→ CDF defined for a cont's r.v:

$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$ → (For Continuous Random variable)

Cumulative distribution Function:

cdf is defined as

$$\boxed{F_X(x) = P(X \leq x) \;;\; -\infty < x < \infty}$$

Properties:

Discrete Random variable

1) $F_X(x) = P(X \leq x) = \sum_{x_i \leq x} f_X(x_i)$

2) $P(a \leq X \leq b) = \cancel{\int_a^b f_X(x)\, dx}\; F_X(b) - F_X(a)$

X = Ran. Var.

x = sample space ($x \in S$)

4) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

3) Probability value always lies betn 0 and 1

(Last ma. 9 mark)

Problem: Find CDF of the following function:

$$f(x) = \begin{cases} \frac{x}{2} & ; \; 0 \le x \le 1 \\ 1/2 & ; \; 1 \le x \le 2 \\ \frac{1}{2}(3-x) & ; \; 2 \le x \le 3 \\ 0 & ; \; \text{otherwise} \end{cases}$$

$$-\infty \quad 0 \quad 1 \quad 2 \quad 3 \quad \infty$$

$0 \le x \le 1$,

As per Formula;

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} dx$$

$$= \int_{0}^{x} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{1}{4}x^2$$

For $x \le 0$,

$$\int_{-\infty}^{0} f\,dx = 0$$

$1 \le x \le 2$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= 0 + \left[\frac{1}{4}x^2\right]_0^1 + \frac{1}{2}[x]_1^x$$

$$= \frac{1}{4}[1-0] + \frac{1}{2}[x-1]$$

$$= 1/4 + \frac{1}{2}x - 1/2$$

$$= 1/2\,x - 1/4$$

$2 \le x \le 3$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^x f(x)\,dx$$

$$= 0 + 1/4 + \frac{1}{2}[x]_1^2 + \int_2^x \frac{1}{2}(3-x)\,dx$$

$$= 1/4 + 1/2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= 1/4 + 1/2 + \frac{1}{2}\left[3x - \frac{x^2}{2} - (6-2)\right]$$

$$= -5/4 + \frac{3x}{2} - \frac{x^2}{4}$$

[Ans]

For $x > 3$, $\int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} f\,dx + \int_0^1 f\,dx + \int_1^2 f\,dx + \int_2^3 f\,dx + \int_3^x f\,dx = 1$

So, C.D.F $= \begin{cases} 0; & x<0 \\ \frac{1}{4}x^2; & 0 \le x < 1 \\ \frac{1}{2}x - \frac{1}{4}; & 1 \le x < 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}; & 2 \le x \le 3 \\ 1; & x > 3 \end{cases}$

Ans.

Previous example: $(16, 18, 19, 20, 21, 30)$

$$P(X \le x) = F(x)$$

$$P(X \le 16) = \frac{1}{20}$$

$$P(X \le 18) = P(X = 16) + P(X = 18) = \frac{1}{20} + \frac{4}{20} = 5/2$$

$$P(X \le 19) = P(X = 16) + P(X = 18) + P(X = 19) = 14/20$$

A college statistics class has 20 students. The age are as follows: One student is 16 y/o, foure are 18 y/o, 3 are 20, two are 21, One is 30, 9 are Nineteen y/o. x = age of any random selected student. Find Pmf and CDF.

$$P(X \le 20) = 17/20$$

$$P(X \le 21) = 19/20$$

$$P(x \le 30) = 20/20 = 1$$

[Ans]

$P(X=16) = \frac{1}{20}$

$P(X=18) = \frac{4}{20}$

$P(X=19) = \frac{9}{20}$

$P(X=20) = \frac{3}{20}$

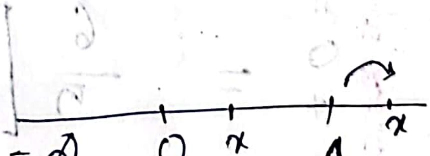
$P(X=21) = \frac{2}{20}$

$P(X=30) = \frac{1}{20}$

Problem: The ran. var. function is given by

$$f(x) = \begin{cases} K(x^2 + x) \,; & \text{if } 0 \le x \le 1 \\ 0 \,; & \text{else} \end{cases}$$

Find CDF.



