Probability and Statistics are the two important concepts in Math's. Probability is all about chance or, possibility. Whereas statistics is more about how we handle various data using different techniques. It helps to represent complicated data in a very easy and understandable way. Statistics and probability are usually introduced in Class 10, Class 11 and Class 12 students are preparing for school exams and competitive examinations. The introduction of these fundamentals is briefly given in your academic books and notes. The statistic has a huge application nowadays in data science professions. The professionals use the stats and do the predictions of the business. It helps them to predict the future **profit or loss** attained by the company.

Introduction:

Probability theory was originated from gambling theory. A large number of problems exist even today which are based on the game of chance, such as coin tossing, dice throwing and playing cards.

The probability is defined in two different ways,

- Mathematical (or a priori) definition
- Statistical (or empirical) definition

SOME IMPORTANT TERMS &CONCEPTS:

- **RANDOM EXPERIMENTS:**
  Experiments of any type where the outcome cannot be predicted are called random experiments.
- **SAMPLE SPACE (S):**
  A set of all possible outcomes from an experiment is called a sample space.
  Example: Consider a random experiment A of throwing **2** coins at a time.
  The possible outcomes are HH, TT, HT, TH.
  These **4** outcomes constitute a sample space denoted by, $S = \{HH, TT, HT, TH\}$.

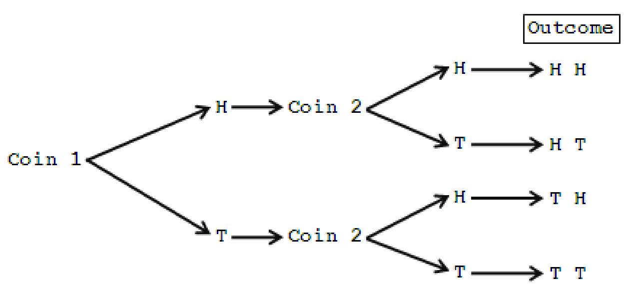

Fig: The outcomes of flipping two coins.

- **TRIAL & EVENT:**

  Consider an experiment of throwing a coin. When tossing a coin, we may get a head ($\boldsymbol{H}$) or tail ($\boldsymbol{T}$). Here tossing of a coin is a trail and getting a hand or tail is an event. In other words, "Every non-empty subset of $\boldsymbol{A}$ of the sample space $\boldsymbol{S}$ is called an event".

- **NULL EVENT:**

  An event having no sample point is called a null event and is denoted by $\emptyset$.

- **EXHAUSTIVE EVENTS:**

  The total number of possible outcomes in any trail is known as exhaustive events.

  Eg: In throwing a die the possible outcomes are getting $\mathbf{1}$ or $\mathbf{2}$ or $\mathbf{3}$ or $\mathbf{4}$ or $\mathbf{5}$ or $\mathbf{6}$. Hence we have $\mathbf{6}$ exhaustive events in throwing a die.

  That means $\boldsymbol{A_1} \cup \boldsymbol{A_2} \cup \ldots \cup \boldsymbol{A_k} = \boldsymbol{S}$

- **MUTUALLY EXCLUSIVE EVENTS:**

  Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if $\boldsymbol{A}$ & $\boldsymbol{B}$ are mutually exclusive events and if $\boldsymbol{A}$ happens then $\boldsymbol{B}$ will not happen and vice-versa.

  That means $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$.

  Eg: In tossing a coin, the events head or, tail are mutually exclusive, since both tail & head cannot appear in the same time.

- **EQUALLY LIKELY EVENTS:**

  Two events are said to be equally likely if one of them cannot be expected in the preference to the other.

  Eg: In throwing a coin, the events head & tail have **equal** chances of occurrence.

- **INDEPENDENT & DEPENDENT EVENTS:**

  Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. Events which are not independent are called dependent events.

  Eg: If we draw a card in a pack of well shuffled cards and again draw a card from the rest of pack of cards (containing $\mathbf{51}$ cards), then the second draw is dependent on the first. But if on the other hand, we draw a second card from the pack by replacing the first card drawn, the second draw is known as independent of the first.

