Chapter – 6.4

Find the maximum likelihood estimator for the Bernoulli distribution $\boldsymbol{f(x;p) = p^x(1-p)^{1-x}}$.

Solution: Let us consider

$$L(p) = \prod_{i=1}^{n} f(x_i\,; p)$$
$$= \prod_{i=1}^{n} p^{x_i}\,(1-p)^{1-x_i}$$
$$= p^{\sum x_i}(1-p)^{n-\sum x_i}$$

$$\ln\big(L(p)\big) = \left(\sum x_i\right)\ln(p) + \left(n - \sum x_i\right)\ln(1-p)$$
$$\frac{d}{dp}\ln\big(L(p)\big) = \frac{\sum x_i}{p} + \frac{n-\sum x_i}{1-p}$$

$$\frac{d}{dp}\ln\big(L(p)\big) = 0$$
$$\Longrightarrow \frac{\sum x_i}{p} = \frac{n-\sum x_i}{1-p}$$
$$\Longrightarrow \sum x_i - p\sum x_i = np - p\sum x_i$$
$$\Longrightarrow np = \sum x_i$$
$$\Longrightarrow p = \frac{\sum x_i}{n}$$

$$\frac{d^2}{dp^2}\ln\big(L(p)\big) = -\frac{\sum x_i}{p^2} - \frac{n-\sum x_i}{(1-p)^2} < 0 \text{ for } p = \frac{\sum x_i}{n}$$

$$\therefore \hat{p} = \frac{\sum x_i}{n} = \bar{x}$$

Example 1: Find the maximum likelihood estimator for the Exponential distribution $\boldsymbol{f(x;\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}}}$.

Solution: Let us consider

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} f(x_i\,;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\theta} e^{\frac{-x_i}{\theta}} \\
&= \frac{1}{\theta^n} e^{\frac{-\sum x_i}{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
\ln\big(L(\theta)\big) &= -n\ln(\theta) - \frac{\sum x_i}{\theta} \\
\frac{d}{d\theta}\ln\big(L(\theta)\big) &= -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2}
\end{aligned}
$$

$$\frac{d}{d\theta}\ln\big(L(\theta)\big) = 0$$

$$\Rightarrow\ -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0$$

$$\Rightarrow\ \frac{\sum x_i}{\theta^2} = \frac{n}{\theta}$$

$$\Rightarrow\ n\theta = \sum x_i$$

$$\Rightarrow\ \theta = \frac{\sum x_i}{n}$$

$$\frac{d^2}{d\theta^2}\ln\big(L(\theta)\big) = -\frac{n}{\theta^2} - \frac{2\sum x_i}{\theta^3} < 0 \text{ for } \theta = \frac{\sum x_i}{n}$$

$$\therefore\ \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

Example 2: Find the maximum likelihood estimator for the Geometric distribution $\boldsymbol{f(x;p) = p(1-p)^{x-1}}$.

Solution: Let us consider

$$L(p) = \prod_{i=1}^{n} f(x_i\,; p)$$

$$= \prod_{i=1}^{n} p\,(1-p)^{x_i - 1}$$

$$= p^n (1-p)^{\sum x_i - n}$$

$$\ln\big(L(p)\big) = n\ln(p) + \left(\sum x_i - n\right)\ln(1-p)$$

$$\frac{d}{dp}\ln\big(L(p)\big) = \frac{n}{p} - \frac{\sum x_i - n}{1-p}$$

$$\frac{d}{dp}\ln\big(L(p)\big) = 0$$

$$\Rightarrow \frac{n}{p} = \frac{\sum x_i - n}{1-p}$$

$$\Rightarrow n - np = p\sum x_i - \mathrm{np}$$

$$\Rightarrow p\sum x_i = n$$

$$\Rightarrow p = \frac{n}{\sum x_i}$$

$$\frac{d^2}{dp^2}\ln\big(L(p)\big) = -\frac{n}{p^2} - \frac{\sum x_i - n}{(1-p)^2} < 0 \text{ for } p = \frac{n}{\sum x_i}$$

$$\therefore \hat{p} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

Example 3: Find the maximum likelihood estimator for the Binomial distribution $\boldsymbol{f(x;\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta_1)^2}{2\theta_2}}}$.