Soln:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow K \int_0^1 (x^2 + x)\,dx = 1$$

$$\Rightarrow \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1/K$$

$$\Rightarrow \frac{1}{3} + \frac{1}{2} = 1/K$$

$\frac{2+3}{6} = 1/K$

$5K = 6$

$K = 6/5$

$$f(x) = \begin{cases} \frac{6}{5}(x^2 + x) & ; \underbrace{0 \le x \le 1} \\ 0 & ; \text{else} \end{cases}$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$0 \le x \le 1$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} \frac{6}{5}(x^2 + x)\,dx$$

$$= \frac{6}{5}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^x$$

$$= \frac{6}{5}\left\{\left[\frac{x^3}{3} + \frac{x^2}{2}\right] - 0\right\}$$

$$= \frac{6}{5}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]$$

$$\text{CDF} = F(x) = \begin{cases} 0 & ; x < 0 \\ \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) & ; 0 \le x \le 1 \\ 1 & ; x > 1 \end{cases}$$

[Ans]

$x > 1$: $\int_{-\infty}^{0} f\,dx + \int_{0}^{1} f\,dx + \int_{1}^{x} f\,dx$ ✓

$= 0 + 1 + 0 = 1$

Mathematical Expectation:

If $f(x)$ is the pmf (prob mass function) of random var. X of $\boxed{\text{discrete}}$ type then, the expected value of X is,

$$E(X) = \sum_{i=1}^{n} x_i f(x)$$

(Probability function)

When, $f(x)$ is prob. density function and X is $\boxed{\text{continuous}}$ Random variable, then expected value of X is

$$E(X) = \int_a^b x f(x)\, dx$$

Mean:

$$\mu = E(X) = \begin{cases} \sum_{i=1}^{n} x_i P(x_i) & x \text{ is discrete} \\ \int_{-\infty}^{\infty} x_i P(x_i)\, dx & x \text{ is continuous} \end{cases}$$

Variance:

$$\sigma^2 = E(x-\bar{x})^2$$
$$= E[x^2] - \{E[\bar{x}]\}^2$$

$E[\bar{x}] \downarrow \mu$ = mean

$$E(x-\mu)^2$$
$$= E(x^2 - 2x\mu + \mu^2)$$
$$= E(x^2) - 2E(x)\mu + \mu^2$$
$$= E[x^2] + \mu^2 - 2E[x]\mu$$
$$= E[x^2] + \mu^2 - 2\mu^2$$
$$= E[x^2] - \mu^2$$
$$= E[x^2] - \{E[x]\}^2$$

$$E[x^2 - 2x\bar{x} + \bar{x}^2]$$
$$= \cancel{E[x^2] + \bar{x}^2 - 2x\bar{x}}$$
$$= E[x^2] - 2E[x]\bar{x} + E[\bar{x}^2]$$
$$= E[x^2] -$$

Standard deviation:

$$\sigma = \sqrt{\text{variance}}$$

Book (Example - $\boxed{2.2-2}$ pg - 51)

Problem Assume, R.V. X have the pmf

$$f(x) = \frac{1}{3} \quad ; \quad x \in S_x$$

where $S_x = \{-1, 0, 1\}$

Find Mean and Variance.

Soln: Mean, $\mu = E(X) = \sum_{i=1}^{n} x_i P_i$

$f(0) = \frac{1}{3}$

$f(-1) = 1/3$

$f(1) = 1/3$

Mean,
Now, $E(X) = \sum x_i P_i$

$= x_1 P_1 + x_2 P_2 + x_3 P_3$

$= \left((-1) \cdot \frac{1}{3}\right) + \left(0 \cdot \frac{1}{3}\right) + \left(1 \cdot \frac{1}{3}\right)$

$= 0$

$E[X^2] = \sum x_i^2 P_i$

$= x_1^2 P_1 + x_2^2 P_2 + x_3^2 P_3$

$= (-1)^2 \cdot 1/3 + (0)^2 \cdot 1/3 + (1)^2 \cdot 1/3$

$= 1/3 + 1/3 = 2/3$

Variance, $\sigma^2 = E[X^2] - \{E[X]\}^2$

$= 2/3$

[Ans.]