- **FAVOURABLE EVENTS:**

  Mathematical or, classical or, a priori definition of probability,
  Probability (of happening an event $\boldsymbol{A}$)

$$= \frac{\textbf{Number of favourable outcomes to happen } \boldsymbol{A}}{\textbf{Total Number of outcomes of the experiment}}$$

PROBLEMS:

1. In tossing a coin, what is the probability of getting a head?
   Sol: Total no. of events = $\boldsymbol{n(\{H,T\}) = 2}$ and, Favourable event = $\boldsymbol{n(\{H\}) = 1}$
   So, Probability = $\frac{1}{2}$
2. In throwing a dice, what is the probability of getting $\mathbf{2}$?
   Sol: Total no. of events = $\boldsymbol{n(\{1,2,3,4,5,6\}) = 6}$ and, Favourable event = $\boldsymbol{n(\{2\}) = 1}$
   So, Probability = $\frac{1}{6}$
3. Find the probability of throwing $\mathbf{7}$ with two dice.
   Sol: Total no. of possible ways of throwing a dice **twice** = $\mathbf{36}$ ways
   Number of ways of getting $\mathbf{7}$ is, $\boldsymbol{n\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = 6}$
   So, Probability = $\frac{6}{36} = \frac{1}{6}$
4. A bag contains $\mathbf{6}$ red & $\mathbf{7}$ black balls. Find the probability of drawing a red ball.
   Sol: Total no. of possible ways of getting $\mathbf{1}$ ball = $\mathbf{6 + 7 = 13}$ and,
   Number of ways of getting $\mathbf{1}$ red ball = $\mathbf{6}$. So, Probability = $\frac{6}{13}$

\# A standard deck of cards contains $\mathbf{52}$ cards, of which $\mathbf{26}$ are red, $\mathbf{26}$ are black, $\mathbf{13}$ are of each suit (hearts, diamonds, spades, clubs) and of which $\mathbf{4}$ are of each suit (aces, jacks, kings and queens).

5. Find the probability of a card drawn at random from an ordinary pack, is a diamond.
   Sol: Total no. of possible ways of getting $\mathbf{1}$ card = $\mathbf{52}$ and,
   Number of ways of getting $\mathbf{1}$ diamond card is $\mathbf{13}$. So, Probability = $\frac{13}{52} = \frac{1}{4}$
6. From a pack of $\mathbf{52}$ cards, $\mathbf{1}$ card is drawn at random. Find the probability of getting a queen.
   Sol: A queen may be chosen in $\mathbf{4}$ ways and, total no. of ways of selecting $\mathbf{1}$ card = $\mathbf{52}$
   So, Probability = $\frac{4}{52} = \frac{1}{13}$
7. Find the probability of throwing: **(a)** $\mathbf{4}$, **(b)** an odd number, **(c)** an even number with an ordinary dice (six faced).
   Sol: (a) When throwing a die there is only one way of getting $\mathbf{4}$. So, Probability = $\frac{1}{6}$
   (b) Number of ways of falling an odd number is $\boldsymbol{\{1,3,5\} = 3}$. So, Probability = $\frac{3}{6} = \frac{1}{2}$
   (c) Number of ways of falling an even number is $\boldsymbol{\{2,4,6\} = 3}$. So, Probability = $\frac{3}{6} = \frac{1}{2}$

OPERATIONS ON SETS:

If $\boldsymbol{A}$ & $\boldsymbol{B}$ are any two sets, then

- **(i) UNION OF TWO SETS:** $\boldsymbol{A \cup B = \{x : x \in A\ or, x \in B\}}$
- **(ii) INTERSECTION OF TWO SETS:** $\boldsymbol{A \cap B = \{x : x \in A\ \&\ x \in B\}}$
- **(iii) COMPLEMENT OF A SET:** $\boldsymbol{A'\ or\ \bar{A} = \{x : x \notin A\}}$
- **(iv) DIFFERENCE OF TWO SETS:** $\boldsymbol{A - B = \{x : x \in A\ but\ x \notin B\}}$

COMMUTATIVE LAW: $A \cup B = B \cup A$ & $A \cap B = B \cap A$

ASSOCIATIVE LAW: $(A \cup B) \cup C = A \cup (B \cup C)$ & $(A \cap B) \cap C = A \cap (B \cap C)$

DISTRIBUTIVE LAW:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

COMPLEMENTARY LAW: $A \cup A' = S$ & $A \cap A' = \emptyset$

De Morgan's Laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

AXIOMATIC APPROACH TO PROBABILITY:

It is a rule which associates to each event a real number $P(A)$ which satisfies the following three axioms.

