Lecture on:

Gauss's Law

Ref book: Fundamentals of Physics - D. Halliday, R. Resnick & J. Walker (10^{th} Ed.)

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GAUSS' LAW

Gauss' law relates the electric fields at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

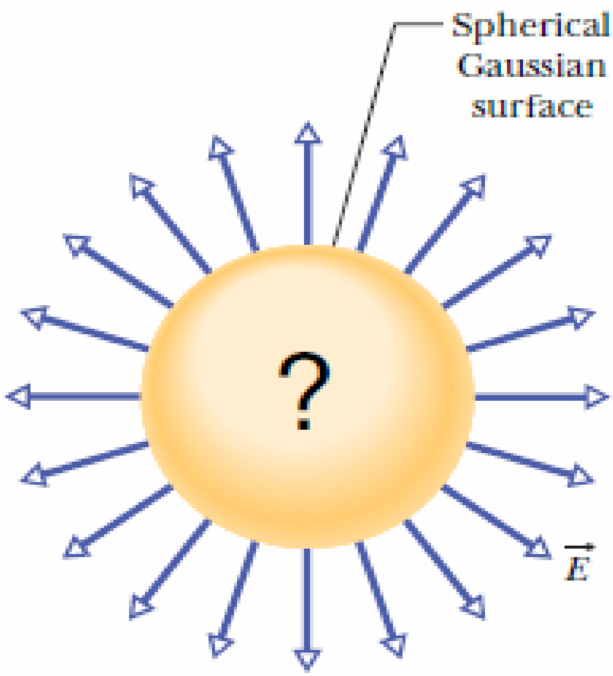


Fig. 23-1 A spherical Gaussian surface. If the electric field vectors are of uniform magnitude and point radially outward at all surface points, you can conclude that a net positive distribution of charge must lie within the surface and have spherical symmetry.

23-2 Flux

Suppose that, as in Fig. 23-2*a*, you aim a wide airstream of uniform velocity $\vec{v}$ at a small square loop of area A. Let Φ represent the *volume flow rate* (volume per unit time) at which air flows through the loop. This rate depends on the angle between $\vec{v}$ and the plane of the loop. If $\vec{v}$ is perpendicular to the plane, the rate Φ is equal to vA.

If $\vec{v}$ is parallel to the plane of the loop, no air moves through the loop, so Φ is zero. For an intermediate angle θ, the rate Φ depends on the component of $\vec{v}$ normal to the plane (Fig. 23-2*b*). Since that component is $v \cos \theta$, the rate of volume flow through the loop is

$$\Phi = (v \cos \theta)A. \qquad (23\text{-}1)$$

This rate of flow through an area is an example of a **flux**—a *volume flux* in this situation.

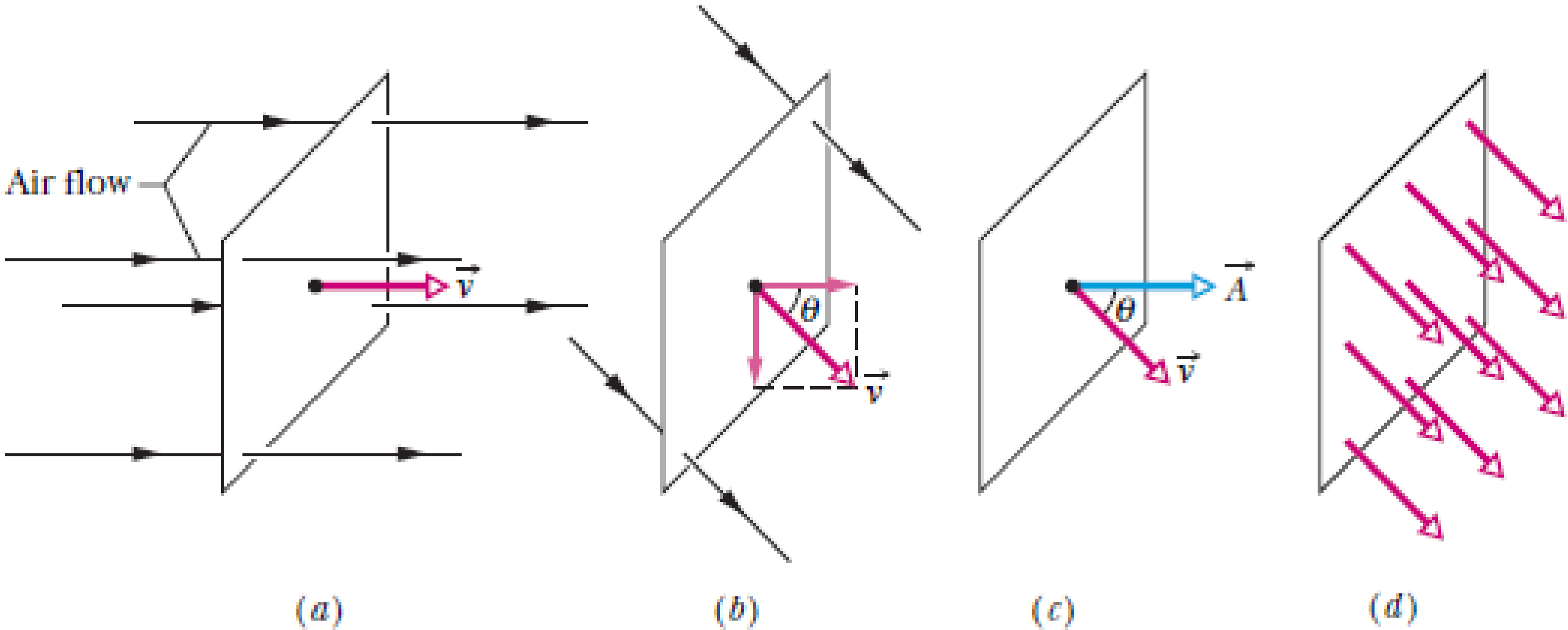


Fig. 23-2 (*a*) A uniform airstream of velocity $\vec{v}$ is perpendicular to the plane of a square loop of area A. (*b*) The component of $\vec{v}$ perpendicular to the plane of the loop is $v \cos \theta$, where θ is the angle between $\vec{v}$ and a normal to the plane. (*c*) The area vector $\vec{A}$ is perpendicular to the plane of the loop and makes an angle θ with $\vec{v}$. (*d*) The velocity field intercepted by the area of the loop.

Before we discuss a flux involved in electrostatics, we need to rewrite Eq. 23-1 in terms of vectors. To do this, we first define an *area vector* $\vec{A}$ as being a vector whose magnitude is equal to an area (here the area of the loop) and whose direction is normal to the plane of the area (Fig. 23-2*c*). We then rewrite Eq. 23-1 as the scalar (or dot) product of the velocity vector $\vec{v}$ of the airstream and the area vector $\vec{A}$ of the loop:

$$\Phi = vA\cos\theta = \vec{v}\cdot\vec{A}, \tag{23-2}$$

where θ is the angle between $\vec{v}$ and $\vec{A}$.