Ch-2.2-2 (Exercise)

$\boxed{\text{Problem:}}$ Find $E(x)$ for the following distribution function.

Let X have pmf. then,

$$f(x) = \frac{(|x|+1)^2}{9} \quad ; \quad x = -1, 0, 1$$

Compute $E(X)$, $E(X^2)$ and $E(3X^2 - 2X + 4)$

<u>Solution:</u> $E(X) = \sum_{i=1}^{n} x_i p_i$

$= x_1 P_1 + x_2 P_2 + x_3 P_3$

$$P(X=-1) = \frac{(|-1|+1)^2}{9} = \frac{4}{9}$$

$$P(X=0) = \frac{(0+1)^2}{9} = \frac{1}{9}$$

$$P(X=1) = \frac{(1+1)^2}{9} = \frac{4}{9}$$

$E(X) = \{(-1) \cdot \frac{4}{9}\} + (0 \cdot \frac{1}{9}) + (1 \times \frac{4}{9})$

$= -\frac{4}{9} + 0 + \frac{4}{9}$

$= 0$

$E(X^2) = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3$

$= (-1)^2 \cdot 4/9 + 0 + 1 \cdot \frac{4}{9}$

$= \frac{4}{9} + \frac{4}{9}$

$= \frac{8}{9}$

$E(3X^2 - 2X + 4)$

$= 3E(X^2) - 2E(X) + 4$

$= 3 \times \frac{8}{9} - (2 \times 0) + 4$

$= \frac{24}{9} + 4$

$= \frac{20}{3}$

[Ans]

Exercise
Ch- 2.2 (4)

$P(X=x) = f(x)$
$\sum P(x) = 1$

Problem: An insurance company sells an automobile policy with a deductible of one unit. Let X be the amount of loss having pmf

$$f(x) = \begin{cases} 0.9 & ; x = 0 \\ \frac{c}{x} & ; x = 1, 2, 3, 4, 5, 6 \end{cases}$$

Determine c and the expected value.

<u>Soln:</u> $P(X = x) = f(x)$ [by defn of pmf]

Now, $P(X = 0) = 0.9$

$P(X = 1) = c$

$P(X = 2) = \frac{c}{2}$

$P(X = 3) = \frac{c}{3}$

$P(X = 4) = c/4$

$P(X = 5) = c/5$

$P(X = 6) = c/6$

Then, $\sum f(x) = 1$

$$\Rightarrow c + \frac{c}{2} + \frac{c}{3} + \frac{c}{4} + \frac{c}{5} + \frac{c}{6} + 0.9 = 1$$

$$\Rightarrow \frac{60c + 30c + 20c + 15c + 12c + 10c + 54}{60} = 1$$

Course Code | Trimester / Semester : Spring / Summer / Fall, 20........

Name of Exam : Class Test / Mid-term / Final | Date: ....

$$\Rightarrow c = \frac{6}{147} = \frac{2}{49}$$

Now, $pmf = \begin{cases} 0.9 & ; x = 0 \\ \frac{2}{49x} & ; x = 1, 2, 3, 4, 5, 6 \end{cases}$

$f(x=0) = 0.9$

$f(x=1) = \frac{2}{49}$

$f(x=2) = \frac{2}{49 \times 2}$

$= 0.0204$

$f(x=3) = \frac{2}{49 \times 3}$

$= 0.01360$

$f(x=4) = \frac{2}{49 \times 4}$

$= 0.0102$

$f(x=5) = \frac{2}{49 \times 5}$

$= 0.0081$

$f(x=6) = \frac{2}{49 \times 6}$

$= 0.0068$

Now, $E(X)$

$$= \sum_{x=0}^{6} x f(x)$$

$$= (0 \times 0.9) + \frac{2}{49} + 0.0204$$

$$+ 0.01360 + 0.01020$$

$$+ 0.0081 + 0.0068$$

$$= 0.0999163$$

[Ans.]

