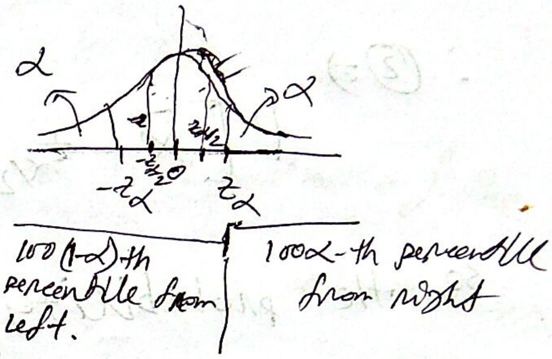
# 7.1

## Confidence intervals for means

If $Z$ is a standard normal distribution $N(0,1)$,

$$P(Z \geq z_\alpha) = \alpha$$




$$\Rightarrow P(Z \leq -z_\alpha) = \alpha$$

$$\Rightarrow P(Z \leq z_\alpha) = 1-\alpha \quad \text{——} \circledast$$

$$\Rightarrow P(|Z| \leq z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1-\alpha \quad \text{——} (1)$$

\# If $X$ is $N(\mu, \sigma^2)$, the distribution of the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is, $N\left(\mu, \frac{\sigma^2}{n}\right)$.

see → (Corollary 5.5-1) → $\boxed{\text{Page} \rightarrow 193}$

$$\therefore (1) \Rightarrow P\left(-z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq z_{\alpha/2}\right) = 1-\alpha \quad \text{——} (2)$$

For example, if $1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$,

$\therefore z_{\alpha/2} = z_{0.025} = 1.96$ (From ~~stan~~ $N(0,1)$ Table)

NOW, $$-z_{\alpha/2} \le \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \le z_{\alpha/2}$$

→ (derivation → Page → 301.)

$$\Rightarrow \bar{x} \cancel{+} z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right).$$

$\therefore$ (2) $\Rightarrow$

$$P\left[\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right] = 1-\alpha$$

so, the probability of the random interval

$$\left[\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \ \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

includes the unknown mean $\mu$ is $1-\alpha$.

If the sample mean computed to $\bar{x}$, the random interval $\left[\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$ which covers $\mu$ (or, briefly $\bar{x} \pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$) is called a ~~%~~ $100(1-\alpha)\%$ "confidence interval" of $\mu$.

For example, $\bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$ is a 95% confidence interval for $\mu$. The number $100(1-\alpha)\% = 1-\alpha$ is called the "confidence coefficient".

For example: $1-\alpha = 0.95$ is the confidence coefficient.

# A shorter confidence interval gives a more precise estimate of $\mu$. If $n$ increases, $z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$ decreases, then we get a shorter confidence interval with the same "confidence coefficient" $1-\alpha$.

For a fixed sample size $n$, the length of the confidence interval is shortened if $(1-\alpha)$ decreases.

<u>Example:- 7.1-1 :-</u>

Here, variance, $\sigma^2 = 1296 \Rightarrow \sigma = 36$

sample size, $n = 27$, ~~$\bar{x} = 1478$~~

sample mean, $\bar{x} = 1478$ hours.

and, $1-\alpha = 95\% = 0.95$

$\Rightarrow \alpha = 1 - 0.95 = 0.05$

$\Rightarrow \frac{\alpha}{2} = 0.025$

$\therefore z_{\alpha/2} = z_{0.025} = 1.96$

Now, the 95% confidence interval for $\mu$ is,

$$\left[\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

$$= \left[1478 - 1.96\left(\frac{36}{\sqrt{27}}\right), 1478 + 1.96\left(\frac{36}{\sqrt{27}}\right)\right]$$

$$= [1464.42, 1491.58]$$

Ans

Example: 7.1-2 :- Here, $\sigma^2 = 16 \Rightarrow \sigma = 4$

$$n = 5$$

and, $1-\alpha = 90\% = 0.9$

$$\Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05$$

$$\therefore z_{\alpha/2} = z_{0.05} = 1.645$$

$\therefore$ the 90% confidence interval for unknow mean $\mu$, is,

$$\left[\bar{x} - 1.645\left(\frac{4}{\sqrt{5}}\right), \bar{x} + 1.645\left(\frac{4}{\sqrt{5}}\right)\right]$$

For a particular sample, this interval either does or, does not contain the mean $\mu$. However, if many such intervals are calculated, about 90% of them should contain $\mu$.

or,

If $W$ is a random variable that counts the number of 90% confidence intervals containing $\mu$, then the distribution of $W$ is $b(\mu, 0.9)$.

# The random interval is also called an approximate $100(1-\alpha)\%$ confidence interval for $\mu$.

see → page (303 & 304).

**Example: 7.1-3 :-** Here, $\sigma = 96$, $n = 576$, $\bar{x} = 133$

and $1-\alpha = 90\% = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow \alpha/2 = 0.05$

$\therefore z_{\alpha/2} = z_{0.05} = 1.645$

$\therefore$ ~~An~~ The approximate 90% confidence interval for $\mu$ is,

$$\left[133 - 1.645 \times \left(\frac{96}{\sqrt{576}}\right),\ 133 + 1.645\left(\frac{96}{\sqrt{576}}\right)\right]$$

$$= [126.42,\ 139.58]$$

**Example: 7.1-4 :-** From data, we have to calculate, mean, $\bar{x} = 19.07$ & variance, $s^2 = 10.6 \Rightarrow s = \sqrt{10.6}$, sample size, $n = 32$. & $1-\alpha = 95\%$

$\Rightarrow z_{\alpha/2} = 1.96$

$\therefore$ The approximate 95% confidence interval for $\mu$ is,

$$\left[19.07 - 1.96\sqrt{\frac{10.6}{32}},\ 19.07 + 1.96\sqrt{\frac{10.6}{32}}\right]$$

$$= [17.94,\ 20.20]$$

# If the random sample arises from a normal distribution, $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t-distribution with $r = n-1$ degrees of freedom (Equation 5.5-2), where, $s^2$ is the usual unbiased estimator of $\sigma^2$.

# If $X$ is a $N(\mu, \sigma^2)$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$,

$$\Rightarrow P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_\alpha\right) = 1-\alpha$$

$$\Rightarrow P\left[\bar{X} - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu\right] = 1-\alpha$$

If $\bar{X} = \bar{x}$, $\left[\bar{x} - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right), \infty\right)$ is a $100(1-\alpha)\%$ one-sided confidence interval for $\mu$.

That is, with the confidence coefficient $1-\alpha$, $\bar{x} - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$ is a lower bound for $\mu$.

Similarly, $\left(-\infty, \bar{x} + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)\right]$ is a one-sided confidence interval for $\mu$ and $\bar{x} + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$ provides an upper bound for $\mu$ with confidence coefficient $1-\alpha$.

Exercise → 7.1-1 ; $\boxed{1-10}$