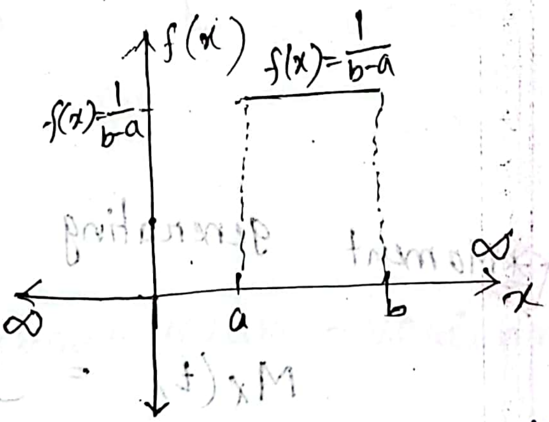
Date: 09.08.2023 Class → 06

Uniform distribution:

A random variable X is said to have a conts uniform distribution over an interval (a, b) if it's p.d.f is given by.

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \quad a < x < b \\ 0 & ; \quad \text{otherwise} \end{cases}$$



Cumulative distn function:

$$F(x) = P(X \leq x)$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{x} f(x)\,dx$$

$$= 0 + \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}[x]_a^x$$

$$= \frac{x-a}{b-a}$$

$\frac{1}{b-a}\int_a^y 2y\,dy$

$\frac{1}{b-a}\times\left[2\frac{y^2}{2}\right]_a^y$

$\frac{1}{b-a}\times\left[y^2\right]_a^y$

$= \frac{1}{b-a}\times\left[y^2-a^2\right]$

$= \frac{y^2-a^2}{b-a}$

$= \frac{y^2-0}{1-0}$

* X is said to have a uniform distribution if it's C.D.F,

$$f(x) = \begin{cases} 0 & ; x < a \\ \frac{x-a}{b-a} & ; a < x < b \\ 1 & ; \cancel{\text{otherwise}}\ b \leq x \end{cases}$$

*/ Example (3.3-1) and 3.3-3) => Pg-88, 91

Problem: Assume, Y be a continuous random variable with P.D.F. $g(y) = 2y$; $0 < y < 1$. Find C.D.F and draw the C.D.F graph. Also, Find Expected value, mean, variance and s.

Soln: by def^n,

$$P(X) = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

Now, for, $0 < y < 1$,

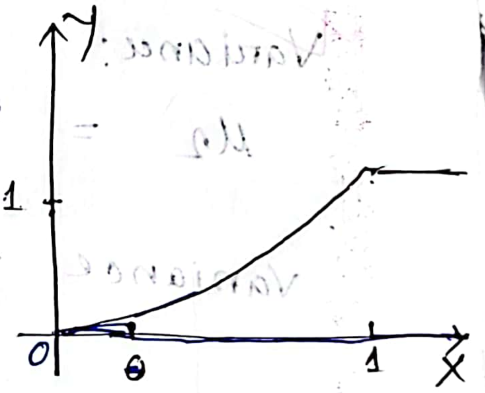
$$F(x) = \int_{\infty}^{0} f(y)\,dy + \int_{0}^{y} f(y)\,dy$$

$$= 0 + \int_{0}^{y} 2y\,dy$$

$$= 2\left[\frac{y^2}{2}\right]_0^y$$

$$= [y^2 - 0]$$

$$\therefore F(y) = y^2$$



C.D.F.

$$\text{C.D.F.} = \begin{cases} 0 & ; \ 0 < y \\ y^2 & ; \ 0 \le y < 1 \\ 1 & ; \ 1 \le y \end{cases}$$

