Big-Oh Notation

Let f and g be functions from positive numbers to positive numbers. $f(n)$ is $O(g(n))$ if there are positive constants C and k such that:

$$f(n) \leq C\, g(n) \quad \text{whenever } n > k$$

$$f(n) \text{ is } O(g(n)) \equiv$$
$$\exists C\, \exists k\, \forall n\, (n > k \rightarrow f(n) \leq C\, g(n))$$

To prove big-Oh, choose values for C and k and prove $n > k$ implies $f(n) \leq C\, g(n)$.

Standard Method to Prove Big-Oh

1. Choose $k = 1$.
2. Assuming $n > 1$, find/derive a C such that

$$\frac{f(n)}{g(n)} \leq \frac{C\, g(n)}{g(n)} = C$$

This shows that $n > 1$ implies $f(n) \leq C\, g(n)$. Keep in mind:

- $n > 1$ implies $1 < n,\ n < n^2,\ n^2 < n^3,\ \ldots$
- "Increase" numerator to "simplify" fraction.

Proving Big-Oh: Example 1

Show that $f(n) = n^2 + 2n + 1$ is $O(n^2)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^2 + 2n + 1}{n^2} < \frac{n^2 + 2n^2 + n^2}{n^2} = 4$$

Choose $C = 4$. Note that $2n < 2n^2$ and $1 < n^2$.

Thus, $n^2 + 2n + 1$ is $O(n^2)$ because $n^2 + 2n + 1 \leq 4n^2$ whenever $n > 1$.

Proving Big-Oh: Example 2

Show that $f(n) = 3n + 7$ is $O(n)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{3n + 7}{n} < \frac{3n + 7n}{n} = \frac{10n}{n} = 10$$

Choose $C = 10$. Note that $7 < 7n$.

Thus, $3n + 7$ is $O(n)$ because $3n + 7 \leq 10n$ whenever $n > 1$.

Proving Big-Oh: Example 3

Show that $f(n) = (n+1)^3$ is $O(n^3)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{(n+1)^3}{n^3} < \frac{(n+n)^3}{n^3} = \frac{8n^3}{n^3} = 8$$

Choose $C = 8$. Note that $n + 1 < n + n$ and $(n+n)^3 = (2n)^3 = 8n^3$. Thus, $(n+1)^3$ is $O(n^3)$ because $(n+1)^3 \leq 8n^3$ whenever $n > 1$.

Proving Big-Oh: Example 4

Show that $f(n) = \sum\limits_{i=1}^{n} i$ is $O(n^2)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{\sum_{i=1}^{n} i}{n^2} \leq \frac{\sum_{i=1}^{n} n}{n^2} = \frac{n^2}{n^2} = 1$$

Choose $C = 1$. Note that $i \leq n$ because n is the upper limit. Thus, $\sum_{i=1}^{n} i$ is $O(n^2)$ because $\sum_{i=1}^{n} i \leq n^2$ whenever $n > 1$.

How to Show Not Big-Oh

$$f(n) \text{ is not } O(g(n)) \equiv$$
$$\forall C\, \forall k\, \exists n\, (n > k \wedge f(n) > C\, g(n))$$

Need to prove for all values of C and k.

C and k cannot be replaced with constants.

Choose n based on C and k.

Prove that this choice implies
$n > k \wedge f(n) > C\, g(n)$

Standard Method to Prove Not-Big-Oh:

1. Assume $n > 1$.
2. Show:
$$\frac{f(n)}{g(n)} \geq \frac{h(n)\, g(n)}{g(n)} = h(n)$$
where $h(n)$ is strictly increasing to ∞.
3. $n > h^{-1}(C)$ implies $h(n) > C$, which implies $f(n) > C\, g(n)$.

So choosing $n > 1$, $n > k$, and $n > h^{-1}(C)$ implies $n > k \wedge f(n) > C\, g(n)$.

Proving Not Big-Oh: Example 1

Show that $f(n) = n^2 - 2n + 1$ is not $O(n)$.

Assume $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^2 - 2n + 1}{n} > \frac{n^2 - 2n}{n} = n - 2$$

$n > C + 2$ implies $n - 2 > C$ and $f(n) > Cn$.

So choosing $n > 1$, $n > k$, and $n > C + 2$ implies $n > k \wedge f(n) > Cn$.

- "Decrease" numerator to "simplify" fraction.

Proving Not Big-Oh: Example 2

Show that $f(n) = (n-1)^3$ is not $O(n^2)$.

Assume $n > 1$, then:

$$\frac{f(n)}{g(n)} = \frac{n^3 - 3n^2 + 3n - 1}{n^2} > \frac{n^3 - 3n^2 - 1}{n^2}$$

$$> \frac{n^3 - 3n^2 - n^2}{n^2} = n - 4$$

$n > C + 4$ implies $n - 4 > C$ and $f(n) > Cn^2$.

Choosing $n > 1$, $n > k$, and $n > C + 4$ implies $n > k \wedge f(n) > Cn^2$.

Proving Not Big-Oh: Example 3

Show that $f(n) = \lfloor n^2/2 \rfloor$ is not $O(n)$.

Assume $n > 1$, then:

$$\frac{f(n)}{g(n)} = \frac{\lfloor n^2/2 \rfloor}{n} > \frac{n^2/2 - 1}{n} > \frac{n^2/2 - n}{n}$$
$$= n/2 - 1$$

$n > 2C + 2 \rightarrow n/2 - 1 > C$ and $f(n) > Cn$.

Choosing $n > 1$, $n > k$, and $n > 2C+2$ implies $n > k \wedge f(n) > C\,n$.