The word "flux" comes from the Latin word meaning "to flow." That meaning makes sense if we talk about the flow of air volume through the loop. However, Eq. 23-2 can be regarded in a more abstract way. To see this different way, note that we can assign a velocity vector to each point in the airstream passing through the loop (Fig. 23-2*d*). Because the composite of all those vectors is a *velocity field*, we can interpret Eq. 23-2 as giving the *flux of the velocity field through the loop.* With this interpretation, flux no longer means the actual flow of something through an area—rather it means the product of an area and the field across that area.

23-3 Flux of an Electric Field

To define the flux of an electric field, consider Fig. 23-3, which shows an arbitrary (asymmetric) Gaussian surface immersed in a nonuniform electric field. Let us divide the surface into small squares of area ΔA, each square being small enough to permit us to neglect any curvature and to consider the individual square to be flat. We represent each such element of area with an area vector $\Delta\vec{A}$, whose magnitude is the area ΔA. Each vector $\Delta\vec{A}$ is perpendicular to the Gaussian surface and directed away from the interior of the surface.

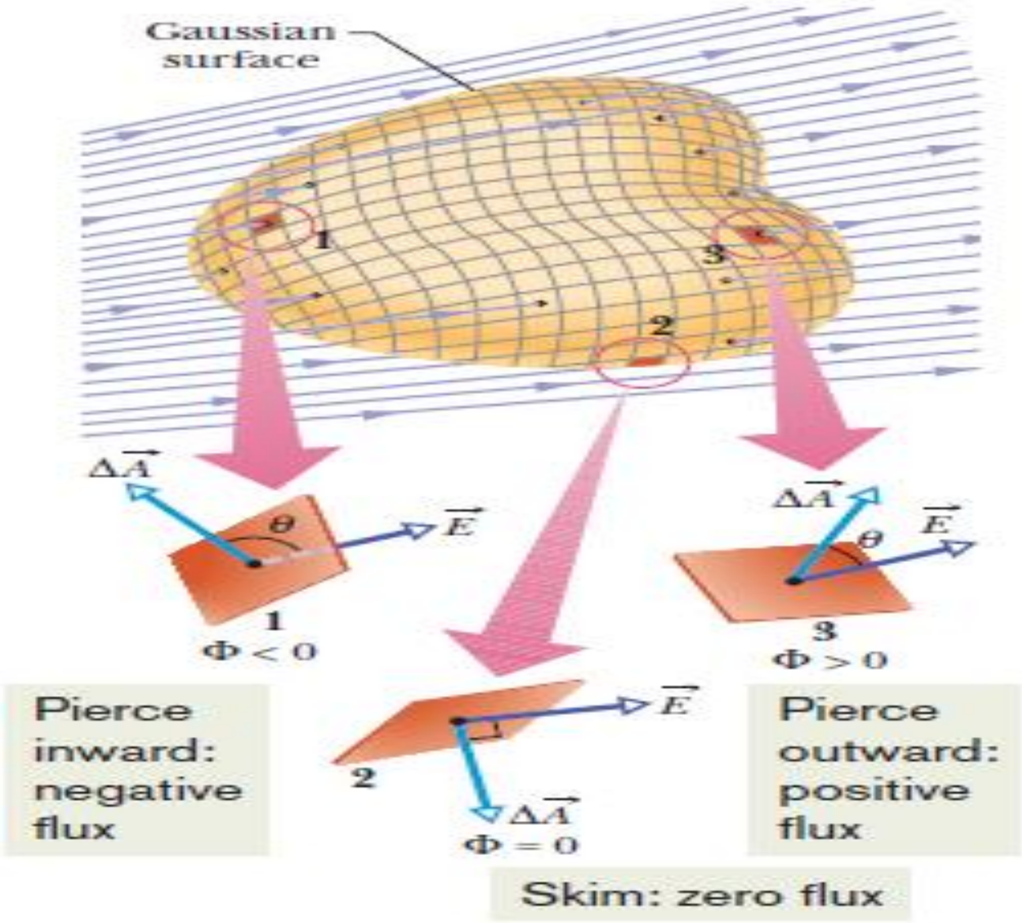


Fig. 23-3 A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area ΔA. The electric field vectors $\vec{E}$ and the area vectors $\Delta\vec{A}$ for three representative squares, marked 1, 2, and 3, are shown.

Because the squares have been taken to be arbitrarily small, the electric field $\vec{E}$ may be taken as constant over any given square. The vectors $\Delta\vec{A}$ and $\vec{E}$ for each square then make some angle θ with each other. Figure 23-3 shows an enlarged view of three squares on the Gaussian surface and the angle θ for each.

A provisional definition for the flux of the electric field for the Gaussian surface of Fig. 23-3 is

$$\Phi = \sum \vec{E} \cdot \Delta\vec{A}. \tag{23-3}$$

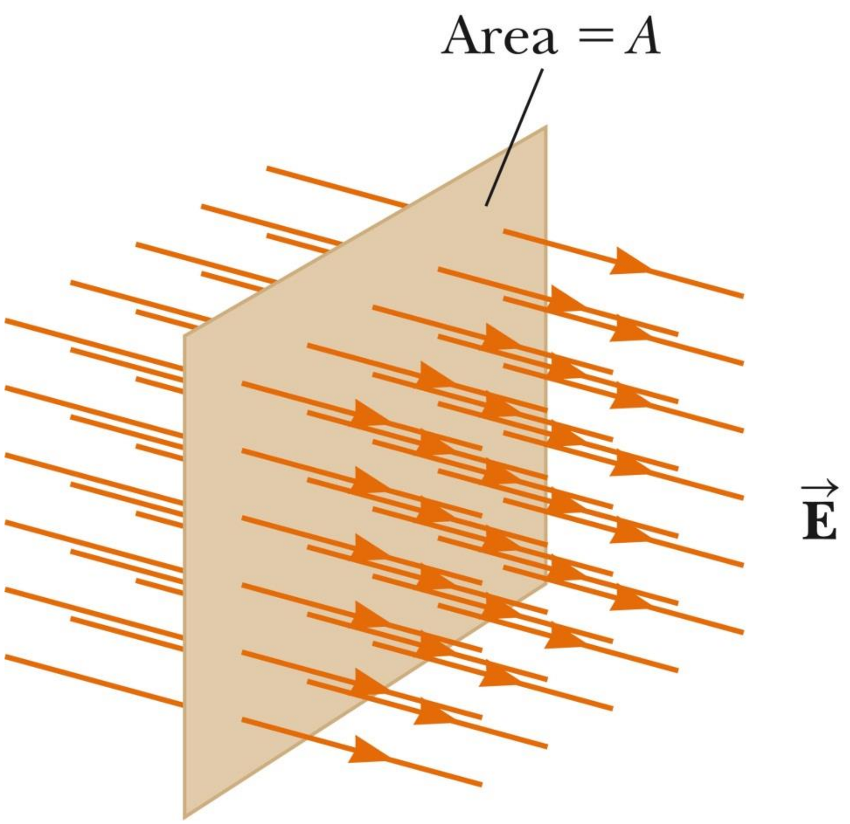
This equation instructs us to visit each square on the Gaussian surface, evaluate the scalar product $\vec{E} \cdot \Delta\vec{A}$ for the two vectors $\vec{E}$ and $\Delta\vec{A}$ we find there, and sum the results algebraically (that is, with signs included) for all the squares that make up the surface. The value of each scalar product (positive, negative, or zero) determines whether the flux through its square is positive, negative, or zero. Squares like square 1 in Fig. 23-3, in which $\vec{E}$ points inward, make a negative contribution to the sum of Eq. 23-3. Squares like 2, in which $\vec{E}$ lies in the surface, make zero contribution. Squares like 3, in which $\vec{E}$ points outward, make a positive contribution.