$-\infty \quad 0 \quad 1/2 \quad 3/4$ $\int_{-\infty}^{\infty} x\,p(x)\,dx = \int_0^1$

$$P\left(\frac{1}{2} < Y \le \frac{3}{4}\right) = P\left(Y = \frac{3}{4}\right) - P\left(Y = 1/2\right)$$

$$= \left(\frac{3}{4}\right)^2 - \left(1/2\right)^2$$

$$= 5/16$$

and $P\left(1/4 \le Y < 2\right) = P(Y=2) - P(Y = 1/4)$

$$= 1 - (1/4)^2 = \frac{15}{16}$$

Expected value, $\underbrace{E(x)}_{\text{Mean}} = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \int_0^1 x\,f(x)\,dx$$

$$= \int_0^1 y.2y\,dy$$

$$= \int_0^1 2y^2\,dy$$

$$= 2\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3}$$

Variance $\sigma^2 = E(x^2) - \{E(x)\}^2$

~~Mean, μ =~~

~~$\mu = E(x^2) =$~~ $\int_0^1 y^2\,2y\,dy - \frac{4}{9}$

$$= 2\left[\frac{y^4}{4}\right]_0^1 - 4/9 = 1/18$$

$S.D. = \sqrt{\text{variance}} = \sqrt{1/18} \approx 0.235$

(Ans.)

Example (3.1-4)

Problem: Assume, X have the p.d.f

$$f(x) = \begin{cases} xe^{-x} & ; 0 \le x < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

Find Moment generating function and Mean. Also, find variance.

Soln: by defn, $M_X(t) = \int_0^\infty e^{tx} f(x)\, dx$

$$= \int_0^\infty e^{tx}\, x e^{-x}\, dx$$

$$= \int_0^\infty x\, e^{tx - x}\, dx$$

$$= \int_0^\infty x\, e^{(t-1)x}\, dx$$

$e^{(t-1)x} = u$

$\Rightarrow (t-1)\, e^{(t-1)x}\, dx = du$

$e^{(t-1)x}\, dx = \frac{du}{t-1}$

$(t-1)x = \ln u$

$x = \frac{\ln u}{t-1}$

$\int_0^\infty \frac{\ln u}{t-1}\, u$

Now, $\int \underset{u}{x}\, \underbrace{e^{(t-1)x}}_{v}\, dx$

$= x \int e^{(t-1)x}\, dx - \int \left\{ \frac{d}{dx}(x) \int e^{(t-1)x}\, dx \right\} dx$

$\frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$= x\frac{e^{(t-1)x}}{t-1} - \int \frac{e^{(t-1)x}}{t-1}\,dx$$

$$= \frac{xe^{(t-1)x}}{t-1} - \frac{1}{t-1}\cdot\frac{e^{(t-1)x}}{t-1}$$

$$= \left[\frac{xe^{-(1-t)x}}{1-t} - \frac{e^{-(1-t)x}}{(1-t)^2}\right]_0^{\infty}$$

$$= \frac{\infty e^{-\infty}}{1-t} - \frac{e^{-\infty}}{(1-t)^2} - \left[0 - \frac{e^0}{(1-t)^2}\right]$$

$$= 0 - 0 - \left[0 - \frac{1}{(1-t)^2}\right]$$

$$= \frac{1}{(t-1)^2}$$

$$= \cancel{-\frac{e^{-(1-t)0}}{(1-t)^2}}$$

$$= \cancel{-\frac{1}{(1-t)^2}}$$

$$\boxed{M(t) = \frac{1}{(t-1)^2}}$$

Then, Mean, $M'(t)\big|_{t=0} = -2\,(t-1)^{-3}\Big|_{t=0}$

$$= -2(0-1)^{-3}$$

$$= -2(-1)$$

$$\therefore M'(0) = 2$$

Now, $M''(t) = 6\,(t-1)^{-4}\Big|_{t=0}$

$$= 6$$

and Variance, $\sigma^2 = M''(0) - \{M'(0)\}^2$

$$= 6 - 2^2 = 2$$

(Exercise)

Problem: (3.1 - 7) $f(x) = 4x^c \; ; 0 \le x \le 1$

Find the constant c for the following funct so that a) $f(x)$ is a p.d.f of a random variable.

