Properties of Mathematical Expectation

- Consider $u = u(X)$ as a function of discrete random variable X with the probability mass function (*pmf*) $f(x)$ then the Mathematical Expectation is defined as $E[u(X)] = \sum_{x\in S} u(x)f(x)$.

- Consider $u = u(X)$ as a function of continuous random variable X with the probability density function (*pdf*) $f(x)$ then the Mathematical Expectation is defined as $E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)$.

- The cumulative density function (*cdf*) $F(x)$ is defined as
  - $F(x) = P(X \le x_t) = \sum_{x=-\infty}^{x_t} f(x)$ (for discrete)
  - $F(x) = P(X \le x_t) = \int_{-\infty}^{x_t} f(x)$ (for continuous)

- For $Y = X^r$ then, we have
  - $E[Y] = E[X^r] = \sum_{x\in S} x^r f(x)$ (for discrete)
  - $E[Y] = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)$ (for continuous)

- If the given function is renamed of a new variable is declared with the same *pmf* the desired mathematical expectation will not be changed.
- If the variable or/and corresponding *pmf* redefined the mathematical expectation will be changed.
- For continuous random variable $P(X \ge a) = P(X > a)$ and $P(X \le a) = P(X < a)$.

- ✓ $E[c] = c$, where c is a constant
- ✓ $E[cu(X)] = cE[u(X)]$
- ✓ $E[u_1(X) \pm u_2(X)] = E[u_1(X)] \pm E[u_2(X)]$
- ✓ $E[c_1u_1(X) \pm c_2u_2(X) \pm c_3] = c_1E[u_1(X)] \pm c_2E[u_2(X)] \pm c_3$

- Mean of any distribution $\mu = E[X] = \sum_{x\in S} xf(x)$ (for discrete)
- Mean of any distribution $\mu = E[X] = \int_{-\infty}^{\infty} xf(x)$ (for continuous)
  - $E[X-\mu] = E[X] - E[\mu] = \mu - \mu = 0$

- Variance of any distribution $\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_{x\in S}(x-\mu)^2 f(x)$ (for discrete)
- Variance of any distribution $\sigma^2 = V(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)$ (for continuous)
  - Shortcut formula $\sigma^2 = E[X^2] - \mu^2 = E[X^2] - \{E[X]\}^2$
  - If $E[X] = \mu = 0, \sigma^2 = E[X^2]$

- Standard deviation $SD = \sigma = +\sqrt{V(x)} = \sqrt{E[X^2] - \{E[X]\}^2}$

- ✓ $V(u(X)) \ge 0$
- ✓ $V(c) = 0$, where c is a constant
- ✓ $V[cu(X)] = c^2V[u(X]$

- ✓ $V[u_1(X) \pm u_2(X)] = V[u_1(X)] \pm V[u_2(X)]$ [Considering $u_1(X)$ and $u_2(X)$ are uncorrelated]
- ✓ $V[c_1 u_1(X) \pm c_2 u_2(X) \pm c_3] = c_1^2 V[u_1(X)] + c_2^2 V[u_2(X)]$

⇨ Moment generating function (*mgf*) is defined as $M(t) = E[e^{tX}] = \sum_{x \in S} e^{tx} f(x)$ (for discrete)

- ➢ $M(0) = E[1] = \sum_{x \in S} f(x) = 1$
- ➢ $M^r(t) = \sum_{x \in S} x^r e^{tx} f(x)$
- ➢ $M^r(0) = \sum_{x \in S} x^r f(x) = E[X^r]$

⇨ Moment generating function (*mgf*) is defined as $M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)$ (for continuous)

- ➢ $M(0) = E[1] = \int_{-\infty}^{\infty} f(x) = 1$
- ➢ $M^r(t) = \int_{-\infty}^{\infty} x^r e^{tx} f(x)$
- ➢ $M^r(0) = \int_{-\infty}^{\infty} x^r f(x) = E[X^r]$

- ✓ Mean $\mu = E[X] = M'(0)$
- ✓ Variance $\sigma^2 = E[X^2] - \{E[X]\}^2 = M''(0) - \{M'(0)\}^2$
- ✓ Standard deviation $SD = \sqrt{M''(0) - \{M'(0)\}^2}$