The exact definition of the flux of the electric field through a closed surface is found by allowing the area of the squares shown in Fig. 23-3 to become smaller and smaller, approaching a differential limit dA. The area vectors then approach a differential limit $d\vec{A}$. The sum of Eq. 23-3 then becomes an integral:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad \text{(electric flux through a Gaussian surface).} \tag{23-4}$$

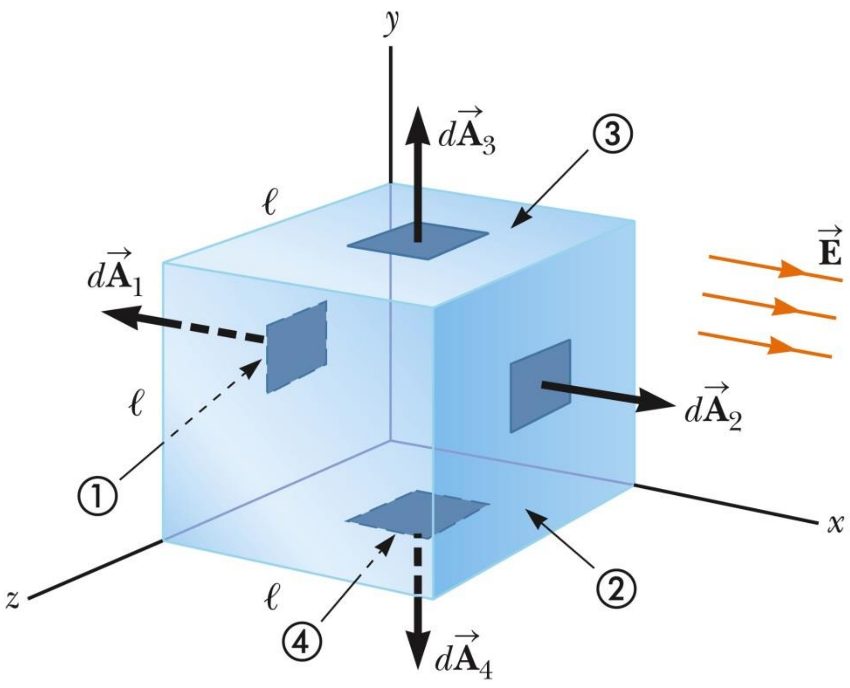
The loop on the integral sign indicates that the integration is to be taken over the entire (closed) surface. The flux of the electric field is a scalar, and its SI unit is the newton–square-meter per coulomb ($N \cdot m^2/C$).

- ***Electric flux*** is the product of the magnitude of the electric field and the surface area, *A*, perpendicular to the field.
- $\Phi_E = EA$
- Units: $N \cdot m^2 / C$



Flux Through a Cube, Example

- The field lines pass through two surfaces perpendicularly and are parallel to the other four surfaces.
- For face 1, $E = -E\ell^2$
- For face 2, $E = E\ell^2$
- For the other sides, $E = 0$
- Therefore, $E_{total} = 0$



Problem-1:

Figure 23-4 shows a Gaussian surface in the form of a cylinder of radius R immersed in a uniform electric field $\vec{E}$, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?

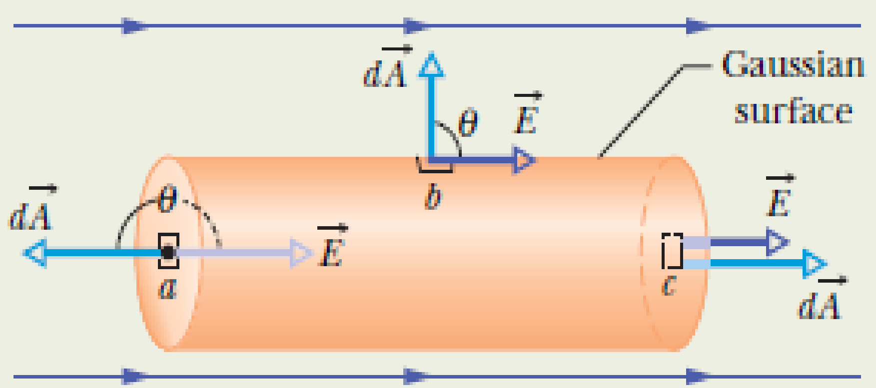


Fig. 23-4 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

Calculations: We can do the integration by writing the flux as the sum of three terms: integrals over the left cylinder cap a, the cylindrical surface b, and the right cap c. Thus, from Eq. 23-4,

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

$$= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \qquad (23\text{-}5)$$

For all points on the left cap, the angle θ between $\vec{E}$ and $d\vec{A}$ is 180° and the magnitude E of the field is uniform. Thus,

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180°)\, dA = -E \int dA = -EA,$$

where $\int dA$ gives the cap's area $A\ (= \pi R^2)$. Similarly, for the right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 0)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

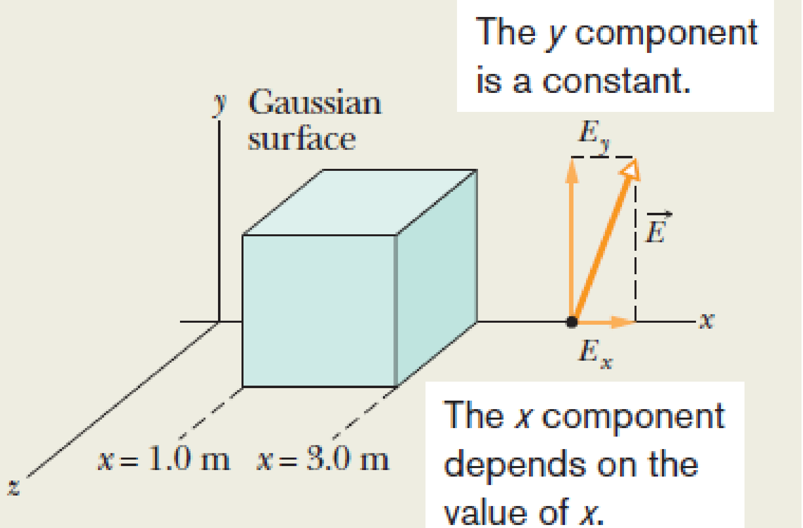
Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \qquad \text{(Answer)}$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.