b) Find the C.D.F, $F(x) = P(X \le x)$.

c) sketch graphs of pdf $f(x)$ and cdf $F(x)$

and

d) Find μ and σ^2

μ → Mean

σ^2 → variance

Soln:

a) For P.d.f,

$$\int_{\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 4x^c \, dx = 1$$

$$\Rightarrow \int_0^1 x^c \, dx = \frac{1}{4}$$

$$\Rightarrow \left[\frac{x^{c+1}}{c+1}\right]_0^1 = \frac{1}{4}$$

$$\Rightarrow \frac{x^2}{2} - \frac{\cancel{2x}}{1} = 1/4$$

$$\Rightarrow \frac{x^2 - 2x}{2} = 1/4$$

$$\Rightarrow 4(x^2 - 2x) = 2$$

$$\Rightarrow x^2 - x = \frac{1}{2}$$

$$\Rightarrow 2x^2 - 2x - 1 = 0$$

$$2x^2 - x - x - 1 = 0$$

$$x^2 - 2x = \frac{1}{2}$$

$$2x^2 + 4x - 1 = 0$$

$$\Rightarrow \cancel{2x^2} \quad x = \frac{2+\sqrt{6}}{2}, \frac{2-\sqrt{6}}{2}$$

$$\Rightarrow \cancel{\tfrac{1}{2}} \frac{4}{c+1} = \frac{1}{\cancel{4}}$$

$$\Rightarrow c + 1 = 4$$

$$\therefore c = 4 - 1 = 3$$

b) To Find C.D.F,

$$F(x) = P(X \le x)$$

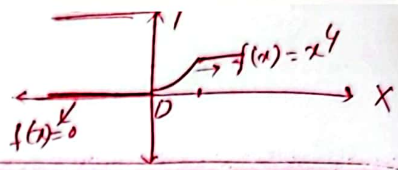
$$= \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 4x^3\,dx + \int_0^x 4x^3\,dx + \int^{\infty} f(x)$$

$$= 4\left[\frac{x^4}{4}\right]_0^x$$

$$= x^4$$

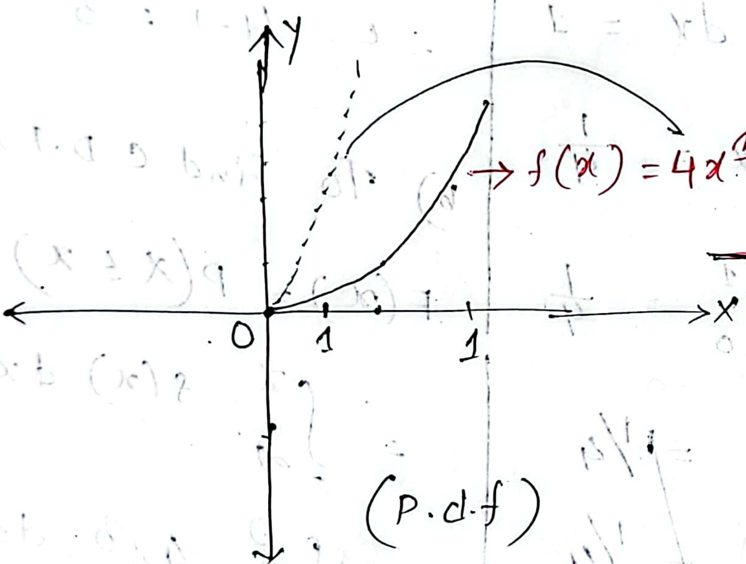
$$\text{C.D.F} = \begin{cases} 0 & ; x < 0 \\ x^4 & ; 0 < x < 1 \\ 1 & ; x > 1 \end{cases}$$