Uniform (Discrete) distribution

Let us consider a distribution having m discrete outcomes with equal probability, and then probability distribution of getting any of the outcomes is called as Uniform distribution of discrete type. The *pmf* of the Uniform distribution is defined as

$$f(x) = \frac{1}{m} \; ; \; x = 1, 2, 3, \dots \dots \dots , m$$

Here, $\sum_{x \in S} f(x) = \sum_{x=1}^{m} \frac{1}{m} = \frac{m}{m} = 1$

- ✓ $E[X] = \sum_{x \in S} x f(x) = \sum_{x=1}^{m} x \frac{1}{m} = \frac{1}{m} \frac{m(m+1)}{2} = \frac{m+1}{2}$
- ✓ $E[X^2] = \sum_{x \in S} x^2 f(x) = \sum_{x=1}^{m} x^2 \frac{1}{m} = \frac{1}{m} \frac{m(m+1)(2m+1)}{6} = \frac{(m+1)(2m+1)}{6}$

- ⇨ Mean $\mu = E[X] = \frac{m+1}{2}$
- ⇨ Variance $\sigma^2 = E[X^2] - \{E[X]\}^2 = \frac{(m+1)(2m+1)}{6} - \left\{\frac{m+1}{2}\right\}^2 = \frac{m^2-1}{12}$
- ⇨ Standard deviation $SD = \sqrt{\frac{m^2-1}{12}}$

- ⇨ Moment generating function $M(t) = \sum_{x \in S} e^{tx} f(x) = \sum_{x=1}^{m} e^{tx} \frac{1}{m} = \sum_{x=1}^{m} \frac{e^{tx}}{m}$
  - ➢ $M(0) = \sum_{x=1}^{m} \frac{1}{m} = 1$

Hypergeometric distribution

Let us consider a distribution having N discrete outcomes can be divided in two well-defined categories with N_1 and N_2 outcomes $N = N_1 + N_2$, respectively. The probability distribution of getting $x \leq N_1$ outcomes from the first category and the rest $n - x \leq N_2$ outcomes from the second category for n selected outcomes is called as Hypergeometric distribution. The *pmf* of the Hypergeometric distribution is defined as

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}} \; ; \; x \leq n$$

Here, $\sum_{x\in S} f(x) = \sum_{x\in S} \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}} = 1$

- ✓ $E[X] = \sum_{x\in S} x f(x) = \sum_{x\in S} x \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}} = n\left(\frac{N_1}{N}\right)$
- ✓ $E[X^2] = \sum_{x\in S} x^2 f(x) = \sum_{x\in S} x^2 \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}} = \frac{n(n-1)N_1(N_1-1)}{N(N-1)} + n\left(\frac{N_1}{N}\right)$

- ⇨ Mean $\mu = E[X] = n\left(\frac{N_1}{N}\right)$
- ⇨ Variance $\sigma^2 = E[X^2] - \{E[X]\}^2 = \frac{n(n-1)N_1(N_1-1)}{N(N-1)} + n\left(\frac{N_1}{N}\right) - \left\{n\left(\frac{N_1}{N}\right)\right\}^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$
- ⇨ Standard deviation $SD = \sqrt{n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)}$

- ⇨ Moment generating function $M(t) = \sum_{x\in S} e^{tx} f(x) = \sum_{x\in S} e^{tx} \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$
    - ➢ $M(0) = \sum_{x\in S} \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}} = 1$

Geometric distribution

Let us consider p and q as the probability of success and failure of any trial with $p+q=1$. The trial will be terminated at the maiden success, and then probability distribution of getting success in x-th trial in successive trials is called as Geometric distribution. The *pmf* of the Geometric distribution is defined as

$$f(x)=q^{x-1}p\ ;\ x=1,2,3,\dots\dots\dots$$

Here, $\sum_{x\in S} f(x)=\sum_{x=1}^{\infty} q^{x-1}p\ =p+qp+q^2p+q^3p+\cdots\dots\dots=\frac{p}{1-q}=1$

- ⇨ Mean $\mu=\sum_{x\in S} xf(x)=\sum_{x=1}^{\infty} xq^{x-1}p\ =\frac{1}{p}$ [Considering $p=1-q$]

- ⇨ Moment generating function $M(t)=\sum_{x\in S} e^{tx}f(x)=\sum_{x=1}^{\infty} e^{tx}\,q^{x-1}p=\frac{pe^t}{1-qe^t}$
    - ➢ $M(0)=\frac{p}{1-q}=1$
    - ➢ $M'(t)=\frac{pe^t}{(1-qe^t)^2}$
    - ➢ $M''(t)=\frac{pe^t(1+qe^t)}{(1-qe^t)^3}$

- ⇨ Mean $\mu=M'(0)=\frac{1}{p}$
- ⇨ Variance $\sigma^2=M''(0)-\{M'(0)\}^2=\frac{1+q}{p^2}-\left\{\frac{1}{p}\right\}^2=\frac{q}{p^2}$
- ⇨ Standard deviation $SD=\sqrt{\frac{q}{p^2}}$

