

Now we have

$$\begin{aligned}
 \langle A, A \rangle &= \text{trace}(A^T A) = \text{trace} \left( \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ \vdots & & \\ - & \vec{v}_m^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix} \right) \\
 &= \text{trace} \left( \begin{bmatrix} \|\vec{v}_1\|^2 & \dots & \dots \\ \dots & \|\vec{v}_2\|^2 & \dots \\ \vdots & \vdots & \ddots \\ \dots & \dots & \|\vec{v}_m\|^2 \end{bmatrix} \right) \\
 &= \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \dots + \|\vec{v}_m\|^2.
 \end{aligned}$$

If  $A$  is nonzero, then at least one of the column vectors  $\vec{v}_i$  is nonzero, so that the sum  $\|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \dots + \|\vec{v}_m\|^2$  is positive, as desired. ■

We can introduce the basic concepts of geometry for an inner product space exactly as we did in  $\mathbb{R}^n$  for the dot product.

### Definition 5.2

#### Norm, orthogonality

The *norm* (or magnitude) of an element  $f$  of an inner product space is

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

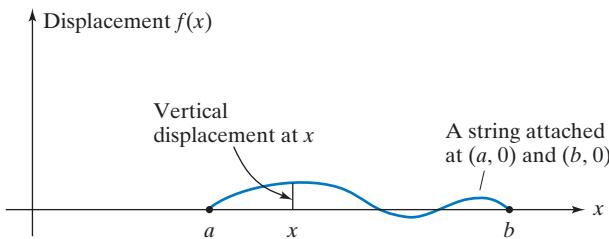
Two elements  $f$  and  $g$  of an inner product space are called *orthogonal* (or perpendicular) if

$$\langle f, g \rangle = 0.$$

We can define the *distance* between two elements of an inner product space as the norm of their difference:

$$\text{dist}(f, g) = \|f - g\|.$$

Consider a function  $f$  in the space  $C[a, b]$ , with the inner product defined in Example 1. In physics, the quantity  $\|f\|^2$  can often be interpreted as *energy*. For example, it describes the acoustic energy of a periodic sound wave  $f(t)$  and the elastic potential energy of a uniform string with vertical displacement  $f(x)$ . See Figure 3. The quantity  $\|f\|^2$  may also measure thermal or electric energy.



**Figure 3**

## Orthogonality and Least Squares

**EXAMPLE 4** In the inner product space  $C[0, 1]$  with  $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$ , find  $\|f\|$  for  $f(t) = t^2$ .

**Solution**

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 t^4 dt} = \sqrt{\frac{1}{5}}$$
■

**EXAMPLE 5** Show that  $f(t) = \sin(t)$  and  $g(t) = \cos(t)$  are orthogonal in the inner product space  $C[0, 2\pi]$  with  $\langle f, g \rangle = \int_0^{2\pi} f(t)g(t) dt$ .

**Solution**

$$\langle f, g \rangle = \int_0^{2\pi} \sin(t) \cos(t) dt = \left[ \frac{1}{2} \sin^2(t) \right]_0^{2\pi} = 0$$
■

**EXAMPLE 6** Find the distance between  $f(t) = t$  and  $g(t) = 1$  in  $C[0, 1]$ .

**Solution**

$$\text{dist}\langle f, g \rangle = \sqrt{\int_0^1 (t - 1)^2 dt} = \sqrt{\left[ \frac{1}{3}(t - 1)^3 \right]_0^1} = \frac{1}{\sqrt{3}}$$
■

The results and procedures discussed for the dot product generalize to arbitrary inner product spaces. For example, the Pythagorean theorem holds; the Gram–Schmidt process can be used to construct an orthonormal basis of a (finite dimensional) inner product space; and the Cauchy–Schwarz inequality tells us that  $|\langle f, g \rangle| \leq \|f\| \|g\|$ , for two elements  $f$  and  $g$  of an inner product space.

### Orthogonal Projections

In an inner product space  $V$ , consider a finite dimensional subspace  $W$  with orthonormal basis  $g_1, \dots, g_m$ . The orthogonal projection  $\text{proj}_W f$  of an element  $f$  of  $V$  onto  $W$  is defined as the unique element of  $W$  such that  $f - \text{proj}_W f$  is orthogonal to  $W$ . As in the case of the dot product in  $\mathbb{R}^n$ , the orthogonal projection is given by the following formula.

**Theorem 5.3**

**Orthogonal projection**

If  $g_1, \dots, g_m$  is an orthonormal basis of a subspace  $W$  of an inner product space  $V$ , then

$$\text{proj}_W f = \langle g_1, f \rangle g_1 + \dots + \langle g_m, f \rangle g_m,$$

for all  $f$  in  $V$ .

(Verify this by checking that  $\langle f - \text{proj}_W f, g_i \rangle = 0$  for  $i = 1, \dots, m$ .)

We may think of  $\text{proj}_W f$  as the element of  $W$  closest to  $f$ . In other words, if we choose another element  $h$  of  $W$ , then the distance between  $f$  and  $h$  will exceed the distance between  $f$  and  $\text{proj}_W f$ .

As an example, consider a subspace  $W$  of  $C[a, b]$ , with the inner product introduced in Example 1. Then  $\text{proj}_W f$  is the function  $g$  in  $W$  that is closest to  $f$ , in the sense that

## Orthogonality and Least Squares

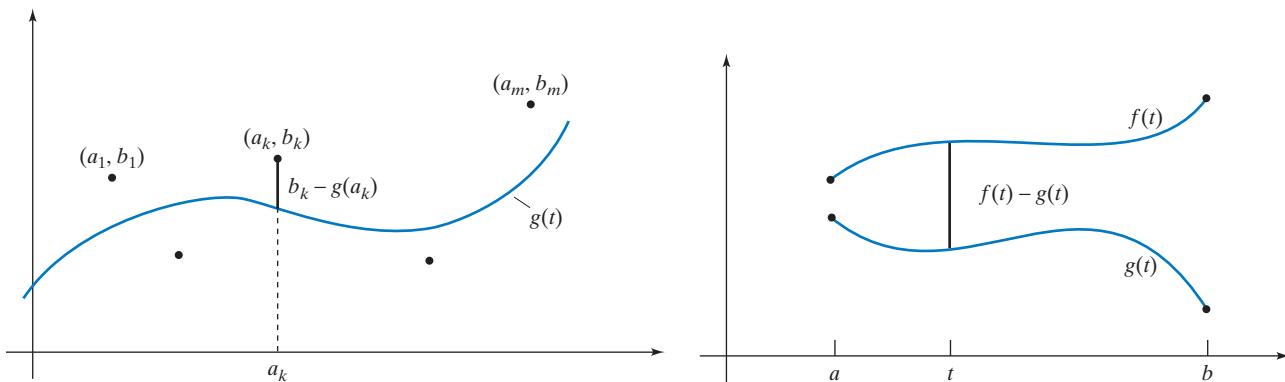
$$\text{dist}(f, g) = \|f - g\| = \sqrt{\int_a^b (f(t) - g(t))^2 dt}$$

is least.

The requirement that

$$\int_a^b (f(t) - g(t))^2 dt$$

be minimal is a *continuous least-squares condition*, as opposed to the discrete least-squares conditions we discussed in Section 4. We can use the discrete least-squares condition to fit a function  $g$  of a certain type to some data points  $(a_k, b_k)$ , while the continuous least-squares condition can be used to fit a function  $g$  of a certain type to a given function  $f$ . (Functions of a certain type are frequently polynomials of a certain degree or trigonometric functions of a certain form.) See Figures 4(a) and 4(b).



**Figure 4** (a) Discrete least-squares condition:  
 $\sum_{k=1}^m (b_k - g(a_k))^2$  is minimal.

(b) Continuous least-squares condition:  
 $\int_a^b (f(t) - g(t))^2 dt$  is minimal.

We can think of the continuous least-squares condition as a limiting case of a discrete least-squares condition by writing

$$\int_a^b (f(t) - g(t))^2 dt = \lim_{m \rightarrow \infty} \sum_{k=1}^m (f(t_k) - g(t_k))^2 \Delta t.$$

**EXAMPLE 7** Find the linear function of the form  $g(t) = a + bt$  that best fits the function  $f(t) = e^t$  over the interval from  $-1$  to  $1$ , in a continuous least-squares sense.

### Solution

We need to find  $\text{proj}_{P_1} f$ . We first find an orthonormal basis of  $P_1$  for the given inner product; then we will use Theorem 5.3. In general, we have to use the Gram–Schmidt process to find an orthonormal basis of an inner product space. Because the two functions  $1, t$  in the standard basis of  $P_1$  are orthogonal already, or

$$\langle 1, t \rangle = \int_{-1}^1 t dt = 0,$$

## Orthogonality and Least Squares

we merely need to divide each function by its norm:

$$\|1\| = \sqrt{\int_{-1}^1 1 dt} = \sqrt{2} \quad \text{and} \quad \|t\| = \sqrt{\int_{-1}^1 t^2 dt} = \sqrt{\frac{2}{3}}.$$

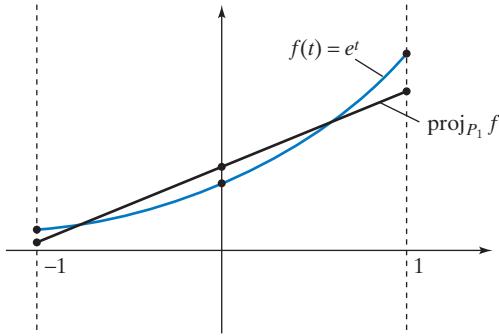
An orthonormal basis of  $P_1$  is

$$\frac{1}{\sqrt{2}}1 \quad \text{and} \quad \sqrt{\frac{3}{2}}t.$$

Now,

$$\begin{aligned} \text{proj}_{P_1} f &= \frac{1}{2}\langle 1, f \rangle 1 + \frac{3}{2}\langle t, f \rangle t \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}t. \quad (\text{We omit the straightforward computations.}) \end{aligned}$$

See Figure 5. ■



**Figure 5**

What follows is one of the major applications of this theory.

## Fourier Analysis<sup>7</sup>

In the space  $C[-\pi, \pi]$ , we introduce an inner product that is a slight modification of the definition given in Example 1:

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt.$$

The factor  $1/\pi$  is introduced to facilitate the computations. Convince yourself that this is indeed an inner product. Compare with Exercise 7.

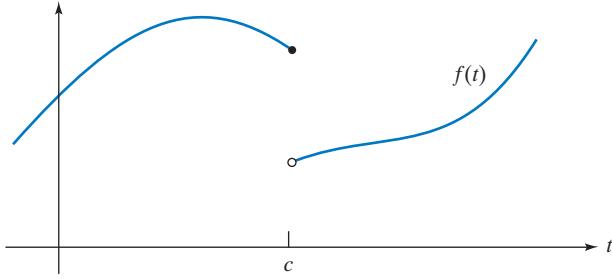
More generally, we can consider this inner product in the space of all *piecewise continuous functions* defined in the interval  $[-\pi, \pi]$ . These are functions  $f(t)$  that are continuous except for a finite number of *jump-discontinuities* [i.e., points  $c$  where the one-sided limits  $\lim_{t \rightarrow c^-} f(t)$  and  $\lim_{t \rightarrow c^+} f(t)$  both exist, but are not equal]. Also, it is required that  $f(c)$  equal one of the two one-sided limits. Let us consider the piecewise continuous functions with  $f(c) = \lim_{t \rightarrow c^-} f(t)$ . See Figure 6.

For a positive integer  $n$ , consider the subspace  $T_n$  of  $C[-\pi, \pi]$  that is defined as the span of the functions  $1, \sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(nt), \cos(nt)$ . The space  $T_n$  consists of all functions of the form

---

<sup>7</sup>Named after the French mathematician Jean-Baptiste-Joseph Fourier (1768–1830), who developed the subject in his *Théorie analytique de la chaleur* (1822), where he investigated the conduction of heat in very thin sheets of metal. Baron Fourier was also an Egyptologist and government administrator; he accompanied Napoléon on his expedition to Egypt in 1798.

## Orthogonality and Least Squares



**Figure 6**  $f(t)$  has a jump-discontinuity at  $t = c$ .

$$f(t) = a + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt),$$

called *trigonometric polynomials* of order  $\leq n$ .

From calculus, you may recall the *Euler identities*:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(pt) \cos(mt) dt &= 0, \quad \text{for integers } p, m \\ \int_{-\pi}^{\pi} \sin(pt) \sin(mt) dt &= 0, \quad \text{for distinct integers } p, m \\ \int_{-\pi}^{\pi} \cos(pt) \cos(mt) dt &= 0, \quad \text{for distinct integers } p, m. \end{aligned}$$

These equations tell us that the functions  $1, \sin(t), \cos(t), \dots, \sin(nt), \cos(nt)$  are orthogonal to one another (and therefore linearly independent).

Another of Euler's identities tells us that

$$\int_{-\pi}^{\pi} \sin^2(mt) dt = \int_{-\pi}^{\pi} \cos^2(mt) dt = \pi,$$

for positive integers  $m$ . This means that the functions  $\sin(t), \cos(t), \dots, \sin(nt), \cos(nt)$  all have norm 1 with respect to the given inner product. This is why we chose the inner product as we did, with the factor  $\frac{1}{\pi}$ .

The norm of the function  $f(t) = 1$  is

$$\|f\| = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} 1 dt} = \sqrt{2};$$

therefore,

$$g(t) = \frac{f(t)}{\|f(t)\|} = \frac{1}{\sqrt{2}}$$

is a function of norm 1.

### Theorem 5.4

#### An orthonormal basis of $T_n$

Let  $T_n$  be the space of all trigonometric polynomials of order  $\leq n$ , with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt.$$

Then the functions

$$\frac{1}{\sqrt{2}}, \sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(nt), \cos(nt)$$

form an orthonormal basis of  $T_n$ .

## Orthogonality and Least Squares

For a piecewise continuous function  $f$ , we can consider

$$f_n = \text{proj}_{T_n} f.$$

As discussed after Theorem 5.3,  $f_n$  is the trigonometric polynomial in  $T_n$  that best approximates  $f$ , in the sense that

$$\text{dist}(f, f_n) < \text{dist}(f, g),$$

for all other  $g$  in  $T_n$ .

We can use Theorems 5.3 and 5.4 to find a formula for  $f_n = \text{proj}_{T_n} f$ .

### Theorem 5.5

#### Fourier coefficients

If  $f$  is a piecewise continuous function defined on the interval  $[-\pi, \pi]$ , then its best approximation  $f_n$  in  $T_n$  is

$$\begin{aligned} f_n(t) &= \text{proj}_{T_n} f(t) \\ &= a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt), \end{aligned}$$

where

$$\begin{aligned} b_k &= \langle f(t), \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt \\ c_k &= \langle f(t), \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \\ a_0 &= \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} f(t) dt. \end{aligned}$$

The  $b_k$ , the  $c_k$ , and  $a_0$  are called the *Fourier coefficients* of the function  $f$ . The function

$$f_n(t) = a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

is called the *n*th-order *Fourier approximation* of  $f$ .

Note that the constant term, written somewhat awkwardly, is

$$a_0 \frac{1}{\sqrt{2}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt,$$

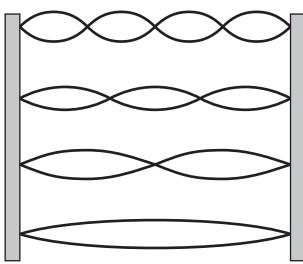
which is the average value of the function  $f$  between  $-\pi$  and  $\pi$ . It makes sense that the best way to approximate  $f(t)$  by a constant function is to take the average value of  $f(t)$ .

The function  $b_k \sin(kt) + c_k \cos(kt)$  is called the *k*th *harmonic* of  $f(t)$ . Using elementary trigonometry, we can write the harmonics alternatively as

$$b_k \sin(kt) + c_k \cos(kt) = A_k \sin(k(t - \delta_k)),$$

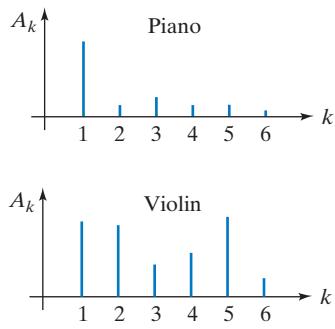
where  $A_k = \sqrt{b_k^2 + c_k^2}$  is the *amplitude* of the harmonic and  $\delta_k$  is the *phase shift*.

Consider the sound generated by a vibrating string, such as in a piano or on a violin. Let  $f(t)$  be the air pressure at your eardrum as a function of time  $t$ . [The function  $f(t)$  is measured as a deviation from the normal atmospheric pressure.] In this case, the harmonics have a simple physical interpretation: They correspond to the various sinusoidal modes at which the string can vibrate. See Figure 7.



**Figure 7**

### Orthogonality and Least Squares



**Figure 8**

The *fundamental frequency* (corresponding to the vibration shown at the bottom of Figure 7) gives us the *first harmonic* of  $f(t)$ , while the *overtones* (with frequencies that are integer multiples of the fundamental frequency) give us the other terms of the harmonic series. The quality of a tone is in part determined by the relative amplitudes of the harmonics. When you play concert A (440 Hertz) on a piano, the first harmonic is much more prominent than the higher ones, but the same tone played on a violin gives prominence to higher harmonics (especially the fifth). See Figure 8. Similar considerations apply to wind instruments; they have a vibrating column of air instead of a vibrating string.

The human ear cannot hear tones whose frequencies exceed 20,000 Hertz. We pick up only finitely many harmonics of a tone. What we hear is the projection of  $f(t)$  onto a certain  $T_n$ .

#### EXAMPLE 8

Find the Fourier coefficients for the function  $f(t) = t$  on the interval  $-\pi \leq t \leq \pi$ :

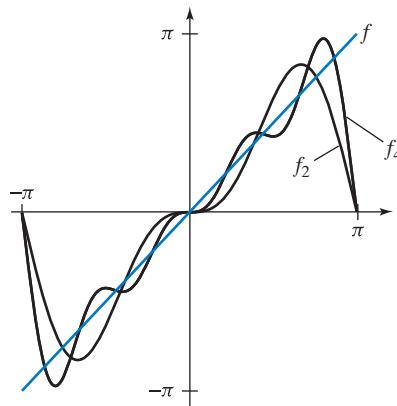
$$\begin{aligned}
 b_k &= \langle f, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kt)t dt \\
 &= \frac{1}{\pi} \left\{ - \left[ \frac{1}{k} \cos(kt)t \right] \Big|_{-\pi}^{\pi} + \frac{1}{k} \int_{-\pi}^{\pi} \cos(kt) dt \right\} \quad (\text{integration by parts}) \\
 &= \begin{cases} -\frac{2}{k} & \text{if } k \text{ is even} \\ \frac{2}{k} & \text{if } k \text{ is odd} \end{cases} \\
 &= (-1)^{k+1} \frac{2}{k}.
 \end{aligned}$$

All  $c_k$  and  $a_0$  are zero, since the integrands are odd functions.

The first few Fourier polynomials are

$$\begin{aligned}
 f_1 &= 2 \sin(t), \\
 f_2 &= 2 \sin(t) - \sin(2t), \\
 f_3 &= 2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t), \\
 f_4 &= 2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t) - \frac{1}{2} \sin(4t).
 \end{aligned}$$

See Figure 9. ■



**Figure 9**

## Orthogonality and Least Squares

How do the errors  $\|f - f_n\|$  and  $\|f - f_{n+1}\|$  of the  $n$ th and the  $(n+1)$ st Fourier approximation compare? We hope that  $f_{n+1}$  will be a better approximation than  $f_n$ , or at least no worse:

$$\|f - f_{n+1}\| \leq \|f - f_n\|.$$

This is indeed the case, by definition:  $f_n$  is a polynomial in  $T_{n+1}$ , since  $T_n$  is contained in  $T_{n+1}$ , and

$$\|f - f_{n+1}\| \leq \|f - g\|,$$

for all  $g$  in  $T_{n+1}$ , in particular for  $g = f_n$ . In other words, as  $n$  goes to infinity, the error  $\|f - f_n\|$  becomes smaller and smaller (or at least not larger). Using somewhat advanced calculus, we can show that this error approaches zero:

$$\lim_{n \rightarrow \infty} \|f - f_n\| = 0.$$

What does this tell us about  $\lim_{n \rightarrow \infty} \|f_n\|$ ? By the theorem of Pythagoras, we have

$$\|f - f_n\|^2 + \|f_n\|^2 = \|f\|^2.$$

As  $n$  goes to infinity, the first summand,  $\|f - f_n\|^2$ , approaches 0, so that

$$\lim_{n \rightarrow \infty} \|f_n\| = \|f\|.$$

We have an expansion of  $f_n$  in terms of an orthonormal basis

$$f_n = a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt),$$

where the  $b_k$ , the  $c_k$ , and  $a_0$  are the Fourier coefficients. We can express  $\|f_n\|$  in terms of these Fourier coefficients, using the Pythagorean theorem:

$$\|f_n\|^2 = a_0^2 + b_1^2 + c_1^2 + \cdots + b_n^2 + c_n^2.$$

Combining the last two “shaded” equations, we get the following identity:

$$a_0^2 + b_1^2 + c_1^2 + \cdots + b_n^2 + c_n^2 + \cdots = \|f\|^2.$$

The infinite series of the squares of the Fourier coefficients of a piecewise continuous function  $f$  converges to  $\|f\|^2$ .

For the function  $f(t)$  studied in Example 8, this means that

$$4 + \frac{4}{4} + \frac{4}{9} + \cdots + \frac{4}{n^2} + \cdots = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{2}{3} \pi^2,$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

an equation discovered by Euler.

Theorem 5.6 has a physical interpretation when  $\|f\|^2$  represents energy. For example, if  $f(x)$  is the displacement of a vibrating string, then  $b_k^2 + c_k^2$  represents the energy of the  $k$ th harmonic, and Theorem 5.6 tells us that the total energy  $\|f\|^2$  is the sum of the energies of the harmonics.

## Orthogonality and Least Squares

There is an interesting application of Fourier analysis in quantum mechanics. In the 1920s, quantum mechanics was presented in two distinct forms: Werner Heisenberg's matrix mechanics and Erwin Schrödinger's wave mechanics. Schrödinger (1887–1961) later showed that the two theories are mathematically equivalent: They use isomorphic inner product spaces. Heisenberg works with the space  $\ell_2$  introduced in Example 2, while Schrödinger works with a function space related to  $C[-\pi, \pi]$ . The isomorphism from Schrödinger's space to  $\ell_2$  is established by taking Fourier coefficients. See Exercise 13.

## EXERCISES 5

**GOAL** Use the idea of an inner product, and apply the basic results derived earlier for the dot product in  $\mathbb{R}^n$  to inner product spaces.

1. In  $C[a, b]$ , define the product

$$\langle f, g \rangle = \int_a^b f(t)g(t) dt.$$

Show that this product satisfies the property

$$\langle f, f \rangle > 0$$

for all nonzero  $f$ .

2. Does the equation

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

hold for all elements  $f, g, h$  of an inner product space? Explain.

3. Consider a matrix  $S$  in  $\mathbb{R}^{n \times n}$ . In  $\mathbb{R}^n$ , define the product

$$\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y}.$$

- a. For matrices  $S$  is this an inner product?
- b. For matrices  $S$  is  $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$  (the dot product)?

4. In  $\mathbb{R}^{n \times m}$ , consider the inner product

$$\langle A, B \rangle = \text{trace}(A^T B)$$

defined in Example 3.

- a. Find a formula for this inner product in  $\mathbb{R}^{n \times 1} = \mathbb{R}^n$ .
  - b. Find a formula for this inner product in  $\mathbb{R}^{1 \times m}$  (i.e., the space of row vectors with  $m$  components).
5. Is  $\langle\langle A, B \rangle\rangle = \text{trace}(AB^T)$  an inner product in  $\mathbb{R}^{n \times m}$ ? (The notation  $\langle\langle A, B \rangle\rangle$  is chosen to distinguish this product from the one considered in Example 3 and Exercise 4.)

6. a. Consider an  $n \times m$  matrix  $P$  and an  $m \times n$  matrix  $Q$ . Show that

$$\text{trace}(PQ) = \text{trace}(QP).$$

- b. Compare the following two inner products in  $\mathbb{R}^{n \times m}$ :

$$\langle A, B \rangle = \text{trace}(A^T B),$$

and

$$\langle\langle A, B \rangle\rangle = \text{trace}(AB^T).$$

See Example 3 and Exercises 4 and 5.

7. Consider an inner product  $\langle v, w \rangle$  in a space  $V$ , and a scalar  $k$ . For which choices of  $k$  is

$$\langle\langle v, w \rangle\rangle = k\langle v, w \rangle$$

an inner product?

8. Consider an inner product  $\langle v, w \rangle$  in a space  $V$ . Let  $w$  be a fixed element of  $V$ . Is the transformation  $T(v) = \langle v, w \rangle$  from  $V$  to  $\mathbb{R}$  linear? What is its image? Give a geometric interpretation of its kernel.

9. Recall that a function  $f(t)$  from  $\mathbb{R}$  to  $\mathbb{R}$  is called

even if  $f(-t) = f(t)$ , for all  $t$ ,

and

odd if  $f(-t) = -f(t)$ , for all  $t$ .

Show that if  $f(x)$  is an odd continuous function and  $g(x)$  is an even continuous function, then functions  $f(x)$  and  $g(x)$  are orthogonal in the space  $C[-1, 1]$  with the inner product defined in Example 1.

10. Consider the space  $P_2$  with inner product

$$\langle f, g \rangle = \frac{1}{2} \int_{-1}^1 f(t)g(t) dt.$$

Find an orthonormal basis of the space of all functions in  $P_2$  that are orthogonal to  $f(t) = t$ .

11. The angle between two nonzero elements  $v$  and  $w$  of an inner product space is defined as

$$\angle(v, w) = \arccos \frac{\langle v, w \rangle}{\|v\|\|w\|}.$$

In the space  $C[-\pi, \pi]$  with inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt,$$

find the angle between  $f(t) = \cos(t)$  and  $g(t) = \cos(t + \delta)$ , where  $0 \leq \delta \leq \pi$ . Hint: Use the formula  $\cos(t + \delta) = \cos(t)\cos(\delta) - \sin(t)\sin(\delta)$ .

## Orthogonality and Least Squares

12. Find all Fourier coefficients of the absolute value function

$$f(t) = |t|.$$

13. For a function  $f$  in  $C[-\pi, \pi]$ , consider the sequence of all its Fourier coefficients,

$$(a_0, b_1, c_1, b_2, c_2, \dots, b_n, c_n, \dots).$$

Is this infinite sequence in  $\ell_2$ ? If so, what is the relationship between

$$\|f\| \quad (\text{the norm in } C[-\pi, \pi])$$

and

$$\|(a_0, b_1, c_1, b_2, c_2, \dots)\| \quad (\text{the norm in } \ell_2)?$$

The inner product space  $\ell_2$  was introduced in Example 2.

14. Which of the following is an inner product in  $P_2$ ? Explain.

- a.  $\langle f, g \rangle = f(1)g(1) + f(2)g(2)$
- b.  $\langle\langle f, g \rangle\rangle = f(1)g(1) + f(2)g(2) + f(3)g(3)$

15. For which values of the constants  $b, c$ , and  $d$  is the following an inner product in  $\mathbb{R}^2$ ?

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$$

*Hint:* Be prepared to complete a square.

16. a. Find an orthonormal basis of the space  $P_1$  with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

- b. Find the linear polynomial  $g(t) = a + bt$  that best approximates the function  $f(t) = t^2$  on the interval  $[0, 1]$  in the (continuous) least-squares sense. Draw a sketch.

17. Consider a linear space  $V$ . For which linear transformations  $T$  from  $V$  to  $\mathbb{R}^n$  is

$$\langle v, w \rangle = T(v) \cdot T(w)$$

↑  
Dot product

an inner product in  $V$ ?

18. Consider an orthonormal basis  $\mathfrak{B}$  of the inner product space  $V$ . For an element  $f$  of  $V$ , what is the relationship between  $\|f\|$  and  $\|[f]_{\mathfrak{B}}\|$  (the norm in  $\mathbb{R}^n$  defined by the dot product)?

19. For which  $2 \times 2$  matrices  $A$  is

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A \vec{w}$$

an inner product in  $\mathbb{R}^2$ ? *Hint:* Be prepared to complete a square.

20. Consider the inner product

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \vec{w}$$

in  $\mathbb{R}^2$ . See Exercise 19.

- a. Find all vectors in  $\mathbb{R}^2$  that are perpendicular to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  with respect to this inner product.
- b. Find an orthonormal basis of  $\mathbb{R}^2$  with respect to this inner product.

21. If  $\|\vec{v}\|$  denotes the standard norm in  $\mathbb{R}^n$ , does the formula

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v} + \vec{w}\|^2 - \|\vec{v}\|^2 - \|\vec{w}\|^2$$

define an inner product in  $\mathbb{R}^n$ ?

22. If  $f(t)$  is a continuous function, what is the relationship between

$$\int_0^1 (f(t))^2 dt \quad \text{and} \quad \left( \int_0^1 f(t) dt \right)^2 ?$$

*Hint:* Use the Cauchy–Schwarz inequality.

23. In the space  $P_1$  of the polynomials of degree  $\leq 1$ , we define the inner product

$$\langle f, g \rangle = \frac{1}{2}(f(0)g(0) + f(1)g(1)).$$

Find an orthonormal basis for this inner product space.

24. Consider the linear space  $P$  of all polynomials, with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

For three polynomials  $f, g$ , and  $h$  we are given the following inner products:

$\langle \cdot \rangle$	$f$	$g$	$h$
$f$	4	0	8
$g$	0	1	3
$h$	8	3	50

For example,  $\langle f, f \rangle = 4$  and  $\langle g, h \rangle = \langle h, g \rangle = 3$ .

- a. Find  $\langle f, g + h \rangle$ .
- b. Find  $\|g + h\|$ .
- c. Find  $\text{proj}_E h$ , where  $E = \text{span}(f, g)$ . Express your solution as linear combinations of  $f$  and  $g$ .
- d. Find an orthonormal basis of  $\text{span}(f, g, h)$ . Express the functions in your basis as linear combinations of  $f, g$ , and  $h$ .

## Orthogonality and Least Squares

25. Find the norm  $\|\vec{x}\|$  of

$$\vec{x} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \quad \text{in } \ell_2.$$

( $\ell_2$  is defined in Example 2.)

26. Find the Fourier coefficients of the piecewise continuous function

$$f(t) = \begin{cases} -1 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Sketch the graphs of the first few Fourier polynomials.

27. Find the Fourier coefficients of the piecewise continuous function

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases}$$

28. Apply Theorem 5.6 to your answer in Exercise 26.

29. Apply Theorem 5.6 to your answer in Exercise 27.

30. Consider an ellipse  $E$  in  $\mathbb{R}^2$  centered at the origin. Show that there is an inner product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}^2$  such that  $E$  consists of all vectors  $\vec{x}$  with  $\|\vec{x}\| = 1$ , where the norm is taken with respect to the inner product  $\langle \cdot, \cdot \rangle$ .

31. *Gaussian integration.* In an introductory calculus course, you may have seen approximation formulas for integrals of the form

$$\int_a^b f(t) dt \approx \sum_{i=1}^n w_i f(a_i),$$

where the  $a_i$  are equally spaced points on the interval  $(a, b)$ , and the  $w_i$  are certain “weights” (Riemann sums, trapezoidal sums, and Simpson’s rule). Gauss has shown that, with the same computational effort, we can get better approximations if we drop the requirement that the  $a_i$  be equally spaced. Next, we outline his approach.

Consider the space  $P_n$  with the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(t)g(t) dt.$$

Let  $f_0, f_1, \dots, f_n$  be an orthonormal basis of this space, with  $\text{degree}(f_k) = k$ . (To construct such a basis, apply the Gram–Schmidt process to the standard basis  $1, t, \dots, t^n$ .) It can be shown that  $f_n$  has  $n$  distinct roots  $a_1, a_2, \dots, a_n$  on the interval  $(-1, 1)$ . We can find “weights”  $w_1, w_2, \dots, w_n$  such that

$$\int_{-1}^1 f(t) dt = \sum_{i=1}^n w_i f(a_i),$$

for all polynomials of degree less than  $n$ . In fact, much more is true: This formula holds for all polynomials  $f(t)$  of degree less than  $2n$ .

You are not asked to prove the foregoing assertions for arbitrary  $n$ , but work out the case  $n = 2$ : Find  $a_1, a_2$  and  $w_1, w_2$ , and show that the formula

$$\int_{-1}^1 f(t) dt = w_1 f(a_1) + w_2 f(a_2)$$

holds for all cubic polynomials.

32. In the space  $C[-1, 1]$ , we introduce the inner product

$$\langle f, g \rangle = \frac{1}{2} \int_{-1}^1 f(t)g(t) dt.$$

- a. Find  $\langle t^n, t^m \rangle$ , where  $n$  and  $m$  are positive integers.
- b. Find the norm of  $f(t) = t^n$ , where  $n$  is a positive integer.
- c. Applying the Gram–Schmidt process to the standard basis  $1, t, t^2, t^3$  of  $P_3$ , construct an orthonormal basis  $g_0(t), \dots, g_3(t)$  of  $P_3$  for the given inner product.
- d. Find the polynomials  $\frac{g_0(t)}{g_0(1)}, \dots, \frac{g_3(t)}{g_3(1)}$ . (Those are the first few *Legendre polynomials*, named after the great French mathematician Adrien-Marie Legendre, 1752–1833. These polynomials have a wide range of applications in math, physics, and engineering. Note that the Legendre polynomials are normalized so that their value at 1 is 1.)

- e. Find the polynomial  $g(t)$  in  $P_3$  that best approximates the function  $f(t) = \frac{1}{1+t^2}$  on the interval  $[-1, 1]$ , for the inner product introduced in this exercise. Draw a sketch.

33. a. Let  $w(t)$  be a positive-valued function in  $C[a, b]$ , where  $b > a$ . Verify that the rule  $\langle f, g \rangle = \int_a^b w(t)f(t)g(t) dt$  defines an inner product on  $C[a, b]$ .

- b. If we chose the weight function  $w(t)$  so that  $\int_a^b w(t) dt = 1$ , what is the norm of the constant function  $f(t) = 1$  in this inner product space?

34. In the space  $C[-1, 1]$ , we define the inner product  $\langle f, g \rangle = \int_{-1}^1 \frac{2}{\pi} \sqrt{1-t^2} f(t)g(t) dt = \frac{2}{\pi} \int_{-1}^1 \sqrt{1-t^2} f(t)g(t) dt$ . See Exercise 33; here we let  $w(t) = \frac{2}{\pi} \sqrt{1-t^2}$ . [This function  $w(t)$  is called a *Wigner semicircle distribution*, after the Hungarian physicist and mathematician E. P. Wigner (1902–1995), who won the 1963 Nobel Prize in Physics.] Since this is not a course in calculus, here are some inner products that will turn out to be useful:  $\langle 1, t^2 \rangle = 1/4$ ,  $\langle t, t^3 \rangle = 1/8$ , and  $\langle t^2, t^3 \rangle = 5/64$ .

- a. Find  $\int_{-1}^1 w(t) dt$ . Sketch a rough graph of the weight function  $w(t)$ .
- b. Find the norm of the constant function  $f(t) = 1$ .
- c. Find  $\langle t^2, t^3 \rangle$ ; explain. More generally, find  $\langle t^n, t^m \rangle$  for positive integers  $n$  and  $m$  whose sum is odd.
- d. Find  $\langle t, t \rangle$  and  $\langle t^2, t^2 \rangle$ . Also, find the norms of the functions  $t$  and  $t^2$ .

- e. Applying the Gram–Schmidt process to the standard basis  $1, t, t^2, t^3$  of  $P_3$ , construct an orthonormal basis  $g_0(t), \dots, g_3(t)$  of  $P_3$  for the given inner product. [The polynomials  $g_0(t), \dots, g_3(t)$  are the first few *Chebyshev polynomials of the second kind*, named after the Russian mathematician Pafnuty Chebyshev (1821–1894). They have a wide range of applications in math, physics, and engineering.]
- f. Find the polynomial  $g(t)$  in  $P_3$  that best approximates the function  $f(t) = t^4$  on the interval  $[-1, 1]$ , for the inner product introduced in this exercise.
35. In this exercise, we compare the inner products and norms introduced in Problems 32 and 34. Let's denote the two norms by  $\|f\|_{32}$  and  $\|f\|_{34}$ , respectively.
- a. Compute  $\|\cdot\|_{32}$  and  $\|\cdot\|_{34}$ . Which is larger? Explain the answer conceptually. Graph the weight functions  $w_{32}(t) = \frac{1}{2}$  and  $w_{34}(t) = \frac{2}{\pi} \sqrt{1 - t^2}$  on the same axes. Then graph the functions  $w_{32}(t)t^2$  and  $w_{34}(t)t^2$  on the same axes.
- b. Give an example of a continuous function  $f(t)$  such that  $\|f\|_{34} > \|f\|_{32}$ .

## Exercises

### TRUE OR FALSE?

1. If  $T$  is a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that  $T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n)$  are all unit vectors, then  $T$  must be an orthogonal transformation.
2. If  $A$  is an invertible matrix, then the equation  $(A^T)^{-1} = (A^{-1})^T$  must hold.
3. If matrix  $A$  is orthogonal, then matrix  $A^2$  must be orthogonal as well.
4. The equation  $(AB)^T = A^T B^T$  holds for all  $n \times n$  matrices  $A$  and  $B$ .
5. If  $A$  and  $B$  are symmetric  $n \times n$  matrices, then  $A + B$  must be symmetric as well.
6. If matrices  $A$  and  $S$  are orthogonal, then  $S^{-1}AS$  is orthogonal as well.
7. All nonzero symmetric matrices are invertible.
8. If  $A$  is an  $n \times n$  matrix such that  $AA^T = I_n$ , then  $A$  must be an orthogonal matrix.
9. If  $\vec{u}$  is a unit vector in  $\mathbb{R}^n$ , and  $L = \text{span}(\vec{u})$ , then  $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$  for all vectors  $\vec{x}$  in  $\mathbb{R}^n$ .
10. If  $A$  is a symmetric matrix, then  $7A$  must be symmetric as well.
11. If  $\vec{x}$  and  $\vec{y}$  are two vectors in  $\mathbb{R}^n$ , then the equation  $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$  must hold.
12. The equation  $\det(A^T) = \det(A)$  holds for all  $2 \times 2$  matrices  $A$ .
13. If matrix  $A$  is orthogonal, then  $A^T$  must be orthogonal as well.
14. If  $A$  and  $B$  are symmetric  $n \times n$  matrices, then  $AB$  must be symmetric as well.
15. If matrices  $A$  and  $B$  commute, then  $A$  must commute with  $B^T$  as well.
16. If  $A$  is any matrix with  $\ker(A) = \{\vec{0}\}$ , then the matrix  $AA^T$  represents the orthogonal projection onto the image of  $A$ .
17. If  $A$  and  $B$  are symmetric  $n \times n$  matrices, then  $ABBA$  must be symmetric as well.
18. If matrices  $A$  and  $B$  commute, then matrices  $A^T$  and  $B^T$  must commute as well.
19. There exists a subspace  $V$  of  $\mathbb{R}^5$  such that  $\dim(V) = \dim(V^\perp)$ , where  $V^\perp$  denotes the orthogonal complement of  $V$ .
20. Every invertible matrix  $A$  can be expressed as the product of an orthogonal matrix and an upper triangular matrix.
21. The determinant of all orthogonal  $2 \times 2$  matrices is 1.
22. If  $A$  is any square matrix, then matrix  $\frac{1}{2}(A - A^T)$  is skew-symmetric.
23. The entries of an orthogonal matrix are all less than or equal to 1.
24. Every nonzero subspace of  $\mathbb{R}^n$  has an orthonormal basis.
25.  $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$  is an orthogonal matrix.
26. If  $V$  is a subspace of  $\mathbb{R}^n$  and  $\vec{x}$  is a vector in  $\mathbb{R}^n$ , then vector  $\text{proj}_V \vec{x}$  must be orthogonal to vector  $\vec{x} - \text{proj}_V \vec{x}$ .
27. If  $A$  and  $B$  are orthogonal  $2 \times 2$  matrices, then  $AB = BA$ .
28. If  $A$  is a symmetric matrix, vector  $\vec{v}$  is in the image of  $A$ , and  $\vec{w}$  is in the kernel of  $A$ , then the equation  $\vec{v} \cdot \vec{w} = 0$  must hold.
29. The formula  $\ker(A) = \ker(A^T A)$  holds for all matrices  $A$ .
30. If  $A^T A = AA^T$  for an  $n \times n$  matrix  $A$ , then  $A$  must be orthogonal.
31. There exist orthogonal  $2 \times 2$  matrices  $A$  and  $B$  such that  $A + B$  is orthogonal as well.
32. If  $\|A\vec{x}\| \leq \|\vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^n$ , then  $A$  must represent the orthogonal projection onto a subspace  $V$  of  $\mathbb{R}^n$ .

## Orthogonality and Least Squares

33. If  $A$  is an invertible matrix such that  $A^{-1} = A$ , then  $A$  must be orthogonal.
34. If the entries of two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are all positive, then  $\vec{v}$  and  $\vec{w}$  must enclose an acute angle.
35. The formula  $(\ker B)^\perp = \text{im}(B^T)$  holds for all matrices  $B$ .
36. The matrix  $A^T A$  is symmetric for all matrices  $A$ .
37. If matrix  $A$  is similar to  $B$  and  $A$  is orthogonal, then  $B$  must be orthogonal as well.
38. The formula  $\text{im}(B) = \text{im}(B^T B)$  holds for all square matrices  $B$ .
39. If matrix  $A$  is symmetric and matrix  $S$  is orthogonal, then matrix  $S^{-1}AS$  must be symmetric.
40. If  $A$  is a square matrix such that  $A^T A = AA^T$ , then  $\ker(A) = \ker(A^T)$ .
41. Any square matrix can be written as the sum of a symmetric and a skew-symmetric matrix.
42. If  $x_1, x_2, \dots, x_n$  are any real numbers, then the inequality

$$\left( \sum_{k=1}^n x_k \right)^2 \leq n \sum_{k=1}^n (x_k^2)$$

must hold.

## Answers to Odd-Numbered Exercises

1 Answers to more theoretical questions are omitted.

1.  $\sqrt{170}$

3.  $\sqrt{54}$

5.  $\arccos\left(\frac{20}{\sqrt{406}}\right) \approx 0.12$  (radians)

7. obtuse

9. acute

11.  $\arccos\left(\frac{1}{\sqrt{n}}\right) \rightarrow \frac{\pi}{2}$  (as  $n \rightarrow \infty$ )

13.  $2\arccos(0.8) \approx 74^\circ$

15.  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

17.  $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

19. a. Orthogonal projection onto  $L^\perp$   
 b. Reflection about  $L^\perp$   
 c. Reflection about  $L$

43. If  $AA^T = A^2$  for a  $2 \times 2$  matrix  $A$ , then  $A$  must be symmetric.
44. If  $V$  is a subspace of  $\mathbb{R}^n$  and  $\vec{x}$  is a vector in  $\mathbb{R}^n$ , then the inequality  $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$  must hold.
45. If  $A$  is an  $n \times n$  matrix such that  $\|A\vec{u}\| = 1$  for all unit vectors  $\vec{u}$ , then  $A$  must be an orthogonal matrix.
46. If  $A$  is any symmetric  $2 \times 2$  matrix, then there must exist a real number  $x$  such that matrix  $A - xI_2$  fails to be invertible.
47. There exists a basis of  $\mathbb{R}^{2 \times 2}$  that consists of orthogonal matrices.
48. If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ , then the matrix  $Q$  in the  $QR$  factorization of  $A$  is a rotation matrix.
49. There exists a linear transformation  $L$  from  $\mathbb{R}^{3 \times 3}$  to  $\mathbb{R}^{2 \times 2}$  whose kernel is the space of all skew-symmetric  $3 \times 3$  matrices.
50. If a  $3 \times 3$  matrix  $A$  represents the orthogonal projection onto a plane  $V$  in  $\mathbb{R}^3$ , then there must exist an orthogonal  $3 \times 3$  matrix  $S$  such that  $S^T AS$  is diagonal.

21. For example:  $b = d = e = g = 0$ ,  $a = \frac{1}{2}$ ,

$$c = \frac{\sqrt{3}}{2}, f = -\frac{\sqrt{3}}{2}$$

25. a.  $\|k\vec{v}\| = \sqrt{(k\vec{v}) \cdot (k\vec{v})} = \sqrt{k^2(\vec{v} \cdot \vec{v})} = \sqrt{k^2}\sqrt{\vec{v} \cdot \vec{v}} = |k|\|\vec{v}\|$

b. By part a,  $\left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$ .

27.  $\begin{bmatrix} 8 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

29. By Pythagoras,  $\|\vec{x}\| = \sqrt{49 + 9 + 4 + 1 + 1} = 8$ .

31.  $p \leq \|\vec{x}\|^2$ . Equality holds if (and only if)  $\vec{x}$  is a linear combination of the vectors  $\vec{u}_i$ .

33. The vector whose  $n$  components are all  $1/n$

35.  $-\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

### Orthogonality and Least Squares

37.  $R(\vec{x}) = 2(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + 2(\vec{u}_2 \cdot \vec{x})\vec{u}_2 - \vec{x}$

39. No; if  $\vec{u}$  is a unit vector in  $L$ , then  $\vec{x} \cdot \text{proj}_L \vec{x} = \vec{x} \cdot (\vec{u} \cdot \vec{x})\vec{u} = (\vec{u} \cdot \vec{x})^2 \geq 0$ .

41.  $\arccos(20/21) \approx 0.31$  radians

43.  $\frac{5}{9}\vec{v}_2$

45.  $\frac{25}{41}\vec{v}_2 - \frac{1}{41}\vec{v}_3$

**2** Answers to more theoretical questions are omitted.

1.  $\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

3.  $\begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix}$

5.  $\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

7.  $\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

9.  $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/10 \\ 7/10 \\ -7/10 \\ 1/10 \end{bmatrix}$

11.  $\begin{bmatrix} 4/5 \\ 0 \\ 0 \\ 3/5 \end{bmatrix}, \begin{bmatrix} -3/15 \\ 2/15 \\ 14/15 \\ 4/15 \end{bmatrix}$

13.  $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

15.  $\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} [3]$

17.  $\begin{bmatrix} 4/5 & 3/5 \\ 0 & 0 \\ 3/5 & -4/5 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 0 & 35 \end{bmatrix}$

19.  $\begin{bmatrix} 2/3 & -1/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{18} \end{bmatrix}$

21.  $\frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$

23.  $\begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 10 \end{bmatrix}$

25.  $\begin{bmatrix} 4/5 & -3/15 \\ 0 & 2/15 \\ 0 & 14/15 \\ 3/5 & 4/15 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 0 & 15 \end{bmatrix}$

27.  $\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

29.  $\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

31.  $\vec{e}_1, \vec{e}_2, \vec{e}_3$

33.  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

35.  $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

37.  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$

39.  $\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$

41.  $Q$  is diagonal with  $q_{ii} = 1$  if  $a_{ii} > 0$  and  $q_{ii} = -1$  if  $a_{ii} < 0$ . You can get  $R$  from  $A$  by multiplying the  $i$ th row of  $A$  with  $-1$  whenever  $a_{ii}$  is negative.

43. Write the  $QR$  factorization of  $A$  in partitioned form as  $A = [A_1 \ A_2] = [Q_1 \ Q_2] \begin{bmatrix} R_1 & R_2 \\ 0 & R_4 \end{bmatrix}$ .

Then  $A_1 = Q_1 R_1$  is the  $QR$  factorization of  $A_1$ .

45. Yes

**3** Answers to more theoretical questions are omitted.

1. Not orthogonal      3. Orthogonal

5. Not orthogonal      7. Orthogonal

9. Orthogonal      11. Orthogonal

13. Symmetric

15. Not necessarily symmetric

17. Symmetric      19. Symmetric

21. Symmetric

23. Not necessarily symmetric

25. Symmetric

27.  $(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T \vec{w} = \vec{v}^T A^T \vec{w} = \vec{v} \cdot (A^T \vec{w})$

29.  $\angle(L(\vec{v}), L(\vec{w})) = \arccos \frac{L(\vec{v}) \cdot L(\vec{w})}{\|L(\vec{v})\| \|L(\vec{w})\|} =$

$\arccos \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \angle(\vec{v}, \vec{w})$  [The equation

$L(\vec{v}) \cdot L(\vec{w}) = \vec{v} \cdot \vec{w}$  is shown in Exercise 28.]

31. Yes, since  $AA^T = I_n$ .

## Orthogonality and Least Squares

33. The first column is a unit vector; we can write it as  $\vec{v}_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  for some  $\theta$ . The second column is a unit vector orthogonal to  $\vec{v}_1$ ; there are two choices:  $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$  and  $\begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}$ . Solution:  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  and  $\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$ , for arbitrary  $\theta$ .

35. For example,  $T(\vec{x}) = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \vec{x}$

37. No, by Theorem 3.2

39. ( $ij$ th entry of  $A$ ) =  $u_i u_j$

41. All entries of  $A$  are  $\frac{1}{n}$ .

43.  $A$  represents the reflection about the line spanned by  $\vec{u}$  (compare with Example 2), and  $B$  represents the reflection about the plane with normal vector  $\vec{u}$ .

45.  $\dim(\ker A) = m - \text{rank } A$  and  $\dim(\ker(A^T)) = n - \text{rank}(A^T) = n - \text{rank } A$  (by Theorem 3.9c). Therefore, the dimensions of the two kernels are equal if (and only if)  $m = n$ , that is, if  $A$  is a square matrix.

47.  $A^T A = (QR)^T QR = R^T Q^T QR = R^T R$

49. By Exercise 2.45, we can write  $A^T = QL$ , where  $Q$  is orthogonal and  $L$  is lower triangular. Then  $A = (QL)^T = L^T Q^T$  does the job.

51. a.  $I_m = Q_1^T Q_1 = S^T Q_2^T Q_2 S = S^T S$ , so that  $S$  is orthogonal.

b.  $R_2 R_1^{-1}$  is both orthogonal (by part a) and upper triangular, with positive diagonal entries. By Exercise 50a, we have  $R_2 R_1^{-1} = I_m$ , so that  $R_2 = R_1$  and  $Q_1 = Q_2$ , as claimed.

53.  $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix};$   
dimension 3

55.  $\frac{n(n+1)}{2}$

57. Yes, and yes

59. The kernel consists of the symmetric  $n \times n$  matrices, and the image consists of the skew-symmetric matrices.

61.  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

63. If  $A = LDU$ , then  $A^T = U^T D L^T$  is the  $LDU$  factorization of  $A^T$ . Since  $A = A^T$  the two factorizations are identical, so that  $U = L^T$ , as claimed.

65.  $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \\ 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}, \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \\ 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}, \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \\ 0.6 & 0.8 \\ 0.6 & -0.6 \end{bmatrix}$ , and

69.  $P_W = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$

71.  $\dim V = 2n - 1$

73. a. Note that  $ab = -1$ . Now  $\vec{a} \cdot \vec{b} = \sum_{k=0}^{n-1} (ab)^k = \frac{1 - (ab)^n}{1 - ab} = 0$  for even  $n$ .

b. By Theorem 3.10 and the preceding paragraph,  $P$  is a linear combination of the matrices  $M = \vec{a}\vec{a}^T$  and  $N = \vec{b}\vec{b}^T$ . It suffices to show that  $M$  and  $N$  are Hankel matrices. Indeed,  $m_{ij} = a^{i+j-2} = m_{i+1,j-1}$  and  $n_{ij} = b^{i+j-2} = n_{i+1,j-1}$  for all  $i = 1, \dots, n-1$  and all  $j = 2, \dots, n$ .

4 Answers to more theoretical questions are omitted.

1.  $\text{im } A = \text{span} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and  $\ker(A^T) = \text{span} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

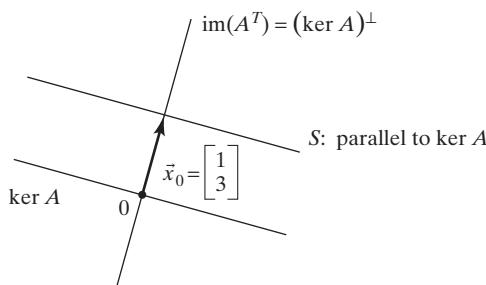
3. The vectors form a basis of  $\mathbb{R}^n$ .

5.  $V^\perp = (\ker A)^\perp = \text{im}(A^T)$ , where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 \end{bmatrix}$$

$$\text{Basis of } V^\perp: \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}$$

7.  $\text{im } A = (\ker A)^\perp$



9.  $\|\vec{x}_0\| < \|\vec{x}\|$  for all other vectors  $\vec{x}$  in  $S$

11. b.  $L(L^+(\vec{y})) = \vec{y}$

c.  $L^+(L(\vec{x})) = \text{proj}_V \vec{x}$ , where  $V = (\ker A)^\perp = \text{im}(A^T)$

d.  $\text{im}(L^+) = \text{im}(A^T)$  and  $\ker(L^+) = \{\vec{0}\}$

e.  $L^+(\vec{y}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{y}$

## Orthogonality and Least Squares

13. b.  $L^+(L(\vec{x})) = \text{proj}_V \vec{x}$ , where  $V = (\ker A)^\perp = \text{im}(A^T)$   
c.  $L(L^+(\vec{y})) = \text{proj}_W \vec{y}$ , where  $W = \text{im } A = (\ker(A^T))^\perp$   
d.  $\text{im}(L^+) = \text{im}(A^T)$  and  $\ker(L^+) = \ker(A^T)$

e.  $L^+(\vec{y}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{y}$

15. Let  $B = (A^T A)^{-1} A^T$ .

17. Yes; note that  $\ker A = \ker(A^T A)$ .

19.  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

21.  $\vec{x}^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ,  $\|\vec{b} - A\vec{x}^*\| = 42$

23.  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

25.  $\begin{bmatrix} 1 - 3t \\ t \end{bmatrix}$ , for arbitrary  $t$

27.  $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$       29.  $x_1^* = x_2^* \approx \frac{1}{2}$

31.  $3 + 1.5t$

33. approximately  $1.5 + 0.1 \sin(t) - 1.41 \cos(t)$

37. a. Try to solve the system  $\begin{cases} c_0 + 35c_1 = \log(35) \\ c_0 + 46c_1 = \log(46) \\ c_0 + 59c_1 = \log(77) \\ c_0 + 69c_1 = \log(133) \end{cases}$ .  
Least-squares solution  $\begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} \approx \begin{bmatrix} 0.915 \\ 0.017 \end{bmatrix}$ . Use approximation  $\log(d) = 0.915 + 0.017t$ .

b. Exponentiate the equation in part (a):  $d = 10^{\log d} = 10^{0.915+0.017t} \approx 8.221 \cdot 10^{0.017t} \approx 8.221 \cdot 1.04^t$ .

c. Predicts 259 displays for the A320; there are much fewer since the A320 is highly computerized.

39. a. Try to solve the system  $\begin{cases} c_0 + \log(600,000)c_1 = \log(250) \\ c_0 + \log(200,000)c_1 = \log(60) \\ c_0 + \log(60,000)c_1 = \log(25) \\ c_0 + \log(10,000)c_1 = \log(12) \\ c_0 + \log(2,500)c_1 = \log(5) \end{cases}$

5 Answers to more theoretical questions are omitted.

3. a. If  $S$  is invertible  
b. If  $S$  is orthogonal

5. Yes      7. For positive  $k$

9. True      11. The angle is  $\delta$

13. The two norms are equal, by Theorem 5.6.

15. If  $b = c$  and  $b^2 < d$       17. If  $\ker T = \{0\}$

19. The matrices  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $b = c$ ,  $a > 0$ , and  $b^2 < ad$ .

21. Yes,  $\langle \vec{v}, \vec{w} \rangle = 2(\vec{v} \cdot \vec{w})$

23.  $1, 2t - 1$

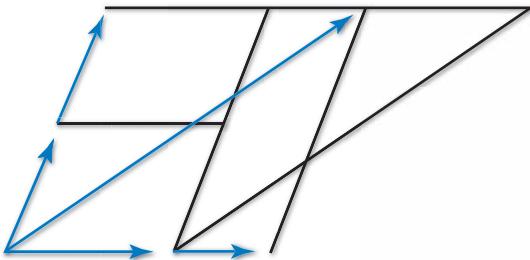
25.  $\sqrt{1 + \frac{1}{4} + \frac{1}{9} + \dots} = \frac{\pi}{\sqrt{6}}$

27.  $a_0 = \frac{1}{\sqrt{2}} \cdot c_k = 0$  for all  $k$   
 $b_k = \begin{cases} \frac{2}{k\pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$

29.  $\sum_{k \text{ odd}} \frac{1}{k^2} = \frac{\pi^2}{8}$

33. b.  $\|f\|^2 = \langle f, f \rangle = \int_a^b w(t)dt = 1$ , so that  $\|f\| = 1$

35. a.  $\|t\|_{32} = \sqrt{\frac{1}{2} \int_{-1}^1 t^2 dt} = \sqrt{\frac{1}{3}}$  and  $\|t\|_{34} = \sqrt{\frac{2}{\pi} \int_{-1}^1 \sqrt{1-t^2} t^2 dt} = \sqrt{\frac{2}{\pi} \cdot \frac{\pi}{8}} = \frac{1}{2}$   
b. For  $f(t) = \sqrt{1-t^2}$  we have  $\|f\|_{32} = \sqrt{\frac{2}{3}}$  and  $\|f\|_{34} = \sqrt{\frac{3}{4}}$



# Determinants

## I Introduction to Determinants

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if (and only if)

$$\det A = ad - bc \neq 0$$

You may wonder whether the concept of a determinant can be generalized to square matrices of arbitrary size. Can we assign a number  $\det A$  to a square matrix  $A$ , expressed in terms of the entries of  $A$ , such that  $A$  is invertible if (and only if)  $\det A \neq 0$ ?

### The Determinant of a $3 \times 3$ Matrix

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$$

(we denote the three column vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ ). See Figure 1.

The matrix  $A$  fails to be invertible if the image of  $A$  isn't all of  $\mathbb{R}^3$ , meaning that the three column vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are contained in some plane  $V$ . In this case, the cross product<sup>1</sup>  $\vec{v} \times \vec{w}$ , being perpendicular to  $V$ , is perpendicular to vector  $\vec{u}$ , so that

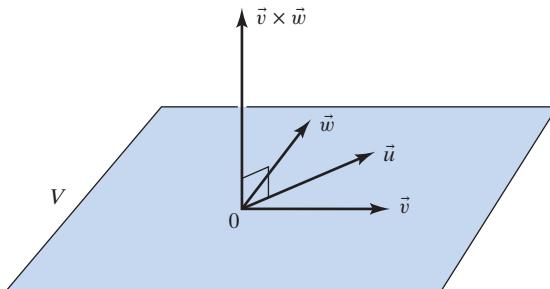
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 0.$$

If  $A$  is invertible, on the other hand, then  $\vec{v} \times \vec{w}$  fails to be perpendicular to  $\vec{u}$ , so that  $\vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0$ .

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<sup>1</sup>This note has been intentionally omitted for this edition.

## Determinants



**Figure I**

Thus, the quantity  $\vec{u} \cdot (\vec{v} \times \vec{w})$  has the property we expect from the determinant: It is nonzero if (and only if) matrix

$$A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$$

is invertible. This motivates the following definition.

**Definition 1.1**

**Determinant of a  $3 \times 3$  matrix, in terms of the columns**

If  $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ , then

$$\det A = \vec{u} \cdot (\vec{v} \times \vec{w}).$$

A  $3 \times 3$  matrix  $A$  is invertible if (and only if)  $\det A \neq 0$ .

Let's express the determinant  $\det A = \vec{u} \cdot (\vec{v} \times \vec{w})$  in terms of the entries of matrix  $A$ :

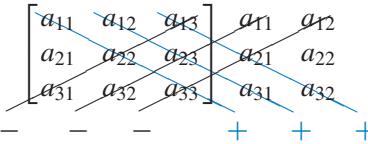
$$\begin{aligned} \det A &= \vec{u} \cdot (\vec{v} \times \vec{w}) \\ &= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \left( \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} \\ a_{32}a_{13} - a_{12}a_{33} \\ a_{12}a_{23} - a_{22}a_{13} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}. \end{aligned}$$

Here is a memory aid for the determinant of a  $3 \times 3$  matrix.

**Theorem 1.2**

**Sarrus's rule<sup>2</sup>**

To find the determinant of a  $3 \times 3$  matrix  $A$ , write the first two columns of  $A$  to the right of  $A$ . Then multiply the entries along the six diagonals shown below.



Add or subtract these diagonal products, as shown in the diagram:

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

<sup>2</sup>Stated by Pierre Frédéric Sarrus (1798–1861) of Strasbourg, c. 1820.

## Determinants

**EXAMPLE 1** Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

**Solution**

By Sarrus's rule,  $\det A = 1 \cdot 5 \cdot 10 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 10 = -3$ . Matrix  $A$  is invertible. ■

**EXAMPLE 2** Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

**Solution**

We find that  $\det A = adf$ , since all other terms in Sarrus's formula are zero. The determinant of an upper (or lower) triangular  $3 \times 3$  matrix is the product of the diagonal entries. Thus, a triangular  $3 \times 3$  matrix is invertible if (and only if) all its diagonal entries are nonzero. ■

**EXAMPLE 3** For which values of the scalar  $\lambda$  (lambda) is the matrix

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & 1 & \lambda \end{bmatrix}$$

invertible?

**Solution**

$$\begin{aligned} \det A &= \lambda^3 - 1 + 1 + \lambda - \lambda - \lambda = \lambda^3 - \lambda \\ &= \lambda(\lambda^2 - 1) = \lambda(\lambda - 1)(\lambda + 1). \end{aligned}$$

The determinant is 0 if  $\lambda = 0$ ,  $\lambda = 1$ , or  $\lambda = -1$ . Matrix  $A$  is invertible if  $\lambda$  is any real number other than 0, 1, and  $-1$ . ■

**EXAMPLE 4** For three column vectors  $\vec{u}, \vec{v}, \vec{w}$  in  $\mathbb{R}^3$ , what is the relationship between the determinants of  $A = [\vec{u} \ \vec{v} \ \vec{w}]$  and  $B = [\vec{u} \ \vec{w} \ \vec{v}]$ ? Note that matrix  $B$  is obtained by swapping the last two columns of  $A$ .

**Solution**

$$\det B = \det [\vec{u} \ \vec{w} \ \vec{v}] = \vec{u} \cdot (\vec{w} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{w}) = -\det [\vec{u} \ \vec{v} \ \vec{w}] = -\det A.$$

We have used the fact that the cross product is anticommutative:  $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$ . ■

It turns out that  $\det B = -\det A$  if  $B$  is obtained by swapping any two columns or any two rows of a  $3 \times 3$  matrix  $A$ ; we can verify this by direct computation. This is referred to as the *alternating property* of the determinant on the columns and on the rows. The  $2 \times 2$  determinant is alternating on the rows and columns as well (verify this!), and we will see that this property generalizes to determinants of square matrices of any size.

### Linearity Properties of the Determinant

#### **EXAMPLE 5**

Is the function  $F(A) = \det A$  from the linear space  $\mathbb{R}^{3 \times 3}$  to  $\mathbb{R}$  a linear transformation?

#### Solution

The answer is negative. For example,  $F(I_3 + I_3) = F(2I_3) = 8$ , while  $F(I_3) + F(I_3) = 1 + 1 = 2$ . ■

However, the determinant does have some noteworthy linearity properties.

#### **EXAMPLE 6**

Is the function

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \det \begin{bmatrix} 2 & x_1 & 5 \\ 3 & x_2 & 6 \\ 4 & x_3 & 7 \end{bmatrix}$$

from  $\mathbb{R}^3$  to  $\mathbb{R}$  a linear transformation? Here we are placing the input variables  $x_1, x_2, x_3$  in the second column, choosing arbitrary constants for all the other entries.

#### Solution

Note that

$$\begin{aligned} T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \det \begin{bmatrix} 2 & x_1 & 5 \\ 3 & x_2 & 6 \\ 4 & x_3 & 7 \end{bmatrix} = (6 \cdot 4 - 3 \cdot 7)x_1 + (2 \cdot 7 - 5 \cdot 4)x_2 + (5 \cdot 3 - 2 \cdot 6)x_3 \\ &= 3x_1 - 6x_2 + 3x_3. \end{aligned}$$

Therefore,  $T$  is a linear transformation, since the output is a linear combination of the input variables. ■

We say that the  $3 \times 3$  determinant is *linear in the second column*. Likewise, the determinant is linear in the two other columns and in all three rows. For example, linearity in the third row means that

$$L(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{bmatrix}$$

is linear on row vectors  $\vec{x}$  with three components, for any two fixed row vectors  $\vec{v}_1$  and  $\vec{v}_2$ .

Alternatively, we can express the linearity of  $L$  by the equations

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) \quad \text{and} \quad L(k\vec{x}) = kL(\vec{x})$$

or

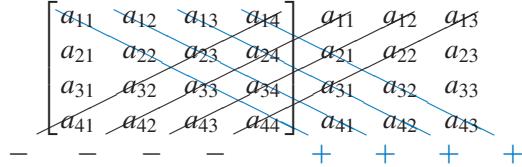
$$\begin{aligned} \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{x} + y \end{bmatrix} &= \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{bmatrix} + \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{y} \end{bmatrix} \quad \text{and} \\ \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & k\vec{x} \end{bmatrix} &= k \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{bmatrix}. \end{aligned}$$

## Determinants

### The Determinant of an $n \times n$ Matrix

We may be tempted to define the determinant of an  $n \times n$  matrix by generalizing Sarrus's rule (see Theorem 1.2). For a  $4 \times 4$  matrix, a naive generalization of Sarrus's rule produces the expression

$$a_{11}a_{22}a_{33}a_{44} + \cdots + a_{14}a_{21}a_{32}a_{43} - a_{14}a_{23}a_{32}a_{41} - \cdots - a_{13}a_{22}a_{31}a_{44}.$$



For example, for the *invertible* matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the expression given by this generalization of Sarrus's rule is 0. This shows that we cannot define the determinant by generalizing Sarrus's rule in this way: Recall that we want the determinant of an invertible matrix to be nonzero.

We have to look for a more subtle structure in the formula

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

for the determinant of a  $3 \times 3$  matrix. Note that each of the six terms in this expression is a product of three factors involving exactly one entry from each row and each column of the matrix:

$$\begin{aligned} & \begin{bmatrix} \textcircled{a_{11}} & a_{12} & a_{13} \\ a_{21} & \textcircled{a_{22}} & a_{23} \\ a_{31} & a_{32} & \textcircled{a_{33}} \end{bmatrix} \begin{bmatrix} a_{11} & \textcircled{a_{12}} & a_{13} \\ a_{21} & a_{22} & \textcircled{a_{23}} \\ \textcircled{a_{31}} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \textcircled{a_{13}} \\ \textcircled{a_{21}} & a_{22} & a_{23} \\ a_{31} & \textcircled{a_{32}} & a_{33} \end{bmatrix} \\ & \begin{bmatrix} a_{11} & a_{12} & \textcircled{a_{13}} \\ a_{21} & \textcircled{a_{22}} & a_{23} \\ \textcircled{a_{31}} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \textcircled{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & \textcircled{a_{23}} \\ a_{31} & \textcircled{a_{32}} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & \textcircled{a_{12}} & a_{13} \\ \textcircled{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & \textcircled{a_{33}} \end{bmatrix}. \end{aligned}$$

For lack of a better word, we call such a choice of a number in each row and column of a square matrix a *pattern* in the matrix.<sup>3</sup> The simplest pattern is the *diagonal pattern*, where we choose all numbers  $a_{ii}$  on the main diagonal. For you chess players, a pattern in an  $8 \times 8$  matrix corresponds to placing 8 rooks on a chessboard so that none of them can attack another.

How many patterns are there in an  $n \times n$  matrix? Let us see how we can construct a pattern column by column. In the first column we have  $n$  choices. For each of these, we then have  $n - 1$  choices left in the second column. Therefore, we have  $n(n - 1)$  choices for the numbers in the first two columns. For each of these, there are  $n - 2$  choices in the third column, and so on. When we come to the last column, we have no choice, because there is only one row left. We conclude that there are  $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$  patterns in an  $n \times n$  matrix. The quantity  $1 \cdot 2 \cdot 3 \cdots (n - 2) \cdot (n - 1) \cdot n$  is written  $n!$  (read “ $n$  factorial”).

---

<sup>3</sup>This theory is usually phrased in the language of *permutations*. Here we attempt a less technical presentation, without sacrificing content.

## Determinants

For a pattern  $P$  in a  $3 \times 3$  matrix, we consider the product of all the entries in the pattern, denoted  $\text{prod } P$ . For example, for the pattern  $P = (a_{12}, a_{23}, a_{31})$ , we have  $\text{prod } P = a_{12}a_{23}a_{31}$ . Then we can write

$$\det A = \sum \pm \text{prod } P,$$

where the sum is taken over all six patterns  $P$  in a  $3 \times 3$  matrix  $A$ . Next we need to examine how the signs of the six summands are chosen.

It turns out that these signs are related to the alternating property of the determinant we discussed after Example 4:

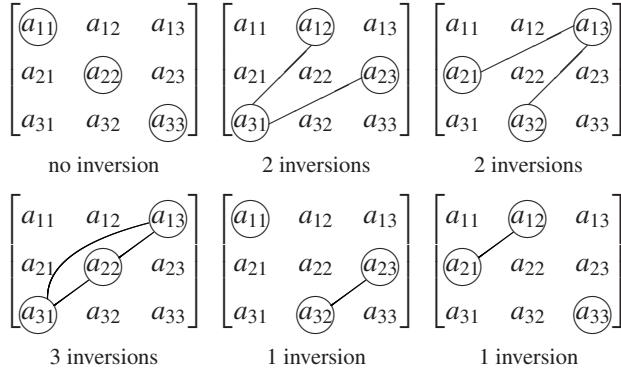
$$\begin{aligned} \det \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} &= -\det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{bmatrix} \\ &= \det \begin{bmatrix} a_{31} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \end{bmatrix} = a_{31}a_{12}a_{23} = a_{12}a_{23}a_{31}, \end{aligned}$$

since we perform two row swaps to bring the matrix into diagonal form, while

$$\det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} a_{31} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{13} \end{bmatrix} = -a_{31}a_{22}a_{13} = -a_{13}a_{22}a_{31}.$$

There is an equivalent way to predict this sign without actually counting row swaps. We say that two numbers in a pattern are *inverted* if one of them is to the right and above the other. Let's indicate the number of inversions for each of the six patterns in a  $3 \times 3$  matrix.

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$



We see that the sign of  $\text{prod } P$  in the formula  $\det A = \sum \pm \text{prod } P$  depends on the number of inversions in  $P$ . We get the plus sign if the number of inversions is even and the minus sign if that number is odd. We can write

$$\det A = \sum (-1)^{\text{(number of inversions in } P\text{)}} \text{prod } P$$

If we define the *signature* of a pattern  $P$  as  $\text{sgn } P = (-1)^{\text{(number of inversions in } P\text{)}}$ , then we can write more succinctly

$$\det A = \sum (\text{sgn } P)(\text{prod } P),$$

where the sum is taken over all six patterns  $P$  in the matrix  $A$ .

Alternatively, we can describe the signature in terms of row swaps: If we can bring a pattern  $P$  into diagonal form by means of  $p$  row swaps, then  $\text{sgn } P = (-1)^p$ . See Theorem 2.3b.

## Determinants

Using these definitions and observations as a guide, we are now ready to define the determinant of an  $n \times n$  matrix.

### Definition 1.3

#### Patterns, inversions, and determinants<sup>4</sup>

A *pattern* in an  $n \times n$  matrix  $A$  is a way to choose  $n$  entries of the matrix so that there is one chosen entry in each row and in each column of  $A$ .

With a pattern  $P$  we associate the product of all its entries, denoted  $\text{prod } P$ .

Two entries in a pattern are said to be *inverted* if one of them is located to the right and above the other in the matrix.

The *signature* of a pattern  $P$  is defined as  $\text{sgn } P = (-1)^{(\text{number of inversions in } P)}$ .

The determinant of  $A$  is defined as

$$\det A = \sum (\text{sgn } P)(\text{prod } P),$$

where the sum is taken over all  $n!$  patterns  $P$  in the matrix  $A$ . Thus, we are summing up the products associated with all patterns with an even number of inversions, and we are subtracting the products associated with the patterns with an odd number of inversions.

**EXAMPLE 7** Apply Definition 1.3 to a  $2 \times 2$  matrix, and verify that the result agrees with the  $2 \times 2$  matrix formula.

#### Solution

There are two patterns in the  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ :

$$\begin{bmatrix} @ & b \\ c & @ \end{bmatrix} \quad \begin{bmatrix} a & \cancel{b} \\ \cancel{c} & d \end{bmatrix}.$$

No inversions      One inversion

Therefore,  $\det A = (-1)^0 ad + (-1)^1 bc = ad - bc$ . ■

**EXAMPLE 8** Find  $\det A$  for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

<sup>4</sup>It appears that determinants were first considered by the Japanese mathematician Seki Kowa (1642–1708). Seki may have known that the determinant of an  $n \times n$  matrix has  $n!$  terms and that rows and columns are interchangeable. See Theorem 2.1. The French mathematician Alexandre-Théophile Vandermonde (1735–1796) was the first to give a coherent and systematic exposition of the theory of determinants. Throughout the 19th century, determinants were considered the ultimate tool in linear algebra, used extensively by Cauchy, Jacobi, Kronecker, and others. Recently, determinants have gone somewhat out of fashion, and some people would like to see them eliminated altogether from linear algebra. See, for example, Sheldon Axler's article "Down with Determinants" in *The American Mathematical Monthly*, February 1995, where we read, "This paper will show how linear algebra can be done better without determinants." Read it and see what you think.