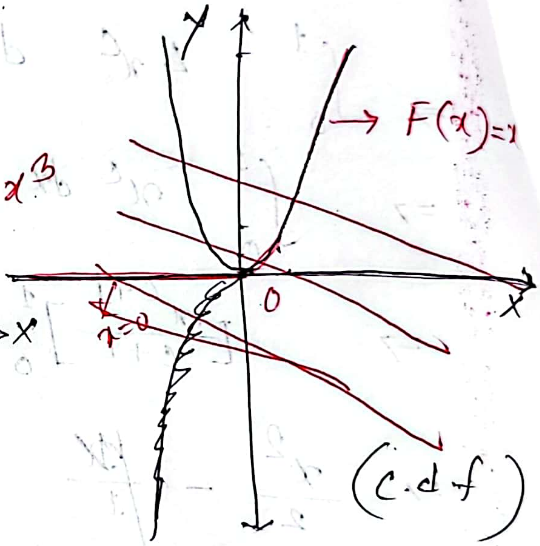


c) The C.D.F =>

$F(x) = x^4$

P.d.f ; $f(x) = 4x^3$; $0 \le x \le 1$





d) $E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^1 x \cdot 4x^3 \, \cancel{f(x)}\, dx$

$= \int_0^1 4x^4 \, dx$

$= \frac{4}{5} = 0.8$

Variance, $\sigma^2 = E(X^2) - \mu^2$

$$E(X^2) = \int_0^1 x^2 \cdot 4x^3 \, dx$$

$$= \int_0^1 4x^5 \, dx$$

$$= \left[\frac{4x^6}{6}\right]_0^1$$

$$= 2/3$$

$$\therefore \sigma^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} \approx 0.0267$$

[Ans.]

Exercise

Problem: (3.1-10)

The pdf of x is $f(x) = \frac{c}{x^2}$; $1 < x < \infty$

a) calculate the value of c so that $f(x)$ is pdf.

b) show that $E(X)$ is not finite.

Solⁿ: a) $\int_{-\infty}^{\infty} f \, dx = 1$

$$= \int_1^{\infty} \frac{c}{x^2} \, dx = 1$$

$$\Rightarrow C\left[\frac{x^{-2+1}}{-2+1}\right]_1^\infty = 1$$

$$\Rightarrow \left(\frac{x^{-1}}{-1}\right)_1^\infty = 1/c$$

$$\Rightarrow -\left(\frac{1}{\infty} - \frac{1}{0}\right) = \frac{1}{c}$$

$$\Rightarrow -(0-\infty) = \frac{1}{c}$$

$$\Rightarrow \infty = \frac{1}{c}$$

$$\Rightarrow c = 0$$

$$\Rightarrow -\left(\frac{1}{\infty} - 1\right) = \frac{1}{c}$$

$$\Rightarrow -(0-1) = \frac{1}{c}$$

$$\Rightarrow 1 = 1/c$$

$$\therefore \boxed{c = 1}$$

So, $f(x) = \frac{1}{x^2}$; $1 < x < \infty$

b) $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_1^\infty x\, x^{-2}\, dx$$

$$= \int_1^\infty x^{-1}\, dx$$

$$= [\ln x]_1^\infty$$

$$= \infty \quad \text{[showed]}$$

(Date: 13.08.2023) (Ch → 2.4 and 2.6) Date: ...../...../..... (Class-07)

Binomial Distribution: (For big sample space)

⇒ All the trials are independent. $X = (0, 1, 2, \dots n)$ / Discrete Random Variable)

⇒ No of trial is finite

trial ⟶ /outcome (success / Failure)

$\boxed{\text{p.m.f :}}$

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$

↓

Binomial Prob^y dist^n) and $\boxed{p + q = 1}$

Mean, $\mu = E(x)$

$= \sum_{x=0}^{n} x \, P(x)$

$= np$ → Total sample space; → Prob^y (constant)

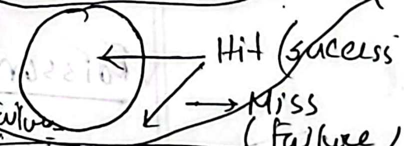
Variance, $\sigma^2 = np(1-p)$

Suppose, 10 trial.