Problem-2:

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ pierces the Gaussian cube shown in Fig. 23-5*a*. (*E* is in newtons per coulomb and *x* is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)



(*a*)

Right face: An area vector $\vec{A}$ is always perpendicular to its surface and always points away from the interior of a Gaussian surface. Thus, the vector $d\vec{A}$ for any area element (small section) on the right face of the cube must point in the positive direction of the x axis. An example of such an element is shown in Figs. 23-5b and c, but we would have an identical vector for any other choice of an area element on that face. The most convenient way to express the vector is in unit-vector notation,

$$d\vec{A} = dA\hat{\mathrm{i}}.$$

From Eq. 23-4, the flux Φ_r through the right face is then

$$\begin{aligned}\Phi_r &= \int \vec{E} \cdot d\vec{A} = \int (3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}) \cdot (dA\hat{\mathrm{i}}) \\ &= \int [(3.0x)(dA)\hat{\mathrm{i}} \cdot \hat{\mathrm{i}} + (4.0)(dA)\hat{\mathrm{j}} \cdot \hat{\mathrm{i}}] \\ &= \int (3.0x\, dA + 0) = 3.0 \int x\, dA.\end{aligned}$$

We are about to integrate over the right face, but we note that x has the same value everywhere on that face—namely, $x = 3.0$ m. This means we can substitute that constant value

The integral $\int dA$ merely gives us the area $A = 4.0\ \text{m}^2$ of the right face; so

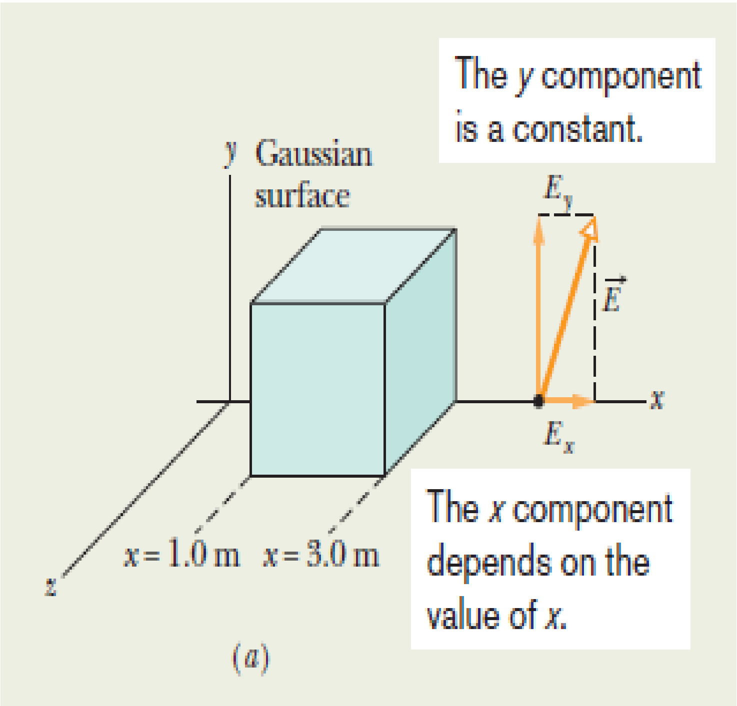
$$\Phi_r = (9.0\ \text{N/C})(4.0\ \text{m}^2) = 36\ \text{N}\cdot\text{m}^2/\text{C}. \qquad \text{(Answer)}$$

Left face: The procedure for finding the flux through the left face is the same as that for the right face. However, two factors change. (1) The differential area vector $d\vec{A}$ points in the negative direction of the x axis, and thus $d\vec{A} = -dA\hat{\mathrm{i}}$ (Fig. 23-5*d*). (2) The term x again appears in our integration, and it is again constant over the face being considered. However, on the left face, $x = 1.0$ m. With these two changes, we find that the flux Φ_l through the left face is

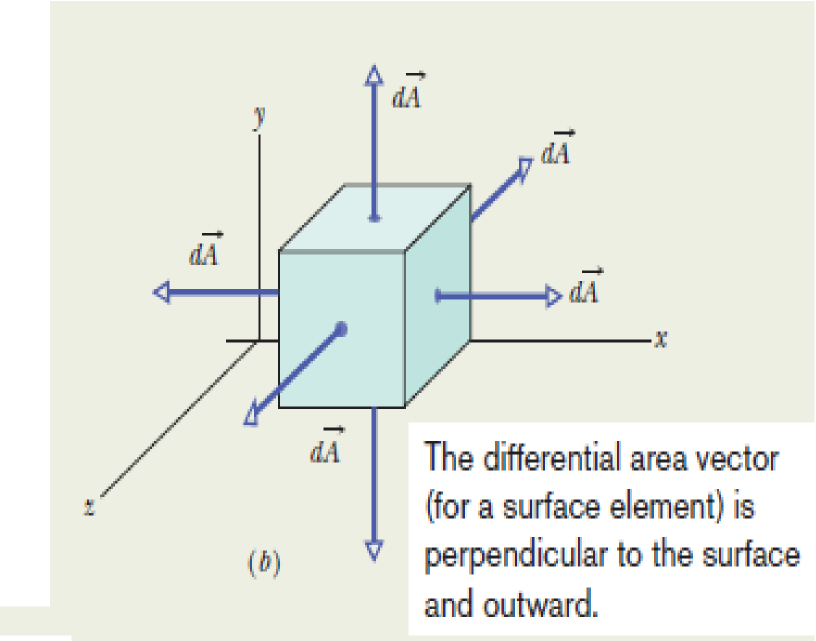
$$\Phi_l = -12\ \text{N}\cdot\text{m}^2/\text{C}. \qquad \text{(Answer)}$$

Top face: The differential area vector $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = dA\hat{\mathrm{j}}$ (Fig. 23-5*e*). The flux Φ_t through the top face is then