## Determinants

### Solution

Only one pattern  $P$  makes a nonzero contribution toward the determinant:

$$\begin{bmatrix} 0 & \textcircled{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{2} \\ \textcircled{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{5} & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 0 \end{bmatrix}.$$

7 inversions

Thus,  $\det A = (\text{sgn } P)(\text{prod } P) = (-1)^7 2 \cdot 8 \cdot 2 \cdot 3 \cdot 5 \cdot 1 = -480$ . ■

**EXAMPLE 9** Find  $\det A$  for

$$A = \begin{bmatrix} 6 & 0 & 1 & 0 & 0 \\ 9 & 3 & 2 & 3 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{bmatrix}.$$

### Solution

Again, let's look for patterns with a nonzero product. We pick the entries column by column this time. In the second column, we must choose the second component, 3. Then, in the fourth column, we must choose the third component, 2. Next, think about the last column, and so on. It turns out that there is only one pattern  $P$  with a nonzero product.

$$\begin{bmatrix} \textcircled{6} & 0 & 1 & 0 & 0 \\ 9 & \textcircled{3} & 2 & 3 & 7 \\ 8 & 0 & 3 & \textcircled{2} & 9 \\ 0 & 0 & \textcircled{4} & 0 & 0 \\ 5 & 0 & 5 & 0 & \textcircled{1} \end{bmatrix}$$

1 inversion

$\det A = (\text{sgn } P)(\text{prod } P) = (-1)^1 6 \cdot 3 \cdot 2 \cdot 4 \cdot 1 = -144$ . ■

**EXAMPLE 10** Find  $\det A$  for

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

### Solution

Note that  $A$  is an upper triangular matrix. To have a nonzero product, a pattern must contain the first component of the first column, then the second component of the second column, and so on. Thus, only the diagonal pattern  $P$  makes a nonzero contribution. We conclude that

$\det A = (\text{sgn } P)(\text{prod } P) = (-1)^0 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5! = 120$ . ■

## Determinants

We can generalize this result:

### Theorem 1.4

#### Determinant of a triangular matrix

The determinant of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.

### The Determinant of a Block Matrix (optional)

**EXAMPLE 11** Find  $\det M$  for

$$M = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix}.$$

#### Solution

It is natural to partition the  $4 \times 4$  matrix  $M$  into four  $2 \times 2$  blocks, one of which is zero:

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

Let's see whether we can express  $\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$  in terms of  $\det A$ ,  $\det B$ , and  $\det C$ .

Let's find the patterns in  $M$  that may have a nonzero product.

$$\begin{bmatrix} \textcircled{a}_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & \textcircled{a}_{22} & b_{21} & b_{22} \\ 0 & 0 & \textcircled{c}_{11} & c_{12} \\ 0 & 0 & c_{21} & \textcircled{c}_{22} \end{bmatrix}, \quad \begin{bmatrix} \textcircled{a}_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & \textcircled{a}_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & \textcircled{c}_{12} \\ 0 & 0 & \textcircled{c}_{21} & c_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & \textcircled{a}_{12} & b_{11} & b_{12} \\ \textcircled{a}_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & \textcircled{c}_{11} & c_{12} \\ 0 & 0 & c_{21} & \textcircled{c}_{22} \end{bmatrix},$$
  

$$\begin{bmatrix} a_{11} & \textcircled{a}_{12} & b_{11} & b_{12} \\ \textcircled{a}_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & \textcircled{c}_{12} \\ 0 & 0 & \textcircled{c}_{21} & c_{22} \end{bmatrix}.$$

Thus

$$\begin{aligned} \det M &= a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21} \\ &= a_{11}a_{22}(c_{11}c_{22} - c_{12}c_{21}) - a_{12}a_{21}(c_{11}c_{22} - c_{12}c_{21}) \\ &= (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21}) = (\det A)(\det C). \end{aligned}$$

In summary,

$$\det M = \det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = (\det A)(\det C).$$

■

It turns out that the formula we derived in Example 11 holds for block matrices of any size.

**Theorem 1.5**

**Determinant of a block matrix**

If  $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ , where  $A$  and  $C$  are square matrices (not necessarily of the same size), then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = (\det A)(\det C).$$

Likewise,

$$\det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = (\det A)(\det C).$$

However, the formula

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det D) - (\det B)(\det C)$$

does not always hold. See Exercise 48.

**Proof** Let's outline a proof for Theorem 1.5. As you follow this somewhat technical presentation, use Example 11 as a guide.

If  $P_A$  is a pattern in  $A$  and  $P_C$  is a pattern in  $C$ , then their concatenation,  $P_M = (P_A, P_C)$ , will be a pattern in  $M$ , with  $\text{prod } P_M = (\text{prod } P_A)(\text{prod } P_C)$  and  $\text{sgn } P_M = (\text{sgn } P_A)(\text{sgn } P_C)$ , since the number of inversions in  $P_M$  will be the sum of those in  $P_A$  and  $P_C$ . Conversely, any pattern  $P_M$  in  $M$  with a nonzero product will be of this form,  $P_M = (P_A, P_C)$ , since the pattern entries cannot be taken from the zero block in matrix  $M$ . Now

$$\begin{aligned} (\det A)(\det C) &= \left( \sum_{P_A} (\text{sgn } P_A)(\text{prod } P_A) \right) \left( \sum_{P_C} (\text{sgn } P_C)(\text{prod } P_C) \right) \\ &= \sum_{(P_A, P_C)} (\text{sgn } P_A)(\text{sgn } P_C)(\text{prod } P_A)(\text{prod } P_C) \\ &= \sum_{P_M} (\text{sgn } P_M)(\text{prod } P_M) = \det M. \end{aligned}$$

Here is another example illustrating this proof:

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 0 \\ \textcircled{4} & 5 & 6 & 0 & 0 & 0 \\ \hline 7 & 8 & \textcircled{7} & 0 & 0 & 0 \\ 6 & 5 & 4 & 3 & 2 & \textcircled{1} \\ 2 & 3 & 4 & \textcircled{5} & 6 & 7 \\ 8 & 7 & 6 & 5 & \textcircled{4} & 3 \end{array} \right].$$

Here,  $\text{prod } P_M = 2 \cdot 4 \cdot 7 \cdot 1 \cdot 5 \cdot 4 = (2 \cdot 4 \cdot 7)(1 \cdot 5 \cdot 4) = (\text{prod } P_A)(\text{prod } P_C)$ . There is one inversion in  $P_A$  and there are two inversions in  $P_C$ , for a total of three inversions in  $P_M$ . Thus,  $\text{sgn } P_M = (-1)^3 = (-1)^1(-1)^2 = (\text{sgn } P_A)(\text{sgn } P_C)$ . ■

## Determinants

**EXAMPLE 12** Find

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 & 0 & 0 \\ 3 & 8 & 6 & 0 & 0 & 0 \\ 4 & 9 & 5 & 2 & 1 & 4 \\ 5 & 8 & 4 & 0 & 2 & 5 \\ 6 & 7 & 3 & 0 & 3 & 6 \end{bmatrix}.$$

**Solution**

$$\begin{aligned} \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 & 0 & 0 \\ 3 & 8 & 6 & 0 & 0 & 0 \\ 4 & 9 & 5 & 2 & 1 & 4 \\ 5 & 8 & 4 & 0 & 2 & 5 \\ 6 & 7 & 3 & 0 & 3 & 6 \end{bmatrix} &= \det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 3 & 8 & 6 \end{bmatrix} \det \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \\ &= (1 \cdot 7 \cdot 6)(2 \cdot 2 \cdot 6 - 2 \cdot 5 \cdot 3) \\ &= 42(-6) = -252 \quad \blacksquare \end{aligned}$$

### EXERCISES I

Find the determinants of the matrices  $A$  in Exercises 1 through 10, and find out which of these matrices are invertible.

1.  $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

2.  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

3.  $\begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

5.  $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

6.  $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

8.  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

9.  $\begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$

10.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

In Exercises 11 through 22, use the determinant to find out for which values of the constant  $k$  the given matrix  $A$  is invertible.

11.  $\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}$

12.  $\begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix}$

13.  $\begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

14.  $\begin{bmatrix} 4 & 0 & 0 \\ 3 & k & 0 \\ 2 & 1 & 0 \end{bmatrix}$

15.  $\begin{bmatrix} 0 & k & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

17.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & k & -1 \\ 1 & k^2 & 1 \end{bmatrix}$

19.  $\begin{bmatrix} 1 & 1 & k \\ 1 & k & k \\ k & k & k \end{bmatrix}$

21.  $\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$

23.  $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$

25.  $\begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}$

27.  $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 6 & 4 \end{bmatrix}$

29.  $\begin{bmatrix} 3 & 5 & 6 \\ 0 & 4 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

16.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 5 \\ 6 & 7 & 8 \end{bmatrix}$

18.  $\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}$

20.  $\begin{bmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{bmatrix}$

22.  $\begin{bmatrix} \cos k & 1 & -\sin k \\ 0 & 2 & 0 \\ \sin k & 0 & \cos k \end{bmatrix}$

24.  $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

26.  $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$

28.  $\begin{bmatrix} 5 & 7 & 11 \\ 0 & 3 & 13 \\ 0 & 0 & 2 \end{bmatrix}$

30.  $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix}$

In Exercises 23 through 30, use the determinant to find out for which values of the constant  $\lambda$  the matrix  $A - \lambda I_n$  fails to be invertible.

## Determinants

*Find the determinants of the matrices A in Exercises 31 through 42.*

31. 
$$\begin{bmatrix} 1 & 9 & 8 & 7 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

32. 
$$\begin{bmatrix} 2 & 5 & 7 & 11 \\ 0 & 3 & 5 & 13 \\ 0 & 0 & 5 & 11 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

33. 
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 7 & 5 \end{bmatrix}$$

34. 
$$\begin{bmatrix} 4 & 5 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 2 & 7 & 1 & 4 \\ 1 & 8 & 2 & 3 \end{bmatrix}$$

35. 
$$\begin{bmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{bmatrix}$$

36. 
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

37. 
$$\begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

38. 
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 0 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

39. 
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

40. 
$$\begin{bmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

41. 
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{bmatrix}$$

42. 
$$\begin{bmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 9 & 7 & 9 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix}$$

43. If  $A$  is an  $n \times n$  matrix, what is the relationship between  $\det A$  and  $\det(-A)$ ?

44. If  $A$  is an  $n \times n$  matrix and  $k$  is an arbitrary constant, what is the relationship between  $\det A$  and  $\det(kA)$ ?

45. If  $A$  is a  $2 \times 2$  matrix, what is the relationship between  $\det A$  and  $\det(A^T)$ ?

46. If  $A$  is an invertible  $2 \times 2$  matrix, what is the relationship between  $\det A$  and  $\det(A^{-1})$ ?

47. Find nonzero numbers  $a, b, c, d, e, f, g, h$  such that the

matrix  $\begin{bmatrix} a & b & c \\ d & k & e \\ f & g & h \end{bmatrix}$  is invertible for all real numbers

$k$ , or explain why no such matrix exists.

48. Find  $2 \times 2$  matrices  $A, B, C, D$  such that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq (\det A)(\det D) - (\det B)(\det C).$$

49. For two nonparallel vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$ , consider the linear transformation

$$T(\vec{x}) = \det \begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}$$

from  $\mathbb{R}^3$  to  $\mathbb{R}$ . Describe the kernel of  $T$  geometrically. What is the image of  $T$ ?

50. If  $\vec{u}, \vec{v}, \vec{w}$  are three unit vectors in  $\mathbb{R}^3$ , what are the possible values of  $\det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ ?

51. Explain why any pattern  $P$  in a matrix  $A$ , other than the diagonal pattern, contains at least one entry below the diagonal and at least one entry above the diagonal.

52. Consider two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$ . Form the matrix

$$A = \begin{bmatrix} \vec{v} \times \vec{w} & \vec{v} & \vec{w} \end{bmatrix}.$$

Express  $\det A$  in terms of  $\|\vec{v} \times \vec{w}\|$ . For which choices of  $\vec{v}$  and  $\vec{w}$  is  $A$  invertible?

53. Find the determinant of the  $(2n) \times (2n)$  matrix

$$A = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

54. Is the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 7 & 1000 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1000 \\ 1 & 2 & 1000 & 3 & 4 \end{bmatrix}$$

positive or negative? How can you tell? Do not use technology.

55. Does the following matrix have an LU factorization?

$$A = \begin{bmatrix} 7 & 4 & 2 \\ 5 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

56. Let  $M_n$  be the  $n \times n$  matrix with all 1's along “the other diagonal,” and 0's everywhere else. For example,

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find  $\det(M_n)$  for  $n = 2, 3, 4, 5, 6, 7$ .

b. Find a formula for  $\det(M_n)$ , in terms of  $n$ .

57. A square matrix is called a *permutation matrix* if each row and each column contains exactly one entry 1, with

all other entries being 0. Examples are  $I_n$ ,  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ ,

and the matrices considered in Exercises 53 and 56. What are the possible values of the determinant of a permutation matrix?

## Determinants

- 58.** a. Find a noninvertible  $2 \times 2$  matrix whose entries are four distinct prime numbers, or explain why no such matrix exists.

- b. Find a noninvertible  $3 \times 3$  matrix whose entries are nine distinct prime numbers, or explain why no such matrix exists.

- 59.** Consider the function  $F(A) = F[\vec{v} \quad \vec{w}] = \vec{v} \cdot \vec{w}$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$ , the dot product of the column vectors of  $A$ .

- a. Is  $F$  linear in both columns of  $A$ ? See Example 6.  
 b. Is  $F$  linear in both rows of  $A$ ?  
 c. Is  $F$  alternating on the columns of  $A$ ? See Example 4.

- 60.** Which of the following functions  $F$  of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  are linear in both columns? Which are linear in both rows? Which are alternating on the columns?

- a.  $F(A) = bc$     b.  $F(A) = cd$     c.  $F(A) = ac$   
 d.  $F(A) = bc - ad$     e.  $F(A) = c$

- 61.** Show that the function

$$F \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = bfg$$

is linear in all three columns and in all three rows. See Example 6. Is  $F$  alternating on the columns? See Example 4.

*In Exercises 62 through 64, consider a function  $D$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$  that is linear in both columns and alternating on the columns. See Examples 4 and 6 and the subsequent discussions. Assume that  $D(I_2) = 1$ .*

- 62.** Show that  $D(A) = 0$  for any  $2 \times 2$  matrix  $A$  whose two columns are equal.

- 63.** Show that  $D \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$ . Hint: Write  $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}$  and use linearity in the second column:  $D \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = D \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + D \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = ab D \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \dots$  Use Exercise 62.

- 64.** Using Exercises 62 and 63 as a guide, show that  $D(A) = ad - bc = \det A$  for all  $2 \times 2$  matrices  $A$ .

- 65.** Consider a function  $D$  from  $\mathbb{R}^{3 \times 3}$  to  $\mathbb{R}$  that is linear in all three columns and alternating on the columns. Assume that  $D(I_3) = 1$ . Using Exercises 62 through 64 as a guide, show that  $D(A) = \det A$  for all  $3 \times 3$  matrices  $A$ .

- 66.** a. Let  $V$  be the linear space of all functions  $F$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$  that are linear in both columns. Find a basis of  $V$ , and thus determine the dimension of  $V$ .  
 b. Let  $W$  be the linear space of all functions  $D$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}$  that are linear in both columns and alternating on the columns. Find a basis of  $W$ , and thus determine the dimension of  $W$ .

## 2 Properties of the Determinant

The main goal of this section is to show that a square matrix of any size is invertible if (and only if) its determinant is nonzero. As we work toward this goal, we will discuss a number of other remarkable properties of the determinant.

### The Determinant of the Transpose<sup>5</sup>

**EXAMPLE I** Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}.$$

Express  $\det(A^T)$  in terms of  $\det A$ . You need not compute  $\det A$ .

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<sup>5</sup>This note has been intentionally omitted for this edition.

## Determinants

### Solution

For each pattern  $P$  in  $A$ , we can consider the corresponding (transposed) pattern  $P^T$  in  $A^T$ ; for example,

$$A = \begin{bmatrix} 1 & 2 & (3) & 4 & 5 \\ (6) & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & (3) \\ 2 & (1) & 2 & 3 & 4 \\ 5 & 6 & 7 & (8) & 9 \end{bmatrix}, \quad P \quad A^T = \begin{bmatrix} 1 & (6) & 7 & 2 & 5 \\ 2 & 7 & 6 & (1) & 6 \\ (3) & 8 & 5 & 2 & 7 \\ 4 & 9 & 4 & 3 & (8) \\ 5 & 8 & (3) & 4 & 9 \end{bmatrix}. \quad P^T$$

The two patterns  $P$  and  $P^T$  involve the same numbers, and they contain the same number of inversions, but the role of the two numbers in each inversion is reversed. Therefore, the two patterns make the same contributions to the respective determinants  $(\text{sgn } P)(\text{prod } P) = (\text{sgn } P^T)(\text{prod } P^T)$ . Since these observations apply to all patterns of  $A$ , we can conclude that  $\det(A^T) = \det A$ . ■

Since we have not used any special properties of the matrix  $A$  in Example 1, we can state more generally:

### Theorem 2.1

#### Determinant of the transpose

If  $A$  is a square matrix, then

$$\det(A^T) = \det A.$$

This symmetry property will prove very useful. Any property of the determinant expressed in terms of the rows holds for the columns as well, and vice versa.

### Linearity Properties of the Determinant

In Section 1 we observed that the  $3 \times 3$  determinant is linear in the rows and in the columns; take another look at Example 6 of Section 1. It turns out that these linearity properties generalize to the determinant of  $n \times n$  matrices.

### Theorem 2.2

#### Linearity of the determinant in the rows and columns

Consider fixed row vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$  with  $n$  components. Then the function

$$T(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 & & \vec{v}_i \\ \vdots & & \vdots \\ \vec{v}_{i-1} & & \vec{v}_i \\ \vec{x} & & \vec{v}_{i+1} \\ \vec{v}_{i+1} & & \vdots \\ \vec{v}_n & & \vec{v}_i \end{bmatrix} \quad \text{from } \mathbb{R}^{1 \times n} \text{ to } \mathbb{R}$$

is a linear transformation. This property is referred to as *linearity of the determinant in the  $i$ th row*. Likewise, the determinant is *linear in all the columns*.

To prove Theorem 2.2, observe that the product  $\text{prod } P$  associated with a pattern  $P$  is linear in all the rows and columns, since this product contains exactly one factor from each row and one from each column. Thus, the determinant itself is linear in all the rows and columns, being a linear combination of pattern products.

## Determinants

We can express the linearity of the transformation  $T$  in Theorem 2.2 in terms of the equations  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$  and  $T(k\vec{x}) = kT(\vec{x})$ , or

$$\det \begin{bmatrix} - & \vec{v}_1 & - \\ \vdots & & \\ - & \vec{x} + \vec{y} & - \\ \vdots & & \\ - & \vec{v}_n & - \end{bmatrix} = \det \begin{bmatrix} - & \vec{v}_1 & - \\ \vdots & & \\ - & \vec{x} & - \\ \vdots & & \\ - & \vec{v}_n & - \end{bmatrix} + \det \begin{bmatrix} - & \vec{v}_1 & - \\ \vdots & & \\ - & \vec{y} & - \\ \vdots & & \\ - & \vec{v}_n & - \end{bmatrix} \quad \text{and}$$

$$\det \begin{bmatrix} - & \vec{v}_1 & - \\ \vdots & & \\ - & k\vec{x} & - \\ \vdots & & \\ - & \vec{v}_n & - \end{bmatrix} = k \det \begin{bmatrix} - & \vec{v}_1 & - \\ \vdots & & \\ - & \vec{x} & - \\ \vdots & & \\ - & \vec{v}_n & - \end{bmatrix}.$$

In these equations, all rows except the  $i$ th are fixed,  $\vec{x}$  and  $\vec{y}$  are arbitrary row vectors with  $n$  components, and  $k$  is an arbitrary real number.

## Determinants and Gauss–Jordan Elimination

Consider a  $30 \times 30$  matrix  $A$ , a rather small matrix by the standards of contemporary scientific and engineering applications. Then  $29 \cdot 30! \approx 7 \cdot 10^{33}$  multiplications are required to compute the determinant of  $A$  by Definition 1.3, using patterns. If a super computer performs 10 trillion ( $10^{13}$ ) multiplications a second, it will take over a trillion years to carry out these computations; our universe might long be gone by then. Clearly, we have to look for more efficient ways to compute the determinant.

To use the language of computer science, is there an algorithm for the determinant that runs on polynomial rather than exponential time?

Gauss–Jordan elimination is a powerful tool for solving numerical problems in linear algebra. If we could understand what happens to the determinant of a matrix as we row-reduce it, we could use Gauss–Jordan elimination to compute determinants as well. We have to understand how the three elementary row operations affect the determinant:

- a. Row division: dividing a row by a nonzero scalar  $k$ ,
- b. Row swap: swapping two rows, and
- c. Row addition: adding a multiple of a row to another row.

Let's look at the case of a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  first, with  $\det A = ad - bc$ .

- a. If  $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$ , then  $\det B = \frac{a}{k}d - \frac{b}{k}c = \frac{1}{k} \det A$ .

Verify that  $\det B = \frac{1}{k} \det A$  if  $B$  is obtained from  $A$  by dividing the *second* row by  $k$ .

- b. If  $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ , then  $\det B = cb - da = -\det A$ .
- c. If  $B = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$ , then  $\det B = (a + kc)d - (b + kd)c = ad + kcd - bc - kdc = \det A$ . Verify that  $\det B = \det A$  if  $B$  is obtained from  $A$  by adding  $k$  times the *first* row to the *second* row.

Next, we will examine the effect of the elementary row operations on the determinant of square matrices of arbitrary size.

## Determinants

- a. Row division: If

$$A = \begin{bmatrix} \cdots & \vec{v}_1 & \cdots \\ \vdots & & \vdots \\ \cdots & \vec{v}_i & \cdots \\ \vdots & & \vdots \\ \cdots & \vec{v}_n & \cdots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cdots & \vec{v}_1 & \cdots \\ \vdots & & \vdots \\ \cdots & \vec{v}_i/k & \cdots \\ \vdots & & \vdots \\ \cdots & \vec{v}_n & \cdots \end{bmatrix},$$

then  $\det B = (1/k) \det A$ , by linearity in the  $i$ th row, Theorem 2.2.

- b. Row swap: Refer to Example 2.

### EXAMPLE 2

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 7 & 8 & 9 & 8 \\ 1 & 2 & 3 & 4 & 5 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}.$$

Note that  $B$  is obtained from  $A$  by swapping the first two rows. Express  $\det B$  in terms of  $\det A$ .

### Solution

For each pattern  $P$  in  $A$ , we can consider the corresponding pattern  $P_{\text{swap}}$  in  $B$ ; for example,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 7 & 8 & 9 & 8 \\ 1 & 2 & 3 & 4 & 5 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}_{P_{\text{swap}}}.$$

These two patterns  $P$  and  $P_{\text{swap}}$  involve the same numbers, but the number of inversions in  $P_{\text{swap}}$  is one less than in  $P$ , since we are losing the inversion formed by the entries in the first two rows of  $A$ . Thus,  $\prod P_{\text{swap}} = \prod P$ , but  $\text{sgn } P_{\text{swap}} = -\text{sgn } P$ , so that the two patterns make opposite contributions to the respective determinants. Since these remarks apply to all patterns in  $A$ , we can conclude that

$$\det B = -\det A.$$

(If  $P$  is a pattern in  $A$  such that the entries in the first two rows do not form an inversion, then an additional inversion is created in  $P_{\text{swap}}$ ; again,  $\text{sgn } P_{\text{swap}} = -\text{sgn } P$ .) ■

What if  $B$  is obtained from  $A$  by swapping any two rows, rather than the first two? If we swap two *adjacent* rows, then everything works the same way as in Example 2, and  $\det B = -\det A$ . But what if  $B$  is obtained from  $A$  by swapping two arbitrary rows? Observe that swapping any two rows amounts to an odd number of swaps of adjacent rows. See Exercise 60. Since the determinant changes its sign with each swap of adjacent rows, the equation  $\det B = -\det A$  still holds.

### EXAMPLE 3

If a matrix  $A$  has two equal rows, what can you say about  $\det A$ ?

### Solution

Swap the two equal rows and call the resulting matrix  $B$ . Since we have swapped two *equal* rows, we have  $A = B$ . Now

## Determinants

$$\det A = \det B = -\det A,$$

so that

$$\det A = 0.$$



- c.** Row addition: Finally, what happens to the determinant if we add  $k$  times the  $i$ th row to the  $j$ th row?

$$A = \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j & - \\ & \vdots & \end{bmatrix} \rightarrow B = \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j + k\vec{v}_i & - \\ & \vdots & \end{bmatrix}$$

By linearity in the  $j$ th row, Theorem 2.2, we find that

$$\det B = \det \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j & - \\ & \vdots & \end{bmatrix} + k \det \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \end{bmatrix} = \det A,$$

by Example 3.

### Theorem 2.3

#### Elementary row operations and determinants

- a.** If  $B$  is obtained from  $A$  by dividing a row of  $A$  by a scalar  $k$ , then

$$\det B = (1/k) \det A.$$

- b.** If  $B$  is obtained from  $A$  by a row swap, then

$$\det B = -\det A.$$

We say that the determinant is *alternating* on the rows.

- c.** If  $B$  is obtained from  $A$  by adding a multiple of a row of  $A$  to another row, then

$$\det B = \det A.$$

Analogous results hold for elementary column operations.

Now that we understand how elementary row operations affect determinants, we can analyze the relationship between the determinant of a square matrix  $A$  and that of  $\text{rref } A$ . Suppose that in the course of Gauss–Jordan elimination we swap rows  $s$  times and divide various rows by the scalars  $k_1, k_2, \dots, k_r$ . Then

$$\det(\text{rref } A) = (-1)^s \frac{1}{k_1 k_2 \cdots k_r} (\det A),$$

or

$$\det A = (-1)^s k_1 k_2 \cdots k_r \det(\text{rref } A),$$

by Theorem 2.3.

Let us examine the cases when  $A$  is invertible and when it is not.

## Determinants

If  $A$  is invertible, then  $\text{rref } A = I_n$ , so that  $\det(\text{rref } A) = \det(I_n) = 1$ , and

$$\det A = (-1)^s k_1 k_2 \cdots k_r \neq 0.$$

Note that  $\det A$  fails to be zero since all the scalars  $k_i$  are nonzero.

If  $A$  is noninvertible, then the last row of  $\text{rref } A$  contains all zeros, so that  $\det(\text{rref } A) = 0$  (by linearity in the last row). It follows that  $\det A = 0$ .

We have established the following fundamental result.

### Theorem 2.4

#### Invertibility and determinant

A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

The foregoing discussion provides us with an efficient method for computing the determinant, using Gauss–Jordan elimination.

### Algorithm 2.5 Using Gauss–Jordan elimination to compute the determinant

- a. Consider an invertible  $n \times n$  matrix  $A$ . Suppose you swap rows  $s$  times as you compute  $\text{rref } A = I_n$ , and you divide various rows by the scalars  $k_1, k_2, \dots, k_r$ . Then

$$\det A = (-1)^s k_1 k_2 \cdots k_r.$$

- b. In fact, it is not always sensible to reduce  $A$  all the way to  $\text{rref } A$ . Suppose you can use elementary row operations to transform  $A$  into some matrix  $B$  whose determinant is easy to compute ( $B$  might be a triangular matrix, for example). Suppose you swap rows  $s$  times as you transform  $A$  into  $B$ , and you divide various rows by the scalars  $k_1, k_2, \dots, k_r$ . Then

$$\det A = (-1)^s k_1 k_2 \cdots k_r \det B.$$

### EXAMPLE 4

Find

$$\det \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

#### Solution

We go through the elimination process, keeping a note of all the row swaps and row divisions we perform (if any). In view of part b of Algorithm 2.5, we realize that it suffices to reduce  $A$  to an upper triangular matrix: There is no need to eliminate entries above the diagonal, or to make the diagonal entries equal to 1.

$$A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-\text{(I)}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-\text{(I)}} B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

## Determinants

We have performed two row swaps, so that  $\det A = (-1)^2(\det B) = 7(-1)(-2) = 14$ . We have used Theorem 1.4: The determinant of the triangular matrix  $B$  is the product of its diagonal entries. ■

### Determinant of a Product

If  $A$  and  $B$  are two  $n \times n$  matrices, what is the relationship among  $\det A$ ,  $\det B$ , and  $\det(AB)$ ? The answer is as simple as could be:

#### Theorem 2.6

##### Determinants of products and powers

If  $A$  and  $B$  are  $n \times n$  matrices and  $m$  is a positive integer, then

- a.  $\det(AB) = (\det A)(\det B)$ , and
- b.  $\det(A^m) = (\det A)^m$ .

**Proof** a. Let's first consider the case when  $A$  is invertible. In Exercise 34 the reader is asked to show that

$$\text{rref} [A \mid AB] = [I_n \mid B].$$

Suppose we swap rows  $s$  times, and we divide various rows by  $k_1, k_2, \dots, k_r$  as we perform this elimination.

Considering the left and right halves of the matrices  $[A \mid AB]$  and  $[I_n \mid B]$  separately, and using Algorithm 2.5, we conclude that

$$\det(A) = (-1)^s k_1 k_2 \cdots k_r$$

and

$$\det(AB) = (-1)^s k_1 k_2 \cdots k_r (\det B) = (\det A)(\det B),$$

as claimed. If  $A$  is not invertible, then neither is  $AB$  (think about the image), so that  $(\det A)(\det B) = 0(\det B) = 0 = \det(AB)$ , as claimed.

b. We have

$$\det(A^m) = \det(\underbrace{A \cdot A \cdots A}_{m \text{ times}}) = \underbrace{(\det A)(\det A) \cdots (\det A)}_{m \text{ times}} = (\det A)^m,$$

as claimed. ■

**EXAMPLE 5** If matrix  $A$  is similar to  $B$ , what is the relationship between  $\det A$  and  $\det B$ ?

#### Solution

By the definition of similar matrices, there exists an invertible matrix  $S$  such that  $AS = SB$ . By Theorem 2.6, we have

$$(\det A)(\det S) = (\det S)(\det B).$$

Dividing both sides by the nonzero scalar  $\det S$ , we find that

$$\det A = \det B.$$
 ■

**Theorem 2.7**

**Determinants of similar matrices**

If matrix  $A$  is similar to  $B$ , then  $\det A = \det B$ .

Conversely, if  $\det A = \det B$ , are the matrices  $A$  and  $B$  necessarily similar? See Exercise 59.

**The Determinant of an Inverse**

If  $A$  is an invertible  $n \times n$  matrix, what is the relationship between  $\det A$  and  $\det(A^{-1})$ ? By definition of an inverse, the equation  $I_n = AA^{-1}$  holds. By taking determinants of both sides and using Theorem 2.6, we find that

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

so that

$$\det(A^{-1}) = \frac{1}{\det A}.$$

It turns out that  $\det(A^{-1})$  is the reciprocal of  $\det A$ .

**Theorem 2.8**

**Determinant of an inverse**

If  $A$  is an invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}.$$

**Minors and Laplace Expansion<sup>6</sup> (Optional)**

Recall the formula

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

for the determinant of a  $3 \times 3$  matrix. See Theorem 1.2. Collecting the two terms involving  $a_{11}$  and then those involving  $a_{21}$  and  $a_{31}$ , we can write

$$\begin{aligned}\det A &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) \\ &\quad + a_{21}(a_{32}a_{13} - a_{12}a_{33}) \\ &\quad + a_{31}(a_{12}a_{23} - a_{22}a_{13}).\end{aligned}$$

(Where have we seen this formula before?)

Note that computing the determinant this way requires only 9 multiplications, compared with the 12 for Sarrus's formula. Let's analyze the structure of this formula more closely. The terms  $a_{22}a_{33} - a_{32}a_{23}$ ,  $a_{32}a_{13} - a_{12}a_{33}$ , and  $a_{12}a_{23} - a_{22}a_{13}$  are the determinants of submatrices of  $A$ , up to the signs. The expression  $a_{22}a_{33} - a_{32}a_{23}$  is the determinant of the matrix we get when we omit the first row and the first column of  $A$ :

$$\left[ \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right].$$

---

<sup>6</sup>Named after the French mathematician Pierre-Simon Marquis de Laplace (1749–1827). Laplace is perhaps best known for his investigation into the stability of the solar system. He was also a prominent member of the committee that aided in the organization of the metric system.

## Determinants

Likewise for the other summands:

$$\det A = a_{11} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - a_{21} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ + a_{31} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{32} & a_{33} \end{bmatrix}.$$

To state these observations more succinctly, we introduce some terminology.

### Definition 2.9

#### Minors

For an  $n \times n$  matrix  $A$ , let  $A_{ij}$  be the matrix obtained by omitting the  $i$ th row and the  $j$ th column of  $A$ . The determinant of the  $(n-1) \times (n-1)$  matrix  $A_{ij}$  is called a *minor* of  $A$ .

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix},$$

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

We can now represent the determinant of a  $3 \times 3$  matrix more succinctly:

$$\det A = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}).$$

This representation of the determinant is called the *Laplace expansion* (or *cofactor expansion*) of  $\det A$  down the first column. Likewise, we can expand along the first row [since  $\det(A^T) = \det A$ ]:

$$\det A = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}).$$

In fact, we can expand along any row or down any column. (We can verify this directly or argue in terms of row or column swaps.) For example, the Laplace expansion down the second column is

$$\det A = -a_{12} \det(A_{12}) + a_{22} \det(A_{22}) - a_{32} \det(A_{32}),$$

and the Laplace expansion along the third row is

$$\det A = a_{31} \det(A_{31}) - a_{32} \det(A_{32}) + a_{33} \det(A_{33}).$$

The rule for the signs is as follows: The summand  $a_{ij} \det(A_{ij})$  has a negative sign if the sum of the two indices,  $i + j$ , is odd. The signs follow a checkerboard pattern:

## Determinants

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

We can generalize the Laplace expansion to  $n \times n$  matrices.

We will focus on the expansion down the  $j$ th column of  $A$ . The formula for the expansion along the  $i$ th row then follows from the fact that  $\det A = \det(A^T)$ , Theorem 2.1.

Consider a pattern  $P$  in an  $n \times n$  matrix  $A$ . For a fixed  $j$ , the pattern  $P$  will contain exactly one entry  $a_{ij}$  in the  $j$ th column of  $A$ . Let  $P_{ij}$  be the pattern in  $A_{ij}$  that contains the same entries as  $P$ , except for the omitted entry  $a_{ij}$ . See the example below, where  $j = 4$  and  $i = 3$ .

Note that  $\text{prod } P = a_{ij} \text{prod}(P_{ij})$ . In Exercise 68 we see that  $\text{sgn } P = (-1)^{i+j} \text{sgn}(P_{ij})$ , so that  $(\text{sgn } P)(\text{prod } P) = (-1)^{i+j} a_{ij} (\text{sgn } P_{ij})(\text{prod } P_{ij})$ . Verify this formula in the example above, where  $(\text{sgn } P)(\text{prod } P) = 7! = 5,040$ . Now we can compute the determinant of  $A$ , collecting the patterns containing  $a_{1j}$ , then those containing  $a_{2j}$ , and so forth, just as we did earlier in the case of a  $3 \times 3$  matrix, with  $j = 1$ :

$$\begin{aligned} \det A &= \sum (\text{sgn } P)(\text{prod } P) = \sum_{i=1}^n \sum_{P \text{ contains } a_{ij}} (\text{sgn } P)(\text{prod } P) \\ &= \sum_{i=1}^n \sum_{P \text{ contains } a_{ij}} (-1)^{i+j} a_{ij} (\text{sgn } P_{ij})(\text{prod } P_{ij}) \\ &= \sum_{i=1}^n (-1)^{i+j} a_{ij} \sum_{P \text{ contains } a_{ij}} (\text{sgn } P_{ij})(\text{prod } P_{ij}) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}). \end{aligned}$$

### Theorem 2.10

#### Laplace expansion (or cofactor expansion)

We can compute the determinant of an  $n \times n$  matrix  $A$  by Laplace expansion down any column or along any row.

Expansion down the  $j$ th column:

$$\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Expansion along the  $i$ th row:

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

## Determinants

Again, the signs follow a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**EXAMPLE 6** Use Laplace expansion to compute  $\det A$  for

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix}.$$

### Solution

Looking for rows or columns with as many zeros as possible, we choose the second column:

$$\begin{aligned} \det A &= -a_{12} \det(A_{12}) + a_{22} \det(A_{22}) - a_{32} \det(A_{32}) + a_{42} \det(A_{42}) \\ &= 1 \det \begin{bmatrix} 1 & 0 & 1 & 2 \\ \cancel{9} & \cancel{1} & \cancel{3} & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ \cancel{9} & \cancel{2} & \cancel{2} & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 5 & 0 & 3 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 9 & 2 \\ 5 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & 1 \\ 9 & 2 \end{bmatrix} - 2 \left( 2 \det \begin{bmatrix} 9 & 3 \\ 5 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & 1 \\ 9 & 3 \end{bmatrix} \right) \\ &\quad \nearrow \\ &\text{Expand down} \\ &\text{the last} \\ &\text{column} \\ &= -20 - 21 - 2(-30 - 18) = 55. \quad \blacksquare \end{aligned}$$

Computing the determinant using Laplace expansion is a bit more efficient than using the definition of the determinant, but a lot less efficient than Gauss–Jordan elimination.

### The Determinant of a Linear Transformation (Optional)

If  $T(\vec{x}) = A\vec{x}$  is a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , then it is natural to define the determinant of  $T$  as the determinant of matrix  $A$ :

$$\det T = \det A.$$

This definition makes sense in view of the fact that an  $n \times n$  matrix is essentially the same thing as a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

If  $T$  is a linear transformation from  $V$  to  $V$ , where  $V$  is a finite-dimensional linear space, then we can introduce coordinates to define the determinant of  $T$ . If  $\mathfrak{B}$  is a basis of  $V$  and  $B$  is the  $\mathfrak{B}$ -matrix of  $T$ , then we define

$$\det T = \det B.$$

We need to think about one issue though. If you pick another basis,  $\mathfrak{A}$ , of  $V$  and consider the  $\mathfrak{A}$ -matrix  $A$  of  $T$ , will you end up with the same determinant; that is, will  $\det A$  equal  $\det B$ ?

Fortunately, there is no reason to worry. We know that matrix  $A$  is similar to  $B$ , so that determinants  $\det A$  and  $\det B$  are indeed equal, by Theorem 2.7.

### Definition 2.11

#### The determinant of a linear transformation

Consider a linear transformation  $T$  from  $V$  to  $V$ , where  $V$  is a finite-dimensional linear space. If  $\mathfrak{B}$  is a basis of  $V$  and  $B$  is the  $\mathfrak{B}$ -matrix of  $T$ , then we define

$$\det T = \det B.$$

This determinant is independent of the basis  $\mathfrak{B}$  we choose.

### EXAMPLE 7

Let  $V$  be the space spanned by functions  $\cos(2x)$  and  $\sin(2x)$ . Find the determinant of the linear transformation  $D(f) = f'$  from  $V$  to  $V$ .

#### Solution

The matrix  $B$  of  $D$  with respect to the basis  $\cos(2x), \sin(2x)$  is

$$B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},$$

so that

$$\det D = \det B = 4.$$



### Determinants: Focus on History

Most of the results of this and the preceding section (with the notable exception of the product rule, Theorem 2.6) were known to Gottfried Wilhelm von Leibniz (1646–1716). In 1678, while studying the solutions of systems of three equations in three unknowns, he used a method that amounts to expanding the determinant of a  $3 \times 3$  matrix down the third column. Later that year he attempted the same for  $4 \times 4$  matrices but made a sign error in his computations. In a manuscript of 1684, however, Leibniz states the sign rule for determinants in the correct, general form. His work remained unpublished and was discovered only after 1850, through careful examination of his manuscripts.

Meanwhile, the greatest mathematician of ancient Japan, Seki Kowa (1642–1708), came up with remarkably similar results, in his manuscript *Kai Fukudai no Ho*. It appears that he found the correct sign rule for determinants of  $4 \times 4$  matrices. However, it is hard to assess his work, as he was an extremely secretive fellow. Florian Cajori, the eminent Swiss historian of mathematics, puts it this way:

Seki was a great teacher who attracted many gifted pupils. Like Pythagoras, he discouraged divulgence of mathematical discoveries made by himself and his school. For that reason it is difficult to determine with certainty the exact origin and nature of some of the discoveries attributed to him. He is said to have left hundreds of manuscripts; the translations of only a few of them still remain. (Cajori, *A History of Mathematics*, 1919)

Apparently without detailed knowledge of Leibniz's work, the Swiss mathematician Gabriel Cramer (1704–1752) developed the general theory of determinants (still without the product rule, though) and published his results in the *Introduction*

## Determinants

*à l'analyse des lignes courbes algébriques* (1750). The mathematical community quickly discovered the power of this new technique, and during the next 100 years many mathematicians made important advances: Bézout, Vandermonde, Laplace, Binet, and Cayley, to name just a few. In 1812, Augustin Louis Cauchy (1789–1857) contributed the product rule. In the 1880s, Karl Weierstrass (1817–1897) offered an axiomatic definition of the determinant that allows a more elegant exposition of the theory. See Exercise 55.

## EXERCISES 2

Use Gaussian elimination to find the determinant of the matrices  $A$  in Exercises 1 through 10.

1. 
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 8 \\ -2 & -4 & 0 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

9. 
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

10. 
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

Consider a  $4 \times 4$  matrix  $A$  with rows  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ . If  $\det(A) = 8$ , find the determinants in Exercises 11 through 16.

11. 
$$\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ -9\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$$

12. 
$$\det \begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix}$$

13. 
$$\det \begin{bmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix}$$

14. 
$$\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$$

15. 
$$\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + \vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{bmatrix}$$

16. 
$$\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$$

Find the determinants of the linear transformations in Exercises 17 through 28.

17.  $T(f) = 2f + 3f'$  from  $P_2$  to  $P_2$

18.  $T(f(t)) = f(3t - 2)$  from  $P_2$  to  $P_2$

19.  $T(f(t)) = f(-t)$  from  $P_2$  to  $P_2$

20.  $L(A) = A^T$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$

21.  $T(f(t)) = f(-t)$  from  $P_3$  to  $P_3$

22.  $T(f(t)) = f(-t)$  from  $P_n$  to  $P_n$

23.  $L(A) = A^T$  from  $\mathbb{R}^{n \times n}$  to  $\mathbb{R}^{n \times n}$

24.  $T(z) = (2 + 3i)z$  from  $\mathbb{C}$  to  $\mathbb{C}$

25.  $T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M$  from the space  $V$  of upper triangular  $2 \times 2$  matrices to  $V$

26.  $T(M) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} M + M \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$  from the space  $V$  of symmetric  $2 \times 2$  matrices to  $V$

27.  $T(f) = af' + bf''$ , where  $a$  and  $b$  are arbitrary constants, from the space  $V$  spanned by  $\cos(x)$  and  $\sin(x)$  to  $V$

28.  $T(\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{v}$  from the plane  $V$  given by

$$x_1 + 2x_2 + 3x_3 = 0$$
 to  $V$

29. Let  $P_n$  be the  $n \times n$  matrix whose entries are all ones, except for zeros directly below the main diagonal; for example,

$$P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Find the determinant of  $P_n$ .

30. Consider two distinct real numbers,  $a$  and  $b$ . We define the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}.$$

## Determinants

- a. Show that  $f(t)$  is a quadratic function. What is the coefficient of  $t^2$ ?
- b. Explain why  $f(a) = f(b) = 0$ . Conclude that  $f(t) = k(t - a)(t - b)$ , for some constant  $k$ . Find  $k$ , using your work in part (a).
- c. For which values of  $t$  is the matrix invertible?
- 31.** *Vandermonde determinants* (introduced by Alexandre-Théophile Vandermonde). Consider distinct real numbers  $a_0, a_1, \dots, a_n$ . We define the  $(n + 1) \times (n + 1)$  matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that

$$\det(A) = \prod_{i>j} (a_i - a_j),$$

the product of all differences  $(a_i - a_j)$ , where  $i$  exceeds  $j$ .

- a. Verify this formula in the case of  $n = 1$ .
- b. Suppose the Vandermonde formula holds for  $n - 1$ . You are asked to demonstrate it for  $n$ . Consider the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}.$$

Explain why  $f(t)$  is a polynomial of  $n$ th degree. Find the coefficient  $k$  of  $t^n$  using Vandermonde's formula for  $a_0, \dots, a_{n-1}$ . Explain why

$$f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0.$$

Conclude that

$$f(t) = k(t - a_0)(t - a_1) \cdots (t - a_{n-1})$$

for the scalar  $k$  you found above. Substitute  $t = a_n$  to demonstrate Vandermonde's formula.

- 32.** Use Exercise 31 to find

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}.$$

Do not use technology.

- 33.** For  $n$  distinct scalars  $a_1, a_2, \dots, a_n$ , find

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$

- 34. a.** For an invertible  $n \times n$  matrix  $A$  and an arbitrary  $n \times n$  matrix  $B$ , show that

$$\text{rref} [A \mid AB] = [I_n \mid B].$$

*Hint:* The left part of  $\text{rref} [A \mid AB]$  is  $\text{rref}(A) = I_n$ . Write  $\text{rref} [A \mid AB] = [I_n \mid M]$ ; we have to show that  $M = B$ . To demonstrate this, note that the columns of matrix

$$\begin{bmatrix} B \\ -I_n \end{bmatrix}$$

are in the kernel of  $[A \mid AB]$  and therefore in the kernel of  $[I_n \mid M]$ .

- b.** What does the formula

$$\text{rref} [A \mid AB] = [I_n \mid B]$$

tell you if  $B = A^{-1}$ ?

- 35.** Consider two distinct points  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  in the plane. Explain why the solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  of the equation

$$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \end{bmatrix} = 0$$

form a line and why this line goes through the two points  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ .

- 36.** Consider three distinct points  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  in the plane. Describe the set of all points  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  satisfying the equation

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & a_1 & b_1 & c_1 \\ x_2 & a_2 & b_2 & c_2 \\ x_1^2 + x_2^2 & a_1^2 + a_2^2 & b_1^2 + b_2^2 & c_1^2 + c_2^2 \end{bmatrix} = 0.$$

- 37.** Consider an  $n \times n$  matrix  $A$  such that both  $A$  and  $A^{-1}$  have integer entries. What are the possible values of  $\det A$ ?

- 38.** If  $\det A = 3$  for some  $n \times n$  matrix, what is  $\det(A^T A)$ ?

- 39.** If  $A$  is an invertible matrix, what can you say about the sign of  $\det(A^T A)$ ?

## Determinants

- 40.** If  $A$  is an orthogonal matrix, what are the possible values of  $\det A$ ?

- 41.** Consider a skew-symmetric  $n \times n$  matrix  $A$ , where  $n$  is odd. Show that  $A$  is noninvertible, by showing that  $\det A = 0$ .

- 42.** Consider an  $n \times m$  matrix

$$A = QR,$$

where  $Q$  is an  $n \times m$  matrix with orthonormal columns and  $R$  is an upper triangular  $m \times m$  matrix with positive diagonal entries  $r_{11}, \dots, r_{mm}$ . Express  $\det(A^T A)$  in terms of the scalars  $r_{ii}$ . What can you say about the sign of  $\det(A^T A)$ ?

- 43.** Consider two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$ . Form the matrix  $A = [\vec{v} \ \vec{w}]$ . Express  $\det(A^T A)$  in terms of  $\|\vec{v}\|$ ,  $\|\vec{w}\|$ , and  $\vec{v} \cdot \vec{w}$ . What can you say about the sign of the result?

- 44.** *The cross product in  $\mathbb{R}^n$ .* Consider the vectors  $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$  in  $\mathbb{R}^n$ . The transformation

$$T(\vec{x}) = \det \begin{bmatrix} | & | & | & & | \\ \vec{x} & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}$$

is linear. Therefore, there exists a unique vector  $\vec{u}$  in  $\mathbb{R}^n$  such that

$$T(\vec{x}) = \vec{x} \cdot \vec{u}$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ . This vector  $\vec{u}$  is called the *cross product* of  $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$ , written as

$$\vec{u} = \vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n.$$

In other words, the cross product is defined by the fact that

$$\begin{aligned} \vec{x} \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \\ = \det \begin{bmatrix} | & | & | & & | \\ \vec{x} & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}, \end{aligned}$$

for all  $\vec{x}$  in  $\mathbb{R}^n$ . Note that the cross product in  $\mathbb{R}^n$  is defined for  $n - 1$  vectors only. (For example, you cannot form the cross product of just two vectors in  $\mathbb{R}^4$ .) Since the  $i$ th component of a vector  $\vec{w}$  is  $\vec{e}_i \cdot \vec{w}$ , we can find the cross product by components as follows:

$$\begin{aligned} i\text{th component of } \vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \\ = \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) \\ = \det \begin{bmatrix} | & | & | & & | \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}. \end{aligned}$$

- a.** When is  $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = \vec{0}$ ? Give your answer in terms of linear independence.

- b.** Find  $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n$ .

- c.** Show that  $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n$  is orthogonal to all the vectors  $\vec{v}_i$ , for  $i = 2, \dots, n$ .

- d.** What is the relationship between  $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n$  and  $\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$ ? (We swap the first two factors.)

- e.** Express  $\det [\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \ \vec{v}_2 \ \vec{v}_3 \ \cdots \ \vec{v}_n]$  in terms of  $\|\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n\|$ .

- f.** How do we know that the cross product of two vectors in  $\mathbb{R}^3$ , as defined here, is the same as the standard cross product in  $\mathbb{R}^3$ ?

- 45.** Find the derivative of the function

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

- 46.** Given some numbers  $a, b, c, d, e$ , and  $f$  such that

$$\det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 7 \quad \text{and} \quad \det \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} = 11,$$

- a.** Find

$$\det \begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix}.$$

- b.** Find

$$\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix}.$$

- 47.** Is the function

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + bc$$

linear in the rows and columns of the matrix?

- 48.** Consider the linear transformation

$$T(\vec{x}) = \det \begin{bmatrix} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{x} \\ | & | & & | & | \end{bmatrix}$$

from  $\mathbb{R}^n$  to  $\mathbb{R}$ , where  $\vec{v}_1, \dots, \vec{v}_{n-1}$ , are linearly independent vectors in  $\mathbb{R}^n$ . Describe image and kernel of this transformation, and determine their dimensions.

- 49.** Give an example of a  $3 \times 3$  matrix  $A$  with all nonzero entries such that  $\det A = 13$ .

## Determinants

- 50.** Find the determinant of the matrix

$$M_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

for arbitrary  $n$ . (The  $ij$ th entry of  $M_n$  is the minimum of  $i$  and  $j$ .)

- 51.** Find the determinant of the  $(2n) \times (2n)$  matrix

$$A = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

- 52.** Consider a  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with column vectors

$$\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

We define the linear transformation

$$T(\vec{x}) = \begin{bmatrix} \det[\vec{x} & \vec{w}] \\ \det[\vec{v} & \vec{x}] \end{bmatrix}$$

from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

- a. Find the standard matrix  $B$  of  $T$ . (Write the entries of  $B$  in terms of the entries  $a, b, c, d$  of  $A$ .)
  - b. What is the relationship between the determinants of  $A$  and  $B$ ?
  - c. Show that  $BA$  is a scalar multiple of  $I_2$ . What about  $AB$ ?
  - d. If  $A$  is noninvertible (but nonzero), what is the relationship between the image of  $A$  and the kernel of  $B$ ? What about the kernel of  $A$  and the image of  $B$ ?
  - e. If  $A$  is invertible, what is the relationship between  $B$  and  $A^{-1}$ ?
- 53.** Consider an invertible  $2 \times 2$  matrix  $A$  with integer entries.
- a. Show that if the entries of  $A^{-1}$  are integers, then  $\det A = 1$  or  $\det A = -1$ .
  - b. Show the converse: If  $\det A = 1$  or  $\det A = -1$ , then the entries of  $A^{-1}$  are integers.

- 54.** Let  $A$  and  $B$  be  $2 \times 2$  matrices with integer entries such that  $A, A + B, A + 2B, A + 3B$ , and  $A + 4B$  are all invertible matrices whose inverses have integer entries. Show that  $A + 5B$  is invertible and that its inverse has integer entries. This question was in the William Lowell Putnam Mathematical Competition in 1994. Hint: Consider the function  $f(t) = (\det(A+tB))^2 - 1$ . Show that this is a polynomial; what can you say about its degree? Find the values  $f(0), f(1), f(2), f(3), f(4)$ , using Exercise 53. Now you can determine  $f(t)$  by using a

familiar result: If a polynomial  $f(t)$  of degree  $\leq m$  has more than  $m$  zeros, then  $f(t) = 0$  for all  $t$ .

- 55.** For a fixed positive integer  $n$ , let  $D$  be a function which assigns to any  $n \times n$  matrix  $A$  a number  $D(A)$  such that

- a.  $D$  is linear in the rows (see Theorem 2.2),
- b.  $D(B) = -D(A)$  if  $B$  is obtained from  $A$  by a row swap, and
- c.  $D(I_n) = 1$ .

Show that  $D(A) = \det(A)$  for all  $n \times n$  matrices  $A$ .

Hint: Consider  $E = \text{rref } A$ . Think about the relationship between  $D(A)$  and  $D(E)$ , mimicking Algorithm 2.5.

The point of this exercise is that  $\det(A)$  can be characterized by the three properties a, b, and c; the determinant can, in fact, be *defined* in terms of these properties. Ever since this approach was first presented in the 1880s by the German mathematician Karl Weierstrass (1817–1897), this definition has been generally used in advanced linear algebra courses because it allows a more elegant presentation of the theory of determinants.

- 56.** Use the characterization of the determinant given in Exercise 55 to show that

$$\det(AM) = (\det A)(\det M).$$

Hint: For a fixed invertible matrix  $M$ , consider the function

$$D(A) = \frac{\det(AM)}{\det M}.$$

Show that this function has the three properties a, b, and c listed in Exercise 55, and therefore  $D(A) = \det A$ .

- 57.** Consider a linear transformation  $T$  from  $\mathbb{R}^{m+n}$  to  $\mathbb{R}^m$ . The matrix  $A$  of  $T$  can be written in block form as  $A = [A_1 \ A_2]$ , where  $A_1$  is  $m \times m$  and  $A_2$  is  $m \times n$ . Suppose that  $\det(A_1) \neq 0$ . Show that for every vector  $\vec{x}$  in  $\mathbb{R}^n$  there exists a unique  $\vec{y}$  in  $\mathbb{R}^m$  such that

$$T \begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix} = \vec{0}.$$

Show that the transformation

$$\vec{x} \rightarrow \vec{y}$$

from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is linear, and find its matrix  $M$  (in terms of  $A_1$  and  $A_2$ ). (This is the linear version of the *implicit function theorem* of multivariable calculus.)

- 58.** Find the matrix  $M$  introduced in Exercise 57 for the linear transformation

$$T(\vec{v}) = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 7 & 4 & 3 \end{bmatrix} \vec{v}.$$

You can either follow the approach outlined in Exercise 57 or use Gaussian elimination, expressing the leading variables  $y_1, y_2$  in terms of the free variables  $x_1, x_2$ , where

## Determinants

$$\vec{v} = \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \end{bmatrix}.$$

Note that this procedure amounts to finding the kernel of  $T$ , in the familiar way; we just interpret the result somewhat differently.

- 59.** If the equation  $\det A = \det B$  holds for two  $n \times n$  matrices  $A$  and  $B$ , is  $A$  necessarily similar to  $B$ ?
- 60.** Consider an  $n \times n$  matrix  $A$ . Show that swapping the  $i$ th and  $j$ th rows of  $A$  (where  $i < j$ ) amounts to performing  $2(j-i)-1$  swaps of adjacent rows.
- 61.** Consider  $n \times n$  matrices  $A$ ,  $B$ ,  $C$ , and  $D$ , where  $A$  is invertible and commutes with  $C$ . Show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB).$$

*Hint:* Consider the product

$$\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

- 62.** Consider  $n \times n$  matrices  $A$ ,  $B$ ,  $C$ , and  $D$  such that

$$\text{rank}(A) = \text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n.$$

Show that

a.  $D = CA^{-1}B$ , and

b. The  $2 \times 2$  matrix  $\begin{bmatrix} \det(A) & \det(B) \\ \det(C) & \det(D) \end{bmatrix}$  is noninvertible. *Hint:* Consider the product

$$\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

- 63.** Show that more than  $n! = 1 \cdot 2 \cdot 3 \cdots \cdots \cdot n$  multiplications are required to compute the determinant of an  $n \times n$  matrix by Laplace expansion (for  $n > 2$ ).

- 64.** Show that fewer than  $e \cdot n!$  algebraic operations (additions and multiplications) are required to compute the determinant of an  $n \times n$  matrix by Laplace expansion.

*Hint:* Let  $L_n$  be the number of operations required to compute the determinant of a “general”  $n \times n$  matrix by Laplace expansion. Find a formula expressing  $L_n$  in terms of  $L_{n-1}$ . Use this formula to show, by induction, that

$$\frac{L_n}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}.$$

Use the Taylor series of  $e^x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , to show that the right-hand side of this equation is less than  $e$ .

- 65.** Let  $M_n$  be the  $n \times n$  matrix with 1's on the main diagonal and directly above the main diagonal, -1's directly below the main diagonal, and 0's elsewhere. For example,

$$M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Let  $d_n = \det(M_n)$ .

- a. For  $n \geq 3$ , find a formula expressing  $d_n$  in terms of  $d_{n-1}$  and  $d_{n-2}$ .
- b. Find  $d_1, d_2, d_3, d_4$ , and  $d_{10}$ .
- c. For which positive integers  $n$  is the matrix  $M_n$  invertible?
- 66.** Let  $M_n$  be the matrix with all 1's along the main diagonal, directly above the main diagonal, and directly below the diagonal, and 0's everywhere else. For example,
- $$M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$
- Let  $d_n = \det(M_n)$ .

a. Find a formula expressing  $d_n$  in terms of  $d_{n-1}$  and  $d_{n-2}$ , for positive integers  $n \geq 3$ .

b. Find  $d_1, d_2, \dots, d_8$ .

c. What is the relationship between  $d_n$  and  $d_{n+3}$ ? What about  $d_n$  and  $d_{n+6}$ ?

d. Find  $d_{100}$ .

**67.** Consider a pattern  $P$  in an  $n \times n$  matrix, and choose an entry  $a_{ij}$  in this pattern. Show that the number of inversions involving  $a_{ij}$  is even if  $(i+j)$  is even and odd if  $(i+j)$  is odd. *Hint:* Suppose there are  $k$  entries in the pattern to the left and above  $a_{ij}$ . Express the number of inversions involving  $a_{ij}$  in terms of  $k$ .

**68.** Using the terminology introduced in the proof of Theorem 2.10, show that  $\text{sgn}P = (-1)^{i+j}\text{sgn}(P_{ij})$ . See Exercise 67.

**69.** Let  $G$  be the set of all integers  $x$  that can be written as the sum of the squares of two integers,  $x = a^2 + b^2$ . For example,  $13 = 3^2 + 2^2$  is in  $G$ , while 7 fails to be in  $G$ .

a. List all integers  $x \leq 10$  that are in  $G$ .

b. Show that  $G$  is closed under multiplication: If  $x = a^2 + b^2$  and  $y = c^2 + d^2$  are in  $G$ , then so is their product  $xy$ . *Hint:* Consider the matrices  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ ,  $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ , their product, and their determinants.

c. Given that  $2642 = 31^2 + 41^2$  and  $3218 = 37^2 + 43^2$ , write  $8,501,956 = 2642 \cdot 3218$  as the sum of the squares of two positive integers. You may use technology.

**70.** Throughout this exercise, consider the Fibonacci sequence  $f_0, f_1, f_2, \dots$  recursively defined by  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_{n+2} = f_n + f_{n+1}$  for all  $n = 0, 1, 2, \dots$
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- a. Find the Fibonacci numbers  $f_0, f_1, \dots, f_8$ .
- b. Consider the matrix  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ . Prove by induction that  $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$  for all  $n = 1, 2, \dots$
- c. Show that  $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ . This equation is known as *Cassini's identity*; it was discovered by the Italian/French mathematician and astronomer Giovanni Domenico Cassini, 1625–1712.

### 3 Geometrical Interpretations of the Determinant; Cramer's Rule

We now present several ways to think about the determinant in geometrical terms. Here is a preliminary exercise.

**EXAMPLE 1** What are the possible values of the determinant of an orthogonal matrix  $A$ ?

#### Solution

We know that

$$A^T A = I_n$$

Taking the determinants of both sides and using Theorems 2.1 and 2.6, we find that

$$\det(A^T A) = \det(A^T) \det A = (\det A)^2 = 1.$$

Therefore,  $\det A$  is either 1 or  $-1$ . ■

#### Theorem 3.1

The determinant of an orthogonal matrix is either 1 or  $-1$ .

For example,

$$\det \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} = 1,$$

representing a rotation, and

$$\det \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} = -1,$$

representing a reflection about a line.

#### Definition 3.2

#### Rotation matrices

An orthogonal  $n \times n$  matrix  $A$  with  $\det A = 1$  is called a *rotation matrix*, and the linear transformation  $T(\vec{x}) = A\vec{x}$  is called a *rotation*.

### The Determinant as Area and Volume

The geometrical interpretation of the determinant of a  $2 \times 2$  matrix  $A$  is based on the formula

$$\det A = \det [\vec{v}_1 \quad \vec{v}_2] = \|\vec{v}_1\| \sin \theta \|\vec{v}_2\|,$$

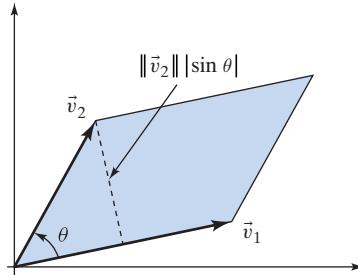
where  $\theta$  is the oriented angle from  $\vec{v}_1$  to  $\vec{v}_2$ .

Figure 1a illustrates the fact that

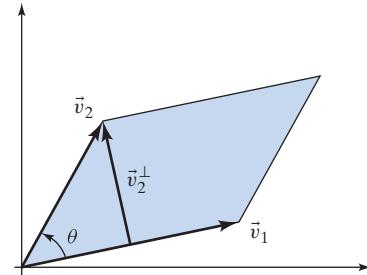
$$|\det A| = |\det [\vec{v}_1 \quad \vec{v}_2]| = \|\vec{v}_1\| |\sin \theta| \|\vec{v}_2\|$$

## Determinants

is the area of the parallelogram spanned by the vectors  $\vec{v}_1$  and  $\vec{v}_2$ . We will focus on interpreting the absolute value.



**Figure 1a**



**Figure 1b**

Alternatively, we can write  $|\det A|$  in terms of the Gram–Schmidt process. Observe that  $|\sin \theta| \|\vec{v}_2\| = \|\vec{v}_2^\perp\|$ , where  $\vec{v}_2^\perp$  denotes the component of  $\vec{v}_2$  perpendicular to  $\vec{v}_1$ . See Figure 1b. Thus

$$|\det A| = \|\vec{v}_1\| \|\vec{v}_2^\perp\|.$$

More generally, consider an invertible  $n \times n$  matrix

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}.$$

We can write  $A = QR$ , where  $Q$  is an orthogonal matrix and  $R$  is an upper triangular matrix whose diagonal entries are

$$r_{11} = \|\vec{v}_1\| \quad \text{and} \quad r_{jj} = \|\vec{v}_j^\perp\|, \quad \text{for } j \geq 2.$$

We conclude that

$$|\det A| = |\det Q| |\det R| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_n^\perp\|.$$

Indeed,  $|\det Q| = 1$  by Theorem 3.1, and the determinant of  $R$  is the product of its diagonal entries, by Theorem 1.4.

### Theorem 3.3

#### The determinant in terms of the columns

If  $A$  is an  $n \times n$  matrix with columns  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ , then

$$|\det A| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_n^\perp\|,$$

where  $\vec{v}_k^\perp$  is the component of  $\vec{v}_k$  perpendicular to  $\text{span}(\vec{v}_1, \dots, \vec{v}_{k-1})$ .

The proof of Theorem 3.3 in the case of a noninvertible matrix  $A$  is left as Exercise 8.

As an example, consider the  $3 \times 3$  matrix

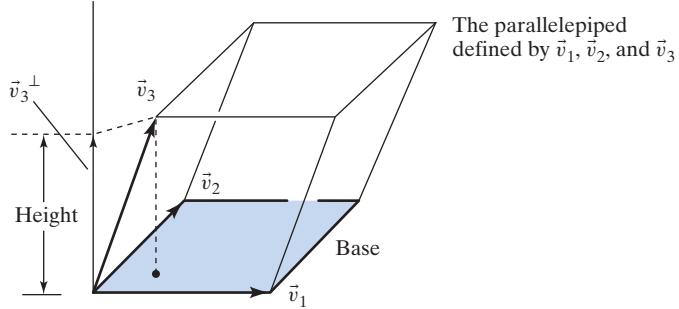
$$A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix},$$

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with

$$|\det A| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \|\vec{v}_3^\perp\|.$$

As in Figure 1b,  $\|\vec{v}_1\| \|\vec{v}_2^\perp\|$  is the area of the parallelogram defined by  $\vec{v}_1$  and  $\vec{v}_2$ . Now consider the parallelepiped defined by  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  (i.e., the set of all vectors of the form  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ , where the  $c_i$  are between 0 and 1, as shown in Figure 2).



**Figure 2**

The volume of this parallelepiped is

$$\text{Volume} = \underbrace{\|\vec{v}_1\| \|\vec{v}_2^\perp\|}_{\text{Base area}} \underbrace{\|\vec{v}_3^\perp\|}_{\text{Height}} = |\det A|$$

(by Theorem 3.3).

### Theorem 3.4

#### Volume of a parallelepiped in $\mathbb{R}^3$

Consider a  $3 \times 3$  matrix  $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ . Then the volume of the parallelepiped defined by  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  is  $|\det A|$ .

For a geometrical interpretation of the *sign* of  $\det A$ , see Exercises 19 through 21.

Let us generalize these observations to higher dimensions.

### Definition 3.5

#### Parallelepipeds in $\mathbb{R}^n$

Consider the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . The  $m$ -parallelepiped defined by the vectors  $\vec{v}_1, \dots, \vec{v}_m$  is the set of all vectors in  $\mathbb{R}^n$  of the form  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$ , where  $0 \leq c_i \leq 1$ . The  $m$ -volume  $V(\vec{v}_1, \dots, \vec{v}_m)$  of this  $m$ -parallelepiped is defined recursively by  $V(\vec{v}_1) = \|\vec{v}_1\|$  and

$$V(\vec{v}_1, \dots, \vec{v}_m) = V(\vec{v}_1, \dots, \vec{v}_{m-1}) \|\vec{v}_m^\perp\|.$$

Note that this formula for the  $m$ -volume generalizes the formula

$$(\text{base})(\text{height})$$

we used to compute the area of a parallelogram (i.e., a 2-parallelepiped) and the volume of a 3-parallelepiped in  $\mathbb{R}^3$ . Take another look at Figures 1 and 2.

## Determinants

Alternatively, we can write the formula for the  $m$ -volume as

$$V(\vec{v}_1, \dots, \vec{v}_m) = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_m^\perp\|.$$

Let  $A$  be the  $n \times m$  matrix whose columns are  $\vec{v}_1, \dots, \vec{v}_m$ . If the columns of  $A$  are linearly independent, we can consider the  $QR$  factorization  $A = QR$ . Then,  $A^T A = R^T Q^T QR = R^T R$ , because  $Q^T Q = I_m$  (since the columns of  $Q$  are orthonormal). Therefore,

$$\begin{aligned} \det(A^T A) &= \det(R^T R) = (\det R)^2 = (r_{11} r_{22} \cdots r_{mm})^2 \\ &= (\|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_m^\perp\|)^2 = (V(\vec{v}_1, \dots, \vec{v}_m))^2. \end{aligned}$$

We can conclude

### Theorem 3.6

#### Volume of a parallelepiped in $\mathbb{R}^n$

Consider the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ . Then the  $m$ -volume of the  $m$ -parallelepiped defined by the vectors  $\vec{v}_1, \dots, \vec{v}_m$  is

$$\sqrt{\det(A^T A)},$$

where  $A$  is the  $n \times m$  matrix with columns  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ .

In particular, if  $m = n$ , this volume is

$$|\det A|.$$

Compare this with Theorem 3.3.

We leave it to the reader to verify Theorem 3.6 for linearly dependent vectors  $\vec{v}_1, \dots, \vec{v}_m$ . See Exercise 15.

As a simple example, consider the 2-volume (i.e., area) of the 2-parallelepiped (i.e., parallelogram) defined by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in  $\mathbb{R}^3$ . By Theorem 3.6, this area is

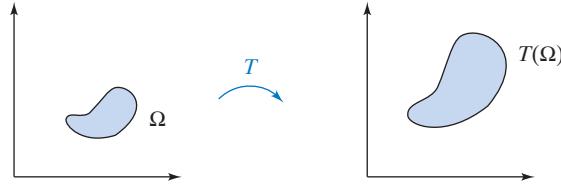
$$\sqrt{\det \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \right)} = \sqrt{\det \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}} = \sqrt{6}.$$

In this special case, we can also determine the area as the norm  $\|\vec{v}_1 \times \vec{v}_2\|$  of the cross product of the two vectors.

### The Determinant as Expansion Factor

Consider a linear transformation  $T$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . A linear transformation  $T$  affects various geometric quantities such as lengths and angles. We can ask how a linear transformation  $T$  affects the *area* of a region  $\Omega$  in the plane. See Figure 3.

## Determinants

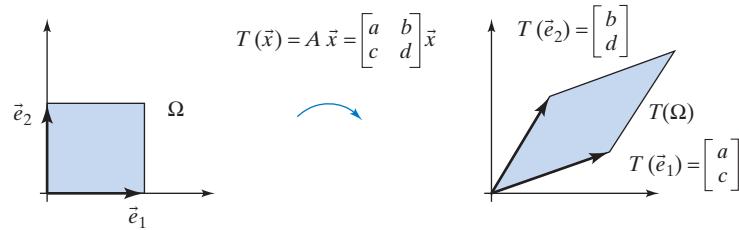


**Figure 3**

We might be interested in finding the *expansion factor*, the ratio

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}.$$

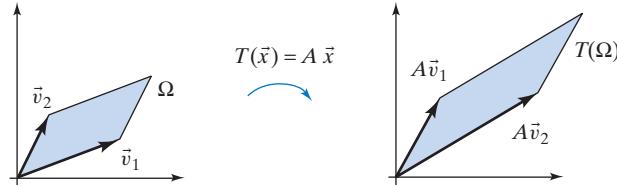
The simplest example is the unit square  $\Omega$  shown in Figure 4.



**Figure 4**

Since the area of  $\Omega$  is 1 here, the expansion factor is simply the area of the parallelogram  $T(\Omega)$ , which is  $|\det A|$ .

More generally, let  $\Omega$  be the parallelogram defined by  $\vec{v}_1$  and  $\vec{v}_2$ , as shown in Figure 5.



**Figure 5**

Let  $B = [\vec{v}_1 \quad \vec{v}_2]$ . Then

$$\text{area of } \Omega = |\det B|,$$

and

$$\text{area of } T(\Omega) = |\det [A\vec{v}_1 \quad A\vec{v}_2]| = |\det(AB)| = |\det A||\det B|,$$

and the expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det A||\det B|}{|\det B|} = |\det A|.$$

It is remarkable that the linear transformation  $T(\vec{x}) = A\vec{x}$  expands the area of *all* parallelograms by the same factor, namely,  $|\det A|$ .

**Theorem 3.7**

**Expansion factor**

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Then  $|\det A|$  is the *expansion factor*

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}$$

of  $T$  on parallelograms  $\Omega$ .

Likewise, for a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ,  $|\det A|$  is the expansion factor of  $T$  on  $n$ -parallelepipeds:

$$V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A| V(\vec{v}_1, \dots, \vec{v}_n),$$

for all vectors  $\vec{v}_1, \dots, \vec{v}_n$  in  $\mathbb{R}^n$ .

This interpretation allows us to think about the formulas  $\det(A^{-1}) = 1/\det A$  and  $\det(AB) = (\det A)(\det B)$  from a geometric point of view. See Figures 6 and 7.

