INTERVAL ESTIMATION

A Confidence Interval (CI) is a range of values that's likely to include a population value with a certain degree of confidence. It is often expressed a percentage whereby a population means lies between an upper and lower interval.

Chapter-7.1 (Confidence intervals for Means)

Consider, n is the number of samples, α is the level of significance and hence $1-\alpha$ is the level of confidence, $\bar{x}$ is the average of the sample data, σ is the standard deviation of the target data, and μ is the required mean of the population.

To find the $z\alpha/_2$, we need to assume

$$P\left(-z\alpha/_2 \le Z \le z\alpha/_2\right) = 1-\alpha$$
$$or, \quad 2\varphi\left(z\alpha/_2\right) - 1 = 1-\alpha$$
$$or, \quad 2\varphi\left(z\alpha/_2\right) = 2-\alpha$$
$$or, \quad \varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

To find the required confidence interval, we need to assume

$$-z\alpha/_2 \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le z\alpha/_2$$
$$or, \quad -z\alpha/_2\, \sigma/\sqrt{n} \le \bar{x}-\mu \le z\alpha/_2\, \sigma/\sqrt{n}$$
$$or, \quad -\bar{x} - z\alpha/_2\, \sigma/\sqrt{n} \le -\mu \le -\bar{x} + z\alpha/_2\, \sigma/\sqrt{n}$$
$$or, \quad \bar{x} + z\alpha/_2\, \sigma/\sqrt{n} \ge \mu \ge \bar{x} - z\alpha/_2\, \sigma/\sqrt{n}$$
$$or, \quad \bar{x} - z\alpha/_2\, \sigma/\sqrt{n} \le \mu \le \bar{x} + z\alpha/_2\, \sigma/\sqrt{n}$$

INTERVAL ESTIMATION

1. Let X equal to the amount of food (in pound per day) consumed by a student. Suppose the variance σ^2 of X is 1.21. To estimate the mean μ of X, an agency took a random sample of 1000 people and found they consumed in total 4200 lb food per day. Find an approximate 95% confidence interval for μ.

Solution:

Here, Sample size $n = 1000$

Mean consumption $\bar{x} = \frac{4200}{1000} = 4.2$ lb

Standard deviation $\sigma = 1.1$

Confidence $1 - \alpha = 0.95$

Significance $\alpha = 0.05$

Estimate the $z_{\alpha/2}$ as

$$\varphi\left(z_{\alpha/2}\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z_{\alpha/2}\right) = \frac{2-0.05}{2}$$

$$or, \qquad \varphi\left(z_{\alpha/2}\right) = 0.975$$

$$or, \qquad z_{\alpha/2} = 1.96$$

Now, the required confidence interval is

$$\bar{x} - z_{\alpha/2}\, {}^{\sigma}\!/\!_{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\, {}^{\sigma}\!/\!_{\sqrt{n}}$$

$$or, \qquad 4.2 - 1.96 * \frac{1.1}{\sqrt{1000}} \le \mu \le 4.2 + 1.96 * \frac{1.1}{\sqrt{1000}}$$

$$or, \qquad 4.2 - 0.068 \le \mu \le 4.2 + 0.068$$

$$or, \qquad 4.132 \le \mu \le 4.268$$

INTERVAL ESTIMATION

2. Let X equal to the amount of juice in milliliter per day consumed by a student. Suppose the variance of X is 36. To estimate the mean μ of X, a survey team took a random sample of 50 students and found they consumed on average 0.5 litter juice per day. Find an approximate 90% confidence interval for μ.