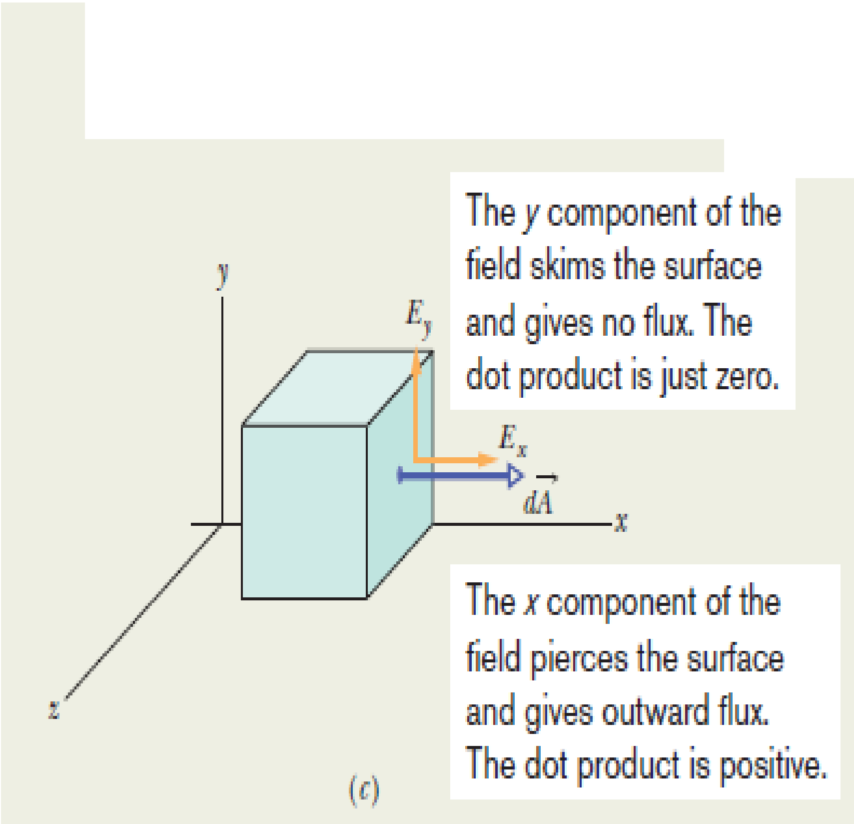
$$\begin{aligned}\Phi_t &= \int (3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}})\cdot(dA\hat{\mathrm{j}})\\ &= \int [(3.0x)(dA)\hat{\mathrm{i}}\cdot\hat{\mathrm{j}} + (4.0)(dA)\hat{\mathrm{j}}\cdot\hat{\mathrm{j}}]\\ &= \int (0 + 4.0\,dA) = 4.0\int dA\\ &= 16\ \text{N}\cdot\text{m}^2/\text{C}. \qquad \text{(Answer)}\end{aligned}$$

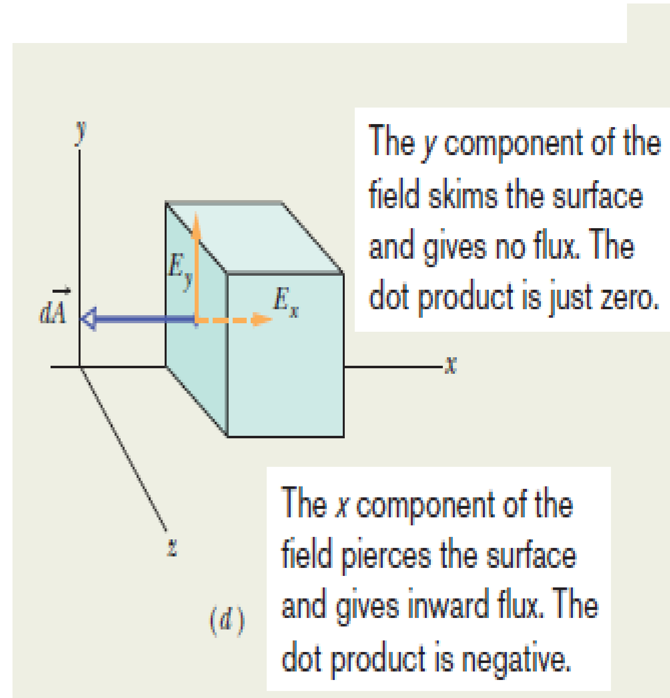

The y component is a constant.
y Gaussian surface
E_y
$\vec{E}$
x
E_x
z
$x = 1.0$ m
$x = 3.0$ m
The x component depends on the value of x.

(a)


$d\vec{A}$
y
$d\vec{A}$
$d\vec{A}$
$d\vec{A}$
x
z
$d\vec{A}$
The differential area vector (for a surface element) is perpendicular to the surface and outward.

(b)


The y component of the field skims the surface and gives no flux. The dot product is just zero.
The x component of the field pierces the surface and gives outward flux. The dot product is positive.
E_y
E_x
$d\vec{A}$
y
x
z
(c)


The y component of the field skims the surface and gives no flux. The dot product is just zero.
The x component of the field pierces the surface and gives inward flux. The dot product is negative.
E_y
E_x
$d\vec{A}$
y
x
z
(d)