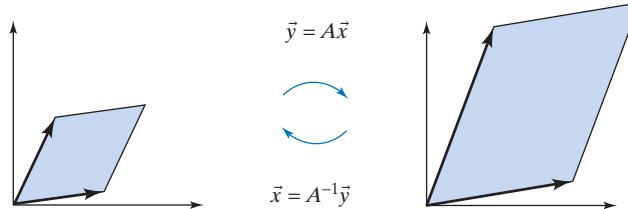


Figure 6

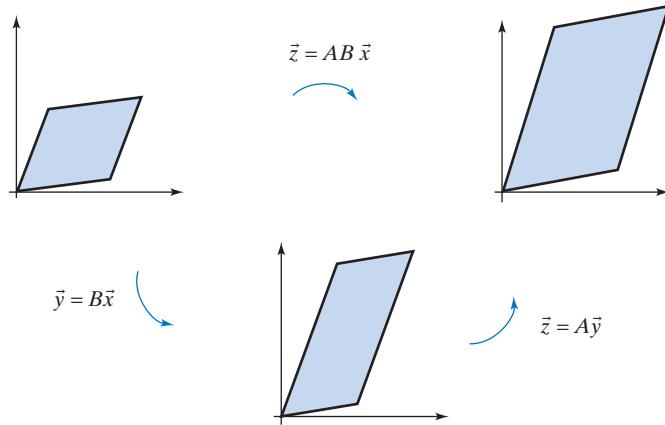


Figure 7

The expansion factor  $|\det(A^{-1})|$  is the reciprocal of the expansion factor  $|\det A|$ :

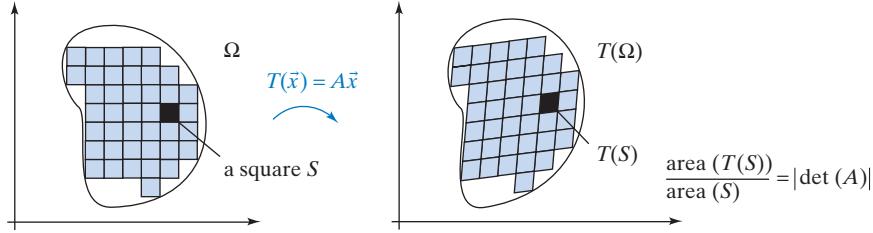
$$|\det(A^{-1})| = \frac{1}{|\det A|}.$$

The expansion factor  $|\det(AB)|$  of the composite transformation is the product of the expansion factors  $|\det A|$  and  $|\det B|$ :

$$|\det(AB)| = |\det A||\det B|.$$

## Determinants

Using techniques of calculus, you can verify that  $|\det A|$  gives us the expansion factor of the transformation  $T(\vec{x}) = A\vec{x}$  on any region  $\Omega$  in the plane. The approach uses inscribed parallelograms (or even squares) to approximate the area of the region, as shown in Figure 8. Note that the expansion factor of  $T$  on each of these squares is  $|\det A|$ . Choosing smaller and smaller squares and applying calculus, you can conclude that the expansion factor of  $T$  on  $\Omega$  itself is  $|\det A|$ .



**Figure 8**

We will conclude this chapter with the discussion of a *closed-form solution* for the linear system  $A\vec{x} = \vec{b}$  in the case when the coefficient matrix  $A$  is invertible.

### Cramer's Rule

If a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible, we can express its inverse in terms of its determinant:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

This formula can be used to find a *closed-formula solution* for a linear system

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{vmatrix}$$

when the coefficient matrix is invertible. We write the system as  $A\vec{x} = \vec{b}$ , where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x} = A^{-1}\vec{b} &= \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ a_{11}b_2 - a_{21}b_1 \end{bmatrix}. \end{aligned}$$

To write this formula more succinctly, we observe that

$$\begin{aligned} a_{22}b_1 - a_{12}b_2 &= \det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix} && \leftarrow \text{replace the first column of } A \text{ by } \vec{b}. \\ a_{11}b_2 - a_{21}b_1 &= \det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix} && \leftarrow \text{replace the second column of } A \text{ by } \vec{b}. \end{aligned}$$

## Determinants

Let  $A_{\vec{b},i}$  be the matrix obtained by replacing the  $i$ th column of  $A$  by  $\vec{b}$ :

$$A_{\vec{b},1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}, \quad A_{\vec{b},2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}.$$

The solution of the system  $A\vec{x} = \vec{b}$  can now be written as

$$x_1 = \frac{\det(A_{\vec{b},1})}{\det A}, \quad x_2 = \frac{\det(A_{\vec{b},2})}{\det A}.$$

**EXAMPLE 2** Use the preceding formula to solve the system

$$\begin{vmatrix} 2x_1 + 3x_2 & 7 \\ 4x_1 + 5x_2 & 13 \end{vmatrix}.$$

**Solution**

$$x_1 = \frac{\det \begin{bmatrix} 7 & 3 \\ 13 & 5 \end{bmatrix}}{\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} = 2, \quad x_2 = \frac{\det \begin{bmatrix} 2 & 7 \\ 4 & 13 \end{bmatrix}}{\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} = 1$$
■

This method is not particularly helpful for solving numerically given linear systems; Gauss–Jordan elimination is preferable in this case. However, in many applications we have to deal with systems whose coefficients contain parameters. Often we want to know how the solution changes as we change the parameters. The closed-formula solution given before is well suited to deal with questions of this kind.

**EXAMPLE 3** Solve the system

$$\begin{vmatrix} (b-1)x_1 + ax_2 & 0 \\ -ax_1 + (b-1)x_2 & C \end{vmatrix},$$

where  $a, b, C$  are arbitrary positive constants.

**Solution**

$$x_1 = \frac{\det \begin{bmatrix} 0 & a \\ C & b-1 \end{bmatrix}}{\det \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}} = \frac{-aC}{(b-1)^2 + a^2}$$

$$x_2 = \frac{\det \begin{bmatrix} b-1 & 0 \\ -a & C \end{bmatrix}}{\det \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}} = \frac{(b-1)C}{(b-1)^2 + a^2}$$
■

**EXAMPLE 4** Consider the linear system

$$\begin{vmatrix} ax + by = 1 \\ cx + dy = 1 \end{vmatrix}, \quad \text{where } d > b > 0 \text{ and } a > c > 0.$$

This system always has a unique solution, since the determinant  $ad - bc$  is positive (note that  $ad > bc$ ). Thus, we can think of the solution vector  $\begin{bmatrix} x \\ y \end{bmatrix}$  as a (nonlinear) function of the vector

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

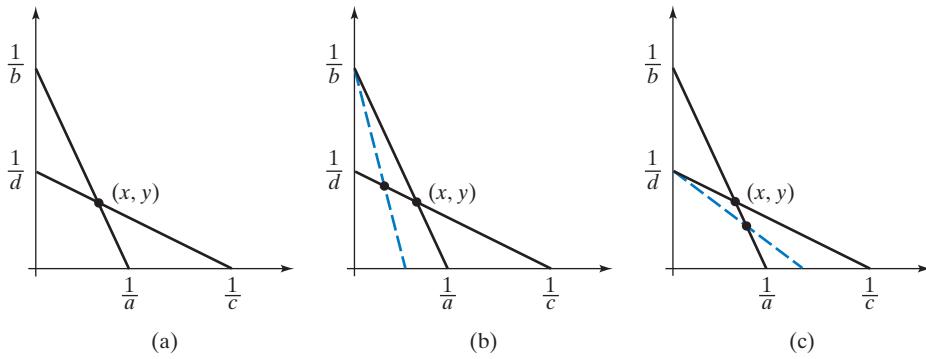
of the parameters. How does  $x$  change as we change the parameters  $a$  and  $c$ ? More precisely, find  $\partial x/\partial a$  and  $\partial x/\partial c$ , and determine the signs of these quantities.

**Solution**

$$x = \frac{\det \begin{bmatrix} 1 & b \\ 1 & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{d-b}{ad-bc} > 0, \quad \frac{\partial x}{\partial a} = \frac{-d(d-b)}{(ad-bc)^2} < 0,$$

$$\frac{\partial x}{\partial c} = \frac{b(d-b)}{(ad-bc)^2} > 0$$

See Figure 9. ■



**Figure 9** (a) Both components  $x$  and  $y$  of the solution are positive. (b)  $\partial x/\partial a < 0$ : As  $a$  increases, the component  $x$  of the solution decreases. (c)  $\partial x/\partial c > 0$ : As  $c$  increases, the component  $x$  of the solution increases.

An interesting application of these simple results in biology is to the study of castes.<sup>7</sup>

The closed formula for solving linear systems of two equations with two unknowns generalizes easily to larger systems.

**Theorem 3.8**

**Cramer's rule**

Consider the linear system

$$A\vec{x} = \vec{b},$$

where  $A$  is an invertible  $n \times n$  matrix. The components  $x_i$  of the solution vector  $\vec{x}$  are

$$x_i = \frac{\det(A_{\vec{b},i})}{\det A},$$

where  $A_{\vec{b},i}$  is the matrix obtained by replacing the  $i$ th column of  $A$  by  $\vec{b}$ .

<sup>7</sup>See E. O. Wilson, "The Ergonomics of Caste in the Social Insects," *American Naturalist*, 102, 923 (1968): 41–66.

## Determinants

This result is due to the Swiss mathematician Gabriel Cramer (1704–1752). The rule appeared in an appendix to his 1750 book, *Introduction à l'analyse des lignes courbes algébriques*.

**Proof** Write  $A = [\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_i \ \cdots \ \vec{w}_n]$ . If  $\vec{x}$  is the solution of the system  $A\vec{x} = \vec{b}$ , then

$$\begin{aligned}\det(A_{\vec{b},i}) &= \det[\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{b} \ \cdots \ \vec{w}_n] \\ &= \det[\vec{w}_1 \ \vec{w}_2 \ \cdots \ A\vec{x} \ \cdots \ \vec{w}_n] \\ &= \det[\vec{w}_1 \ \vec{w}_2 \ \cdots \ (x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_i\vec{w}_i + \cdots + x_n\vec{w}_n) \ \cdots \ \vec{w}_n] \\ &= \det[\vec{w}_1 \ \vec{w}_2 \ \cdots \ x_i\vec{w}_i \ \cdots \ \vec{w}_n] \\ &= x_i \det[\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_i \ \cdots \ \vec{w}_n] \\ &= x_i \det A.\end{aligned}$$

Note that we have used the linearity of the determinant in the  $i$ th column (Theorem 2.2).

Therefore,

$$x_i = \frac{\det(A_{\vec{b},i})}{\det A}.$$

■

Cramer's rule allows us to find a closed formula for  $A^{-1}$ , generalizing the result

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for  $2 \times 2$  matrices.

Consider an invertible  $n \times n$  matrix  $A$  and write

$$A^{-1} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nj} & \cdots & m_{nn} \end{bmatrix}.$$

We know that  $AA^{-1} = I_n$ . Picking out the  $j$ th column of  $A^{-1}$ , we find that

$$A \begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{bmatrix} = \vec{e}_j.$$

By Cramer's rule,  $m_{ij} = \det(A_{\vec{e}_j,i}) / \det A$ .

$$A_{\vec{e}_j,i} = \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix} \quad \leftarrow j\text{th row}$$

$\uparrow$   
ith column

Now  $\det(A_{\vec{e}_j,i}) = (-1)^{i+j} \det(A_{ji})$  by Laplace expansion down the  $i$ th column, so that

## Determinants

$$m_{ij} = (-1)^{i+j} \frac{\det(A_{ji})}{\det A}.$$

We have shown the following result.

### Theorem 3.9

#### Adjoint and inverse of a matrix

Consider an invertible  $n \times n$  matrix  $A$ . The *classical adjoint*  $\text{adj}(A)$  is the  $n \times n$  matrix whose  $ij$ th entry is  $(-1)^{i+j} \det(A_{ji})$ . Then

$$A^{-1} = \frac{1}{\det A} \text{adj}(A).$$

For an invertible  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , we find

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For an invertible  $3 \times 3$  matrix

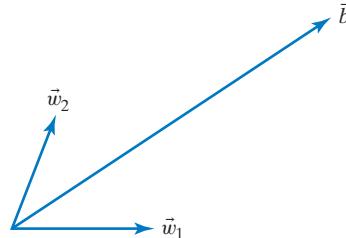
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

the formula is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} ek - fh & ch - bk & bf - ce \\ fg - dk & ak - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}.$$

We can interpret Cramer's rule geometrically.

**EXAMPLE 5** For the vectors  $\vec{w}_1$ ,  $\vec{w}_2$ , and  $\vec{b}$  shown in Figure 10, consider the linear system  $A\vec{x} = \vec{b}$ , where  $A = [\vec{w}_1 \ \vec{w}_2]$ .



**Figure 10**

Using the terminology introduced in Cramer's rule, let  $A_{\vec{b},2} = [\vec{w}_1 \ \vec{b}]$ . Note that  $\det(A)$  and  $\det(A_{\vec{b},2})$  are both positive. Cramer's rule tells us that

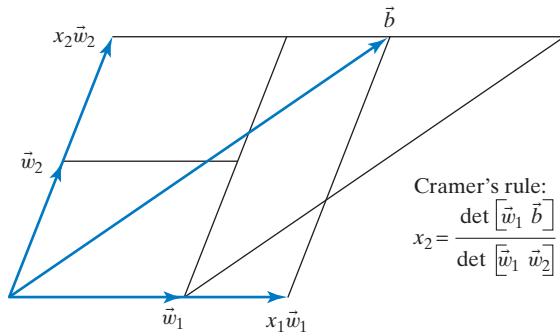
$$x_2 = \frac{\det(A_{\vec{b},2})}{\det A} \quad \text{or} \quad \det(A_{\vec{b},2}) = x_2 \det A.$$

Explain this last equation geometrically, in terms of areas of parallelograms.

#### Solution

We can write the system  $A\vec{x} = \vec{b}$  as  $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{b}$ . The geometrical solution is shown in Figure 11.

## Determinants



**Figure 11**

Now,

$$\begin{aligned}
 \det(A_{\vec{b},2}) &= \det [\vec{w}_1 \vec{b}] = \text{area of the parallelogram defined by } \vec{w}_1 \text{ and } \vec{b} \\
 &= \text{area of the parallelogram}^8 \text{ defined by } \vec{w}_1 \text{ and } x_2 \vec{w}_2 \\
 &= x_2 (\text{area of the parallelogram}^9 \text{ defined by } \vec{w}_1 \text{ and } \vec{w}_2) \\
 &= x_2 \det A,
 \end{aligned}$$

as claimed. Note that this geometrical proof mimics the algebraic proof of Cramer's rule, Theorem 3.8. █

The ambitious and artistically inclined reader is encouraged to draw an analogous figure illustrating Cramer's rule for a system of three linear equations with three unknowns.

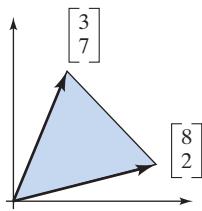
<sup>8</sup>The two parallelograms have the same base and the same height.

<sup>9</sup>Again, think about base and height.

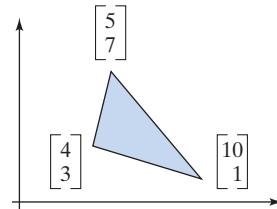
## EXERCISES 3

**GOAL** Interpret the determinant as an area or volume and as an expansion factor. Use Cramer's rule.

1. Find the area of the parallelogram defined by  $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$  and  $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$ .
2. Find the area of the triangle defined by  $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$  and  $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$ .



3. Find the area of the triangle below.



4. Consider the area  $A$  of the triangle with vertices  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ,  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . Express  $A$  in terms of

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}.$$

## Determinants

5. The tetrahedron defined by three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^3$  is the set of all vectors of the form  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ , where  $c_i \geq 0$  and  $c_1 + c_2 + c_3 \leq 1$ . Explain why the volume of this tetrahedron is one sixth of the volume of the parallelepiped defined by  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .
6. What is the relationship between the volume of the tetrahedron defined by the vectors

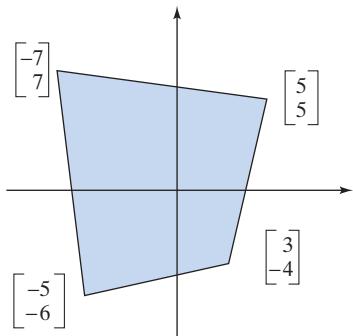
$$\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$$

and the area of the triangle with vertices

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}?$$

See Exercises 4 and 5. Explain this relationship geometrically. Hint: Consider the top face of the tetrahedron.

7. Find the area of the following region:



8. Demonstrate the equation

$$|\det A| = \left\| \vec{v}_1 \right\| \left\| \vec{v}_2^\perp \right\| \cdots \left\| \vec{v}_n^\perp \right\|$$

for a noninvertible  $n \times n$  matrix  $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$  (Theorem 3.3).

9. If  $\vec{v}_1$  and  $\vec{v}_2$  are linearly independent vectors in  $\mathbb{R}^2$ , what is the relationship between  $\det[\vec{v}_1 \ \vec{v}_2]$  and  $\det[\vec{v}_1 \ \vec{v}_2^\perp]$ , where  $\vec{v}_2^\perp$  is the component of  $\vec{v}_2$  orthogonal to  $\vec{v}_1$ ?
10. Consider an  $n \times n$  matrix  $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$ . What is the relationship between the product  $\|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_n\|$  and  $|\det A|$ ? When is  $|\det A| = \|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_n\|$ ?

11. Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . Suppose for two vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^2$  we have  $T(\vec{v}_1) = 3\vec{v}_1$  and  $T(\vec{v}_2) = 4\vec{v}_2$ . What can you say about  $\det A$ ? Justify your answer carefully.

12. Consider those  $4 \times 4$  matrices whose entries are all 1, -1, or 0. What is the maximal value of the determinant of a matrix of this type? Give an example of a matrix whose determinant has this maximal value.

13. Find the area (or 2-volume) of the parallelogram (or 2-parallelepiped) defined by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

14. Find the 3-volume of the 3-parallelepiped defined by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

15. Demonstrate Theorem 3.6 for linearly dependent vectors  $\vec{v}_1, \dots, \vec{v}_m$ .

16. True or false? If  $\Omega$  is a parallelogram in  $\mathbb{R}^3$  and  $T(\vec{x}) = A\vec{x}$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , then

$$\text{area of } T(\Omega) = |\det A|(\text{area of } \Omega).$$

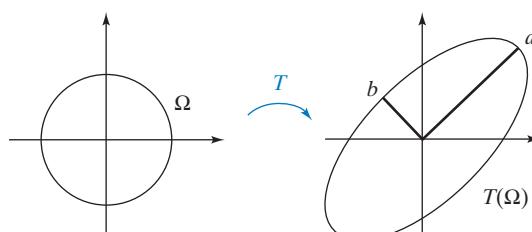
17. Consider three linearly independent vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^4$ .

- a. What is the relationship between  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3)$  and  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3)$ ? See Definition 3.5. Exercise 2.44c is helpful.
- b. Express  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3)$  in terms of  $\|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$ .
- c. Use parts (a) and (b) to express  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3)$  in terms of  $\|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$ . Is your result still true when the  $\vec{v}_i$  are linearly dependent?

(Note the analogy to the fact that for two vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^3$ ,  $\|\vec{v}_1 \times \vec{v}_2\|$  is the area of the parallelogram defined by  $\vec{v}_1$  and  $\vec{v}_2$ .)

18. If  $T(\vec{x}) = A\vec{x}$  is an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , then the image  $T(\Omega)$  of the unit circle  $\Omega$  is an ellipse.

- a. Sketch this ellipse when  $A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ , where  $p$  and  $q$  are positive. What is its area?
- b. For an arbitrary invertible transformation  $T(\vec{x}) = A\vec{x}$ , denote the lengths of the semimajor and semiminor axes of  $T(\Omega)$  by  $a$  and  $b$ , respectively. What is the relationship among  $a$ ,  $b$ , and  $\det(A)$ ?



## Determinants

- c. For the transformation  $T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \vec{x}$ , sketch this ellipse and determine its axes. Hint: Consider  $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $T \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
19. A basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  of  $\mathbb{R}^3$  is called *positively oriented* if  $\vec{v}_1$  encloses an acute angle with  $\vec{v}_2 \times \vec{v}_3$ . Illustrate this definition with a sketch. Show that the basis is positively oriented if (and only if)  $\det [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  is positive.
20. We say that a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  preserves orientation if it transforms any positively oriented basis into another positively oriented basis. See Exercise 19. Explain why a linear transformation  $T(\vec{x}) = A\vec{x}$  preserves orientation if (and only if)  $\det A$  is positive.
21. Arguing geometrically, determine whether the following orthogonal transformations from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  preserve or reverse orientation. See Exercise 20.
- Reflection about a plane
  - Reflection about a line
  - Reflection about the origin

*Use Cramer's rule to solve the systems in Exercises 22 through 24.*

$$22. \begin{vmatrix} 3x + 7y = 1 \\ 4x + 11y = 3 \end{vmatrix} \quad 23. \begin{vmatrix} 5x_1 - 3x_2 = 1 \\ -6x_1 + 7x_2 = 0 \end{vmatrix}$$

$$24. \begin{vmatrix} 2x + 3y = 8 \\ 4y + 5z = 3 \\ 6x + 7z = -1 \end{vmatrix}$$

25. Find the classical adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

and use the result to find  $A^{-1}$ .

26. Consider an  $n \times n$  matrix  $A$  with integer entries such that  $\det A = 1$ . Are the entries of  $A^{-1}$  necessarily integers? Explain.
27. Consider two positive numbers  $a$  and  $b$ . Solve the following system:

$$\begin{vmatrix} ax - by = 1 \\ bx + ay = 0 \end{vmatrix}.$$

What are the signs of the solutions  $x$  and  $y$ ? How does  $x$  change as  $b$  increases?

28. In an economics text,<sup>10</sup> we find the following system:
- $$\begin{aligned} sY + ar &= I^\circ + G \\ mY - hr &= M_s - M^\circ. \end{aligned}$$

Solve for  $Y$  and  $r$ .

---

<sup>10</sup>Simon and Blume, *Mathematics for Economists*, Norton, 1994.

29. In an economics text<sup>11</sup> we find the following system:

$$\begin{bmatrix} -R_1 & R_1 & -(1-\alpha) \\ \alpha & 1-\alpha & -(1-\alpha)^2 \\ R_2 & -R_2 & \frac{-(1-\alpha)^2}{\alpha} \end{bmatrix} \begin{bmatrix} dx_1 \\ dy_1 \\ dp \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -R_2 de_2 \end{bmatrix}.$$

Solve for  $dx_1$ ,  $dy_1$ , and  $dp$ . In your answer, you may refer to the determinant of the coefficient matrix as  $D$ . (You need not compute  $D$ .) The quantities  $R_1$ ,  $R_2$ , and  $D$  are positive, and  $\alpha$  is between zero and one. If  $de_2$  is positive, what can you say about the signs of  $dy_1$  and  $dp$ ?

30. Find the classical adjoint of  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ .

31. Find the classical adjoint of  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 6 & 6 \end{bmatrix}$ .

32. Find the classical adjoint of  $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ .

33. Find the classical adjoint of  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ .

34. For an invertible  $n \times n$  matrix  $A$ , find the product  $A(\text{adj } A)$ . What about  $(\text{adj } A)(A)$ ?

35. For an invertible  $n \times n$  matrix  $A$ , what is the relationship between  $\det(A)$  and  $\det(\text{adj } A)$ ?

36. For an invertible  $n \times n$  matrix  $A$ , what is  $\text{adj}(\text{adj } A)$ ?

37. For an invertible  $n \times n$  matrix  $A$ , what is the relationship between  $\text{adj}(A)$  and  $\text{adj}(A^{-1})$ ?

38. For two invertible  $n \times n$  matrices  $A$  and  $B$ , what is the relationship between  $\text{adj}(A)$ ,  $\text{adj}(B)$ , and  $\text{adj}(AB)$ ?

39. If  $A$  and  $B$  are invertible  $n \times n$  matrices, and if  $A$  is similar to  $B$ , is  $\text{adj}(A)$  necessarily similar to  $\text{adj}(B)$ ?

40. For an invertible  $n \times n$  matrix  $A$ , consider the linear transformation

$$T(\vec{x}) = \begin{bmatrix} \det(A_{\vec{x},1}) \\ \det(A_{\vec{x},2}) \\ \vdots \\ \det(A_{\vec{x},n}) \end{bmatrix} \quad \text{from } \mathbb{R}^n \text{ to } \mathbb{R}^n.$$

Express the standard matrix of  $T$  in terms of  $\text{adj}(A)$ .

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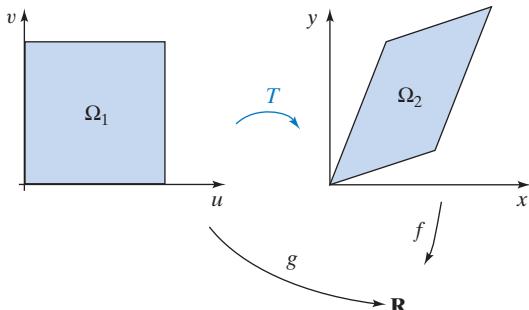
<sup>11</sup>Simon and Blume, *op. cit.*

## Determinants

41. Show that an  $n \times n$  matrix  $A$  has at least one nonzero minor if (and only if)  $\text{rank}(A) \geq n - 1$ .
42. Even if an  $n \times n$  matrix  $A$  fails to be invertible, we can define the adjoint  $\text{adj}(A)$  as in Theorem 3.9. The  $i,j$ th entry of  $\text{adj}(A)$  is  $(-1)^{i+j} \det(A_{ji})$ . For which  $n \times n$  matrices  $A$  is  $\text{adj}(A) = 0$ ? Give your answer in terms of the rank of  $A$ . See Exercise 41.
43. Show that  $A(\text{adj}A) = 0 = (\text{adj}A)A$  for all noninvertible  $n \times n$  matrices  $A$ . See Exercise 42.
44. If  $A$  is an  $n \times n$  matrix of rank  $n - 1$ , what is the rank of  $\text{adj}(A)$ ? See Exercises 42 and 43.
45. Find all  $2 \times 2$  matrices  $A$  such that  $\text{adj}(A) = A^T$ .
46. (For those who have studied multivariable calculus.) Let  $T$  be an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , represented by the matrix  $M$ . Let  $\Omega_1$  be the unit square in  $\mathbb{R}^2$  and  $\Omega_2$  its image under  $T$ . Consider a continuous function  $f(x, y)$  from  $\mathbb{R}^2$  to  $\mathbb{R}$ , and define the function  $g(u, v) = f(T(u, v))$ . What is the relationship between the following two double integrals?

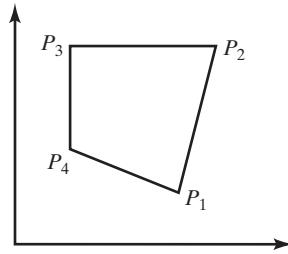
$$\iint_{\Omega_2} f(x, y) dA \quad \text{and} \quad \iint_{\Omega_1} g(u, v) dA$$

Your answer will involve the matrix  $M$ . Hint: What happens when  $f(x, y) = 1$ , for all  $x, y$ ?

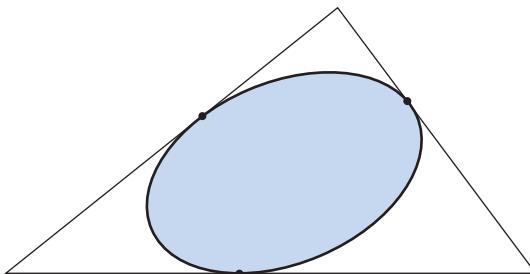


47. Consider the quadrilateral in the accompanying figure, with vertices  $P_i = (x_i, y_i)$ , for  $i = 1, 2, 3, 4$ . Show that the area of this quadrilateral is

$$\frac{1}{2} \left( \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \det \begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix} + \det \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix} + \det \begin{bmatrix} x_4 & x_1 \\ y_4 & y_1 \end{bmatrix} \right).$$



48. What is the area of the largest ellipse you can inscribe into a triangle with side lengths 3, 4, and 5? Hint: The largest ellipse you can inscribe into an equilateral triangle is a circle.



49. What are the lengths of the semiaxes of the largest ellipse you can inscribe into a triangle with sides 3, 4, and 5? See Exercise 48.

## Exercises

### TRUE OR FALSE?

1. If  $B$  is obtained by multiplying a column of  $A$  by 9, then the equation  $\det B = 9 \det A$  must hold.
2.  $\det(A^{10}) = (\det A)^{10}$  for all  $10 \times 10$  matrices  $A$ .
3. The determinant of any diagonal  $n \times n$  matrix is the product of its diagonal entries.
4. If matrix  $B$  is obtained by swapping two rows of an  $n \times n$  matrix  $A$ , then the equation  $\det B = -\det A$  must hold.
5. If  $A = [\vec{u} \quad \vec{v} \quad \vec{w}]$  is any  $3 \times 3$  matrix, then  $\det A = \vec{u} \cdot (\vec{v} \times \vec{w})$ .
6.  $\det(4A) = 4 \det A$  for all  $4 \times 4$  matrices  $A$ .

7.  $\det(A + B) = \det A + \det B$  for all  $5 \times 5$  matrices  $A$  and  $B$ .
8. The equation  $\det(-A) = \det A$  holds for all  $6 \times 6$  matrices.
9. If all the entries of a  $7 \times 7$  matrix  $A$  are 7, then  $\det A$  must be  $7^7$ .
10. An  $8 \times 8$  matrix fails to be invertible if (and only if) its determinant is nonzero.
11. The matrix  $\begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$  is invertible for all positive constants  $k$ .

## Determinants

12.  $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 1.$

13. Matrix  $\begin{bmatrix} 9 & 100 & 3 & 7 \\ 5 & 4 & 100 & 8 \\ 100 & 9 & 8 & 7 \\ 6 & 5 & 4 & 100 \end{bmatrix}$  is invertible.

14. If  $A$  is an invertible  $n \times n$  matrix, then  $\det(A^T)$  must equal  $\det(A^{-1})$ .

15. If the determinant of a  $4 \times 4$  matrix  $A$  is 4, then its rank must be 4.

16. There exists a nonzero  $4 \times 4$  matrix  $A$  such that  $\det A = \det(4A)$ .

17. If two  $n \times n$  matrices  $A$  and  $B$  are similar, then the equation  $\det A = \det B$  must hold.

18. The determinant of all orthogonal matrices is 1.

19. If  $A$  is any  $n \times n$  matrix, then  $\det(AA^T) = \det(A^TA)$ .

20. There exists an invertible matrix of the form

$$\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}.$$

21. If all the entries of a square matrix are 1 or 0, then  $\det A$  must be 1, 0, or  $-1$ .

22. If all the entries of a square matrix  $A$  are integers and  $\det A = 1$ , then the entries of matrix  $A^{-1}$  must be integers as well.

23. If all the columns of a square matrix  $A$  are unit vectors, then the determinant of  $A$  must be less than or equal to 1.

24. If  $A$  is any noninvertible square matrix, then  $\det A = \det(\text{rref } A)$ .

25. If the determinant of a square matrix is  $-1$ , then  $A$  must be an orthogonal matrix.

26. If all the entries of an invertible matrix  $A$  are integers, then the entries of  $A^{-1}$  must be integers as well.

27. There exists a  $4 \times 4$  matrix  $A$  whose entries are all 1 or  $-1$ , and such that  $\det A = 16$ .

28. If the determinant of a  $2 \times 2$  matrix  $A$  is 4, then the inequality  $\|A\vec{v}\| \leq 4\|\vec{v}\|$  must hold for all vectors  $\vec{v}$  in  $\mathbb{R}^2$ .

29. If  $A = [\vec{u} \quad \vec{v} \quad \vec{w}]$  is a  $3 \times 3$  matrix, then the formula  $\det(A) = \vec{v} \cdot (\vec{u} \times \vec{w})$  must hold.

30. There exist invertible  $2 \times 2$  matrices  $A$  and  $B$  such that  $\det(A + B) = \det A + \det B$ .

31. There exist real invertible  $3 \times 3$  matrices  $A$  and  $S$  such that  $S^{-1}AS = 2A$ .

32. There exist real invertible  $3 \times 3$  matrices  $A$  and  $S$  such that  $S^TAS = -A$ .

33. If  $A$  is any symmetric matrix, then  $\det A = 1$  or  $\det A = -1$ .

34. If  $A$  is any skew-symmetric  $4 \times 4$  matrix, then  $\det A = 0$ .

35. If  $\det A = \det B$  for two  $n \times n$  matrices  $A$  and  $B$ , then  $A$  must be similar to  $B$ .

36. Suppose  $A$  is an  $n \times n$  matrix and  $B$  is obtained from  $A$  by swapping two rows of  $A$ . If  $\det B < \det A$ , then  $A$  must be invertible.

37. If an  $n \times n$  matrix  $A$  is invertible, then there must be an  $(n-1) \times (n-1)$  submatrix of  $A$  (obtained by deleting a row and a column of  $A$ ) that is invertible as well.

38. If all the entries of matrices  $A$  and  $A^{-1}$  are integers, then the equation  $\det A = \det(A^{-1})$  must hold.

39. If a square matrix  $A$  is invertible, then its classical adjoint  $\text{adj}(A)$  is invertible as well.

40. There exists a real  $3 \times 3$  matrix  $A$  such that  $A^2 = -I_3$ .

41. If all the diagonal entries of an  $n \times n$  matrix  $A$  are odd integers and all the other entries are even integers, then  $A$  must be an invertible matrix.<sup>12</sup>

42. If all the diagonal entries of an  $n \times n$  matrix  $A$  are even integers and all the other entries are odd integers, then  $A$  must be an invertible matrix.<sup>12</sup>

43. For every nonzero  $2 \times 2$  matrix  $A$  there exists a  $2 \times 2$  matrix  $B$  such that  $\det(A + B) \neq \det A + \det B$ .

44. If  $A$  is a  $4 \times 4$  matrix whose entries are all 1 or  $-1$ , then  $\det A$  must be divisible by 8 (i.e.,  $\det A = 8k$  for some integer  $k$ ).

45. If  $A$  is an invertible  $n \times n$  matrix, then  $A$  must commute with its adjoint,  $\text{adj}(A)$ .

46. There exists a real number  $k$  such that the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & k & 7 \\ 8 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

is invertible.

47. If  $A$  and  $B$  are orthogonal  $n \times n$  matrices such that  $\det A = \det B = 1$ , then matrices  $A$  and  $B$  must commute.

---

<sup>12</sup>The even integers are  $0, \pm 2, \pm 4, \dots$ , while the odd integers are  $\pm 1, \pm 3, \dots$

## Answers to Odd-Numbered Exercises

**1** Answers to more theoretical questions are omitted.

- |                                                                                                                                                                                                                          |                            |                  |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|------------------|
| 1. 0                                                                                                                                                                                                                     | 3. -2                      | 5. 110           |
| 7. 0                                                                                                                                                                                                                     | 9. -36                     | 11. $k \neq 3/2$ |
| 13. $k \neq 0$                                                                                                                                                                                                           | 15. $k \neq 1/2$           |                  |
| 17. If $k$ is neither 1 nor $-1$                                                                                                                                                                                         |                            |                  |
| 19. If $k$ is neither 0 nor 1                                                                                                                                                                                            |                            |                  |
| 21. If $k$ is neither 1 nor $-2$                                                                                                                                                                                         |                            |                  |
| 23. If $\lambda$ is 1 or 4                                                                                                                                                                                               | 25. If $\lambda$ is 2 or 8 |                  |
| 27. If $\lambda$ is 2, 3, or 4                                                                                                                                                                                           | 29. If $\lambda$ is 3 or 8 |                  |
| 31. 24                                                                                                                                                                                                                   | 33. 99                     | 35. 18           |
| 37. 55                                                                                                                                                                                                                   | 39. 120                    | 41. 24           |
| 43. $\det(-A) = (-1)^n \det A$                                                                                                                                                                                           |                            |                  |
| 45. They are the same.                                                                                                                                                                                                   |                            |                  |
| 49. The kernel is the plane $\text{span}(\vec{v}, \vec{w})$ and the image is $\mathbb{R}$ .                                                                                                                              |                            |                  |
| 51. Let $a_{ii}$ be the first diagonal entry that does not belong to the pattern. The pattern must contain an entry in the $i$ th row to the right of $a_{ii}$ as well as an entry in the $i$ th column below $a_{ii}$ . |                            |                  |
| 53. Only one pattern has a nonzero product, and that product is 1. Since there are $n^2$ inversions in that pattern, we have $\det A = (-1)^{n^2} = (-1)^n$ .                                                            |                            |                  |
| 55. Yes, since the determinants of all principal submatrices are nonzero.                                                                                                                                                |                            |                  |
| 57. Only one pattern has a nonzero product, and that product is 1. Thus, $\det A = 1$ or $\det A = -1$ .                                                                                                                 |                            |                  |
| 59. a. Yes    b. No    c. No                                                                                                                                                                                             |                            |                  |

61. Fails to be alternating, since  $F \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 1$   
 but  $F \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 0$ .

**2** Answers to more theoretical questions are omitted.

- |                                                                                                     |        |         |
|-----------------------------------------------------------------------------------------------------|--------|---------|
| 1. 6                                                                                                | 3. -24 | 5. -24  |
| 7. 1                                                                                                | 9. 24  | 11. -72 |
| 13. 8                                                                                               | 15. 8  | 17. 8   |
| 19. -1                                                                                              | 21. 1  |         |
| 23. $(-1)^{n(n-1)/2}$ . This is 1 if either $n$ or $(n - 1)$ is divisible by 4, and $-1$ otherwise. |        |         |

25. 16

27.  $a^2 + b^2$

29.  $\det(P_1) = 1$  and  $\det(P_n) = \det(P_{n-1})$ , by expansion down the first column, so  $\det(P_n) = 1$  for all  $n$ .

$$31. \text{a. } \det \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix} = a_1 - a_0$$

b. Use Laplace expansion down the last column to see that  $f(t)$  is a polynomial of degree  $\leq n$ . The coefficient  $k$  of  $t^n$  is  $\prod_{n-1 \geq i > j} (a_i - a_j)$ . Now  $\det A = f(a_n) = k(a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1}) = \prod_{n \geq i > j} (a_i - a_j)$ , as claimed.

33.  $\prod_{i=1}^n a_i \cdot \prod_{i>j} (a_i - a_j)$  (use linearity in the columns and Exercise 31)

35.  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$  and  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  are solutions. The equation is of the form  $px_1 + qx_2 + b = 0$ ; that is, it defines a line.

37.  $\pm 1$

39.  $\det(A^T A) = (\det A)^2 > 0$

41.  $\det A = \det(A^T) = \det(-A) = (-1)^n \det A = -\det A$ , so  $\det A = 0$

$$43. A^T A = \begin{bmatrix} \|\vec{v}\|^2 & \vec{v} \cdot \vec{w} \\ \vec{v} \cdot \vec{w} & \|\vec{w}\|^2 \end{bmatrix},$$

so  $\det(A^T A) = \|\vec{v}\|^2 \|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2 \geq 0$ , by the Cauchy-Schwarz inequality. Equality holds only if  $\vec{v}$  and  $\vec{w}$  are parallel.

45. Expand down the first column:  $f(x) = -x \det(A_{41}) + \text{constant}$ , so  $f'(x) = -\det(A_{41}) = -24$ .

47.  $T$  is linear in the rows and columns.

49.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 14 \end{bmatrix}$ , for example. Start with a

triangular matrix with determinant 13, such as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix}, \text{ and add the first row to the second}$$

and to the third to make all entries nonzero.

51.  $\det A = (-1)^n$

53. a. Note that  $\det(A) \det(A^{-1}) = 1$ , and both factors are integers.

b. Use the formula for the inverse of a  $2 \times 2$  matrix.

59. No

## Determinants

61. Take the determinant of both sides of

$$\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & AD - CB \end{bmatrix},$$

and divide by  $\det A$ .

65. a.  $d_n = d_{n-1} + d_{n-2}$ , a Fibonacci sequence  
 b.  $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, \dots, d_{10} = 89$   
 c. Invertible for all positive integers  $n$

3 Answers to more theoretical questions are omitted.

1. 50                  3. 13                  7. 110

11.  $|\det A| = 12$ , the expansion factor of  $T$  on the parallelogram defined by  $\vec{v}_1$  and  $\vec{v}_2$

13.  $\sqrt{20}$

15. We need to show that if  $\vec{v}_1, \dots, \vec{v}_m$  are linearly dependent, then (a)  $V(\vec{v}_1, \dots, \vec{v}_m) = 0$  and (b)  $\det(A^T A) = 0$ .

- a. One of the  $\vec{v}_i$  is redundant, so that  $\vec{v}_i^\perp = \vec{0}$  and  $V(\vec{v}_1, \dots, \vec{v}_m) = 0$ , by Definition 3.5.  
 b.  $\ker A \neq \{\vec{0}\}$  and  $\ker A \subseteq \ker(A^T A)$ , so that  $\ker(A^T A) \neq \{\vec{0}\}$ . Thus,  $A^T A$  fails to be invertible.

17. a.  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3) = V(\vec{v}_1, \vec{v}_2, \vec{v}_3) \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$  because  $\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3$  is orthogonal to  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$   
 b.  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3) = |\det[\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3 \quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]| = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|^2$ , by definition of the cross product  
 c.  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$ , by parts (a) and (b)
19.  $\det[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$  is positive if (and only if)  $\vec{v}_1$  and  $\vec{v}_2 \times \vec{v}_3$  enclose an acute angle.

21. a. Reverses      b. Preserves      c. Reverses

$$23. x_1 = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{\det \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}} = \frac{7}{17};$$

$$x_2 = \frac{\det \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{\det \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}} = \frac{6}{17}$$

$$25. \text{adj } A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix};$$

$$A^{-1} = \frac{1}{\det A} \text{adj } A = -\text{adj } A \\ = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$27. x = \frac{a}{a^2 + b^2} > 0; y = \frac{-b}{a^2 + b^2} < 0; x \text{ decreases as } b \text{ increases.}$$

$$29. dx_1 = -D^{-1} R_2 (1 - R_1)(1 - \alpha)^2 de_2 \\ dy_1 = D^{-1} (1 - \alpha) R_2 (R_1(1 - \alpha) + \alpha) de_2 > 0 \\ dp = D^{-1} R_1 R_2 de_2 > 0$$

$$31. \begin{bmatrix} -6 & 0 & 1 \\ -3 & 5 & -2 \\ 4 & -5 & 1 \end{bmatrix} \quad 33. \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$35. \det(\text{adj } A) = (\det A)^{n-1}$$

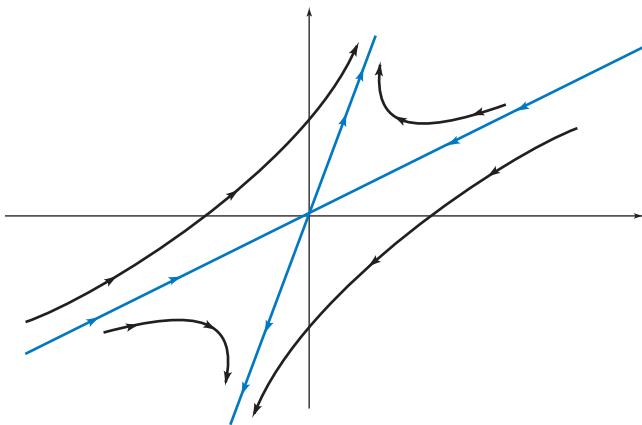
$$37. \text{adj}(A^{-1}) = (\text{adj } A)^{-1} = (\det A)^{-1} A$$

$$39. \text{Yes. Use Exercises 38: If } AS = SB, \text{ then } (\text{adj } S)(\text{adj } A) = (\text{adj } B)(\text{adj } S).$$

$$43. A(\text{adj } A) = (\text{adj } A)A = (\det A)I_n = 0$$

$$45. \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$





# Eigenvalues and Eigenvectors

## I Diagonalization

Here is an introductory example to motivate the main theme of this chapter.

**EXAMPLE I** Consider the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}.$$

Find  $A^5$ ,  $B^5$ , rank  $A$ , rank  $B$ ,  $\det A$ , and  $\det B$ . Find bases of  $\ker A$  and  $\ker B$ .

**(Partial) Solution**

The parts pertaining to the *diagonal* matrix  $A$  are easy to do, by inspection:

$$A^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix},$$

where  $\text{rank } A = 3$ , the number of nonzero entries on the diagonal;  $\det A = 0$ , the product of the diagonal entries; and a basis of  $\ker A$  is  $\vec{e}_2$  since the second column of  $A$  is zero.

The corresponding computations for matrix  $B$  are doable, of course, but they are much more tedious. We could compute rref  $B$  to find  $\text{rank } B$ ,  $\det B$ , and  $\ker B$ , but then we still have to compute  $B^5$ . ■

Example 1 illustrates why we like to work with diagonal matrices whenever possible: They are easy to handle.

Let's put in another plug for diagonal matrices.

## Eigenvalues and Eigenvectors

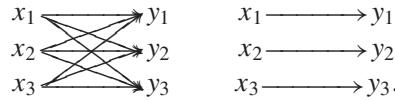
Consider the linear transformations

$$\vec{y} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \vec{x} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x},$$

represented by the equations

$$\begin{array}{lll} y_1 = x_1 + 2x_2 + 3x_3 & & y_1 = x_1 \\ y_2 = 4x_1 + 5x_2 + 6x_3 & \text{and} & y_2 = 2x_2 \\ y_3 = 7x_1 + 8x_2 + 9x_3 & & y_3 = 3x_3, \end{array}$$

respectively. If we use arrows to represent the functional dependencies between the variables, we see a tangled web of dependencies in the case of the first transformation. In the case of the diagonal matrix, however, the web is untangled since the three equations are *uncoupled*: We can deal with the three simple equations  $y_1 = x_1$ ,  $y_2 = 2x_2$ , and  $y_3 = 3x_3$  separately:



The idea behind diagonalization is to untangle the web of functional dependencies we see on the left.

When dealing with a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , where  $A$  fails to be a diagonal matrix, we may be interested in finding a basis of  $\mathbb{R}^n$  such that the matrix of  $T$  with respect to that basis is diagonal. In this context, the following definition is useful.

### Definition 1.1

#### Diagonalizable matrices

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Then  $A$  (or  $T$ ) is said to be *diagonalizable* if the matrix  $B$  of  $T$  with respect to some basis is diagonal.

Matrix  $A$  is diagonalizable if (and only if)  $A$  is similar to some diagonal matrix  $B$ , meaning that there exists an invertible matrix  $S$  such that  $S^{-1}AS = B$  is diagonal.

To *diagonalize* a square matrix  $A$  means to find an invertible matrix  $S$  and a diagonal matrix  $B$  such that  $S^{-1}AS = B$ .

The reflection about a line in  $\mathbb{R}^2$  is diagonalizable, while the rotation  $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$  fails to be diagonalizable.

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  and a basis  $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$  for  $\mathbb{R}^n$ . The  $\mathcal{B}$ -matrix  $B$  of  $T$  is diagonal if (and only if)  $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$  for some scalars  $\lambda_1, \dots, \lambda_n$ . (In this context, it is customary to use the Greek letter lambda,  $\lambda$ , to denote the scalars.)

The diagram below reminds us why this result holds; nothing “deep” is going on here.

$$B = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \dots & A\vec{v}_n \\ \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}$$

We can conclude that an  $n \times n$  matrix  $A$  is diagonalizable if (and only if) there exists a basis  $\vec{v}_1, \dots, \vec{v}_n$  of  $\mathbb{R}^n$  such that  $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$  for some scalars  $\lambda_1, \dots, \lambda_n$ .

## Eigenvalues and Eigenvectors

If we wish to diagonalize a matrix  $A$ , we need to find a basis of  $\mathbb{R}^n$  consisting of vectors  $\vec{v}$  such that

$$A\vec{v} = \lambda\vec{v}, \text{ for some scalar } \lambda.$$

It is sensible to introduce a name for vectors with this property.

### Definition 1.2

#### Eigenvectors, eigenvalues, and eigenbases<sup>1</sup>

Consider a linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

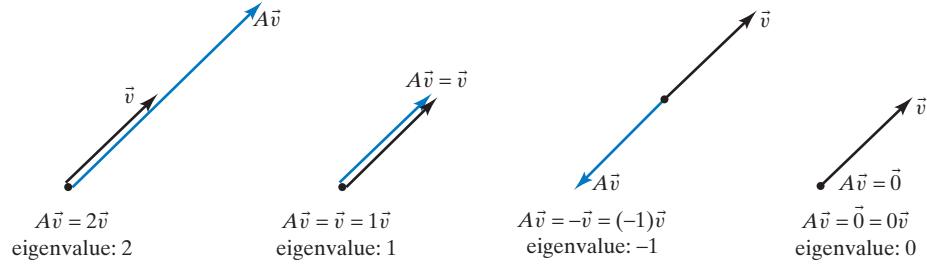
A nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  is called an **eigenvector** of  $A$  (or  $T$ ) if

$$A\vec{v} = \lambda\vec{v}$$

for some scalar  $\lambda$ . This  $\lambda$  is called the **eigenvalue** associated with eigenvector  $\vec{v}$ .

A basis  $\vec{v}_1, \dots, \vec{v}_n$  of  $\mathbb{R}^n$  is called an **eigenbasis** for  $A$  (or  $T$ ) if the vectors  $\vec{v}_1, \dots, \vec{v}_n$  are eigenvectors of  $A$ , meaning that  $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$  for some scalars  $\lambda_1, \dots, \lambda_n$ .

A nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  is an eigenvector of  $A$  if  $A\vec{v}$  is parallel to  $\vec{v}$ , as shown in Figure 1.



**Figure 1**

If  $\vec{v}$  is an eigenvector of matrix  $A$ , with associated eigenvalue  $\lambda$ , then  $\vec{v}$  is an eigenvector of the matrices  $A^2, A^3, \dots$  as well, with

$$A^2\vec{v} = \lambda^2\vec{v}, \quad A^3\vec{v} = \lambda^3\vec{v}, \quad \dots, \quad A^m\vec{v} = \lambda^m\vec{v},$$

for all positive integers  $m$ . The proof, by induction on  $m$ , is straightforward. The base case,  $A^1\vec{v} = A\vec{v} = \lambda\vec{v} = \lambda^1\vec{v}$ , is given. As for the induction step, we have  $A^{m+1}\vec{v} = A(A^m\vec{v}) \stackrel{\text{step 2}}{=} A(\lambda^m\vec{v}) = \lambda^m(A\vec{v}) = \lambda^m\lambda\vec{v} = \lambda^{m+1}\vec{v}$ . In step 2, we are

using the induction hypothesis,  $A^m\vec{v} = \lambda^m\vec{v}$ .

The important result below follows from our work after Definition 1.1.

### Theorem 1.3

#### Eigenbases and diagonalization

The matrix  $A$  is diagonalizable if (and only if) there exists an eigenbasis for  $A$ . If  $\vec{v}_1, \dots, \vec{v}_n$  is an eigenbasis for  $A$ , with  $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$ , then the matrices

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \text{ and } B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

will diagonalize  $A$ , meaning that  $S^{-1}AS = B$ .

<sup>1</sup>From German *eigen*: proper, characteristic. (The words *eigen* and *own* have a common Indo-European root.)

**Theorem 1.3**
**Eigenbases and diagonalization (Continued)**

Conversely, if the matrices  $S$  and  $B$  diagonalize  $A$ , then the column vectors of  $S$  will form an eigenbasis for  $A$ , and the diagonal entries of  $B$  will be the associated eigenvalues.

Above, we explained Theorem 1.3 in terms of a change of basis. Equivalently, we can prove the result by using matrix products. Here we will prove just one implication, leaving the other as Exercise 23 to the reader.

Suppose there exists an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$ , and define  $S$  and  $B$  as in Theorem 1.3. Then

$$\begin{aligned} AS &= A \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\vec{v}_1 & A\vec{v}_2 & \dots & A\vec{v}_n \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \dots & \lambda_n\vec{v}_n \\ | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = SB, \end{aligned}$$

showing that  $S^{-1}AS = B$ , as claimed.

**EXAMPLE 2**

Find all eigenvectors and eigenvalues of the identity matrix  $I_n$ . Is there an eigenbasis for  $I_n$ ? Is  $I_n$  diagonalizable?

**Solution**

Since  $I_n\vec{v} = \vec{v} = 1\vec{v}$  for all  $\vec{v}$  in  $\mathbb{R}^n$ , all nonzero vectors in  $\mathbb{R}^n$  are eigenvectors of  $I_n$ , with associated eigenvalue 1.

Thus, all bases of  $\mathbb{R}^n$  are eigenbases for  $I_n$ .

Clearly,  $I_n$  is diagonalizable; it's a diagonal matrix already. If  $S$  is any invertible matrix, then  $S^{-1}I_nS = I_n$  is diagonal. ■

**EXAMPLE 3**

Consider the linear transformation  $T(\vec{x}) = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}\vec{x}$ . We are told that  $T(\vec{x}) = A\vec{x}$  represents the orthogonal projection onto the line  $L = \text{span} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ . Describe the eigenvectors of  $A$  geometrically, and find all eigenvalues of  $A$ . Is there an eigenbasis for  $A$ ? Diagonalize matrix  $A$  if you can.

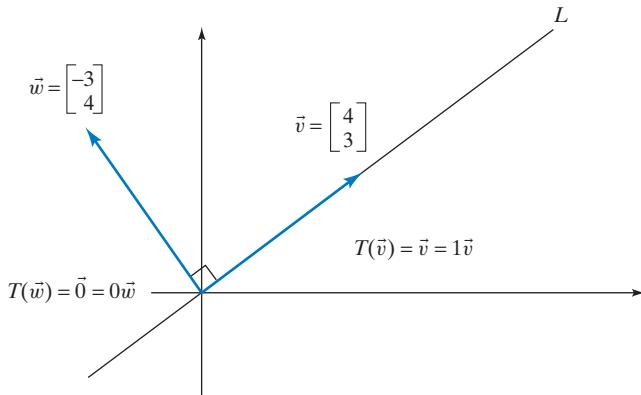
**Solution**

We find the eigenvectors of  $A$  by inspection. Can you think of any nonzero vectors  $\vec{v}$  in  $\mathbb{R}^2$  such that  $T(\vec{v}) = A\vec{v}$  is a scalar multiple of  $\vec{v}$ ? Clearly, any nonzero vector  $\vec{v}$  parallel to  $L$  will do, with  $A\vec{v} = \vec{v} = 1\vec{v}$ , as well as any nonzero vector  $\vec{w}$  perpendicular to  $L$ , with  $A\vec{w} = \vec{0} = 0\vec{w}$ . The eigenvalues of  $A$  are 1 and 0. Note that the eigenvectors with eigenvalue 0 are the nonzero vectors in the kernel of  $A$ .

For example,  $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$  is an eigenvector with eigenvalue 1, and  $\vec{w} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$  is an eigenvector with eigenvalue 0. See Figure 2. Thus,  $\mathcal{B} = (\vec{v}, \vec{w})$  will be an eigenbasis for  $A$ , and the  $\mathcal{B}$ -matrix  $B$  of  $T$  will be the diagonal matrix

$$B = \begin{bmatrix} T(\vec{v}) & T(\vec{w}) \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}.$$

## Eigenvalues and Eigenvectors



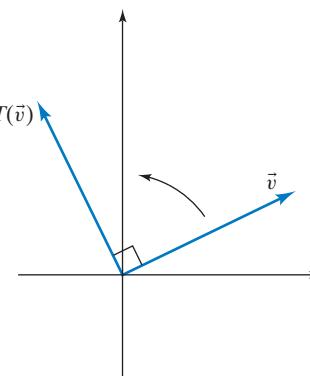
**Figure 2**

The matrices  $S = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  will diagonalize  $A$ , meaning that  $S^{-1}AS = B$ . There is no need to compute  $S^{-1}AS$ , since the equation  $S^{-1}AS = B$  follows from the theory derived earlier (take another look at the proof of Theorem 1.3). To check our work, we may want to verify that  $AS = SB$ . ■

**EXAMPLE 4** Let  $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$  be the rotation through an angle of  $90^\circ$  in the counterclockwise direction. Find all eigenvectors and eigenvalues of  $A$ . Is there an eigenbasis for  $A$ ? Is  $A$  diagonalizable?

### Solution

If  $\vec{v}$  is any nonzero vector in  $\mathbb{R}^2$ , then  $T(\vec{v}) = A\vec{v}$  fails to be parallel to  $\vec{v}$  (it's perpendicular). See Figure 3. There are no real eigenvectors and eigenvalues here,<sup>2</sup> there is no eigenbasis for  $A$ , and  $A$  fails to be diagonalizable. ■



**Figure 3**

---

<sup>2</sup>In Section 5, we will consider complex eigenvalues, but through Section 4, all eigenvalues and eigenvectors are assumed to be real unless stated otherwise.

## Eigenvalues and Eigenvectors

**EXAMPLE 5** What are the possible real eigenvalues of an orthogonal<sup>3</sup>  $n \times n$  matrix  $A$ ?

### Solution

If  $A$  is an orthogonal matrix, then the linear transformation  $T(\vec{x}) = A\vec{x}$  preserves length: We have  $\|T(\vec{x})\| = \|A\vec{x}\| = \|\vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^n$ . Consider an eigenvector  $\vec{v}$  of  $A$ , with associated eigenvalue  $\lambda$ ,

$$A\vec{v} = \lambda\vec{v}.$$

Then

$$\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|,$$

so that  $|\lambda| = 1$ . We conclude that  $\lambda = 1$  or  $\lambda = -1$ .

For example, consider the reflection matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , with  $A\vec{e}_1 = \vec{e}_1 = 1\vec{e}_1$  and  $A\vec{e}_2 = -\vec{e}_2 = (-1)\vec{e}_2$ , so that 1 and  $-1$  are the eigenvalues of  $A$ . ■

### Theorem 1.4

The possible real eigenvalues of an orthogonal matrix are 1 and  $-1$ .

**EXAMPLE 6** For which  $n \times n$  matrices  $A$  is 0 an eigenvalue? Give your answer in terms of the kernel of  $A$  and also in terms of the invertibility of  $A$ .

### Solution

By definition, 0 is an eigenvalue of  $A$  if there exists a nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  such that  $A\vec{v} = 0\vec{v} = \vec{0}$ , meaning that there exists a nonzero vector in the kernel of  $A$ . Thus, 0 is an eigenvalue of  $A$  if (and only if)  $\ker A \neq \{\vec{0}\}$ , meaning that  $A$  fails to be invertible. ■

Example 6 allows us to update our “Summary on invertible matrices.” Compare this with Summary 3.10.

### SUMMARY 1.5

#### Various characterizations of invertible matrices

For an  $n \times n$  matrix  $A$ , the following statements are equivalent.

- i.  $A$  is invertible.
- ii. The linear system  $A\vec{x} = \vec{b}$  has a unique solution  $\vec{x}$ , for all  $\vec{b}$  in  $\mathbb{R}^n$ .
- iii.  $\text{rref } A = I_n$ .
- iv.  $\text{rank } A = n$ .
- v.  $\text{im } A = \mathbb{R}^n$ .
- vi.  $\ker A = \{\vec{0}\}$ .
- vii. The column vectors of  $A$  form a basis of  $\mathbb{R}^n$ .
- viii. The column vectors of  $A$  span  $\mathbb{R}^n$ .
- ix. The column vectors of  $A$  are linearly independent.
- x.  $\det A \neq 0$ .
- xi. 0 fails to be an eigenvalue of  $A$ .

<sup>3</sup>This note has been intentionally omitted for this edition.

## Dynamical Systems and Eigenvectors (optional)

Dynamical systems provide a powerful application of eigenvalues and eigenvectors. We start with a somewhat lighthearted example, but a wealth of relevant examples can be found in applied mathematics, physics, chemistry, biology, engineering, computer science, economics, and the theory of finance. In Section 4 we will recast the theory of transition matrices in terms of a dynamical system.

### EXAMPLE 7

A stretch of desert in northwestern Mexico is populated mainly by two species of animals: coyotes and roadrunners. We wish to model the populations  $c(t)$  and  $r(t)$  of coyotes and roadrunners  $t$  years from now if the current populations  $c_0$  and  $r_0$  are known.<sup>4</sup>

For this habitat, the following equations model the transformation of this system from one year to the next, from time  $t$  to time  $(t + 1)$ :

$$\begin{cases} c(t+1) = 0.86c(t) + 0.08r(t) \\ r(t+1) = -0.12c(t) + 1.14r(t) \end{cases}.$$

Why is the coefficient of  $c(t)$  in the first equation less than 1, while the coefficient of  $r(t)$  in the second equation exceeds 1? What is the practical significance of the signs of the other two coefficients, 0.08 and  $-0.12$ ?

The two equations can be written in matrix form, as

$$\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix} = \begin{bmatrix} 0.86c(t) + 0.08r(t) \\ -0.12c(t) + 1.14r(t) \end{bmatrix} = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}.$$

The vector

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

is called the *state vector* of the system at time  $t$ , because it completely describes this system at time  $t$ . If we let

$$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix},$$

we can write the preceding matrix equation more succinctly as

$$\vec{x}(t+1) = A\vec{x}(t).$$

The transformation the system undergoes over the period of one year is linear, represented by the matrix  $A$ :

$$\vec{x}(t) \xrightarrow{A} \vec{x}(t+1).$$

Suppose we know the initial state

$$\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix}.$$

We wish to find  $\vec{x}(t)$ , for an arbitrary positive integer  $t$ :

$$\vec{x}(0) \xrightarrow{A} \vec{x}(1) \xrightarrow{A} \vec{x}(2) \xrightarrow{A} \vec{x}(3) \xrightarrow{A} \cdots \xrightarrow{A} \vec{x}(t) \xrightarrow{A} \cdots.$$

---

<sup>4</sup>The point of this lighthearted story is to present an introductory example where neither messy data nor a complicated scenario distracts us from the mathematical ideas we wish to develop.

## Eigenvalues and Eigenvectors

We can find  $\vec{x}(t)$  by applying the transformation  $t$  times to  $\vec{x}(0)$ :

$$\vec{x}(t) = A^t \vec{x}(0) = A^t \vec{x}_0.$$

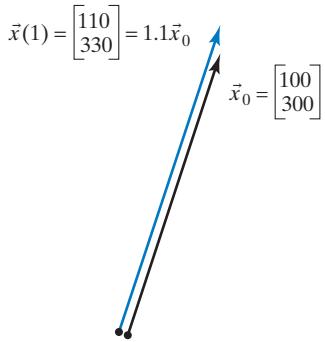
Although it is extremely tedious to find  $\vec{x}(t)$  with paper and pencil for large  $t$ , we can easily compute  $\vec{x}(t)$  using technology. For example, given

$$\vec{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix},$$

we find that

$$\vec{x}(10) = A^{10} \vec{x}_0 \approx \begin{bmatrix} 80 \\ 170 \end{bmatrix}.$$

To understand the long-term behavior of this system and how it depends on the initial values, we must go beyond numerical experimentation. It would be useful to have *closed formulas* for  $c(t)$  and  $r(t)$ , expressing these quantities as functions of  $t$ . We will first do this for certain (carefully chosen) initial state vectors.



**Figure 4**

We keep multiplying the state vector by 1.1 each time we multiply with matrix  $A$ .

Recall that our goal is to find closed formulas for  $c(t)$  and  $r(t)$ . We have

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = 1.1^t \vec{x}_0 = 1.1^t \begin{bmatrix} 100 \\ 300 \end{bmatrix},$$

so that

$$c(t) = 100(1.1)^t \quad \text{and} \quad r(t) = 300(1.1)^t.$$

Both populations will grow exponentially, by 10% each year.

**Case 2 ■** Suppose we have  $c_0 = 200$  and  $r_0 = 100$ . Then

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = 0.9\vec{x}_0.$$

In this case,  $\vec{x}_0$  turns out to be an eigenvector of  $A$  with associated eigenvalue 0.9. Both populations decline by 10% in the first year and will therefore decline another 10% each subsequent year. Thus,

$$\vec{x}(t) = A^t \vec{x}_0 = 0.9^t \vec{x}_0,$$

so that

$$c(t) = 200(0.9)^t \quad \text{and} \quad r(t) = 100(0.9)^t.$$

The initial populations are mismatched: Too many coyotes are chasing too few roadrunners, a bad state of affairs for both species.

## Eigenvalues and Eigenvectors

**Case 3 ■** Suppose we have  $c_0 = r_0 = 1,000$ . Then

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix} = \begin{bmatrix} 940 \\ 1,020 \end{bmatrix}.$$

Things are not working out as nicely as in the first two cases we considered: The initial state vector  $\vec{x}_0$  fails to be an eigenvector of  $A$  in this case. Just by computing  $\vec{x}(2), \vec{x}(3), \dots$ , we could not easily detect a trend that would allow us to generate closed formulas for  $c(t)$  and  $r(t)$ . We have to look for another approach.

The idea is to work with the eigenbasis

$$\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

considered in the first two cases. Any vector in  $\mathbb{R}^2$  can be written uniquely as a linear combination of the eigenvectors  $\vec{v}_1$  and  $\vec{v}_2$ . This holds in particular for the initial state vector

$$\vec{x}_0 = \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix}$$

of the coyote–roadrunner system:

$$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2.$$

A straightforward computation shows that the coordinates are  $c_1 = 2$  and  $c_2 = 4$ :

$$\vec{x}_0 = 2\vec{v}_1 + 4\vec{v}_2.$$

Recall that  $A^t \vec{v}_1 = (1.1)^t \vec{v}_1$  and  $A^t \vec{v}_2 = (0.9)^t \vec{v}_2$ . Therefore,

$$\begin{aligned} \vec{x}(t) &= A^t \vec{x}_0 = A^t(2\vec{v}_1 + 4\vec{v}_2) = 2A^t \vec{v}_1 + 4A^t \vec{v}_2 \\ &= 2(1.1)^t \vec{v}_1 + 4(0.9)^t \vec{v}_2 \\ &= 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}. \end{aligned}$$

Considering the components of this vector equation, we can now find closed formulas for  $c(t)$  and  $r(t)$ :

$$\begin{aligned} c(t) &= 200(1.1)^t + 800(0.9)^t \\ r(t) &= 600(1.1)^t + 400(0.9)^t. \end{aligned}$$

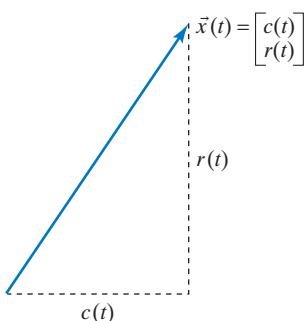
Since the terms involving  $0.9^t$  approach zero as  $t$  increases, both populations eventually grow by about 10% a year, and their ratio  $r(t)/c(t)$  approaches  $600/200 = 3$ .

Note that the ratio  $r(t)/c(t)$  can be interpreted as the slope of the state vector  $\vec{x}(t)$ , as shown in Figure 5.

Alternatively, we can derive the formula

$$\vec{x}(t) = 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

by means of *diagonalization*. We can use the eigenbasis  $\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$  to form the invertible matrix  $S = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix}$ . Then  $S^{-1}AS = B = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$ . See Theorem 1.3. Thus,  $A = SBS^{-1}$  and  $A^t = (SBS^{-1})^t =$



**Figure 5**

## Eigenvalues and Eigenvectors

$\underbrace{(SBS^{-1})(SBS^{-1}) \cdots (SBS^{-1})}_{t \text{ times}} = SB^t S^{-1}$ . Now  $\vec{x}(t) = SB^t S^{-1} \vec{x}_0$ . We leave it

to the reader to verify that  $S^{-1} \vec{x}_0 = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix}^{-1} \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ . This is just the coordinate vector of the initial state vector  $\vec{x}_0$  with respect to the given eigenbasis. Thus,

$$\begin{aligned} \vec{x}(t) &= SB^t S^{-1} \vec{x}_0 = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix} \begin{bmatrix} 1.1^t & 0 \\ 0 & 0.9^t \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix} \begin{bmatrix} 2(1.1)^t \\ 4(0.9)^t \end{bmatrix} \\ &= 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}, \end{aligned}$$

as expected.

How can we represent the preceding computations graphically?

Figure 6 shows the representation  $\vec{x}_0 = 2\vec{v}_1 + 4\vec{v}_2$  of  $\vec{x}_0$  as the sum of a vector on  $L_1 = \text{span}(\vec{v}_1)$  and a vector on  $L_2 = \text{span}(\vec{v}_2)$ . The formula

$$\vec{x}(t) = (1.1)^t 2\vec{v}_1 + (0.9)^t 4\vec{v}_2$$

now tells us that the component in  $L_1$  grows by 10% each year, while the component in  $L_2$  shrinks by 10%. The component  $(0.9)^t 4\vec{v}_2$  in  $L_2$  approaches  $\vec{0}$ , which means that the tip of the state vector  $\vec{x}(t)$  approaches the line  $L_1$ , so that the slope of the state vector will approach 3, the slope of  $L_1$ .

To show the evolution of the system more clearly, we can sketch just the endpoints of the state vectors  $\vec{x}(t)$ . Then the changing state of the system will be traced out as a sequence of points in the  $c$ - $r$ -plane.

It is natural to connect the dots to create the illusion of a continuous *trajectory*. (Although, of course, we do not know what really happens between times  $t$  and  $t+1$ .)

Sometimes we are interested in the state of the system in the past, at times  $-1, -2, \dots$ . Note that  $\vec{x}(0) = A\vec{x}(-1)$ , so that  $\vec{x}(-1) = A^{-1}\vec{x}_0$  if  $A$  is invertible (as in our example). Likewise,  $\vec{x}(-t) = (A^t)^{-1}\vec{x}_0$ , for  $t = 2, 3, \dots$ . The trajectory (future and past) for our coyote-roadrunner system is shown in Figure 7.

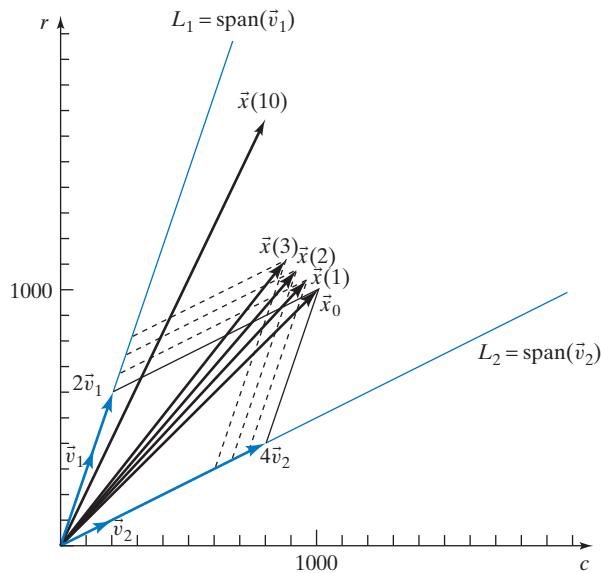
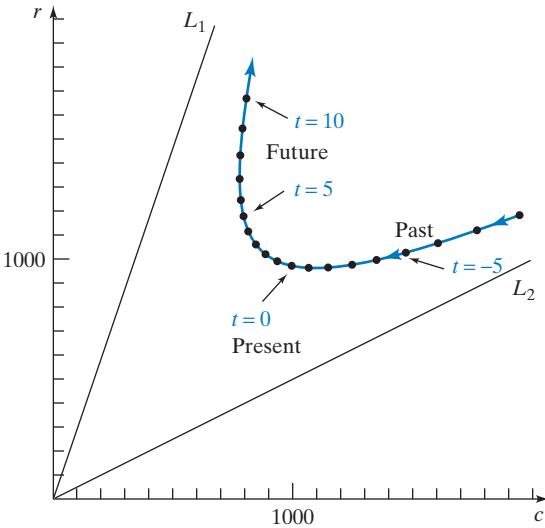


Figure 6

## Eigenvalues and Eigenvectors



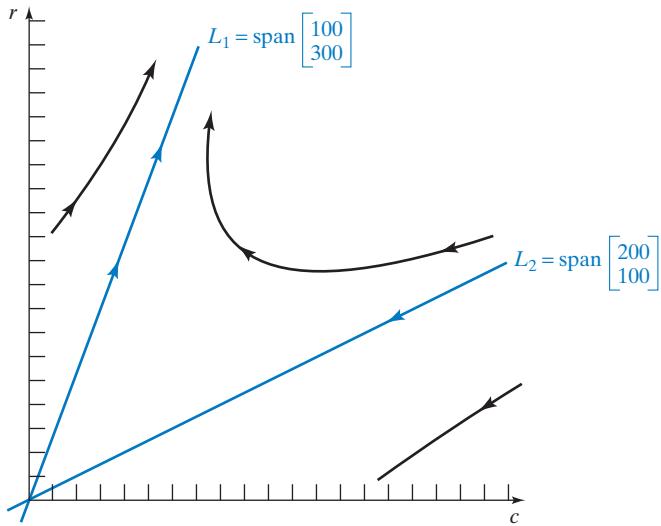
**Figure 7**

To get a sense for the long-term behavior of this system and how it depends on the initial state, we can draw a rough sketch that shows a number of different trajectories, representing the various qualitative types of behavior. Such a sketch is called a *phase portrait* of the system. In our example, a phase portrait might show the trajectory drawn in Figure 7, trajectories moving along the lines  $L_1$  and  $L_2$ , as well as a trajectory that starts above  $L_1$  and one that starts below  $L_2$ . See Figure 8.

To sketch these trajectories, express the initial state vector  $\vec{x}_0$  as the sum of a vector  $\vec{w}_1$  on  $L_1$  and a vector  $\vec{w}_2$  on  $L_2$ . Then see how these two vectors change over time. If  $\vec{x}_0 = \vec{w}_1 + \vec{w}_2$ , then

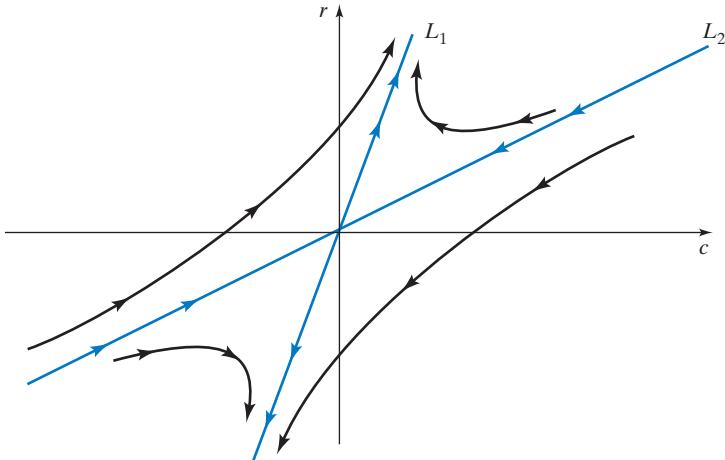
$$\vec{x}(t) = (1.1)^t \vec{w}_1 + (0.9)^t \vec{w}_2.$$

We see that the two populations will prosper over the long term if the ratio  $r_0/c_0$  of the initial populations exceeds  $1/2$ ; otherwise, both populations will die out.