$n-1=2$
$n=3$

Variance: $E(X^2) = 2\int_0^\infty x^2 e^{-2x}\,dx$

$= 2\frac{\Gamma 3}{2^3}$

$= \frac{(3-1)!}{4}$

$= \frac{2!}{4} = \frac{1}{2}$

Variance,
$\sigma^2 = E(X^2) - \{E(X)\}^2$
$= 1/2 - 1/4$
$= \frac{2-1}{4}$
$\Rightarrow 1/4$

Problem: A continuous R.V. X has density function, given by

$$f(x) = \begin{cases} 2e^{-2x} & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find Expected value and variance of X.

Soln: For conts R.V.,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x\,2e^{-2x}\,dx$$

$$= 2\int_0^\infty x e^{-2x}\,dx$$

$$= 2\left(\frac{\Gamma 2}{2^2}\right) = \frac{\Gamma 2}{2} = \frac{1!}{2} = 1/2$$

[Ans]

$n-1=1$
$\Rightarrow n = 1+1$
$\therefore n=2$
$-a=-2$
$\therefore a=2$

$[\Gamma n = (n-1)!]$

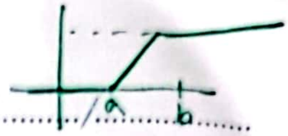
Gamma

$\left[\because \int_0^\infty x^{n-1} e^{-ax}\,dx = \frac{\Gamma n}{a^n}\right]$

$=1+1=2$

Problem: A cont^s. random variable X has prob. density function $f(x) = 6(\sqrt{x} - x)$;

$0 \le x \le 1$:



Date : ....../....../......

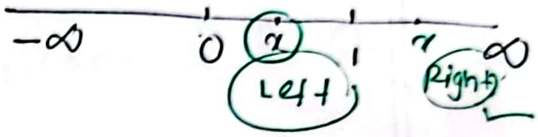
Soln: $P(X \le x) = F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx \quad ; 0 \le x \le 1$$

For ; $x < 0$, $\int_{-\infty}^{0} f_X(x)\,dx = 0$

For $0 \le x \le 1$,

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$



$$= 0 + \int_{0}^{x} 6(\sqrt{x} - x)\,dx$$

$$= 4x^{\frac{3}{2}} - 3x^2$$

For $x \ge 1$, , $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$

$$= 0 + 1 + 0$$

$$= 1$$

$$\therefore \quad \text{C.D.F.} = \begin{cases} 4x^{\frac{3}{2}} - 3x^2 & ; 0 \le x \le 1 \\ 0 & ; x < 0 \\ 1 & ; x \ge 1 \end{cases}$$

→ Example → (2 ...)

Problem: Let X have a distn on the first m positive integers. ~~The mean of X is~~

$$f(x) = \frac{1}{m}$$

Find mean and variance.

Soln: Mean, $\mu = E(X) = \sum x f(x)$

$$= \sum x \frac{1}{m}$$

$$= \cancel{x \sum \frac{1}{m}}$$

$$= \frac{1}{m} \sum_{x=1}^{m} x = \frac{1}{\cancel{m}} \frac{\cancel{m}(m+1)}{2}$$

$$\therefore \text{Mean}, \mu = \frac{m+1}{2}$$

$$\sum_{x=1}^{n} x^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Variance, $\sigma^2 = E(X^2) - \{E(\bar{X})\}^2$

$$E(X^2) = \sum x^2 \frac{1}{m}$$

$$= \frac{1}{m} \sum_{x=1}^{m} x^2$$

$$= \frac{1}{\cancel{m}} \cdot \frac{\cancel{m}(m+1)(2m+1)}{6}$$

$$= \frac{(m+1)(2m+1)}{6}$$

Variance, $= \frac{(m+1)(2m+1)}{6} - \left(\frac{m+1}{2}\right)^2$

$$= \frac{m+1}{2}\left[\frac{2m+1}{3} - \frac{m+1}{2}\right]$$

$$= \frac{m+1}{2}\left[\frac{4m+2-3m-3}{6}\right]$$

$$= \frac{m+1}{2}\left[\frac{m-1}{6}\right]$$

$$= \frac{m^2-1}{12}$$

[Ans.]

