Chapter-5

Moments, skewness & kurtosis

Moments :- If $X_1, X_2, \dots, X_N$ are the N values assumed by the variable X,

Example 5.1

$$\overline{X^r} = \frac{\sum_{j=1}^{N} X_j^r}{N} \quad \text{——} (1)$$

is called the "r-th moment".

The 1st moment with $r=1$ is the arithmetic mean $\bar{X}$.

The r-th "moment about the mean $\bar{X}$" is

Example 5.2

$$m_r = \frac{\sum_{j=1}^{N} (X_j - \bar{X})^r}{N} \quad \text{——} (2)$$

If $r=1$, (2) $\Rightarrow m_1 = 0$

If $r=2$, (2) $\Rightarrow m_2 = s^2 =$ variance.

The r-th moment about any origin A is,

Example 5.3

$$m_r' = \frac{\sum_{j=1}^{N} (X_j - A)^r}{N} = \frac{\sum_{j=1}^{N} d_j^r}{N} \quad \text{——} (3)$$

where, $d_j = X_j - A$ are the deviations of X from A.

If $A = 0$, (3) $\Rightarrow$ (1).

so, (1) is often called the "r-th moment about zero."

If $x_1, x_2, \dots, x_k$ occur with frequencies $f_1, f_2, \dots, f_k$, respectively,

① ⇒ $$\overline{X^r} = \frac{\sum_{j=1}^{k} f_j x_j^r}{N = \sum_{j=1}^{k} f_j} \quad \text{—— (4)}$$

② ⇒ $$m_r = \frac{\sum_{j=1}^{k} f_j (x_j - \bar{x})^r}{N} \quad \text{—— (5)}$$

where, $N = \sum_{j=1}^{k} f_j$

& ③ ⇒ $$m_r' = \frac{\sum_{j=1}^{k} f_j (x_j - A)^r}{N} \quad \text{—— (6)}$$

(4), (5) & (6) are used for grouped data.

The relations between Moments :-

The relations between "moments about the mean m_r" and "moment about an arbitrary origin m_r'" are:

$$\left.\begin{aligned} m_2 &= m_2' - m_1'^2 \\ m_3 &= m_3' - 3m_1' m_2' + 2m_1'^3 \\ m_4 &= m_4' - 4m_1' m_3' + 6m_1'^2 m_2' - 3m_1'^4 \end{aligned}\right\} \text{—— (7)}$$

$\boxed{\text{Example 5.4, 5.6}}$

... etc..

where, $m_1' = \bar{X} - A$

If c is equal size of all class intervals,

$d_j = x_j - A = c\,u_j$; where, $u_j = \frac{x_j - A}{c}$

$\boxed{\text{Example 5.6, 5.7}}$

Then, (6) $\Rightarrow$ $m_r' = c^r \dfrac{\sum_{j=1}^{k} f_j u_j^r}{N}$ —— (8)

Moments in dimensionless form:-

The "dimensionless moments about the mean" is,

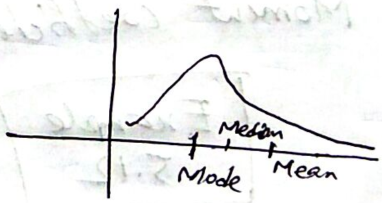
$$a_r = \frac{m_r}{s^r} = \frac{m_r}{(\sqrt{m_2})^r} = \frac{m_r}{\sqrt{m_2^r}} \qquad \text{—— (9)}$$

where, $s = \sqrt{m_2}$ = S.D. since, $m_1 = 0$ & $m_2 = s^2$,

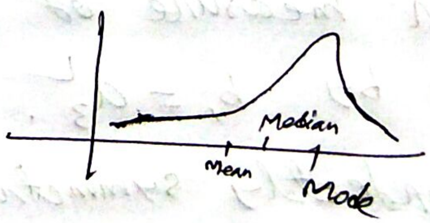
$\therefore$ (9) $\Rightarrow$ $a_1 = 0$ & $a_2 = 1$.

Skewness:- Skewness is the degree of asymmetry, or departure from symmetry of a distribution.

Positive skewness:-



Negative skewness:



$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{Standard deviation}} = \frac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

Pearson's 1st coefficient of skewness.

Pearson's 2nd coefficient of skewness

Example 5.10

In terms of quartiles & percentiles are:

$$\text{Quartile coefficient of skewness} = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{Q_3 - Q_1}$$

$$= \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

Example 5.11

$$\text{10–90 percentile " " "} = \frac{(P_{90} - P_{50}) - (P_{50} - P_{10})}{P_{90} - P_{10}}$$

In dimensionless form,

$$\text{Moment coefficient of skewness, } a_3 = \frac{m_3}{s^3} = \frac{m_3}{\sqrt{m_2^3}}$$

Example 5.12

Another measure of skewness is sometimes given by $b_1 = a_3^2$.

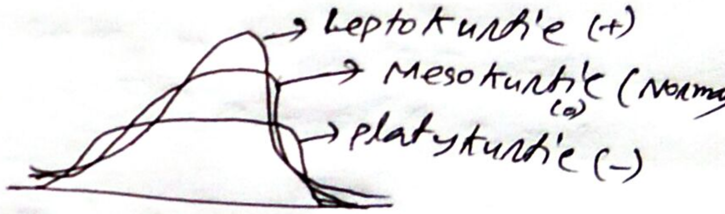
For perfectly symmetrical curves (Normal curves) a_3 and b_1 are zero.

Kurtosis : Kurtosis is the degree of peakedness of a distribution, usually taken relatively to a normal distribution.

Leptokurtic :- A distribution having a relatively high peak.

Platykurtic :- Having a flat-topped.

Mesokurtic :- A normal distribution, which is not very peaked or very flat topped.



In dimensionless form,

Moment coefficient of kurtosis, $a_4 = \frac{m_4}{s^4}$

$= \frac{m_4}{m_2^2}$

$= b_2$ (denoted by b_2)

$\boxed{\text{Example: 5.13}}$

For normal distribution,

$b_2 = a_4 = 3$.

For this reason, the kurtosis is sometimes defined by $(b_2 - 3)$, which is $(+) \rightarrow$ for leptokurtic distribution

$(-) \rightarrow$ " platykurtic "

& $(0) \rightarrow$ " Normal ".

Based on both quartiles and percentiles,

$$K\ (\text{kappa}) = \frac{Q}{P_{90}-P_{10}} \quad \text{where, } Q = \frac{1}{2}(Q_3-Q_1)$$

For the normal distribution,

$$K = 0.263.$$