**Figure 8**

## Eigenvalues and Eigenvectors



**Figure 9**

From a mathematical point of view, it is informative to sketch a phase portrait of this system in the whole  $c$ - $r$ -plane, even though the trajectories outside the first quadrant are meaningless in terms of our population study. See Figure 9. ■

Let's discuss the ideas and techniques developed in Example 7 more systematically. Consider a physical system whose state at any given time  $t$  is described by some quantities  $x_1(t), x_2(t), \dots, x_n(t)$ . [In Example 7, there were two such quantities, the populations  $c(t)$  and  $r(t)$ .] We can represent the quantities  $x_1(t), x_2(t), \dots, x_n(t)$  by the state vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Suppose that the state of the system at time  $t + 1$  is determined by the state at time  $t$  and that the transformation of the system from time  $t$  to time  $t + 1$  is linear, represented by an  $n \times n$  matrix  $A$ :

$$\vec{x}(t + 1) = A\vec{x}(t).$$

Then

$$\vec{x}(t) = A^t \vec{x}_0.$$

Such a system is called a *discrete linear dynamical system*. (*Discrete* indicates that we model the change of the system from time  $t$  to time  $t + 1$ , rather than modeling the *continuous* rate of change, which would be described by differential equations.)

For an initial state  $\vec{x}_0$ , it is often our goal to find *closed formulas* for  $x_1(t), x_2(t), \dots, x_n(t)$  [i.e., formulas expressing  $x_i(t)$  as a function of  $t$  alone, as opposed to a recursive formula, for example, which would merely express  $x_i(t + 1)$  in terms of  $x_1(t), x_2(t), \dots, x_n(t)$ ].

### Theorem 1.6

#### Discrete dynamical systems

Consider the dynamical system

$$\vec{x}(t + 1) = A\vec{x}(t) \quad \text{with} \quad \vec{x}(0) = \vec{x}_0.$$

**Theorem 1.6**
**Discrete dynamical systems (*Continued*)**

Then  $\vec{x}(t) = A^t \vec{x}_0$ . Suppose we can find an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$ , with  $A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$ . Find the coordinates  $c_1, \dots, c_n$  of the vector  $\vec{x}_0$  with respect to the eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$ :

$$\vec{x}_0 = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n.$$

Then

$$\vec{x}(t) = A^t \vec{x}_0 = c_1 A^t \vec{v}_1 + \dots + c_n A^t \vec{v}_n = c_1 \lambda_1^t \vec{v}_1 + \dots + c_n \lambda_n^t \vec{v}_n.$$

Alternatively, we can derive the formula  $\vec{x}(t) = c_1 \lambda_1^t \vec{v}_1 + \dots + c_n \lambda_n^t \vec{v}_n$  by means of diagonalization. See Example 7. If we let

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}, B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \text{ and } S^{-1} \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \text{ then}$$

$S^{-1}AS = B$ ,  $A = SBS^{-1}$ , and  $A^t = (SBS^{-1})^t = \underbrace{(SBS^{-1})(SBS^{-1}) \cdots (SBS^{-1})}_{t \text{ times}} = SB^t S^{-1}$ , so that

$$\begin{aligned} \vec{x}(t) &= A^t \vec{x}_0 = SB^t S^{-1} \vec{x}_0 = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 & \dots & 0 \\ 0 & \lambda_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \lambda_1^t \\ c_2 \lambda_2^t \\ \vdots \\ c_n \lambda_n^t \end{bmatrix} = c_1 \lambda_1^t \vec{v}_1 + c_2 \lambda_2^t \vec{v}_2 + \dots + c_n \lambda_n^t \vec{v}_n. \end{aligned}$$

We are left with two questions: How can we find the eigenvalues and eigenvectors of an  $n \times n$  matrix  $A$ ? When is there an eigenbasis for  $A$ , i.e., when is  $A$  diagonalizable? These issues are central to linear algebra; they will keep us busy for the rest of this long chapter.

**Definition 1.7**
**Discrete trajectories and phase portraits**

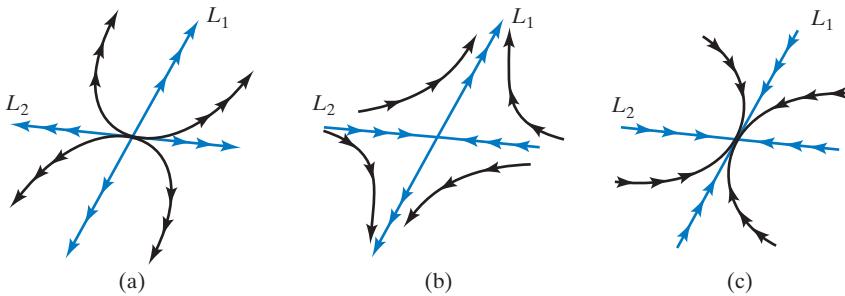
Consider a discrete dynamical system

$$\vec{x}(t+1) = A\vec{x}(t) \quad \text{with initial value } \vec{x}(0) = \vec{x}_0,$$

where  $A$  is a  $2 \times 2$  matrix. In this case, the state vector  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  can be represented geometrically in the  $x_1$ - $x_2$ -plane.

The endpoints of state vectors  $\vec{x}(0) = \vec{x}_0, \vec{x}(1) = A\vec{x}_0, \vec{x}(2) = A^2\vec{x}_0, \dots$  form the (discrete) *trajectory* of this system, representing its evolution in the future. Sometimes we are interested in the past states  $\vec{x}(-1) = A^{-1}\vec{x}_0, \vec{x}(-2) = (A^2)^{-1}\vec{x}_0, \dots$  as well. It is suggestive to “connect the dots” to create the illusion of a continuous trajectory. Take another look at Figure 7.

A (discrete) *phase portrait* of the system  $\vec{x}(t+1) = A\vec{x}(t)$  shows trajectories for various initial states, capturing all the qualitatively different scenarios (as in Figure 9).



**Figure 10** (a)  $\lambda_1 > \lambda_2 > 1$ . (b)  $\lambda_1 > 1 > \lambda_2 > 0$ . (c)  $1 > \lambda_1 > \lambda_2 > 0$ .

In Figure 10, we sketch phase portraits for the case when  $A$  has two eigenvalues  $\lambda_1 > \lambda_2 > 0$  with associated eigenvectors  $\vec{v}_1$  and  $\vec{v}_2$ . We leave out the special case when one of the eigenvalues is 1. Start by sketching the trajectories along the lines  $L_1 = \text{span}(\vec{v}_1)$  and  $L_2 = \text{span}(\vec{v}_2)$ . As you sketch the other trajectories

$$\vec{x}(t) = c_1 \lambda_1^t \vec{v}_1 + c_2 \lambda_2^t \vec{v}_2,$$

think about the summands  $c_1 \lambda_1^t \vec{v}_1$  and  $c_2 \lambda_2^t \vec{v}_2$ . Note that for a large positive  $t$ , the vector  $\vec{x}(t)$  will be almost parallel to  $L_1$ , since  $\lambda_1^t$  will be much larger than  $\lambda_2^t$ . Likewise, for large negative  $t$  the vector  $\vec{x}(t)$  will be almost parallel to  $L_2$ .

## EXERCISES I

**GOAL** Apply the concepts of eigenvalues, eigenvectors, eigenbases, and diagonalization. Use eigenvectors to analyze discrete dynamical systems.

In Exercises 1 through 4, let  $A$  be an invertible  $n \times n$  matrix and  $\vec{v}$  an eigenvector of  $A$  with associated eigenvalue  $\lambda$ .

1. Is  $\vec{v}$  an eigenvector of  $A^3$ ? If so, what is the eigenvalue?
2. Is  $\vec{v}$  an eigenvector of  $A^{-1}$ ? If so, what is the eigenvalue?
3. Is  $\vec{v}$  an eigenvector of  $A + 2I_n$ ? If so, what is the eigenvalue?
4. Is  $\vec{v}$  an eigenvector of  $7A$ ? If so, what is the eigenvalue?
5. If a vector  $\vec{v}$  is an eigenvector of both  $A$  and  $B$ , is  $\vec{v}$  necessarily an eigenvector of  $A + B$ ?
6. If a vector  $\vec{v}$  is an eigenvector of both  $A$  and  $B$ , is  $\vec{v}$  necessarily an eigenvector of  $AB$ ?
7. If  $\vec{v}$  is an eigenvector of the  $n \times n$  matrix  $A$  with associated eigenvalue  $\lambda$ , what can you say about

$$\ker(A - \lambda I_n)?$$

Is the matrix  $A - \lambda I_n$  invertible?

8. Find all  $2 \times 2$  matrices for which  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is an eigenvector with associated eigenvalue 5.
9. Find all  $2 \times 2$  matrices for which  $\vec{e}_1$  is an eigenvector.

10. Find all  $2 \times 2$  matrices for which  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an eigenvector with associated eigenvalue 5.

11. Find all  $2 \times 2$  matrices for which  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$  is an eigenvector with associated eigenvalue -1.

12. Consider the matrix  $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$ . Show that 2 and 4 are eigenvalues of  $A$  and find all corresponding eigenvectors. Find an eigenbasis for  $A$  and thus diagonalize  $A$ .

13. Show that 4 is an eigenvalue of  $A = \begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix}$  and find all corresponding eigenvectors.

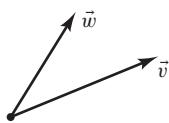
14. Find all  $4 \times 4$  matrices for which  $\vec{e}_2$  is an eigenvector.

Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformations in Exercises 15 through 22. In each case, find an eigenbasis if you can, and thus determine whether the given transformation is diagonalizable.

15. Reflection about a line  $L$  in  $\mathbb{R}^2$
16. Rotation through an angle of  $180^\circ$  in  $\mathbb{R}^2$
17. Counterclockwise rotation through an angle of  $45^\circ$  followed by a scaling by 2 in  $\mathbb{R}^2$

## Eigenvalues and Eigenvectors

18. Reflection about a plane  $V$  in  $\mathbb{R}^3$
19. Orthogonal projection onto a line  $L$  in  $\mathbb{R}^3$
20. Rotation about the  $\vec{e}_3$ -axis through an angle of  $90^\circ$ , counterclockwise as viewed from the positive  $\vec{e}_3$ -axis in  $\mathbb{R}^3$
21. Scaling by 5 in  $\mathbb{R}^3$
22. The linear transformation with  $T(\vec{v}) = \vec{v}$  and  $T(\vec{w}) = \vec{v} + \vec{w}$  for the vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^2$  sketched below

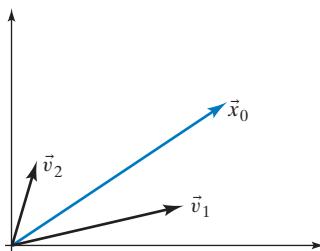


23. Use matrix products to prove the following: If  $S^{-1}AS = B$  for an invertible matrix  $S$  and a diagonal matrix  $B$ , then the column vectors of  $S$  will form an eigenbasis for  $A$ , and the diagonal entries of  $B$  will be the associated eigenvalues. See Theorem 1.3.

*In Exercises 24 through 29, consider a dynamical system*

$$\vec{x}(t+1) = A\vec{x}(t)$$

*with two components. The accompanying sketch shows the initial state vector  $\vec{x}_0$  and two eigenvectors,  $\vec{v}_1$  and  $\vec{v}_2$ , of  $A$  (with eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively). For the given values of  $\lambda_1$  and  $\lambda_2$ , sketch a rough trajectory. Consider the future and the past of the system.*



24.  $\lambda_1 = 1.1, \lambda_2 = 0.9$       25.  $\lambda_1 = 1, \lambda_2 = 0.9$   
 26.  $\lambda_1 = 1.1, \lambda_2 = 1$       27.  $\lambda_1 = 0.9, \lambda_2 = 0.8$   
 28.  $\lambda_1 = 1.2, \lambda_2 = 1.1$       29.  $\lambda_1 = 0.9, \lambda_2 = 0.9$

*In Exercises 30 through 32, consider the dynamical system*

$$\vec{x}(t+1) = \begin{bmatrix} 1.1 & 0 \\ 0 & \lambda \end{bmatrix} \vec{x}(t).$$

*Sketch a phase portrait of this system for the given values of  $\lambda$ :*

30.  $\lambda = 1.2$       31.  $\lambda = 1$       32.  $\lambda = 0.9$

33. Find a  $2 \times 2$  matrix  $A$  such that

$$\vec{x}(t) = \begin{bmatrix} 2^t - 6^t \\ 2^t + 6^t \end{bmatrix}$$

is a trajectory of the dynamical system

$$\vec{x}(t+1) = A\vec{x}(t).$$

34. Suppose  $\vec{v}$  is an eigenvector of the  $n \times n$  matrix  $A$ , with eigenvalue 4. Explain why  $\vec{v}$  is an eigenvector of  $A^2 + 2A + 3I_n$ . What is the associated eigenvalue?

35. Show that similar matrices have the same eigenvalues.  
*Hint:* If  $\vec{v}$  is an eigenvector of  $S^{-1}AS$ , then  $S\vec{v}$  is an eigenvector of  $A$ .

36. Find a  $2 \times 2$  matrix  $A$  such that  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  are eigenvectors of  $A$ , with eigenvalues 5 and 10, respectively.

37. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

- a. Use the geometric interpretation of this transformation as a reflection combined with a scaling to find the eigenvalues of  $A$ .

- b. Find an eigenbasis for  $A$ .  
 c. Diagonalize  $A$ .

38. We are told that  $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$  is an eigenvector of the matrix

$$\begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix}; \text{ what is the associated eigenvalue?}$$

39. Find a basis of the linear space  $V$  of all  $2 \times 2$  matrices  $A$  for which  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is an eigenvector, and thus determine the dimension of  $V$ .

40. Find a basis of the linear space  $V$  of all  $2 \times 2$  matrices  $A$  for which  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$  is an eigenvector, and thus determine the dimension of  $V$ .

41. Find a basis of the linear space  $V$  of all  $2 \times 2$  matrices  $A$  for which both  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  are eigenvectors, and thus determine the dimension of  $V$ .

42. Find a basis of the linear space  $V$  of all  $3 \times 3$  matrices  $A$  for which both  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  are eigenvectors, and thus determine the dimension of  $V$ .

## Eigenvalues and Eigenvectors

43. Consider the linear space  $V$  of all  $n \times n$  matrices for which all the vectors  $\vec{e}_1, \dots, \vec{e}_n$  are eigenvectors. Describe the space  $V$  (the matrices in  $V$  “have a name”), and determine the dimension of  $V$ .
44. For  $m \leq n$ , find the dimension of the space of all  $n \times n$  matrices  $A$  for which all the vectors  $\vec{e}_1, \dots, \vec{e}_m$  are eigenvectors.
45. If  $\vec{v}$  is any nonzero vector in  $\mathbb{R}^2$ , what is the dimension of the space  $V$  of all  $2 \times 2$  matrices for which  $\vec{v}$  is an eigenvector?
46. If  $\vec{v}$  is an eigenvector of matrix  $A$  with associated eigenvalue 3, show that  $\vec{v}$  is in the image of matrix  $A$ .
47. If  $\vec{v}$  is an eigenvector of matrix  $A$ , show that  $\vec{v}$  is in the image of  $A$  or in the kernel of  $A$ . Hint: Exercise 46 is helpful.
48. If  $A$  is a matrix of rank 1, show that any nonzero vector in the image of  $A$  is an eigenvector of  $A$ .
49. Give an example of a matrix  $A$  of rank 1 that fails to be diagonalizable.

*Find an eigenbasis for each of the matrices  $A$  in Exercises 50 through 54, and thus diagonalize  $A$ . Hint: Exercise 48 is helpful.*

50.  $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$     51.  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$     52.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

53.  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$     54.  $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

*Arguing geometrically, find an eigenbasis for each of the matrices  $A$  in Exercises 55 through 63, and thus diagonalize  $A$ . Use Example 3 as a guide.*

55.  $A = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$ , representing the orthogonal projection onto the line  $L = \text{span} \left[ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right]$

56.  $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ , representing the reflection about the line  $L = \text{span} \left[ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]$

57.  $A = \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$     58.  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

59.  $A = \frac{1}{9} \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$ , representing the orthogonal projection onto the plane  $x - 2y + 2z = 0$

60.  $A = \frac{1}{9} \begin{bmatrix} 7 & 4 & -4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix}$ , representing the reflection about the plane  $x - 2y + 2z = 0$

61.  $A = \frac{1}{7} \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}$ , representing the reflection about a plane  $E$

62.  $A = \frac{1}{14} \begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$ , representing the orthogonal projection onto a plane  $E$

63.  $A = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

64. In all parts of this problem, let  $V$  be the linear space of all  $2 \times 2$  matrices for which  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an eigenvector.

a. Find a basis of  $V$  and thus determine the dimension of  $V$ .

b. Consider the linear transformation  $T(A) = A \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  from  $V$  to  $\mathbb{R}^2$ . Find a basis of the image of  $T$  and a basis of the kernel of  $T$ . Determine the rank of  $T$ .

c. Consider the linear transformation  $L(A) = A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  from  $V$  to  $\mathbb{R}^2$ . Find a basis of the image of  $L$  and a basis of the kernel of  $L$ . Determine the rank of  $L$ .

65. Consider an  $n \times n$  matrix  $A$ . A subspace  $V$  of  $\mathbb{R}^n$  is said to be  $A$ -invariant if  $A\vec{v}$  is in  $V$  for all  $\vec{v}$  in  $V$ . Describe all the one-dimensional  $A$ -invariant subspaces of  $\mathbb{R}^n$ , in terms of the eigenvectors of  $A$ .

66. a. Give an example of a  $3 \times 3$  matrix  $A$  with as many nonzero entries as possible such that both  $\text{span}(\vec{e}_1)$  and  $\text{span}(\vec{e}_1, \vec{e}_2)$  are  $A$ -invariant subspaces of  $\mathbb{R}^3$ . See Exercise 65.

b. Consider the linear space  $V$  of all  $3 \times 3$  matrices  $A$  such that both  $\text{span}(\vec{e}_1)$  and  $\text{span}(\vec{e}_1, \vec{e}_2)$  are  $A$ -invariant subspaces of  $\mathbb{R}^3$ . Describe the space  $V$  (the matrices in  $V$  “have a name”), and determine the dimension of  $V$ .

67. Consider the coyotes–roadrunner system discussed in Example 7. Find closed formulas for  $c(t)$  and  $r(t)$ , for the initial populations  $c_0 = 100$ ,  $r_0 = 800$ .

68. Two interacting populations of hares and foxes can be modeled by the recursive equations

$$h(t+1) = 4h(t) - 2f(t) \\ f(t+1) = h(t) + f(t).$$

For each of the initial populations given in parts (a) through (c), find closed formulas for  $h(t)$  and  $f(t)$ .

## Eigenvalues and Eigenvectors

- a.  $h(0) = f(0) = 100$   
 b.  $h(0) = 200, f(0) = 100$   
 c.  $h(0) = 600, f(0) = 500$
69. Two interacting populations of coyotes and roadrunners can be modeled by the recursive equations
- $$\begin{aligned}c(t+1) &= && 0.75r(t) \\r(t+1) &= -1.5c(t) + 2.25r(t).\end{aligned}$$
- For each of the initial populations given in parts (a) through (c), find closed formulas for  $c(t)$  and  $r(t)$ .
- a.  $c(0) = 100, r(0) = 200$   
 b.  $c(0) = r(0) = 100$   
 c.  $c(0) = 500, r(0) = 700$
70. Imagine that you are diabetic and have to pay close attention to how your body metabolizes glucose. Let  $g(t)$  represent the *excess glucose concentration* in your blood, usually measured in milligrams of glucose per 100 milliliters of blood. (*Excess* means that we measure how much the glucose concentration deviates from your fasting level, i.e., the level your system approaches after many hours of fasting.) A negative value of  $g(t)$  indicates that the glucose concentration is below fasting level at time  $t$ . Shortly after you eat a heavy meal, the function  $g(t)$  will reach a peak, and then it will slowly return to 0. Certain hormones help regulate glucose, especially the hormone *insulin*. Let  $h(t)$  represent the excess hormone concentration in your blood. Researchers have developed mathematical models for the glucose regulatory system. The following is one such model, in slightly simplified form (these formulas apply between meals; obviously, the system is disturbed during and right after a meal):
- $$\begin{cases} g(t+1) = ag(t) - bh(t) \\ h(t+1) = cg(t) + dh(t) \end{cases},$$
- where time  $t$  is measured in minutes;  $a$  and  $d$  are constants slightly less than 1; and  $b$  and  $c$  are small positive constants. For your system, the equations might be
- $$\begin{cases} g(t+1) = 0.978g(t) - 0.006h(t) \\ h(t+1) = 0.004g(t) + 0.992h(t) \end{cases}.$$
- The term  $-0.006h(t)$  in the first equation is negative, because insulin helps your body absorb glucose. The term  $0.004g(t)$  is positive, because glucose in your blood stimulates the cells of the pancreas to secrete insulin. (For a more thorough discussion of this model, read E. Ackerman et al., “Blood glucose regulation and diabetes,” Chapter 4 in *Concepts and Models of Biomathematics*, Marcel Dekker, 1969.)
- Consider the coefficient matrix
- $$A = \begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix}$$
- of this dynamical system.
- a. We are told that  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$  are eigenvectors of  $A$ . Find the associated eigenvalues.  
 b. After you have consumed a heavy meal, the concentrations in your blood are  $g_0 = 100$  and  $h_0 = 0$ . Find closed formulas for  $g(t)$  and  $h(t)$ . Sketch the trajectory. Briefly describe the evolution of this system in practical terms.  
 c. For the case discussed in part (b), how long does it take for the glucose concentration to fall below fasting level? (This quantity is useful in diagnosing diabetes: A period of more than four hours may indicate mild diabetes.)
71. Three holy men (let’s call them Anselm, Benjamin, and Caspar) put little stock in material things; their only earthly possession is a small purse with a bit of gold dust. Each day they get together for the following bizarre bonding ritual: Each of them takes his purse and gives his gold away to the two others, in equal parts. For example, if Anselm has 4 ounces one day, he will give 2 ounces each to Benjamin and Caspar.
- a. If Anselm starts out with 6 ounces, Benjamin with 1 ounce, and Caspar with 2 ounces, find formulas for the amounts  $a(t)$ ,  $b(t)$ , and  $c(t)$  each will have after  $t$  distributions.  
*Hint:* The vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  will be useful.  
 b. Who will have the most gold after one year, that is, after 365 distributions?
72. Consider the growth of a lilac bush. The state of this lilac bush for several years (at year’s end) is shown in the accompanying sketch. Let  $n(t)$  be the number of new branches (grown in the year  $t$ ) and  $a(t)$  the number of old branches. In the sketch, the new branches are represented by shorter lines. Each old branch will grow two new branches in the following year. We assume that no branches ever die.
- |            |            |            |            |            |
|------------|------------|------------|------------|------------|
| year 0     | year 1     | year 2     | year 3     | year 4     |
|            |            |            |            |            |
| $n(0) = 1$ | $n(1) = 0$ | $n(2) = 2$ | $n(3) = 2$ | $n(4) = 6$ |
| $a(0) = 0$ | $a(1) = 1$ | $a(2) = 1$ | $a(3) = 3$ | $a(4) = 5$ |
- a. Find the matrix  $A$  such that
- $$\begin{bmatrix} n(t+1) \\ a(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ a(t) \end{bmatrix}.$$
- b. Verify that  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  are eigenvectors of  $A$ . Find the associated eigenvalues.  
 c. Find closed formulas for  $n(t)$  and  $a(t)$ .

## 2 Finding the Eigenvalues of a Matrix

In the previous section, we used eigenvalues to diagonalize a matrix  $A$  and to analyze the dynamical system

$$\vec{x}(t+1) = A\vec{x}(t).$$

Now we will see how we can actually find those eigenvalues.

Consider an  $n \times n$  matrix  $A$  and a scalar  $\lambda$ . By Definition 1.2,  $\lambda$  is an eigenvalue of  $A$  if there exists a nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  such that

$$A\vec{v} = \lambda\vec{v} \quad \text{or} \quad A\vec{v} - \lambda\vec{v} = \vec{0} \quad \text{or} \quad A\vec{v} - \lambda I_n \vec{v} = \vec{0} \quad \text{or} \quad (A - \lambda I_n)\vec{v} = \vec{0}.$$

This means, by definition of the kernel, that

$$\ker(A - \lambda I_n) \neq \{\vec{0}\}.$$

(That is, there are other vectors in the kernel besides the zero vector.) This is the case if (and only if) the matrix  $A - \lambda I_n$  fails to be invertible, that is, if  $\det(A - \lambda I_n) = 0$ .

### Theorem 2.1

#### Eigenvalues and determinants; characteristic equation

Consider an  $n \times n$  matrix  $A$  and a scalar  $\lambda$ . Then  $\lambda$  is an eigenvalue<sup>5</sup> of  $A$  if (and only if)

$$\det(A - \lambda I_n) = 0.$$

This is called the *characteristic equation* (or the *secular equation*) of matrix  $A$ .

Let's write the observations we made previously as a string of equivalent statements.

$\lambda$  is an eigenvalue of  $A$ .

$\Updownarrow$

There exists a nonzero vector  $\vec{v}$  such that

$$A\vec{v} = \lambda\vec{v} \quad \text{or} \quad (A - \lambda I_n)\vec{v} = \vec{0}.$$

$\Updownarrow$

$$\ker(A - \lambda I_n) \neq \{\vec{0}\}.$$

$\Updownarrow$

Matrix  $A - \lambda I_n$  fails to be invertible.

$\Updownarrow$

$$\det(A - \lambda I_n) = 0.$$

The idea of the characteristic equation is implicit in the work of Jean d'Alembert (1717–1783), in his *Traité de Dynamique* of 1743. Joseph Louis Lagrange (1736–1813) was the first to study the equation systematically (naming it *équation séculaire*), in his works on gravitational attraction between heavenly bodies. Augustin-Louis Cauchy (1789–1857) wrote the equation in its modern form, involving a determinant. It appears that the term *eigenvalue* (*Eigenwert* in German)

---

<sup>5</sup>Alternatively, the eigenvalues are the solutions of the equation  $\det(\lambda I_n - A) = 0$ . Our formula  $\det(A - \lambda I_n) = 0$  is usually more convenient for numerical work.

## Eigenvalues and Eigenvectors

was first used by David Hilbert in 1904, based perhaps on Helmholtz's notion of an *Eigenton* in acoustics.

**EXAMPLE 1** Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

### Solution

By Theorem 2.1, we have to solve the characteristic equation  $\det(A - \lambda I_2) = 0$ . Now

$$\begin{aligned} \det(A - \lambda I_2) &= \det \left( \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det \begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix} \\ &= (1 - \lambda)(3 - \lambda) - 2 \cdot 4 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0. \end{aligned}$$

The equation  $\det(A - \lambda I_2) = (\lambda - 5)(\lambda + 1) = 0$  holds for  $\lambda_1 = 5$  and  $\lambda_2 = -1$ . These two scalars are the eigenvalues of  $A$ . In Section 3 we will find the corresponding eigenvectors. ■

**EXAMPLE 2** Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

### Solution

We have to solve the characteristic equation  $\det(A - \lambda I_3) = 0$ .

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 2 - \lambda & 3 & 4 \\ 0 & 3 - \lambda & 4 \\ 0 & 0 & 4 - \lambda \end{bmatrix} \xrightarrow{\text{step 2}} (2 - \lambda)(3 - \lambda)(4 - \lambda) = 0$$

In step 2 we use the fact that the determinant of a triangular matrix is the product of its diagonal entries. The solutions of the characteristic equation are 2, 3, and 4; these are the eigenvalues of  $A$ . ■

### Theorem 2.2

#### Eigenvalues of a triangular matrix

The eigenvalues of a triangular matrix are its diagonal entries.

**EXAMPLE 3** Find the characteristic equation for an arbitrary  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

### Solution

$$\begin{aligned} \det(A - \lambda I_2) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \end{aligned}$$

This is a quadratic equation. The constant term of  $\det(A - \lambda I_2)$  is  $ad - bc = \det A$ , the value of  $\det(A - \lambda I_2)$  at  $\lambda = 0$ . The coefficient of  $\lambda$  is  $-(a + d)$ , the opposite of the sum of the diagonal entries  $a$  and  $d$  of  $A$ . Since this sum is important in many other contexts as well, we introduce a name for it. ■

## Eigenvalues and Eigenvectors

### Definition 2.3

#### Trace

The sum of the diagonal entries of a square matrix  $A$  is called the *trace* of  $A$ , denoted by  $\text{tr } A$ .

Let us highlight the result of Example 3.

### Theorem 2.4

#### Characteristic equation of a $2 \times 2$ matrix $A$

$$\det(A - \lambda I_2) = \lambda^2 - (\text{tr } A)\lambda + \det A = 0$$

For the matrix  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ , we have  $\text{tr } A = 1+3 = 4$  and  $\det A = 3-8 = -5$ , so that the characteristic equation is

$$\lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 4\lambda - 5 = 0,$$

as we found in Example 1.

If  $A$  is a  $3 \times 3$  matrix, what does the characteristic equation  $\det(A - \lambda I_3) = 0$  look like?

$$\begin{aligned} & \det \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) + (\text{a polynomial of degree } \leq 1) \\ &= (\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22})(a_{33} - \lambda) + (\text{a polynomial of degree } \leq 1) \\ &= -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 + (\text{a polynomial of degree } \leq 1) \\ &= -\lambda^3 + (\text{tr } A)\lambda^2 + (\text{a polynomial of degree } \leq 1) \\ &= 0, \end{aligned}$$

a cubic equation. Again, the constant term is  $\det A$ , so that the characteristic equation has the form

$$\det(A - \lambda I_3) = -\lambda^3 + (\text{tr } A)\lambda^2 - c\lambda + \det A = 0, \quad \text{for some scalar } c.$$

It is possible to give a formula for  $c$ , in terms of the entries of  $A$ , but this formula is complicated.

Based on the quadratic and cubic cases, we might conjecture that the characteristic equation of any  $n \times n$  matrix  $A$  is a polynomial equation of degree  $n$ , of the form

$$\det(A - \lambda I_n) = (-1)^n \lambda^n + (-1)^{n-1}(\text{tr } A)\lambda^{n-1} + \cdots + \det A = 0.$$

It makes sense to write this equation in terms of  $-\lambda$  rather than  $\lambda$ :

$$\det(A - \lambda I_n) = (-\lambda)^n + (\text{tr } A)(-\lambda)^{n-1} + \cdots + \det A = 0.$$

Let us state and then prove that the characteristic equation is indeed of this form.

### Theorem 2.5

#### Characteristic polynomial

If  $A$  is an  $n \times n$  matrix, then  $\det(A - \lambda I_n)$  is a polynomial of degree  $n$ , of the form

$$\begin{aligned} & (-\lambda)^n + (\text{tr } A)(-\lambda)^{n-1} + \cdots + \det A \\ &= (-1)^n \lambda^n + (-1)^{n-1}(\text{tr } A)\lambda^{n-1} + \cdots + \det A. \end{aligned}$$

This is called the *characteristic polynomial* of  $A$ , denoted by  $f_A(\lambda)$ .

## Eigenvalues and Eigenvectors

### Proof

$$f_A(\lambda) = \det(A - \lambda I_n) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

The product associated with any pattern in the matrix  $A - \lambda I_n$  is a polynomial of degree less than or equal to  $n$ . This implies that  $\det(A - \lambda I_n)$  is a polynomial of degree less than or equal to  $n$ .

We can be more precise: The diagonal pattern gives the product

$$\begin{aligned} & (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) \\ &= (-\lambda)^n + (a_{11} + a_{22} + \cdots + a_{nn})(-\lambda)^{n-1} + (\text{a polynomial of degree } \leq n-2) \\ &= (-\lambda)^n + (\text{tr } A)(-\lambda)^{n-1} + (\text{a polynomial of degree } \leq n-2). \end{aligned}$$

Any other pattern involves at least two scalars off the diagonal, and its product is therefore a polynomial of degree less than or equal to  $n-2$ . This implies that

$$f_A(\lambda) = (-\lambda)^n + (\text{tr } A)(-\lambda)^{n-1} + (\text{a polynomial of degree } \leq n-2).$$

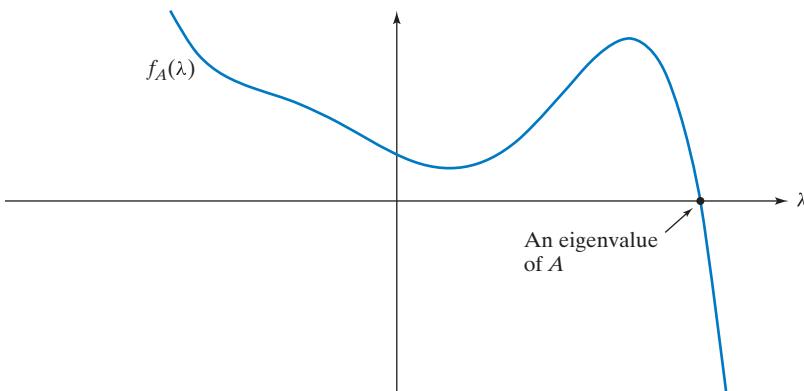
The constant term is  $f_A(0) = \det A$ . ■

Note that Theorem 2.4 represents a special case of Theorem 2.5, when  $n = 2$ .

What does Theorem 2.5 tell us about the number of eigenvalues of an  $n \times n$  matrix  $A$ ? We know from elementary algebra that a polynomial of degree  $n$  has at most  $n$  zeros. Therefore, an  $n \times n$  matrix has at most  $n$  eigenvalues. If  $n$  is odd, then  $f_A(\lambda)$  has at least one zero, by the intermediate value theorem, since

$$\lim_{\lambda \rightarrow \infty} f_A(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \rightarrow -\infty} f_A(\lambda) = \infty.$$

See Figure 1.



**Figure 1**

## Eigenvalues and Eigenvectors

**EXAMPLE 4** Find all eigenvalues of

$$A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

### Solution

The characteristic polynomial is  $f_A(\lambda) = (5 - \lambda)^3(4 - \lambda)^2$ , so that the eigenvalues are 4 and 5. Since 5 is a root of multiplicity 3 of the characteristic polynomial, we say that the eigenvalue 5 has algebraic multiplicity 3. Likewise, the eigenvalue 4 has algebraic multiplicity 2. ■

**Definition 2.6**

### Algebraic multiplicity of an eigenvalue

We say that an eigenvalue  $\lambda_0$  of a square matrix  $A$  has *algebraic multiplicity*  $k$  if  $\lambda_0$  is a root of multiplicity  $k$  of the characteristic polynomial  $f_A(\lambda)$ , meaning that we can write

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

for some polynomial  $g(\lambda)$  with  $g(\lambda_0) \neq 0$ . We write  $\text{almu}(\lambda_0) = k$ .

In Example 4,  $\text{almu}(5) = 3$  since

$$f_A(\lambda) = (5 - \lambda)^3 \underbrace{(4 - \lambda)^2}_{g(\lambda)}.$$

**EXAMPLE 5** Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

with their algebraic multiplicities.

### Solution

We leave it to the reader to verify that

$$f_A(\lambda) = \lambda^2(3 - \lambda) = (0 - \lambda)^2(3 - \lambda).$$

We have two distinct eigenvalues, 0 and 3, with algebraic multiplicities 2 and 1, respectively. We can write, more succinctly, that the eigenvalues are 0, 0, 3. ■

Let us summarize.

**Theorem 2.7**

### Number of eigenvalues

An  $n \times n$  matrix has *at most*  $n$  real eigenvalues, even if they are counted with their algebraic multiplicities.

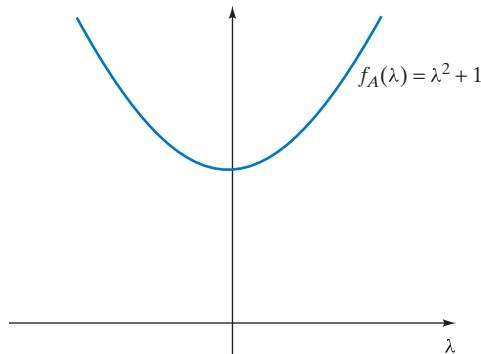
If  $n$  is odd, then an  $n \times n$  matrix has *at least* one real eigenvalue.

## Eigenvalues and Eigenvectors

If  $n$  is even, an  $n \times n$  matrix  $A$  need not have any real eigenvalues. Consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

with  $f_A(\lambda) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$ . See Figure 2.



**Figure 2**

Recall that the transformation  $T(\vec{x}) = A\vec{x}$  is a counterclockwise rotation through an angle of  $90^\circ$ . Geometrically, it makes sense that  $A$  has no real eigenvalues: Compare with Example 1.4.

**EXAMPLE 6** Describe all possible cases for the number of real eigenvalues (with their algebraic multiplicities) of a  $3 \times 3$  matrix  $A$ .

### Solution

Either the characteristic polynomial factors completely,

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda),$$

or it has a quadratic factor without real zeros:

$$f_A(\lambda) = (\lambda_1 - \lambda)p(\lambda), \quad \text{where } p(\lambda) \neq 0 \text{ for all real } \lambda.$$

In the first case, the  $\lambda_i$  could all be distinct, two of them could be equal, or they could all be equal. This leads to the following possibilities.

Case	No. of Distinct Eigenvalues	Algebraic Multiplicities
1	3	1 each
2	2	2 and 1
3	1	3
4	1	1

## Eigenvalues and Eigenvectors

Examples for each case follow.

**Case 1 ■** (see Figure 3)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad f_A(\lambda) = (1 - \lambda)(2 - \lambda)(3 - \lambda), \quad \text{Eigenvalues } 1, 2, 3$$

**Case 2 ■** (see Figure 4)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad f_A(\lambda) = (1 - \lambda)^2(2 - \lambda), \quad \text{Eigenvalues } 1, 1, 2$$

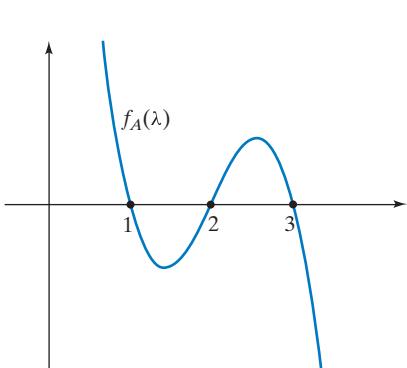
**Case 3 ■** (see Figure 5)

$$A = I_3, \quad f_A(\lambda) = (1 - \lambda)^3, \quad \text{Eigenvalues } 1, 1, 1$$

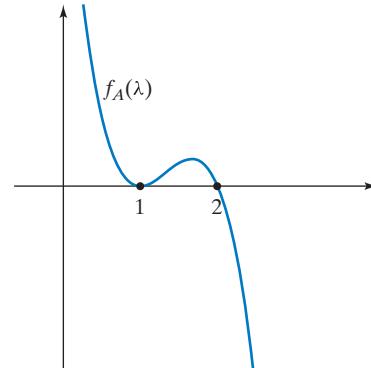
**Case 4 ■** (see Figure 6)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad f_A(\lambda) = (1 - \lambda)(\lambda^2 + 1), \quad \text{Eigenvalue } 1$$

You can recognize an eigenvalue  $\lambda_0$  whose algebraic multiplicity exceeds 1 on the graph of  $f_A(\lambda)$  by the fact that  $f_A(\lambda_0) = f'_A(\lambda_0) = 0$  (the derivative is zero, so that the tangent is horizontal). The verification of this observation is left as Exercise 37. ■



**Figure 3**



**Figure 4**

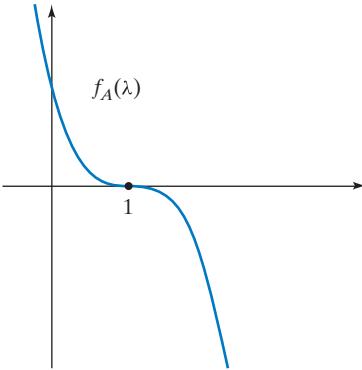


Figure 5

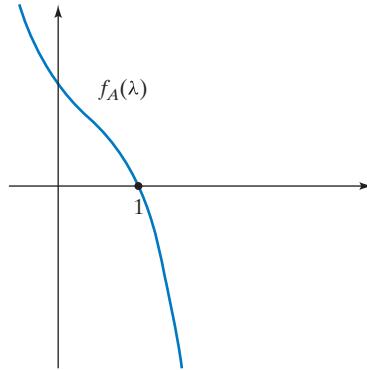


Figure 6

**EXAMPLE 7**

Suppose  $A$  is a  $2 \times 2$  matrix with eigenvalues  $\lambda_1$  and  $\lambda_2$  (we allow  $\lambda_1 = \lambda_2$  if  $\lambda_1$  has algebraic multiplicity 2). Explore the relationship among the sum  $\lambda_1 + \lambda_2$ , the product  $\lambda_1\lambda_2$ , the determinant  $\det A$ , and the trace  $\text{tr } A$ . You may want to consider a numerical example first, such as  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ , with eigenvalues  $\lambda_1 = 5$  and  $\lambda_2 = -1$ . See Example 1.

**Solution**

In the case of  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ , we observe that  $\det A = \lambda_1\lambda_2 = -5$  and  $\text{tr } A = \lambda_1 + \lambda_2 = 4$ . To see that these results hold in general, write the characteristic polynomial in two ways, as

$$f_A(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A$$

and as

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Comparing coefficients, we conclude that  $\det A = \lambda_1\lambda_2$  and  $\text{tr } A = \lambda_1 + \lambda_2$ . ■

It turns out that the observations we made in Example 7 generalize to  $n \times n$  matrices.

**Theorem 2.8**
**Eigenvalues, determinant, and trace**

If an  $n \times n$  matrix  $A$  has the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , listed with their algebraic multiplicities, then

$$\det A = \lambda_1\lambda_2 \cdots \lambda_n, \quad \text{the product of the eigenvalues}$$

and

$$\text{tr } A = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad \text{the sum of the eigenvalues.}$$

We will prove the claim concerning the determinant and leave the case of the trace as Exercise 21 for the reader.

Since the characteristic polynomial factors completely in this case, we can write

$$f_A(\lambda) = \det(A - \lambda I_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

## Eigenvalues and Eigenvectors

Now

$$f_A(0) = \det A = \lambda_1 \lambda_2 \cdots \lambda_n,$$

as claimed.

Note that the claims of Theorem 2.8 are trivial in the case of a triangular matrix, since the eigenvalues are the diagonal entries in this case.

### Finding the Eigenvalues of a Matrix in Practice

To find the eigenvalues of an  $n \times n$  matrix  $A$  with the method developed in this section, we have to find the zeros of  $f_A(\lambda)$ , a polynomial of degree  $n$ . For  $n = 2$ , this is a trivial matter: We can either factor the polynomial by inspection or use the quadratic formula (this formula was known over 3500 years ago in Mesopotamia, the present-day Iraq). The problem of finding the zeros of a polynomial of higher degree is nontrivial; it has been of considerable interest throughout the history of mathematics. In the early 1500s, Italian mathematicians found formulas in the cases  $n = 3$  and  $n = 4$ , published in the *Ars Magna* by Gerolamo Cardano.<sup>6</sup> See Exercise 50 for the case  $n = 3$ . During the next 300 years, people tried hard to find a general formula to solve the quintic (a polynomial equation of fifth degree). In 1824, the Norwegian mathematician Niels Henrik Abel (1802–1829) showed that no such general solution is possible, putting an end to the long search. The French mathematician Evariste Galois (1811–1832) was the first to give a numerical example of a quintic that cannot be solved by radicals. (Note the short life spans of these two brilliant mathematicians. Abel died from tuberculosis, and Galois died in a duel.)

When finding the eigenvalues of a matrix by means of the characteristic polynomial, it may be worth trying out a few small integers, such as  $\pm 1$  and  $\pm 2$ . The matrices considered in introductory linear algebra texts often just happen to have such eigenvalues.

In light of the preceding discussion, it is usually impossible to find the exact eigenvalues of a matrix. To find approximations for the eigenvalues, you could graph the characteristic polynomial, using technology. The graph may give you an idea of the number of eigenvalues and their approximate values. Numerical analysts tell us that this is not a very efficient way to go about finding the eigenvalues of large matrices; other techniques are used in practice. See Exercise 5.33 for an example; another approach uses the *QR* factorization. There is a lot of ongoing research in this area. A text in numerical linear algebra<sup>7</sup> characterizes the eigenvalue problem as “the third major problem area in matrix computations,” after linear equations and least squares, dedicating some 200 pages to this topic.

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<sup>6</sup>Cardano (1501–1576) was a Renaissance man with a wide range of interests. In his book *Liber de ludo aleae*, he presents the first systematic computations of probabilities. Trained as a physician, he gave the first clinical description of typhoid fever. In his book *Somniorum Synesiorum* (Basel, 1562), he explores the meaning of dreams. He was also a leading astrologer of his day, whose predictions won him access to some of the most powerful people in sixteenth-century Europe. Still, he is best known today as the most outstanding mathematician of his time and the author of the *Ars Magna*. In 1570, he was arrested on accusation of heresy; he lost his academic position and the right to publish.

To learn more about this fascinating fellow, read the award-winning biography, *Cardano's Cosmos*, by Anthony Grafton (Harvard University Press, 2000), focusing on Cardano's work as an astrologer.

For an English translation of part XI of the *Ars Magna* (dealing with cubic equations), see D. J. Struik (editor), *A Source Book in Mathematics 1200–1800*, Princeton University Press, 1986.

<sup>7</sup>G. H. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, 1996.

## EXERCISES 2

**GOAL** Use the characteristic polynomial  $f_A(\lambda) = \det(A - \lambda I_n)$  to find the eigenvalues of a matrix  $A$ , with their algebraic multiplicities.

For each of the matrices in Exercises 1 through 13, find all real eigenvalues, with their algebraic multiplicities. Show your work. Do not use technology.

1.  $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

2.  $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

4.  $\begin{bmatrix} 0 & 4 \\ -1 & 4 \end{bmatrix}$

5.  $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$

6.  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

7.  $I_3$

8.  $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

9.  $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

10.  $\begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$

11.  $\begin{bmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{bmatrix}$

12.  $\begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$

13.  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

14. Consider a  $4 \times 4$  matrix  $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ , where  $B, C$ , and  $D$  are  $2 \times 2$  matrices. What is the relationship among the eigenvalues of  $A, B, C$ , and  $D$ ?

15. Consider the matrix  $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$ , where  $k$  is an arbitrary constant. For which values of  $k$  does  $A$  have two distinct real eigenvalues? When is there no real eigenvalue?

16. Consider the matrix  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ , where  $a, b$ , and  $c$  are nonzero constants. For which values of  $a, b$ , and  $c$  does  $A$  have two distinct eigenvalues?

17. Consider the matrix  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a$  and  $b$  are arbitrary constants. Find all eigenvalues of  $A$ . Explain in terms of the geometric interpretation of the linear transformation  $T(\vec{x}) = A\vec{x}$ .

18. Consider the matrix  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ , where  $a$  and  $b$  are arbitrary constants. Find all eigenvalues of  $A$ .

19. True or false? If the determinant of a  $2 \times 2$  matrix  $A$  is negative, then  $A$  has two distinct real eigenvalues.

20. If a  $2 \times 2$  matrix  $A$  has two distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , show that  $A$  is diagonalizable.

21. Prove the part of Theorem 2.8 that concerns the trace: If an  $n \times n$  matrix  $A$  has  $n$  eigenvalues  $\lambda_1, \dots, \lambda_n$ , listed with their algebraic multiplicities, then  $\text{tr } A = \lambda_1 + \dots + \lambda_n$ .

22. Consider an arbitrary  $n \times n$  matrix  $A$ . What is the relationship between the characteristic polynomials of  $A$  and  $A^T$ ? What does your answer tell you about the eigenvalues of  $A$  and  $A^T$ ?

23. Suppose matrix  $A$  is similar to  $B$ . What is the relationship between the characteristic polynomials of  $A$  and  $B$ ? What does your answer tell you about the eigenvalues of  $A$  and  $B$ ?

24. Find all eigenvalues of the positive transition matrix

$$A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}.$$

25. Consider a positive transition matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

meaning that  $a, b, c$ , and  $d$  are positive numbers such that  $a + c = b + d = 1$ . (The matrix in Exercise 24 has this form.) Verify that  $\begin{bmatrix} b \\ c \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  are eigenvectors of  $A$ . What are the associated eigenvalues? Is the absolute value of these eigenvalues more or less than 1? Sketch a phase portrait.

26. Based on your answers in Exercises 24 and 25, sketch a phase portrait of the dynamical system

$$\vec{x}(t+1) = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \vec{x}(t).$$

27. a. Based on your answers in Exercises 24 and 25, find closed formulas for the components of the dynamical system

$$\vec{x}(t+1) = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \vec{x}(t),$$

with initial value  $\vec{x}_0 = \vec{e}_1$ . Then do the same for the initial value  $\vec{x}_0 = \vec{e}_2$ . Sketch the two trajectories.

## Eigenvalues and Eigenvectors

- b.** Consider the matrix

$$A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}.$$

Using technology, compute some powers of the matrix  $A$ , say,  $A^2, A^5, A^{10}, \dots$ . What do you observe? Diagonalize matrix  $A$  to prove your conjecture.

- c.** If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is an arbitrary positive transition matrix, what can you say about the powers  $A^t$  as  $t$  goes to infinity?
- 28.** Consider the isolated Swiss town of Andelfingen, inhabited by 1,200 families. Each family takes a weekly shopping trip to the only grocery store in town, run by Mr. and Mrs. Wipf, until the day when a new, fancier (and cheaper) chain store, Migros, opens its doors. It is not expected that everybody will immediately run to the new store, but we do anticipate that 20% of those shopping at Wipf's each week switch to Migros the following week. Some people who do switch miss the personal service (and the gossip) and switch back: We expect that 10% of those shopping at Migros each week go to Wipf's the following week. The state of this town (as far as grocery shopping is concerned) can be represented by the vector

$$\vec{x}(t) = \begin{bmatrix} w(t) \\ m(t) \end{bmatrix},$$

where  $w(t)$  and  $m(t)$  are the numbers of families shopping at Wipf's and at Migros, respectively,  $t$  weeks after Migros opens. Suppose  $w(0) = 1,200$  and  $m(0) = 0$ .

- a.** Find a  $2 \times 2$  matrix  $A$  such that  $\vec{x}(t+1) = A\vec{x}(t)$ . Verify that  $A$  is a positive transition matrix. See Exercise 25.
- b.** How many families will shop at each store after  $t$  weeks? Give closed formulas.
- c.** The Wipfs expect that they must close down when they have less than 250 customers a week. When does that happen?
- 29.** Consider an  $n \times n$  matrix  $A$  such that the sum of the entries in each row is 1. Show that the vector

$$\vec{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

in  $\mathbb{R}^n$  is an eigenvector of  $A$ . What is the corresponding eigenvalue?

- 30.** In all parts of this problem, consider an  $n \times n$  matrix  $A$  such that all entries are positive and the sum of the entries in each row is 1 (meaning that  $A^T$  is a positive transition matrix).

- a.** Consider an eigenvector  $\vec{v}$  of  $A$  with positive components. Show that the associated eigenvalue is less than or equal to 1. Hint: Consider the largest entry  $v_i$  of  $\vec{v}$ . What can you say about the  $i$ th entry of  $A\vec{v}$ ?

- b.** Now we drop the requirement that the components of the eigenvector  $\vec{v}$  be positive. Show that the associated eigenvalue is less than or equal to 1 in absolute value.

- c.** Show that  $\lambda = -1$  fails to be an eigenvalue of  $A$ , and show that the eigenvectors with eigenvalue 1 are the vectors of the form

$$\begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix},$$

where  $c$  is nonzero.

- 31.** Consider a positive transition matrix  $A$ . Explain why 1 is an eigenvalue of  $A$ . What can you say about the other eigenvalues? Is

$$\vec{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

necessarily an eigenvector? Hint: Consider Exercises 22, 29, and 30.

- 32.** Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k & 3 & 0 \end{bmatrix},$$

where  $k$  is an arbitrary constant. For which values of  $k$  does  $A$  have three distinct real eigenvalues? For which  $k$  does  $A$  have two distinct eigenvalues? Hint: Graph the function  $g(\lambda) = \lambda^3 - 3\lambda$ . Find its local maxima and minima.

- 33. a.** Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}.$$

- b.** Can you find a  $3 \times 3$  matrix  $M$  whose characteristic polynomial is

$$-\lambda^3 + 17\lambda^2 - 5\lambda + \pi?$$

- 34.** Suppose a certain  $4 \times 4$  matrix  $A$  has two distinct real eigenvalues. What could the algebraic multiplicities of these eigenvalues be? Give an example for each possible case and sketch the characteristic polynomial.

## Eigenvalues and Eigenvectors

- 35.** Give an example of a  $4 \times 4$  matrix  $A$  without real eigenvalues.
- 36.** For an arbitrary positive integer  $n$ , give a  $2n \times 2n$  matrix  $A$  without real eigenvalues.
- 37.** Consider an eigenvalue  $\lambda_0$  of an  $n \times n$  matrix  $A$ . We are told that the algebraic multiplicity of  $\lambda_0$  exceeds 1. Show that  $f'_A(\lambda_0) = 0$  (i.e., the derivative of the characteristic polynomial of  $A$  vanishes at  $\lambda_0$ ).
- 38.** If  $A$  is a  $2 \times 2$  matrix with  $\text{tr } A = 5$  and  $\det A = -14$ , what are the eigenvalues of  $A$ ?
- 39.** If  $A$  and  $B$  are  $2 \times 2$  matrices, show that  $\text{tr}(AB) = \text{tr}(BA)$ .
- 40.** If  $A$  and  $B$  are  $n \times n$  matrices, show that  $\text{tr}(AB) = \text{tr}(BA)$ .
- 41.** If matrix  $A$  is similar to  $B$ , show that  $\text{tr } B = \text{tr } A$ . *Hint:* Exercise 40 is helpful.
- 42.** Consider two  $n \times n$  matrices  $A$  and  $B$  such that  $BA = 0$ . Show that  $\text{tr}((A + B)^2) = \text{tr}(A^2) + \text{tr}(B^2)$ . *Hint:* Exercise 40 is helpful.
- 43.** Do there exist  $n \times n$  matrices  $A$  and  $B$  such that  $AB - BA = I_n$ ? Explain. *Hint:* Exercise 40 is helpful.
- 44.** Do there exist invertible  $n \times n$  matrices  $A$  and  $B$  such that  $AB - BA = A$ ? Explain.
- 45.** For which value of the constant  $k$  does the matrix  $A = \begin{bmatrix} -1 & k \\ 4 & 3 \end{bmatrix}$  have 5 as an eigenvalue?
- 46.** In all the parts of this problem, consider a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with the eigenvalues  $\lambda_1$  and  $\lambda_2$ .
- Show that  $\lambda_1^2 + \lambda_2^2 = a^2 + d^2 + 2bc$ .
  - Show that  $\lambda_1^2 + \lambda_2^2 \leq a^2 + b^2 + c^2 + d^2$ .
  - For which matrices  $A$  does the equality  $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$  hold?
- 47.** For which  $2 \times 2$  matrices  $A$  does there exist a nonzero matrix  $M$  such that  $AM = MD$ , where  $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ ? Give your answer in terms of the eigenvalues of  $A$ .
- 48.** For which  $2 \times 2$  matrices  $A$  does there exist an invertible matrix  $S$  such that  $AS = SD$ , where  $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ ? Give your answer in terms of the eigenvalues of  $A$ .
- 49.** For which  $3 \times 3$  matrices  $A$  does there exist a nonzero matrix  $M$  such that  $AM = MD$ , where  $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ ? Give your answer in terms of the eigenvalues of  $A$ .
- 50.** In his groundbreaking text *Ars Magna* (Nuremberg, 1545), the Italian mathematician Gerolamo Cardano explains how to solve cubic equations. In Chapter XI, he considers the following example:
- $$x^3 + 6x = 20.$$
- Explain why this equation has exactly one (real) solution. Here, this solution is easy to find by inspection. The point of the exercise is to show a systematic way to find it.
  - Cardano explains his method as follows (we are using modern notation for the variables): “I take two cubes  $v^3$  and  $u^3$  whose difference shall be 20, so that the product  $vu$  shall be 2, that is, a third of the coefficient of the unknown  $x$ . Then, I say that  $v - u$  is the value of the unknown  $x$ .” Show that if  $v$  and  $u$  are chosen as stated by Cardano, then  $x = v - u$  is indeed the solution of the equation  $x^3 + 6x = 20$ .
  - Solve the system
- $$\left| \begin{array}{l} v^3 - u^3 = 20 \\ vu = 2 \end{array} \right|$$
- to find  $u$  and  $v$ .
- Consider the equation
- $$x^3 + px = q,$$
- where  $p$  is positive. Using your work in parts (a), (b), and (c) as a guide, show that the unique solution of this equation is
- $$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$
- This solution can also be written as
- $$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$
- What can go wrong when  $p$  is negative?
- Consider an arbitrary cubic equation
- $$x^3 + ax^2 + bx + c = 0.$$
- Show that the substitution  $x = t - (a/3)$  allows you to write this equation as
- $$t^3 + pt = q.$$

### 3 Finding the Eigenvectors of a Matrix

Having found an eigenvalue  $\lambda$  of an  $n \times n$  matrix  $A$ , we will now turn our attention to the corresponding eigenvectors. We have to find the vectors  $\vec{v}$  in  $\mathbb{R}^n$  such that

$$A\vec{v} = \lambda\vec{v}, \quad \text{or} \quad (A - \lambda I_n)\vec{v} = \vec{0}.$$

In other words, we have to find the kernel of the matrix  $A - \lambda I_n$ . In this context, the following definition is useful.

#### Definition 3.1

##### Eigenspaces

Consider an eigenvalue  $\lambda$  of an  $n \times n$  matrix  $A$ . Then the kernel of the matrix  $A - \lambda I_n$  is called the *eigenspace* associated with  $\lambda$ , denoted by  $E_\lambda$ :

$$E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}.$$

Note that the eigenvectors with eigenvalue  $\lambda$  are the *nonzero* vectors in the eigenspace  $E_\lambda$ .

**EXAMPLE 1** Let  $T(\vec{x}) = A\vec{x}$  be the orthogonal projection onto a plane  $V$  in  $\mathbb{R}^3$ . Describe the eigenspaces  $E_1$  and  $E_0$  geometrically. Is matrix  $A$  diagonalizable?

#### Solution

Eigenspace  $E_1$  consists of the solutions of the equation  $A\vec{v} = 1\vec{v} = \vec{v}$ ; those are the vectors on plane  $V$ . Thus,  $E_1 = V$ .

Eigenspace  $E_0 = \ker A$  consists of the solutions of the equation  $A\vec{v} = 0\vec{v} = \vec{0}$ ; those are the vectors on the line  $V^\perp$  perpendicular to plane  $V$ . See Figure 1.

We can construct an eigenbasis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  for  $A$  by choosing two linearly independent vectors  $\vec{v}_1, \vec{v}_2$  in  $E_1 = V$  and a nonzero vector  $\vec{v}_3$  in  $E_0 = V^\perp$ . Thus,

matrix  $A$  is diagonalizable; namely,  $A$  is similar to  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . ■

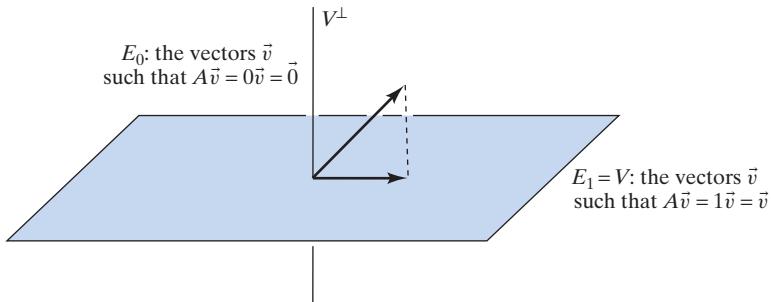


Figure 1

To find the eigenvectors associated with a known eigenvalue  $\lambda$  algebraically, we seek a basis of the eigenspace  $E_\lambda = \ker(A - \lambda I_n)$ , a problem we can handle.

**EXAMPLE 2** Find the eigenspaces of the matrix  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ . Diagonalize matrix  $A$  if you can.

**Solution**

In Example 1 of Section 2, we saw that the eigenvalues are 5 and  $-1$ . Now

$$E_5 = \ker(A - 5I_2) = \ker \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}.$$

Finding the kernel amounts to finding the relations between the columns. In the case of a  $2 \times 2$  matrix, this can be done by inspection. Consider the Kyle numbers,

$$\begin{bmatrix} 1 & 2 \\ -4 & 2 \\ 4 & -2 \end{bmatrix},$$

so that

$$E_5 = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Similarly,

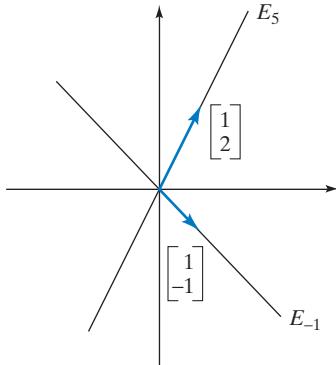
$$E_{-1} = \ker(A + I_2) = \ker \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 4 & 4 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can (and should) check that the vectors we found are indeed eigenvectors of  $A$ , with the eigenvalues we claim:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad \checkmark$$



**Figure 2**

Both eigenspaces are lines, as shown in Figure 2. The vectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  form an eigenbasis for  $A$ , so that  $A$  is diagonalizable, with  $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ . For peace of mind, we may want to check that  $AS = SB$ . ■

Geometrically, the matrix  $A$  represents a scaling by a factor of 5 along the line spanned by vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , while the line spanned by  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is flipped over the origin.

**EXAMPLE 3** Find the eigenspaces of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Diagonalize matrix  $A$  if you can.

**Solution**

The eigenvalues are 1 and 0, the diagonal entries of the upper triangular matrix  $A$ , with algebraic multiplicities 2 and 1, respectively. Now

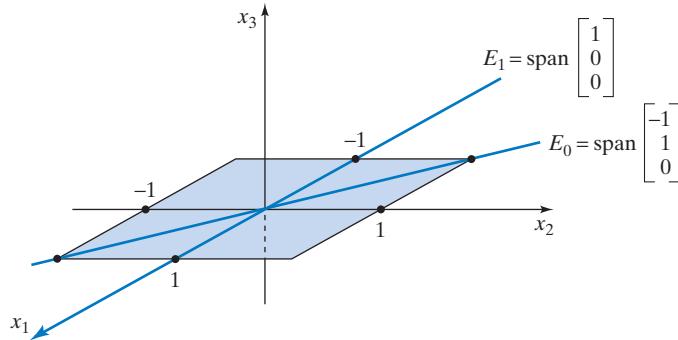
$$E_1 = \ker(A - I_3) = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

## Eigenvalues and Eigenvectors

and

$$E_0 = \ker A = \ker \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \text{span} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Both eigenspaces are lines in the  $x_1$ - $x_2$ -plane, as shown in Figure 3.



**Figure 3**

We can find only two linearly independent eigenvectors, one in each of the eigenspaces  $E_0$  and  $E_1$ , so that we are unable to construct an eigenbasis for  $A$ . Thus, matrix  $A$  fails to be diagonalizable. ■

Note that Example 3 is qualitatively different from Example 1, where we studied the orthogonal projection onto a plane in  $\mathbb{R}^3$ . There, too, we had two eigenvalues, 1 and 0, but one of the eigenspaces,  $E_1$ , was a plane, while both eigenspaces in Example 3 turn out to be lines.

To discuss these different cases, it is useful to introduce the following terminology.

### Definition 3.2

#### Geometric multiplicity

Consider an eigenvalue  $\lambda$  of an  $n \times n$  matrix  $A$ . The dimension of eigenspace  $E_\lambda = \ker(A - \lambda I_n)$  is called the *geometric multiplicity* of eigenvalue  $\lambda$ , denoted  $\text{gemu}(\lambda)$ . Thus,

$$\text{gemu}(\lambda) = \text{nullity}(A - \lambda I_n) = n - \text{rank}(A - \lambda I_n)$$

Example 3 shows that the geometric multiplicity of an eigenvalue may be different from the algebraic multiplicity (but see Theorem 3.6). We have

$$\text{almu}(1) = 2,$$

but

$$\text{gemu}(1) = \dim(E_1) = 1.$$

Based on our findings in Examples 1 through 3, we can now tackle the main questions of this chapter.

- a. Which square matrices  $A$  are diagonalizable; that is, when does there exist an eigenbasis for  $A$ ?
- b. If eigenbases exist, how can we find one?

Consider an  $n \times n$  matrix  $A$ . If the sum  $s$  of the geometric multiplicities of the eigenvalues is less than  $n$  (as in Example 3, where  $s = 2$  and  $n = 3$ ), then there are not enough linearly independent eigenvectors to form an eigenbasis. In fact, we can find no more than  $s$  linearly independent eigenvectors.

Conversely, suppose that the geometric multiplicities of the eigenvalues do add up to  $n$ , as in Examples 1 and 2. Can we construct an eigenbasis for  $A$  by finding a basis of each eigenspace and *concatenating*<sup>8</sup> these bases? This method does work in Examples 1 and 2. Next we will state and prove that this approach works in general.

### Theorem 3.3

#### Eigenbases and geometric multiplicities

- a. Consider an  $n \times n$  matrix  $A$ . If we find a basis of each eigenspace of  $A$  and concatenate all these bases, then the resulting eigenvectors  $\vec{v}_1, \dots, \vec{v}_s$  will be linearly independent. (Note that  $s$  is the sum of the geometric multiplicities of the eigenvalues of  $A$ .) This result implies that  $s \leq n$ .
- b. Matrix  $A$  is diagonalizable if (and only if) the geometric multiplicities of the eigenvalues add up to  $n$  (meaning that  $s = n$  in part a).

#### Proof

- a. We will argue indirectly, assuming that the eigenvectors  $\vec{v}_1, \dots, \vec{v}_s$  are linearly dependent. Let  $\vec{v}_m$  be the *first* redundant vector in this list, with  $\vec{v}_m = c_1\vec{v}_1 + \dots + c_{m-1}\vec{v}_{m-1}$ . Suppose that  $A\vec{v}_i = \lambda_i\vec{v}_i$ . There must be at least one *nonzero* coefficient  $c_k$  such that  $\lambda_k \neq \lambda_m$ , since  $\vec{v}_m$  cannot be expressed as a linear combination of vectors  $\vec{v}_i$  that are all in the same eigenspace  $E_{\lambda_m}$ . Multiplying the equation  $\vec{v}_m = c_1\vec{v}_1 + \dots + c_k\vec{v}_k + \dots + c_{m-1}\vec{v}_{m-1}$  by  $A - \lambda_m I_n$  from the left, and realizing that  $(A - \lambda_m I_n)\vec{v}_i = (\lambda_i - \lambda_m)\vec{v}_i$ , we find that

$$\begin{aligned} &(\lambda_m - \lambda_m)\vec{v}_m = \vec{0} \\ &= (\lambda_1 - \lambda_m)c_1\vec{v}_1 + \dots + \underbrace{(\lambda_k - \lambda_m)c_k\vec{v}_k}_{\neq 0} + \dots + (\lambda_{m-1} - \lambda_m)c_{m-1}\vec{v}_{m-1}. \end{aligned}$$

This is a nontrivial relation among vectors  $\vec{v}_1, \dots, \vec{v}_{m-1}$ , contradicting our assumption that  $\vec{v}_m$  is the *first* redundant vector in the list.

- b. This claim follows directly from part a. There exists an eigenbasis if (and only if)  $s = n$  in part a. ■

Here is an important special case of Theorem 3.3.

### Theorem 3.4

#### An $n \times n$ matrix with $n$ distinct eigenvalues

If an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable. We can construct an eigenbasis by finding an eigenvector for each eigenvalue.

<sup>8</sup> The *concatenation* of two lists  $(a_1, a_2, \dots, a_p)$  and  $(b_1, b_2, \dots, b_q)$  is the list  $(a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q)$ .

## Eigenvalues and Eigenvectors

**EXAMPLE 4** Is the following matrix  $A$  diagonalizable?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

### Solution

Yes, since the  $6 \times 6$  matrix  $A$  has six distinct eigenvalues, namely, the diagonal entries 1, 2, 3, 4, 5, 6. ■

## Eigenvalues and Similarity

If matrix  $A$  is similar to  $B$ , what is the relationship between the eigenvalues of  $A$  and  $B$ ? The following theorem shows that this relationship is very close indeed.

### Theorem 3.5

#### The eigenvalues of similar matrices

Suppose matrix  $A$  is similar to  $B$ . Then

- a. Matrices  $A$  and  $B$  have the same characteristic polynomial; that is,  $f_A(\lambda) = f_B(\lambda)$ .
- b.  $\text{rank } A = \text{rank } B$  and  $\text{nullity } A = \text{nullity } B$ .
- c. Matrices  $A$  and  $B$  have the same eigenvalues, with the same algebraic and geometric multiplicities. (However, the eigenvectors need not be the same.)
- d. Matrices  $A$  and  $B$  have the same determinant and the same trace:  
 $\det A = \det B$  and  $\text{tr } A = \text{tr } B$ .

### Proof

- a. If  $B = S^{-1}AS$ , then  $f_B(\lambda) = \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n) = \det(S^{-1}(A - \lambda I_n)S) = (\det S)^{-1} \det(A - \lambda I_n) \det(S) = (\det S)^{-1}(\det S) \det(A - \lambda I_n) = \det(A - \lambda I_n) = f_A(\lambda)$  for all scalars  $\lambda$ .
- b. An alternative proof is suggested in Exercise 34 of this section.
- c. It follows from part a that matrices  $A$  and  $B$  have the same eigenvalues, with the same algebraic multiplicities. See Theorem 2.1 and Definition 2.6. As for the geometric multiplicity, note that  $A - \lambda I_n$  is similar to  $B - \lambda I_n$  for all  $\lambda$  (see Exercise 33), so that  $\text{nullity}(A - \lambda I_n) = \text{nullity}(B - \lambda I_n)$  for all eigenvalues  $\lambda$ , by part b. See Definition 3.2.
- d. These equations follow from part a and Theorem 2.5: Trace and determinant are coefficients of the characteristic polynomial, up to signs. ■

**EXAMPLE 5** Is the matrix  $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$  similar to  $B = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ ?

### Solution

No, since  $\text{tr } A = 9$  and  $\text{tr } B = 8$ . See Theorem 3.5d. ■

Earlier in this section we observed that the algebraic and the geometric multiplicity of an eigenvalue are not necessarily the same. However, the following inequality always holds.

**Theorem 3.6**
**Algebraic versus geometric multiplicity**

If  $\lambda$  is an eigenvalue of a square matrix  $A$ , then

$$\text{gemu}(\lambda) \leq \text{almu}(\lambda).$$

**Proof** Suppose  $\lambda_0$  is an eigenvalue of an  $n \times n$  matrix  $A$ , with geometric multiplicity  $m$ , meaning that the dimension of eigenspace  $E_{\lambda_0}$  is  $m$ . Let  $\vec{v}_1, \dots, \vec{v}_m$  be a basis of  $E_{\lambda_0}$ , and consider an invertible  $n \times n$  matrix  $S$  whose first  $m$  columns are  $\vec{v}_1, \dots, \vec{v}_m$ . (How would you find such an  $S$ ?) Let  $B = S^{-1}AS$ , a matrix similar to  $A$ . Now compute  $B\vec{e}_i$ , for  $i = 1, \dots, m$ , keeping in mind that  $S\vec{e}_i = \vec{v}_i$ , and therefore  $S^{-1}\vec{v}_i = \vec{e}_i$ : ( $i$ th column of  $B$ ) =  $B\vec{e}_i = S^{-1}AS\vec{e}_i = S^{-1}A\vec{v}_i = S^{-1}(\lambda_0\vec{v}_i) = \lambda_0(S^{-1}\vec{v}_i) = \lambda_0\vec{e}_i$ . This computation shows that the first  $m$  columns of  $B$  look like those of  $\lambda_0 I_n$ .

$$B = \left[ \begin{array}{cccc|ccc} \overbrace{\lambda_0 & 0 & \cdots & 0}^m & * & \cdots & * \\ 0 & \lambda_0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_0 & * & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{array} \right] = \begin{bmatrix} \lambda_0 I_m & P \\ 0 & Q \end{bmatrix}$$

Since  $B$  is similar to  $A$ , we have

$$\overset{\text{step 1}}{f_A(\lambda)} = f_B(\lambda) = \det(B - \lambda I_n) \overset{\text{step 3}}{=} (\lambda_0 - \lambda)^m f_Q(\lambda),$$

showing that the algebraic multiplicity of eigenvalue  $\lambda_0$  is at least  $m$ , as claimed. ■

We conclude this section with a summary on diagonalization.

**Theorem 3.7**
**Strategy for Diagonalization**

Suppose we are asked to determine whether a given  $n \times n$  matrix  $A$  is diagonalizable. If so, we wish to find an invertible matrix  $S$  such that  $S^{-1}AS = B$  is diagonal.

We can proceed as follows.

- a. Find the eigenvalues of  $A$  by solving the characteristic equation

$$f_A(\lambda) = \det(A - \lambda I_n) = 0.$$

- b. For each eigenvalue  $\lambda$ , find a basis of the eigenspace

$$E_\lambda = \ker(A - \lambda I_n).$$

- c. Matrix  $A$  is diagonalizable if (and only if) the dimensions of the eigenspaces add up to  $n$ . In this case, we find an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$

## Eigenvalues and Eigenvectors

for  $A$  by concatenating the bases of the eigenspaces we found in part b. Let

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix}. \text{ Then } S^{-1}AS = B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

where  $\lambda_j$  is the eigenvalue associated with  $\vec{v}_j$ .