1

Mean, σ, Variance

Problems:

⇒ Compute moments of a random variable.

Moment generating Function (M.g.f):

When, x is $\boxed{\text{discrete}}$ Random variable,

then,

$$M_X(t) = E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

M_X → Ran. Var.

S → sample space

This $M_X(t)$ is defined as M.g.f.

When X is $\boxed{\text{continuous}}$ R.V, then,

$$M_X(t) = \int e^{tx} f(x)\, dx$$

Particular form of M.g.f for Probab. distribution

Now,

$$M_X(t) = E(e^{tX})$$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^r}{r!}X^r + \cdots\right)$$

$$= 1 + t\underbrace{E(X)} + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

(General form of derivation)

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

μ_1' ↓ First moment about origin

μ_2' ↓ 2nd moment about origin

$r = 1$, 1st moment about origin...

$r = 2$, 2nd moment about origin
(variance)

To find μ'_r, [M.g.f about origin]

$$\boxed{\mu'_r = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}}$$

$$\boxed{\text{Variance}, \ \sigma^2 = M''(0) - \{M'(0)\}^2}$$

Example (2.3-7)

Problem: Assume, X has the geometric distribution, the p.m.f of X is

$$f(x) = q^{x-1} p \quad ; \quad x = 1, 2, 3, \ldots$$

Find Moment generating function. Use the result to compute $E(X)$ and $Var(X)$. [Here, $q = 1-p$]

Soln: By def^n,

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} e^{tx} q^x$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

Geometric Series

$$\cancel{\sum_{x=1}^{\infty} \frac{(tx)^n}{n!}}$$

$$= \cancel{\frac{p}{q}\left[q^x \frac{(tx)^n}{n!}\right]}$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3\right.$$

$$\frac{(qe^t)^2}{qe^t} = qe^t \qquad \frac{a}{1-r}$$

$$M(t) = \frac{\cancel{q}e^t}{1-qe^t} \cdot \frac{p}{\cancel{q}} \qquad qe^t < 1$$

$$= \frac{pe^t}{1-qe^t}$$

$\ln(qe^t) < \ln(1)$

$\ln(q) + t\ln e < 0$

$t < -\ln q$

We've to find Mean and Variance,

Mean, $E(X) = M'_X(t)\Big|_{t=0}$ [ ]

Thus, $E(X) = \left[\frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\right]_{t=0}$

$$= \left[\frac{(1-qe^t)\cdot pe^t - pe^t(-e^t q)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

The second moment is,

$$E(X^2) = M''_X(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[M'(t)\right]$$

$$= \frac{(1-qe^t)^2\, pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-2qe^t+q^2e^{2t})pe^t + 2pqe^{2t}(1-qe^t)}{(1-qe^t)^4}$$

$$= \frac{pe^t - \cancel{2pqe^{2t}} + pq^2e^{3t} + \cancel{2pqe^{2t}} - 2pq^2e^{3t}}{(1-qe^t)^4}$$

$$\cancel{= \frac{pe^t - 4pqe^{2t} + 3pq^2e^{3t}}{(1-qe^t)^4}}$$

$$= \frac{pe^t - pq^2e^{3t}}{(1-qe^t)^4}$$

$$= \frac{pe^t(1-q^2e^{2t})}{(1-qe^t)^4}$$

$$= \frac{pe^t(1-(qe^t)^2)}{\{(1-qe^t)^2\}^2}$$

$$= \frac{pe^t(1+qe^t)(1-qe^t)}{(1-qe^t)^2(1-qe^t)^2}$$

$$= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$E(x^2) - \{E(x)\}^2$$

$$M''_x(t)\Big|_{t=0} = \frac{p(1+q)}{(1-q)^3}$$

$$= \frac{p(1+q)}{p^3} \quad \left[\begin{aligned}1-p &= q\\ \therefore 1-q &= p\end{aligned}\right]$$

$$= \frac{1+q}{p^2}$$

Now, Variance, $\sigma^2 = M''(0) - \{M'(0)\}^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2}$$

$$= \frac{q}{p^2}$$

$\therefore$ ~~Std~~ standard deviation, $\sigma = \sqrt{q/p^2}$

$$= \frac{\sqrt{q}}{p}$$

$$= \frac{q}{p\sqrt{q}} \quad \text{[Ans.]}$$

Exercise (9.3-2)