Prob^y of shoot target

success, $= \frac{1}{2}$

Prob^y of failure $= \frac{1}{2}$



$\boxed{\text{Ques:}}$

Prove that $P(x)$ is p.m.f.

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} {}^nC_x \, p^x \, q^{n-x}$$

$$= q^n + {}^nC_1 \, p^1 \, q^{n-1} + \dots + p^n$$

$$= (p+q)^n$$

$\left[(x+y)^n = x^n + n x^{n-1} y + \dots + y^n\right]$

$$= (1)^n$$

n = no of trials

x = number of success $0, 1, 2, \dots n$

P = Prob^y of success $1-p = q$ = prob^y of failure

$$\sum_{x=0}^{n} P(x) = 1$$

[Ans.]

M.g.f = $\boxed{(pe^t + q)^n}$

Policy holder

Machine manufacturer

Defective

Poisson Distribution: (Discrete distribution)

⇒ Number of trials is infinite. (very large)

Prob^y of success ⇒ $p \to 0$ (very small)

Defined as, $P(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$; $x = 0, 1, 2, \ldots, \infty$

$n = 10000$
(write word)

$P = 0.001$

p.m.f , $P(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

Mean , $E(x) = \sum_{x=0}^{\infty} x\, p(x)$

$= \sum_{x=0}^{\infty} x \dfrac{\lambda^x e^{-\lambda}}{x!}$

$= \sum_{x=0}^{\infty} \dfrac{\cancel{x} \lambda^x e^{-\lambda}}{\cancel{x}(x-1)!}$

$= e^{-\lambda} \sum_{x} \dfrac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \lambda e^{\lambda}$

$= e^0 = 1 \cdot \lambda = \lambda$

and spreads with decreasing prob)

$\frac{\lambda^{1-1}}{0!} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots$

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$\boxed{\text{Mean, } \mu = \lambda = E(x)}$

~~Variance,~~ $E\{x(x-1)\} = \sum_x x(x-1)\, p(x)$

$= \sum_x x(x-1) \frac{\lambda^x e^{-\lambda}}{x!}$

$= \sum_x \cancel{x}(x-1) \frac{\lambda^x e^{-\lambda}}{\cancel{x}(x-1)!}$

$= e^{-\lambda} \lambda^2 \sum_x \frac{\cancel{(x-1)}\, \lambda^{x-2}}{\cancel{(x-1)}(x-2)!}$

$= e^{-\lambda} \lambda^2 \sum_x \frac{\lambda^{x-2}}{(x-2)!}$

$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda}$

$E(x(x-1)) = \lambda^2$

~~Now,~~ And ~~Variance,~~ $E(x^2) = E(x(x-1)) + E(x)$

$= \lambda^2 + \lambda$

Variance, $\sigma^2 = E(x^2) - \{E(x)\}^2$

$= \lambda^2 + \lambda - \lambda^2$

$\boxed{\sigma^2 = \lambda}$

S.D., $\boxed{\sigma = \sqrt{\lambda}}$

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(Example-07) **Problem:** The prob^y of germination of beet seed is 0.8 and the germination of a seed is called success. If we plant $\underbrace{10 \text{ seeds}}$ and can assume that the germination of one seed is independent of the germination of another seed.

Find a) $P(X \le 8)$

b) $P(X \le 6)$

Soln: by binomial dist^n,

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x} \; ; \quad x = 0,1,2,\ldots,10 \quad [p+q=1]$$

$$= {}^{10}C_{x} (0.8)^{x} (1-0.8)^{10-x}$$

Total, $n=10$
prob^y, $p=0.8$

Now, a) $P(X \le 8) = 1 - P(X > 8)$

$= 1 - P(X=9) - P(X=10)$

$= 1 - {}^{10}C_9 (0.8)^9 (1-0.8)^{10-9} - {}^{10}C_{10} (0.8)^{10} (1-0.8)^{10-10}$

$= 1 - 0.268 - 0.1073$

$= 0.6242$

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b) $P(X \le 6) = \sum_{x=0}^{6} {}^{10}C_x (0.8)^x (0.$

$= 0.1209$ [Ans.]