### EXERCISES 3

**GOAL** For a given eigenvalue, find a basis of the associated eigenspace. Use the geometric multiplicities of the eigenvalues to determine whether a matrix is diagonalizable.

For each of the matrices  $A$  in Exercises 1 through 20, find all (real) eigenvalues. Then find a basis of each eigenspace, and diagonalize  $A$ , if you can. Do not use technology.

1.  $\begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

2.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$

4.  $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

5.  $\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$

6.  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8.  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

9.  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10.  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

12.  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

13.  $\begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$

14.  $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

15.  $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$

16.  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

17.  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

18.  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

19.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

20.  $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

21. Find a  $2 \times 2$  matrix  $A$  for which

$$E_1 = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad E_2 = \text{span} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

How many such matrices are there?

22. Find all  $2 \times 2$  matrices  $A$  for which

$$E_7 = \mathbb{R}^2.$$

23. Find all eigenvalues and eigenvectors of  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

Is there an eigenbasis? Interpret your result geometrically.

24. Find a  $2 \times 2$  matrix  $A$  for which

$$E_1 = \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the only eigenspace.

25. What can you say about the geometric multiplicity of the eigenvalues of a matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix},$$

where  $a, b, c$  are arbitrary constants?

26. Show that if a  $6 \times 6$  matrix  $A$  has a negative determinant, then  $A$  has at least one positive eigenvalue. Hint: Sketch the graph of the characteristic polynomial.

27. Consider a  $2 \times 2$  matrix  $A$ . Suppose that  $\text{tr } A = 5$  and  $\det A = 6$ . Find the eigenvalues of  $A$ .

28. Consider the matrix

$$J_n(k) = \begin{bmatrix} k & 1 & 0 & \dots & 0 & 0 \\ 0 & k & 1 & \dots & 0 & 0 \\ 0 & 0 & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k & 1 \\ 0 & 0 & 0 & \dots & 0 & k \end{bmatrix}$$

(with all  $k$ 's on the diagonal and 1's directly above), where  $k$  is an arbitrary constant. Find the eigenvalue(s) of  $J_n(k)$ , and determine their algebraic and geometric multiplicities.

## Eigenvalues and Eigenvectors

- 29.** Consider a diagonal  $n \times n$  matrix  $A$  with rank  $A = r < n$ . Find the algebraic and the geometric multiplicity of the eigenvalue 0 of  $A$  in terms of  $r$  and  $n$ .
- 30.** Consider an upper triangular  $n \times n$  matrix  $A$  with  $a_{ii} \neq 0$  for  $i = 1, 2, \dots, m$  and  $a_{ii} = 0$  for  $i = m+1, \dots, n$ . Find the algebraic multiplicity of the eigenvalue 0 of  $A$ . Without using Theorem 3.6, what can you say about the geometric multiplicity?
- 31.** Suppose there is an eigenbasis for a matrix  $A$ . What is the relationship between the algebraic and geometric multiplicities of its eigenvalues?
- 32.** Consider an eigenvalue  $\lambda$  of an  $n \times n$  matrix  $A$ . We know that  $\lambda$  is an eigenvalue of  $A^T$  as well (since  $A$  and  $A^T$  have the same characteristic polynomial). Compare the geometric multiplicities of  $\lambda$  as an eigenvalue of  $A$  and  $A^T$ .
- 33.** Show that if matrix  $A$  is similar to  $B$ , then  $A - \lambda I_n$  is similar to  $B - \lambda I_n$ , for all scalars  $\lambda$ .
- 34.** Suppose that  $B = S^{-1}AS$  for some  $n \times n$  matrices  $A$ ,  $B$ , and  $S$ .
- Show that if  $\vec{x}$  is in  $\ker B$ , then  $S\vec{x}$  is in  $\ker A$ .
  - Show that the linear transformation  $T(\vec{x}) = S\vec{x}$  from  $\ker B$  to  $\ker A$  is an isomorphism.
  - Show that  $\text{nullity } A = \text{nullity } B$  and  $\text{rank } A = \text{rank } B$ .
- 35.** Is matrix  $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$  similar to  $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ ?
- 36.** Is matrix  $\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$  similar to  $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ ?
- 37.** Consider a symmetric  $n \times n$  matrix  $A$ .
- Show that if  $\vec{v}$  and  $\vec{w}$  are two vectors in  $\mathbb{R}^n$ , then
- $$A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}.$$
- Show that if  $\vec{v}$  and  $\vec{w}$  are two eigenvectors of  $A$ , with distinct eigenvalues, then  $\vec{w}$  is orthogonal to  $\vec{v}$ .
- 38.** Consider a rotation  $T(\vec{x}) = A\vec{x}$  in  $\mathbb{R}^3$ . (That is,  $A$  is an orthogonal  $3 \times 3$  matrix with determinant 1.) Show that  $T$  has a nonzero fixed point [i.e., a vector  $\vec{v}$  with  $T(\vec{v}) = \vec{v}$ ]. This result is known as *Euler's theorem*, after the great Swiss mathematician Leonhard Euler (1707–1783). Hint: Consider the characteristic polynomial  $f_A(\lambda)$ . Pay attention to the intercepts with both axes. Use Theorem 1.4.
- 39.** Consider a subspace  $V$  of  $\mathbb{R}^n$  with  $\dim(V) = m$ .
- Suppose the  $n \times n$  matrix  $A$  represents the orthogonal projection onto  $V$ . What can you say about the eigenvalues of  $A$  and their algebraic and geometric multiplicities?
  - Suppose the  $n \times n$  matrix  $B$  represents the reflection about  $V$ . What can you say about the

eigenvalues of  $B$  and their algebraic and geometric multiplicities?

For which values of constants  $a$ ,  $b$ , and  $c$  are the matrices in Exercises 40 through 50 diagonalizable?

**40.**  $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

**41.**  $\begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}$

**43.**  $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$

**45.**  $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$

**47.**  $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

**49.**  $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

**42.**  $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$

**44.**  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

**46.**  $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$

**48.**  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix}$

**50.**  $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

**51.** Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}, \text{ where } a, b, \text{ and } c \text{ are arbitrary constants.}$$

**52.** Find the characteristic polynomial of the  $n \times n$  matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}.$$

Note that the  $i$ th column of  $A$  is  $\vec{e}_{i+1}$ , for  $i = 1, \dots, n-1$ , while the last column has the arbitrary entries  $a_0, \dots, a_{n-1}$ . See Exercise 51 for the special case  $n = 3$ .

**53.** Consider a  $5 \times 5$  matrix  $A$  and a vector  $\vec{v}$  in  $\mathbb{R}^5$ . Suppose the vectors  $\vec{v}$ ,  $A\vec{v}$ ,  $A^2\vec{v}$  are linearly independent, while  $A^3\vec{v} = a\vec{v} + bA\vec{v} + cA^2\vec{v}$  for some scalars  $a, b, c$ . We can take the linearly independent vectors  $\vec{v}, A\vec{v}, A^2\vec{v}$  and expand them to a basis  $\mathcal{B} = (\vec{v}, A\vec{v}, A^2\vec{v}, \vec{w}_4, \vec{w}_5)$  of  $\mathbb{R}^5$ .

**a.** Consider the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\mathcal{B}$ . Write the entries of the first three columns of  $B$ . (Note that we do not know anything about the entries of the last two columns of  $B$ .)

**b.** Explain why  $f_A(\lambda) = f_B(\lambda) = h(\lambda)(-\lambda^3 + c\lambda^2 + b\lambda + a)$ , for some quadratic polynomial  $h(\lambda)$ . See Exercise 51.

## Eigenvalues and Eigenvectors

- c. Explain why  $f_A(A)\vec{v} = \vec{0}$ . Here,  $f_A(A)$  is the characteristic polynomial evaluated at  $A$ , that is, if  $f_A(\lambda) = c_n\lambda^n + \dots + c_1\lambda + c_0$ , then  $f_A(A) = c_nA^n + \dots + c_1A + c_0I_n$ .
54. Consider an  $n \times n$  matrix  $A$  and a vector  $\vec{v}$  in  $\mathbb{R}^n$ . Form the vectors  $\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}, \dots$ , and let  $A^m\vec{v}$  be the first redundant vector in this list. Then the  $m$  vectors  $\vec{v}, A\vec{v}, A^2\vec{v}, \dots, A^{m-1}\vec{v}$  are linearly independent; note that  $m \leq n$ . Since  $A^m\vec{v}$  is redundant, we can write  $A^m\vec{v} = a_0\vec{v} + a_1A\vec{v} + a_2A^2\vec{v} + \dots + a_{m-1}A^{m-1}\vec{v}$  for some scalars  $a_0, \dots, a_{m-1}$ . Form a basis  $\mathfrak{B} = (\vec{v}, A\vec{v}, A^2\vec{v}, \dots, A^{m-1}\vec{v}, \tilde{w}_{m+1}, \dots, \tilde{w}_n)$  of  $\mathbb{R}^n$ .
- a. Consider the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\mathfrak{B}$ . Write  $B$  in block form,  $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ , where  $B_{11}$  is an  $m \times m$  matrix. Describe  $B_{11}$  column by column, paying particular attention to the  $m$ th column. What can you say about  $B_{21}$ ? (Note that we do not know anything about the entries of  $B_{12}$  and  $B_{22}$ .)
- b. Explain why  $f_A(\lambda) = f_B(\lambda) = f_{B_{22}}(\lambda)f_{B_{11}}(\lambda) = (-1)^m f_{B_{22}}(\lambda)(\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0)$ . See Exercise 52.
- c. Explain why  $f_A(A)\vec{v} = \vec{0}$ . See Exercise 53.
- d. Explain why  $f_A(A) = 0$ .
- The equation  $f_A(A) = 0$  is referred to as the *Cayley–Hamilton theorem*: A square matrix satisfies its characteristic polynomial. The English mathematician Arthur Cayley (1821–1895) played a leading role in the development of the algebra of matrices, and the Irish mathematician Sir William Rowan Hamilton (1805–1865) is best remembered today for his discovery of the quaternions. See Exercise 5.37.
55. Give an example of a  $3 \times 3$  matrix  $A$  with nonzero integer entries such that 7 is an eigenvalue of  $A$ .
56. Give an example of a  $3 \times 3$  matrix  $A$  with nonzero integer entries such that 1, 2, and 3 are the eigenvalues of  $A$ .

## 4 More on Dynamical Systems

In Section 1 we learned how to use eigenbases and diagonalization to analyze the dynamical system

$$\vec{x}(t+1) = A\vec{x}(t), \text{ or } \vec{x}(t) = A^t\vec{x}_0.$$

Take another look at Theorem 1.6.

Now that we know how to find the eigenvalues and the eigenvectors of a matrix, let us work two more examples of dynamical systems.

**EXAMPLE 1** Consider a “mini-Web” with only three pages, labeled 1, 2, and 3. Initially there is an equal number of surfers on each page, meaning that the initial distribution vector is

$$\vec{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

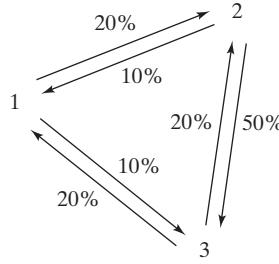
At the blow of a whistle, some surfers will move on to a different page, in a way described by the transition matrix

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}.$$

For example, the entries of the first column of  $A$  tell us that 20% of those who are initially on Page 1 will move to Page 2, while 10% will move to Page 3 and 70% will stay on Page 1. (These are not the rules of transition we considered when defining PageRank.)

## Eigenvalues and Eigenvectors

We can represent the rules of transition in a diagram:



After one transition, the distribution of the surfers will be

$$A\vec{x}_0 = \begin{bmatrix} 1/3 \\ 4/15 \\ 2/5 \end{bmatrix} \approx \begin{bmatrix} 0.333 \\ 0.267 \\ 0.4 \end{bmatrix}.$$

If we iterate this transition  $t$  times, the final distribution will be  $A^t\vec{x}_0$ .

- a. Find a closed formula for  $A^t\vec{x}_0$ , expressing the vector  $A^t\vec{x}_0$  as a function of  $t$ .
- b. What happens in the long run? Find  $\lim_{t \rightarrow \infty} A^t\vec{x}_0$  if it exists.

### Solution

- a. Following the strategy outlined in Theorem 1.6, we wish to construct an eigenbasis for  $A$ . Using technology, we find the eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 0.5$ , and  $\lambda_3 = 0.2$  of  $A$ .

At this point, we know that matrix  $A$  is diagonalizable, by Theorem 3.4.

A straightforward but tedious computation, involving nothing more than finding some reduced row-echelon forms, reveals that

$$E_1 = \text{span} \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}, \quad E_{0.5} = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad E_{0.2} = \text{span} \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

$$\text{Thus, we have the eigenbasis } \vec{v}_1 = \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

Next we need to find the coordinates  $c_1, c_2, c_3$  of the initial state vector  $\vec{x}_0$  with respect to the given eigenbasis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . It turns out that

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \frac{1}{20}\vec{v}_1 - \frac{2}{45}\vec{v}_2 - \frac{1}{36}\vec{v}_3.$$

Using the formula derived in Theorem 1.6, we have

$$\begin{aligned} A^t\vec{x}_0 &= c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + c_3\lambda_3^t\vec{v}_3 \\ &= \frac{1}{20}\begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix} - \frac{2}{45}(0.5)^t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{36}(0.2)^t\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}. \end{aligned}$$

For example, the proportion of surfers on Page 1 after  $t$  iterations will be

$$\frac{7}{20} - \frac{2}{45}(0.5)^t + \frac{1}{36}(0.2)^t.$$

## Eigenvalues and Eigenvectors

In the long run, as we let  $t$  go to infinity, this proportion will approach  $\frac{7}{20} = 35\%$ , since the other two terms,  $\frac{2}{45}(0.5)^t$  and  $\frac{1}{36}(0.2)^t$ , decay exponentially.

- b.  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \frac{1}{20} \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 35\% \\ 25\% \\ 40\% \end{bmatrix}$ , since the other two terms go to  $\vec{0}$ . See our work at the end of part a. In the long run, the proportions of surfers on the three pages will approach 35%, 25%, and 40%, respectively. Note that this limit is the unique distribution vector that is an eigenvector of  $A$  with eigenvalue 1. This is the *equilibrium distribution* for  $A$ ,

$$\vec{x}_{\text{equ}} = \frac{1}{20} \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}.$$

In summary,  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{x}_{\text{equ}}$ . ■

We will now state and prove that the equation  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{x}_{\text{equ}}$  holds in fact for all diagonalizable regular transition matrices  $A$ , regardless of the initial distribution  $\vec{x}_0$ .

### Theorem 4.1

#### Equilibria for regular transition matrices

Let  $A$  be a regular transition matrix of size  $n \times n$ .

- a. There exists exactly one distribution vector  $\vec{x}$  in  $\mathbb{R}^n$  such that  $A\vec{x} = \vec{x}$ , meaning that  $\vec{x}$  is an eigenvector with eigenvalue 1. This is called the *equilibrium distribution* for  $A$ , denoted  $\vec{x}_{\text{equ}}$ . All components of  $\vec{x}_{\text{equ}}$  are positive.
- b. If  $\vec{x}_0$  is any distribution vector in  $\mathbb{R}^n$ , then  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{x}_{\text{equ}}$ .
- c.  $\lim_{t \rightarrow \infty} A^t = \begin{bmatrix} | & | \\ \vec{x}_{\text{equ}} & \dots & \vec{x}_{\text{equ}} \\ | & | \end{bmatrix}$ , the matrix whose columns are all  $\vec{x}_{\text{equ}}$ .

**Proof** We will prove Theorem 4.1 for diagonalizable matrices only.<sup>9</sup> We will prove parts a and b together. Part c follows easily from part b since  $\lim_{t \rightarrow \infty} A^t = \lim_{t \rightarrow \infty} \begin{bmatrix} | & | \\ A^t \vec{e}_1 & \dots & A^t \vec{e}_n \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}_{\text{equ}} & \dots & \vec{x}_{\text{equ}} \\ | & | \end{bmatrix}$ . Note that  $\vec{e}_1, \dots, \vec{e}_n$  are distribution vectors, so that  $\lim_{t \rightarrow \infty} (A^t \vec{e}_j) = \vec{x}_{\text{equ}}$  by part b.

In Exercises 2.30, 2.31, and 3.32, we have seen the following facts concerning the eigenvalues of a positive transition matrix  $A$ :

- $\lambda = 1$  is an eigenvalue of  $A$  with  $\text{gemu}(1) = 1$ .
- If  $\lambda \neq 1$  is an eigenvalue of  $A$ , then  $|\lambda| < 1$ , so that  $\lim_{t \rightarrow \infty} \lambda^t = 0$ .

---

<sup>9</sup>Here we consider a matrix  $A$  that is diagonalizable over  $\mathbb{R}$ . However, our proof can easily be adapted to the case of a matrix that is diagonalizable over  $\mathbb{C}$ , as illustrated in Exercises 5.31 and 5.32. In Section 5, we will see that “most” square matrices are diagonalizable over  $\mathbb{C}$ .

## Eigenvalues and Eigenvectors

In Exercise 53, you will be asked to prove the analogous results for the eigenvalues of a regular transition matrix.

Since  $A$  is assumed to be diagonalizable, there exists an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$ , with associated eigenvalues  $\lambda_1, \dots, \lambda_n$ . We can order the eigenvectors so that  $\lambda_1 = 1$  and  $|\lambda_j| < 1$  for  $j = 2, \dots, n$ . Now we can write

$$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n.$$

Then

$$A^t \vec{x}_0 = c_1 \lambda_1^t \vec{v}_1 + c_2 \lambda_2^t \vec{v}_2 + \cdots + c_n \lambda_n^t \vec{v}_n = c_1 \vec{v}_1 + c_2 \lambda_2^t \vec{v}_2 + \cdots + c_n \lambda_n^t \vec{v}_n$$

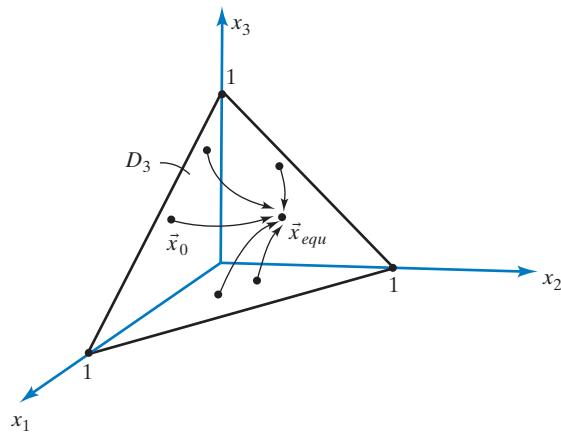
and

$$\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \lim_{t \rightarrow \infty} (c_1 \vec{v}_1 + \underbrace{c_2 \lambda_2^t \vec{v}_2 + \cdots + c_n \lambda_n^t \vec{v}_n}_{\vec{0}}) = c_1 \vec{v}_1.$$

Since  $A^t \vec{x}_0$  is a distribution vector for all  $t$ , so is  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = c_1 \vec{v}_1$ , by the sum rule for limits. Also,  $c_1 \vec{v}_1$  is an eigenvector with eigenvalue 1, so that it is an equilibrium distribution. Since the eigenspace  $E_1$  is one-dimensional, this equilibrium distribution is in fact unique, and we can denote it by  $\vec{x}_{equ}$ . All the components of  $\vec{x}_{equ}$  are positive. ■

We can interpret part b of Theorem 4.1 geometrically. Let  $D_n$  be the set of all distribution vectors in  $\mathbb{R}^n$ , that is, vectors  $\vec{x}$  with components  $x_1, \dots, x_n$  such that  $x_1 + \cdots + x_n = 1$  and  $x_j \geq 0$  for all  $j$ . For example, for  $n = 3$ , the set  $D_3$  is the portion of the plane  $x_1 + x_2 + x_3 = 1$  in the first octant, the triangle with its vertices at  $(1,0,0)$ ,  $(0,1,0)$ , and  $(0,0,1)$ . See Figure 1. We know that a transition matrix  $A$  maps  $D_n$  into  $D_n$ .

For a regular transition matrix  $A$ , the equilibrium distribution  $\vec{x}_{equ}$  sits somewhere on this set  $D_n$ , and all the other trajectories  $A^t \vec{x}_0$ , for any other point  $\vec{x}_0$  on  $D_n$ , will approach  $\vec{x}_{equ}$  as we let  $t$  go to infinity. The phase portrait in Figure 1 illustrates the case  $n = 3$ .



**Figure 1**

## Eigenvalues and Eigenvectors

Let's work one more example to show a different approach to dynamical systems.

- EXAMPLE 2** Consider the positive transition matrix  $A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$ .
- Use diagonalization to find a closed formula for  $A^t$ . Compute  $\lim_{t \rightarrow \infty} A^t$ .
  - Find a closed formula for the dynamical system  $A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$  and compute  $\lim_{t \rightarrow \infty} \left( A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \right)$ .

### Solution

- a. The characteristic polynomial of  $A$  is  $f_A(\lambda) = \lambda^2 - 1.1\lambda + 0.1 = (\lambda - 1)(\lambda - 0.1)$ , so that the eigenvalues are 1 and 0.1.

Now  $E_1 = \ker \begin{bmatrix} -0.6 & 0.3 \\ 0.6 & -0.3 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $E_{0.1} = \ker \begin{bmatrix} 0.3 & 0.3 \\ 0.6 & 0.6 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . We can diagonalize  $A$  with  $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$ . Then  $S^{-1}AS = B$  and  $A = SBS^{-1}$ , so that

$$\begin{aligned} A^t &= SB^tS^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (0.1)^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 + 2(0.1)^t & 1 - (0.1)^t \\ 2 - 2(0.1)^t & 2 + (0.1)^t \end{bmatrix}. \end{aligned}$$

$$\text{Now } \lim_{t \rightarrow \infty} A^t = \lim_{t \rightarrow \infty} \left( \frac{1}{3} \begin{bmatrix} 1 + 2(0.1)^t & 1 - (0.1)^t \\ 2 - 2(0.1)^t & 2 + (0.1)^t \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- b.  $A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 + 2(0.1)^t & 1 - (0.1)^t \\ 2 - 2(0.1)^t & 2 + (0.1)^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 + (0.1)^t \\ 4 - (0.1)^t \end{bmatrix}$  and  $\lim_{t \rightarrow \infty} \left( A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \right) = \lim_{t \rightarrow \infty} \left( \frac{1}{6} \begin{bmatrix} 2 + (0.1)^t \\ 4 - (0.1)^t \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .



Let us summarize the technique we used in Example 2a.

### Theorem 4.2

#### Powers of a diagonalizable matrix

If

$$S^{-1}AS = B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

then

$$A^t = SB^tS^{-1} = S \begin{bmatrix} \lambda_1^t & 0 & \dots & 0 \\ 0 & \lambda_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^t \end{bmatrix} S^{-1}.$$

## The Eigenvalues of a Linear Transformation

In the preceding three sections, we developed the theory of eigenvalues and eigenvectors for  $n \times n$  matrices, or, equivalently, for linear transformations  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . These concepts can be generalized to linear transformations from  $V$  to  $V$ , where  $V$  is any linear space. In the case of a finite dimensional space  $V$ , we can generalize the idea of diagonalization as well.

### Definition 4.3

#### The eigenvalues of a linear transformation

Consider a linear transformation  $T$  from  $V$  to  $V$ , where  $V$  is a linear space. A scalar  $\lambda$  is called an *eigenvalue* of  $T$  if there exists a nonzero element  $f$  of  $V$  such that

$$T(f) = \lambda f.$$

Such an  $f$  is called an *eigenfunction* if  $V$  consists of functions, an *eigenmatrix* if  $V$  consists of matrices, and so on. In theoretical work, the inclusive term *eigenvector* is often used for  $f$ .

Now suppose that  $V$  is finite dimensional. Then a basis  $\mathcal{B}$  of  $V$  consisting of eigenvectors of  $T$  is called an *eigenbasis* for  $T$ . We say that transformation  $T$  is *diagonalizable* if the matrix of  $T$  with respect to some basis is diagonal. Transformation  $T$  is diagonalizable if (and only if) there exists an eigenbasis for  $T$ . See Theorem 1.3.

### EXAMPLE 3

Consider the linear transformation  $D(f) = f'$  (the derivative) from  $C^\infty$  to  $C^\infty$ . Show that all real numbers are eigenvalues of  $D$ . Hint: Apply  $D$  to exponential functions.

#### Solution

Following the hint, we observe that  $D(e^x) = (e^x)' = e^x = 1e^x$ . This shows that  $e^x$  is an eigenfunction of  $D$ , with associated eigenvalue 1. More generally,

$$D(e^{kx}) = (e^{kx})' = k(e^{kx}) \quad (\text{use the chain rule}),$$

showing that  $e^{kx}$  is an eigenfunction of  $D$  with associated eigenvalue  $k$ . Here  $k$  can be any real number, proving our claim. ■

### EXAMPLE 4

Consider the linear transformation  $L(A) = A^T$  (the transpose<sup>10</sup>) from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ . Is transformation  $L$  diagonalizable? If so, find an eigenbasis for  $L$ . Hint: Consider symmetric and skew-symmetric matrices.

#### Solution

If  $A$  is symmetric, then  $L(A) = A^T = A = 1A$ , so that  $A$  is an eigenmatrix with eigenvalue 1. The symmetric  $2 \times 2$  matrices  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  form a three-dimensional space, with basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We need only one more matrix to form an eigenbasis for  $L$ , since  $\mathbb{R}^{2 \times 2}$  is four-dimensional.

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<sup>10</sup>This note has been intentionally omitted for this edition.

## Eigenvalues and Eigenvectors

If  $A$  is skew symmetric, then  $L(A) = A^T = -A = (-1)A$ , so that  $A$  is an eigenmatrix with eigenvalue  $-1$ . The skew-symmetric  $2 \times 2$  matrices  $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$  form a one-dimensional space, with basis  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

We have found enough eigenmatrices to form an eigenbasis for  $L$ :

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus,  $L$  is diagonalizable. ■

**EXAMPLE 5** Consider the linear transformation  $T(f(x)) = f(2x - 1)$  from  $P_2$  to  $P_2$ . Is transformation  $T$  diagonalizable? If so, find an eigenbasis  $\mathfrak{B}$  and the  $\mathfrak{B}$ -matrix  $B$  of  $T$ .

### Solution

Here it would be hard to find eigenvalues and eigenfunctions “by inspection”; we need a systematic approach. The idea is to find the matrix  $A$  of  $T$  with respect to some convenient basis  $\mathfrak{A}$ . Then we can determine whether  $A$  is diagonalizable, and, if so, we can find an eigenbasis for  $A$ . Finally we can transform this basis back into  $P_2$  to find an eigenbasis  $\mathfrak{B}$  for  $T$ .

We will use a commutative diagram to find the matrix  $A$  of  $T$  with respect to the standard basis  $\mathfrak{A} = (1, x, x^2)$ .

$$\begin{array}{ccc} a + bx + cx^2 & \xrightarrow{T} & T(a + bx + cx^2) \\ & \downarrow L_{\mathfrak{A}} & \downarrow L_{\mathfrak{A}} \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}} & \begin{bmatrix} a - b + c \\ 2b - 4c \\ 4c \end{bmatrix} \end{array}$$

The upper triangular matrix  $A$  has the three distinct eigenvalues, 1, 2, and 4, so that  $A$  is diagonalizable, by Theorem 3.4. A straightforward computation produces the eigenbasis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

for  $A$ . Transforming these vectors back into  $P_2$ , we find the eigenbasis  $\mathfrak{B}$  for  $T$  consisting of

$$1, \quad x - 1, \quad x^2 - 2x + 1 = (x - 1)^2.$$

To check our work, we can verify that these are indeed eigenfunctions of  $T$ :

$$\begin{aligned} T(1) &= 1 \\ T(x - 1) &= (2x - 1) - 1 = 2x - 2 = 2(x - 1) \\ T((x - 1)^2) &= ((2x - 1) - 1)^2 = (2x - 2)^2 = 4(x - 1)^2. \quad \checkmark \end{aligned}$$

The  $\mathfrak{B}$ -matrix of  $T$  is

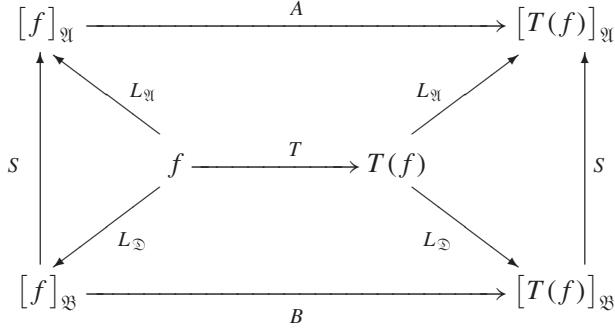
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

### Eigenvalues and Eigenvectors

Consider Figure 2, where

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

is the change of basis matrix from  $\mathfrak{B}$  to  $\mathfrak{A}$ . ■



**Figure 2**

**EXAMPLE 6** Let  $V$  be the space of all infinite sequences of real numbers. We define the linear transformation

$$T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_1, x_2, x_3, x_4, \dots)$$

from  $V$  to  $V$  (we omit the first term,  $x_0$ ). Find all the eigenvalues and eigensequences of  $T$ .

#### Solution

Since  $V$  is infinite dimensional, we cannot use the matrix techniques of Example 5 here. We have to go back to the definition of an eigenvalue: For a fixed scalar  $\lambda$ , we are looking for the infinite sequences  $(x_0, x_1, x_2, x_3, \dots)$  such that

$$T(x_0, x_1, x_2, x_3, \dots) = \lambda(x_0, x_1, x_2, x_3, \dots)$$

or

$$(x_1, x_2, x_3, \dots) = \lambda(x_0, x_1, x_2, x_3, \dots)$$

or

$$x_1 = \lambda x_0, \quad x_2 = \lambda x_1 = \lambda^2 x_0, \quad x_3 = \lambda x_2 = \lambda^3 x_0, \dots$$

The solutions are the *geometric sequences* of the form

$$(x_0, \lambda x_0, \lambda^2 x_0, \lambda^3 x_0, \dots) = x_0(1, \lambda, \lambda^2, \lambda^3, \dots).$$

Thus, all real numbers  $\lambda$  are eigenvalues of  $T$ , and the eigenspace  $E_\lambda$  is one-dimensional for all  $\lambda$ , with the geometric sequence  $(1, \lambda, \lambda^2, \lambda^3, \dots)$  as a basis.

For example, when  $\lambda = 3$ , we have

$$T(1, 3, 9, 27, 81, \dots) = (3, 9, 27, 81, \dots) = 3(1, 3, 9, 27, \dots),$$

demonstrating that  $(1, 3, 9, 27, 81, \dots)$  is an eigensequence of  $T$  with eigenvalue 3. ■

## EXERCISES 4

For the matrices  $A$  in Exercises 1 through 12, find closed formulas for  $A^t$ , where  $t$  is an arbitrary positive integer. Follow the strategy outlined in Theorem 4.2 and illustrated in Example 2. In Exercises 9 through 12, feel free to use technology.

1.  $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

2.  $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

3.  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

4.  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

5.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

6.  $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

7.  $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$

8.  $A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

9.  $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

10.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

11.  $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 3 \end{bmatrix}$

12.  $A = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.4 & 0.6 & 0.4 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$

For the matrices  $A$  and the vectors  $\vec{x}_0$  in Exercises 13 through 19, find closed formulas for  $A^t \vec{x}_0$ , where  $t$  is an arbitrary positive integer. Follow the strategy outlined in Theorem 1.6 and illustrated in Example 1. In Exercises 16 through 19, feel free to use technology.

13.  $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

14.  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

15.  $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

16.  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

17.  $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

18.  $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

19.  $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$

For the matrices  $A$  in Exercises 20 through 24, find  $\lim_{t \rightarrow \infty} A^t$ . Feel free to use Theorem 4.1.

20.  $A = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$

21.  $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$

22.  $A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

23.  $A = \begin{bmatrix} 0 & 0.5 & 0.4 \\ 1 & 0 & 0.6 \\ 0 & 0.5 & 0 \end{bmatrix}$

24.  $A = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.4 & 0.6 & 0.4 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$

For the matrices  $A$  and the vectors  $\vec{x}_0$  in Exercises 25 through 29, find  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0)$ . Feel free to use Theorem 4.1.

25.  $A = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$

26.  $A = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix}$

27.  $A = \begin{bmatrix} 0 & 0.5 & 0.4 \\ 1 & 0 & 0.6 \\ 0 & 0.5 & 0 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$

28.  $A = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.4 & 0.6 & 0.4 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$

29.  $A = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.3 \end{bmatrix}$

30. a. Sketch a phase portrait for the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ , where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

- b. In his paper “On the Measurement of the Circle,” the great Greek mathematician Archimedes (c. 280–210 B.C.) uses the approximation

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

to estimate  $\cos(30^\circ)$ . He does not explain how he arrived at these estimates. Explain how we can obtain these approximations from the dynamical system in part a. Hint:

$$A^4 = \begin{bmatrix} 97 & 56 \\ 168 & 97 \end{bmatrix}, \quad A^6 = \begin{bmatrix} 1351 & 780 \\ 2340 & 1351 \end{bmatrix}.$$

- c. Without using technology, explain why

$$\frac{1351}{780} - \sqrt{3} < 10^{-6}.$$

Hint: Consider  $\det(A^6)$ .

## Eigenvalues and Eigenvectors

- d. Based on the data in part b, give an underestimate of the form  $p/q$  of  $\sqrt{3}$  that is better than the one given by Archimedes.
31. Let  $x(t)$  and  $y(t)$  be the annual defense budgets of two antagonistic nations (expressed in billions of U.S. dollars). The change of these budgets is modeled by the equations

$$\begin{aligned}x(t+1) &= ax(t) + by(t) \\y(t+1) &= bx(t) + ay(t),\end{aligned}$$

where  $a$  is a constant slightly less than 1, expressing the fact that defense budgets tend to decline when there is no perceived threat. The constant  $b$  is a small positive number. You may assume that  $a$  exceeds  $b$ .

Suppose  $x(0) = 3$  and  $y(0) = 0.5$ . What will happen in the long term? There are three possible cases, depending on the numerical values of  $a$  and  $b$ . Sketch a trajectory for each case, and discuss the outcome in practical terms. Include the eigenspaces in all your sketches.

32. A street magician at Montmartre begins to perform at 11:00 P.M. on Saturday night. He starts out with no onlookers, but he attracts passersby at a rate of 10 per minute. Some get bored and wander off: Of the people present  $t$  minutes after 11:00 P.M., 20% will have left a minute later (but everybody stays for at least a minute). Let  $C(t)$  be the size of the crowd  $t$  minutes after 11:00 P.M. Find a  $2 \times 2$  matrix  $A$  such that

$$\begin{bmatrix} C(t+1) \\ 1 \end{bmatrix} = A \begin{bmatrix} C(t) \\ 1 \end{bmatrix}.$$

Find a closed formula for  $C(t)$ , and graph this function. What is the long-term behavior of  $C(t)$ ?

33. Three friends, Alberich, Brunnhilde, and Carl, play a number game together: Each thinks of a (real) number and announces it to the others. In the first round, each player finds the average of the numbers chosen by the two others; that is his or her new score. In the second round, the corresponding averages of the scores in the first round are taken, and so on. Here is an example:

	A	B	C
Initial choice	7	11	5
After 1st round	8	6	9
After 2nd round	7.5	8.5	7

Whoever is ahead after 1,001 rounds wins.

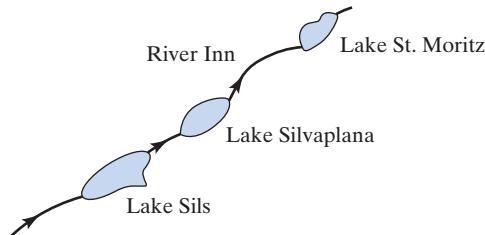
- a. The state of the game after  $t$  rounds can be represented as a vector:

$$\vec{x}(t) = \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix} \quad \begin{array}{l} \text{Alberich's score} \\ \text{Brunnhilde's score} \\ \text{Carl's score} \end{array}$$

Find the matrix  $A$  such that  $\vec{x}(t+1) = A\vec{x}(t)$ .

- b. With the initial values mentioned earlier ( $a_0 = 7$ ,  $b_0 = 11$ ,  $c_0 = 5$ ), what is the score after 10 rounds? After 50 rounds? Use technology.
- c. Now suppose that Alberich and Brunnhilde initially pick the numbers 1 and 2, respectively. If Carl picks the number  $c_0$ , what is the state of the game after  $t$  rounds? [Find closed formulas for  $a(t)$ ,  $b(t)$ ,  $c(t)$ , in terms of  $c_0$ .] For which choices of  $c_0$  does Carl win the game?

34. In an unfortunate accident involving an Austrian truck, 100 kg of a highly toxic substance are spilled into Lake Silvaplana, in the Swiss Engadine Valley. The river Inn carries the pollutant down to Lake Silvaplana and later to Lake St. Moritz.



This sorry state,  $t$  weeks after the accident, can be described by the vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \begin{array}{l} \text{pollutant in Lake Sils} \\ \text{pollutant in Lake Silvaplana} \\ \text{pollutant in Lake St. Moritz} \end{array} \quad \left. \right\} \text{ (in kg)}.$$

Suppose that

$$\vec{x}(t+1) = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \vec{x}(t).$$

- a. Explain the significance of the entries of the transformation matrix in practical terms.  
 b. Find closed formulas for the amount of pollutant in each of the three lakes  $t$  weeks after the accident. Graph the three functions against time (on the same axes). When does the pollution in Lake Silvaplana reach a maximum?

35. Consider a dynamical system

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

whose transformation from time  $t$  to time  $t+1$  is given by the following equations:

$$\begin{aligned}x_1(t+1) &= 0.1x_1(t) + 0.2x_2(t) + 1 \\x_2(t+1) &= 0.4x_1(t) + 0.3x_2(t) + 2.\end{aligned}$$

Such a system, with constant terms in the equations, is not linear, but *affine*.

## Eigenvalues and Eigenvectors

- a. Find a  $2 \times 2$  matrix  $A$  and a vector  $\vec{b}$  in  $\mathbb{R}^2$  such that

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}.$$

- b. Introduce a new state vector

$$\vec{y}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ 1 \end{bmatrix},$$

with a “dummy” 1 in the last component. Find a  $3 \times 3$  matrix  $B$  such that

$$\vec{y}(t+1) = B\vec{y}(t).$$

How is  $B$  related to the matrix  $A$  and the vector  $\vec{b}$  in part (a)? Can you write  $B$  as a block matrix involving  $A$  and  $\vec{b}$ ?

- c. What is the relationship between the eigenvalues of  $A$  and  $B$ ? What about eigenvectors?  
d. For arbitrary values of  $x_1(0)$  and  $x_2(0)$ , what can you say about the long-term behavior of  $x_1(t)$  and  $x_2(t)$ ?  
**36.** A machine contains the grid of wires shown in the accompanying sketch. At the seven indicated points, the temperature is kept fixed at the given values (in  $^\circ\text{C}$ ). Consider the temperatures  $T_1(t)$ ,  $T_2(t)$ , and  $T_3(t)$  at the other three mesh points. Because of heat flow along the wires, the temperatures  $T_i(t)$  changes according to the formula

$$T_i(t+1) = T_i(t) - \frac{1}{10} \sum (T_i(t) - T_{\text{adj}}(t)),$$

where the sum is taken over the four adjacent points in the grid and time is measured in minutes. For example,

$$\begin{aligned} T_2(t+1) &= T_2(t) - \frac{1}{10}(T_2(t) - T_1(t)) - \frac{1}{10}(T_2(t) - 200) \\ &\quad - \frac{1}{10}(T_2(t) - 0) - \frac{1}{10}(T_2(t) - T_3(t)). \end{aligned}$$

Note that each of the four terms we subtract represents the cooling caused by heat flowing along one of the wires. Let

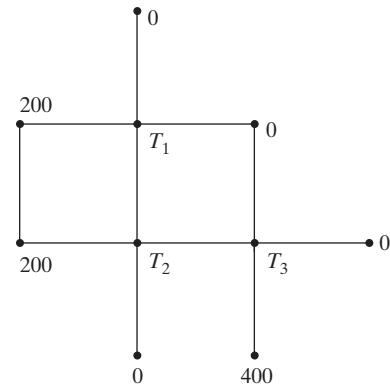
$$\vec{x}(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix}.$$

- a. Find a  $3 \times 3$  matrix  $A$  and a vector  $\vec{b}$  in  $\mathbb{R}^3$  such that

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}.$$

- b. Introduce the state vector

$$\vec{y}(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \\ 1 \end{bmatrix},$$



with a “dummy” 1 as the last component. Find a  $4 \times 4$  matrix  $B$  such that

$$\vec{y}(t+1) = B\vec{y}(t).$$

(This technique for converting an affine system into a linear system is introduced in Exercise 35; see also Exercise 32.)

- c. Suppose the initial temperatures are  $T_1(0) = T_2(0) = T_3(0) = 0$ . Using technology, find the temperatures at the three points at  $t = 10$  and  $t = 30$ . What long-term behavior do you expect?  
d. Using technology, find numerical approximations for the eigenvalues of the matrix  $B$ . Find an eigenvector for the largest eigenvalue. Use the results to confirm your conjecture in part (c).  
**37.** The color of snapdragons is determined by a pair of genes, which we designate by the letters  $A$  and  $a$ . The pair of genes is called the flower’s *genotype*. Genotype  $AA$  produces red flowers, genotype  $Aa$  pink ones, and genotype  $aa$  white ones. A biologist undertakes a breeding program, starting with a large population of flowers of genotype  $AA$ . Each flower is fertilized with pollen from a plant of genotype  $Aa$  (taken from another population), and one offspring is produced. Since it is a matter of chance which of the genes a parent passes on, we expect half of the flowers in the next generation to be red (genotype  $AA$ ) and the other half pink (genotype  $Aa$ ). All the flowers in this generation are now fertilized with pollen from plants of genotype  $Aa$  (taken from another population), and so on.

- a. Find closed formulas for the fractions of red, pink, and white flowers in the  $t$ th generation. We know that  $r(0) = 1$  and  $p(0) = w(0) = 0$ , and we found that  $r(1) = p(1) = \frac{1}{2}$  and  $w(1) = 0$ .

## Eigenvalues and Eigenvectors

- b.** What is the proportion  $r(t) : p(t) : w(t)$  in the long run?
- 38.** *Leonardo of Pisa: The rabbit problem.* Leonardo of Pisa (c. 1170–1240), also known as Fibonacci, was the first outstanding European mathematician after the ancient Greeks. He traveled widely in the Islamic world and studied Arabic mathematical writing. His work is in the spirit of the Arabic mathematics of his day. Fibonacci brought the decimal-position system to Europe. In his book *Liber abaci* (1202),<sup>11</sup> Fibonacci discusses the following problem:

How many pairs of rabbits can be bred from one pair in one year? A man has one pair of rabbits at a certain place entirely surrounded by a wall. We wish to know how many pairs can be bred from it in one year, if the nature of these rabbits is such that they breed every month one other pair and begin to breed in the second month after their birth. Let the first pair breed a pair in the first month, then duplicate it and there will be 2 pairs in a month. From these pairs one, namely, the first, breeds a pair in the second month, and thus there are 3 pairs in the second month. From these, in one month, two will become pregnant, so that in the third month 2 pairs of rabbits will be born. Thus, there are 5 pairs in this month. From these, in the same month, 3 will be pregnant, so that in the fourth month there will be 8 pairs. From these pairs, 5 will breed 5 other pairs, which, added to the 8 pairs, gives 13 pairs in the fifth month, from which 5 pairs (which were bred in that same month) will not conceive in that month, but the other 8 will be pregnant. Thus, there will be 21 pairs in the sixth month. When we add to these the 13 pairs that are bred in the seventh month, then there will be in that month 34 pairs [and so on, 55, 89, 144, 233, 377, ...]. Finally, there will be 377, and this number of pairs has been born from the first-mentioned pair at the given place in one year.

Let  $j(t)$  be the number of juvenile pairs and  $a(t)$  the number of adult pairs after  $t$  months. Fibonacci starts his thought experiment in rabbit breeding with one adult pair, so  $j(0) = 0$  and  $a(0) = 1$ . At  $t = 1$ , the adult pair will have bred a (juvenile) pair, so  $a(1) = 1$  and  $j(1) = 1$ . At  $t = 2$ , the initial adult pair will have bred another (juvenile) pair, and last month's juvenile pair will have grown up, so  $a(2) = 2$  and  $j(2) = 1$ .

- a.** Find formulas expressing  $a(t + 1)$  and  $j(t + 1)$  in terms of  $a(t)$  and  $j(t)$ . Find the matrix  $A$  such that

$$\vec{x}(t + 1) = A\vec{x}(t),$$

where

$$\vec{x}(t) = \begin{bmatrix} a(t) \\ j(t) \end{bmatrix}.$$

- b.** Find closed formulas for  $a(t)$  and  $j(t)$ . (Note: You will have to deal with irrational quantities here.)
- c.** Find the limit of the ratio  $a(t)/j(t)$  as  $t$  approaches infinity. The result is known as the *golden section*. The golden section of a line segment  $AB$  is given by the point  $P$  such that

$$\frac{\overline{AB}}{\overline{AP}} = \frac{\overline{AP}}{\overline{PB}}.$$



*Find all the eigenvalues and “eigenvectors” of the linear transformations in Exercises 39 through 52.*

- 39.**  $T(f) = f' - f$  from  $C^\infty$  to  $C^\infty$
- 40.**  $T(f) = 5f' - 3f$  from  $C^\infty$  to  $C^\infty$
- 41.**  $L(A) = A + A^T$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ . Is  $L$  diagonalizable?
- 42.**  $L(A) = A - A^T$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ . Is  $L$  diagonalizable?
- 43.**  $T(x + iy) = x - iy$  from  $\mathbb{C}$  to  $\mathbb{C}$ . Is  $T$  diagonalizable?
- 44.**  $T(x_0, x_1, x_2, \dots) = (x_2, x_3, \dots)$  from the space  $V$  of infinite sequences into  $V$ . (We drop the first two terms of the sequence.)
- 45.**  $T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)$  from the space  $V$  of infinite sequences into  $V$ . (We insert a zero at the beginning.)
- 46.**  $T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots)$  from the space  $V$  of infinite sequences into  $V$ . (We drop every other term.)
- 47.**  $T(f(x)) = f(-x)$  from  $P_2$  to  $P_2$ . Is  $T$  diagonalizable?
- 48.**  $T(f(x)) = f(2x)$  from  $P_2$  to  $P_2$ . Is  $T$  diagonalizable?
- 49.**  $T(f(x)) = f(3x - 1)$  from  $P_2$  to  $P_2$ . Is  $T$  diagonalizable?
- 50.**  $T(f(x)) = f(x - 3)$  from  $P_2$  to  $P_2$ . Is  $T$  diagonalizable?
- 51.**  $T(f) = f'$  from  $P$  to  $P$
- 52.**  $T(f(x)) = x(f'(x))$  from  $P$  to  $P$

<sup>11</sup>For a translation into modern English, see Laurence E. Sigler, *Fibonacci's Liber Abaci*, Springer-Verlag, 2002.

## Eigenvalues and Eigenvectors

53. For a regular transition matrix  $A$ , prove the following:

a.  $\lambda = 1$  is an eigenvalue of  $A$  with geometric multiplicity 1, and

b. If  $\lambda$  is any real eigenvalue of  $A$ , then  $-1 < \lambda \leq 1$ .

*Hint:* There exists an odd positive integer  $m$  such that  $A^m$  is a positive transition matrix.

54. Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*Hint:* Compute  $A^2$  and  $B^2$ .

55. Find two  $2 \times 2$  matrices  $A$  and  $B$  such that  $AB$  fails to be similar to  $BA$ . *Hint:* It can be arranged that  $AB$  is zero, but  $BA$  isn't.

56. Show that if  $A$  and  $B$  are two  $n \times n$  matrices, then the matrices  $AB$  and  $BA$  have the same characteristic polynomial, and thus the same eigenvalues (matrices  $AB$  and  $BA$  need not be similar though; see Exercise 55).

*Hint:*

$$\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_n & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_n & A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}.$$

57. Consider an  $m \times n$  matrix  $A$  and an  $n \times m$  matrix  $B$ . Using Exercise 56 as a guide, show that matrices  $AB$  and  $BA$  have the same nonzero eigenvalues, with the same algebraic multiplicities. What about eigenvalue 0?

58. Consider a nonzero  $3 \times 3$  matrix  $A$  such that  $A^2 = 0$ .

a. Show that the image of  $A$  is a subspace of the kernel of  $A$ .

b. Find the dimensions of the image and kernel of  $A$ .

c. Pick a nonzero vector  $\vec{v}_1$  in the image of  $A$ , and write  $\vec{v}_1 = A\vec{v}_2$  for some  $\vec{v}_2$  in  $\mathbb{R}^3$ . Let  $\vec{v}_3$  be a vector in the kernel of  $A$  that fails to be a scalar multiple of  $\vec{v}_1$ . Show that  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$  is a basis of  $\mathbb{R}^3$ .

d. Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$  with respect to basis  $\mathfrak{B}$ .

59. If  $A$  and  $B$  are two nonzero  $3 \times 3$  matrices such that  $A^2 = B^2 = 0$ , is  $A$  necessarily similar to  $B$ ? *Hint:* Exercise 58 is useful.

60. For the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$ , find an invertible

matrix  $S$  such that  $S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . *Hint:* Exercise 58 is useful.

61. Consider an  $n \times n$  matrix  $A$  such that  $A^2 = 0$ , with rank  $A = r$ . (In Example 58 we consider the case when

$n = 3$  and  $r = 1$ .) Show that  $A$  is similar to the block matrix

$$B = \begin{bmatrix} J & 0 & \cdots & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & J & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad \text{where } J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrix  $B$  has  $r$  blocks of the form  $J$  along the diagonal, with all other entries being 0. *Hint:* Mimic the approach outlined in Exercise 58. Pick a basis  $\vec{v}_1, \dots, \vec{v}_r$  of the image of  $A$ , write  $\vec{v}_i = A\vec{w}_i$  for  $i = 1, \dots, r$ , and expand  $\vec{v}_1, \dots, \vec{v}_r$  to a basis  $\vec{v}_1, \dots, \vec{v}_r, \vec{u}_1, \dots, \vec{u}_m$  of the kernel of  $A$ . Show that  $\vec{v}_1, \vec{w}_1, \vec{v}_2, \vec{w}_2, \dots, \vec{v}_r, \vec{w}_r, \vec{u}_1, \dots, \vec{u}_m$  is a basis of  $\mathbb{R}^n$ , and show that  $B$  is the matrix of  $T(\vec{x}) = A\vec{x}$  with respect to this basis.

62. This Exercise has been intentionally omitted for this edition.

63. Consider the linear transformation  $T(f) = f''$  from  $C^\infty$  to  $C^\infty$ . For each of the following eigenvalues, find a basis of the associated eigenspace. See Exercise 62.

a.  $\lambda = 1$    b.  $\lambda = 0$    c.  $\lambda = -1$    d.  $\lambda = -4$

64. If  $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ , find a basis of the linear space  $V$  of all  $2 \times 2$  matrices  $S$  such that  $AS = SB$ , where  $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ . Find the dimension of  $V$ .

65. If  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ , find a basis of the linear space  $V$  of all  $2 \times 2$  matrices  $S$  such that  $AS = SB$ , where  $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ . Find the dimension of  $V$ .

66. If  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ , find a basis of the linear space  $V$  of all  $3 \times 3$  matrices  $S$  such that  $AS = SB$ , where  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ . Find the dimension of  $V$ .

67. Consider a  $5 \times 5$  matrix  $A$  with two distinct eigenvalues,  $\lambda_1$  and  $\lambda_2$ , with geometric multiplicities 3 and 2, respectively. What is the dimension of the linear space of all  $5 \times 5$  matrices  $S$  such that  $AS = SB$ , where  $B$  is the diagonal matrix with the diagonal entries  $\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2$ ?

68. If  $A$  is an  $n \times n$  matrix with  $n$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , what is the dimension of the linear space of

## Eigenvalues and Eigenvectors

all  $n \times n$  matrices  $S$  such that  $AS = SB$ , where  $B$  is the diagonal matrix with the diagonal entries  $\lambda_1, \dots, \lambda_n$ ? Use Exercises 64 and 65 as a guide.

- 69.** We say that two  $n \times n$  matrices  $A$  and  $B$  are *simultaneously diagonalizable* if there exists an invertible  $n \times n$  matrix  $S$  such that  $S^{-1}AS$  and  $S^{-1}BS$  are both diagonal.

- a. Are the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

simultaneously diagonalizable? Explain.

- b. Show that if  $A$  and  $B$  are simultaneously diagonalizable then  $AB = BA$ .
- c. Give an example of two  $n \times n$  matrices such that  $AB = BA$ , but  $A$  and  $B$  are not simultaneously diagonalizable.
- d. Let  $D$  be a diagonal  $n \times n$  matrix with  $n$  distinct entries on the diagonal. Find all  $n \times n$  matrices  $B$  that commute with  $D$ .
- e. Show that if  $AB = BA$  and  $A$  has  $n$  distinct eigenvalues, then  $A$  and  $B$  are simultaneously diagonalizable. *Hint:* Part d is useful.
- 70.** Consider an  $n \times n$  matrix  $A$  with  $m$  distinct eigenvalues  $\lambda_1, \dots, \lambda_m$ . Show that matrix  $A$  is diagonalizable if (and only if)  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$ . *Hint:* If  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$ , show that the sum of the dimensions of the eigenspaces is  $n$ .

- 71.** Use the method outlined in Exercise 70 to check whether the matrix  $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$  is diagonalizable.

- 72.** Use the method outlined in Exercise 70 to check for which values of the constants  $a$ ,  $b$ , and  $c$  the matrix  $A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$  is diagonalizable.

- 73.** Prove the Cayley–Hamilton theorem,  $f_A(A) = 0$ , for diagonalizable matrices  $A$ . See Exercise 3.54.

- 74.** In both parts of this problem, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

with eigenvalues  $\lambda_1 = 5$  and  $\lambda_2 = -1$ . See Example 1.

- a. Are the column vectors of the matrices  $A - \lambda_1 I_2$  and  $A - \lambda_2 I_2$  eigenvectors of  $A$ ? Explain. Does this work for other  $2 \times 2$  matrices? What about diagonalizable  $n \times n$  matrices with two distinct eigenvalues, such as projections or reflections? *Hint:* Exercise 70 is helpful.

- b. Are the column vectors of

$$A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

eigenvectors of  $A$ ? Explain.

## 5 Complex Eigenvalues

Imagine that you are diabetic and have to pay close attention to how your body metabolizes glucose. After you eat a heavy meal, the glucose concentration will reach a peak, and then it will slowly return to the fasting level. Certain hormones help regulate the glucose metabolism, especially the hormone insulin. Compare with Exercise 1.70. Let  $g(t)$  be the excess glucose concentration in your blood, usually measured in milligrams of glucose per 100 milliliters of blood. (*Excess* means that we measure how much the glucose concentration deviates from the fasting level.) A negative value of  $g(t)$  indicates that the glucose concentration is below fasting level at time  $t$ . Let  $h(t)$  be the excess insulin concentration in your blood. Researchers have developed mathematical models for the glucose regulatory system. The following is one such model, in slightly simplified (linearized) form.

$$\begin{aligned} g(t+1) &= ag(t) - bh(t) \\ h(t+1) &= cg(t) + dh(t) \end{aligned}$$

(These formulas apply between meals; obviously, the system is disturbed during and right after a meal.)

## Eigenvalues and Eigenvectors

In these formulas,  $a$ ,  $b$ ,  $c$ , and  $d$  are positive constants; constants  $a$  and  $d$  will be less than 1. The term  $-bh(t)$  expresses the fact that insulin helps your body absorb glucose, and the term  $cg(t)$  represents the fact that the glucose in your blood stimulates the pancreas to secrete insulin.

For your system, the equations might be

$$\begin{aligned} g(t+1) &= 0.9g(t) - 0.4h(t) \\ h(t+1) &= 0.1g(t) + 0.9h(t), \end{aligned}$$

with initial values  $g(0) = 100$  and  $h(0) = 0$ , after a heavy meal. Here, time  $t$  is measured in hours.

After one hour, the values will be  $g(1) = 90$  and  $h(1) = 10$ . Some of the glucose has been absorbed, and the excess glucose has stimulated the pancreas to produce 10 extra units of insulin.

The rounded values of  $g(t)$  and  $h(t)$  in the following table give you some sense for the evolution of this dynamical system.

$t$	0	1	2	3	4	5	6	7	8	15	22	29
$g(t)$	100	90	77	62.1	46.3	30.6	15.7	2.3	-9.3	-29	1.6	9.5
$h(t)$	0	10	18	23.9	27.7	29.6	29.7	28.3	25.7	-2	-8.3	0.3

We can “connect the dots” to sketch a rough trajectory, visualizing the long-term behavior. See Figure 1.

We see that after 7 hours the excess glucose is almost gone, but now there are about 30 units of excess insulin in the system. Since this excess insulin helps to reduce glucose further, the glucose concentration will now fall below fasting level, reaching about  $-30$  after 15 hours. (You will feel awfully hungry by now.) Under normal circumstances, you would have taken another meal in the meantime, of course, but let’s consider the case of (voluntary or involuntary) fasting.

We leave it to the reader to explain the concentrations after 22 and 29 hours, in terms of how glucose and insulin concentrations influence each other, according to our model. The *spiralizing trajectory* indicates an *oscillatory behavior* of the system: Both glucose and insulin levels will swing back and forth around the fasting level,

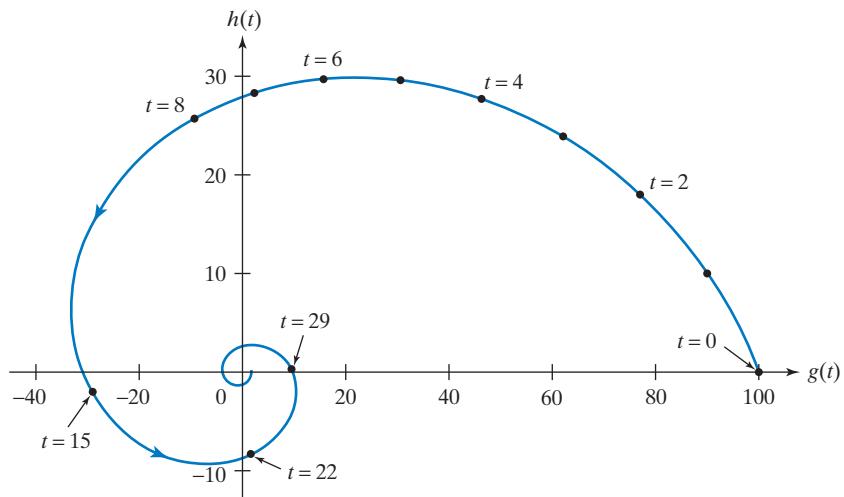
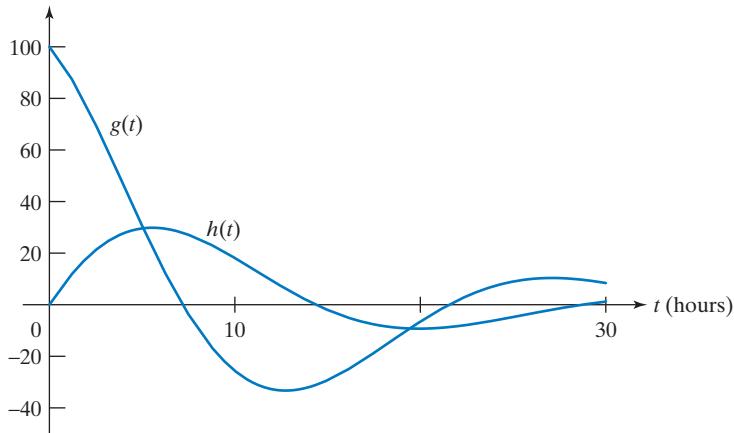


Figure 1

## Eigenvalues and Eigenvectors



**Figure 2**

like a damped pendulum. Both concentrations will approach the fasting level (thus the name).

Another way to visualize this oscillatory behavior is to graph the functions  $g(t)$  and  $h(t)$  against time, using the values from our table. See Figure 2.

Next, we try to use the tools developed in the last four sections to analyze this system. We can introduce the transformation matrix

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

and the state vector

$$\vec{x}(t) = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}.$$

Then

$$\vec{x}(t+1) = A\vec{x}(t) \quad \text{and thus} \quad \vec{x}(t) = A^t \vec{x}(0) = A^t \begin{bmatrix} 100 \\ 0 \end{bmatrix}.$$

To find formulas for  $g(t)$  and  $h(t)$ , we need to know the eigenvalues and eigenvectors of matrix  $A$ . The characteristic polynomial of  $A$  is

$$f_A(\lambda) = \lambda^2 - 1.8\lambda + 0.85,$$

so that

$$\lambda_{1,2} = \frac{1.8 \pm \sqrt{3.24 - 3.4}}{2} = \frac{1.8 \pm \sqrt{-0.16}}{2}.$$

Since the square of a real number cannot be negative, there are no *real* eigenvalues here. However, if we allow complex solutions, then we have the eigenvalues

$$\lambda_{1,2} = \frac{1.8 \pm \sqrt{-0.16}}{2} = \frac{1.8 \pm i\sqrt{0.16}}{2} = 0.9 \pm 0.2i.$$

In this section, we will first review some basic facts on complex numbers. Then we will examine how the theory of eigenvalues and eigenvectors developed in Sections 1 through 4 can be adapted to the complex case. In Section 6 we will apply this work to dynamical systems. A great many dynamical systems, in physics, chemistry, biology, and economics, show oscillatory behavior; we will see that we can expect complex eigenvalues in this case.

## Eigenvalues and Eigenvectors

These tools will enable you to find formulas for  $g(t)$  and  $h(t)$ . See Exercise 6.32.

### Complex Numbers: A Brief Review

Let us review some basic facts about complex numbers. We trust that you have at least a fleeting acquaintance with complex numbers. Without attempting a formal definition, we recall that a complex number can be expressed as

$$z = a + ib,$$

where  $a$  and  $b$  are real numbers.<sup>12</sup> Addition of complex numbers is defined in a natural way, by the rule

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

and multiplication is defined by the rule

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc);$$

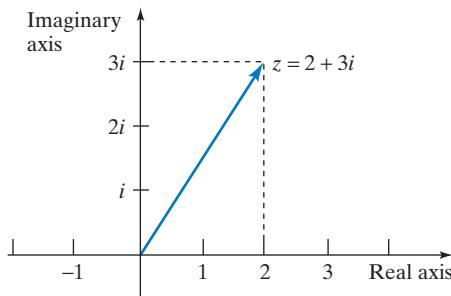
that is, we let  $i \cdot i = -1$  and distribute.

If  $z = a + ib$  is a complex number, we call  $a$  its *real part*, denoted by  $\operatorname{Re} z$  and  $b$  its *imaginary part*, denoted by  $\operatorname{Im} z$ . A complex number of the form  $ib$  (with  $a = 0$ ) is called *imaginary*.

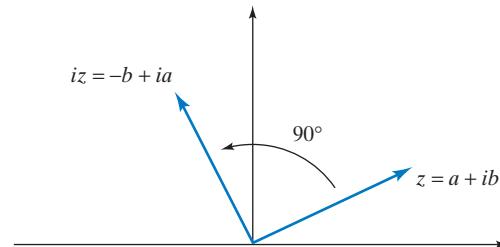
The set of all complex numbers is denoted by  $\mathbb{C}$ . The real numbers,  $\mathbb{R}$ , form a subset of  $\mathbb{C}$  (namely, those complex numbers with imaginary part 0).

Complex numbers can be represented as vectors (or points) in the complex plane,<sup>13</sup> as shown in Figure 3. This is a graphical representation of the isomorphism

$$T \begin{bmatrix} a \\ b \end{bmatrix} = a + ib \quad \text{from } \mathbb{R}^2 \text{ to } \mathbb{C}.$$



**Figure 3**



**Figure 4**

**EXAMPLE 1** Consider a nonzero complex number  $z$ . What is the geometric relationship between  $z$  and  $iz$  in the complex plane?

---

<sup>12</sup> The letter  $i$  for the imaginary unit was introduced by Leonhard Euler, the most prolific mathematician in history. For a fascinating glimpse at the history of the complex numbers, see Tobias Dantzig, *Number: The Language of Science*, Macmillan, 1954. For another intriguing introduction, full of poetry, history, and philosophy, see Barry Mazur, *Imagining Numbers (particularly the square root of minus fifteen)*, Farrar, Straus, and Giroux, 2003.

<sup>13</sup> Also called “Argand plane,” after the Swiss mathematician Jean Robert Argand (1768–1822). The representation of complex numbers in the plane was introduced independently by Argand, by Gauss, and by the Norwegian mathematician Caspar Wessel (1745–1818).

## Eigenvalues and Eigenvectors

### Solution

If  $z = a + ib$ , then  $iz = -b + ia$ . We obtain the vector  $\begin{bmatrix} -b \\ a \end{bmatrix}$  (representing  $iz$ ) by rotating the vector  $\begin{bmatrix} a \\ b \end{bmatrix}$  (representing  $z$ ) through an angle of  $90^\circ$  in the counter-clockwise direction. See Figure 4. ■

The *conjugate* of a complex number  $z = a + ib$  is defined by

$$\bar{z} = a - ib.$$

(The sign of the imaginary part is reversed.) We say that  $z$  and  $\bar{z}$  form a *conjugate pair* of complex numbers. Geometrically, the conjugate  $\bar{z}$  is the reflection of  $z$  about the real axis, as shown in Figure 5.

Sometimes it is useful to describe a complex number in *polar coordinates*, as shown in Figure 6.

The length  $r$  of the vector is called the *modulus* of  $z$ , denoted by  $|z|$ . The polar angle  $\theta$  is called an *argument* of  $z$ ; note that the argument is determined only up to a multiple of  $2\pi$ . (Mathematicians say “modulo  $2\pi$ .”) For example, for  $z = -1$ , we can choose the argument  $\pi$ ,  $-\pi$ , or  $3\pi$ .

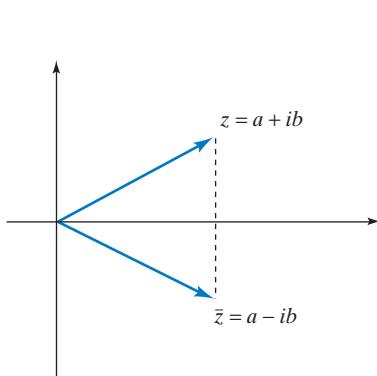


Figure 5

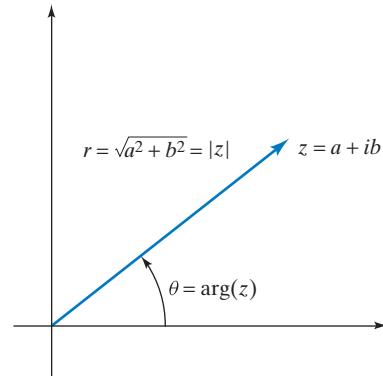


Figure 6

**EXAMPLE 2** Find the modulus and an argument of  $z = -2 + 2i$ .

### Solution

$|z| = \sqrt{2^2 + 2^2} = \sqrt{8}$ . Representing  $z$  in the complex plane, we see that  $\frac{3}{4}\pi$  is an argument of  $z$ . See Figure 7. ■

If  $z$  is a complex number with modulus  $r$  and argument  $\theta$ , we can write  $z$  as

$$z = r(\cos \theta + i \sin \theta) = r(\cos \theta + i \sin \theta),$$

as shown in Figure 8.

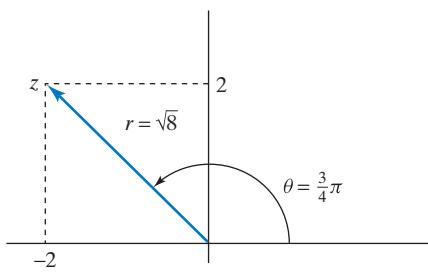


Figure 7

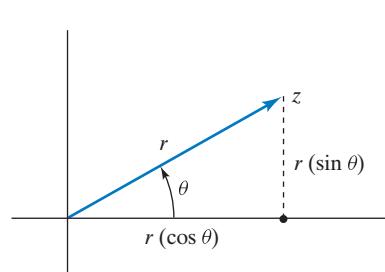


Figure 8

## Eigenvalues and Eigenvectors

The representation

$$z = r(\cos \theta + i \sin \theta)$$

is called the *polar form* of the complex number  $z$ .

**EXAMPLE 3** Consider the complex numbers  $z = \cos \alpha + i \sin \alpha$  and  $w = \cos \beta + i \sin \beta$ . Find the polar form of the product  $zw$ .

### Solution

Apply the addition formulas from trigonometry:

$$\begin{aligned} zw &= (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \\ &= \cos(\alpha + \beta) + i \sin(\alpha + \beta). \end{aligned}$$

We conclude that the modulus of  $zw$  is 1, and  $\alpha + \beta$  is an argument of  $zw$ . See Figure 9. ■

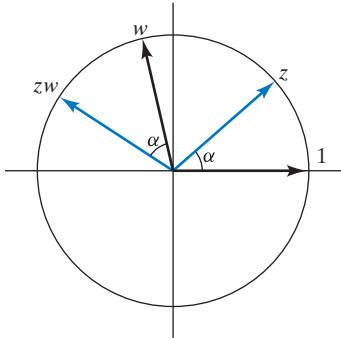


Figure 9

In general, if  $z = r(\cos \alpha + i \sin \alpha)$  and  $w = s(\cos \beta + i \sin \beta)$ , then

$$zw = rs(\cos(\alpha + \beta) + i \sin(\alpha + \beta)).$$

When we multiply two complex numbers, we multiply the moduli, and we add the arguments:

$$\begin{aligned} |zw| &= |z||w| \\ \arg(zw) &= \arg z + \arg w \quad (\text{modulo } 2\pi). \end{aligned}$$

**EXAMPLE 4** Describe the transformation  $T(z) = (3 + 4i)z$  from  $\mathbb{C}$  to  $\mathbb{C}$  geometrically.

### Solution

$$\begin{aligned} |T(z)| &= |3 + 4i||z| = 5|z| \\ \arg(T(z)) &= \arg(3 + 4i) + \arg(z) = \arctan\left(\frac{4}{3}\right) + \arg(z) \approx 53^\circ + \arg(z) \end{aligned}$$

The transformation  $T$  is a rotation combined with a scaling in the complex plane. See Figure 10.

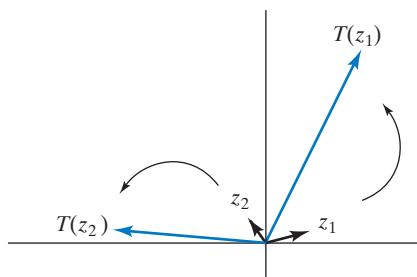


Figure 10 Rotate through about  $53^\circ$  and stretch the vector by a factor of 5.

## Eigenvalues and Eigenvectors

Alternatively, we observe that the matrix of the linear transformation  $T$  with respect to the basis  $1, i$  is  $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ , representing a rotation combined with a scaling.

The polar form is convenient for finding *powers* of a complex number  $z$ : If

$$z = r(\cos \theta + i \sin \theta),$$

then

$$\begin{aligned} z^2 &= r^2(\cos(2\theta) + i \sin(2\theta)), \\ &\vdots \\ z^n &= r^n(\cos(n\theta) + i \sin(n\theta)), \end{aligned}$$

for any positive integer  $n$ . Each time we multiply by  $z$ , the modulus is multiplied by  $r$  and the argument increases by  $\theta$ . The preceding formula was found by the French mathematician Abraham de Moivre (1667–1754). ■

### Theorem 5.1

#### De Moivre's formula

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

### EXAMPLE 5

Consider the complex number  $z = 0.5 + 0.8i$ . Represent the powers  $z^2, z^3, \dots$  in the complex plane. What is  $\lim_{n \rightarrow \infty} z^n$ ?

#### Solution

To study the powers, write  $z$  in polar form:

$$z = r(\cos \theta + i \sin \theta).$$

Here

$$r = \sqrt{0.5^2 + 0.8^2} \approx 0.943$$

and

$$\theta = \arctan \frac{0.8}{0.5} \approx 58^\circ.$$

We have

$$z^n = r^n(\cos(n\theta) + i \sin(n\theta)).$$

The vector representation of  $z^{n+1}$  is a little shorter than that of  $z^n$  (by about 5.7%), and  $z^{n+1}$  makes an angle  $\theta \approx 58^\circ$  with  $z^n$ . If we connect the tips of consecutive vectors, we see a trajectory that spirals in toward the origin, as shown in Figure 11. Note that  $\lim_{n \rightarrow \infty} z^n = 0$ , since  $r = |z| < 1$ . ■

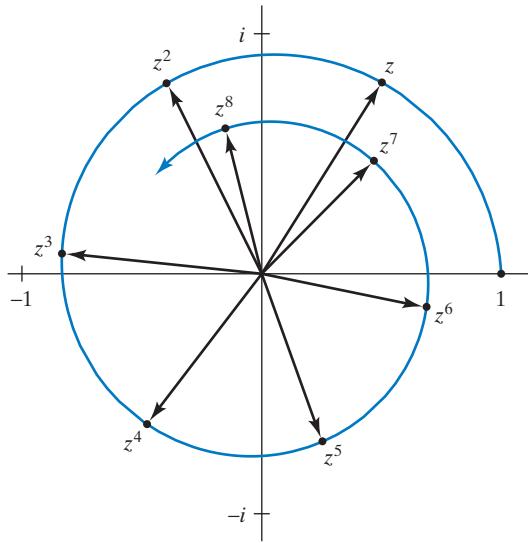


Figure 11

Perhaps the most remarkable property of the complex numbers is expressed in the fundamental theorem of algebra, first demonstrated by Carl Friedrich Gauss (in his thesis, at age 22).