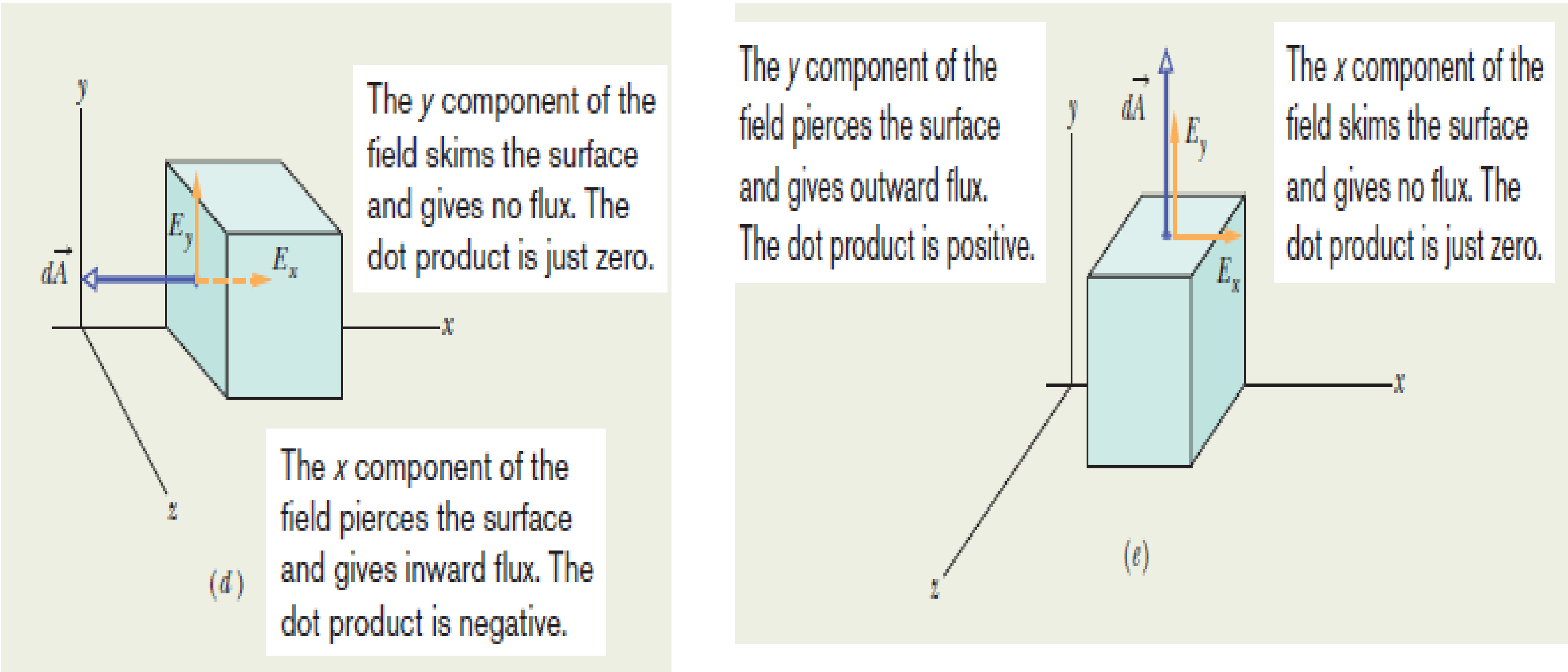

y
x
z
$d\vec{A}$
E_y
E_x
The y component of the field skims the surface and gives no flux. The dot product is just zero.
The x component of the field pierces the surface and gives inward flux. The dot product is negative.
(d)
The y component of the field pierces the surface and gives outward flux. The dot product is positive.
The x component of the field skims the surface and gives no flux. The dot product is just zero.
(e)

Fig. 23-5 (*a*) A Gaussian cube with one edge on the x axis lies within a nonuniform electric field that depends on the value of x. (*b*) Each differential area element has an outward vector that is perpendicular to the area. (*c*) Right face: the x component of the field pierces the area and produces positive (outward) flux. The y component does not pierce the area and thus does not produce any flux. (*d*) Left face: the x component of the field produces negative (inward) flux. (*e*) Top face: the y component of the field produces positive (outward) flux.

23-4 Gauss' Law

Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is *enclosed* by that surface. It tells us that

$$\varepsilon_0 \Phi = q_{enc} \quad \text{(Gauss' law).} \tag{23-6}$$

By substituting Eq. 23-4, the definition of flux, we can also write Gauss' law as

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \quad \text{(Gauss' law).} \tag{23-7}$$

Equations 23-6 and 23-7 hold only when the net charge is located in a vacuum or (what is the same for most practical purposes) in air

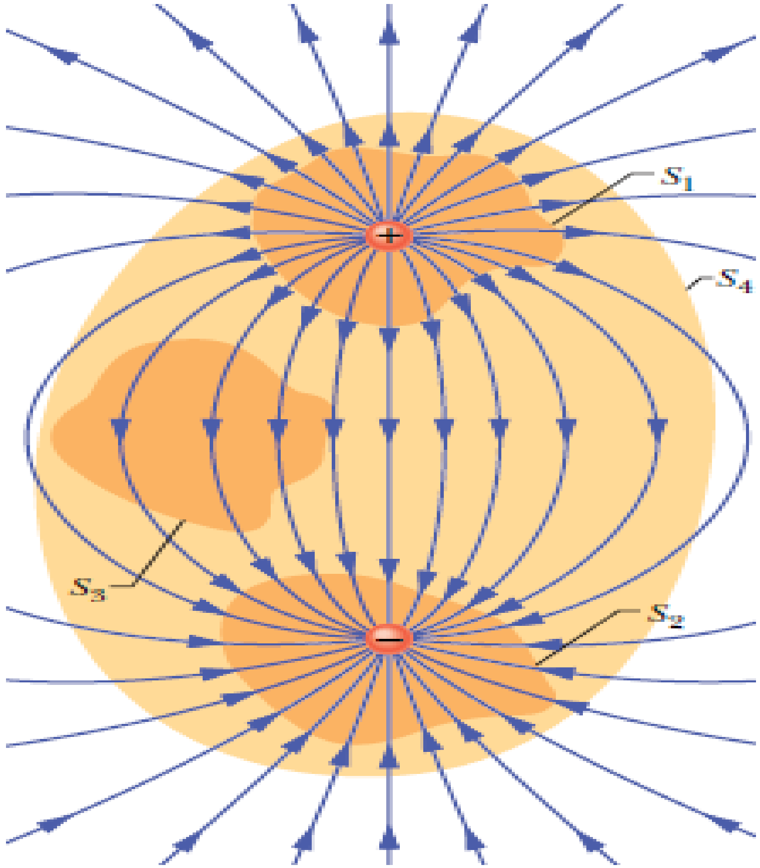


Fig. 23-6 Two point charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section. Surface S_1 encloses the positive charge. Surface S_2 encloses the negative charge. Surface S_3 encloses no charge. Surface S_4 encloses both charges and thus no net charge.

Let us apply these ideas to Fig. 23-6, which shows two point charges, equal in magnitude but opposite in sign, and the field lines describing the electric fields the charges set up in the surrounding space. Four Gaussian surfaces are also shown, in cross section. Let us consider each in turn.

Surface *S*1. The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires. (That is, in Eq. 23-6, if is positive, *q*enc must be also.)
Surface *S*2. The electric field is inward for all points on this surface. Thus, the flux of the electric field through this surface is negative and so is the enclosed charge, as
Gauss' law requires.

Surface *S*3. This surface encloses no charge, and thus *q*enc
0. Gauss' law (Eq.
23-6) requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

Surface *S*4. This surface encloses no *net* charge, because the enclosed positive
and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface *S*4 as entering it.

What would happen if we were to bring an enormous charge *Q* up close to surface *S*4 in Fig. 23-6? The pattern of the field lines would certainly change, but the net flux for each of the four Gaussian surfaces would not change. We can understand this because the field lines associated with the added *Q* would pass entirely through each of the four Gaussian surfaces, making no contribution to the net flux through any of them.The value of *Q* would not enter Gauss' law in any way, because *Q* lies outside all four of the Gaussian surfaces that we are considering.

Problem-3:

Figure 23-7 shows five charged lumps of plastic and an electrically neutral coin. The cross section of a Gaussian surface *S* is indicated. What is the net electric flux through the surface if $q1 = q4 = 3.1$ nC, $q2 = q5 = 5.9$ nC, and $q3 = -3.1$ nC?

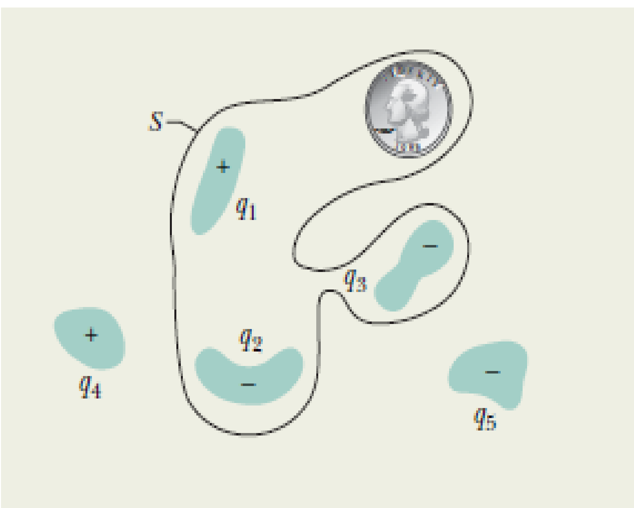


Fig. 23-7 Five plastic objects, each with an electric charge, and a coin, which has no net charge.A Gaussian surface, shown in cross section, encloses three of the plastic objects and the coin.