AXIOM I: For any event A, $P(A) \geq 0$
AXIOM II: $P(S) = 1$
AXIOM III: If $A_1, A_2, \dots, A_n$ are finite number of disjoint event of S, then

$$P(A_1 \cup A_2 \cup, \dots, \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = \sum P(A_i)$$

THEOREMS ON PROBABILITY:

THEOREM 1: Probability of an impossible event is zero. i.e. $P(\emptyset) = 0$
THEOREM 2: Probability of the complementary event $\bar{A}$ of A is given by, $P(\bar{A}) = 1 - P(A)$.
THEOREM 3: For any two events A & B, $P(\bar{A} \cap B) = P(B) - P(A \cap B)$.
THEOREM 4: If A and B are two events such that $A \subset B$, then $P(B \cap \bar{A}) = P(B) - P(A)$.
THEOREM 5: If $B \subset A$, then $P(A) \geq P(B)$.
THEOREM 6: If $A \cap B = \emptyset$, then $P(A) \leq P(\bar{B})$.

LAW OF ADDITION OF PROBABILITIES:

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where A & B are any two events and are not disjoint.

(ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$, where A, B, and C are any three events.

PROBLEMS:

1. If from a pack of cards a single card is drawn. What is the probability that it is either a spade or a king?
   Sol: $P(A) = P(\text{a spade card}) = \frac{13}{52} = \frac{1}{4}$, $P(B) = P(\text{a king card}) = \frac{4}{52} = \frac{1}{13}$ and,
   $P(A \cap B) = \frac{1}{52}$. So, $P(\text{either a spade or, a king card}) = P(A \text{ or}, B) = P(A \cup B)$
   $$= P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{4}{13}$$

2. If $P(A) = 0.35$, $P(B) = 0.73$ and, $P(A \cap B) = 0.14$. Find $P(A' \cup B')$.
   Sol: Using De Morgan's Law, $A' \cup B' = (A \cap B)' \Rightarrow P(A' \cup B') = P(A \cap B)'$
   $$= 1 - P(A \cap B) = 1 - 0.14 = 0.86$$
   or,

$$P\,(A' \cup B') = P(A') + P(B') - P(A' \cap B') = 1 - P(A) + 1 - P(B) - P(A \cup B)'$$
$$= 2 - P(A) - P(B) - \{1 - P(A \cup B)\} = 1 - \{\,P(A) + P(B) - P(A \cup B)\}$$
$$= 1 - P(A \cap B) = 1 - 0.14 = 0.86$$

Example 1.1-2: A disk $\mathbf{2}$ inches in diameter is thrown at random on a tiled floor, where each tile is a square with sides $\mathbf{4}$ inches in length. Find the probability of region that if the disk is thrown, it lies entirely on the tile.
Sol: Let $\boldsymbol{A}$ be the event that the disk will land entirely on the tile. If we draw a picture, it should be clear that the center must lie within a square having sides of length $\mathbf{2}$ and with its center coincident with the center of a tile. Since the area of this square is $\mathbf{4}$ and the area of a tile is $\mathbf{16}$, so the probability, $P(A) = \frac{4}{16} = \frac{1}{4}$.

Example 1.1-3: A fair coin is flipped successively until the same face is observed on successive flips. Let $A = \{x: x = 3, 4, 5, \dots\}$; that is, $\boldsymbol{A}$ is the event that it will take three or more flips of the coin to observe the same face on two consecutive flips. Find the probability of $\boldsymbol{A}$.
Sol: Here, $A' = \{x: x = 2\}$, the complement of $\boldsymbol{A}$. In two flips of a coin, the possible outcomes are $\{HH, HT, TH, TT\}$. So, $P(A') = P(\{HH, TT\}) = \frac{2}{4} = \frac{1}{2}$.

