Hypothesis testing is a form of statistical inference, which is the process by which we make a decision (or "infer") about the value of an unknown population parameter.

Hypothesis tests are widely used in translational areas, including biomedicine, epidemiology, engineering, and the social sciences. They are commonly taught in introductory courses, so most students have at least heard of them. Hypothesis tests are useful because they can shed insight on answers to interesting questions. For example,

- Have starting salaries of USC graduates increased over the last 5 years?
- Is a new drug or intervention superior when compared to the standard method of treatment?
- How do vaccination rates compare among different races/ethnicities?

Problems:

Example-01: Design a decision rule to test the hypothesis that a die is fair if we take a sample of 150 trials of the die to get even/odd faces and use 0.05 as the significance level. Predict the acceptance and critical region.

Solution:

Here, $H_0: p = 0.5$ and $H_1: p \neq 0.5$. So, we will use two-tailed test here.

Significance level $\alpha = 0.05$

From figure, $P\left(Z < -z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

$$\Rightarrow \phi\left(-z_{\frac{\alpha}{2}}\right) = 0.025$$

$$\Rightarrow \phi\left(z_{\frac{\alpha}{2}}\right) = 0.975 = \phi(1.96)$$

$$\Rightarrow z_{\frac{\alpha}{2}} = 1.96$$

Here, Null hypothesis H_0 is true for the interval:

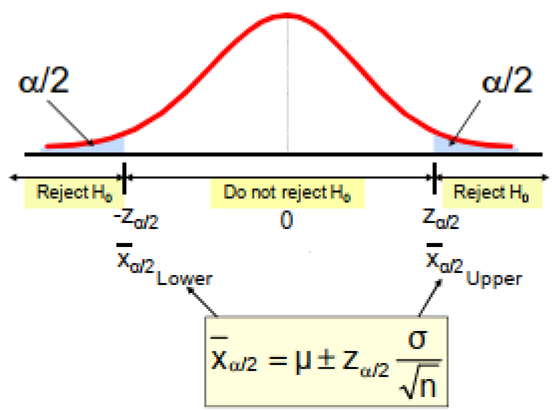
$$-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}$$

$$\Rightarrow -1.96 < \frac{X-\mu}{\sigma} < 1.96$$

$$\Rightarrow \mu - 1.96\sigma < X < \mu + 1.96\sigma$$

$$\Rightarrow 75 - 1.96 * 6.124 < X < 75 + 1.96 * 6.124$$

$$\Rightarrow 62.997 < X < 87.003$$



Here,

We can use binomial distribution for being $p = 0.5, q = 0.5$ and $n = 150$

So, $\mu = np = 75$ and $\sigma = \sqrt{npq} = 6.124$

So, the claim of the die is fair will be accepted if the number of even/odd faces will appear within 63 and 87 (faces of the die cannot be measured by a fraction) and rejected otherwise.

Example-02: A pharmaceutical company produces a new medicine and they claimed that it will reduce the migraine pain very fast with 95% accuracy. Now design decision rule for the process with significance 0.01 by applying the medicine to 120 people. Make a decision if 115 people get relief from the migraine pain by using the medicine.

Solution:

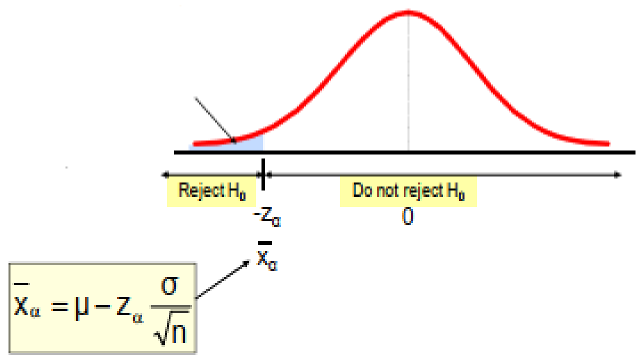
Here, $H_0: p = 0.95$ and $H_1: p < 0.95$. So, we will use left-tailed test here.
Significance level $\alpha = 0.01$