Calculation: The coin does not contribute to because it is neutral and thus contains equal amounts of positive and negative charge. We could include those equal amounts, but they would simply sum to be zero when we calculate the *net* charge enclosed by the surface. So, let’s not bother. Charges $q4$ and $q5$ do not contribute because they are outside surface S. They certainly send electric field lines through the surface, but as much enters as leaves and no net flux is contributed.Thus, $q$enc is only the sum $q1$ $q2$ $q3$ and Eq. 23-6 gives us

$$\Phi = \frac{q_{enc}}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

$$= \frac{+3.1 \times 10^{-9}\,\mathrm{C} - 5.9 \times 10^{-9}\,\mathrm{C} - 3.1 \times 10^{-9}\,\mathrm{C}}{8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2}}$$

$$= -670\,\mathrm{N \cdot m^2/C}. \qquad \text{(Answer)}$$

The minus sign shows that the net flux through the surface is inward and thus that the net charge within the surface is negative.

Problem-4:

What is the net charge enclosed by the Gaussian cube of Fig. 23-5, which lies in the electric field $\vec{E} = 3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}$? ($E$ is in newtons per coulomb and x is in meters.)

The net charge enclosed by a (real or mathematical) closed surface is related to the total electric flux through the surface by Gauss' law as given by Eq. 23-6 ($\varepsilon_0\Phi = q_{enc}$).

Flux: To use Eq. 23-6, we need to know the flux through all six faces of the cube. We already know the flux through the right face ($\Phi_r = 36\ \mathrm{N \cdot m^2/C}$), the left face ($\Phi_l = -12\ \mathrm{N \cdot m^2/C}$), and the top face ($\Phi_t = 16\ \mathrm{N \cdot m^2/C}$).

For the bottom face, our calculation is just like that for the top face *except* that the differential area vector $d\vec{A}$ is now directed downward along the y axis (recall, it must be *outward* from the Gaussian enclosure). Thus, we have

$d\vec{A} = -dA\hat{\mathrm{j}}$, and we find

$$\Phi_b = -16\ \mathrm{N \cdot m^2/C}.$$

For the front face we have $d\vec{A} = dA\hat{\mathrm{k}}$, and for the back face, $d\vec{A} = -dA\hat{\mathrm{k}}$. When we take the dot product of the given electric field $\vec{E} = 3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}$ with either of these expressions for $d\vec{A}$, we get 0 and thus there is no flux through those faces. We can now find the total flux through the six sides of the cube:

$$\begin{aligned}\Phi &= (36 - 12 + 16 - 16 + 0 + 0)\ \mathrm{N \cdot m^2/C} \\ &= 24\ \mathrm{N \cdot m^2/C}.\end{aligned}$$

Enclosed charge: Next, we use Gauss' law to find the charge q_{enc} enclosed by the cube:

$$\begin{aligned}q_{enc} &= \varepsilon_0\Phi = (8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(24\ \mathrm{N \cdot m^2/C}) \\ &= 2.1 \times 10^{-10}\ \mathrm{C}. \qquad \text{(Answer)}\end{aligned}$$

Thus, the cube encloses a *net* positive charge.

23-5 Gauss' Law and Coulomb's Law

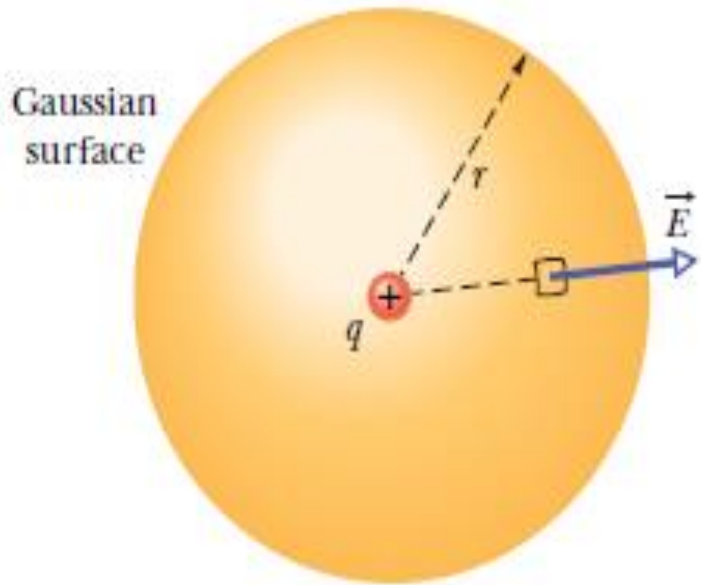


Fig. 23-8 A spherical Gaussian surface centered on a point charge q.

Figure 23-8 shows a positive point charge *q*, around which we have drawn a concentric spherical Gaussian surface of radius *r*. Let us divide this surface into differential areas *dA*. By definition, the area vector at any point is perpendicular to the surface and directed outward from the interior. From the symmetry of the situation, we know that at any point the electric field is also perpendicular to the surface and directed outward from the interior. Thus, since the angle u between and is zero, we can rewrite Eq. 23-7 for Gauss' law as

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 \oint E\, dA = q_{enc}. \tag{23-8}$$

Here $q_{enc} = q$. Although E varies radially with distance from q, it has the same value everywhere on the spherical surface. Since the integral in Eq. 23-8 is taken over that surface, E is a constant in the integration and can be brought out in front of the integral sign. That gives us

$$\varepsilon_0 E \oint dA = q. \tag{23-9}$$

The integral is now merely the sum of all the differential areas dA on the sphere and thus is just the surface area, $4\pi r^2$. Substituting this, we have

$$\varepsilon_0 E(4\pi r^2) = q$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \tag{23-10}$$

This is exactly Eq. 22-3, which we found using Coulomb's law.

Problem-5:

Figure 23-10*a* shows, in cross section, a plastic, spherical shell with uniform charge $Q = -16e$ and radius $R = 10$ cm. A particle with charge $q = +5e$ is at the center. What is the electric field (magnitude and direction) at (a) point P_1 at radial distance $r_1 = 6.00$ cm and (b) point P_2 at radial distance $r_2 = 12.0$ cm?