Thus, $P(A) = 1 - P(A') = 1 - \frac{1}{2} = \frac{1}{2}$

Exercises: 1.1-1 to 1.1-6 & 1.1-9 (Try yourself)

Exercise 1.1-7: Given that, $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.

Sol: we have, $P(A \cup B') = 1 - P(A \cup B')' = 1 - P(A' \cap B) = 1 - \{P(B) - P(A \cap B)\}$
$$= 1 - \{P(B) - P(A) - P(B) + P(A \cup B)\} = 1 + P(A) - P(A \cup B)$$
$$\Rightarrow P(A) = P(A \cup B') + P(A \cup B) - 1 = 0.87 + 0.76 - 1 = 0.63$$
Or, $P(A \cup B') = P(A) + P(B') - P(A \cap B') = P(A) + 1 - P(B) - \{P(A) - P(A \cap B)\}$
$$= P(A) + 1 - \{P(A) + P(B) - P(A \cap B)\} = 1 + P(A) - P(A \cup B)$$
$$\Rightarrow P(A) = P(A \cup B') + P(A \cup B) - 1 = 0.87 + 0.76 - 1 = 0.63$$

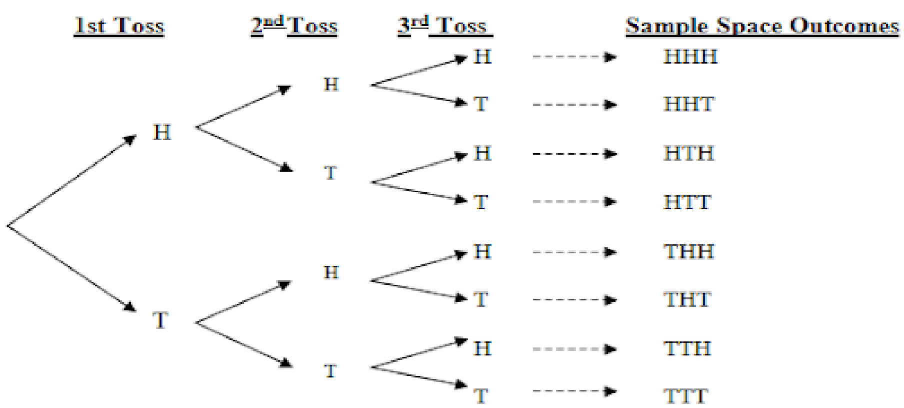

Fig: The outcomes of flipping three coins.

Example: A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If $\boldsymbol{E}$ is the event that a number less than $\mathbf{4}$ occurs on a single toss of the die, find $\boldsymbol{P(E)}$.
Sol: The sample space is $\boldsymbol{S = \{1, 2, 3, 4, 5, 6\}}$. We assign a probability of $\boldsymbol{w}$ to each odd number and a probability of $\boldsymbol{2w}$ to each even number. Since the sum of the probabilities must be $\mathbf{1}$, we have $\boldsymbol{9w = 1}$ or, $\boldsymbol{w = \frac{1}{9}}$. Hence, probabilities of $\boldsymbol{\frac{1}{9}}$ and $\boldsymbol{\frac{2}{9}}$are assigned to each odd and even number, respectively. Therefore, $\boldsymbol{E = \{1, 2, 3\}}$ and $\boldsymbol{P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}}$.

Example: In the previous example, let $\boldsymbol{A}$ be the event that an even number turns up and let $\boldsymbol{B}$ be the event that a number divisible by $\mathbf{3}$ occurs. Find $\boldsymbol{P(A \cup B)}$ and $\boldsymbol{P(A \cap B)}$.
Sol: For the events $\boldsymbol{A = \{2, 4, 6\}}$ and $\boldsymbol{B = \{3, 6\}}$, we have $\boldsymbol{A \cup B = \{2, 3, 4, 6\}}$ and $\boldsymbol{A \cap B = \{6\}}$. By assigning a probability of $\boldsymbol{\frac{1}{9}}$ to each odd number and $\boldsymbol{\frac{2}{9}}$ to each even number, we have
$\boldsymbol{P(A \cup B) = \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{7}{9}}$ and, $\boldsymbol{P(A \cap B) = \frac{2}{9}}$.