From figure, $P(Z < -z_\alpha) = \alpha$

$$\Rightarrow \phi(-z_\alpha) = 0.01$$
$$\Rightarrow \phi(z_\alpha) = 0.99 = \phi(2.33)$$
$$\Rightarrow z_\alpha = 2.33$$

Here, Null hypothesis H_0 is true for the interval:

$$Z > -z_\alpha$$
$$\Rightarrow \frac{X - \mu}{\sigma} > -2.33$$
$$\Rightarrow X > \mu - 2.33\sigma$$
$$\Rightarrow X > 114 - 2.33 * 2.39$$
$$\Rightarrow X > 108.43$$



Here,
We can use binomial distribution for being $p = 0.95, q = 0.05$ and $n = 120$
So, $\mu = np = 114$ and $\sigma = \sqrt{npq} = 2.39$

So, the claim of the pharmaceutical company will be accepted if more than 109 people (people cannot be measured by a fraction) get relief from the migraine pain and rejected otherwise. Since 115 people relieved from the migraine pain, the claim of the pharmaceutical company should be accepted.

Example-03: A company produces electric bulbs whose average lifetime is 180 days and an average variation of 15 days. It is claimed that by using a newly developed process the mean lifetime of the bulbs can be increased.

(i) Design a decision rule for the process at the 0.01 significance level to test 100 bulbs. Also, what about the decision if the average lifetime of a bulb 182 days and 197 days.

(ii) If the new process has increased the mean lifetime to 192 days and assuming the estimated sample mean 187 days, find α and β for 30 samples.

(iii) Suppose for 50 samples average lifetime is observed as 185 days, estimate the p-value of the claim of the manufacturer.

Solution:

Here, $H_0: \mu = 180$ and $H_1: \mu > 180$. So, we will use right-tailed test here.

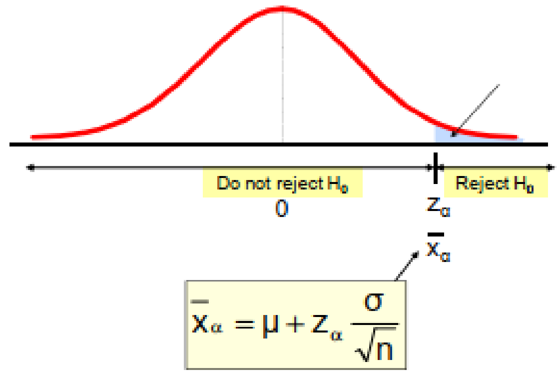
(i) Significance level $\alpha = 0.01$

From figure, $P(Z > z_\alpha) = \alpha$

$$\Rightarrow P(Z < -z_\alpha) = \alpha$$
$$\Rightarrow \phi(-z_\alpha) = 0.01$$
$$\Rightarrow \phi(z_\alpha) = 0.99 = \phi(2.33)$$
$$\Rightarrow z_\alpha = 2.33$$

For making a decision rule and the mean lifetime of the bulbs can be increased,
the alternating hypothesis H_1 is true for the interval:

$$Z > z_\alpha$$
$$\Rightarrow \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > 2.33$$
$$\Rightarrow \bar{X} > \mu + 2.33 * \frac{\sigma}{\sqrt{n}}$$
$$\Rightarrow \bar{X} > 180 + 2.33 * \frac{15}{\sqrt{100}}$$
$$\Rightarrow \bar{X} > 183.495$$



Here,

$$\sigma = 15 \text{ and } n = 100$$

So, the claim of the company will be accepted if the lifetime increased more than 183.495 days and rejected otherwise. Now, the claim will be rejected and accepted for the average sample lifetime 182 days and 197 days, respectively.