**Theorem 5.2**
**Fundamental theorem of algebra**

Any polynomial  $p(\lambda)$  with complex coefficients splits; that is, it can be written as a product of linear factors

$$p(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n),$$

for some complex numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$ , and  $k$ . (The  $\lambda_i$  need not be distinct.)

Therefore, a polynomial  $p(\lambda)$  of degree  $n$  has precisely  $n$  complex roots if they are properly counted with their multiplicities.

For example, the polynomial

$$p(\lambda) = \lambda^2 + 1,$$

which does not have any real zeros, splits over  $\mathbb{C}$ :

$$p(\lambda) = (\lambda + i)(\lambda - i).$$

More generally, for a quadratic polynomial

$$q(\lambda) = \lambda^2 + b\lambda + c,$$

where  $b$  and  $c$  are real, we can find the complex roots

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

and

$$q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

Proving the fundamental theorem of algebra would lead us too far afield. Read any introduction to complex analysis or check Gauss's original proof.<sup>14</sup>

## Complex Eigenvalues and Eigenvectors

The complex numbers share some basic algebraic properties with the real numbers.<sup>15</sup> Mathematicians summarize these properties by saying that both the real numbers  $\mathbb{R}$  and the complex numbers  $\mathbb{C}$  form a *field*. The rational numbers  $\mathbb{Q}$  are another important example of a field; the integers  $\mathbb{Z}$ , on the other hand, don't form a field. (Which of the 10 properties listed in the footnote fail to hold in this case?)

We observe that everything works the same way except for those geometrical concepts that are defined in terms of the dot product (length, angles, orthogonality, and so on). The whole body of "core linear algebra" can be generalized without difficulty, however: echelon form, linear transformation, kernel, image, linear independence, basis, dimension, coordinates, linear spaces, determinant, eigenvalues, eigenvectors, and diagonalization.

### EXAMPLE 6

Diagonalize the rotation-scaling matrix  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  "over  $\mathbb{C}$ ." Here,  $a$  and  $b$  are real numbers, and  $b$  is nonzero.

#### Solution

We will find the eigenvalues of  $A$  first:

$$f_A(\lambda) = \det \begin{bmatrix} a - \lambda & -b \\ b & a - \lambda \end{bmatrix} = (a - \lambda)^2 + b^2 = 0$$

when

$$(a - \lambda)^2 = -b^2 \quad \text{or} \quad a - \lambda = \pm ib \quad \text{or} \quad \lambda = a \pm ib.$$

Now we find the eigenvectors:

$$\begin{aligned} E_{a+ib} &= \ker \begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix} = \text{span} \begin{bmatrix} i \\ 1 \end{bmatrix} \\ E_{a-ib} &= \ker \begin{bmatrix} ib & -b \\ b & ib \end{bmatrix} = \text{span} \begin{bmatrix} -i \\ 1 \end{bmatrix}. \end{aligned}$$

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<sup>14</sup>C. F. Gauss, *Werke*, III, 3–56. For an English translation, see D. J. Struik (editor), *A Source Book in Mathematics 1200–1800*, Princeton University Press, 1986.

<sup>15</sup>Here is a list of these properties:

1. Addition is commutative.
2. Addition is associative.
3. There exists a unique number  $n$  such that  $a + n = a$ , for all numbers  $a$ . This number  $n$  is denoted by 0.
4. For each number  $a$  there exists a unique number  $b$  such that  $a + b = 0$ . This number  $b$  is denoted by  $-a$ . (*Comment*: This property says that we can subtract in this number system.)
5. Multiplication is commutative.
6. Multiplication is associative.
7. There is a unique number  $e$  such that  $ea = a$ , for all numbers  $a$ . This number  $e$  is denoted by 1.
8. For each nonzero number  $a$  there exists a unique number  $b$  such that  $ab = 1$ . This number  $b$  is denoted by  $a^{-1}$ . (*Comment*: This property says that we can divide by a nonzero number.)
9. Multiplication distributes over addition:  $a(b + c) = ab + ac$ .
10. The numbers 0 and 1 are not equal.

### Eigenvalues and Eigenvectors

Thus,

$$R^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} R = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}, \quad \text{where } R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}. \quad \blacksquare$$

**EXAMPLE 7** Let  $A$  be a real  $2 \times 2$  matrix with eigenvalues  $a \pm ib$  (where  $b \neq 0$ ). Show that  $A$  is similar (over  $\mathbb{R}$ ) to the matrix  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , representing a rotation combined with a scaling.

#### Solution

Let  $\vec{v} \pm i\vec{w}$  be eigenvectors of  $A$  with eigenvalues  $a \pm ib$ . See Exercise 42. Matrix  $A$  is similar to  $\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$ ; more precisely,

$$\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix} = P^{-1}AP,$$

where  $P = \begin{bmatrix} \vec{v} + i\vec{w} & \vec{v} - i\vec{w} \end{bmatrix}$ . By Example 6, matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is similar to

$$\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

as well, with

$$\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix} = R^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} R, \quad \text{where } R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}.$$

Thus,

$$P^{-1}AP = R^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} R,$$

and

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = RP^{-1}APR^{-1} = S^{-1}AS,$$

where  $S = PR^{-1}$  and  $S^{-1} = (PR^{-1})^{-1} = RP^{-1}$ .

A straightforward computation shows that

$$S = PR^{-1} = \frac{1}{2i} \begin{bmatrix} \vec{v} + i\vec{w} & \vec{v} - i\vec{w} \end{bmatrix} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix};$$

note that  $S$  has real entries, as claimed. ■

#### Theorem 5.3

##### Complex eigenvalues and rotation-scaling matrices

If  $A$  is a real  $2 \times 2$  matrix with eigenvalues  $a \pm ib$  (where  $b \neq 0$ ), and if  $\vec{v} + i\vec{w}$  is an eigenvector of  $A$  with eigenvalue  $a + ib$ , then

$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$

## Eigenvalues and Eigenvectors

We see that matrix  $A$  is similar to a rotation-scaling matrix. If we introduce the inner product  $\langle \vec{x}, \vec{y} \rangle = (S^{-1}\vec{x}) \cdot (S^{-1}\vec{y})$  in  $\mathbb{R}^2$  and define the length of vectors and the angle between vectors with respect to this inner product, then the transformation  $T(\vec{x}) = A\vec{x}$  is a rotation combined with a scaling in that inner product space. (Think about it!)

**EXAMPLE 8** For  $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$ , find an invertible  $2 \times 2$  matrix  $S$  such that  $S^{-1}AS$  is a rotation-scaling matrix.

### Solution

We will use the method outlined in Theorem 5.3:

$$f_A(\lambda) = \lambda^2 - 2\lambda + 2, \quad \text{so that} \quad \lambda_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Now

$$E_{1+i} = \ker \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} = \text{span} \begin{bmatrix} -5 \\ -2+i \end{bmatrix},$$

and

$$\begin{bmatrix} -5 \\ -2+i \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{so that} \quad \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}.$$

Therefore,

$$S^{-1}AS = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{where} \quad S = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix}. \quad \blacksquare$$

The great advantage of complex eigenvalues is that there are so many of them. By the fundamental theorem of algebra, Theorem 5.2, the characteristic polynomial always splits:

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

### Theorem 5.4

A complex  $n \times n$  matrix has  $n$  complex eigenvalues if they are counted with their algebraic multiplicities.

Although a complex  $n \times n$  matrix may have fewer than  $n$  distinct complex eigenvalues (examples are  $I_n$  or  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ), this is literally a coincidence: Some of the  $\lambda_i$  in the factorization of the characteristic polynomial  $f_A(\lambda)$  coincide. “Most” complex  $n \times n$  matrices do have  $n$  distinct eigenvalues, so that *most complex  $n \times n$  matrices are diagonalizable* (by Theorem 3.4). An example of a matrix that fails to be diagonalizable over  $\mathbb{C}$  is  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

**EXAMPLE 9**

Consider an  $n \times n$  matrix  $A$  with complex eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , listed with their algebraic multiplicities. What is the relationship between the  $\lambda_i$  and the determinant of  $A$ ? Compare with Theorem 2.8.

**Solution**

$$f_A(\lambda) = \det(A - \lambda I_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

$$f_A(0) = \det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

so that

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

Can you interpret this result geometrically when  $A$  is a  $3 \times 3$  matrix with a real eigenbasis? Hint: Think about the expansion factor. See Exercise 18.

In Example 9, we found that the determinant of a matrix is the product of its complex eigenvalues. Likewise, the trace is the *sum* of the eigenvalues. The verification is left as Exercise 35.

**Theorem 5.5**
**Trace, determinant, and eigenvalues**

Consider an  $n \times n$  matrix  $A$  with complex eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , listed with their algebraic multiplicities. Then

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

and

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Note that this result is obvious for a triangular matrix: In this case, the eigenvalues are the diagonal entries.

## EXERCISES 5

**GOAL** Use the basic properties of complex numbers. Write products and powers of complex numbers in polar form. Apply the fundamental theorem of algebra.

1. Write the complex number  $z = 3 - 3i$  in polar form.
2. Find all complex numbers  $z$  such that  $z^4 = 1$ . Represent your answers graphically in the complex plane.
3. For an arbitrary positive integer  $n$ , find all complex numbers  $z$  such that  $z^n = 1$  (in polar form). Represent your answers graphically.
4. Show that if  $z$  is a nonzero complex number, then there are exactly two complex numbers  $w$  such that  $w^2 = z$ . If  $z$  is in polar form, describe  $w$  in polar form.
5. Show that if  $z$  is a nonzero complex number, then there exist exactly  $n$  complex numbers  $w$  such that  $w^n = z$ . If  $z$  is in polar form, write  $w$  in polar form. Represent the vectors  $w$  in the complex plane.
6. If  $z$  is a nonzero complex number in polar form, describe  $1/z$  in polar form. What is the relationship between the complex conjugate  $\bar{z}$  and  $1/z$ ? Represent the numbers  $z$ ,  $\bar{z}$ , and  $1/z$  in the complex plane.
7. Describe the transformation  $T(z) = (1 - i)z$  from  $\mathbb{C}$  to  $\mathbb{C}$  geometrically.

8. Use de Moivre's formula to express  $\cos(3\theta)$  and  $\sin(3\theta)$  in terms of  $\cos \theta$  and  $\sin \theta$ .
9. Consider the complex number  $z = 0.8 - 0.7i$ . Represent the powers  $z^2, z^3, \dots$  in the complex plane and explain their long-term behavior.
10. Prove the fundamental theorem of algebra for cubic polynomials with real coefficients.
11. Express the polynomial  $f(\lambda) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$  as a product of linear factors over  $\mathbb{C}$ .
12. Consider a polynomial  $f(\lambda)$  with real coefficients. Show that if a complex number  $\lambda_0$  is a root of  $f(\lambda)$ , then so is its complex conjugate,  $\bar{\lambda}_0$ .

For the matrices  $A$  listed in Exercises 13 through 17, find an invertible matrix  $S$  such that  $S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , where  $a$  and  $b$  are real numbers.

13.  $\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$

14.  $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

15.  $\begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$

16.  $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$

## Eigenvalues and Eigenvectors

17.  $\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$

18. Consider a real  $2 \times 2$  matrix  $A$  with two distinct real eigenvalues,  $\lambda_1$  and  $\lambda_2$ . Explain the formula  $\det A = \lambda_1\lambda_2$  geometrically, thinking of  $|\det A|$  as an expansion factor. Illustrate your explanation with a sketch. Is there a similar geometric interpretation for a  $3 \times 3$  matrix?

19. Consider a subspace  $V$  of  $\mathbb{R}^n$ , with  $\dim(V) = m < n$ .
- If the  $n \times n$  matrix  $A$  represents the orthogonal projection onto  $V$ , what is  $\text{tr } A$ ? What is  $\det A$ ?
  - If the  $n \times n$  matrix  $B$  represents the reflection about  $V$ , what is  $\text{tr } B$ ? What is  $\det B$ ?

*Find all complex eigenvalues of the matrices in Exercises 20 through 26 (including the real ones, of course). Do not use technology. Show all your work.*

20.  $\begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$

21.  $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$

22.  $\begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}$

23.  $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

24.  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$

25.  $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

26.  $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

27. Suppose a real  $3 \times 3$  matrix  $A$  has only two distinct eigenvalues. Suppose that  $\text{tr } A = 1$  and  $\det A = 3$ . Find the eigenvalues of  $A$  with their algebraic multiplicities.

28. Suppose a  $3 \times 3$  matrix  $A$  has the real eigenvalue 2 and two complex conjugate eigenvalues. Also, suppose that  $\det A = 50$  and  $\text{tr } A = 8$ . Find the complex eigenvalues.

29. Consider a matrix of the form

$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix},$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive real numbers. Suppose the matrix  $A$  has three distinct real eigenvalues. What can you say about the signs of the eigenvalues? (How many of them are positive, negative, zero?) Is the eigenvalue with the largest absolute value positive or negative?

30. a. If  $2i$  is an eigenvalue of a real  $2 \times 2$  matrix  $A$ , find  $A^2$ .  
 b. Give an example of a real  $2 \times 2$  matrix  $A$  such that all the entries of  $A$  are nonzero and  $2i$  is an eigen-

value of  $A$ . Compute  $A^2$  and check that your answer agrees with part a.

31. Consider the regular transition matrix

$$A = \frac{1}{15} \begin{bmatrix} 4 & 2 & 5 & 1 & 3 \\ 1 & 3 & 4 & 5 & 2 \\ 3 & 5 & 1 & 2 & 4 \\ 2 & 1 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{bmatrix}.$$

Note that the matrix  $15A$  contains each of the integers 1, 2, 3, 4, and 5 once in every row and in every column.

- Using technology, compute a high power of  $A$ , such as  $A^{20}$ . What do you observe? Make a conjecture for  $\lim_{t \rightarrow \infty} A^t$ . (In part e, you will prove this conjecture.)
- Use technology to find the complex eigenvalues of  $A$ . Is matrix  $A$  diagonalizable over  $\mathbb{C}$ ?
- Find the equilibrium distribution  $\vec{x}_{equ}$  for  $A$ , that is, the unique distribution vector in the eigenspace  $E_1$ .
- Without using Theorem 4.1 (which was proven only for matrices that are diagonalizable over  $\mathbb{R}$ ), show that  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{x}_{equ}$  for any distribution vector  $\vec{x}_0$ . Hint: Adapt the proof of Theorem 4.1 to the complex case.
- Find  $\lim_{t \rightarrow \infty} A^t$ , proving your conjecture from part a.

32. Consider the dynamical system

$$\vec{x}(t+1) = A\vec{x}(t), \quad \text{where } A = \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.4 \end{bmatrix},$$

perhaps modeling the way people surf a mini-Web, as in Exercise 4.1.

- Using technology, compute a high power of  $A$ , such as  $A^{20}$ . What do you observe? Make a conjecture for  $\lim_{t \rightarrow \infty} A^t$ . (In part e, you will prove this conjecture.)
- Use technology to find the complex eigenvalues of  $A$ . Is matrix  $A$  diagonalizable over  $\mathbb{C}$ ?
- Find the equilibrium distribution  $\vec{x}_{equ}$  for  $A$ , that is, the unique distribution vector in the eigenspace  $E_1$ .
- Without using Theorem 4.1 (which was proven only for matrices that are diagonalizable over  $\mathbb{R}$ ), show that  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{x}_{equ}$  for any distribution vector  $\vec{x}_0$ . Hint: Adapt the proof of Theorem 4.1 to the complex case.
- Find  $\lim_{t \rightarrow \infty} A^t$ , proving your conjecture from part a.

33. *The power method for finding eigenvalues.* Using technology, generate a random  $5 \times 5$  matrix  $A$  with non-negative entries. (Depending on the technology you are using, the entries could be integers between zero and

## Eigenvalues and Eigenvectors

nine, or numbers between zero and one.) Using technology, compute  $B = A^{20}$  (or another high power of  $A$ ). We wish to compare the columns of  $B$ . This is hard to do by inspection, particularly because the entries of  $B$  may get rather large.

To get a better hold on  $B$ , form the diagonal  $5 \times 5$  matrix  $D$  whose  $i$ th diagonal element is  $b_{1i}$ , the  $i$ th entry of the first row of  $B$ . Compute  $C = BD^{-1}$ .

- a. How is  $C$  obtained from  $B$ ? Give your answer in terms of elementary row or column operations.
- b. Take a look at the columns of the matrix  $C$  you get. What do you observe? What does your answer tell you about the columns of  $B = A^{20}$ ?
- c. Explain the observations you made in part b. You may assume that  $A$  has five distinct (complex) eigenvalues and that the eigenvalue with maximal modulus is real and positive. (We cannot explain here why this will usually be the case.)
- d. Compute  $AC$ . What is the significance of the entries in the top row of this matrix in terms of the eigenvalues of  $A$ ? What is the significance of the columns of  $C$  (or  $B$ ) in terms of the eigenvectors of  $A$ ?

34. Exercise 33 illustrates how you can use the powers of a matrix to find its dominant eigenvalue (i.e., the eigenvalue with maximal modulus), at least when this eigenvalue is real. But what about the other eigenvalues?

- a. Consider an  $n \times n$  matrix  $A$  with  $n$  distinct complex eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , where  $\lambda_1$  is real. Suppose you have a good (real) approximation  $\lambda$  of  $\lambda_1$  (good in that  $|\lambda - \lambda_1| < |\lambda - \lambda_i|$ , for  $i = 2, \dots, n$ ). Consider the matrix  $A - \lambda I_n$ . What are its eigenvalues? Which has the smallest modulus? Now consider the matrix  $(A - \lambda I_n)^{-1}$ . What are its eigenvalues? Which has the largest modulus? What is the relationship between the eigenvectors of  $A$  and those of  $(A - \lambda I_n)^{-1}$ ? Consider higher and higher powers of  $(A - \lambda I_n)^{-1}$ . How does this help you to find an eigenvector of  $A$  with eigenvalue  $\lambda_1$ , and  $\lambda_1$  itself? Use the results of Exercise 33.

- b. As an example of part a, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

We wish to find the eigenvectors and eigenvalues of  $A$  without using the corresponding commands on the computer (which is, after all, a “black box”). First, we find approximations for the eigenvalues by graphing the characteristic polynomial (use technology). Approximate the three real eigenvalues of  $A$  to the nearest integer. One of the three eigenvalues of  $A$  is negative. Find a good approximation for this eigenvalue and a corresponding eigenvector by using the procedure outlined in part a. You

are not asked to do the same for the two other eigenvalues.

35. Demonstrate the formula

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

where the  $\lambda_i$  are the complex eigenvalues of the matrix  $A$ , counted with their algebraic multiplicities. Hint: Consider the coefficient of  $\lambda^{n-1}$  in  $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$ , and compare the result with Theorem 2.5.

36. In 1990, the population of the African country Benin was about 4.6 million people. Its composition by age was as follows:

Age Bracket	0–15	15–30	30–45	45–60	60–75	75–90
Percent of Population	46.6	25.7	14.7	8.4	3.8	0.8

We represent these data in a state vector whose components are the populations in the various age brackets, in millions:

$$\vec{x}(0) = 4.6 \begin{bmatrix} 0.466 \\ 0.257 \\ 0.147 \\ 0.084 \\ 0.038 \\ 0.008 \end{bmatrix} \approx \begin{bmatrix} 2.14 \\ 1.18 \\ 0.68 \\ 0.39 \\ 0.17 \\ 0.04 \end{bmatrix}.$$

We measure time in increments of 15 years, with  $t = 0$  in 1990. For example,  $\vec{x}(3)$  gives the age composition in the year 2035 ( $1990 + 3 \cdot 15$ ). If current age-dependent birth and death rates are extrapolated, we have the following model:

$$\vec{x}(t+1) = \begin{bmatrix} 1.1 & 1.6 & 0.6 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.53 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.29 & 0 \end{bmatrix} \vec{x}(t) \\ = A\vec{x}(t).$$

- a. Explain the significance of all the entries in the matrix  $A$  in terms of population dynamics.
- b. Find the eigenvalue of  $A$  with the largest modulus and an associated eigenvector (use technology). What is the significance of these quantities in terms of population dynamics? (For a summary on matrix techniques used in the study of age-structured populations, see Dmitrii O. Logofet, *Matrices and Graphs: Stability Problems in Mathematical Ecology*, Chapters 2 and 3, CRC Press, 1993.)

## Eigenvalues and Eigenvectors

- 37.** Consider the set  $\mathbb{H}$  of all complex  $2 \times 2$  matrices of the form

$$A = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix},$$

where  $w$  and  $z$  are arbitrary complex numbers.

- a. Show that  $\mathbb{H}$  is closed under addition and multiplication. (That is, show that the sum and the product of two matrices in  $\mathbb{H}$  are again in  $\mathbb{H}$ .)
- b. Which matrices in  $\mathbb{H}$  are invertible?
- c. If a matrix in  $\mathbb{H}$  is invertible, is the inverse in  $\mathbb{H}$  as well?
- d. Find two matrices  $A$  and  $B$  in  $\mathbb{H}$  such that  $AB \neq BA$ .

$\mathbb{H}$  is an example of a *skew field*: It satisfies all axioms for a field, except for the commutativity of multiplication. [The skew field  $\mathbb{H}$  was introduced by the Irish mathematician Sir William Hamilton (1805–1865); its elements are called the *quaternions*.]

- 38.** Consider the matrix

$$C_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- a. Find the powers  $C_4^2, C_4^3, C_4^4, \dots$
- b. Find all complex eigenvalues of  $C_4$ , and construct a complex eigenbasis.
- c. A  $4 \times 4$  matrix  $M$  is called *circulant* if it is of the form

$$M = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}.$$

Circulant matrices play an important role in statistics. Show that any circulant  $4 \times 4$  matrix  $M$  can be expressed as a linear combination of  $I_4, C_4, C_4^2, C_4^3$ . Use this representation to find an eigenbasis for  $M$ . What are the eigenvalues (in terms of  $a, b, c, d$ )?

- 39.** Consider the  $n \times n$  matrix  $C_n$  which has ones directly below the main diagonal and in the right upper corner, and zeros everywhere else. See Exercise 38 for a discussion of  $C_4$ .
- a. Describe the powers of  $C_n$ .
  - b. Find all complex eigenvalues of  $C_n$ , and construct a complex eigenbasis.
  - c. Generalize part c of Exercise 38.
- 40.** Consider a cubic equation

$$x^3 + px = q,$$

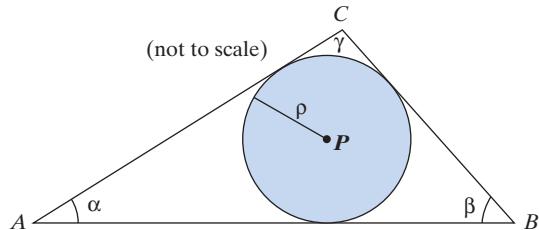
where  $(p/3)^3 + (q/2)^2$  is negative. Show that this equation has three real solutions; write the solutions in the form  $x_j = A \cos(\theta_j)$  for  $j = 1, 2, 3$ , expressing  $A$  and  $\theta_j$  in terms of  $p$  and  $q$ . How many of the solutions are in the interval  $(\sqrt{-p/3}, 2\sqrt{-p/3})$ ? Can there be solutions larger than  $2\sqrt{-p/3}$ ? Hint: Cardano's formula derived in Exercise 2.50 is useful.

- 41.** In his high school final examination (Aarau, Switzerland, 1896), young Albert Einstein (1879–1955) was given the following problem: In a triangle  $ABC$ , let  $P$  be the center of the inscribed circle. We are told that  $\overline{AP} = 1$ ,  $\overline{BP} = \frac{1}{2}$ , and  $\overline{CP} = \frac{1}{3}$ . Find the radius  $\rho$  of the inscribed circle. Einstein worked through this problem as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \rho$$

$$\sin\left(\frac{\beta}{2}\right) = 2\rho$$

$$\sin\left(\frac{\gamma}{2}\right) = 3\rho.$$



For every triangle the following equation holds:

$$\begin{aligned} \sin^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) + \sin^2\left(\frac{\gamma}{2}\right) \\ + 2 \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) = 1. \end{aligned}$$

In our case

$$14\rho^2 + 12\rho^3 - 1 = 0.$$

Now let

$$\rho = \frac{1}{x}.$$

At this point we interrupt Einstein's work and ask you to finish the job. Hint: Exercise 40 is helpful. Find the exact solution (in terms of trigonometric and inverse trigonometric functions), and give a numerical approximation as well. (By the way, Einstein, who was allowed to use a logarithm table, solved the problem correctly.) Source: *The Collected Papers of Albert Einstein*, Vol. 1, Princeton University Press, 1987.

- 42.** Consider a complex  $n \times m$  matrix  $A$ . The conjugate  $\bar{A}$  is defined by taking the conjugate of each entry of  $A$ . For example, if

## Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 2+3i & 5 \\ 2i & 9 \end{bmatrix}, \quad \text{then} \quad \overline{A} = \begin{bmatrix} 2-3i & 5 \\ -2i & 9 \end{bmatrix}.$$

- a. Show that if  $A$  and  $B$  are complex  $n \times p$  and  $p \times m$  matrices, respectively, then
- $$\overline{AB} = \overline{A}\overline{B}.$$
- b. Let  $A$  be a real  $n \times n$  matrix and  $\vec{v} + i\vec{w}$  an eigenvector of  $A$  with eigenvalue  $p + iq$ . Show that the vector  $\vec{v} - i\vec{w}$  is an eigenvector of  $A$  with eigenvalue  $p - iq$ .
43. Consider two real  $n \times n$  matrices  $A$  and  $B$  that are “similar over  $\mathbb{C}$ ”: That is, there is a complex invertible  $n \times n$  matrix  $S$  such that  $B = S^{-1}AS$ . Show that  $A$  and  $B$  are in fact “similar over  $\mathbb{R}$ ”: That is, there is a real  $R$  such that  $B = R^{-1}AR$ . Hint: Write  $S = S_1 + iS_2$ , where  $S_1$  and  $S_2$  are real. Consider the function  $f(z) = \det(S_1 + zS_2)$ , where  $z$  is a complex variable. Show that  $f(z)$  is a nonzero polynomial. Conclude that there is a real number  $x$  such that  $f(x) \neq 0$ . Show that  $R = S_1 + xS_2$  does the job.
44. Show that every complex  $2 \times 2$  matrix is similar to an upper triangular  $2 \times 2$  matrix. Can you generalize this result to square matrices of larger size? Hint: Argue by induction.

For which values of the real constant  $a$  are the matrices in Exercises 45 through 50 diagonalizable over  $\mathbb{C}$ ?

45.  $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$       46.  $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$       47.  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix}$

48.  $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$       49.  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1-a & a \end{bmatrix}$

50.  $\begin{bmatrix} -a & a & -a \\ -a-1 & a+1 & -a-1 \\ 0 & 0 & 0 \end{bmatrix}$

For Exercises 51 through 55, state whether the given set is a field (with the customary addition and multiplication).

51. The rational numbers  $\mathbb{Q}$   
 52. The integers  $\mathbb{Z}$   
 53. This Exercise is intentionally omitted for this edition.  
 54. The rotation-scaling matrices of the form  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ , where  $p$  and  $q$  are real numbers  
 55. This Exercise is intentionally omitted for this edition.

## 6 Stability

In applications, the long-term behavior is often the most important qualitative feature of a dynamical system. We are frequently faced with the following situation: The state  $\vec{0}$  represents an equilibrium of the system (in physics, ecology, or economics, for example). If the system is disturbed (moved into another state, away from the equilibrium  $\vec{0}$ ) and then left to its own devices, will it always return to the equilibrium state  $\vec{0}$ ?

**EXAMPLE 1** Consider a dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ , where  $A$  is an  $n \times n$  matrix. Suppose an initial state vector  $\vec{x}_0$  is given. We are told that  $A$  has  $n$  distinct complex eigenvalues,  $\lambda_1, \dots, \lambda_n$ , and that  $|\lambda_i| < 1$  for  $i = 1, \dots, n$ . What can you say about the long-term behavior of the system, that is, about  $\lim_{t \rightarrow \infty} \vec{x}(t)$ ?

### Solution

For each complex eigenvalue  $\lambda_i$ , we can choose a complex eigenvector  $\vec{v}_i$ . Then the  $\vec{v}_i$  form a complex eigenbasis for  $A$  (by Theorem 3.4). We can write  $\vec{x}_0$  as a complex linear combination of the  $\vec{v}_i$ :

$$\vec{x}_0 = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$

Then

$$\vec{x}(t) = A^t\vec{x}_0 = c_1\lambda_1^t\vec{v}_1 + \cdots + c_n\lambda_n^t\vec{v}_n.$$

By Example 5 of Section 5,

$$\lim_{t \rightarrow \infty} \lambda_i^t = 0, \quad \text{since} \quad |\lambda_i| < 1.$$

Therefore,

$$\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{0}. \quad \blacksquare$$

For the discussion of the long-term behavior of a dynamical system, the following definition is useful:

### Definition 6.1

#### Stable equilibrium

Consider a dynamical system

$$\vec{x}(t+1) = A\vec{x}(t).$$

We say that  $\vec{0}$  is an (asymptotically) *stable equilibrium* for this system if

$$\lim_{t \rightarrow \infty} \vec{x}(t) = \lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \vec{0}$$

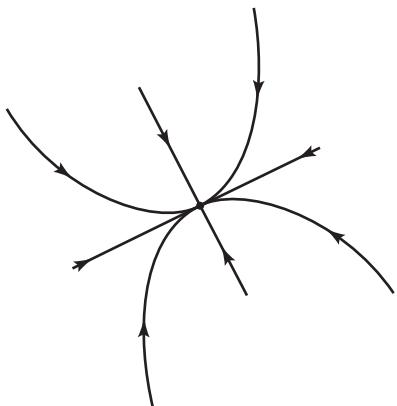
for all vectors  $\vec{x}_0$  in  $\mathbb{R}^n$ .<sup>16</sup>

Note that the zero state is stable if (and only if)

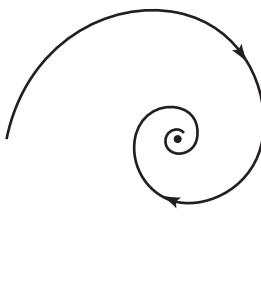
$$\lim_{t \rightarrow \infty} A^t = 0$$

(meaning that all entries of  $A^t$  approach zero). See Exercise 36.

Consider the examples shown in Figure 1.



**Figure 1(a)** Asymptotically stable.



**Figure 1(b)** Not asymptotically stable.

Generalizing Example 1, we have the following result:

### Theorem 6.2

#### Stability and eigenvalues

Consider a dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ . The zero state is asymptotically stable if (and only if) the modulus of all the complex eigenvalues of  $A$  is less than 1.

Example 1 illustrates this fact only when  $A$  is diagonalizable (i.e., when there is a complex eigenbasis for  $A$ ); recall that this is the case for most matrices  $A$ .

---

<sup>16</sup>The term *stable* means “asymptotically stable.” Several other notions of stability are used in applied mathematics.

## Eigenvalues and Eigenvectors

For an illustration of Theorem 6.2, see Figure 10 of Section 1, where we sketched the phase portraits of  $2 \times 2$  matrices with two distinct positive eigenvalues.

We will now turn our attention to the phase portraits of  $2 \times 2$  matrices with complex eigenvalues  $p \pm iq$  (where  $q \neq 0$ ).

**EXAMPLE 2** Consider the dynamical system

$$\vec{x}(t+1) = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \vec{x}(t),$$

where  $p$  and  $q$  are real, and  $q$  is nonzero. Examine the stability of this system. Sketch phase portraits. Discuss your results in terms of Theorem 6.2.

### Solution

We can write

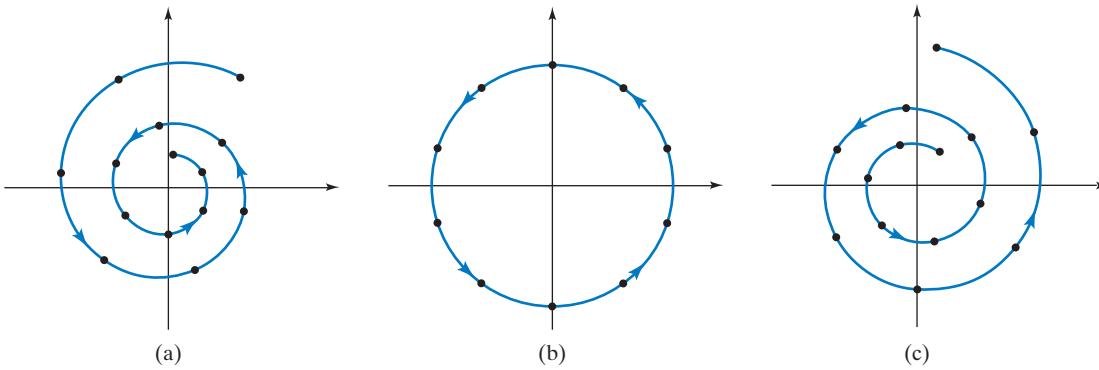
$$\begin{bmatrix} p & -q \\ q & p \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

representing the transformation as a rotation through an angle  $\theta$  combined with a scaling by  $r = \sqrt{p^2 + q^2}$ . Then

$$\vec{x}(t) = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}^t \vec{x}_0 = r^t \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \vec{x}_0,$$

representing a rotation through an angle  $\theta t$  combined with a scaling by  $r^t$ .

Figure 2 illustrates that the zero state is stable if  $r = \sqrt{p^2 + q^2} < 1$ .



**Figure 2** (a)  $r < 1$ : trajectories spiral inward. (b)  $r = 1$ : trajectories are circles. (c)  $r > 1$ : trajectories spiral outward.

Alternatively, we can use Theorem 6.2 to examine the stability of the system.

From Example 6 of Section 5, we know that the eigenvalues of  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$  are  $\lambda_{1,2} = p \pm iq$ , with  $|\lambda_1| = |\lambda_2| = \sqrt{p^2 + q^2}$ . By Theorem 6.2, the zero state is stable if  $\sqrt{p^2 + q^2} < 1$ . ■

Let us generalize Example 2. If  $A$  is any  $2 \times 2$  matrix with eigenvalues  $\lambda_{1,2} = p \pm iq$ , what does the phase portrait of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$  look like? Let  $\vec{v} + i\vec{w}$  be an eigenvector of  $A$  with eigenvalue  $p + iq$ . From Theorem 5.3, we know that  $A$  is similar to the rotation-scaling matrix  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ , with

## Eigenvalues and Eigenvectors

$$S^{-1}AS = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \quad \text{or} \quad A = S \begin{bmatrix} p & -q \\ q & p \end{bmatrix} S^{-1}, \quad \text{where} \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$

Using the terminology introduced in Example 2, we find that

$$\vec{x}(t) = A^t \vec{x}_0 = S \begin{bmatrix} p & -q \\ q & p \end{bmatrix}^t S^{-1} \vec{x}_0 = r^t S \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} S^{-1} \vec{x}_0.$$

### Theorem 6.3

#### Dynamical systems with complex eigenvalues

Consider the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ , where  $A$  is a real  $2 \times 2$  matrix with eigenvalues

$$\lambda_{1,2} = p \pm iq = r(\cos(\theta) \pm i \sin(\theta)), \quad \text{where} \quad q \neq 0.$$

Let  $\vec{v} + i\vec{w}$  be an eigenvector of  $A$  with eigenvalue  $p + iq$ .

Then

$$\vec{x}(t) = r^t S \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} S^{-1} \vec{x}_0, \quad \text{where} \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$

Note that  $S^{-1}\vec{x}_0$  is the coordinate vector of  $\vec{x}_0$  with respect to basis  $\vec{w}, \vec{v}$ .

### EXAMPLE 3

Consider the dynamical system

$$\vec{x}(t+1) = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} \vec{x}(t) \quad \text{with initial state} \quad \vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use Theorem 6.3 to find a closed formula for  $\vec{x}(t)$ , and sketch the trajectory.

#### Solution

In Example 8 of Section 5, we found the eigenvalues

$$\lambda_{1,2} = 1 \pm i.$$

The polar coordinates of eigenvalue  $1 + i$  are  $r = \sqrt{2}$  and  $\theta = \frac{\pi}{4}$ . Furthermore, we found that

$$S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix}.$$

Since

$$S^{-1} \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Theorem 6.3 gives

$$\begin{aligned} \vec{x}(t) &= (\sqrt{2})^t \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (\sqrt{2})^t \begin{bmatrix} -5 \sin\left(\frac{\pi}{4}t\right) \\ \cos\left(\frac{\pi}{4}t\right) - 2 \sin\left(\frac{\pi}{4}t\right) \end{bmatrix}. \end{aligned}$$

We leave it to the reader to work out the details of this computation.

Next, let's think about the trajectory. We will develop the trajectory step by step:

## Eigenvalues and Eigenvectors

- The points

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{for } t = 0, 1, 2, \dots)$$

are located on the unit circle, as shown in Figure 3a. Note that at  $t = 8$  the system returns to its initial position,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ; the period of this system is 8.

- An invertible linear transformation maps the unit circle into an ellipse. Thus, the points

$$\begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

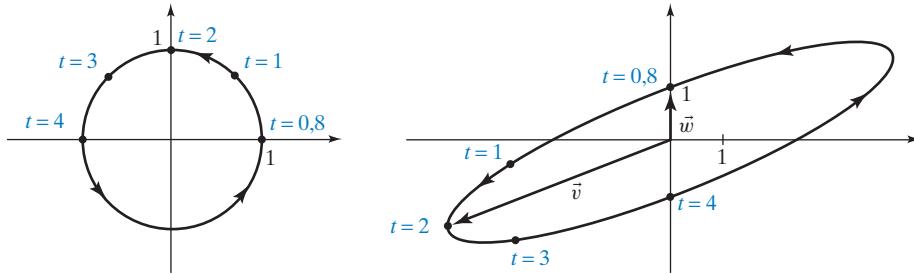
are located on an ellipse, as shown in Figure 3b. The two column vectors of

$$S = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

are shown in that figure as well. Again, the period of this system is 8.

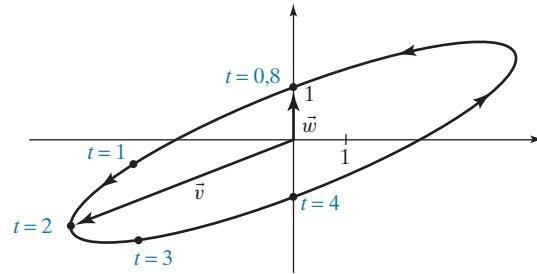
- The exponential growth factor  $(\sqrt{2})^t$  will produce longer and longer vectors

$$\vec{x}(t) = (\sqrt{2})^t \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



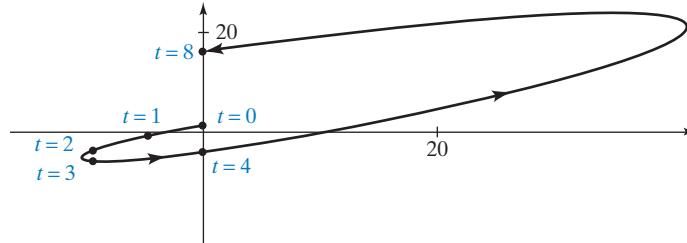
$$\begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a)



$$\begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b)



$$\vec{x}(t) = (\sqrt{2})^t \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c)

**Figure 3**

Thus, the trajectory *spirals outward*, as shown in Figure 3c. (We are using different scales in Figures 3a, b, and c.) Note that  $\vec{x}(8) = (\sqrt{2})^8 \vec{x}(0) = 16\vec{x}(0)$ . ■

We can generalize our findings in Example 3.

### Theorem 6.4

#### Phase portrait of a system with complex eigenvalues

Consider a dynamical system

$$\vec{x}(t+1) = A\vec{x}(t),$$

where  $A$  is a real  $2 \times 2$  matrix with eigenvalues  $\lambda_{1,2} = p \pm iq$  (where  $q \neq 0$ ). Let

$$r = |\lambda_1| = |\lambda_2| = \sqrt{p^2 + q^2}.$$

If  $r = 1$ , then the points  $\vec{x}(t)$  are located on an ellipse; if  $r$  exceeds 1, then the trajectory spirals outward; and if  $r$  is less than 1, then the trajectory spirals inward, approaching the origin.

Theorem 6.4 provides another illustration of Theorem 6.2: The zero state is stable if (and only if)  $r = |\lambda_1| = |\lambda_2| < 1$ .

If you have to sketch a trajectory of a system with complex eigenvalues without the aid of technology, it helps to compute and plot the first few points  $\vec{x}(0), \vec{x}(1), \vec{x}(2), \dots$ , until you see a trend.

## EXERCISES 6

**GOAL** Use eigenvalues to determine the stability of a dynamical system. Analyze the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ , where  $A$  is a real  $2 \times 2$  matrix with eigenvalues  $p \pm iq$ .

For the matrices  $A$  in Exercises 1 through 10, determine whether the zero state is a stable equilibrium of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ .

$$1. A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}$$

$$2. A = \begin{bmatrix} -1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$$

$$3. A = \begin{bmatrix} 0.8 & 0.7 \\ -0.7 & 0.8 \end{bmatrix}$$

$$4. A = \begin{bmatrix} -0.9 & -0.4 \\ 0.4 & -0.9 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 0.5 & 0.6 \\ -0.3 & 1.4 \end{bmatrix}$$

$$6. A = \begin{bmatrix} -1 & 3 \\ -1.2 & 2.6 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 2.4 & -2.5 \\ 1 & -0.6 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$$

$$9. A = \begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 0.7 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$$

Consider the matrices  $A$  in Exercises 11 through 16. For which real numbers  $k$  is the zero state a stable equilibrium of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ ?

$$11. A = \begin{bmatrix} k & 0 \\ 0 & 0.9 \end{bmatrix} \quad 12. A = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$$

$$13. A = \begin{bmatrix} 0.7 & k \\ 0 & -0.9 \end{bmatrix} \quad 14. A = \begin{bmatrix} k & k \\ k & k \end{bmatrix}$$

$$15. A = \begin{bmatrix} 1 & k \\ 0.01 & 1 \end{bmatrix} \quad 16. A = \begin{bmatrix} 0.1 & k \\ 0.3 & 0.3 \end{bmatrix}$$

For the matrices  $A$  in Exercises 17 through 24, find real closed formulas for the trajectory  $\vec{x}(t+1) = A\vec{x}(t)$ , where  $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Draw a rough sketch.

$$17. A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \quad 18. A = \begin{bmatrix} -0.8 & 0.6 \\ -0.8 & -0.8 \end{bmatrix}$$

$$19. A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad 20. A = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$$

$$21. A = \begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix} \quad 22. A = \begin{bmatrix} 7 & -15 \\ 6 & -11 \end{bmatrix}$$

$$23. A = \begin{bmatrix} -0.5 & 1.5 \\ -0.6 & 1.3 \end{bmatrix} \quad 24. A = \begin{bmatrix} 1 & -3 \\ 1.2 & -2.6 \end{bmatrix}$$

## Eigenvalues and Eigenvectors

Consider an invertible  $n \times n$  matrix  $A$  such that the zero state is a stable equilibrium of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ . What can you say about the stability of the systems listed in Exercises 25 through 30?

25.  $\vec{x}(t+1) = A^{-1}\vec{x}(t)$     26.  $\vec{x}(t+1) = A^T\vec{x}(t)$   
 27.  $\vec{x}(t+1) = -A\vec{x}(t)$     28.  $\vec{x}(t+1) = (A - 2I_n)\vec{x}(t)$   
 29.  $\vec{x}(t+1) = (A + I_n)\vec{x}(t)$     30.  $\vec{x}(t+1) = A^2\vec{x}(t)$

31. Let  $A$  be a real  $2 \times 2$  matrix. Show that the zero state is a stable equilibrium of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$  if (and only if)

$$|\operatorname{tr} A| - 1 < \det A < 1.$$

32. Let's revisit the introductory example of Section 5: The glucose regulatory system of a certain patient can be modeled by the equations

$$\begin{aligned} g(t+1) &= 0.9g(t) - 0.4h(t) \\ h(t+1) &= 0.1g(t) + 0.9h(t). \end{aligned}$$

Find closed formulas for  $g(t)$  and  $h(t)$ , and draw the trajectory. Does your trajectory look like the one on page 361?

33. Consider a real  $2 \times 2$  matrix  $A$  with eigenvalues  $p \pm iq$  and corresponding eigenvectors  $\vec{v} \pm i\vec{w}$ . Show that if a real vector  $\vec{x}_0$  is written as  $\vec{x}_0 = c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})$ , then  $c_2 = \bar{c}_1$ .
34. Consider a dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ , where  $A$  is a real  $n \times n$  matrix.
- a. If  $|\det A| \geq 1$ , what can you say about the stability of the zero state?
  - b. If  $|\det A| < 1$ , what can you say about the stability of the zero state?

35. a. Consider a real  $n \times n$  matrix with  $n$  distinct real eigenvalues  $\lambda_1, \dots, \lambda_n$ , where  $|\lambda_i| \leq 1$  for all  $i = 1, \dots, n$ . Let  $\vec{x}(t)$  be a trajectory of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$ . Show that this trajectory is *bounded*; that is, there is a positive number  $M$  such that  $\|\vec{x}(t)\| \leq M$  for all positive integers  $t$ .
- b. Are all trajectories of the dynamical system

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}(t)$$

bounded? Explain.

36. Show that the zero state is a stable equilibrium of the dynamical system  $\vec{x}(t+1) = A\vec{x}(t)$  if (and only if)

$$\lim_{t \rightarrow \infty} A^t = 0$$

(meaning that all entries of  $A^t$  approach zero).

37. Consider the national income of a country, which consists of consumption, investment, and government expenditures. Here we assume the government

expenditure to be constant, at  $G_0$ , while the national income  $Y(t)$ , consumption  $C(t)$ , and investment  $I(t)$  change over time. According to a simple model, we have

$$\left| \begin{array}{l} Y(t) = C(t) + I(t) + G_0 \\ C(t+1) = \gamma Y(t) \\ I(t+1) = \alpha(C(t+1) - C(t)) \end{array} \right| \quad \begin{array}{l} (0 < \gamma < 1), \\ (\alpha > 0) \end{array}$$

where  $\gamma$  is the marginal propensity to consume and  $\alpha$  is the acceleration coefficient. (See Paul E. Samuelson, "Interactions between the Multiplier Analysis and the Principle of Acceleration," *Review of Economic Statistics*, May 1939, pp. 75–78.)

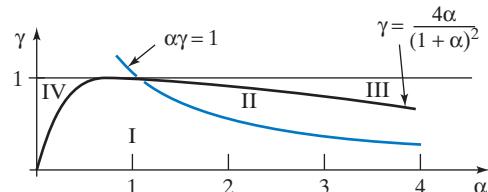
- a. Find the equilibrium solution of these equations, when  $Y(t+1) = Y(t)$ ,  $C(t+1) = C(t)$ , and  $I(t+1) = I(t)$ .
- b. Let  $y(t)$ ,  $c(t)$ , and  $i(t)$  be the deviations of  $Y(t)$ ,  $C(t)$ , and  $I(t)$ , respectively, from the equilibrium state you found in part (a). These quantities are related by the equations

$$\left| \begin{array}{l} y(t) = c(t) + i(t) \\ c(t+1) = \gamma y(t) \\ i(t+1) = \alpha(c(t+1) - c(t)) \end{array} \right|.$$

(Verify this!) By substituting  $y(t)$  into the second equation, set up equations of the form

$$\left| \begin{array}{l} c(t+1) = pc(t) + qi(t) \\ i(t+1) = rc(t) + si(t) \end{array} \right|.$$

- c. When  $\alpha = 5$  and  $\gamma = 0.2$ , determine the stability of the zero state of this system.
- d. When  $\alpha = 1$  (and  $\gamma$  is arbitrary,  $0 < \gamma < 1$ ), determine the stability of the zero state.
- e. For each of the four sectors in the  $\alpha-\gamma$ -plane, determine the stability of the zero state.



Discuss the various cases, in practical terms.

38. Consider an affine transformation

$$T(\vec{x}) = A\vec{x} + \vec{b},$$

where  $A$  is an  $n \times n$  matrix and  $\vec{b}$  is a vector in  $\mathbb{R}^n$ . Compare this with Exercise 4.35. Suppose that 1 is not an eigenvalue of  $A$ .

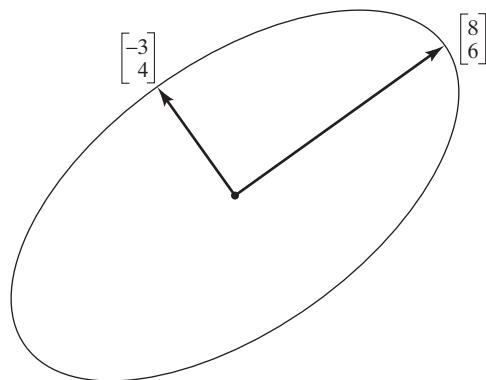
- a. Find the vector  $\vec{v}$  in  $\mathbb{R}^n$  such that  $T(\vec{v}) = \vec{v}$ ; this vector is called the *equilibrium state* of the dynamical system  $\vec{x}(t+1) = T(\vec{x}(t))$ .
- b. When is the equilibrium  $\vec{v}$  in part (a) stable (meaning that  $\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{v}$  for all trajectories)?

39. Consider the dynamical system

$$\begin{aligned}x_1(t+1) &= 0.1x_1(t) + 0.2x_2(t) + 1, \\x_2(t+1) &= 0.4x_1(t) + 0.3x_2(t) + 2.\end{aligned}$$

See Exercise 4.35. Find the equilibrium state of this system and determine its stability. See Exercise 38. Sketch a phase portrait.

40. This Exercise is intentionally omitted for this edition.
41. Find a  $2 \times 2$  matrix  $A$  without real eigenvalues and a vector  $\vec{x}_0$  in  $\mathbb{R}^2$  such that for all positive integers  $t$ , the point  $A^t \vec{x}_0$  is located on the ellipse in the accompanying sketch.



42. We quote from a text on computer graphics (M. Beeler et al., "HAKMEM," MIT Artificial Intelligence Report AIM-239, 1972):

Here is an elegant way to draw almost circles on a point-plotting display.

#### CIRCLE ALGORITHM:

```
NEW X = OLD X - K*OLD Y;
NEW Y = OLD Y + K*NEW X.
```

This makes a very round ellipse centered at the origin with its size determined by the initial point. The circle algorithm was invented by mistake when I tried to save a register in a display hack!

(In the preceding formula,  $k$  is a small number.) Here, a dynamical system is defined in "computer lingo." In our terminology, the formulas are

$$\begin{aligned}x(t+1) &= x(t) - ky(t), \\y(t+1) &= y(t) + kx(t+1).\end{aligned}$$

- a. Find the matrix of this transformation. [Note the entry  $x(t+1)$  in the second formula.]  
b. Explain why the trajectories are ellipses, as claimed.

## Exercises

### TRUE OR FALSE?

1. If 0 is an eigenvalue of a matrix  $A$ , then  $\det A = 0$ .
2. The eigenvalues of a  $2 \times 2$  matrix  $A$  are the solutions of the equation  $\lambda^2 - (\text{tr } A)\lambda + (\det A) = 0$ .
3. The eigenvalues of any triangular matrix are its diagonal entries.
4. The trace of any square matrix is the sum of its diagonal entries.
5. The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.
6. If an  $n \times n$  matrix  $A$  is diagonalizable (over  $\mathbb{R}$ ), then there must be a basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .
7. If the standard vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$  are eigenvectors of an  $n \times n$  matrix  $A$ , then  $A$  must be diagonal.
8. If  $\vec{v}$  is an eigenvector of  $A$ , then  $\vec{v}$  must be an eigenvector of  $A^3$  as well.
9. There exists a diagonalizable  $5 \times 5$  matrix with only two distinct eigenvalues (over  $\mathbb{C}$ ).

10. There exists a real  $5 \times 5$  matrix without any real eigenvalues.
11. If matrices  $A$  and  $B$  have the same eigenvalues (over  $\mathbb{C}$ ), with the same algebraic multiplicities, then matrices  $A$  and  $B$  must have the same trace.
12. If a real matrix  $A$  has only the eigenvalues 1 and  $-1$ , then  $A$  must be orthogonal.
13. Any rotation-scaling matrix in  $\mathbb{R}^{2 \times 2}$  is diagonalizable over  $\mathbb{C}$ .
14. If  $A$  is a noninvertible  $n \times n$  matrix, then the geometric multiplicity of eigenvalue 0 is  $n - \text{rank}(A)$ .
15. If matrix  $A$  is diagonalizable, then its transpose  $A^T$  must be diagonalizable as well.
16. If  $A$  and  $B$  are two  $3 \times 3$  matrices such that  $\text{tr } A = \text{tr } B$  and  $\det A = \det B$ , then  $A$  and  $B$  must have the same eigenvalues.
17. If 1 is the only eigenvalue of an  $n \times n$  matrix  $A$ , then  $A$  must be  $I_n$ .

## Eigenvalues and Eigenvectors

18. If  $A$  and  $B$  are  $n \times n$  matrices, if  $\alpha$  is an eigenvalue of  $A$ , and if  $\beta$  is an eigenvalue of  $B$ , then  $\alpha\beta$  must be an eigenvalue of  $AB$ .
19. If 3 is an eigenvalue of an  $n \times n$  matrix  $A$ , then 9 must be an eigenvalue of  $A^2$ .
20. The matrix of any orthogonal projection onto a subspace  $V$  of  $\mathbb{R}^n$  is diagonalizable.
21. All diagonalizable matrices are invertible.
22. If vector  $\vec{v}$  is an eigenvector of both  $A$  and  $B$ , then  $\vec{v}$  must be an eigenvector of  $A + B$ .
23. If matrix  $A^2$  is diagonalizable, then matrix  $A$  must be diagonalizable as well.
24. The determinant of a matrix is the product of its eigenvalues (over  $\mathbb{C}$ ), counted with their algebraic multiplicities.
25. All lower triangular matrices are diagonalizable (over  $\mathbb{C}$ ).
26. If two  $n \times n$  matrices  $A$  and  $B$  are diagonalizable, then  $AB$  must be diagonalizable as well.
27. If an invertible matrix  $A$  is diagonalizable, then  $A^{-1}$  must be diagonalizable as well.
28. If  $\det(A) = \det(A^T)$ , then matrix  $A$  must be symmetric.
29. If matrix  $A = \begin{bmatrix} 7 & a & b \\ 0 & 7 & c \\ 0 & 0 & 7 \end{bmatrix}$  is diagonalizable, then  $a, b$ , and  $c$  must all be zero.
30. If two  $n \times n$  matrices  $A$  and  $B$  are diagonalizable, then  $A + B$  must be diagonalizable as well.
31. If  $\vec{u}, \vec{v}, \vec{w}$  are eigenvectors of a  $4 \times 4$  matrix  $A$ , with associated eigenvalues 3, 7, and 11, respectively, then vectors  $\vec{u}, \vec{v}, \vec{w}$  must be linearly independent.
32. If a  $4 \times 4$  matrix  $A$  is diagonalizable, then the matrix  $A + 4I_4$  must be diagonalizable as well.
33. If an  $n \times n$  matrix  $A$  is diagonalizable, then  $A$  must have  $n$  distinct eigenvalues.
34. If two  $3 \times 3$  matrices  $A$  and  $B$  both have the eigenvalues 1, 2, and 3, then  $A$  must be similar to  $B$ .
35. If  $\vec{v}$  is an eigenvector of  $A$ , then  $\vec{v}$  must be an eigenvector of  $A^T$  as well.
36. All invertible matrices are diagonalizable.
37. If  $\vec{v}$  and  $\vec{w}$  are linearly independent eigenvectors of matrix  $A$ , then  $\vec{v} + \vec{w}$  must be an eigenvector of  $A$  as well.
38. If a  $2 \times 2$  matrix  $R$  represents a reflection about a line  $L$ , then  $R$  must be diagonalizable.
39. If  $A$  is a  $2 \times 2$  matrix such that  $\text{tr } A = 1$  and  $\det A = -6$ , then  $A$  must be diagonalizable.
40. If a matrix is diagonalizable, then the algebraic multiplicity of each of its eigenvalues  $\lambda$  must equal the geometric multiplicity of  $\lambda$ .
41. All orthogonal matrices are diagonalizable (over  $\mathbb{R}$ ).
42. If  $A$  is an  $n \times n$  matrix and  $\lambda$  is an eigenvalue of the block matrix  $M = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ , then  $\lambda$  must be an eigenvalue of matrix  $A$ .
43. If two matrices  $A$  and  $B$  have the same characteristic polynomials, then they must be similar.
44. If  $A$  is a diagonalizable  $4 \times 4$  matrix with  $A^4 = 0$ , then  $A$  must be the zero matrix.
45. If an  $n \times n$  matrix  $A$  is diagonalizable (over  $\mathbb{R}$ ), then every vector  $\vec{v}$  in  $\mathbb{R}^n$  can be expressed as a sum of eigenvectors of  $A$ .
46. If vector  $\vec{v}$  is an eigenvector of both  $A$  and  $B$ , then  $\vec{v}$  is an eigenvector of  $AB$ .
47. Similar matrices have the same characteristic polynomials.
48. If a matrix  $A$  has  $k$  distinct eigenvalues, then  $\text{rank}(A) \geq k$ .
49. If the rank of a square matrix  $A$  is 1, then all the nonzero vectors in the image of  $A$  are eigenvectors of  $A$ .
50. If the rank of an  $n \times n$  matrix  $A$  is 1, then  $A$  must be diagonalizable.
51. If  $A$  is a  $4 \times 4$  matrix with  $A^4 = 0$ , then 0 is the only eigenvalue of  $A$ .
52. If two  $n \times n$  matrices  $A$  and  $B$  are both diagonalizable, then they must commute.
53. If  $\vec{v}$  is an eigenvector of  $A$ , then  $\vec{v}$  must be in the kernel of  $A$  or in the image of  $A$ .
54. All symmetric  $2 \times 2$  matrices are diagonalizable (over  $\mathbb{R}$ ).
55. If  $A$  is a  $2 \times 2$  matrix with eigenvalues 3 and 4 and if  $\vec{u}$  is a unit eigenvector of  $A$ , then the length of vector  $A\vec{u}$  cannot exceed 4.
56. If  $\vec{u}$  is a nonzero vector in  $\mathbb{R}^n$ , then  $\vec{u}$  must be an eigenvector of matrix  $\vec{u}\vec{u}^T$ .
57. If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  is an eigenbasis for both  $A$  and  $B$ , then matrices  $A$  and  $B$  must commute.
58. If  $\vec{v}$  is an eigenvector of a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $\vec{v}$  must be an eigenvector of its classical adjoint  $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  as well.

## Answers to Odd-Numbered Exercises

1 Answers to more theoretical questions are omitted.

1. Yes; the eigenvalue is  $\lambda^3$ .

3. Yes; the eigenvalue is  $\lambda + 2$ .

5. Yes

7.  $\ker(A - \lambda I_n) \neq \{\vec{0}\}$  because  $(A - \lambda I_n)\vec{v} = \vec{0}$ . The matrix  $A - \lambda I_n$  fails to be invertible.

9.  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

11.  $\begin{bmatrix} a & \frac{-2-2a}{3} \\ c & \frac{-3-2c}{3} \end{bmatrix}$

13. All vectors of the form  $\begin{bmatrix} 3t \\ 5t \end{bmatrix}$ , where  $t \neq 0$  (solve the linear system  $A\vec{x} = 4\vec{x}$ )

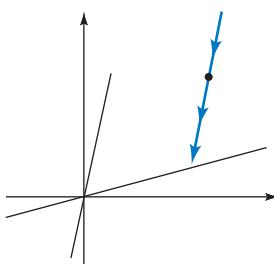
15. The nonzero vectors in  $L$  are the eigenvectors with eigenvalue 1, and the nonzero vectors in  $L^\perp$  have eigenvalue  $-1$ . Construct an eigenbasis by picking one of each. This transformation is diagonalizable.

17. There are no eigenvectors and eigenvalues (compare with Example 4). There is no eigenbasis, and the transformation fails to be diagonalizable.

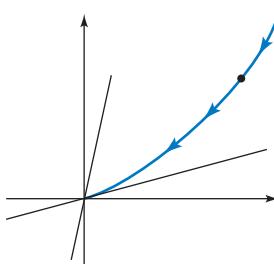
19. The nonzero vectors in  $L$  are the eigenvectors with eigenvalue 1, and the nonzero vectors in the plane  $L^\perp$  have eigenvalue 0. Construct an eigenbasis by picking one nonzero vector in  $L$  and two linearly independent vectors in  $L^\perp$ . (Compare with Example 3). This transformation is diagonalizable.

21. All nonzero vectors in  $\mathbb{R}^3$  are eigenvectors with eigenvalue 5. Any basis of  $\mathbb{R}^3$  is an eigenbasis, so that the transformation is diagonalizable.

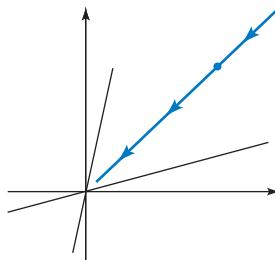
25.



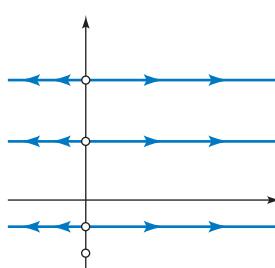
27.



29.



31.



33.  $\vec{x}(t) = 2^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ . We need a matrix  $A$  with eigenvectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , with associated eigenvalues 2 and 6, respectively. Let  $A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 6 \end{bmatrix}$  and solve for  $A$ . We find  $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$ .

35. If  $\lambda$  is an eigenvalue of  $S^{-1}AS$ , with corresponding eigenvector  $\vec{v}$ , then

$$S^{-1}AS\vec{v} = \lambda\vec{v}, \text{ so}$$

$$AS\vec{v} = S\lambda\vec{v} = \lambda S\vec{v},$$

and  $\lambda$  is an eigenvalue of  $A$  ( $S\vec{v}$  is an eigenvector). Likewise, if  $\vec{w}$  is an eigenvector of  $A$ , then  $S^{-1}\vec{w}$  is an eigenvector of  $S^{-1}AS$  with the same eigenvalue.

37. a.  $A$  represents a reflection about a line followed by a scaling by a factor of  $\sqrt{3^2 + 4^2} = 5$ . Therefore, the eigenvalues are 5 and  $-5$ .

b. Solving the linear systems  $A\vec{x} = 5\vec{x}$  and  $A\vec{x} = -5\vec{x}$  we find the eigenbasis  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ .

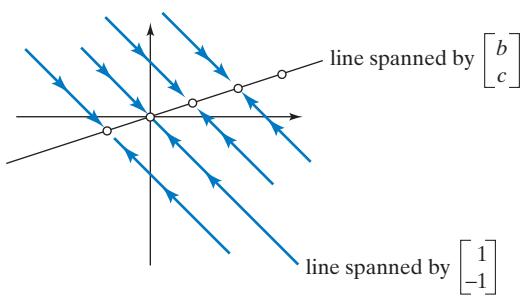
c.  $S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

39.  $V$  consists of all lower triangular  $2 \times 2$  matrices, and  $\dim V = 3$ .

41. A basis is  $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ , and  $\dim V = 2$ .

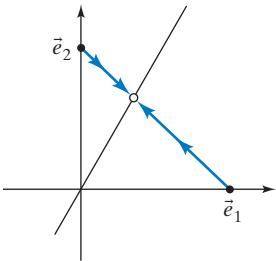
## Eigenvalues and Eigenvectors

43.  $V$  consists of all diagonal matrices, so that  $\dim V = n$ .
45.  $\dim V = 3$
47. Let  $A\vec{v} = \lambda\vec{v}$ . If  $\lambda = 0$ , then  $\vec{v}$  is in the kernel of  $A$ . If  $\lambda \neq 0$ , then  $\vec{v} = A\left(\frac{1}{\lambda}\vec{v}\right)$ , so that  $\vec{v}$  is in the image of  $A$ .
49.  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
51. Eigenbasis  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$
53. To form an eigenbasis for this matrix of rank 1, concatenate a basis of the kernel with a nonzero vector in the image, for example,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . The matrices  $S = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$  diagonalize  $A$ .
55.  $S = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
57. Matrix  $A$  represents a reflection, with eigenvalues 1 and  $-1$ . Solve the equation  $A\vec{x} = \vec{x}$  to find an eigenvector with eigenvalue 1. For example,  $S = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .
59. Pick two linearly independent vectors  $\vec{v}_1, \vec{v}_2$  on the plane and one perpendicular vector  $\vec{v}_3$ , for example,  $S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .
61. Pick two linearly independent solutions  $\vec{v}_1, \vec{v}_2$  of the equation  $A\vec{x} = \vec{x}$ , let  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ , and make  $S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .
63. Matrix  $A$  represents the orthogonal projection onto  $\text{span} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . We can let  $S = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .
65. The subspaces spanned by an eigenvector
67.  $c(t) = 300(1.1)^t - 200(0.9)^t$   
 $r(t) = 900(1.1)^t - 100(0.9)^t$
69. a.  $c(t) = 100(1.5)^t, r(t) = 200(1.5)^t$   
b.  $c(t) = 100(0.75)^t, r(t) = 100(0.75)^t$   
c.  $c(t) = 300(0.75)^t + 200(1.5)^t,$   
 $r(t) = 300(0.75)^t + 400(1.5)^t$
71.  $\begin{bmatrix} a(t+1) \\ b(t+1) \\ c(t+1) \end{bmatrix} = \underbrace{\frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_A \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix}$ . The three given vectors are eigenvectors of  $A$ , with eigenvalues  $1, -\frac{1}{2}, -\frac{1}{2}$ , respectively.
- a.  $a(t) = 3 + 3 \left(-\frac{1}{2}\right)^t, b(t) = 3 - 2 \left(-\frac{1}{2}\right)^t, c(t) = 3 - \left(-\frac{1}{2}\right)^t$
- b. Benjamin will have the most.
- 2 Answers to more theoretical questions are omitted.
1. 1, 3  
5. None  
9. 1, 2, 2  
13. 1  
15. Eigenvalues  $\lambda_{1,2} = 1 \pm \sqrt{k}$ . Two distinct real eigenvalues if  $k$  is positive; none, if  $k$  is negative.  
17.  $A$  represents a reflection followed by a scaling, with a scaling factor of  $\sqrt{a^2 + b^2}$ . The eigenvalues are  $\pm \sqrt{a^2 + b^2}$ .  
19. True [the discriminant  $(\text{tr } A)^2 - 4 \det A$  is positive]  
21. Write  $f_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$  to show that the coefficient of  $(-\lambda)^{n-1}$  is  $\lambda_1 + \cdots + \lambda_n$ . But that coefficient is  $\text{tr } A$ , by Theorem 2.5.  
23.  $A$  and  $B$  have the same characteristic polynomial and the same eigenvalues, with the same algebraic multiplicities.  
25.  $A \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$  and  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (a-b) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . Note that  $|a-b| < 1$ . Phase portrait when  $a > b$ :



## Eigenvalues and Eigenvectors

27. a.  $\vec{x}(t) = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3} \left(\frac{1}{4}\right)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  for  $\vec{x}_0 = \vec{e}_1$   
 b.  $\vec{x}(t) = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3} \left(\frac{1}{4}\right)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  for  $\vec{x}_0 = \vec{e}_2$



- b.  $A^t = [A^t \vec{e}_1 \quad A^t \vec{e}_2]$  approaches  $\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ , by part a.  
 c.  $A^t \rightarrow \frac{1}{b+c} \begin{bmatrix} b & b \\ c & c \end{bmatrix}$

29.  $A\vec{e} = \vec{e}$ , so that  $\vec{e}$  is an eigenvector with associated eigenvalue 1.

31.  $A$  and  $A^T$  have the same eigenvalues, by Exercise 22. Since the row sums of  $A^T$  are 1, we can use the results of Exercises 29 and 30: 1 is an eigenvalue of  $A$ ; if  $\lambda$  is an eigenvalue of  $A$ , then  $-1 < \lambda \leq 1$ .  $\vec{e}$  need not be an eigenvector of  $A$ ; consider  $A = \begin{bmatrix} 0.9 & 0.9 \\ 0.1 & 0.1 \end{bmatrix}$ .

33. a.  $f_A(\lambda) = -\lambda^3 + c\lambda^2 + b\lambda + a$

$$\text{b. } M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & -5 & 17 \end{bmatrix}$$

$$\text{35. } A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

37. We can write  $f_A(\lambda) = (\lambda - \lambda_0)^2 g(\lambda)$ . By the product rule,  $f'_A(\lambda) = 2(\lambda - \lambda_0)g(\lambda) + (\lambda - \lambda_0)^2 g'(\lambda)$ , so that  $f'_A(\lambda_0) = 0$ .

39. It's a straightforward computation.

41.  $\text{tr}(S^{-1}(AS)) = \text{tr}((AS)S^{-1}) = \text{tr } A$

43. No, since  $\text{tr}(AB - BA) = 0$  and  $\text{tr}(I_n) = n$

45. For  $k = 3$

47. If we write  $M = [\vec{v} \quad \vec{w}]$ , then it is required that  $A\vec{v} = 2\vec{v}$  and  $A\vec{w} = 3\vec{w}$ . Thus, a nonzero  $M$  with the given property exists if 2 or 3 is an eigenvalue of  $A$ .

49. If 2, 3, or 4 is an eigenvalue of  $A$

3 Answers to more theoretical questions are omitted.

1. Eigenbasis:  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ , with eigenvalues 7, 9

$$3. S = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

5. No real eigenvalues.

7. Eigenbasis:  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ , with eigenvalues 1, 2, 3

$$9. S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$11. \text{Eigenbasis: } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \text{ with eigenvalues } 3, 0, 0$$

$$13. S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & -1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$15. \text{Eigenvectors: } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \text{ with eigenvalues } 0, 1; \text{ no eigenbasis}$$

17. Eigenbasis:  $\vec{e}_2, \vec{e}_4, \vec{e}_1, \vec{e}_3 - \vec{e}_2$ , with eigenvalues 1, 1, 0, 0

$$19. \text{Eigenvectors: } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ with eigenvalues } 0, 1. \text{ Matrix } A \text{ fails to be diagonalizable.}$$

$$21. \text{We want } A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; \text{ that is, } A \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}. \text{ The unique solution is } A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}.$$

23. The only eigenvalue of  $A$  is 1, with  $E_1 = \text{span}(\vec{e}_1)$ . There is no eigenbasis.  $A$  represents a horizontal shear.

25. The geometric multiplicity is always 1.

27.  $f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$ , so that the eigenvalues are 2, 3.

29. Both multiplicities are  $n - r$ .

31. They are the same.

33. If  $B = S^{-1}AS$ , then  $B - \lambda I_n = S^{-1}(A - \lambda I_n)S$ .

35. No (consider the eigenvalues)

$$37. \text{a. } A\vec{v} \cdot \vec{w} = (A\vec{v})^T \vec{w} = \vec{v}^T A^T \vec{w} = \vec{v}^T A \vec{w} = \vec{v} \cdot A\vec{w}$$

- b. Suppose  $A\vec{v} = \lambda\vec{v}$  and  $A\vec{w} = \mu\vec{w}$ . Then  $A\vec{v} \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$  and  $\vec{v} \cdot A\vec{w} = \mu(\vec{v} \cdot \vec{w})$ . By part a,  $\lambda(\vec{v} \cdot \vec{w}) = \mu(\vec{v} \cdot \vec{w})$ , so that  $(\lambda - \mu)(\vec{v} \cdot \vec{w}) = 0$ .

## Eigenvalues and Eigenvectors

Since  $\lambda \neq \mu$ , it follows that  $\vec{v} \cdot \vec{w} = 0$ , as claimed.

39. a.  $E_1 = V$  and  $E_0 = V^\perp$ , so that the geometric multiplicity of 1 is  $m$  and that of 0 is  $n - m$ . The algebraic multiplicities are the same. See Exercise 31.  
 b.  $E_1 = V$  and  $E_{-1} = V^\perp$ , so that the multiplicity of 1 is  $m$  and that of  $-1$  is  $n - m$ .