*// → Example → (2.4-9)

Problem: Assume, We are in one of those rare times when 65% of American public approve the way the president is handling Job. Take random sample $n = 8$ and Let Y = number who gives disapproval.

Find: a) $P(Y \geq 6)$

b) $P(Y \le 5)$ and $P(Y = 5)$

Soln: Total, $n = 8$

Prob^y, $P = 0.65$

Now, $P(X = x) = {}^{n}C_x \, p^x (1-p)^{n-x}$

$= {}^{8}C_x (0.65)^x (1 - 0.65)^{8-x}$

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a) $P(Y \geq 6)$

$= P(Y=6) + P(Y=7) + P(Y=8)$

$= {}^{8}C_{6}(0.65)^{6}(1-0.65)^{8-6} + {}^{8}C_{7}(0.65)^{7}(1-0.65)^{8-7} + {}^{8}C_{8}(0.65)^{8}(1-0.65)^{8-8}$

$= 0.4278$

b) $P(Y \leq 5) = 1 - P(Y>5)$

$= 1 - P(Y=6) - P(Y=7) - P(Y=8)$

$= 0.574$

c) $P(Y=5) = {}^{8}C_{5}(0.65)^{5}(1-0.65)^{3}$

$= 0.2785$

[Ans.]

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7

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CamScanner

Exercise → (2.4-1)

Date : ....../......

Problem: A bag contains 7 red and 11 ~~bla~~ white balls. Draw one ball at random. Let $X = 1$ if a red ball is drawn and let $X = 0$ if a white ball is drawn. Find pmf, mean and variance of X.

Soln: Total ball $= 11 + 7 = 18$

$$X = \begin{cases} 1 & ; \text{ red ball} \\ 0 & ; \text{ white ball} \end{cases}$$

Now, $P(X=1) = \frac{7}{18}$

$P(X=0) = \frac{11}{18}$

~~Now, p.m.f, $P(X=x) = P(X=0) + P(X=1)$~~

~~$= \frac{11}{18} + \frac{7}{18}$~~

$= 1$

So, The pmf of X is

$$f(x) = P(X=x) = \begin{cases} \frac{7}{18} & ; x = 1 \\ \frac{11}{18} & ; x = 0 \end{cases}$$

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~~Problem:~~

~~For mean,~~

For binomial distn, mean, $\mu = np = 1 \times \frac{7}{18}$

≈ 0.3889

and Variance, $\sigma^2 = npq$

$= np(1-p)$

$= 1 \times \frac{7}{18}\left(1 - \frac{7}{18}\right) \approx 0.2377$

[Ans.]

→ Exercise → (2.4-3)

Problem: On a six question multiple-choice test there are five possible answers for each question; of which one is correct (c) and four are incorrect (I). If a student guesses randomly ~~and indly~~, find prob of

a) Being correct only on ques 1 and 4 (i.e. scoring $\underset{1}{C}$, I, I, $\underset{4}{C}$, I, I)

b) Being correct on two questions

Date :........./........./.........

a) possible outcome = 5

Prob^y of correct ans = $P(C) = \frac{1}{5}$

and prob^y incorrect ans = $P(I) = 1 - P(C)$

$= 1 - \frac{1}{5}$

$= \frac{4}{5}$

$P(C, I, I, C, I, I) = P(C) \times P(I) \times P(I) \times P(C) \times P(I) \times P(I)$

$= \left(\frac{1}{5}\right)^2 \times \left(4/5\right)^4$

≈ 0.0163

b) being $\boxed{\text{correct}}$ on two questions

exactly 2

Total = 6 Question

P = Correct possibility

$= {}^{n}C_{x}\, p^x (1-p)^{n-x}$

$= {}^{6}C_{2} \left(\frac{1}{5}\right)^2 \left(1 - 1/5\right)^{6-2}$

≈ 0.2458

[Ans.]