(ii) Here, $H_1 = 192,\ n = 30$ and the estimated sample mean $\bar{X} = 187$

$$\alpha = \text{Type I Error } (\mathrm{H_0} \text{ reject, but true})$$
$$= P(\bar{X} \geq 187;\ \mu = 180)$$
$$= P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{187 - 180}{\frac{15}{\sqrt{30}}}\right)$$
$$= P(Z \geq 2.56) = \phi(-2.56) = 0.0052$$

$$\beta = \text{Type II Error } (\mathrm{H_0} \text{ accept, but false})$$
$$= P(\bar{X} < 187;\ \mu = 192)$$
$$= P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{187 - 192}{\frac{15}{\sqrt{30}}}\right)$$
$$= P(Z < -1.83) = \phi(-1.83) = 0.0336$$

(iii) Here, $n = 50$ and the observed value $= 185$
For right-tailed test,

$$p - \text{value} = P(\bar{X} > 185;\ \mu = 180)$$
$$= P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{185 - 180}{\frac{15}{\sqrt{50}}}\right)$$
$$= P(Z > 2.36) = \phi(-2.36) = 0.0091$$

Example-04: A Telecom service provider claims that individual customers pay on an average 400 rs. per month with standard deviation of 25 rs. A random sample of 50 customers' bills during a given month is taken with a mean of 250 and standard deviation of 15. What to say with respect to the claim made by the service provider?

Solution:
Here, $H_0: \mu = 400$ and $H_1: \mu \neq 400$. So, we will use two-tailed test here.
Significance level $\alpha = 0.05$ (consider), $\sigma = 25, n = 50, \bar{X} = 250$ and $s = 15$
Z-test will be used here for being $n \geq 30$

From figure, $P\left(Z < -z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

$$\Rightarrow \phi\left(-z_{\frac{\alpha}{2}}\right) = 0.025$$

$$\Rightarrow \phi\left(z_{\frac{\alpha}{2}}\right) = 0.975 = \phi(1.96)$$

$$\Rightarrow z_{\frac{\alpha}{2}} = 1.96$$

Here, Null hypothesis H_0 is true for the interval:

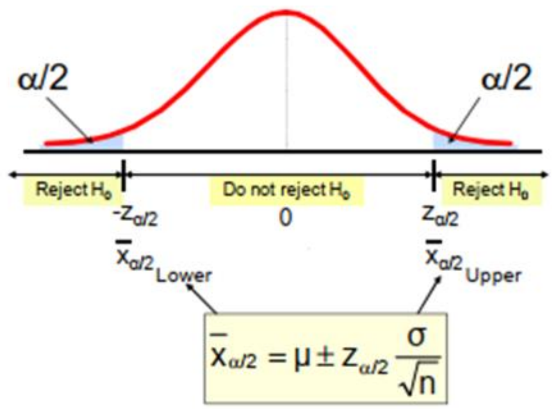
$$-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}$$

$$\Rightarrow -1.96 < Z < 1.96$$

Now, Z-value

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{250 - 400}{\frac{25}{\sqrt{50}}}$$

$$= -42.42 < -1.96$$



which is outside of the interval of H_0. So, null hypothesis is rejected here.

Example-05: It is claimed that the mean of the population is 67 at 5% level of significance. Mean obtained from a random sample of size 100 is 64 with SD 3. Validate the claim.

Solution:
Here, $H_0: \mu = 67$ and $H_1: \mu \neq 67$. So, we will use two-tailed test here.
Significance level $\alpha = 0.05, n = 100, \bar{X} = 64$ and $s = 3$
Z-test will be used here for being $n \geq 30$

From figure, $P\left(Z < -z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

$$\Rightarrow \phi\left(-z_{\frac{\alpha}{2}}\right) = 0.025$$

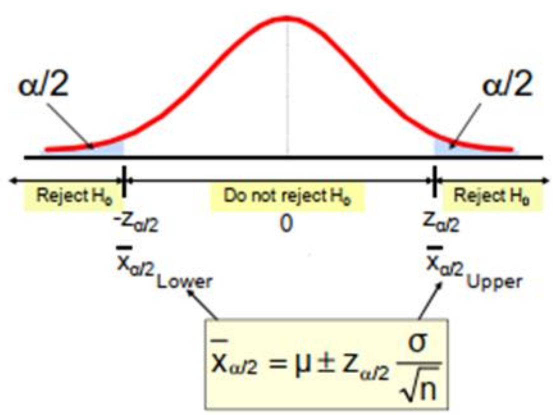
$$\Rightarrow \phi\left(z_{\frac{\alpha}{2}}\right) = 0.975 = \phi(1.96)$$

$$\Rightarrow z_{\frac{\alpha}{2}} = 1.96$$

Here, Null hypothesis H_0 is true for the interval:

$$-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}$$

$$\Rightarrow -1.96 < Z < 1.96$$



Now, Z-value

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{64 - 67}{\frac{3}{\sqrt{100}}}$$
$$= -10 < -1.96$$

which is outside of the interval of H_0. So, null hypothesis is rejected here.