Problem: For ~~each of~~ the following distribution

Find, $\mu = E(X)$, $E[X(X-1)]$ and $\sigma^2 = E[X(X-1)] + E(X) - \mu^2$

*//

$$f(x) = \frac{3!}{x!(3-x)!} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x} \quad ; x = 0, 1, 2,$$

Soln: Mean, $\mu = E(X)$

$$= \Sigma x_i P_i$$

$$= x_1 P_1 + x_2 P_2 + x_3 P_3 + x_4 P_4$$

$$= 0 + \left(1 \times \frac{3!}{(3-1)!} \cdot \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{3-1}\right)$$

$$+ \left(2 \times \frac{3!}{2!(3-2)!} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{3-2}\right) +$$

$$\left(3 \times \frac{\cancel{3!}}{\cancel{3!}(3-3)!} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{3-3}\right)$$

$$= \cancel{\frac{27}{128}} \cancel{+ \frac{9}{32}} \cancel{+ \frac{3}{64}}$$

$$= \left\{\frac{3 \times \cancel{2}}{\cancel{2} \times 1} \times \frac{1}{4} \times \left(\frac{3}{4}\right)^2\right\} + \left(2 \times \frac{6}{2 \times 1! \times \frac{3}{4}} \times \frac{1}{16}\right)$$

$+ \left(3 \times \frac{1}{64} \times \left(\frac{3}{4}\right)^0 \right)$

$= \frac{27}{64} + \frac{9}{32} + \frac{3}{64}$

$= \frac{3}{4}$

And $E(x(x-1)) = \sum x_i P_i$

$= \sum x(x-1) P_i$

$= \cancel{x_1(x_1-1) P_1 + x}$

$= \sum (x^2 - x) P_i$

$= (x_1^2 - x_1) P_1 + (x_2^2 - x_2) P_2 + (x_3^2 - x_3) P_3 + (x_4^2 - x_4) P_4$

$= \frac{3}{8}$

$x_1 = 0 \; ; P_1 =$
$x_2 = 1 \; ; P_2 =$
$x_3 = 2 \; ; P_3 =$
$x_4 = 3 \; ; P_4 =$

$\sigma^2 = E(x(x-1)) + E(x) - \mu^2$

$= \frac{3}{8} + \frac{3}{4} - \left(\frac{3}{4}\right)^2$

$= \frac{9}{16}$ [Ans]

Exercise (2.3-8)

Problem: Let, X equal the larger outco[me] when a pair of fair four sided dice is rolled. The pmf of X is,

$$f(x) = \frac{2x-1}{16}, \quad x = 1,2,3,4 \text{ (Discrete)}$$

Find mean, variance and S.D. of X

Solⁿ: To Find mean, $\mu = E(X)$

$$= \sum x_i P_i$$

$$= x_1 P_1 + x_2 P_2 + x_3 P_3 + x_4 P_4$$

$$= \left(1 \times \frac{1}{16}\right) + \left(2 \times \frac{3}{16}\right) + \left(3 \times \frac{5}{16}\right) + \left(4 \times \frac{7}{16}\right)$$

$$= \frac{1}{16} + \frac{3}{8} + \frac{15}{16} + \frac{28}{16}$$

Mean, $\mu = \frac{25}{8}$

Variance, $\sigma^2 = E(X^2) - \{E(X)\}^2$

$$E(x^2) = \Sigma x^2 P(x)$$
$$= x_1^2 P(1) + x_2^2 P(2) + x_3^2 P(3) + x_4^2 P(4)$$
$$= \frac{1}{16} + \left(4 \times \frac{3}{16}\right) + \left(9 \times \frac{5}{16}\right) + \left(16 \times \frac{7}{16}\right)$$
$$= \frac{85}{8}$$