Solution:

Here, Sample size $n = 50$

Mean consumption $\bar{x} = 0.5$ litter or $\bar{x} = 500$ milliliter

Standard deviation $\sigma = 6$

Confidence $1 - \alpha = 0.90$

Significance $\alpha = 0.10$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.10}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.95$$

$$or, \qquad z\alpha/_2 = 1.645$$

Now, the required confidence interval is

$$\bar{x} - z\alpha/_2\,{}^{\sigma}/_{\sqrt{n}} \le \mu \le \bar{x} + z\alpha/_2\,{}^{\sigma}/_{\sqrt{n}}$$

$$or, \qquad 500 - 1.645 * \frac{6}{\sqrt{50}} \le \mu \le 500 + 1.645 * \frac{6}{\sqrt{50}}$$

$$or, \qquad 500 - 1.396 \le \mu \le 500 + 1.396$$

$$or, \qquad 498.604 \le \mu \le 501.396$$

INTERVAL ESTIMATION

3. Let X equal to the amount of food (in pound per day) consumed by a student having the standard deviation $\sigma = 1.2$. To estimate the mean μ of X, a random sample of 50 people has been taken and found they consumed in total 230 lb food per day. Find the confidence interval for μ with a 5% significance level.

Solution:

Here, Sample size $n = 50$

Mean consumption $\bar{x} = \frac{230}{50} = 4.6$ pound

Standard deviation $\sigma = 1.2$

Significance $\alpha = 0.05$

Confidence $1 - \alpha = 0.95$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.05}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.975$$

$$or, \qquad z\alpha/_2 = 1.96$$

Now, the required confidence interval is

$$\bar{x} - z\alpha/_2\, {}^{\sigma}/_{\sqrt{n}} \le \mu \le \bar{x} + z\alpha/_2\, {}^{\sigma}/_{\sqrt{n}}$$

$$or, \qquad 4.6 - 1.96 * \frac{1.2}{\sqrt{50}} \le \mu \le 4.6 + 1.96 * \frac{1.2}{\sqrt{50}}$$

$$or, \qquad 4.6 - 0.333 \le \mu \le 4.6 + 0.333$$

$$or, \qquad 4.267 \le \mu \le 4.933$$

INTERVAL ESTIMATION

4. Let X equal the weight of fruits in kg per day consumed by a student. Suppose the standard deviation of X is 0.1 kg. To estimate the mean μ of X, an agency took a random sample of 20 students and found they consumed 10 kg of fruits per day. Find an approximate 90% confidence interval for μ.

Solution:

Here, Sample size $n = 20$

Mean consumption $\bar{x} = \frac{10}{20} = 0.5$ kg

Standard deviation $\sigma = 0.1$

Confidence $1 - \alpha = 0.90$

Significance $\alpha = 0.10$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.10}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.95$$

$$or, \qquad z\alpha/_2 = 1.645$$

Now, the required confidence interval is

$$\bar{x} - z\alpha/_2 \, \sigma/\sqrt{n} \le \mu \le \bar{x} + z\alpha/_2 \, \sigma/\sqrt{n}$$

$$or, \qquad 0.5 - 1.645 * \frac{0.1}{\sqrt{20}} \le \mu \le 0.5 + 1.645 * \frac{0.1}{\sqrt{20}}$$

$$or, \qquad 0.5 - 0.037 \le \mu \le 0.5 + 0.037$$

$$or, \qquad 0.463 \le \mu \le 0.537$$

INTERVAL ESTIMATION

5. Let X equal to the amount of coffee (in milliliter per day) consumed by a man. Suppose the variance σ^2 of X is 32. To estimate the mean μ of X, an agency took a random sample of 10 men and found they consumed on average 150 milliliters coffee per day. Find an approximate 80% confidence interval for μ.

Solution:

Here, Sample size $n = 10$

Mean consumption $\bar{x} = 150$ milliliter

Standard deviation $\sigma = \sqrt{32}$

Confidence $1 - \alpha = 0.80$

Significance $\alpha = 0.20$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.20}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.90$$

$$or, \qquad z\alpha/_2 = 1.28$$

Now, the required confidence interval is

$$\bar{x} - z\alpha/_2 \, {}^{\sigma}/_{\sqrt{n}} \leq \mu \leq \bar{x} + z\alpha/_2 \, {}^{\sigma}/_{\sqrt{n}}$$

$$or, \qquad 150 - 1.28 * \frac{\sqrt{32}}{\sqrt{10}} \leq \mu \leq 150 + 1.28 * \frac{\sqrt{32}}{\sqrt{10}}$$

$$or, \qquad 150 - 2.29 \leq \mu \leq 150 + 2.29$$

$$or, \qquad 147.71 \leq \mu \leq 152.29$$

INTERVAL ESTIMATION

Chapter-7.3 (Confidence intervals for Proportions)

Consider, n is the number of target samples, Y is the number of the samples in favor of an event, α is the level of significance and hence $1-\alpha$ is the level of confidence, and p is the required proportion of the population.