Conclusion:

with given data it is significantly proven that population mean is not equal to 67.

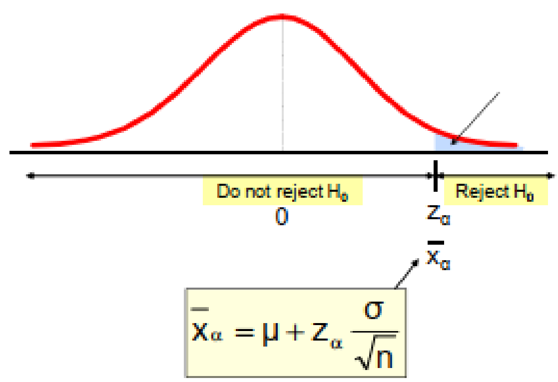
Example-06: In the past, the mean height of plants of a particular species has been 2.3m. A random sample of 60 plants of this species was treated with fertilizer and the mean height of these 60 plants was found to be 2.4m. Assume that the standard deviation of the heights of plants treated with fertilizer is 0.4m.

Carry out a test at the 2.5% significance level of whether the mean height of plants treated with fertilizer is greater than 2.3m.

Solution:

Here, $H_0: \mu = 2.3 \; and \; H_1: \mu > 2.3$. So, we will use right-tailed test here.

Significance level $\alpha = 2.5\% = 0.025, \;\; \sigma = 0.4, \bar{X} = 2.4$ and $n = 60$



From figure, $P(Z > z_\alpha) = \alpha$

$$\Rightarrow P(Z < -z_\alpha) = \alpha$$
$$\Rightarrow \phi(-z_\alpha) = 0.025$$
$$\Rightarrow \phi(z_\alpha) = 0.975 = \phi(1.96)$$
$$\Rightarrow z_\alpha = 1.96$$

Now, Z-value

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{2.4 - 2.3}{\frac{0.4}{\sqrt{60}}}$$
$$= 1.936 < 1.96$$

which means that the null hypothesis is not rejected here.

No evidence that (mean) height (with fertilizer) is more than without.

Example-07: Arvind uses an ordinary fair 6-sided die to play a game. He believes he has a system to predict the score when the die is thrown. Before each throw of the die, he writes down what he thinks the score will be. He claims that he can write the correct score more often than he would if he were just guessing. His friend Laxmi tests his claim by asking him to write down the score before each of 15 throws of the die. Arvind writes the correct score on exactly 5 out of 15 throws. Test Arvind's claim at the 10% significance level.

Solution:

Here, $H_0: P(\text{correct}) = \frac{1}{6}$ and $H_1: P(\text{correct}) > \frac{1}{6}$, Significance level $\alpha = 10\% = 0.1$

For his claim that he can write the correct score more often,

$$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} 15_{C_x} \left(\frac{1}{6}\right)^x \left(1 - \frac{1}{6}\right)^{15-x}$$
$$= 0.0898 < 0.1$$

which means that the null hypothesis is rejected here. So, there is evidence (at the 10% significance level) that Arvind can predict scores.

Example-08: The percentage of people having a different medical condition is thought to be 30%. A researcher suspects that the true percentage is less than 30%. In a medical trial a random sample of 28 people was selected and 4 people were found to have this condition. Use a binomial distribution to test the researcher's suspicion at the 2% significance level.

Solution:

Here, $H_0: p = 0.3$ and $H_1: p < 0.3$, Significance level $\alpha = 2\% = 0.02$

For the true percentage is less than 30%,

$$P(X \leq 4) = \sum_{x=0}^{4} 28_{C_x} (0.3)^x (1 - 0.3)^{28-x} = 0.0474 > 0.02$$

which means that the null hypothesis is not rejected here. So, there is no evidence that suspicion is true.

