3.1 Continuous Random Variables and Probability Density Functions:

A continuous random variable takes a "range of values", which may be finite or infinite in extent. Here are a few examples of ranges: $[\boldsymbol{0}, \boldsymbol{1}]$, $[\boldsymbol{0}, \infty)$, $(-\infty, \infty)$, $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition: A random variable $\boldsymbol{X}$ is continuous if there is a function $\boldsymbol{f(x)}$ such that for any $\boldsymbol{a \leq b}$ we have

$$\boldsymbol{P(a \leq X \leq b) = \int_a^b f(x)dx}$$

The function $\boldsymbol{f(x)}$ is called the "probability density function" (**pdf**).

The pdf always satisfies the following properties:

- **(i)** $\boldsymbol{f(x) \geq 0}$; for all $\boldsymbol{x}$ ($\boldsymbol{f(x)}$ is non-negative).
- **(ii)** $\boldsymbol{\int_{-\infty}^{\infty} f(x)dx = 1}$ (This is equivalent to: $\boldsymbol{P(-\infty \leq X \leq \infty) = 1}$).

Note: The pdf $\boldsymbol{f(x)}$ is ***not*** a probability. We have to integrate it to get probability. Since $\boldsymbol{f(x)}$ is not a probability, there is no restriction that $\boldsymbol{f(x)}$ be less than or equal to $\boldsymbol{1}$.

The cumulative distribution function (**cdf**) for random variable $\boldsymbol{X}$ is

$$\boldsymbol{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt\,;\ -\infty < x < \infty}$$

and has the following properties:

- **(i)** $\boldsymbol{\lim_{x \to -\infty} F(x) = 0}$
- **(ii)** $\boldsymbol{\lim_{x \to \infty} F(x) = 1}$
- **(iii)** If $\boldsymbol{x_1 < x_2}$, then $\boldsymbol{F(x_1) \leq F(x_2)}$; that is, $\boldsymbol{F}$ is non-decreasing,
- **(iv)** $\boldsymbol{P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a) = \int_a^b f(x)dx}$
- **(v)** $\boldsymbol{F'(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)dt = f(x)}$.

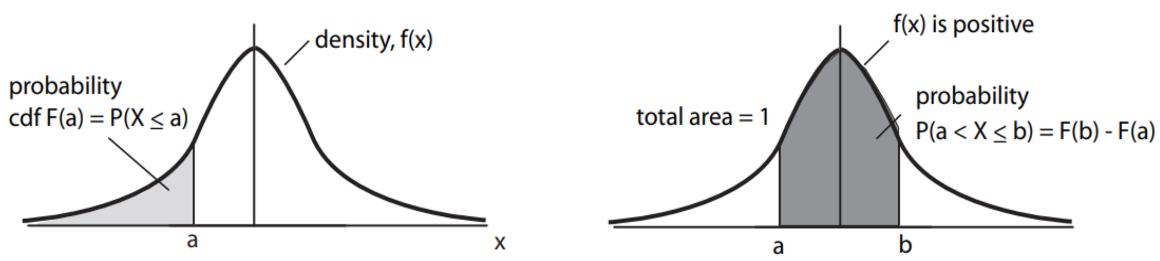

Figure: Continuous distribution

The ***expected value*** or ***mean*** of random variable $\boldsymbol{X}$ is given by

$$\boldsymbol{\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx}$$

The ***variance*** is

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$

with associated ***standard deviation***, $\sigma = \sqrt{\sigma^2}$.

The ***moment-generating function*** (**mgf**) is

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

for values of t for which this integral exists.

Expected value, assuming it ***exists***, of a function u of X is

$$E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

The **(100p)th** ***percentile*** is a value of X denoted π_p where

$$p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p)$$

and where π_p is also called the ***quantile of order*** p. The **25th**, **50th** and **75th** percentiles are also called ***first***, ***second*** and ***third quartiles***, respectively and are denoted by $q_1 = \pi_{0.25}$, $q_2 = \pi_{0.50}$ and $q_3 = \pi_{0.75}$ where also **50th** percentile is called the ***median*** and denoted by $m = q_2 = \pi_{0.50}$.

The ***mode*** is the value x where $f(x)$ is maximum.

Uniform distribution:

A continuous random variable X is said to have a **uniform distribution** on the interval $[a, b]$ if the pdf of X is

$$f(x) = \begin{cases} \dfrac{1}{b-a}; & \text{for } a \le x \le b \\ 0; & \text{otherwise} \end{cases}$$

The **cdf** of X is defined as

$$F(x) = \begin{cases} 0; & x < a \\ \displaystyle\int_a^x f(t)dt; & a \le x < b \\ 1; & x \ge b \end{cases}$$

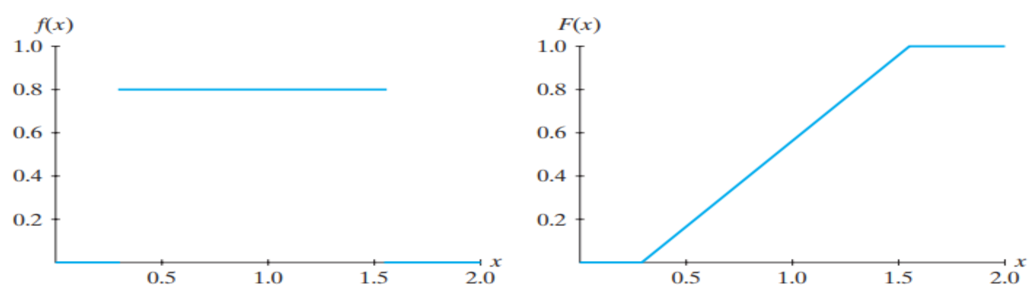

Figure: Uniform pdf and cdf

Examples: 3.1-1 to 3.1-6 (See yourself)

Exercises: 3.1-1 to 3.1-16 (Try yourself)