Solution: Let us consider

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} f(x_i\,;\theta_1,\theta_2)$$

$$= \prod_{i=1}^{n} 2\pi\theta_2{}^{-\frac{1}{2}}\, e^{\frac{-(x_i-\theta_1)^2}{2\theta_2}}$$

$$= (2\pi\theta_2)^{-\frac{n}{2}}\, e^{\frac{-\sum(x_i-\theta_1)^2}{2\theta_2}}$$

$$\ln\,(L(\theta_1,\theta_2) = -\frac{n}{2}\ln(2\pi\theta_2) - \frac{\sum(x_i-\theta_1)^2}{2\theta_2}$$

$$\frac{d}{d\theta_1}\ln\,(L(\theta_1,\theta_2) = -\frac{2\sum(x_i-\theta_1)^2}{2\theta_2}(-1)$$

$$\frac{d}{d\theta_1}\ln\big(L(\theta_1,\theta_2)\big) = 0$$

$$\Rightarrow -\frac{\sum(x_i-\theta_1)^2}{\theta_2} = 0$$

$$\Rightarrow \sum x_i - n\theta_1 = 0$$

$$\Rightarrow n\theta_1 = \sum x_i$$

$$\Rightarrow \theta_1 = \frac{\sum x_i}{n}$$

$$\frac{d^2}{d\theta_1{}^2}\ln\,(L(\theta_1,\theta_2) = -\frac{n}{\theta_2} < 0 \;\text{ for }\; \theta_1 = \frac{\sum x_i}{n}$$

$$\therefore\ \widehat{\theta_1} = \frac{\sum x_i}{n} = \mu$$

$$\ln\left(L(\theta_1, \theta_2\right) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta_2) - \frac{\sum(x_i - \mu)^2}{2\theta_2}$$

$$\frac{d}{d\theta_2}\ln\left(L(\theta_1, \theta_2\right) = -\frac{n}{2\theta_2} + \frac{\sum(x_i - \mu)^2}{2{\theta_2}^2}$$

$$\frac{d}{d\theta_2}\ln\big(L(\theta_1, \theta_2)\big) = 0$$

$$\Longrightarrow \frac{n}{2\theta_2} = \frac{\sum(x_i - \mu)^2}{2{\theta_2}^2}$$

$$\Longrightarrow \theta_2 = \frac{\sum(x_i - \mu)^2}{n}$$

$$\frac{d^2}{d{\theta_2}^2}\ln\left(L(\theta_1, \theta_2\right) = \frac{n}{2{\theta_2}^2} - \frac{\sum(x_i - \mu)^2}{{\theta_2}^3} < 0 \text{ for } \theta_2 = \frac{\sum(x_i - \mu)^2}{n}$$

$$\therefore \widehat{\theta_2} = \frac{\sum(x_i - \mu)^2}{n} = \sigma^2$$

Exercise 3: Find the maximum likelihood estimator for the Poisson distribution $f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$.

Solution: Let us consider

$$
\begin{aligned}
L(\lambda) &= \prod_{i=1}^{n} f(x_i\,;\lambda) \\
&= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\
&= \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n}(x_i\,!)} \\
&= \frac{e^{-n\lambda}\lambda^{\sum x_i}}{k}
\end{aligned}
$$

$$\ln\big(L(\lambda)\big) = -n\lambda + \left(\ln(\lambda)\sum x_i\right) - \ln(k)$$

$$\frac{d}{d\lambda}\ln\big(L(\lambda)\big) = -n + \frac{\sum x_i}{\lambda}$$

$$\frac{d}{d\lambda}\ln\big(L(\lambda)\big) = 0$$

$$\Longrightarrow -n + \frac{\sum x_i}{\lambda} = 0$$

$$\Longrightarrow \frac{\sum x_i}{\lambda} = n$$

$$\Longrightarrow \lambda = \frac{\sum x_i}{n}$$

$$\frac{d^2}{d\lambda^2}\ln\big(L(\lambda)\big) = -\frac{\sum x_i}{\lambda^2} < 0 \text{ for } \lambda = \frac{\sum x_i}{n}$$

$$\therefore\ \hat{\lambda} = \frac{\sum x_i}{n} = \bar{x}$$