41. Diagonalizable for all  $a$

43. Diagonalizable for positive  $a$

45. Diagonalizable for all  $a, b, c$

47. Diagonalizable only if  $a = b = c = 0$

49. Never diagonalizable

51.  $f_A(\lambda) = -\lambda^3 + c\lambda^2 + b\lambda + a$

$$53. \text{ a. } B = \left[ \begin{array}{ccc|cc} 0 & 0 & a & * & * \\ 1 & 0 & b & * & * \\ 0 & 1 & c & * & * \\ \hline 0 & 0 & 0 & w & x \\ 0 & 0 & 0 & y & z \end{array} \right] = \left[ \begin{array}{cc} B_1 & B_2 \\ 0 & B_3 \end{array} \right]$$

b. Note that  $A$  is similar to  $B$ . Thus,

$$\begin{aligned} f_A(\lambda) &= f_B(\lambda) = f_{B_3}(\lambda)f_{B_1}(\lambda) \\ &= h(\lambda)(-\lambda^3 + c\lambda^2 + b\lambda + a), \text{ where} \\ &h(\lambda) = f_{B_3}(\lambda). \text{ See Exercise 51.} \end{aligned}$$

c.  $f_A(A)\vec{v} = h(A)(-A^3 + cA^2 + bA + aI_5)\vec{v} = h(A)\underbrace{(-A^3\vec{v} + cA^2\vec{v} + bA\vec{v} + a\vec{v})}_0 = \vec{0}$

55. We want  $A - 7I_3$  to be noninvertible. For example,

$$A = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}.$$

4 Answers to more theoretical questions are omitted.

1.  $A^t = \begin{bmatrix} 1 & 3^t - 1 \\ 0 & 3^t \end{bmatrix}$

3.  $A^t = \frac{1}{3} \begin{bmatrix} 5^t + 2(-1)^t & 5 - (-1)^t \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{bmatrix}$

5.  $A^t = 7^{t-1} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

7.  $A^t = \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t & 1 - (0.25)^t \\ 2 - 2(0.25)^t & 2 + (0.25)^t \end{bmatrix}$

9.  $A^t = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ -2(-1)^t & 2(-1)^t & 0 \\ 1 + (-1)^t & 1 - (-1)^t & 2 \end{bmatrix}$

11.  $A^t = \frac{1}{2} \begin{bmatrix} 4 - 2^t & 2 - 2^t & 4 - 3 \cdot 2^t \\ -4 & -2 & -4 \\ 2^t & 2^t & 3 \cdot 2^t \end{bmatrix}$

13.  $A^t \vec{x}_0 = \begin{bmatrix} 1 + 2 \cdot 3^t \\ 2 \cdot 3^t \end{bmatrix}$

15.  $A^t \vec{x}_0 = \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t \\ 2 - 2(0.25)^t \end{bmatrix}$

17.  $A^t \vec{x}_0 = \begin{bmatrix} 4 - 2^t \\ -4 \\ 2^t \end{bmatrix}$

19.  $A^t \vec{x}_0 = \begin{bmatrix} 1 + 2^t + 3 \cdot 6^t \\ -2^t + 5 \cdot 6^t \\ -1 + 2 \cdot 6^t \end{bmatrix}$

21.  $\lim_{t \rightarrow \infty} A^t = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

23.  $\lim_{t \rightarrow \infty} A^t = \frac{1}{22} \begin{bmatrix} 7 & 7 & 7 \\ 10 & 10 & 10 \\ 5 & 5 & 5 \end{bmatrix}$

25.  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \frac{1}{17} \begin{bmatrix} 10 \\ 7 \end{bmatrix}$

27.  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \frac{1}{22} \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$

29.  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \frac{1}{84} \begin{bmatrix} 24 \\ 29 \\ 31 \end{bmatrix}$

33. a.  $A = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

c.  $\vec{x}(t) = \left(1 + \frac{c_0}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \frac{c_0}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Carl wins if he chooses  $c_0 < 1$ .

35. a.  $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b.  $B = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$

c. The eigenvalues of  $A$  are 0.5 and  $-0.1$ , those of  $B$  are  $0.5, -0.1, 1$ . If  $\vec{v}$  is an eigenvector of  $A$ ,

then  $\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}$  is an eigenvector of  $B$ . Furthermore,

$$\begin{bmatrix} (I_2 - A)^{-1} \vec{b} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \text{ is an eigenvector of } B$$

with eigenvalue 1.

d. Will approach  $(I_2 - A)^{-1} \vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ , for any initial value

## Eigenvalues and Eigenvectors

37. Let  $\vec{x}(t) = \begin{bmatrix} r(t) \\ p(t) \\ w(t) \end{bmatrix}$ . Then  $\vec{x}(t+1) = A\vec{x}(t)$  where

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \text{ Eigenbasis for } A: \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ with eigenvalues } 1, \frac{1}{2}, 0.$$

$$\vec{x}_0 = \vec{e}_1 = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\text{so } \vec{x}(t) = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right)^{t+1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ for } t > 0.$$

The proportion in the long run is 1:2:1.

39. All real numbers  $\lambda$  are eigenvalues, with corresponding eigenfunctions  $Ce^{(\lambda+1)t}$ .

41. The symmetric matrices are eigenmatrices with eigenvalue 2, and the skew-symmetric matrices have eigenvalue 0. Yes,  $L$  is diagonalizable, since the sum of the dimensions of the eigenspaces is 4.

43. 1 and  $i$  are “eigenvectors” with eigenvalues 1 and  $-1$ , respectively. Yes,  $T$  is diagonalizable; 1,  $i$  is an eigenbasis.

45. No eigensequences

47. The nonzero polynomials of the form  $a + cx^2$  are eigenfunctions with eigenvalue 1, and  $bx$  (with  $b \neq 0$ ) has eigenvalue  $-1$ . Yes,  $T$  is diagonalizable, with eigenbasis 1,  $x$ ,  $x^2$ .

49. 1,  $2x - 1$ , and  $(2x - 1)^2$  are eigenfunctions with eigenvalues 1, 3, and 9, respectively. These functions form an eigenbasis, so that  $T$  is indeed diagonalizable.

51. The only eigenfunctions are the nonzero constant functions, with eigenvalue 0.

$$55. A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ for example}$$

59. Exercise 58 implies that  $A$  and  $B$  are both similar to the matrix  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , so that  $A$  is similar to  $B$ .

65. A basis of  $V$  is  $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ , and  $\dim V = 2$ .

67. The dimension is  $3^2 + 2^2 = 13$ .

71. The eigenvalues are 1 and 2, and  $(A - I_3)(A - 2I_3) = 0$ . Thus  $A$  is diagonalizable.

73. If  $\lambda_1, \dots, \lambda_m$  are the distinct eigenvalues of  $A$ , then  $f_A(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_m)h(\lambda)$  for some polynomial  $h(\lambda)$ , so that  $f_A(A) = \underbrace{(A - \lambda_1 I_n) \cdots (A - \lambda_m I_n)}_0 h(A) = 0$ , by Exercise 70.

**5** Answers to more theoretical questions are omitted.

$$1. \sqrt{18} \left( \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$$

$$3. \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right), \text{ for } k = 0, \dots, n-1$$

$$5. \text{ If } z = r(\cos \phi + i \sin \phi), \text{ then}$$

$$w = \sqrt[n]{r} \left( \cos\left(\frac{\phi + 2\pi k}{n}\right) + i \sin\left(\frac{\phi + 2\pi k}{n}\right) \right),$$

for  $k = 0, \dots, n-1$ .

7. Clockwise rotation through an angle of  $\frac{\pi}{4}$  followed by a scaling by a factor of  $\sqrt{2}$

9. Spirals outward since  $|z| > 1$

$$11. f(\lambda) = (\lambda - 1)(\lambda - 1 - 2i)(\lambda - 1 + 2i)$$

$$13. S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ for example}$$

$$15. S = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \text{ for example}$$

$$17. S = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}, \text{ for example}$$

$$19. \text{ a. } \text{tr } A = m, \det A = 0$$

b.  $\text{tr } B = 2m - n, \det B = (-1)^{n-m}$ . Compare with Exercise 3.39

$$21. 2 \pm 3i$$

$$23. 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$25. \pm 1, \pm i$$

$$27. -1, -1, 3$$

29.  $\text{tr } A = \lambda_1 + \lambda_2 + \lambda_3 = 0$  and  $\det A = \lambda_1 \lambda_2 \lambda_3 = bcd > 0$ . Therefore, there are one positive and two negative eigenvalues; the positive one is largest in absolute value.

31. b. Eigenvalues  $\lambda_1 = 1, \lambda_{2,3} \approx -0.2 \pm 0.136i, \lambda_{4,5} \approx 0.134 \pm 0.132i$ . Note that  $|\lambda_j| < 1$  for  $j = 2, 3, 4, 5$ . Having five distinct eigenvalues, matrix  $A$  is diagonalizable.

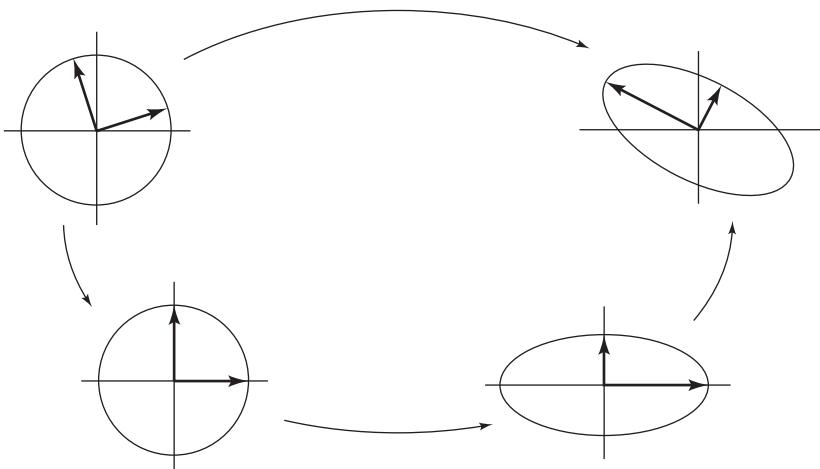
$$\text{c. } \vec{x}_{\text{equ}} = [0.2 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.2]^T$$

d. The proof is analogous to the proof of Theorem 4.1. Note that  $\lim_{t \rightarrow \infty} (\lambda_j^t) = 0$  for  $j = 2, 3, 4, 5$  since  $|\lambda_j| < 1$ . See Example 5.

e.  $\lim_{t \rightarrow \infty} A^t$  is the  $5 \times 5$  matrix whose entries are all 0.2.

## Eigenvalues and Eigenvectors

33. c. *Hint:* Let  $\lambda_1, \lambda_2, \dots, \lambda_5$  be the eigenvalues, with  $\lambda_1 > |\lambda_j|$ , for  $j = 2, \dots, 5$ . Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_5$  be corresponding eigenvectors. Write  $\vec{e}_i = c_1\vec{v}_1 + \dots + c_5\vec{v}_5$ . Then  $i$ th column of  $A^T = A^T\vec{e}_i = c_1\lambda_1^T\vec{v}_1 + \dots + c_5\lambda_5^T\vec{v}_5$  is nearly parallel to  $\vec{v}_1$  for large  $t$ .
45. If  $a$  is nonzero.
47. If  $a$  is nonzero.
49. If  $a$  is neither 1 nor 2.
51.  $\mathbb{Q}$  is a field.
53. The binary digits form a field.
55.  $H$  is not a field (multiplication is noncommutative).
- 6** Answers to more theoretical questions are omitted.
1. Stable                    3. Not stable  
 5. Not stable              7. Not stable  
 9. Not stable              11. For  $|k| < 1$   
 13. For all  $k$               15. Never stable  
 17.  $\vec{x}(t) = \begin{bmatrix} -\sin(\phi t) \\ \cos(\phi t) \end{bmatrix}$ , where  $\phi = \arctan\left(\frac{4}{3}\right)$ ; a circle  
 19.  $\vec{x}(t) = \sqrt{13}^t \begin{bmatrix} -\sin(\phi t) \\ \cos(\phi t) \end{bmatrix}$ , where  $\phi = \arctan\left(\frac{3}{2}\right)$ ; spirals outward  
 21.  $\vec{x}(t) = \sqrt{17}^t \begin{bmatrix} 5 \sin(\phi t) \\ \cos(\phi t) + 3 \sin(\phi t) \end{bmatrix}$ , where  $\phi = \arctan\left(\frac{1}{4}\right)$ ; spirals outward
23.  $\vec{x}(t) = \left(\frac{1}{2}\right)^t \begin{bmatrix} 5 \sin(\phi t) \\ \cos(\phi t) + 3 \sin(\phi t) \end{bmatrix}$ , where  $\phi = \arctan\left(\frac{3}{4}\right)$ ; spirals inward
25. Not stable              27. Stable
29. May or may not be stable; consider  $A = \pm \frac{1}{2}I_2$ .
33. The matrix represents a rotation followed by a scaling with a scaling factor of  $\sqrt{0.99^2 + 0.01^2} < 1$ . Trajectory spirals inward.
35. a. Choose an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  and write  $\vec{x}_0 = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ . Then  $\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + \dots + c_n\lambda_n^t\vec{v}_n$  and  $\|\vec{x}(t)\| \leq |c_1|\|\vec{v}_1\| + \dots + |c_n|\|\vec{v}_n\| = M$  (use the triangle inequality  $\|\vec{u} + \vec{w}\| \leq \|\vec{u}\| + \|\vec{w}\|$ , and observe that  $|\lambda_i^t| \leq 1$ ).  
 b. The trajectory  $\vec{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$  is not bounded. This does not contradict part a, since there is no eigenbasis for the matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .  
 39.  $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$  is a stable equilibrium.



# Symmetric Matrices and Quadratic Forms

## I Symmetric Matrices

In this chapter we will work with real numbers throughout, except for a brief digression into  $\mathbb{C}$  in the discussion of Theorem 1.3.

Consider the following central question:

When is a given square matrix  $A$  *diagonalizable*? That is, when is there an *eigenbasis* for  $A$ ?

In geometry, we prefer to work with *orthonormal* bases, which raises the following question:

For which matrices is there an *orthonormal* eigenbasis? Or, equivalently, for which matrices  $A$  is there an *orthogonal* matrix  $S$  such that  $S^{-1}AS = S^TAS$  is diagonal?

(Note that  $S^{-1} = S^T$  for orthogonal matrices.) We say that  $A$  is *orthogonally diagonalizable* if there exists an orthogonal matrix  $S$  such that  $S^{-1}AS = S^TAS$  is diagonal. Then, the question is:

Which matrices are orthogonally diagonalizable?

Simple examples of orthogonally diagonalizable matrices are diagonal matrices (we can let  $S = I_n$ ) and the matrices of orthogonal projections and reflections.

**EXAMPLE I** If  $A$  is orthogonally diagonalizable, what is the relationship between  $A^T$  and  $A$ ?

### Solution

We have

$$S^{-1}AS = D \quad \text{or} \quad A = SDS^{-1} = SDS^T,$$

for an orthogonal  $S$  and a diagonal  $D$ . Then

$$A^T = (SDS^T)^T = SD^T S^T = SDS^T = A.$$

We find that  $A$  is symmetric:

$$A^T = A. \quad \blacksquare$$

Surprisingly, the converse is true as well:

### Theorem 1.1

**Spectral theorem**  
A matrix  $A$  is *orthogonally diagonalizable* (i.e., there exists an orthogonal  $S$  such that  $S^{-1}AS = S^TAS$  is diagonal) if and only if  $A$  is *symmetric* (i.e.,  $A^T = A$ ).

We will prove this theorem later in this section, based on two preliminary results, Theorems 1.2 and 1.3. First, we will illustrate the spectral theorem with an example.

**EXAMPLE 2** For the symmetric matrix  $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$ , find an orthogonal  $S$  such that  $S^{-1}AS$  is diagonal.

### Solution

We will first find an eigenbasis. The eigenvalues of  $A$  are 3 and 8, with corresponding eigenvectors  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , respectively. See Figure 1.

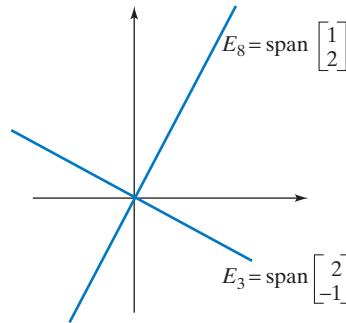


Figure 1

Note that the two eigenspaces,  $E_3$  and  $E_8$ , are perpendicular. (This is no coincidence, as we will see in Theorem 1.2.) Therefore, we can find an orthonormal eigenbasis simply by dividing the given eigenvectors by their lengths:

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

If we define the orthogonal matrix

$$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

then  $S^{-1}AS$  will be diagonal; namely,  $S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$ . ■

## Symmetric Matrices and Quadratic Forms

The key observation we made in Example 2 generalizes as follows:

### Theorem 1.2

Consider a symmetric matrix  $A$ . If  $\vec{v}_1$  and  $\vec{v}_2$  are eigenvectors of  $A$  with *distinct* eigenvalues  $\lambda_1$  and  $\lambda_2$ , then  $\vec{v}_1 \cdot \vec{v}_2 = 0$ ; that is,  $\vec{v}_2$  is orthogonal to  $\vec{v}_1$ .

**Proof** We compute the product

$$\vec{v}_1^T A \vec{v}_2$$

in two different ways:

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T (\lambda_2 \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (A \vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2).$$

Comparing the results, we find

$$\lambda_1 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2),$$

or

$$(\lambda_1 - \lambda_2) (\vec{v}_1 \cdot \vec{v}_2) = 0.$$

Since the first factor in this product,  $\lambda_1 - \lambda_2$ , is nonzero, the second factor,  $\vec{v}_1 \cdot \vec{v}_2$ , must be zero, as claimed. ■

Theorem 1.2 tells us that the eigenspaces of a symmetric matrix are perpendicular to one another. Here is another illustration of this property:

### EXAMPLE 3

For the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

find an orthogonal  $S$  such that  $S^{-1}AS$  is diagonal.

#### Solution

The eigenvalues are 0 and 3, with

$$E_0 = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad E_3 = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that the two eigenspaces are indeed perpendicular to one another, in accordance with Theorem 1.2. See Figure 2.

We can construct an orthonormal eigenbasis for  $A$  by picking an orthonormal basis of each eigenspace (using the Gram–Schmidt process in the case of  $E_0$ ). See Figure 3.

In Figure 3, the vectors  $\vec{v}_1, \vec{v}_2$  form an orthonormal basis of  $E_0$ , and  $\vec{v}_3$  is a unit vector in  $E_3$ . Then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is an orthonormal eigenbasis for  $A$ . We can let  $S = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  to diagonalize  $A$  orthogonally.

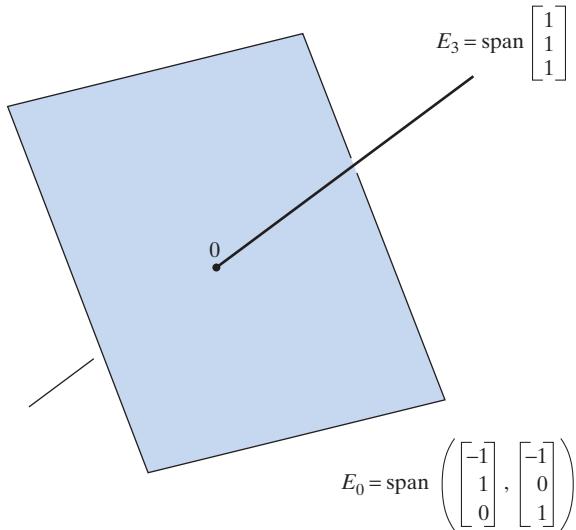
If we apply the Gram–Schmidt<sup>1</sup> process to the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

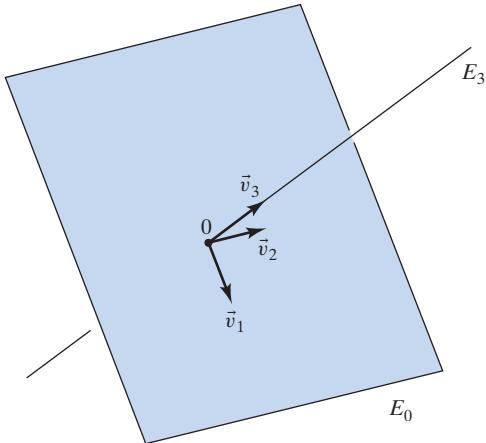
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<sup>1</sup>Alternatively, we could find a unit vector  $\vec{v}_1$  in  $E_0$  and a unit vector  $\vec{v}_3$  in  $E_3$ , and then let  $\vec{v}_2 = \vec{v}_3 \times \vec{v}_1$ .

## Symmetric Matrices and Quadratic Forms



**Figure 2** The eigenspaces  $E_0$  and  $E_3$  are orthogonal complements.



**Figure 3**

spanning  $E_0$ , we find

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

The computations are left as an exercise. For  $E_3$ , we get

$$\vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the orthogonal matrix

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

diagonalizes the matrix  $A$ :

$$S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad \blacksquare$$

By Theorem 1.2, if a symmetric matrix is diagonalizable, then it is orthogonally diagonalizable. We still have to show that symmetric matrices are diagonalizable in the first place (over  $\mathbb{R}$ ). The key point is the following observation:

### Theorem 1.3

A symmetric  $n \times n$  matrix  $A$  has  $n$  real eigenvalues if they are counted with their algebraic multiplicities.

#### Proof

We need to show that all the complex eigenvalues of matrix  $A$  are in fact real. Consider two complex conjugate eigenvalues  $p \pm iq$  of  $A$  with corresponding

## Symmetric Matrices and Quadratic Forms

eigenvectors  $\vec{v} \pm i\vec{w}$ . We wish to show that these eigenvalues are real; that is,  $q = 0$ . Note first that

$$(\vec{v} + i\vec{w})^T (\vec{v} - i\vec{w}) = \|\vec{v}\|^2 + \|\vec{w}\|^2.$$

(Verify this.) Now we compute the product

$$(\vec{v} + i\vec{w})^T A (\vec{v} - i\vec{w})$$

in two different ways:

$$\begin{aligned} (\vec{v} + i\vec{w})^T A (\vec{v} - i\vec{w}) &= (\vec{v} + i\vec{w})^T (p - iq)(\vec{v} - i\vec{w}) \\ &= (p - iq)(\|\vec{v}\|^2 + \|\vec{w}\|^2) \\ (\vec{v} + i\vec{w})^T A (\vec{v} - i\vec{w}) &= (A(\vec{v} + i\vec{w}))^T (\vec{v} - i\vec{w}) = (p + iq)(\vec{v} + i\vec{w})^T (\vec{v} - i\vec{w}) \\ &= (p + iq)(\|\vec{v}\|^2 + \|\vec{w}\|^2). \end{aligned}$$

Comparing the results, we find that  $p + iq = p - iq$ , so that  $q = 0$ , as claimed. ■

The foregoing proof is not very enlightening. A more transparent proof would follow if we were to define the dot product for complex vectors, but to do so would lead us too far afield.

We are now ready to prove Theorem 1.1: Symmetric matrices are orthogonally diagonalizable.

Even though this is not logically necessary, let us first examine the case of a symmetric  $n \times n$  matrix  $A$  with  $n$  distinct real eigenvalues. For each eigenvalue, we can choose an eigenvector of length 1. By Theorem 1.2, these eigenvectors will form an orthonormal eigenbasis; that is, the matrix  $A$  will be orthogonally diagonalizable, as claimed.

**Proof (of Theorem 1.1):** This proof is somewhat technical; it may be skipped in a first reading of this text without harm.

We prove by induction on  $n$  that a symmetric  $n \times n$  matrix  $A$  is orthogonally diagonalizable.

For a  $1 \times 1$  matrix  $A$ , we can let  $P = [1]$ .

Now assume that the claim is true for  $n$ ; we show that it holds for  $n + 1$ . Pick a real eigenvalue  $\lambda$  of  $A$  (this is possible by Theorem 1.3), and choose an eigenvector  $\vec{v}_1$  of length 1 for  $\lambda$ . We can find an orthonormal basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n+1}$  of  $\mathbb{R}^{n+1}$ . (Think about how you could construct such a basis.) Form the orthogonal matrix

$$P = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n+1} \\ | & | & & | \end{bmatrix},$$

and compute

$$P^{-1}AP.$$

The first column of  $P^{-1}AP$  is  $\lambda\vec{e}_1$ . (Why?) Also note that  $P^{-1}AP = P^TAP$  is symmetric:  $(P^TAP)^T = P^TA^TP = P^TAP$ , because  $A$  is symmetric. Combining these two statements, we conclude that  $P^{-1}AP$  has the block form

$$P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}, \quad (\text{I})$$

### Symmetric Matrices and Quadratic Forms

where  $B$  is a symmetric  $n \times n$  matrix. By the induction hypothesis,  $B$  is orthogonally diagonalizable; that is, there exists an orthogonal  $n \times n$  matrix  $Q$  such that

$$Q^{-1}BQ = D$$

is a diagonal  $n \times n$  matrix. Now introduce the orthogonal  $(n+1) \times (n+1)$  matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}.$$

Then

$$R^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} R = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix} \quad (\text{II})$$

is diagonal.

Combining equations (I) and (II), we find that

$$R^{-1}P^{-1}APR = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix} \quad (\text{III})$$

is diagonal. Consider the orthogonal matrix  $S = PR$ . (Note that the product of orthogonal matrices is orthogonal.) Note that  $S^{-1} = (PR)^{-1} = R^{-1}P^{-1}$ . Therefore, equation (III) can be written

$$S^{-1}AS = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix},$$

proving our claim. ■

The method outlined in the proof of Theorem 1.1 is not a sensible way to find the matrix  $S$  in a numerical example. Rather, we can proceed as in Example 3:

#### Theorem 1.4

##### Orthogonal diagonalization of a symmetric matrix $A$

- a. Find the eigenvalues of  $A$ , and find a basis of each eigenspace.
- b. Using the Gram–Schmidt process, find an *orthonormal* basis of each eigenspace.
- c. Form an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  for  $A$  by concatenating the orthonormal bases you found in part b, and let

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & | \end{bmatrix}.$$

$S$  is orthogonal (by Theorem 1.2), and  $S^{-1}AS$  will be diagonal.

We conclude this section with an example of a geometric nature:

#### EXAMPLE 4

Consider an invertible symmetric  $2 \times 2$  matrix  $A$ . Show that the linear transformation  $T(\vec{x}) = A\vec{x}$  maps the unit circle into an ellipse, and find the lengths of the semimajor and the semiminor axes of this ellipse in terms of the eigenvalues of  $A$ .

#### Solution

The spectral theorem tells us that there exists an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $T$ , with associated real eigenvalues  $\lambda_1$  and  $\lambda_2$ . These eigenvalues will be nonzero,

## Symmetric Matrices and Quadratic Forms

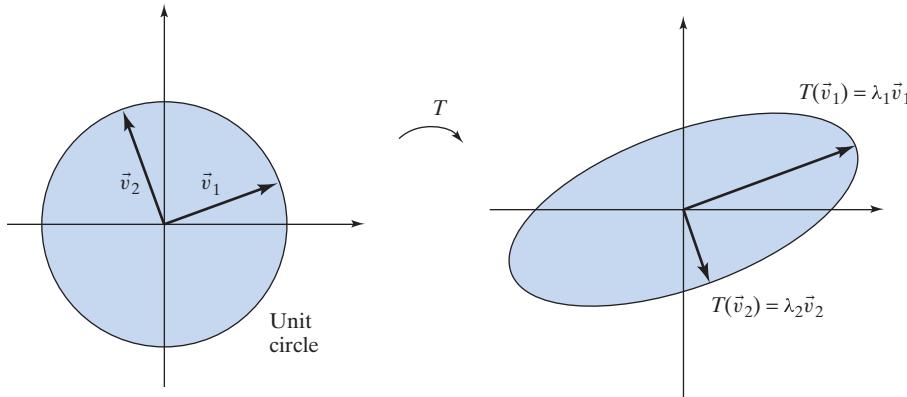
since  $A$  is invertible. Arrange things so that  $|\lambda_1| \geq |\lambda_2|$ . The unit circle in  $\mathbb{R}^2$  consists of all vectors of the form

$$\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2.$$

The image of the unit circle consists of the vectors

$$\begin{aligned} T(\vec{v}) &= \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2) \\ &= \cos(t)\lambda_1\vec{v}_1 + \sin(t)\lambda_2\vec{v}_2, \end{aligned}$$

an ellipse whose semimajor axis  $\lambda_1\vec{v}_1$  has the length  $\|\lambda_1\vec{v}_1\| = |\lambda_1|$ , while the length of the semiminor axis is  $\|\lambda_2\vec{v}_2\| = |\lambda_2|$ . See Figure 4.



**Figure 4**

In the example illustrated in Figure 4, the eigenvalue  $\lambda_1$  is positive, and  $\lambda_2$  is negative. ■

## EXERCISES |

**GOAL** Find orthonormal eigenbases for symmetric matrices. Apply the spectral theorem.

For each of the matrices in Exercises 1 through 6, find an orthonormal eigenbasis. Do not use technology.

1.  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

4.  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5.  $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

6.  $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

For each of the matrices  $A$  in Exercises 7 through 11, find an orthogonal matrix  $S$  and a diagonal matrix  $D$  such that  $S^{-1}AS = D$ . Do not use technology.

7.  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

8.  $A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$

9.  $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

10.  $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$

11.  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

12. Let  $L$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  be the reflection about the line spanned by

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

- a. Find an orthonormal eigenbasis  $\mathcal{B}$  for  $L$ .  
 b. Find the matrix  $B$  of  $L$  with respect to  $\mathcal{B}$ .  
 c. Find the matrix  $A$  of  $L$  with respect to the standard basis of  $\mathbb{R}^3$ .

13. Consider a symmetric  $3 \times 3$  matrix  $A$  with  $A^2 = I_3$ . Is the linear transformation  $T(\vec{x}) = A\vec{x}$  necessarily the reflection about a subspace of  $\mathbb{R}^3$ ?

## Symmetric Matrices and Quadratic Forms

- 14.** In Example 3 of this section, we diagonalized the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

by means of an orthogonal matrix  $S$ . Use this result to diagonalize the following matrices orthogonally (find  $S$  and  $D$  in each case):

$$\begin{array}{ll} \text{a. } \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} & \text{b. } \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\ \text{c. } \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} & \end{array}$$

- 15.** If  $A$  is invertible and orthogonally diagonalizable, is  $A^{-1}$  orthogonally diagonalizable as well?

- 16. a.** Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with their multiplicities. Note that the algebraic multiplicity agrees with the geometric multiplicity. (Why?) Hint: What is the kernel of  $A$ ?

- b.** Find the eigenvalues of the matrix

$$B = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

with their multiplicities. Do not use technology.

- c.** Use your result in part b to find  $\det B$ .

- 17.** Use the approach of Exercise 16 to find the determinant of the  $n \times n$  matrix  $B$  that has  $p$ 's on the diagonal and  $q$ 's elsewhere:

$$B = \begin{bmatrix} p & q & \cdots & q \\ q & p & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & p \end{bmatrix}.$$

- 18.** Consider unit vectors  $\vec{v}_1, \dots, \vec{v}_n$  in  $\mathbb{R}^n$  such that the angle between  $\vec{v}_i$  and  $\vec{v}_j$  is  $60^\circ$  for all  $i \neq j$ . Find the  $n$ -volume of the  $n$ -parallelepiped spanned by  $\vec{v}_1, \dots, \vec{v}_n$ . Hint: Let  $A = [\vec{v}_1 \ \dots \ \vec{v}_n]$ , and think about the matrix  $A^T A$  and its determinant. Exercise 17 is useful.

- 19.** Consider a linear transformation  $L$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Show that there exists an orthonormal basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  of  $\mathbb{R}^m$  such that the vectors  $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_m)$  are orthogonal. Note that some of the vectors  $L(\vec{v}_i)$  may be zero. Hint: Consider an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  for the symmetric matrix  $A^T A$ .

- 20.** Consider a linear transformation  $T$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , where  $m \leq n$ . Show that there exist an orthonormal basis  $\vec{v}_1, \dots, \vec{v}_m$  of  $\mathbb{R}^m$  and an orthonormal basis  $\vec{w}_1, \dots, \vec{w}_n$  of  $\mathbb{R}^n$  such that  $T(\vec{v}_i)$  is a scalar multiple of  $\vec{w}_i$ , for  $i = 1, \dots, m$ . Hint: Exercise 19 is helpful.

- 21.** Consider a symmetric  $3 \times 3$  matrix  $A$  with eigenvalues 1, 2, and 3. How many different orthogonal matrices  $S$  are there such that  $S^{-1}AS$  is diagonal?

- 22.** Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k & 0 & 2 \\ 0 & 0 & k & 0 \end{bmatrix},$$

where  $k$  is a constant.

- a.** Find a value of  $k$  such that the matrix  $A$  is diagonalizable.  
**b.** Find a value of  $k$  such that  $A$  fails to be diagonalizable.

- 23.** If an  $n \times n$  matrix  $A$  is both symmetric and orthogonal, what can you say about the eigenvalues of  $A$ ? What about the eigenspaces? Interpret the linear transformation  $T(\vec{x}) = A\vec{x}$  geometrically in the cases  $n = 2$  and  $n = 3$ .

- 24.** Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find an orthonormal eigenbasis for  $A$ .

- 25.** Consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find an orthogonal  $5 \times 5$  matrix  $S$  such that  $S^{-1}AS$  is diagonal.

- 26.** Let  $J_n$  be the  $n \times n$  matrix with all ones on the “other diagonal” and zeros elsewhere. (In Exercises 24 and 25, we studied  $J_4$  and  $J_5$ , respectively.) Find the eigenvalues of  $J_n$ , with their multiplicities.

## Symmetric Matrices and Quadratic Forms

27. Diagonalize the  $n \times n$  matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

(All ones along both diagonals, and zeros elsewhere.)

28. Diagonalize the  $13 \times 13$  matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

(All ones in the last row and the last column, and zeros elsewhere.)

29. Consider a symmetric matrix  $A$ . If the vector  $\vec{v}$  is in the image of  $A$  and  $\vec{w}$  is in the kernel of  $A$ , is  $\vec{v}$  necessarily orthogonal to  $\vec{w}$ ? Justify your answer.
30. Consider an orthogonal matrix  $R$  whose first column is  $\vec{v}$ . Form the symmetric matrix  $A = \vec{v}\vec{v}^T$ . Find an orthogonal matrix  $S$  and a diagonal matrix  $D$  such that  $S^{-1}AS = D$ . Describe  $S$  in terms of  $R$ .
31. *True or false?* If  $A$  is a symmetric matrix, then  $\text{rank}(A) = \text{rank}(A^2)$ .
32. Consider the  $n \times n$  matrix with all ones on the main diagonal and all  $q$ 's elsewhere. For which values of  $q$  is this matrix invertible? *Hint:* Exercise 17 is helpful.
33. For which angle(s)  $\theta$  can you find three distinct unit vectors in  $\mathbb{R}^2$  such that the angle between any two of them is  $\theta$ ? Draw a sketch.
34. For which angle(s)  $\theta$  can you find four distinct unit vectors in  $\mathbb{R}^3$  such that the angle between any two of them is  $\theta$ ? Draw a sketch.
35. Consider  $n + 1$  distinct unit vectors in  $\mathbb{R}^n$  such that the angle between any two of them is  $\theta$ . Find  $\theta$ .
36. Consider a symmetric  $n \times n$  matrix  $A$  with  $A^2 = A$ . Is the linear transformation  $T(\vec{x}) = A\vec{x}$  necessarily the orthogonal projection onto a subspace of  $\mathbb{R}^n$ ?
37. If  $A$  is any symmetric  $2 \times 2$  matrix with eigenvalues  $-2$  and  $3$ , and  $\vec{u}$  is a unit vector in  $\mathbb{R}^2$ , what are the possible values of  $\|A\vec{u}\|$ ? Explain your answer geometrically, using Example 4 as a guide.
38. If  $A$  is any symmetric  $2 \times 2$  matrix with eigenvalues  $-2$  and  $3$ , and  $\vec{u}$  is a unit vector in  $\mathbb{R}^2$ , what are the possible values of the dot product  $\vec{u} \cdot A\vec{u}$ ? Illustrate

your answer, in terms of the unit circle and its image under  $A$ .

39. If  $A$  is any symmetric  $3 \times 3$  matrix with eigenvalues  $-2$ ,  $3$ , and  $4$ , and  $\vec{u}$  is a unit vector in  $\mathbb{R}^3$ , what are the possible values of the dot product  $\vec{u} \cdot A\vec{u}$ ?
40. If  $A$  is any symmetric  $3 \times 3$  matrix with eigenvalues  $-2$ ,  $3$ , and  $4$ , and  $\vec{u}$  is a unit vector in  $\mathbb{R}^3$ , what are the possible values of  $\|A\vec{u}\|$ ? Explain your answer geometrically, in terms of the unit sphere and its image under  $A$ .
41. Show that for every symmetric  $n \times n$  matrix  $A$ , there exists a symmetric  $n \times n$  matrix  $B$  such that  $B^3 = A$ .
42. Find a symmetric  $2 \times 2$  matrix  $B$  such that  $B^3 = \frac{1}{5} \begin{bmatrix} 12 & 14 \\ 14 & 33 \end{bmatrix}$ .
43. For  $A = \begin{bmatrix} 2 & 11 & 11 \\ 11 & 2 & 11 \\ 11 & 11 & 2 \end{bmatrix}$ , find a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $A\vec{v}$  is orthogonal to  $\vec{v}$ .
44. Consider an invertible symmetric  $n \times n$  matrix  $A$ . When does there exist a nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  such that  $A\vec{v}$  is orthogonal to  $\vec{v}$ ? Give your answer in terms of the signs of the eigenvalues of  $A$ .
45. We say that an  $n \times n$  matrix  $A$  is *triangulizable* if  $A$  is similar to an upper triangular  $n \times n$  matrix  $B$ .
- a. Give an example of a matrix with real entries that fails to be triangulizable over  $\mathbb{R}$ .
  - b. Show that any  $n \times n$  matrix with complex entries is triangulizable over  $\mathbb{C}$ . *Hint:* Give a proof by induction analogous to the proof of Theorem 1.1.
46. a. Consider a complex upper triangular  $n \times n$  matrix  $U$  with zeros on the diagonal. Show that  $U$  is *nilpotent* (i.e., that  $U^n = 0$ ).
- b. Consider a complex  $n \times n$  matrix  $A$  that has zero as its only eigenvalue (with algebraic multiplicity  $n$ ). Use Exercise 45 to show that  $A$  is nilpotent.
47. Let us first introduce two notations. For a complex  $n \times n$  matrix  $A$ , let  $|A|$  be the matrix whose  $ij$ th entry is  $|a_{ij}|$ . For two real  $n \times n$  matrices  $A$  and  $B$ , we write  $A \leq B$  if  $a_{ij} \leq b_{ij}$  for all entries. Show that
- a.  $|AB| \leq |A||B|$ , for all complex  $n \times n$  matrices  $A$  and  $B$ , and
  - b.  $|A^t| \leq |A|^t$ , for all complex  $n \times n$  matrices  $A$  and all positive integers  $t$ .
48. Let  $U \geq 0$  be a real upper triangular  $n \times n$  matrix with zeros on the diagonal. Show that

$$(I_n + U)^t \leq t^n(I_n + U + U^2 + \cdots + U^{n-1})$$

for all positive integers  $t$ . See Exercises 46 and 47.

49. Let  $R$  be a complex upper triangular  $n \times n$  matrix with  $|r_{ii}| < 1$  for  $i = 1, \dots, n$ . Show that

$$\lim_{t \rightarrow \infty} R^t = 0,$$

meaning that the modulus of all entries of  $R^t$  approaches zero. Hint: We can write  $|R| \leq \lambda(I_n + U)$ , for some positive real number  $\lambda < 1$  and an upper triangular matrix  $U \geq 0$  with zeros on the diagonal. Exercises 47 and 48 are helpful.

50. a. Let  $A$  be a complex  $n \times n$  matrix such that  $|\lambda| < 1$  for all eigenvalues  $\lambda$  of  $A$ . Show that

$$\lim_{t \rightarrow \infty} A^t = 0,$$

meaning that the modulus of all entries of  $A^t$  approaches zero.

- b. This part of the Exercise has been intentionally omitted for this edition.

## 2 Quadratic Forms

In this section, we will present an important application of the spectral theorem (Theorem 1.1).

In a multivariable calculus text, we find the following problem:

**EXAMPLE I** Consider the function

$$q(x_1, x_2) = 8x_1^2 - 4x_1x_2 + 5x_2^2$$

from  $\mathbb{R}^2$  to  $\mathbb{R}$ .

Determine whether  $q(0, 0) = 0$  is the global maximum, the global minimum, or neither.

Recall that  $q(0, 0)$  is called the global (or absolute) minimum if  $q(0, 0) \leq q(x_1, x_2)$  for all real numbers  $x_1, x_2$ ; the global maximum is defined analogously.

### Solution

There are a number of ways to do this problem, some of which you may have seen in a previous course. Here we present an approach based on matrix techniques. We will first develop some theory and then do the example.

Note that we can write

$$\begin{aligned} q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 8x_1^2 - 4x_1x_2 + 5x_2^2 \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 8x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix} \quad \text{We "split" the term } -4x_1x_2 \text{ equally between the two components.} \end{aligned}$$

More succinctly, we can write

$$q(\vec{x}) = \vec{x} \cdot A\vec{x}, \quad \text{where } A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix},$$

or

$$q(\vec{x}) = \vec{x}^T A \vec{x}.$$

The matrix  $A$  is symmetric by construction. By the spectral theorem (Theorem 1.1), there exists an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $A$ . We find

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

with associated eigenvalues  $\lambda_1 = 9$  and  $\lambda_2 = 4$ . (Verify this.)

If we write  $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ , we can express the value of the function as follows:

$$q(\vec{x}) = \vec{x} \cdot A\vec{x} = (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot (c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2) = \lambda_1c_1^2 + \lambda_2c_2^2 = 9c_1^2 + 4c_2^2.$$

## Symmetric Matrices and Quadratic Forms

(Recall that  $\vec{v}_1 \cdot \vec{v}_1 = 1$ ,  $\vec{v}_1 \cdot \vec{v}_2 = 0$ , and  $\vec{v}_2 \cdot \vec{v}_2 = 1$ , since  $\vec{v}_1, \vec{v}_2$  is an orthonormal basis of  $\mathbb{R}^2$ .)

The formula  $q(\vec{x}) = 9c_1^2 + 4c_2^2$  shows that  $q(\vec{x}) > 0$  for all nonzero  $\vec{x}$ , because at least one of the terms  $9c_1^2$  and  $4c_2^2$  is positive.

Thus,  $q(0, 0) = 0$  is the global minimum of the function.

The preceding work shows that the  $c_1$ - $c_2$ -coordinate system defined by an orthonormal eigenbasis for  $A$  is “well adjusted” to the function  $q$ . The formula

$$9c_1^2 + 4c_2^2$$

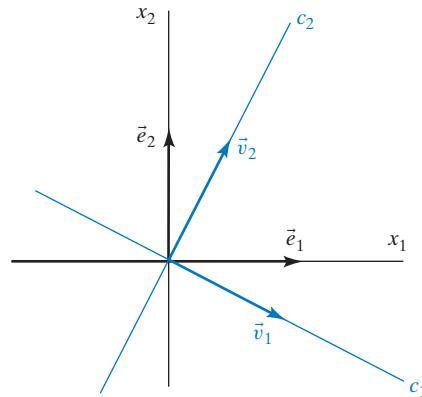
is easier to work with than the original formula

$$8x_1^2 - 4x_1x_2 + 5x_2^2,$$

because no term involves  $c_1c_2$ :

$$\begin{aligned} q(x_1, x_2) &= 8x_1^2 - 4x_1x_2 + 5x_2^2 \\ &= 9c_1^2 + 4c_2^2 \end{aligned}$$

The two coordinate systems are shown in Figure 1. ■



**Figure 1**

Let us present these ideas in greater generality:

### Definition 2.1

#### Quadratic forms

A function  $q(x_1, x_2, \dots, x_n)$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  is called a *quadratic form* if it is a linear combination of functions of the form  $x_i x_j$  (where  $i$  and  $j$  may be equal). A quadratic form can be written as

$$q(\vec{x}) = \vec{x} \cdot A\vec{x} = \vec{x}^T A \vec{x},$$

for a unique symmetric  $n \times n$  matrix  $A$ , called the matrix of  $q$ .

The uniqueness of matrix  $A$  will be shown in Exercise 52.

The set  $Q_n$  of quadratic forms  $q(x_1, x_2, \dots, x_n)$  is a *subspace* of the linear space of all functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ . In Exercise 42 you will be asked to think about the dimension of this space.

### EXAMPLE 2

Consider the quadratic form

$$q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3.$$

Find the matrix of  $q$ .

### Solution

As in Example 1, we let

$$a_{ii} = (\text{coefficient of } x_i^2)$$

$$a_{ij} = a_{ji} = \frac{1}{2} (\text{coefficient of } x_i x_j), \quad \text{if } i \neq j.$$

Therefore,

$$A = \begin{bmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{bmatrix}.$$

■

The observation we made in Example 1 can now be generalized as follows:

### Theorem 2.2

#### Diagonalizing a quadratic form

Consider a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ , where  $A$  is a symmetric  $n \times n$  matrix. Let  $\mathfrak{B}$  be an orthonormal eigenbasis for  $A$ , with associated eigenvalues  $\lambda_1, \dots, \lambda_n$ . Then

$$q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \dots + \lambda_n c_n^2,$$

where the  $c_i$  are the coordinates of  $\vec{x}$  with respect to  $\mathfrak{B}$ .<sup>2</sup>

Again, note that we have been able to get rid of the mixed terms: No summand involves  $c_i c_j$  (with  $i \neq j$ ) in the preceding formula. To justify the formula stated in Theorem 2.2, we can proceed as in Example 1. We leave the details as an exercise.

When we study a quadratic form  $q$ , we are often interested in finding out whether  $q(\vec{x}) > 0$  for all nonzero  $\vec{x}$  (as in Example 1). In this context, it is useful to introduce the following terminology:

### Definition 2.3

#### Definiteness of a quadratic form

Consider a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ , where  $A$  is a symmetric  $n \times n$  matrix.

We say that  $A$  is *positive definite* if  $q(\vec{x})$  is positive for all nonzero  $\vec{x}$  in  $\mathbb{R}^n$ , and we call  $A$  *positive semidefinite* if  $q(\vec{x}) \geq 0$ , for all  $\vec{x}$  in  $\mathbb{R}^n$ .

Negative definite and negative semidefinite symmetric matrices are defined analogously.

Finally, we call  $A$  *indefinite* if  $q$  takes positive as well as negative values.

### EXAMPLE 3

Consider an  $n \times m$  matrix  $A$ . Show that the function  $q(\vec{x}) = \|A\vec{x}\|^2$  is a quadratic form, find its matrix, and determine its definiteness.

### Solution

We can write  $q(\vec{x}) = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T (A\vec{x}) = \vec{x}^T A^T A \vec{x} = \vec{x} \cdot (A^T A \vec{x})$ . This shows that  $q$  is a quadratic form, with matrix  $A^T A$ . This quadratic form is positive semidefinite, because  $q(\vec{x}) = \|A\vec{x}\|^2 \geq 0$  for all vectors  $\vec{x}$  in  $\mathbb{R}^m$ . Note that  $q(\vec{x}) = 0$  if and only if  $\vec{x}$  is in the kernel of  $A$ . Therefore, the quadratic form is positive definite if and only if  $\ker(A) = \{\vec{0}\}$ . ■

---

<sup>2</sup>The basic properties of quadratic forms were first derived by the Dutchman Johan de Witt (1625–1672) in his *Elementa curvarum linearum*. De Witt was one of the leading statesmen of his time, guiding his country through two wars against England. He consolidated his nation's commercial and naval power. De Witt met an unfortunate end when he was literally torn to pieces by an angry mob. (He should have stayed with math!)

## Symmetric Matrices and Quadratic Forms

By Theorem 2.2, the definiteness of a symmetric matrix  $A$  is easy to determine from its eigenvalues:

### Theorem 2.4

#### Eigenvalues and definiteness

A symmetric matrix  $A$  is *positive definite* if (and only if) all of its eigenvalues are positive. The matrix  $A$  is *positive semidefinite* if (and only if) all of its eigenvalues are positive or zero.

These facts follow immediately from the formula

$$q(\vec{x}) = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2. \quad \text{See Theorem 2.2.}$$

The determinant of a positive definite matrix is positive, since the determinant is the product of the eigenvalues. The converse is not true, however: Consider a symmetric  $3 \times 3$  matrix  $A$  with one positive and two negative eigenvalues. Then  $\det A$  is positive, but  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$  is indefinite. In practice, the following criterion for positive definiteness is often used (a proof is outlined in Exercise 34):

### Theorem 2.5

#### Principal submatrices and definiteness

Consider a symmetric  $n \times n$  matrix  $A$ . For  $m = 1, \dots, n$ , let  $A^{(m)}$  be the  $m \times m$  matrix obtained by omitting all rows and columns of  $A$  past the  $m$ th. These matrices  $A^{(m)}$  are called the *principal submatrices* of  $A$ .

The matrix  $A$  is positive definite if (and only if)  $\det(A^{(m)}) > 0$ , for all  $m = 1, \dots, n$ .

Consider the matrix

$$A = \begin{bmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{bmatrix}$$

from Example 2:

$$\begin{aligned} \det(A^{(1)}) &= \det[9] = 9 > 0 \\ \det(A^{(2)}) &= \det \begin{bmatrix} 9 & -1 \\ -1 & 7 \end{bmatrix} = 62 > 0 \\ \det(A^{(3)}) &= \det(A) = 89 > 0. \end{aligned}$$

We can conclude that  $A$  is positive definite.

Alternatively, we could find the eigenvalues of  $A$  and use Theorem 2.4. Using technology, we find that  $\lambda_1 \approx 10.7$ ,  $\lambda_2 \approx 7.1$ , and  $\lambda_3 \approx 1.2$ , confirming our result.

## Principal Axes

When we study a function  $f(x_1, x_2, \dots, x_n)$  from  $\mathbb{R}^n$  to  $\mathbb{R}$ , we are often interested in the solutions of the equations

$$f(x_1, x_2, \dots, x_n) = k,$$

for a fixed  $k$  in  $\mathbb{R}$ , called the *level sets* of  $f$  (*level curves* for  $n = 2$ , *level surfaces* for  $n = 3$ ).

Here we will think about the level curves of a quadratic form  $q(x_1, x_2)$  of two variables. For simplicity, we focus on the level curve  $q(x_1, x_2) = 1$ .

## Symmetric Matrices and Quadratic Forms

Let us first consider the case when there is no mixed term in the formula. We trust that you had at least a brief encounter with those level curves in a previous course. Let us discuss the two major cases:

**Case 1:**  $q(x_1, x_2) = ax_1^2 + bx_2^2 = 1$ , where  $b > a > 0$ . This curve is an *ellipse*, as shown in Figure 2. The lengths of the semimajor and the semiminor axes are  $1/\sqrt{a}$  and  $1/\sqrt{b}$ , respectively. This ellipse can be parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cos t \begin{bmatrix} 1/\sqrt{a} \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1/\sqrt{b} \end{bmatrix}.$$

**Case 2:**  $q(x_1, x_2) = ax_1^2 + bx_2^2 = 1$ , where  $a$  is positive and  $b$  negative. This is a hyperbola, with  $x_1$ -intercepts  $\begin{bmatrix} \pm 1/\sqrt{|a|} \\ 0 \end{bmatrix}$ , as shown in Figure 3. What are the slopes of the asymptotes, in terms of  $a$  and  $b$ ?

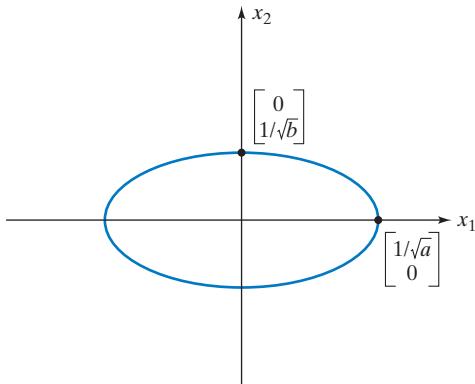


Figure 2

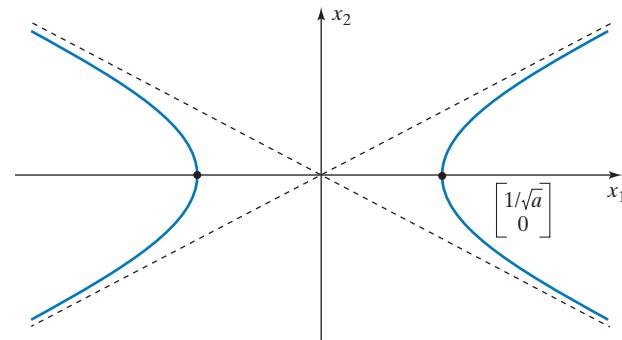


Figure 3

Now consider the level curve

$$q(\vec{x}) = \vec{x} \cdot A\vec{x} = 1,$$

where  $A$  is an invertible symmetric  $2 \times 2$  matrix. By Theorem 2.2, we can write this equation as

$$\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1,$$

where  $c_1, c_2$  are the coordinates of  $\vec{x}$  with respect to an orthonormal eigenbasis for  $A$ , and  $\lambda_1, \lambda_2$  are the associated eigenvalues.

This curve is an ellipse if both eigenvalues are positive and a hyperbola if one eigenvalue is positive and one negative. (What happens when both eigenvalues are negative?)

**EXAMPLE 4** Sketch the curve

$$8x_1^2 - 4x_1x_2 + 5x_2^2 = 1.$$

See Example 1.

**Solution**

In Example 1, we found that we can write this equation as

$$9c_1^2 + 4c_2^2 = 1,$$

where  $c_1, c_2$  are the coordinates of  $\vec{x}$  with respect to the orthonormal eigenbasis

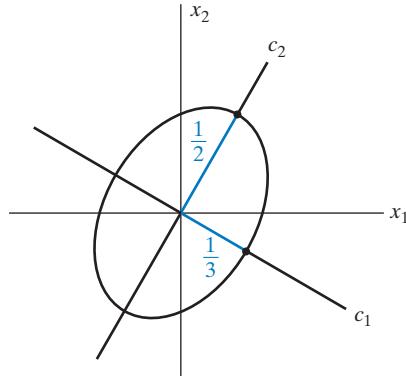
$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

for  $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ . We sketch this ellipse in Figure 4.

The  $c_1$ - and  $c_2$ -axes are called the *principal axes* of the quadratic form  $q(x_1, x_2) = 8x_1^2 - 4x_1x_2 + 5x_2^2$ . Note that these are the eigenspaces of the matrix

$$A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

of the quadratic form. ■



**Figure 4**

**Definition 2.6**
**Principal axes**

Consider a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ , where  $A$  is a symmetric  $n \times n$  matrix with  $n$  distinct eigenvalues. Then the eigenspaces of  $A$  are called the *principal axes* of  $q$ . (Note that these eigenspaces will be one-dimensional.)

Let's return to the case of a quadratic form of two variables. We can summarize our findings as follows:

**Theorem 2.7**
**Ellipses and hyperbolas**

Consider the curve  $C$  in  $\mathbb{R}^2$  defined by

$$q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 = 1.$$

Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of the matrix  $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$  of  $q$ .

If both  $\lambda_1$  and  $\lambda_2$  are positive, then  $C$  is an *ellipse*. If one eigenvalue is positive and the other is negative, then  $C$  is a *hyperbola*.

## EXERCISES 2

**GOAL** Apply the concept of a quadratic form. Use an orthonormal eigenbasis for  $A$  to analyze the quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ .

For each of the quadratic forms  $q$  listed in Exercises 1 through 3, find the matrix of  $q$ .

1.  $q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$

2.  $q(x_1, x_2) = x_1x_2$

3.  $q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$

Determine the definiteness of the quadratic forms in Exercises 4 through 7.

4.  $q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2$

5.  $q(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2$

6.  $q(x_1, x_2) = 2x_1^2 + 6x_1x_2 + 4x_2^2$

7.  $q(x_1, x_2, x_3) = 3x_2^2 + 4x_1x_3$

8. If  $A$  is a symmetric matrix, what can you say about the definiteness of  $A^2$ ? When is  $A^2$  positive definite?

9. Recall that a real square matrix  $A$  is called *skew symmetric* if  $A^T = -A$ .

a. If  $A$  is skew symmetric, is  $A^2$  skew symmetric as well? Or is  $A^2$  symmetric?

b. If  $A$  is skew symmetric, what can you say about the definiteness of  $A^2$ ? What about the eigenvalues of  $A^2$ ?

c. What can you say about the complex eigenvalues of a skew-symmetric matrix? Which skew-symmetric matrices are diagonalizable over  $\mathbb{R}$ ?

10. Consider a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$  on  $\mathbb{R}^n$  and a fixed vector  $\vec{v}$  in  $\mathbb{R}^n$ . Is the transformation

$$L(\vec{x}) = q(\vec{x} + \vec{v}) - q(\vec{x}) - q(\vec{v})$$

linear? If so, what is its matrix?

11. If  $A$  is an invertible symmetric matrix, what is the relationship between the definiteness of  $A$  and  $A^{-1}$ ?

12. Show that a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$  of two variables is indefinite if (and only if)  $\det A < 0$ . Here,  $A$  is a symmetric  $2 \times 2$  matrix.

13. Show that the diagonal elements of a positive definite matrix  $A$  are positive.

14. Consider a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ , where  $a$  and  $\det A$  are both positive. Without using Theorem 2.5, show that  $A$  is positive definite. Hint: Show first that  $c$  is positive, and thus  $\text{tr } A$  is positive. Then think about the signs of the eigenvalues.

Sketch the curves defined in Exercises 15 through 20. In each case, draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

15.  $6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$  16.  $x_1x_2 = 1$

17.  $3x_1^2 + 4x_1x_2 = 1$  18.  $9x_1^2 - 4x_1x_2 + 6x_2^2 = 1$

19.  $x_1^2 + 4x_1x_2 + 4x_2^2 = 1$  20.  $-3x_1^2 + 6x_1x_2 + 5x_2^2 = 1$

21. a. Sketch the following three surfaces:

$$x_1^2 + 4x_2^2 + 9x_3^2 = 1$$

$$x_1^2 + 4x_2^2 - 9x_3^2 = 1$$

$$-x_1^2 - 4x_2^2 + 9x_3^2 = 1.$$

Which of these are bounded? Which are connected? Label the points closest to and farthest from the origin (if there are any).

b. Consider the surface

$$x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + 2x_1x_3 + 3x_2x_3 = 1.$$

Which of the three surfaces in part (a) does this surface qualitatively resemble most? Which points on this surface are closest to the origin? Give a rough approximation; you may use technology.

22. On the surface

$$-x_1^2 + x_2^2 - x_3^2 + 10x_1x_3 = 1,$$

find the two points closest to the origin.

23. Consider an  $n \times n$  matrix  $M$  that is not symmetric, and define the function  $g(\vec{x}) = \vec{x} \cdot M\vec{x}$  from  $\mathbb{R}^n$  to  $\mathbb{R}$ . Is  $g$  necessarily a quadratic form? If so, find the matrix of  $g$ .

24. Consider a quadratic form

$$q(\vec{x}) = \vec{x} \cdot A\vec{x},$$

where  $A$  is a symmetric  $n \times n$  matrix. Find  $q(\vec{e}_1)$ . Give your answer in terms of the entries of the matrix  $A$ .

25. Consider a quadratic form

$$q(\vec{x}) = \vec{x} \cdot A\vec{x},$$

where  $A$  is a symmetric  $n \times n$  matrix. Let  $\vec{v}$  be a unit eigenvector of  $A$ , with associated eigenvalue  $\lambda$ . Find  $q(\vec{v})$ .

26. Consider a quadratic form

$$q(\vec{x}) = \vec{x} \cdot A\vec{x},$$

where  $A$  is a symmetric  $n \times n$  matrix. True or false? If there exists a nonzero vector  $\vec{v}$  in  $\mathbb{R}^n$  such that  $q(\vec{v}) = 0$ , then  $A$  fails to be invertible.

## Symmetric Matrices and Quadratic Forms

27. Consider a quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ , where  $A$  is a symmetric  $n \times n$  matrix with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Let  $S^{n-1}$  be the set of all unit vectors in  $\mathbb{R}^n$ . Describe the image of  $S^{n-1}$  under  $q$ , in terms of the eigenvalues of  $A$ .

28. Show that any positive definite  $n \times n$  matrix  $A$  can be written as  $A = BB^T$ , where  $B$  is an  $n \times n$  matrix with orthogonal columns. *Hint:* There exists an orthogonal matrix  $S$  such that  $S^{-1}AS = S^TAS = D$  is a diagonal matrix with positive diagonal entries. Then  $A = SDS^T$ . Now write  $D$  as the square of a diagonal matrix.

29. For the matrix  $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ , write  $A = BB^T$  as discussed in Exercise 28. See Example 1.

30. Show that any positive definite matrix  $A$  can be written as  $A = B^2$ , where  $B$  is a positive definite matrix.

31. For the matrix  $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ , write  $A = B^2$  as discussed in Exercise 30. See Example 1.

32. *Cholesky factorization for  $2 \times 2$  matrices.* Show that any positive definite  $2 \times 2$  matrix  $A$  can be written uniquely as  $A = LL^T$ , where  $L$  is a lower triangular  $2 \times 2$  matrix with positive entries on the diagonal. *Hint:* Solve the equation

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}.$$

33. Find the Cholesky factorization (discussed in Exercise 32) for

$$A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}.$$

34. A Cholesky factorization of a symmetric matrix  $A$  is a factorization of the form  $A = LL^T$ , where  $L$  is lower triangular with positive diagonal entries.

Show that for a symmetric  $n \times n$  matrix  $A$ , the following are equivalent:

- (i)  $A$  is positive definite.
- (ii) All principal submatrices  $A^{(m)}$  of  $A$  are positive definite. See Theorem 2.5.
- (iii)  $\det(A^{(m)}) > 0$  for  $m = 1, \dots, n$ .
- (iv)  $A$  has a Cholesky factorization  $A = LL^T$ .

*Hint:* Show that (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and (iv) implies (i). The hardest step is the implication from (iii) to (iv): Arguing by induction on  $n$ , you may assume that  $A^{(n-1)}$  has a Cholesky factorization  $A^{(n-1)} = BB^T$ . Now show that there exist a vector  $\vec{x}$  in  $\mathbb{R}^{n-1}$  and a scalar  $t$  such that

$$A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{v}^T & k \end{bmatrix} = \begin{bmatrix} B & 0 \\ \vec{x}^T & 1 \end{bmatrix} \begin{bmatrix} B^T & \vec{x} \\ 0 & t \end{bmatrix}.$$

Explain why the scalar  $t$  is positive. Therefore, we have the Cholesky factorization

$$A = \begin{bmatrix} B & 0 \\ \vec{x}^T & \sqrt{t} \end{bmatrix} \begin{bmatrix} B^T & \vec{x} \\ 0 & \sqrt{t} \end{bmatrix}.$$

This reasoning also shows that the Cholesky factorization of  $A$  is unique. Alternatively, you can use the  $LDL^T$  factorization of  $A$  to show that (iii) implies (iv).

To show that (i) implies (ii), consider a nonzero vector  $\vec{x}$  in  $\mathbb{R}^m$ , and define

$$\vec{y} = \begin{bmatrix} \vec{x} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in  $\mathbb{R}^n$  (fill in  $n - m$  zeros). Then

$$\vec{x}^T A^{(m)} \vec{x} = \vec{y}^T A \vec{y} > 0.$$

35. Find the Cholesky factorization of the matrix

$$A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 13 & 1 \\ 8 & 1 & 26 \end{bmatrix}.$$

36. Consider an invertible  $n \times n$  matrix  $A$ . What is the relationship between the matrix  $R$  in the  $QR$  factorization of  $A$  and the matrix  $L$  in the Cholesky factorization of  $A^T A$ ?

37. Consider the quadratic form

$$q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2.$$

We define

$$q_{11} = \frac{\partial^2 q}{\partial x_1^2}, \quad q_{12} = q_{21} = \frac{\partial^2 q}{\partial x_1 \partial x_2}, \quad q_{22} = \frac{\partial^2 q}{\partial x_2^2}.$$

The discriminant  $D$  of  $q$  is defined as

$$D = \det \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = q_{11}q_{22} - (q_{12})^2.$$

The *second derivative test* tells us that if  $D$  and  $q_{11}$  are both positive, then  $q(x_1, x_2)$  has a minimum at  $(0, 0)$ . Justify this fact, using the theory developed in this section.

38. For which values of the constants  $p$  and  $q$  is the  $n \times n$  matrix

$$B = \begin{bmatrix} p & q & \cdots & q \\ q & p & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & p \end{bmatrix}$$

positive definite? ( $B$  has  $p$ 's on the diagonal and  $q$ 's elsewhere.) *Hint:* Exercise 1.17 is helpful.

39. For which angles  $\theta$  can you find a basis of  $\mathbb{R}^n$  such that the angle between any two vectors in this basis is  $\theta$ ?

## Symmetric Matrices and Quadratic Forms

- 40.** Show that for every symmetric  $n \times n$  matrix  $A$  there exists a constant  $k$  such that matrix  $A + kI_n$  is positive definite.
- 41.** Find the dimension of the space  $Q_2$  of all quadratic forms in two variables.
- 42.** Find the dimension of the space  $Q_n$  of all quadratic forms in  $n$  variables.
- 43.** Consider the transformation  $T(q(x_1, x_2)) = q(x_1, 0)$  from  $Q_2$  to  $P_2$ . Is  $T$  a linear transformation? If so, find the image, rank, kernel, and nullity of  $T$ .
- 44.** Consider the transformation  $T(q(x_1, x_2)) = q(x_1, 1)$  from  $Q_2$  to  $P_2$ . Is  $T$  a linear transformation? Is it an isomorphism?
- 45.** Consider the transformation  $T(q(x_1, x_2, x_3)) = q(x_1, 1, 1)$  from  $Q_3$  to  $P_2$ . Is  $T$  a linear transformation? If so, find the image, rank, kernel, and nullity of  $T$ .
- 46.** Consider the linear transformation  $T(q(x_1, x_2, x_3)) = q(x_1, x_2, x_1)$  from  $Q_3$  to  $Q_2$ . Find the image, kernel, rank, and nullity of this transformation.
- 47.** Consider the function  $T(A)(\vec{x}) = \vec{x}^T A \vec{x}$  from  $\mathbb{R}^{n \times n}$  to  $Q_n$ . Show that  $T$  is a linear transformation. Find the image, kernel, rank, and nullity of  $T$ .
- 48.** Consider the linear transformation  $T(q(x_1, x_2)) = q(x_2, x_1)$  from  $Q_2$  to  $Q_2$ . Find all the eigenvalues and eigenfunctions of  $T$ . Is transformation  $T$  diagonalizable?
- 49.** Consider the linear transformation  $T(q(x_1, x_2)) = q(x_1, 2x_2)$  from  $Q_2$  to  $Q_2$ . Find all the eigenvalues and eigenfunctions of  $T$ . Is transformation  $T$  diagonalizable?
- 50.** Consider the linear transformation
- $$T(q(x_1, x_2)) = x_1 \frac{\partial q}{\partial x_2} + x_2 \frac{\partial q}{\partial x_1}$$
- from  $Q_2$  to  $Q_2$ . Find all the eigenvalues and eigenfunctions of  $T$ . Is transformation  $T$  diagonalizable?
- 51.** What are the signs of the determinants of the principal submatrices of a negative definite matrix? See Theorem 2.5.
- 52.** Consider a quadratic form  $q$ . If  $A$  is a symmetric matrix such that  $q(\vec{x}) = \vec{x}^T A \vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^n$ , show that  $a_{ii} = q(\vec{e}_i)$  and  $a_{ij} = \frac{1}{2}(q(\vec{e}_i + \vec{e}_j) - q(\vec{e}_i) - q(\vec{e}_j))$  for  $i \neq j$ .
- 53.** Consider a quadratic form  $q(x_1, \dots, x_n)$  with symmetric matrix  $A$ . For two integers  $i$  and  $j$  with  $1 \leq i < j \leq n$ , we define the function
- $$p(x, y) = q\left(0, \dots, 0, \underbrace{x}_{ith}, 0, \dots, 0, \underbrace{y}_{jth}, 0, \dots, 0\right).$$
- a.** Show that  $p$  is a quadratic form, with matrix  $\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ .
- b.** If  $q$  is positive definite, show that  $p$  is positive definite as well.
- c.** If  $q$  is positive semidefinite, show that  $p$  is positive semidefinite as well.
- d.** Give an example where  $q$  is indefinite, but  $p$  is positive definite.
- 54.** If  $A$  is a positive semidefinite matrix with  $a_{11} = 0$ , what can you say about the other entries in the first row and in the first column of  $A$ ? Hint: Exercise 53 is helpful.
- 55.** If  $A$  is a positive definite  $n \times n$  matrix, show that the largest entry of  $A$  must be on the diagonal. Hint: Use Exercise 53 to show that  $a_{ij} < a_{ii}$  or  $a_{ij} < a_{jj}$  for all  $1 \leq i < j \leq n$ .
- 56.** If  $A$  is a real symmetric matrix, show that there exists an eigenvalue  $\lambda$  of  $A$  with  $\lambda \geq a_{11}$ . Hint: Exercise 27 is helpful.
- In Exercises 57 through 61, consider a quadratic form  $q$  on  $\mathbb{R}^3$  with symmetric matrix  $A$ , with the given properties. In each case, describe the level surface  $q(\vec{x}) = 1$  geometrically.*
- 57.**  $q$  is positive definite.
- 58.**  $q$  is positive semidefinite and  $\text{rank } A = 2$ .
- 59.**  $q$  is positive semidefinite and  $\text{rank } A = 1$ .
- 60.**  $q$  is indefinite and  $\det A > 0$ .
- 61.**  $q$  is indefinite and  $\det A < 0$ .
- 62.** Consider an indefinite quadratic form  $q$  on  $\mathbb{R}^3$  with symmetric matrix  $A$ . If  $\det A < 0$ , describe the level surface  $q(\vec{x}) = 0$ .
- 63.** Consider a positive definite quadratic form  $q$  on  $\mathbb{R}^n$  with symmetric matrix  $A$ . We know that there exists an orthonormal eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$ , with associated positive eigenvalues  $\lambda_1, \dots, \lambda_n$ . Now consider the orthogonal eigenbasis  $\vec{w}_1, \dots, \vec{w}_n$ , where  $\vec{w}_i = \frac{1}{\sqrt{\lambda_i}}\vec{v}_i$ . Show that  $q(c_1\vec{w}_1 + \dots + c_n\vec{w}_n) = c_1^2 + \dots + c_n^2$ .
- 64.** For the quadratic form  $q(x_1, x_2) = 8x_1^2 - 4x_1x_2 + 5x_2^2$ , find an orthogonal basis  $\vec{w}_1, \vec{w}_2$  of  $\mathbb{R}^2$  such that  $q(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1^2 + c_2^2$ . Use your answer to sketch the level curve  $q(\vec{x}) = 1$ . Compare with Example 4 and Figure 4 in this section. Exercise 63 is helpful.
- 65.** Show that for every indefinite quadratic form  $q$  on  $\mathbb{R}^2$ , there exists an orthogonal basis  $\vec{w}_1, \vec{w}_2$  of  $\mathbb{R}^2$  such that  $q(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1^2 - c_2^2$ . Hint: Modify the approach outlined in Exercise 63.
- 66.** For the quadratic form  $q(x_1, x_2) = 3x_1^2 - 10x_1x_2 + 3x_2^2$ , find an orthogonal basis  $\vec{w}_1, \vec{w}_2$  of  $\mathbb{R}^2$  such that

## Symmetric Matrices and Quadratic Forms

$q(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1^2 - c_2^2$ . Use your answer to sketch the level curve  $q(\vec{x}) = 1$ . Exercise 65 is helpful.

67. Consider a quadratic form  $q$  on  $\mathbb{R}^n$  with symmetric matrix  $A$ , with rank  $A = r$ . Suppose that  $A$  has  $p$  positive eigenvalues, if eigenvalues are counted with their multiplicities. Show that there exists an orthogonal basis  $\vec{w}_1, \dots, \vec{w}_n$  of  $\mathbb{R}^n$  such that  $q(c_1\vec{w}_1 + \dots + c_n\vec{w}_n) = c_1^2 + \dots + c_p^2 - c_{p+1}^2 - \dots - c_r^2$ . Hint: Modify the approach outlined in Exercises 63 and 65.
68. If  $q$  is a quadratic form on  $\mathbb{R}^n$  with symmetric matrix  $A$ , and if  $L(\vec{x}) = R\vec{x}$  is a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , show that the composite function  $p(\vec{x}) = q(L(\vec{x}))$

is a quadratic form on  $\mathbb{R}^m$ . Express the symmetric matrix of  $p$  in terms of  $R$  and  $A$ .

69. If  $A$  is a positive definite  $n \times n$  matrix, and  $R$  is any real  $n \times m$  matrix, what can you say about the definiteness of the matrix  $R^T A R$ ? For which matrices  $R$  is  $R^T A R$  positive definite?
70. If  $A$  is an indefinite  $n \times n$  matrix, and  $R$  is a real  $n \times m$  matrix of rank  $n$ , what can you say about the definiteness of the matrix  $R^T A R$ ?
71. If  $A$  is an indefinite  $n \times n$  matrix, and  $R$  is any real  $n \times m$  matrix, what can you say about the definiteness of the matrix  $R^T A R$ ?

## 3 Singular Values

### EXAMPLE I

Show that if  $L(\vec{x}) = A\vec{x}$  is a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , then there exist two orthogonal unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^2$  such that vectors  $L(\vec{v}_1)$  and  $L(\vec{v}_2)$  are orthogonal as well (although not necessarily unit vectors). See Figure 1. Hint: Consider an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  of the symmetric matrix  $A^T A$ .

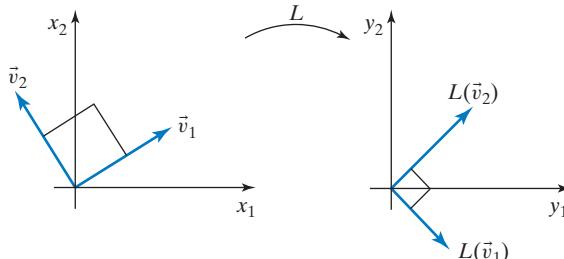


Figure 1

### Solution

This statement is clear for some classes of transformations. For example,

- If  $L$  is an orthogonal transformation, then any two orthogonal unit vectors  $\vec{v}_1$  and  $\vec{v}_2$  will do.
- If  $L(\vec{x}) = A\vec{x}$ , where  $A$  is symmetric, then we can choose two orthogonal unit eigenvectors, by the spectral theorem, Theorem 1.1. See also Example 4 of Section 1.

However, for an arbitrary linear transformation  $L$ , the statement isn't that obvious; think about the case of a shear, for example.

We will present a proof in the spirit of linear algebra that generalizes more easily to higher-dimensional spaces.