To find the $z_{\alpha/2}$, we need to assume

$$P\left(-z_{\alpha/2} \le Z \le z_{\alpha/2}\right) = 1-\alpha$$
$$or, \quad 2\varphi\left(z_{\alpha/2}\right) - 1 = 1-\alpha$$
$$or, \quad 2\varphi\left(z_{\alpha/2}\right) = 2-\alpha$$
$$or, \quad \varphi\left(z_{\alpha/2}\right) = \frac{2-\alpha}{2}$$

To find the required confidence interval, we need to assume

$$-z_{\alpha/2} \le \frac{\frac{Y}{n} - p}{\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}} \le z_{\alpha/2}$$
$$or, \quad -z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le \frac{Y}{n} - p \le z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$
$$or, \quad -\frac{Y}{n} - z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le -p \le -\frac{Y}{n} + z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$
$$or, \quad \frac{Y}{n} + z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \ge p \ge \frac{Y}{n} - z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$
$$or, \quad \frac{Y}{n} - z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le p \le \frac{Y}{n} + z_{\alpha/2}\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$

INTERVAL ESTIMATION

1. In the forest there were 800 plants under extinction, 70% of the plants were transferred for saving from extinction. If 75% of the transferred plants exist after the attempt, find the confidence interval of the proportion with a 10% significance level.

Solution:

Here, Target sample size $n = 70\%\ of\ 800 = 560$

Number of favorable samples $Y = 75\%\ of\ 560 = 420$

Probability of success in the samples $\frac{Y}{n} = 0.75$

Significance $\alpha = 0.10$

Confidence $1 - \alpha = 0.90$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.10}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.95$$

$$or, \qquad z\alpha/_2 = 1.645$$

Now, the required confidence interval is

$$\frac{Y}{n} - z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le p \le \frac{Y}{n} + z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$

$$or, \qquad 0.75 - 1.645 * \sqrt{\frac{0.75 * (1-0.75)}{560}} \le p \le 0.75 + 1.645 * \sqrt{\frac{0.75 * (1-0.75)}{560}}$$

$$or, \qquad 0.75 - 0.03 \le p \le 0.75 + 0.03$$

$$or, \qquad 0.72 \le p \le 0.78$$

INTERVAL ESTIMATION

2. In a certain pollution awareness campaign, one advisor has a survey taken at random among 300 people with 65% of them provided their opinion about the awareness. If the survey secured 170 positive opinions. Find an approximate 95% confidence interval for the fraction p of the people who support the advisor.

Solution:

Here, Target sample size $n = 65\%\ of\ 300 = 195$

Number of favorable samples $Y = 170$

Probability of success in the samples $\frac{Y}{n} = 0.872$

Significance $\alpha = 0.05$

Confidence $1 - \alpha = 0.95$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.05}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.975$$

$$or, \qquad z\alpha/_2 = 1.96$$

Now, the required confidence interval is

$$\frac{Y}{n} - z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le p \le \frac{Y}{n} + z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$

$$or, \qquad 0.872 - 1.96 * \sqrt{\frac{0.872 * (1-0.872)}{195}} \le p \le 0.872 + 1.96 * \sqrt{\frac{0.872 * (1-0.872)}{195}}$$

$$or, \qquad 0.872 - 0.047 \le p \le 0.872 + 0.047$$

$$or, \qquad 0.825 \le p \le 0.919$$

INTERVAL ESTIMATION

3. In a certain political campaign, one candidate has a poll taken at random among 2500 people with 60% of them are a voter. If the candidate secured 55% of casted votes. Find an approximate 95% confidence interval for the fraction p of the voting population that favors the candidate.