Example-09: A machine is supposed to procedure random digits. Bob thinks that the machine is not fair and that the probability of it producing the digit 0 is less than $\frac{1}{10}$. In order to test his suspicion, he notes the number of times the digit 0 occurs in 30 digits produced by the machine. He carries out a test at the 10% significance level.

(a) Sate suitable null and alternative hypotheses.

(b) Find the rejection region for the test.

Solution:

(a) Here, $H_0: P(0) = \frac{1}{10}$ and $H_1: P(0) < \frac{1}{10}$, Significance level $\alpha = 10\% = 0.1$

(b) For no occur of the digit 0 in 30 digits, $P(X = 0) = 30_{C_0} \left(\frac{1}{10}\right)^0 \left(1 - \frac{1}{10}\right)^{30} = 0.9^{30}$
$$= 0.0424 < 0.1$$

which means that the null hypothesis is rejected here

and, for one occur of the digit 0 in 30 digits, $P(X = 0) + P(X = 1)$

$$= 0.9^{30} + 30_{C_1} \left(\frac{1}{10}\right)^1 \left(1 - \frac{1}{10}\right)^{29}$$
$$= 0.184 > 0.1$$

which means that the null hypothesis is not rejected here

Example-10: Accidents on a stretch of road occur at the rate of seven each month. New traffic measures have been put in places to reduce the number of accidents. In the following month, there were only two accidents.

(a) Test at the 5% level of significance if there is evidence that the new traffic measures have significantly reduced the number of accidents.

(b) Over a period of 6 months, there are 32 accidents. It is claimed that new traffic measures are no longer reducing the number of accidents. Test this at 5% level of significance

Solution:

This situation can be modeled as a Poisson distribution, because of the accidents occur at a certain rate per unit of time.

Here, $H_0: \lambda = 7$ and $H_1: \lambda < 7$, Significance level $\alpha = 5\% = 0.05$, $\mu = \lambda = 7, X = 2$ & $\sigma = \sqrt{7}$

(a)

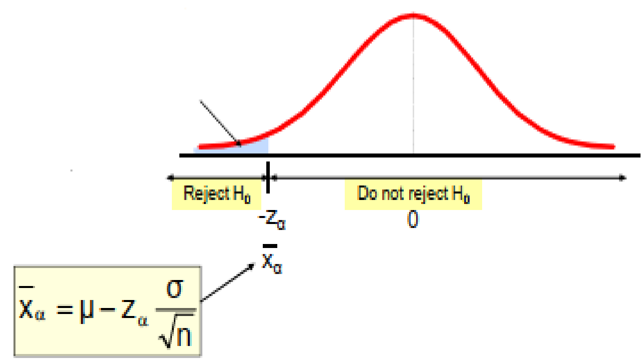
From figure, $P(Z < -z_\alpha) = \alpha$

$$\Rightarrow \phi(-z_\alpha) = 0.05$$
$$\Rightarrow \phi(z_\alpha) = 0.95 = \phi(1.645)$$
$$\Rightarrow z_\alpha = 1.645$$
$$\therefore -z_\alpha = -1.645$$

Now, Z-value

$$Z = \frac{X - \mu}{\sigma} = \frac{2 - 7}{\sqrt{7}}$$
$$= -1.89 < -1.645$$

which means that the null hypothesis is rejected here.



This suggests that there is evidence to conclude that the new traffic measures have significantly reduced the number of accidents.

(b) Mean rate over 6 the month, $\mu = \lambda = 7 * 6 = 42$, $\alpha = 5\% = 0.05$, $X = 32$ & $\sigma = \sqrt{42}$

Now, Z-value

$$Z = \frac{X - \mu}{\sigma} = \frac{32 - 42}{\sqrt{42}}$$
$$= -1.54 > -1.645$$

which means that the null hypothesis is not rejected here.

This concludes that the new traffic measures are no longer reducing the number of accidents.