So, Variance, $\sigma^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2$
$$= \frac{55}{64}$$

and S.D., $\sigma = \sqrt{\frac{55}{64}}$
$$= \frac{\sqrt{55}}{8} \quad [\text{Ans.}]$$

Exercise (2.3-11)

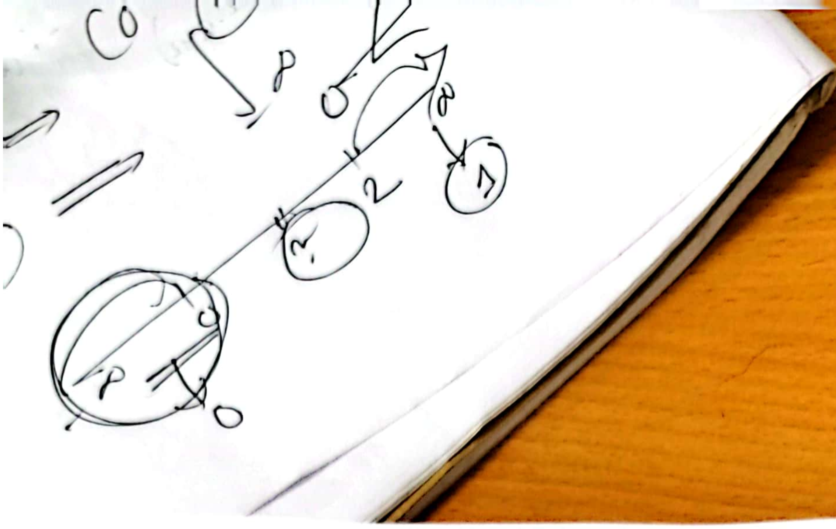
Problem: If M.g.f of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$$

Find the mean, variance and pmf of X.

$\big|_{t=0}$

$t)\big)\big|_{t=0}$



Solm: Given,

$$M_X(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$$

Then, Mean, $\mu_1' = M'(t)$ [Moment ab. origin]

$$= \frac{d}{dt}\left(M(t)\right)$$

$$= \frac{d}{dt}\left[\frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}\right]$$

$$= \left[\frac{2}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}\right]$$

$$M'(t) = 2$$

and $\mu_2' = \frac{d}{dt}\left[M'(t)\right]$

$$= \left[\frac{2}{5}e^{t} + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}\right]_{t}$$

$$= \frac{2}{5} + \frac{4}{5} + \frac{18}{5}$$

$$M''(t) = \frac{24}{5}$$

Variance

Variance, $\sigma^2 = M''(0) - \{M'(0)\}^2$

$$= \frac{24}{5} - 4$$

$$= \frac{24-20}{5}$$

$$= 4/5$$

From $M(t)$, It is ~~sho~~ seen,

$P(X=1) = \frac{2}{5}$

$P(X=2) = \frac{1}{5}$

$P(X=3) = 2/5$

and these are p.m.f.

[Ans.]

Exercise (2.3-16)

Problem: Let, X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.

* a) Find pmf of X
  b) Find m.g.f of X.

c) Use m.g.f to find mean and variance.

d) Find values of (i) $P(X \le 3)$

(ii) $P(X \geq 5)$ and (iii) $P(X = 3)$

Soln: a) Let, $X = \{1, 2, 3, 4, \ldots n\}$

For flipping coin one time,

H, T $\quad P(1) = 0$

HT, TH, TT, HH $= \frac{2}{4} = 1/2 \quad P(2) = \frac{1}{2}$

HHH, TTT, THH, HTT $\quad P(3) = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$4/16 = 1/4$

$$P(n) = \frac{1}{2^{n-1}} \; ; \; n = 2, 3, \ldots$$

Then, Probability mass function, p.m.f

$$f(x) = P(X = x) = \frac{1}{2^{n-1}} \; ; \; n = 2, 3, \ldots$$

$\frac{1}{4} \div \frac{1}{2}$ $\frac{1}{4} \times \frac{2}{1}$ $= \frac{1}{2}$

b) by defn,

$$M_X(t) = \sum e^{tx} f(x)$$

$$= \sum e^{tx} \frac{1}{2^{x-1}} \quad [x = 2, 3, \ldots]$$

$$= e^{2t} \frac{1}{2^{2-1}} + e^{3t} \frac{1}{2^{3-1}} + e^{4t} \frac{1}{2^{4-1}} + \ldots$$