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Problem: In a lab experiment involving organic syntheses of molecular to organometalic ceramics the final step of a five-step reaction involves the formation of a metal-metal bond. The probᵈ of such a bond forming, $p = 0.20$. Let, X = num. of successful reaction out of $n = 25$ experiments.

(a) Find the probᵈ that X is at most 4.

(b) Find the probᵈ that X is at least 5.

(c) Find the probᵈ that X is equal to 6.

(d) Given the mean, variance and S.D. of X

Soln: Given, $n = 25$

$p = 0.20$

(a) $P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= {}^{25}C_0 (0.20)^0 (1-0.20)^{25-0} + \dots$

$+ {}^{25}C_4 (0.20)^4 (1-0.20)^{21}$

$= 0.4207$

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(b) $P(X \geq 5) = 1 - P(X < 5)$

$$= 1 - \left(P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)\right)$$

(c) $P(X=6) = {}^{25}C_6 (0.20)^6 (1-0.20)^{25-6}$

$$\approx 0.1633$$

(d) Mean, $\mu = np = 25 \times 0.20 = 5$

Variance ~~Standard deviation~~, $\sigma^2 = npq$

$$= np(1-p) = 25 \times 0.20 (1-0.20)$$

$$= 4$$

and S.D., $\sigma = \sqrt{4} = 2$

[Ans.]

Date : $\frac{\lambda^x e^{-\lambda}}{x!}$

Exercise → 2.4 (1 - 6)

↳ Book pg - 72

Exercise → (2.6 - 1) *

Problem: Let X have a $\boxed{\text{poisson}}$ distn with a mean of 4. Find,

(a) $P(2 \leq X \leq 5)$

(b) $P(X \geq 3)$

(c) $P(X \leq 3)$

Soln: by defn, poisson distn,

$$P(X = x) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; \lambda > 0$$

$$x = 0, 1, 2, 3, \ldots$$

(b) $P(X \geq 3) = 1 - P(X < 3)$

$\boxed{\lambda = 4}$
Mean

$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$= 1 - \left\{\frac{4^0 e^{-4}}{0!} + \frac{4 \cdot e^{-4}}{1!} + \frac{4^2 \cdot e^{-4}}{2!}\right.$

$= 1 - 0.2381$

$= 0.7619$

[Ans.]

$$\frac{\lambda^x e^{-\lambda}}{x!}$$

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(c) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \frac{4^0 \cdot e^{-4}}{0!} + \cdots + \frac{4^3 e^{-4}}{3!}$$

$$= e^{-4} + 4e^{-4} + 8e^{-4} + \frac{32}{3} e^{-4}$$

[Ans.]

(a) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= ?$ [Ans.]

Exercise → 2.6-3

Problem: Customers arrive at a travel agency at a mean rate of 11 per hour. Assuming that the number of arrivals per hour has a poisson distn, give the probty that more than 10 customers arrive in a given hour.

$\frac{11^x \cdot e^{-11}}{0!}$ $1-P(X \le 10)$ $P(X > 10)$ $1-P(X=11)$

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55:08

Soln: Here, $\lambda = \mu = \cancel{10}\ 11$

Now, $\frac{\lambda^x e^{-\lambda}}{x!}$; $x = 0, \ldots\ldots, 11$

Then, $P(X \geq 10) = 1 - P(X \leq 10)$

$= 1 - \{P(X=0) + P(X=1) + \cdots\cdots + P(X=10)\}$

≈ 0.5401 [Ans.]