Solution:

Here, Target sample size $n = 60\%\ of\ 2500 = 1500$

Number of favorable samples $Y = 55\%\ of\ 1500 = 825$

Probability of success in the samples $\frac{Y}{n} = 0.55$

Significance $\alpha = 0.05$

Confidence $1 - \alpha = 0.95$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.05}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.975$$

$$or, \qquad z\alpha/_2 = 1.96$$

Now, the required confidence interval is

$$\frac{Y}{n} - z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \le p \le \frac{Y}{n} + z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$

$$or, \qquad 0.55 - 1.96 * \sqrt{\frac{0.55*(1-0.55)}{1500}} \le p \le 0.55 + 1.96 * \sqrt{\frac{0.55*(1-0.55)}{1500}}$$

$$or, \qquad 0.55 - 0.025 \le p \le 0.55 + 0.025$$

$$or, \qquad 0.525 \le p \le 0.575$$

INTERVAL ESTIMATION

4. In a forest there are 200 birds under severe trouble of habitats, 75% of the birds are rescued from the forest. If 80% of the rescued birds survived after the attempt, find the confidence interval of the proportion with an 85% confidence level.

Solution:

Here, Target sample size $n = 75\% \; of \; 200 = 150$

Number of favorable samples $Y = 80\% \; of \; 150 = 120$

Probability of success in the samples $\frac{Y}{n} = 0.8$

Confidence $1 - \alpha = 0.85$

Significance $\alpha = 0.15$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2-\alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2-0.15}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.925$$

$$or, \qquad z\alpha/_2 = 1.44$$

Now, the required confidence interval is

$$\frac{Y}{n} - z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}} \leq p \leq \frac{Y}{n} + z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1-\frac{Y}{n}\right)}{n}}$$

$$or, \qquad 0.8 - 1.44 * \sqrt{\frac{0.8*(1-0.8)}{150}} \leq p \leq 0.8 + 1.44 * \sqrt{\frac{0.8*(1-0.8)}{150}}$$

$$or, \qquad 0.8 - 0.047 \leq p \leq 0.8 + 0.047$$

$$or, \qquad 0.753 \leq p \leq 0.847$$

INTERVAL ESTIMATION

5. In a forest there are 1200 animals under severe virus infection, 85% of the animals are rescued from the forest. If half of the total animals survived after the attempt, find the confidence interval of the proportion with a 1% significance level. Is the rescue process effective? Why?

Solution:

Here, Target sample size $n = 85\% \; of \; 1200 = 1020$

Number of favorable samples $Y = 50\% \; of \; 1200 = 600$

Probability of success in the samples $\frac{Y}{n} = 0.588$

Significance $\alpha = 0.01$

Confidence $1 - \alpha = 0.99$

Estimate the $z\alpha/_2$ as

$$\varphi\left(z\alpha/_2\right) = \frac{2 - \alpha}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = \frac{2 - 0.01}{2}$$

$$or, \qquad \varphi\left(z\alpha/_2\right) = 0.995$$

$$or, \qquad z\alpha/_2 = 2.575$$

Now, the required confidence interval is

$$\frac{Y}{n} - z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1 - \frac{Y}{n}\right)}{n}} \le p \le \frac{Y}{n} + z\alpha/_2\sqrt{\frac{\frac{Y}{n}\left(1 - \frac{Y}{n}\right)}{n}}$$

$$or, \qquad 0.588 - 2.575 * \sqrt{\frac{0.588 * (1 - 0.588)}{1020}} \le p \le 0.588 + 2.575 * \sqrt{\frac{0.588 * (1 - 0.588)}{1020}}$$

$$or, \qquad 0.588 - 0.040 \le p \le 0.588 + 0.040$$

$$or, \qquad 0.548 \le p \le 0.628$$

Since both of the limits of the proportion is more than 50%, we can say the rescue process is effective.