Ex-1

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}$$

$$= \frac{1}{\theta^n} e^{-\Sigma x_i/\theta}$$

$$\ln L(\theta) = -n \ln\theta - \frac{\Sigma x_i}{\theta}$$

$$\frac{d \ln L(\theta)}{d\theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0 \quad (\text{Let})$$

$$\Rightarrow \frac{\Sigma x_i}{\theta^2} = \frac{n}{\theta}$$

$$\Rightarrow n\theta = \Sigma x_i$$

$$\Rightarrow \boxed{\theta = \frac{\Sigma x_i}{n}}$$

$$\frac{d^2 \ln L(\theta)}{d\theta^2} = \frac{n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3} < 0$$

for $\theta = \frac{\Sigma x_i}{n}$

$$\boxed{\hat{\theta} = \frac{\Sigma x_i}{n} = \bar{x}}$$

Ex-2

$$f(x;p) = (1-p)^{x-1} p$$

$$L(p) = \prod_{i=1}^{n} f(x_i;p) = \prod_{i=1}^{n} (1-p)^{x_i-1} p$$

$$= (1-p)^{\sum x_i - n} p^n$$

$$\ln L(p) = (\sum x_i - n)\ln(1-p) + n \ln p$$

$$\frac{d \ln L(p)}{dp} = -\frac{\sum x_i - n}{1-p} + \frac{n}{p} = 0 \quad \text{(let)}$$

$$\Rightarrow \frac{\sum x_i - n}{1-p} = \frac{n}{p}$$

$$n - np = p\sum x_i - np$$

$$\Rightarrow p = \frac{n}{\sum x_i}$$

$$\Rightarrow$$

$$\frac{d^2 \ln L(p)}{dp^2} = -\frac{\sum x_i - n}{(1-p)^2} - \frac{n}{p^2} < 0$$

for $p = \frac{n}{\sum x_i}$

$$\hat{p} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

Ex-3

$$f(x;\theta_1,\theta_2)=\frac{1}{\sqrt{2\pi\theta_2}}\, e^{-\frac{(x-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1,\theta_2)=\prod_{i=1}^{n}(2\pi\theta_2)^{-1/2}\, e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$=(2\pi\theta_2)^{-n/2}\, e^{-\frac{\Sigma(x_i-\theta_1)^2}{2\theta_2}}$$

$$\frac{a}{b}+\frac{c}{b}+\frac{d}{b}=\frac{a+c+d}{b}$$

$$\ln L(\theta_1,\theta_2)=-\frac{n}{2}\ln(2\pi\theta_2)-\frac{\Sigma(x_i-\theta_1)^2}{2\theta_2}$$

$$=-\frac{n}{2}\ln(2\pi)-\frac{n}{2}\ln\theta_2-\frac{\Sigma(x_i-\theta_1)^2}{2\theta_2}$$

$$\frac{\partial \ln L(\theta_1,\theta_2)}{\partial\theta_1}=-\frac{2\Sigma(x_i-\theta_1)}{2\theta_2}(-1)$$

$$=\frac{\Sigma(x_i-\theta_1)}{\theta_2}=0 \text{ (let)}$$

$$\Rightarrow \frac{\Sigma x_i-n\theta_1}{\theta_2}=0$$

$$\Rightarrow \Sigma x_i-n\theta_1=0$$

$$\Rightarrow \theta_1=\frac{\Sigma x_i}{n}$$

$$\Rightarrow$$

$$\frac{\partial^2 L(\theta_1,\theta_2)}{\partial \theta_1^2} = -\frac{n}{\theta_2} < 0 \quad \text{for } \theta_1 = \frac{\Sigma x_i}{n}$$

$$\hat{\theta}_1 = \frac{\Sigma x_i}{n} = \mu$$

so, $$\ln L(\theta_1,\theta_2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta_2 - \frac{\Sigma(x_i-\mu)^2}{2\theta_2}$$

$$\frac{\partial \ln L(\theta_1,\theta_2)}{\partial \theta_2} = -\frac{n}{2\theta_2} + \frac{\Sigma(x_i-\mu)^2}{2\theta_2^2} = 0 \text{ (let)}$$

$$\Rightarrow \frac{n}{2\theta_2} = \frac{\Sigma(x_i-\mu)^2}{2\theta_2^2}$$

$$\Rightarrow \theta_2 = \frac{\Sigma(x_i-\mu)^2}{n}$$

$$\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial \theta_2^2} = \frac{n}{2\theta_2^2} - \frac{\Sigma(x_i-\mu)^2}{\theta_2^3} < 0$$

$$\text{for } \theta_2 = \frac{\Sigma(x_i-\mu)^2}{n}$$

$$\hat{\theta}_2 = \frac{\Sigma(x_i-\mu)^2}{n} = \sigma^2$$