Binomial distribution

Let us consider p and q as the probability of success and failure of n independent trials with $p + q = 1$. The value of p and q remain the same on each trial, and then probability distribution of getting x number of success in n trials is called as Binomial distribution. The pmf of the Binomial distribution is defined as

$$f(x) = {}^{n}C_x p^x q^{n-x} \; ; \; x = 0,1,2,3, \dots \dots \dots , n$$

Here, $\sum_{x \in S} f(x) = \sum_{x=0}^{n} {}^{n}C_x p^x q^{n-x} = (q + p)^n = 1$

The Binomial distribution can be represented as $b(n, p)$.

- ⇨ Mean $\mu = \sum_{x \in S} x f(x) = \sum_{x=0}^{n} x \, {}^{n}C_x p^x q^{n-x} = np$ [Considering $p = 1 - q$]

- ⇨ Moment generating function $M(t) = \sum_{x \in S} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \, {}^{n}C_x p^x q^{n-x} = (q + pe^t)^n$
  - ➢ $M(0) = (q + p)^n = 1$
  - ➢ $M'(t) = n(q + pe^t)^{n-1} pe^t$
  - ➢ $M''(t) = n(n-1)(q + pe^t)^{n-2}(pe^t)^2 + n(q + pe^t)^{n-1} pe^t$

- ⇨ Mean $\mu = M'(0) = np$
- ⇨ Variance $\sigma^2 = M''(0) - \{M'(0)\}^2 = n(n-1)p^2 + np - \{np\}^2 = np(1-p) = npq$
- ⇨ Standard deviation $SD = \sqrt{npq}$

Poisson distribution

Let us consider p as the probability of success of n trials with $n \to \infty$ implies $p \to 0$. The probability distribution of getting x number of success is called as Poisson distribution. Then, a finite constant $\lambda > 0$ is known as the Poisson parameter and can be attained as $\lambda = np$. The *pmf* of the Poisson distribution is defined as

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \ ; \ x = 0,1,2,3, \dots \dots \dots$$

Here, $\sum_{x \in S} f(x) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$

- Mean $\mu = \sum_{x \in S} x f(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$

- Moment generating function $M(t) = \sum_{x \in S} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{\lambda(e^t - 1)}$
    - $M(0) = e^{\lambda(1-1)} = 1$
    - $M'(t) = \lambda e^t e^{\lambda(e^t - 1)}$
    - $M''(t) = (\lambda e^t)^2 e^{\lambda(e^t - 1)} + \lambda e^t e^{\lambda(e^t - 1)}$

- Mean $\mu = M'(0) = \lambda$
- Variance $\sigma^2 = M''(0) - \{M'(0)\}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$
- Standard deviation $SD = \sqrt{\lambda}$

Uniform (Continuous) distribution

Let us consider a distribution having constant probability within the interval $[a, b]$, and then distribution of the probability in that interval is called as Uniform distribution of continuous type. The *pdf* of the Uniform distribution is defined as

$$f(x) = \frac{1}{b-a} \;\; ; \; a \le x \le b$$

The *cdf* of Uniform distribution is defined as $F(x) = P(X \le x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$

Here, $\int_{-\infty}^{\infty} f(x)dx = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a}(b-a) = 1$

The Uniform distribution can be represented as $U(a, b)$.

- ✓ $E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_a^b \frac{x}{b-a} dx = \frac{a+b}{2}$
- ✓ $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_a^b \frac{x^2}{b-a} dx = \frac{a^2+ab+b^2}{3}$

- ⇨ Mean $\mu = E[X] = \frac{a+b}{2}$
- ⇨ Variance $\sigma^2 = E[X^2] - \{E[X]\}^2 = \frac{a^2+ab+b^2}{3} - \left\{\frac{a+b}{2}\right\}^2 = \frac{(b-a)^2}{12}$
- ⇨ Standard deviation $SD = \sqrt{\frac{(b-a)^2}{12}}$

- ⇨ Moment generating function $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_a^b \frac{e^{tx}}{b-a} dx = \begin{cases} \frac{e^{bt}-e^{at}}{t(b-a)}, & t \ne 0 \\ 1, & t = 0 \end{cases}$
    - ➢ $M(0) = 1$

Exponential distribution

Let us consider a distribution having average waiting time $\theta = \frac{1}{\lambda} > 0$ for the first occurrence defined in $[0, \infty)$, and then distribution of the probability in that interval is called as Exponential distribution. The *pdf* of the Exponential distribution is defined as