Following the hint, we first note that matrix  $A^T A$  is symmetric, since  $(A^T A)^T = A^T (A^T)^T = A^T A$ . The spectral theorem (Theorem 1.1) tells us that there exists an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $A^T A$ , with associated eigenvalues  $\lambda_1, \lambda_2$ . We can verify that vectors  $L(\vec{v}_1) = A\vec{v}_1$  and  $L(\vec{v}_2) = A\vec{v}_2$  are orthogonal, as claimed:

### Symmetric Matrices and Quadratic Forms

$$(A\vec{v}_1) \cdot (A\vec{v}_2) = (A\vec{v}_1)^T A\vec{v}_2 = \vec{v}_1^T A^T A\vec{v}_2 = \vec{v}_1^T (\lambda_2 \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2) = 0.$$

It is worth mentioning that  $\vec{v}_1$  and  $\vec{v}_2$  need not be eigenvectors of matrix  $A$ . ■

**EXAMPLE 2** Consider the linear transformation  $L(\vec{x}) = A\vec{x}$ , where  $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$ .

- a. Find an orthonormal basis  $\vec{v}_1, \vec{v}_2$  of  $\mathbb{R}^2$  such that vectors  $L(\vec{v}_1)$  and  $L(\vec{v}_2)$  are orthogonal.
- b. Show that the image of the unit circle under transformation  $L$  is an ellipse. Find the lengths of the two semiaxes of this ellipse, in terms of the eigenvalues of matrix  $A^T A$ .

#### Solution

- a. Using the ideas of Example 1, we will find an orthonormal eigenbasis for matrix  $A^T A$ :

$$A^T A = \begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} 85 & -30 \\ -30 & 40 \end{bmatrix}.$$

The characteristic polynomial of  $A^T A$  is

$$\lambda^2 - 125\lambda + 2500 = (\lambda - 100)(\lambda - 25),$$

so that the eigenvalues of  $A^T A$  are  $\lambda_1 = 100$  and  $\lambda_2 = 25$ . Now we can find the eigenspaces of  $A^T A$ :

$$E_{100} = \ker \begin{bmatrix} -15 & -30 \\ -30 & -60 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and

$$E_{25} = \ker \begin{bmatrix} 60 & -30 \\ -30 & 15 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To find an orthonormal basis, we need to multiply these vectors by the reciprocals of their lengths:

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- b. The unit circle consists of the vectors of the form  $\vec{x} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$ , and the image of the unit circle consists of the vectors  $L(\vec{x}) = \cos(t)L(\vec{v}_1) + \sin(t)L(\vec{v}_2)$ . This image is the ellipse whose semimajor and semiminor axes are  $L(\vec{v}_1)$  and  $L(\vec{v}_2)$ , respectively. What are the lengths of these axes?

$$\|L(\vec{v}_1)\|^2 = (A\vec{v}_1) \cdot (A\vec{v}_1) = \vec{v}_1^T A^T A \vec{v}_1 = \vec{v}_1^T (\lambda_1 \vec{v}_1) = \lambda_1 (\vec{v}_1 \cdot \vec{v}_1) = \lambda_1$$

Likewise,

$$\|L(\vec{v}_2)\|^2 = \lambda_2.$$

Thus,

$$\|L(\vec{v}_1)\| = \sqrt{\lambda_1} = \sqrt{100} = 10 \quad \text{and} \quad \|L(\vec{v}_2)\| = \sqrt{\lambda_2} = \sqrt{25} = 5.$$

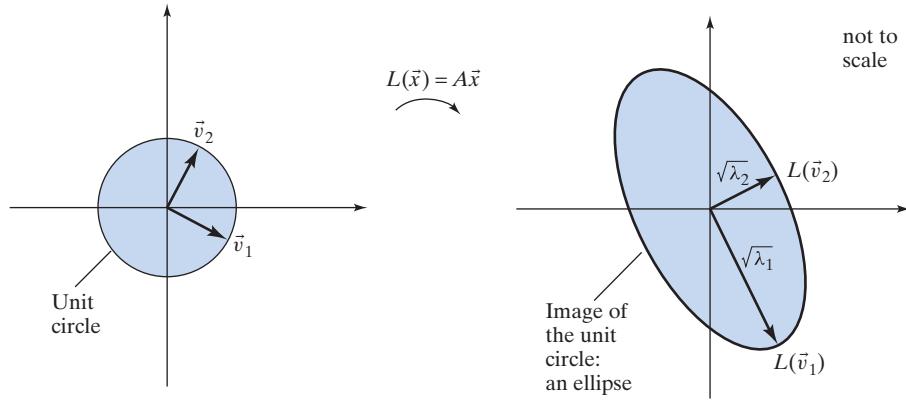
See Figure 2. We can compute the lengths of vectors  $L(\vec{v}_1)$  and  $L(\vec{v}_2)$  directly, of course, but the way we did it before is more informative. For example,

$$L(\vec{v}_1) = A\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 10 \\ -20 \end{bmatrix},$$

so that

$$\|L(\vec{v}_1)\| = \left\| \frac{1}{\sqrt{5}} \begin{bmatrix} 10 \\ -20 \end{bmatrix} \right\| = 10. ■$$

## Symmetric Matrices and Quadratic Forms



**Figure 2**

Part b of Example 2 shows that the square roots of the eigenvalues of matrix  $A^T A$  play an important role in the geometrical interpretation of the transformation  $L(\vec{x}) = A\vec{x}$ . In Example 2.3 we have seen that the symmetric matrix  $A^T A$  is positive semidefinite for any  $n \times m$  matrix  $A$ , meaning that the eigenvalues of  $A^T A$  are positive or zero.

### Definition 3.1

#### Singular values

The *singular values* of an  $n \times m$  matrix  $A$  are the square roots of the eigenvalues of the symmetric  $m \times m$  matrix  $A^T A$ , listed with their algebraic multiplicities. It is customary to denote the singular values by  $\sigma_1, \sigma_2, \dots, \sigma_m$  and to list them in decreasing order:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0.$$

The singular values of the matrix  $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$  considered in Example 2 are  $\sigma_1 = \sqrt{\lambda_1} = 10$  and  $\sigma_2 = \sqrt{\lambda_2} = 5$ , since the eigenvalues of  $A^T A$  are  $\lambda_1 = 100$  and  $\lambda_2 = 25$ .

We can now generalize our work in Example 2.

### Theorem 3.2

#### The image of the unit circle

Let  $L(\vec{x}) = A\vec{x}$  be an invertible linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . The image of the unit circle under  $L$  is an ellipse  $E$ . The lengths of the semimajor and semiminor axes of  $E$  are the singular values  $\sigma_1$  and  $\sigma_2$  of  $A$ , respectively.

Take another look at Figure 2.

We can generalize our findings in Examples 1 and 2 to matrices of arbitrary size.

### Theorem 3.3

Let  $L(\vec{x}) = A\vec{x}$  be a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Then there exists an orthonormal basis  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  of  $\mathbb{R}^m$  such that

- a. Vectors  $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_m)$  are orthogonal, and
- b. The lengths of vectors  $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_m)$  are the singular values  $\sigma_1, \sigma_2, \dots, \sigma_m$  of matrix  $A$ .

**Theorem 3.3**

*(Continued)*

To construct  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ , find an orthonormal eigenbasis for matrix  $A^T A$ . Make sure that the corresponding eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  appear in descending order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0,$$

so that  $\lambda_i = \sigma_i^2$  for  $i = 1, \dots, m$ .

The proof is analogous to the special case  $n = m = 2$  considered in Examples 1 and 2:

- a.  $L(\vec{v}_i) \cdot L(\vec{v}_j) = (A\vec{v}_i) \cdot (A\vec{v}_j) = (A\vec{v}_i)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i^T (\lambda_j \vec{v}_j) = \lambda_j (\vec{v}_i \cdot \vec{v}_j) = 0$  when  $i \neq j$ , and
- b.  $\|L(\vec{v}_i)\|^2 = (A\vec{v}_i) \cdot (A\vec{v}_i) = \vec{v}_i^T A^T A\vec{v}_i = \vec{v}_i^T (\lambda_i \vec{v}_i) = \lambda_i (\vec{v}_i \cdot \vec{v}_i) = \lambda_i = \sigma_i^2$ , so that  $\|L(\vec{v}_i)\| = \sigma_i$ .

**EXAMPLE 3** Consider the linear transformation

$$L(\vec{x}) = A\vec{x}, \quad \text{where } A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- a. Find the singular values of  $A$ .
- b. Find orthonormal vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^3$  such that  $L(\vec{v}_1), L(\vec{v}_2)$ , and  $L(\vec{v}_3)$  are orthogonal.
- c. Sketch and describe the image of the unit sphere under the transformation  $L$ .

**Solution**

$$\text{a. } A^T A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The eigenvalues of  $A^T A$  are  $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$ . The singular values of  $A$  are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = 1, \quad \sigma_3 = \sqrt{\lambda_3} = 0.$$

- b. Find an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  for  $A^T A$  (we omit the details):

$$E_3 = \text{span} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad E_0 = \ker(A^T A) = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We compute  $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$  and check orthogonality:

$$A\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad A\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad A\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can also check that the length of  $A\vec{v}_i$  is  $\sigma_i$ :

$$\|A\vec{v}_1\| = \sqrt{3} = \sigma_1, \quad \|A\vec{v}_2\| = 1 = \sigma_2, \quad \|A\vec{v}_3\| = 0 = \sigma_3.$$

- c. The unit sphere in  $\mathbb{R}^3$  consists of all vectors of the form

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3, \quad \text{where } c_1^2 + c_2^2 + c_3^2 = 1.$$

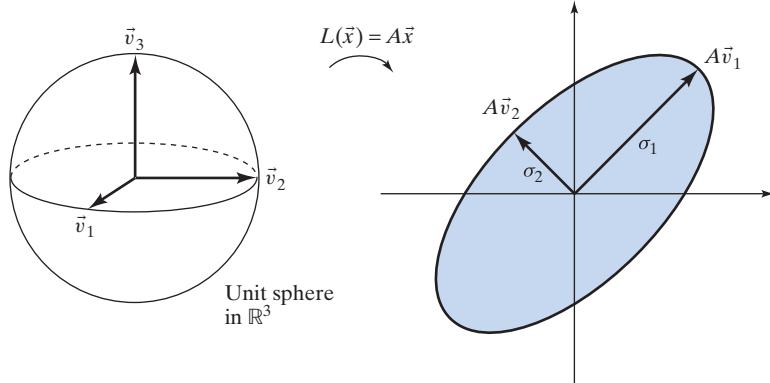
## Symmetric Matrices and Quadratic Forms

The image of the unit sphere consists of the vectors

$$L(\vec{x}) = c_1 L(\vec{v}_1) + c_2 L(\vec{v}_2),$$

where  $c_1^2 + c_2^2 \leq 1$ . [Recall that  $L(\vec{v}_3) = \vec{0}$ .]

The image is the full ellipse shaded in Figure 3. ■



**Figure 3**

Example 3 shows that some of the singular values of a matrix may be zero. Suppose the singular values  $\sigma_1, \dots, \sigma_s$  of an  $n \times m$  matrix  $A$  are nonzero, while  $\sigma_{s+1}, \dots, \sigma_m$  are zero. Choose vectors  $\vec{v}_1, \dots, \vec{v}_s, \vec{v}_{s+1}, \dots, \vec{v}_m$  for  $A$  as introduced in Theorem 3.3. Note that  $\|A\vec{v}_i\| = \sigma_i = 0$  and therefore  $A\vec{v}_i = \vec{0}$  for  $i = s+1, \dots, m$ . We claim that the vectors  $A\vec{v}_1, \dots, A\vec{v}_s$  form a basis of the image of  $A$ . Indeed, these vectors are linearly independent (because they are orthogonal and nonzero), and they span the image, since any vector in the image of  $A$  can be written as

$$\begin{aligned} A\vec{x} &= A(c_1\vec{v}_1 + \dots + c_s\vec{v}_s + \dots + c_m\vec{v}_m) \\ &= c_1A\vec{v}_1 + \dots + c_sA\vec{v}_s. \end{aligned}$$

This shows that  $s = \dim(\text{im } A) = \text{rank } A$ .

### Theorem 3.4

#### Singular values and rank

If  $A$  is an  $n \times m$  matrix of rank  $r$ , then the singular values  $\sigma_1, \dots, \sigma_r$  are nonzero, while  $\sigma_{r+1}, \dots, \sigma_m$  are zero.

## The Singular Value Decomposition

Just as we expressed the Gram–Schmidt process in terms of a matrix decomposition (the  $QR$ -factorization), we will now express Theorem 3.3 in terms of a matrix decomposition.

Consider a linear transformation  $L(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , and choose an orthonormal basis  $\vec{v}_1, \dots, \vec{v}_m$  of  $\mathbb{R}^m$  as in Theorem 3.3. Let  $r = \text{rank } A$ . We know that the vectors  $A\vec{v}_1, \dots, A\vec{v}_r$  are orthogonal and nonzero, with  $\|A\vec{v}_i\| = \sigma_i$ . We introduce the unit vectors

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1, \dots, \vec{u}_r = \frac{1}{\sigma_r}A\vec{v}_r.$$

### Symmetric Matrices and Quadratic Forms

We can expand the sequence  $\vec{u}_1, \dots, \vec{u}_r$  to an orthonormal basis  $\vec{u}_1, \dots, \vec{u}_n$  of  $\mathbb{R}^n$ . Then we can write

$$A\vec{v}_i = \sigma_i \vec{u}_i \quad \text{for } i = 1, \dots, r$$

and

$$A\vec{v}_i = \vec{0} \quad \text{for } i = r+1, \dots, m.$$

We can express these equations in matrix form as follows:

$$\begin{aligned} A \underbrace{\begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \dots & \vec{v}_r & \vec{v}_{r+1} & \dots & \vec{v}_m \\ | & | & | & | \end{bmatrix}}_V &= \begin{bmatrix} | & | & | & | \\ \sigma_1 \vec{u}_1 & \dots & \sigma_r \vec{u}_r & \vec{0} & \dots & \vec{0} \\ | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \dots & \vec{u}_r & \vec{0} & \dots & \vec{0} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & 0 \\ & & \sigma_r & & \\ & & & 0 & 0 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \dots & \vec{u}_r & \vec{u}_{r+1} & \dots & \vec{u}_n \\ | & | & | & | \end{bmatrix}}_U \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & 0 \\ & & \sigma_r & & \\ & & & 0 & 0 \end{bmatrix}}_{\Sigma}, \end{aligned}$$

or, more succinctly,

$$AV = U\Sigma.$$

Note that  $V$  is an orthogonal  $m \times m$  matrix;  $U$  is an orthogonal  $n \times n$  matrix; and  $\Sigma$  is an  $n \times m$  matrix whose first  $r$  diagonal entries are  $\sigma_1, \dots, \sigma_r$ , while all other entries are zero.

Multiplying the equation  $AV = U\Sigma$  with  $V^T$  from the right, we find that  $A = U\Sigma V^T$ .

### Theorem 3.5

#### Singular value decomposition (SVD)

Any  $n \times m$  matrix  $A$  can be written as

$$A = U\Sigma V^T,$$

where  $U$  is an orthogonal  $n \times n$  matrix;  $V$  is an orthogonal  $m \times m$  matrix; and  $\Sigma$  is an  $n \times m$  matrix whose first  $r$  diagonal entries are the nonzero singular values  $\sigma_1, \dots, \sigma_r$  of  $A$ , while all other entries are zero (where  $r = \text{rank } A$ ).

Alternatively, this singular value decomposition can be written as

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T,$$

where the  $\vec{u}_i$  and the  $\vec{v}_i$  are the columns of  $U$  and  $V$ , respectively. See Exercise 29.

## Symmetric Matrices and Quadratic Forms

A singular value decomposition of a  $2 \times 2$  matrix  $A$  is presented in Figure 4.

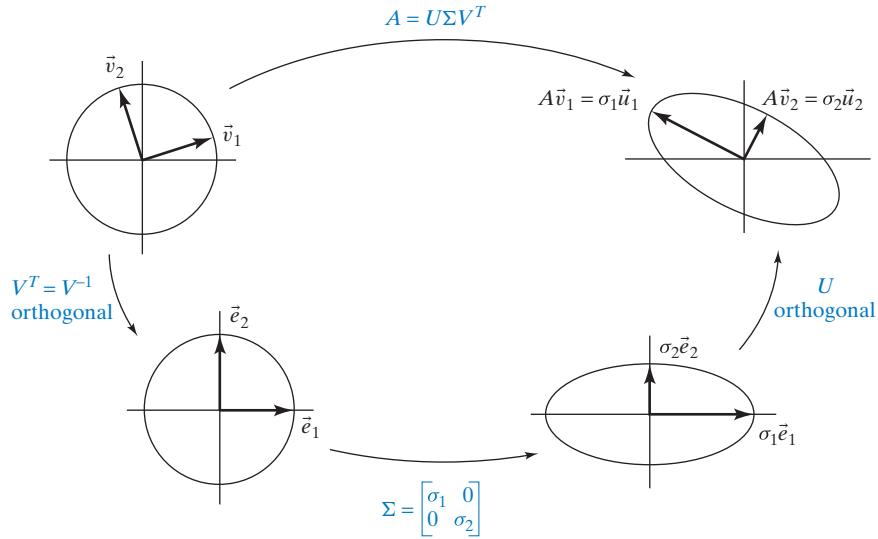


Figure 4

Here are two numerical examples.

**EXAMPLE 4** Find an SVD for  $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$ . Compare with Example 2.

### Solution

In Example 2, we found  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  and  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , so that

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

The columns  $\vec{u}_1$  and  $\vec{u}_2$  of  $U$  are defined as

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and therefore

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Finally,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

You can check that

$$A = U\Sigma V^T.$$

■

### Symmetric Matrices and Quadratic Forms

**EXAMPLE 5** Find an SVD for  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ . Compare with Example 3.

#### Solution

Using our work in Example 3, we find that

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Check that  $A = U\Sigma V^T$ . ■

Consider a singular value decomposition

$$A = U\Sigma V^T,$$

where

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_n \\ | & & | \end{bmatrix}.$$

We know that

$$A\vec{v}_i = \sigma_i \vec{u}_i \quad \text{for } i = 1, \dots, r$$

and

$$A\vec{v}_i = \vec{0} \quad \text{for } i = r+1, \dots, m.$$

These equations tell us that

$$\ker A = \text{span}(\vec{v}_{r+1}, \dots, \vec{v}_m)$$

and

$$\text{im } A = \text{span}(\vec{u}_1, \dots, \vec{u}_r).$$

(Fill in the details.) We see that an SVD provides us with orthonormal bases for the kernel and image of  $A$ .

Likewise, we have

$$A^T = V\Sigma^T U^T \quad \text{or} \quad A^T U = V\Sigma^T.$$

Reading the last equation column by column, we find that

$$A^T \vec{u}_i = \sigma_i \vec{v}_i \quad \text{for } i = 1, \dots, r,$$

and

$$A^T \vec{u}_i = \vec{0} \quad \text{for } i = r+1, \dots, n.$$

(Observe that the roles of vectors  $\vec{u}_i$  and  $\vec{v}_i$  are reversed.)

As before, we have

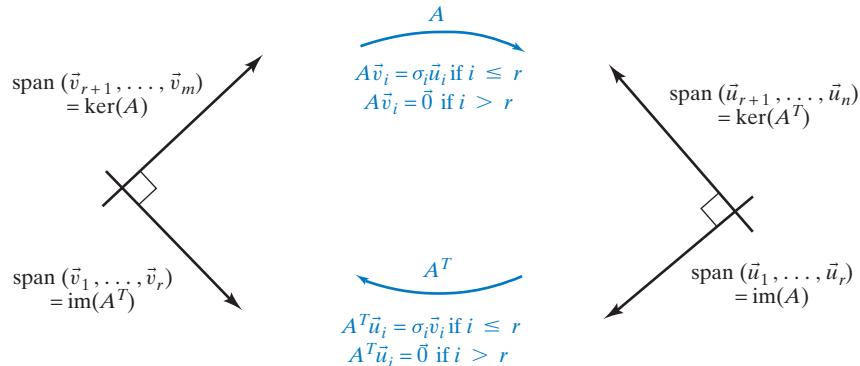
$$\text{im}(A^T) = \text{span}(\vec{v}_1, \dots, \vec{v}_r)$$

## Symmetric Matrices and Quadratic Forms

and

$$\ker(A^T) = \text{span}(\vec{u}_{r+1}, \dots, \vec{u}_n).$$

In Figure 5, we make an attempt to visualize these observations. We represent each of the kernels and images simply as a line.



**Figure 5**

Note that  $\text{im}(A)$  and  $\ker(A^T)$  are orthogonal complements.

We conclude this section with a brief discussion of one of the many applications of the SVD—an application to data compression. We follow the exposition of Gilbert Strang (*Linear Algebra and Its Applications*, 4th ed., Brooks Cole, 2005).

Suppose a satellite transmits a picture containing  $1000 \times 1000$  pixels. If the color of each pixel is digitized, this information can be represented in a  $1000 \times 1000$  matrix  $A$ . How can we transmit the essential information contained in this picture without sending all 1,000,000 numbers?

Suppose we know an SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T.$$

Even if the rank  $r$  of the matrix  $A$  is large, most of the singular values will typically be very small (relative to  $\sigma_1$ ). If we neglect those, we get a good approximation  $A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_s \vec{u}_s \vec{v}_s^T$ , where  $s$  is much smaller than  $r$ . For example, if we choose  $s = 10$ , we need to transmit only the 20 vectors  $\sigma_1 \vec{u}_1, \dots, \sigma_{10} \vec{u}_{10}$  and  $\vec{v}_1, \dots, \vec{v}_{10}$  in  $\mathbb{R}^{1000}$ , that is, 20,000 numbers.

## EXERCISES 3

**GOAL** Find the singular values and a singular value decomposition of a matrix. Interpret the singular values of a  $2 \times 2$  matrix in terms of the image of the unit circle.

1. Find the singular values of  $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ .
2. Let  $A$  be an orthogonal  $2 \times 2$  matrix. Use the image of the unit circle to find the singular values of  $A$ .
3. Let  $A$  be an orthogonal  $n \times n$  matrix. Find the singular values of  $A$  algebraically.
4. Find the singular values of  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

5. Find the singular values of  $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ . Explain your answer geometrically.

6. Find the singular values of  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ . Find a unit vector  $\vec{v}_1$  such that  $\|A\vec{v}_1\| = \sigma_1$ . Sketch the image of the unit circle.

*Find singular value decompositions for the matrices listed in Exercises 7 through 14. Work with paper and pencil. In each case, draw a sketch analogous to Figure 4 in the text, showing the effect of the transformation on the unit circle, in three steps.*

## Symmetric Matrices and Quadratic Forms

7.  $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$     8.  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$     9.  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

10.  $\begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix}$  (See Example 4.)

11.  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$     12.  $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$  (See Example 5.)

13.  $\begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix}$     14.  $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

15. If  $A$  is an invertible  $2 \times 2$  matrix, what is the relationship between the singular values of  $A$  and  $A^{-1}$ ? Justify your answer in terms of the image of the unit circle.

16. If  $A$  is an invertible  $n \times n$  matrix, what is the relationship between the singular values of  $A$  and  $A^{-1}$ ?

17. Consider an  $n \times m$  matrix  $A$  with  $\text{rank}(A) = m$ , and a singular value decomposition  $A = U\Sigma V^T$ . Show that the least-squares solution of a linear system  $A\vec{x} = \vec{b}$  can be written as

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1} \vec{v}_1 + \cdots + \frac{\vec{b} \cdot \vec{u}_m}{\sigma_m} \vec{v}_m.$$

18. Consider the  $4 \times 2$  matrix

$$A = \frac{1}{10} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Use the result of Exercise 17 to find the least-squares solution of the linear system

$$A\vec{x} = \vec{b}, \quad \text{where } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Work with paper and pencil.

19. Consider an  $n \times m$  matrix  $A$  of rank  $r$ , and a singular value decomposition  $A = U\Sigma V^T$ . Explain how you can express the least-squares solutions of a system  $A\vec{x} = \vec{b}$  as linear combinations of the columns  $\vec{v}_1, \dots, \vec{v}_m$  of  $V$ .

20. a. Explain how any square matrix  $A$  can be written as

$$A = QS,$$

where  $Q$  is orthogonal and  $S$  is symmetric positive semidefinite. This is called the *polar decomposition* of  $A$ . Hint: Write  $A = U\Sigma V^T = UV^T V\Sigma V^T$ .

b. Is it possible to write  $A = S_1 Q_1$ , where  $Q_1$  is orthogonal and  $S_1$  is symmetric positive semidefinite?

21. Find a polar decomposition  $A = QS$  as discussed in Exercise 20 for  $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$ . Draw a sketch showing

$S(C)$  and  $A(C) = Q(S(C))$ , where  $C$  is the unit circle centered at the origin. Compare with Examples 2 and 4 and with Figure 4.

22. Consider the standard matrix  $A$  representing the linear transformation

$$T(\vec{x}) = \vec{v} \times \vec{x} \quad \text{from } \mathbb{R}^3 \text{ to } \mathbb{R}^3,$$

where  $\vec{v}$  is a given nonzero vector in  $\mathbb{R}^3$ .

a. Use the geometrical interpretation of the cross product to find an orthogonal projection  $T_1$  onto a plane, a scaling  $T_2$ , and a rotation  $T_3$  about a line such that  $T(\vec{x}) = T_3(T_2(T_1(\vec{x})))$ , for all  $\vec{x}$  in  $\mathbb{R}^3$ . Describe the transformations  $T_1$ ,  $T_2$ , and  $T_3$  as precisely as you can: For  $T_1$  give the plane onto which we project, for  $T_2$  find the scaling factor, and for  $T_3$  give the line about which we rotate and the angle of rotation. All of these answers, except for the angle of rotation, will be in terms of the given vector  $\vec{v}$ . Now let  $A_1$ ,  $A_2$ , and  $A_3$  be the standard matrices of these transformations  $T_1$ ,  $T_2$ , and  $T_3$ , respectively. (You are not asked to find these matrices.) Explain how you can use the factorization  $A = A_3 A_2 A_1$  to write a polar decomposition  $A = QS$  of  $A$ . Express the matrices  $Q$  and  $S$  in terms of  $A_1$ ,  $A_2$ , and  $A_3$ . See Exercise 20.

b. Find the  $A_3 A_2 A_1$  and  $QS$  factorizations discussed in part (a) in the case

$$\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

23. Consider an SVD

$$A = U\Sigma V^T$$

of an  $n \times m$  matrix  $A$ . Show that the columns of  $U$  form an orthonormal eigenbasis for  $AA^T$ . What are the associated eigenvalues? What does your answer tell you about the relationship between the eigenvalues of  $A^T A$  and  $AA^T$ ? Compare this with Exercise 7.4.57.

24. If  $A$  is a symmetric  $n \times n$  matrix, what is the relationship between the eigenvalues of  $A$  and the singular values of  $A$ ?

25. Let  $A$  be a  $2 \times 2$  matrix and  $\vec{u}$  a unit vector in  $\mathbb{R}^2$ . Show that

$$\sigma_2 \leq \|A\vec{u}\| \leq \sigma_1,$$

where  $\sigma_1$ ,  $\sigma_2$  are the singular values of  $A$ . Illustrate this inequality with a sketch, and justify it algebraically.

26. Let  $A$  be an  $n \times m$  matrix and  $\vec{v}$  a vector in  $\mathbb{R}^m$ . Show that

$$\sigma_m \|\vec{v}\| \leq \|A\vec{v}\| \leq \sigma_1 \|\vec{v}\|,$$

where  $\sigma_1$  and  $\sigma_m$  are the largest and the smallest singular values of  $A$ , respectively. Compare this with Exercise 25.

## Symmetric Matrices and Quadratic Forms

27. Let  $\lambda$  be a real eigenvalue of an  $n \times n$  matrix  $A$ . Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1,$$

where  $\sigma_1$  and  $\sigma_n$  are the largest and the smallest singular values of  $A$ , respectively.

28. If  $A$  is an  $n \times n$  matrix, what is the product of its singular values  $\sigma_1, \dots, \sigma_n$ ? State the product in terms of the determinant of  $A$ . For a  $2 \times 2$  matrix  $A$ , explain this result in terms of the image of the unit circle.

29. Show that an SVD

$$A = U \Sigma V^T$$

can be written as

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T.$$

30. Find a decomposition

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

for  $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$ . See Exercise 29 and Example 2.

31. Show that any matrix of rank  $r$  can be written as the sum of  $r$  matrices of rank 1.

32. Consider an  $n \times m$  matrix  $A$ , an orthogonal  $n \times n$  matrix  $S$ , and an orthogonal  $m \times m$  matrix  $R$ . Compare the singular values of  $A$  with those of  $SAR$ .

33. If the singular values of an  $n \times n$  matrix  $A$  are all 1, is  $A$  necessarily orthogonal?

34. For which square matrices  $A$  is there a singular value decomposition  $A = U \Sigma V^T$  with  $U = V$ ?

35. Consider a singular value decomposition  $A = U \Sigma V^T$  of an  $n \times m$  matrix  $A$  with rank  $A = m$ . Let  $\vec{v}_1, \dots, \vec{v}_m$  be the columns of  $V$  and  $\vec{u}_1, \dots, \vec{u}_n$  the columns of  $U$ . Compute  $(A^T A)^{-1} A^T \vec{u}_i$ . Explain the result in terms of least-squares approximations.

36. Consider a singular value decomposition  $A = U \Sigma V^T$  of an  $n \times m$  matrix  $A$  with rank  $A = m$ . Let  $\vec{u}_1, \dots, \vec{u}_n$  be the columns of  $U$ . Compute  $A(A^T A)^{-1} A^T \vec{u}_i$ .

## Exercises

### TRUE OR FALSE?

(Work with real numbers throughout.)

1. The singular values of any diagonal matrix  $D$  are the absolute values of the diagonal entries of  $D$ .
2. The equation  $2x^2 + 5xy + 3y^2 = 1$  defines an ellipse.
3. All symmetric matrices are diagonalizable.
4. If the matrix  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  is positive definite, then  $a$  must be positive.
5. If  $A$  is an orthogonal matrix, then there must exist a symmetric invertible matrix  $S$  such that  $S^{-1}AS$  is diagonal.
6. The singular value of the  $2 \times 1$  matrix  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$  is 5.
7. The function  $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2$  is a quadratic form.
8. The singular values of any matrix  $A$  are the eigenvalues of matrix  $A^T A$ .
9. If matrix  $A$  is positive definite, then all the eigenvalues of  $A$  must be positive.
10. The function  $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \vec{x}$  is a quadratic form.
11. The singular values of any triangular matrix are the absolute values of its diagonal entries.

12. If  $A$  is any matrix, then matrix  $A^T A$  is the transpose of  $AA^T$ .

13. If the singular values of a  $2 \times 2$  matrix  $A$  are 3 and 4, then there must exist a unit vector  $\vec{u}$  in  $\mathbb{R}^2$  such that  $\|A\vec{u}\| = 4$ .

14. The determinant of a negative definite  $4 \times 4$  matrix must be positive.

15. If  $A$  is a symmetric matrix such that  $A\vec{v} = 3\vec{v}$  and  $A\vec{w} = 4\vec{w}$ , then the equation  $\vec{v} \cdot \vec{w} = 0$  must hold.

16. Matrix  $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$  is negative definite.

17. All skew-symmetric matrices are diagonalizable (over  $\mathbb{R}$ ).

18. If  $A$  is any matrix, then matrix  $AA^T$  is diagonalizable.

19. All positive definite matrices are invertible.

20. Matrix  $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  is diagonalizable.

21. If  $A$  is an invertible symmetric matrix, then  $A^2$  must be positive definite.

22. If the two columns  $\vec{v}$  and  $\vec{w}$  of a  $2 \times 2$  matrix  $A$  are orthogonal, then the singular values of  $A$  must be  $\|\vec{v}\|$  and  $\|\vec{w}\|$ .

## Symmetric Matrices and Quadratic Forms

23. If  $A$  and  $S$  are invertible  $n \times n$  matrices, then matrices  $A$  and  $S^T AS$  must be similar.
24. If  $A$  is negative definite, then all the diagonal entries of  $A$  must be negative.
25. If the positive definite matrix  $A$  is similar to the symmetric matrix  $B$ , then  $B$  must be positive definite as well.
26. If  $A$  is a symmetric matrix, then there must exist an orthogonal matrix  $S$  such that  $SAS^T$  is diagonal.
27. If  $\vec{v}$  and  $\vec{w}$  are linearly independent eigenvectors of a symmetric matrix  $A$ , then  $\vec{w}$  must be orthogonal to  $\vec{v}$ .
28. For any  $n \times m$  matrix  $A$  there exists an orthogonal  $m \times m$  matrix  $S$  such that the columns of matrix  $AS$  are orthogonal.
29. If  $A$  is a symmetric  $n \times n$  matrix such that  $A^n = 0$ , then  $A$  must be the zero matrix.
30. If  $q(\vec{x})$  is a positive definite quadratic form, then so is  $kq(\vec{x})$ , for any scalar  $k$ .
31. If  $A$  and  $B$  are  $2 \times 2$  matrices, then the singular values of matrices  $AB$  and  $BA$  must be the same.
32. If  $A$  is any orthogonal matrix, then matrix  $A + A^{-1}$  is diagonalizable (over  $\mathbb{R}$ ).
33. The product of two quadratic forms in three variables must be a quadratic form as well.
34. The function  $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$  is a quadratic form.
35. If the determinants of all the principal submatrices of a symmetric  $3 \times 3$  matrix  $A$  are negative, then  $A$  must be negative definite.
36. If  $A$  and  $B$  are positive definite  $n \times n$  matrices, then matrix  $A + B$  must be positive definite as well.
37. If  $A$  is a positive definite  $n \times n$  matrix and  $\vec{x}$  is a nonzero vector in  $\mathbb{R}^n$ , then the angle between  $\vec{x}$  and  $A\vec{x}$  must be acute.
38. If the  $2 \times 2$  matrix  $A$  has the singular values 2 and 3 and the  $2 \times 2$  matrix  $B$  has the singular values 4 and 5, then both singular values of matrix  $AB$  must be  $\leq 15$ .
39. The equation  $A^T A = AA^T$  holds for all square matrices  $A$ .
40. For every symmetric  $n \times n$  matrix  $A$  there exists a constant  $k$  such that  $A + kI_n$  is positive definite.
41. If matrix  $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$  is positive definite, then  $af$  must exceed  $c^2$ .
42. If  $A$  is positive definite, then all the entries of  $A$  must be positive or zero.
43. If  $A$  is indefinite, then 0 must be an eigenvalue of  $A$ .
44. If  $A$  is a  $2 \times 2$  matrix with singular values 3 and 5, then there must exist a unit vector  $\vec{u}$  in  $\mathbb{R}^2$  such that  $\|A\vec{u}\| = 4$ .
45. If  $A$  is skew symmetric, then  $A^2$  must be negative semidefinite.
46. The product of the  $n$  singular values of an  $n \times n$  matrix  $A$  must be  $|\det A|$ .
47. If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ , then there exist exactly four orthogonal  $2 \times 2$  matrices  $S$  such that  $S^{-1}AS$  is diagonal.
48. The sum of two quadratic forms in three variables must be a quadratic form as well.
49. The eigenvalues of a symmetric matrix  $A$  must be equal to the singular values of  $A$ .
50. Similar matrices must have the same singular values.
51. If  $A$  is a symmetric  $2 \times 2$  matrix with eigenvalues 1 and 2, then the angle between  $\vec{x}$  and  $A\vec{x}$  must be less than  $\pi/6$ , for all nonzero vectors  $\vec{x}$  in  $\mathbb{R}^2$ .
52. If both singular values of a  $2 \times 2$  matrix  $A$  are less than 5, then all the entries of  $A$  must be less than 5.
53. If  $A$  is a positive definite matrix, then the largest entry of  $A$  must be on the diagonal.
54. If  $A$  and  $B$  are real symmetric matrices such that  $A^3 = B^3$ , then  $A$  must be equal to  $B$ .

## Answers to Odd-Numbered Exercises

1 Answers to more theoretical questions are omitted.

1.  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3.  $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

5.  $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

7.  $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

## Symmetric Matrices and Quadratic Forms

9.  $S = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$ ,  $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

11. Same  $S$  as in 9,  $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. Yes (reflection about  $E_1$ )

15. Yes (can use the same orthonormal eigenbasis)

17. Let  $A$  be the  $n \times n$  matrix whose entries are all 1. The eigenvalues of  $A$  are 0 (with multiplicity  $n-1$ ) and  $n$ . Now  $B = qA + (p-q)I_n$ , so that the eigenvalues of  $B$  are  $p-q$  (with multiplicity  $n-1$ ) and  $qn+p-q$ . Therefore,  $\det B = (p-q)^{n-1}(qn+p-q)$ .

21.  $48 = 6 \cdot 4 \cdot 2$  (note that  $A$  has 6 unit eigenvectors)

23. The only possible eigenvalues are 1 and  $-1$  (because  $A$  is orthogonal), and the eigenspaces  $E_1$  and  $E_{-1}$  are orthogonal complements (because  $A$  is symmetric). Hence,  $A$  represents the reflection about a subspace of  $\mathbb{R}^n$ .

25.  $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}$

27. If  $n$  is even, we have the eigenbasis  $\vec{e}_1 - \vec{e}_n, \vec{e}_2 - \vec{e}_{n-1}, \dots, \vec{e}_{n/2} - \vec{e}_{n/2+1}, \vec{e}_1 + \vec{e}_n, \vec{e}_2 + \vec{e}_{n-1}, \dots, \vec{e}_{n/2} + \vec{e}_{n/2+1}$ , with associated eigenvalues 0 ( $n/2$  times) and 2 ( $n/2$  times).

29. Yes    31. True    33.  $\theta = \frac{2}{3}\pi = 120^\circ$

35.  $\theta = \arccos(-\frac{1}{n})$ . Hint: If  $\vec{v}_0, \dots, \vec{v}_n$  are such vectors, let  $A = [\vec{v}_0 \ \dots \ \vec{v}_n]$ . Then the noninvertible matrix  $A^T A$  has 1's on the diagonal and  $\cos \theta$  everywhere else. Now use Exercise 17.

37. In Example 4 we see that the image of the unit circle will be an ellipse with semimajor axis 3 and semiminor axis 2. Thus  $2 \leq \|A\vec{u}\| \leq 3$ .

39. Let  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  be an orthonormal eigenbasis with associated eigenvalues  $-2, 3$ , and  $4$ , respectively. Consider a unit vector  $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ . Then  $A\vec{u} = -2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3$  and  $\vec{u} \cdot A\vec{u} = -2c_1^2 + 3c_2^2 + 4c_3^2 \leq 4c_1^2 + 4c_2^2 + 4c_3^2 = 4$ . Likewise,  $\vec{u} \cdot A\vec{u} = -2c_1^2 + 3c_2^2 + 4c_3^2 \geq -2c_1^2 - 2c_2^2 - 2c_3^2 = -2$ . Thus  $-2 \leq \vec{u} \cdot A\vec{u} \leq 4$ .

41. There exist an orthogonal  $S$  and a diagonal  $D$  such that  $A = SDS^{-1}$ . Taking cube roots of the diagonal entries of  $D$ , we can write  $D = D_0^3$  for some diagonal  $D_0$ . Now  $A = SDS^{-1} = SD_0^3S^{-1} = (SD_0S^{-1})^3 = B^3$ , where  $B = SD_0S^{-1}$ .

43. Consider the eigenvectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ , with eigenvalues 24 and  $-9$ , respectively.

There must exist a nonzero solution of the form  $\vec{v} = a\vec{v}_1 + b\vec{v}_2$ . Now  $\vec{v} \cdot A\vec{v} = (a\vec{v}_1 + b\vec{v}_2) \cdot (24a\vec{v}_1 - 9b\vec{v}_2) = 72a^2 - 18b^2 = 0$  when  $b = \pm 2a$ . Let

$$a = 1 \text{ and } b = 2 \text{ to find the solution } \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

47. a.  $ij$ th entry of  $|AB| = |\sum_{k=1}^n a_{ik}b_{kj}| \leq \sum_{k=1}^n |a_{ik}||b_{kj}| = ij$ th entry of  $|A||B|$

b. By induction on  $t$ , using part a:  $|A^t| = |A^{t-1}A| \leq |A^{t-1}||A| \leq |A|^{t-1}|A| = |A|^t$

49. Let  $\lambda$  be the maximum of all  $|r_{ii}|$ , for  $i = 1, \dots, n$ . Note that  $\lambda < 1$ . Then  $|R| \leq \lambda(I_n + U)$ , where  $U$  is upper triangular with  $u_{ii} = 0$  and  $u_{ij} = |r_{ij}|/\lambda$  if  $j > i$ . Note that  $U^n = 0$ , by Exercise 46a. Now  $|R^t| \leq |R|^t \leq \lambda^t(I_n + U)^t \leq \lambda^t t^n(I_n + U + \dots + U^{n-1})$ . From calculus we know that  $\lim_{t \rightarrow \infty} \lambda^t t^n = 0$ .

2 Answers to more theoretical questions are omitted.

1.  $\begin{bmatrix} 6 & -3.5 \\ -3.5 & 8 \end{bmatrix}$

3.  $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & 3.5 \\ 3 & 3.5 & 5 \end{bmatrix}$

5. Indefinite

7. Indefinite

9. a.  $A^2$  is symmetric.

b.  $A^2 = -A^T A$  is negative semidefinite, so that its eigenvalues are  $\leq 0$ .

c. The eigenvalues of  $A$  are imaginary (that is, of the form  $bi$ , for a real  $b$ ). The zero matrix is the only skew-symmetric matrix that is diagonalizable over  $\mathbb{R}$ .

11. The same (the eigenvalues of  $A$  and  $A^{-1}$  have the same signs).

13.  $a_{ii} = q(\vec{e}_i) > 0$ .

15. Ellipse; principal axes spanned by  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ; equation  $7c_1^2 + 2c_2^2 = 1$

17. Hyperbola; principal axes spanned by  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , equation  $4c_1^2 - c_2^2 = 1$

## Symmetric Matrices and Quadratic Forms

19. A pair of lines; principal axes spanned by  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ; equation  $5c_2^2 = 1$

Note that we can write  $x_1^2 + 4x_1x_2 + 4x_2^2 = (x_1 + 2x_2)^2 = 1$ , so that  $x_1 + 2x_2 = \pm 1$ .

21. a. The first is an ellipsoid, the second a hyperboloid of one sheet, and the third a hyperboloid of two sheets (see any text in multivariable calculus). Only the ellipsoid is bounded, and the first two surfaces are connected.  
 b. The matrix  $A$  of this quadratic form has positive eigenvalues  $\lambda_1 \approx 0.56$ ,  $\lambda_2 \approx 4.44$ , and  $\lambda_3 = 1$ , with corresponding unit eigenvectors

$$\vec{v}_1 \approx \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix}, \quad \vec{v}_2 \approx \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix},$$

$$\vec{v}_3 \approx \begin{bmatrix} 0.41 \\ -0.82 \\ 0.41 \end{bmatrix}.$$

Since all eigenvalues are positive, the surface is an ellipsoid. The points farthest from the origin are

$$\pm \frac{1}{\sqrt{\lambda_1}} \vec{v}_1 \approx \pm \begin{bmatrix} 1.15 \\ 0.26 \\ -0.63 \end{bmatrix}$$

and those closest are

$$\pm \frac{1}{\sqrt{\lambda_2}} \vec{v}_2 \approx \pm \begin{bmatrix} 0.15 \\ 0.26 \\ 0.37 \end{bmatrix}.$$

23. Yes;  $A = \frac{1}{2}(M + M^T)$

25.  $q(\vec{v}) = \vec{v} \cdot \lambda \vec{v} = \lambda$

27. The closed interval  $[\lambda_n, \lambda_1]$

29.  $B = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix}$

31.  $B = \frac{1}{5} \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$

33.  $L = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix}$       35.  $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{bmatrix}$

39. For  $0 < \theta < \arccos\left(-\frac{1}{n-1}\right)$

41. 3

43.  $\text{im } T = \text{span}(x_1^2)$ , rank  $T = 1$ ,  
 $\ker T = \text{span}(x_1x_2, x_2^2)$ , nullity  $T = 2$

45.  $\text{im } T = P_2$ , rank  $T = 3$ ,  
 $\ker T = \text{span}(x_3^2 - x_2^2, x_1x_3 - x_1x_2, x_2x_3 - x_2^2)$ ,  
 nullity  $T = 3$

47. The determinant of the  $m$ th principal submatrix is positive if  $m$  is even, and negative if  $m$  is odd.

55. Note that  $\det \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} = a_{ii}a_{jj} - a_{ij}^2 > 0$ , so that  $a_{ii} > a_{ij}$  or  $a_{jj} > a_{ij}$ .

57.  $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ , with positive  $\lambda_i$ , defines an ellipsoid.

59.  $q(\vec{x}) = \lambda_1 c_1^2 = 1$ , with positive  $\lambda_1$ , defines a pair of parallel planes,  $c_1 = \pm \frac{1}{\sqrt{\lambda_1}}$ .

61.  $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ , with  $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ,  $\lambda_3 < 0$  defines a hyperboloid of one sheet.

63.  $q(c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) = (c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) \cdot (c_1 \lambda_1 \vec{w}_1 + \dots + c_n \lambda_n \vec{w}_n) = c_1^2 \lambda_1 \|\vec{w}_1\|^2 + \dots + c_n^2 \lambda_n \|\vec{w}_n\|^2 = c_1^2 + \dots + c_n^2$  since  $\|\vec{w}_i\|^2 = \frac{1}{\lambda_i}$ , by construction.

65. Adapt the method outlined in Exercise 63. Consider an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $A$  with associated eigenvalues  $\lambda_1 > 0$  and  $\lambda_2 < 0$ . Now let  $\vec{w}_1 = \vec{v}_1/\sqrt{\lambda_1}$  and  $\vec{w}_2 = \vec{v}_2/\sqrt{-\lambda_2}$ , so that  $\|\vec{w}_1\|^2 = 1/\lambda_1$  and  $\|\vec{w}_2\|^2 = -1/\lambda_2$ . Then  $q(c_1 \vec{w}_1 + c_2 \vec{w}_2) = (c_1 \vec{w}_1 + c_2 \vec{w}_2) \cdot (\lambda_1 c_1 \vec{w}_1 + \lambda_2 c_2 \vec{w}_2) = \lambda_1 c_1^2 \|\vec{w}_1\|^2 + \lambda_2 c_2^2 \|\vec{w}_2\|^2 = c_1^2 - c_2^2$ .

67. Adapt the method outlined in Exercises 63 and 65. Consider an orthonormal eigenbasis  $\vec{v}_1, \dots, \vec{v}_p, \dots, \vec{v}_r, \dots, \vec{v}_n$  for  $A$  such that the associated eigenvalues  $\lambda_j$  are positive for  $j = 1, \dots, p$ , negative for  $j = p+1, \dots, r$ , and zero for  $j = r+1, \dots, n$ . Let  $\vec{w}_j = \vec{v}_j/\sqrt{|\lambda_j|}$  for  $j = 1, \dots, r$  and  $\vec{w}_j = \vec{v}_j$  for  $j = r+1, \dots, n$ .

69. Note that  $\vec{x}^T R^T A R \vec{x} = (R\vec{x})^T A (R\vec{x}) \geq 0$  for all  $\vec{x}$  in  $\mathbb{R}^m$ . Thus  $R^T A R$  is positive semidefinite.  $R^T A R$  is positive definite if (and only if)  $\ker R = \{\vec{0}\}$ .

71. Anything can happen: The matrix  $R^T A R$  may be positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

**3** Answers to more theoretical questions are omitted.

1.  $\sigma_1 = 2, \sigma_2 = 1$

3. All singular values are 1 (since  $A^T A = I_n$ ).

5.  $\sigma_1 = \sigma_2 = \sqrt{p^2 + q^2}$

7.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

9.  $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

11.  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

## Symmetric Matrices and Quadratic Forms

13.  $I_2 \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

15. Singular values of  $A^{-1}$  are the reciprocals of those of  $A$ .

21.  $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

23.  $AA^T \vec{u}_i = \begin{cases} \sigma_i^2 \vec{u}_i & \text{for } i = 1, \dots, r \\ \vec{0} & \text{for } i = r+1, \dots, n \end{cases}$

The nonzero eigenvalues of  $A^T A$  and  $AA^T$  are the same.

25. Choose vectors  $\vec{v}_1$  and  $\vec{v}_2$  as in Theorem 3.3. Write

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2.$$

Note that

$$\|\vec{u}\|^2 = c_1^2 + c_2^2 = 1.$$

Now

$$A\vec{u} = c_1 A\vec{v}_1 + c_2 A\vec{v}_2,$$

so that

$$\begin{aligned} \|A\vec{u}\|^2 &= c_1^2 \|A\vec{v}_1\|^2 + c_2^2 \|A\vec{v}_2\|^2 \\ &= c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 \\ &\leq (c_1^2 + c_2^2) \sigma_1^2 \\ &= \sigma_1^2. \end{aligned}$$

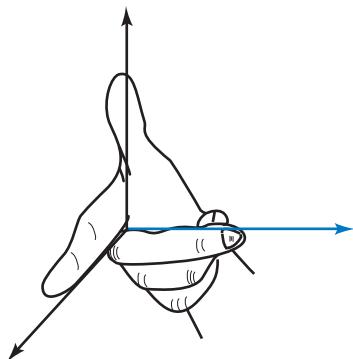
We conclude that  $\|A\vec{u}\| \leq \sigma_1$ . The proof of  $\sigma_2 \leq \|A\vec{u}\|$  is analogous.

27. Apply Exercise 26 to a unit eigenvector  $\vec{v}$  with associated eigenvalue  $\lambda$ .

33. No; consider  $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ .

35.  $(A^T A)^{-1} A^T \vec{u}_i = \begin{cases} \frac{1}{\sigma_i} \vec{v}_i & \text{for } i = 1, \dots, m \\ \vec{0} & \text{for } i = m+1, \dots, n \end{cases}$





# Vectors

Here we will provide a concise summary of basic facts on vectors. Vectors are defined as matrices with only one column:  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ . The scalars

$v_i$  are called the *components* of the vector.<sup>1</sup> The set of all vectors with  $n$  components is denoted by  $\mathbb{R}^n$ .

You may be accustomed to a different notation for vectors. Writing the components in a column is the most convenient notation for linear algebra.

## Vector Algebra

### Definition I

### Vector addition and scalar multiplication

- a. The sum of two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  is defined “componentwise”:

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

- b. The product of a scalar  $k$  and a vector  $\vec{v}$  is defined componentwise as well:

$$k\vec{v} = k \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}.$$

<sup>1</sup>In vector and matrix algebra, the term “scalar” is synonymous with (real) number.

**EXAMPLE 1**  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 3 \end{bmatrix}$  ■

**EXAMPLE 2**  $3 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \\ -3 \end{bmatrix}$  ■

The *negative* or *opposite* of a vector  $\vec{v}$  in  $\mathbb{R}^n$  is defined as

$$-\vec{v} = (-1)\vec{v}.$$

The *difference*  $\vec{v} - \vec{w}$  of two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  is defined componentwise. Alternatively, we can express the difference of two vectors as

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}).$$

The vector in  $\mathbb{R}^n$  that consists of  $n$  zeros is called the *zero vector* in  $\mathbb{R}^n$ :

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

### Theorem 2

#### Rules of vector algebra

The following formulas hold for all vectors  $\vec{u}, \vec{v}, \vec{w}$  in  $\mathbb{R}^n$  and for all scalars  $c$  and  $k$ :

1.  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ : Addition is *associative*.
2.  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ : Addition is *commutative*.
3.  $\vec{v} + \vec{0} = \vec{v}$ .
4. For each  $\vec{v}$  in  $\mathbb{R}^n$ , there exists a unique  $\vec{x}$  in  $\mathbb{R}^n$  such that  $\vec{v} + \vec{x} = \vec{0}$ ; namely,  $\vec{x} = -\vec{v}$ .
5.  $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$ .
6.  $(c + k)\vec{v} = c\vec{v} + k\vec{v}$ .
7.  $c(k\vec{v}) = (ck)\vec{v}$ .
8.  $1\vec{v} = \vec{v}$ .

These rules follow from the corresponding rules for scalars (commutativity, associativity, distributivity); for example:

$$\begin{aligned} \vec{v} + \vec{w} &= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} = \begin{bmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + v_n \end{bmatrix} \\ &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{w} + \vec{v}. \end{aligned}$$

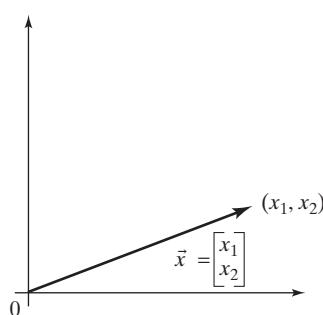
## Geometrical Representation of Vectors

The *standard representation* of a vector

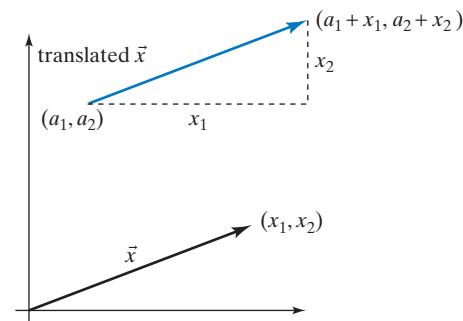
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in the Cartesian coordinate plane is as an *arrow* (a directed line segment) connecting the origin to the point  $(x_1, x_2)$ , as shown in Figure 1.

Occasionally, it is helpful to translate (or shift) the vector in the plane (preserving its direction and length), so that it will connect some point  $(a_1, a_2)$  to the point  $(a_1 + x_1, a_2 + x_2)$ . See Figure 2.



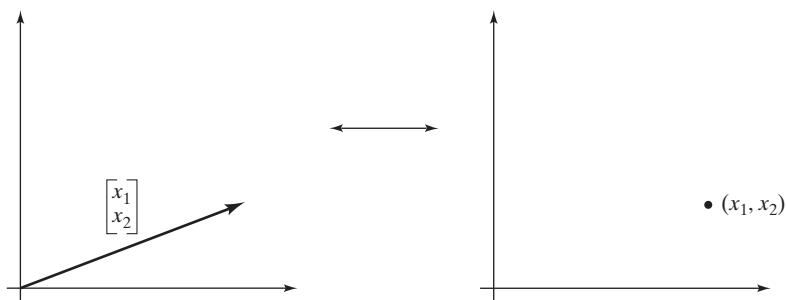
**Figure 1**



**Figure 2**

In this text, we consider the *standard representation* of vectors, unless we explicitly state that the vector has been translated.

A vector in  $\mathbb{R}^2$  (in standard representation) is uniquely determined by its endpoint. Conversely, with each point in the plane we can associate its *position vector*, which connects the origin to the given point. See Figure 3.



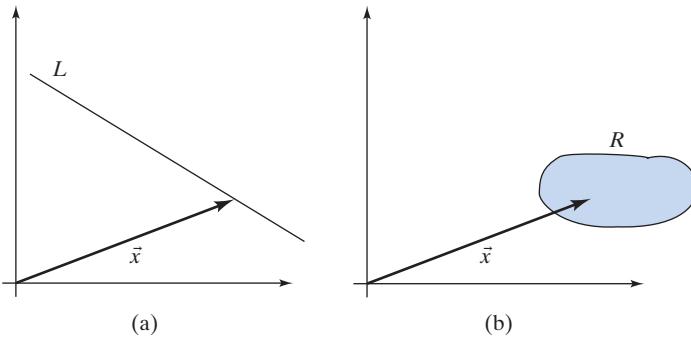
**Figure 3** The components of a vector in standard representation are the coordinates of its endpoint.

We need not clearly distinguish between a vector and its endpoint; we can identify them as long as we consistently use the standard representation of vectors.

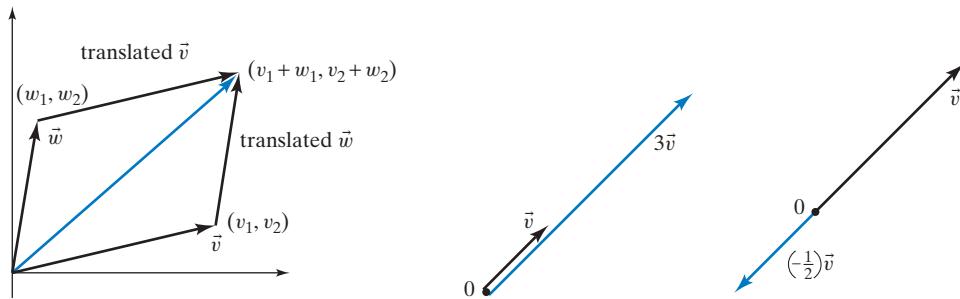
For example, we will talk about “the vectors on a line  $L$ ” when we really mean the vectors whose endpoints are on the line  $L$  (in standard representation). Likewise, we can talk about “the vectors in a region  $R$ ” in the plane. See Figure 4.

Adding vectors in  $\mathbb{R}^2$  can be represented by means of a parallelogram, as shown in Figure 5.

## Vectors



**Figure 4** (a)  $\vec{x}$  is a vector on the line  $L$ . (b)  $\vec{x}$  is a vector in the region  $R$ .



If  $k$  is a positive scalar, then  $k\vec{v}$  is obtained by stretching the vector  $\vec{v}$  by a factor of  $k$ , leaving its direction unchanged. If  $k$  is negative, then the direction is reversed. See Figure 6.

**Definition 3** We say that two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are *parallel* if one of them is a scalar multiple of the other.

**EXAMPLE 3** The vectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 9 \\ 6 \\ -6 \end{bmatrix}$$

are parallel, since

$$\begin{bmatrix} 3 \\ 9 \\ 6 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}.$$

■

**EXAMPLE 4** The vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

are parallel, since

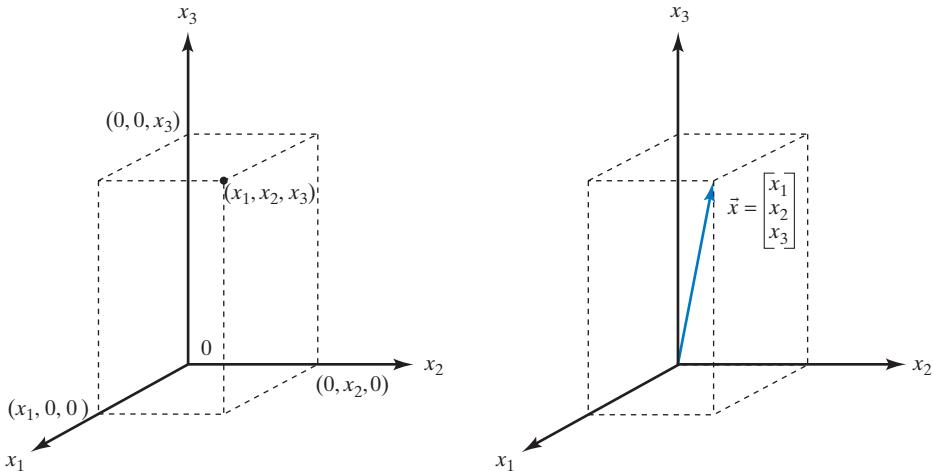
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \quad \blacksquare$$

Let us briefly review Cartesian coordinates in *space*: If we choose an origin 0 and three mutually perpendicular coordinate axes through 0, we can describe any point in space by a triple of numbers,  $(x_1, x_2, x_3)$ . See Figure 7.

The standard representation of the vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is the arrow connecting the origin to the point  $(x_1, x_2, x_3)$ , as shown in Figure 8.



**Figure 7**

**Figure 8**

## Dot Product, Length, Orthogonality

### Definition 4

Consider two vectors  $\vec{v}$  and  $\vec{w}$  with components  $v_1, v_2, \dots, v_n$  and  $w_1, w_2, \dots, w_n$ , respectively. Here  $\vec{v}$  and  $\vec{w}$  may be column or row vectors, and they need not be of the same type (these conventions are convenient in linear algebra). The *dot product* of  $\vec{v}$  and  $\vec{w}$  is defined as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

We can interpret the dot product geometrically: If  $\vec{v}$  and  $\vec{w}$  are two nonzero vectors in  $\mathbb{R}^n$ , then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cos \theta \|\vec{w}\|,$$

where  $\theta$  is the angle enclosed by vectors  $\vec{v}$  and  $\vec{w}$ .

Note that the dot product of two vectors is a *scalar*.

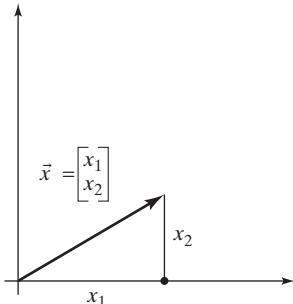
**EXAMPLE 5**  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = 1 \cdot 3 + 2 \cdot (-1) + 1 \cdot (-1) = 0$  ■

**EXAMPLE 6**  $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 + 2 + 0 - 4 = 1$  ■

**Theorem 5****Rules for dot products**

The following equations hold for all column or row vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  with  $n$  components, and for all scalars  $k$ :

1.  $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ .
2.  $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ .
3.  $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$ .
4.  $\vec{v} \cdot \vec{v} > 0$  for all nonzero  $\vec{v}$ .

**Figure 9**

The verification of these rules is straightforward. Let us justify Rule 4: Since  $\vec{v}$  is nonzero, at least one of the components  $v_i$  is nonzero, so that  $v_i^2$  is positive. Then

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_i^2 + \cdots + v_n^2$$

is positive as well.

Let us think about the *length* of a vector. The length of a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in  $\mathbb{R}^2$  is  $\sqrt{x_1^2 + x_2^2}$  by the Pythagorean theorem. See Figure 9.

This length is often denoted by  $\|\vec{x}\|$ . Note that we have

$$\vec{x} \cdot \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 = \|\vec{x}\|^2;$$

therefore,

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}.$$

Verify that this formula holds for vectors  $\vec{x}$  in  $\mathbb{R}^3$  as well.

We can use this formula to *define* the length of a vector in  $\mathbb{R}^n$ :

**Definition 6**

The *length* (or *norm*)  $\|\vec{x}\|$  of a vector  $\vec{x}$  in  $\mathbb{R}^n$  is

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

**EXAMPLE 7** Find  $\|\vec{x}\|$  for

$$\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ -1 \end{bmatrix}.$$

**Solution**

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{49 + 1 + 49 + 1} = 10$$
 ■

**Definition 7** A vector  $\vec{u}$  in  $\mathbb{R}^n$  is called a *unit vector* if  $\|\vec{u}\| = 1$ ; that is, the length of the vector  $\vec{u}$  is 1.

Consider two perpendicular vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^2$ , as shown in Figure 10.

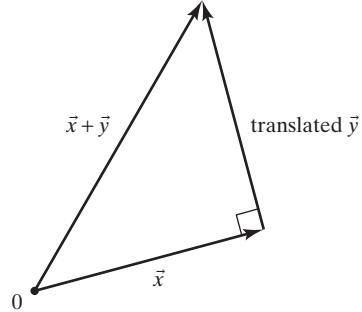


Figure 10

By the theorem of Pythagoras,

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2,$$

or

$$(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y}.$$

By Theorem 5,

$$\vec{x} \cdot \vec{x} + 2(\vec{x} \cdot \vec{y}) + \vec{y} \cdot \vec{y} = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y},$$

or

$$\vec{x} \cdot \vec{y} = 0.$$

You can read these equations backward to show that  $\vec{x} \cdot \vec{y} = 0$  if and only if  $\vec{x}$  and  $\vec{y}$  are perpendicular. This reasoning applies to vectors in  $\mathbb{R}^3$  as well.

We can use this characterization to *define* perpendicular vectors in  $\mathbb{R}^n$ :

**Definition 8** Two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are called *perpendicular* (or *orthogonal*) if  $\vec{v} \cdot \vec{w} = 0$ .

## Cross Product

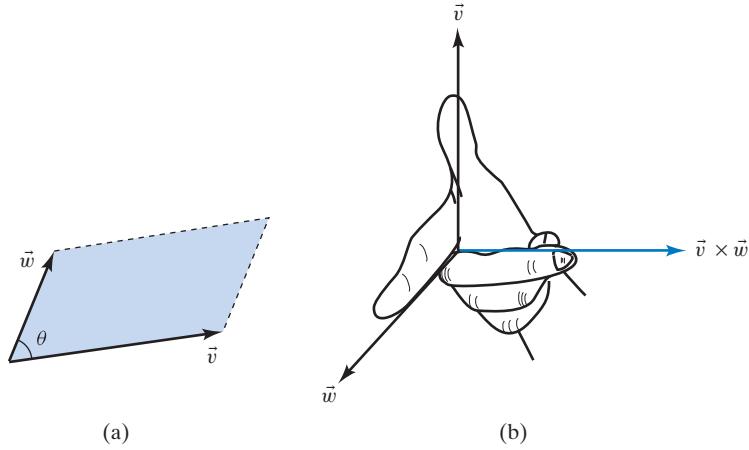
Here we present the cross product for vectors in  $\mathbb{R}^3$  only.

**Definition 9**

### Cross product in $\mathbb{R}^3$

The cross product  $\vec{v} \times \vec{w}$  of two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  is the vector in  $\mathbb{R}^3$  with the following three properties:

- $\vec{v} \times \vec{w}$  is orthogonal to both  $\vec{v}$  and  $\vec{w}$ .
- $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \sin \theta \|\vec{w}\|$ , where  $\theta$  is the angle between  $\vec{v}$  and  $\vec{w}$ , with  $0 \leq \theta \leq \pi$ . This means that the magnitude of the vector  $\vec{v} \times \vec{w}$  is the area of the parallelogram spanned by  $\vec{v}$  and  $\vec{w}$ , as illustrated in Figure 11a.
- The direction of  $\vec{v} \times \vec{w}$  follows the *right-hand rule*, as illustrated in Figure 11b.



**Figure 11** (a)  $\|\vec{v} \times \vec{w}\|$  is the shaded area. (b) A right-handed system.

### Theorem 10

#### Properties of the cross product

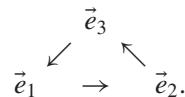
The following equations hold for all vectors  $\vec{u}, \vec{v}, \vec{w}$  in  $\mathbb{R}^3$  and for all scalars  $k$ .

- a.  $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$ : The cross product is *anticommutative*.
- b.  $(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w})$ .
- c.  $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$ .
- d.  $\vec{v} \times \vec{w} = \vec{0}$  if (and only if)  $\vec{v}$  is parallel to  $\vec{w}$ .
- e.  $\vec{v} \times \vec{v} = \vec{0}$ .
- f.  $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \quad \vec{e}_3 \times \vec{e}_1 = \vec{e}_2$   
(and  $\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3, \quad \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1, \quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$ ).

Note that the cross product fails to be associative:  $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$ , in general. For example,  $(\vec{e}_1 \times \vec{e}_1) \times \vec{e}_2 = \vec{0}$ , but  $\vec{e}_1 \times (\vec{e}_1 \times \vec{e}_2) = -\vec{e}_2$ .

Properties b and c stated in Theorem 10 imply that the function  $T(\vec{x}) = \vec{v} \times \vec{x}$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , for any fixed vector  $\vec{v}$  in  $\mathbb{R}^3$ .

The following diagram can serve as a memory aid for property f:



We can use the properties stated in Theorem 10 to express the cross products in components.

$$\begin{aligned}
 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= (v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3) \times (w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3) \\
 &= (v_2w_3 - v_3w_2)\vec{e}_1 + (v_3w_1 - v_1w_3)\vec{e}_2 + (v_1w_2 - v_2w_1)\vec{e}_3 \\
 &= \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}
 \end{aligned}$$

**Theorem 11****The cross product in components**

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

**EXAMPLE 8**

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \cdot 7 - 4 \cdot 6 \\ 4 \cdot 5 - 2 \cdot 7 \\ 2 \cdot 6 - 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

■



# Two Techniques of Proof: Induction and Contraposition

Here is the principle of induction, succinctly stated (you may have to work an example or two before fully understanding the wording).

Proof Technique 1

## Principle of mathematical induction

The truth of a sequence of statements  $P_1, P_2, \dots, P_n, \dots$  is established if<sup>2</sup>

1.  $P_1$  is true, and
2.  $P_n$  implies  $P_{n+1}$  for all  $n = 1, 2, \dots$

Alternatively, we can write property 2 as  $P_n \Rightarrow P_{n+1}$ , with the arrow “ $\Rightarrow$ ” signifying implication.

For example, if properties 1 and 2 hold, then  $P_5$  is true since

$$P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_5.$$

First we are using property 1 and then property 2 for  $n = 1, 2, 3$ , and 4.

The following terminology is useful for proofs by induction:

- $P_1$  is referred to as the **base case**,
- The implication  $P_n \Rightarrow P_{n+1}$  is called the **induction step**,
- When proving the implication  $P_n \Rightarrow P_{n+1}$ , statement  $P_n$  is called the **induction hypothesis**.

**EXAMPLE 1** Let's consider the *Fibonacci sequence*  $f_0, f_1, f_2, \dots$ , recursively defined by  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_{n+2} = f_n + f_{n+1}$  for all  $n = 0, 1, 2, \dots$ . The first few terms are  $f_0 = 0$ ,  $f_1 = 1$ ,  $f_2 = 1$ ,  $f_3 = 2$ ,  $f_4 = 3$ ,  $f_5 = 5$ ,  $f_6 = 8$ ,  $f_7 = 13$ ,  $f_8 = 21$ ,  $\dots$ .

Let's prove by induction that

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$$

for all  $n = 1, 2, \dots$  (Verify this equation for  $n = 6$ .)

<sup>2</sup>“Statement  $P_n$ ” is written as an equation or a sentence involving an arbitrary positive integer  $n$ .

## Two Techniques of Proof: Induction and Contraposition

Here, the “Statement  $P_n$ ” in 1 is the equation  $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ .

**1. Base Case:** We need to show that  $P_1$  is true, meaning that  $f_1 = f_3 - 1$ . This equation does hold since  $f_1 = 1$  and  $f_3 = 2$ .

**2. Induction Step:** Then we need to show that  $P_n$  implies  $P_{n+1}$ , meaning that the equation  $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$  implies  $f_1 + f_2 + \cdots + f_{n+1} = f_{n+3} - 1$ ; here we are replacing  $n$  by  $n + 1$  throughout.

Indeed,  $f_1 + f_2 + \cdots + f_{n+1} = (f_1 + f_2 + \cdots + f_n) + f_{n+1} \stackrel{\text{step 2}}{=} f_{n+2} - 1$

$1 + f_{n+1} \stackrel{\text{step 3}}{=} f_{n+3} - 1$ . In step 2, we are using the induction hypothesis,  $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ , and step 3,  $f_{n+3} = f_{n+1} + f_{n+2}$ , follows from the definition of the Fibonacci sequence. ■

As a broad *strategy for a proof by induction*, we suggest you proceed systematically, as illustrated in the table below. In the right column, we show the implementation of this strategy in Example 1.

Write down statement $P_n$	$f_1 + f_2 + \cdots + f_n \stackrel{?}{=} f_{n+2} - 1$
Let $n = 1$ and write down the base case $P_1$	$f_1 \stackrel{?}{=} f_3 - 1$
Verify the base case	$1 = 2 - 1 \quad \checkmark$
Write down the induction step, $P_n \Rightarrow P_{n+1}$	$(f_1 + f_2 + \cdots + f_n = f_{n+2} - 1)$ $\stackrel{?}{\Rightarrow} (f_1 + f_2 + \cdots + f_{n+1} = f_{n+3} - 1)$
Do the induction step: Using the induction hypothesis $P_n$ , prove $P_{n+1}$	$\begin{aligned} f_1 + f_2 + \cdots + f_{n+1} \\ = (f_1 + f_2 + \cdots + f_n) + f_{n+1} \\ = f_{n+2} - 1 + f_{n+1} = f_{n+3} - 1 \quad \checkmark \end{aligned}$

Let’s move on to a second important technique of proof.

### Proof Technique 2

#### Contraposition

The contrapositive of the statement “ $P$  implies  $Q$ ” is “Not  $Q$  implies not  $P$ ,” where,  $P$  and  $Q$  are two statements.

An implication “ $P$  implies  $Q$ ” and its contrapositive “Not  $Q$  implies not  $P$ ” are logically equivalent: They are either both true or both false.

The technique of “proof by contraposition” takes advantage of the logical equivalence of an implication with its contrapositive. Instead of proving an implication, we can prove the contrapositive if that turns out to be easier.

### EXAMPLE 2

On a plumber’s service truck we read the boast “If we can’t fix it, it ain’t broken”. The contrapositive of this claim is the logically equivalent (but less catchy) phrase “If it is broken, we can fix it”. (The lesson to be learned from this example: If your story is too dull, contrapose it!) ■

## Two Techniques of Proof: Induction and Contraposition

**EXAMPLE 3** Prove the following proposition by contraposition:

If  $x \leq y + \varepsilon$  for all positive  $\varepsilon$ , then  $x \leq y$ .

Here,  $\varepsilon$  denotes the Greek letter epsilon.

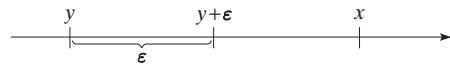
Let's state and then prove the contrapositive:

If  $x > y$ , then  $x > y + \varepsilon$  for some positive  $\varepsilon$ .

We assume that  $x > y$ , and we draw the two numbers  $x$  and  $y$  on the number line. What positive number  $\varepsilon$  can we add to  $y$  so that  $x > y + \varepsilon$ ? Clearly, any number  $0 < \varepsilon < x - y$  will work, for example,  $\varepsilon = \frac{x-y}{2}$ . Let's verify the inequality  $x > y + \varepsilon$  algebraically:

$$y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} \underset{\text{step3}}{<} \frac{x+x}{2} = x. \text{ In step 3 we use the fact that } y < x.$$

■





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