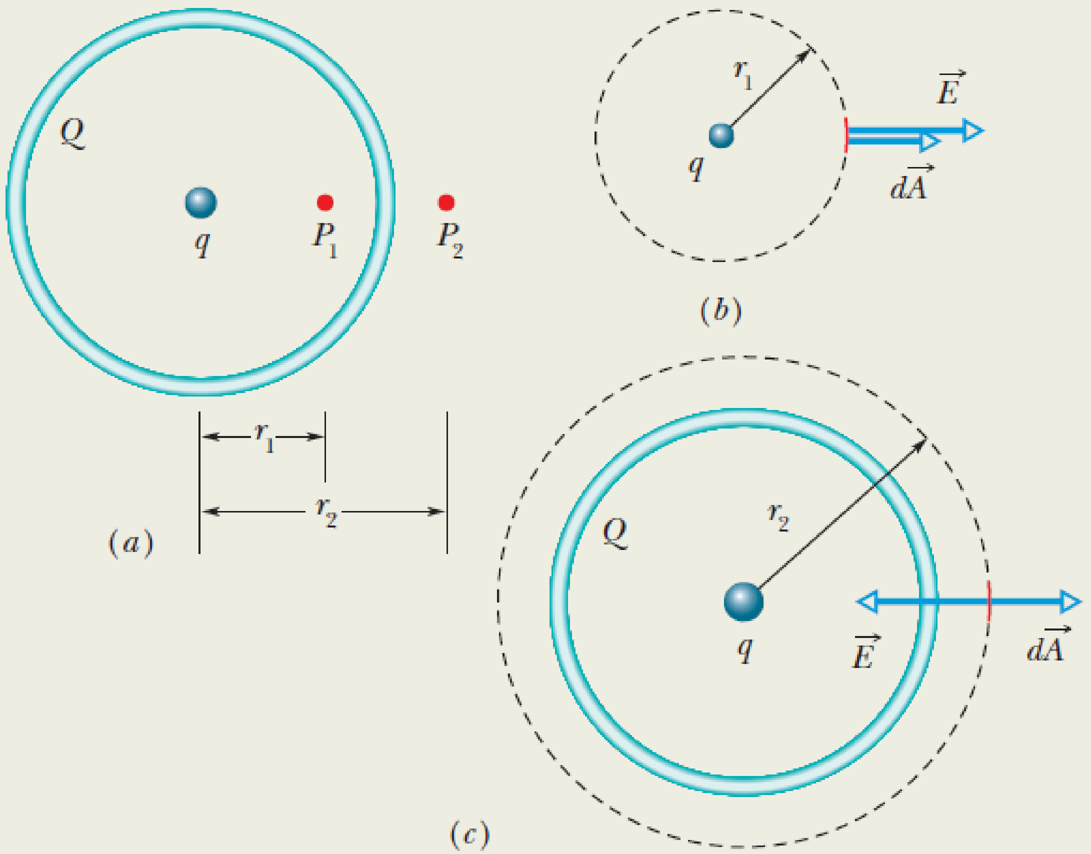


Figure 23-10 (*a*) A charged plastic spherical shell encloses a charged particle. (*b*) To find the electric field at P_1, arrange for the point to be on a Gaussian sphere. The electric field pierces outward. The area vector for the patch element is outward. (*c*) P_2 is on a Gaussian sphere, $\vec{E}$ is inward, and $d\vec{A}$ is still outward.

Calculation:

(a) Electric field at P_1 : From Gauss's Law,

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 \oint E \cos 0 \, dA = \varepsilon_0 \oint E \, dA = \varepsilon_0 E \oint dA,$$

$$\varepsilon_0 E 4\pi r^2 = q_{\text{enc}}.$$

$$\begin{aligned} E &= \frac{q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \\ &= \frac{5(1.60 \times 10^{-19}\,\text{C})}{4\pi(8.85 \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.0600\,\text{m})^2} \\ &= 2.00 \times 10^{-6}\,\text{N/C}. \qquad \text{(Answer)} \end{aligned}$$

(b) Electric field at P_2 :   To find the electric field at P_2,

$$q_{enc} = q + Q = 5e + (-16e) = -11e.$$

$$E = \frac{-q_{enc}}{4\pi\varepsilon_0 r^2}$$

$$= \frac{-[-11(1.60 \times 10^{-19}\,\mathrm{C})]}{4\pi(8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2})(0.120\,\mathrm{m})^2}$$

$$= 1.10 \times 10^{-6}\,\mathrm{N/C}. \qquad \text{(Answer)}$$

23-7 Applying Gauss' Law: Cylindrical Symmetry

Figure 23-12 shows a section of an infinitely long cylindrical plastic rod with a uniform positive linear charge density λ. Let us find an expression for the magnitude of the electric field $\vec{E}$ at a distance r from the axis of the rod.

Our Gaussian surface should match the symmetry of the problem, which is cylindrical. We choose a circular cylinder of radius r and length h, coaxial with the rod. Because the Gaussian surface must be closed, we include two end caps as part of the surface.

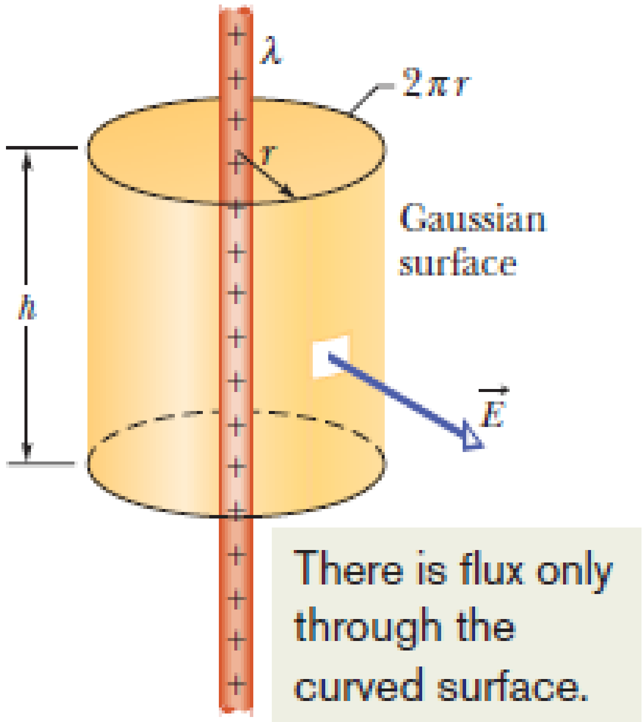


Fig. 23-12 A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

Since $2\pi r$ is the cylinder's circumference and h is its height, the area A of the cylindrical surface is $2\pi rh$. The flux of $\vec{E}$ through this cylindrical surface is then

$$\Phi = EA\cos\theta = E(2\pi rh)\cos 0 = E(2\pi rh).$$

There is no flux through the end caps because $\vec{E}$, being radially directed, is parallel to the end caps at every point.

The charge enclosed by the surface is λh, which means Gauss' law,

$$\varepsilon_0\Phi = q_{enc},$$

reduces to

$$\varepsilon_0 E(2\pi rh) = \lambda h,$$

yielding

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge).} \tag{23-12}$$