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \; ; \; 0 \le x < \infty$$

The *cdf* of Exponential distribution is defined as $F(x) = P(X \le x) = 1 - e^{-\frac{x}{\theta}} \; ; \; 0 \le x < \infty$

- $P(X > x) = 1 - F(x) = e^{-\frac{x}{\theta}} \; ; \; 0 \le x < \infty$

Here, $\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \frac{0-1}{-\frac{1}{\theta}} = 1$

- Median $m = \theta \log 2$
- Moment generating function $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{0}^{\infty} e^{tx} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = \frac{1}{1-\theta t}$
  - $M(0) = \frac{1}{1-0} = 1$
  - $M'(t) = \frac{\theta}{(1-\theta t)^2}$
  - $M''(t) = \frac{2\theta^2}{(1-\theta t)^2}$

- Mean $\mu = M'(0) = \theta$
- Variance $\sigma^2 = M''(0) - \{M'(0)\}^2 = 2\theta^2 - \theta^2 = \theta^2$
- Standard deviation $SD = \theta$

Gamma distribution

Let us consider a distribution having average waiting time $\theta = \frac{1}{\lambda} > 0$ defined in $[0, \infty)$, and then distribution of the probability waiting for the α-th occurrence in that interval is called as Gamma distribution. The *pdf* of the Gamma distribution is defined as

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} \ ; \ 0 \leq x < \infty$$

The *cdf* of Gamma distribution is defined as $F(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_{-\infty}^{x_t} x^{\alpha-1} e^{-\frac{x}{\theta}} dx \ ; \ 0 \leq x_t < \infty$

Here, $\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{1}{\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} dx = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) = 1$

- ⇨ Moment generating function $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{0}^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} dx = \frac{1}{(1-\theta t)^{\alpha}}$
  - ➢ $M(0) = \frac{1}{(1-0)^{\alpha}} = 1$
  - ➢ $M'(t) = \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}$
  - ➢ $M''(t) = \frac{\alpha(\alpha+1)\theta^2}{(1-\theta t)^{\alpha+2}}$

- ⇨ Mean $\mu = M'(0) = \alpha\theta$
- ⇨ Variance $\sigma^2 = M''(0) - \{M'(0)\}^2 = \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \alpha\theta^2$
- ⇨ Standard deviation $SD = \sqrt{\alpha\theta^2}$

Normal distribution

Let us consider a symmetric distribution (Mean, Median, Mode coincide) having mean $-\infty < \mu < \infty$ and variance σ^2 ; $0 < \sigma < \infty$, defined in $(-\infty, \infty)$ is called the Normal distribution. The *pdf* of the Normal distribution is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \ ; \ -\infty < x < \infty$$

For the standardized variable $z = \frac{x-\mu}{\sigma}$ *cdf* of the Normal distribution is defined as

$$\phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \ ; \ -\infty < z < \infty$$

Here, $\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$

The Normal distribution can be represented as $N(\mu, \sigma^2)$.

- Moment generating function $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
    - $M(0) = e^0 = 1$
    - $M'(t) = (\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
    - $M''(t) = [(\mu + \sigma^2 t)^2 + \sigma^2] e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

- Mean $\mu = M'(0) = \mu$
- Variance $\sigma^2 = M''(0) - \{M'(0)\}^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$
- Standard deviation $SD = \sqrt{\sigma^2}$

- $P(Z \le z) = \phi(z)$
- $P(z_1 \le Z \le z_2) = \phi(z_2) - \phi(z_1)$
- $P(-z_2 \le Z \le -z_1) = P(z_1 \le Z \le z_2) = \phi(z_2) - \phi(z_1)$
- $P(Z \ge z) = 1 - P(Z \le z) = 1 - \phi(z)$
- $P(Z \ge -z) = P(Z \le z) = \phi(z)$
- $P(Z \le -z) = P(Z \ge z) = 1 - \phi(z)$
- $P(|Z| \le z) = P(-z \le Z \le z) = \phi(z) - \phi(-z) = \phi(z) - [1 - \phi(z)] = 2\phi(z) - 1$
- $P(|Z| \ge z) = P(Z \le -z) + P(Z \ge z) = [1 - \phi(z)] + [1 - \phi(z)] = 2 - 2\phi(z)$
- $P(Z \le 0) = \phi(0) = 0.5$
- $P(Z \ge 0) = P(Z \le 0) = \phi(0) = 0.5$
- $P(Z = z) = 0$ or undefined