$$= \frac{1}{2} e^{2t} + \frac{1}{2^2} e^{3t} + \frac{1}{2^3} e^{4t} + \ldots$$

Common Ratio,
$r = \frac{1}{2} e^t$

$$\therefore M_X(t) = \frac{\frac{1}{2} e^{2t}}{1 - \frac{1}{2} e^t}$$

$$= \frac{\frac{1}{2} e^{2t}}{\frac{2 - e^t}{2}}$$

$$= \frac{1}{\cancel{2}} e^{2t} \times \frac{\cancel{2}}{2 - e^t}$$

$$= e^{2t} (2 - e^t)^{-1}$$

$$M_X(t) = \frac{e^{2t}}{2 - e^t}$$

$\frac{e^{3t}}{e^{2t}} = e^x$

$\frac{1}{4} \div \frac{1}{2}$ $\frac{1}{4} \times \frac{2}{1}$ $= \frac{1}{2}$

2
2
t
e^{3t}
$2^{4-1} \cdot 3t$
e^t
$2t$
$= e^t$

c) Mean, $\mu'_r = \frac{d^r}{dt^r}(M(t))$

then,

$$M'(t) = \frac{d}{dt}\left(\frac{e^{2t}}{2-e^t}\right)$$

$$= \frac{(2-e^t)\cdot 2e^{2t} - e^{2t}(-e^t)}{(2-e^t)^2}$$

$$= \frac{4e^{2t} - 2e^{3t} + e^{3t}}{(2-e^t)^2}$$

$$= \frac{4e^{2t} - e^{3t}}{(2-e^t)^2}$$

$$M''(t) = \frac{d}{dt}(M'(t))$$

$$= \frac{(2-e^t)^2(8e^{2t} - 3e^{3t}) - (4e^{2t} - e^{3t})\cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}$$

$$= \frac{(4 - 4e^t + e^{2t})(8e^{2t} - 3e^{3t}) + e^t(8e^{2t} - 2e^{3t})(2-e^t)}{(2-e^t)^4}$$

$+(8e^{3t} - 2e^{4t})(2-e^t)$

$$= \frac{32e^{2t} - 12e^{3t} - 32e^{3t} + 12e^{4t} + 8e^{4t} - 3e^{5t}}{(2-e^t)^4}$$

$$+16e^{3t} - 4e^{4t} - 8e^{4t} + 2e^{5t}$$

$$M''(t) = \frac{-44e^{3t} + 20e^{4t} - 3e^{5t} + 32e^{2t}}{(2-e^t)^4}$$

$$= \frac{-28e^{3t} + 8e^{4t} \cancel{\ 100}\, e^{5t} + 32e^{2t}}{(2-e^t)^4}$$

$$M'(0) = \frac{4-1}{(2-1)^2} = \frac{3}{1} = 3$$

$$M''(0) = \frac{11}{1} = 11$$

$$\text{Variance}\quad \sigma^2 = M''(0) - \{M'(0)\}^2$$

$$= 11 - (3)^2$$

$$= 11 - 9$$

$$\therefore \sigma^2 = 2$$

$$F(x) = \frac{1}{2^{x-1}} \quad ; x = 2, 3, 4$$

(Cumulative dist^n function)

d) (i) $P(x \leq 3)$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= 0 + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{2+1}{4}$$

$$= 3/4$$

(ii) $P(X > 5)$

$$= \cancel{P(X=6)+} \; 1 - P(X \leq 5)$$

$$= 1 - [P(X=2) + P(X=3) + P(X=4)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right]$$

$$= 1 - \frac{4+2+1}{8}$$

$$= 1 - \frac{7}{8}$$

$$= \frac{8-7}{8} = 1/8$$

$$P(X=3) = \frac{1}{2^{3-1}}$$

$$= \cancel{\frac{1}{2^{8}4}} \qquad \frac{1}{2^2}$$

$$= 1/4$$

[Ans.]