1.5: BAYES' THEOREM

Bayes' theorem is a pillar of both probability and statistics. The general form of Bayes' theorem is:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) \times P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)}$$

Each term in Bayes' theorem has a conventional name:

1. $P(A_i)$ are the **prior or marginal probabilities** corresponding of A_i. These are "**prior**" in the sense that they don't take into account any information about B.
2. $P(A_i|B)$ are the **conditional probabilities** corresponding of A_i, given B. These are also called the **posterior probabilities** because these are derived from or depends upon the specified value of B.
3. $P(B|A_i)$ are the **conditional probabilities** of B, given A_i.
4. $P(B)$ is the **prior or marginal probability** of B, and acts as a normalizing constant, which is also called "**the law of total probability**".

Problems:

1. At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio $3:2$ in favour of women. If a student is selected at random from among all those over **six** feet tall, what is the probability that the student is a **woman**?

Solution:

Let $M = \{\text{Student is Male}\}$, $F = \{\text{Student is Female}\}$ and $T = \{\text{Student is over 6 feet tall}\}$.

Here, $P(M) = \frac{2}{5}$, $P(F) = \frac{3}{5}$, $P(T|M) = \frac{4}{100}$ and $P(T|F) = \frac{1}{100}$.

We have to find $P(F|T)$. Using Bayes' theorem we have:

$$P(F|T) = \frac{P(T|F)P(F)}{P(T|F)P(F) + P(T|M)P(M)} = \frac{\frac{1}{100} \times \frac{3}{5}}{\frac{1}{100} \times \frac{3}{5} + \frac{4}{100} \times \frac{2}{5}} = \frac{3}{11}$$

2. A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from **(a)** machine A, **(b)** machine B and **(c)** machine C?

Solution:

Let $D = \{\text{bolt is defective}\}$, $A = \{\text{bolt is from machine } A\}$, $B = \{\text{bolt is from machine } B\}$ and $C = \{\text{bolt is from machine } C\}$.

Here, $P(A) = 0.25, P(B) = 0.35, P(C) = 0.4, P(D|A) = 0.05, P(D|B) = 0.04$ and

$$P(D|C) = 0.02$$

So, by the law of total probability: $P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$

$$= 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4 = 0.0345$$

Now, $P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{0.05 \times 0.25}{0.0345} = 0.362$, $P(B|D) = \frac{P(D|B)P(B)}{P(D)} = \frac{0.04 \times 0.35}{0.0345} = 0.406$

and $P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{0.02 \times 0.4}{0.0345} = 0.232$

3. An engineering company advertises a job in three newspapers, A, B and C. It is known that these papers attract undergraduate engineering readerships in the proportions $2:3:1$. The probabilities that an engineering undergraduate sees and replies to the job advertisement in these papers are 0.002, 0.001 and 0.005 respectively. Assume that the undergraduate sees only one job advertisement.
   - **a)** If the engineering company receives only one reply to it advertisements, calculate the probability that the applicant has seen the job advertised in place: **(i)** A, **(ii)** B and **(iii)** C.
   - **b)** If the company receives two replies, what is the probability that both applicants saw the job advertised in paper A?

Solution:

Let $A = \{\text{Person is a reader of paper } A\}$, $B = \{\text{Person is a reader of paper } B\}$, $C = \{\text{Person is a reader of paper } C\}$ and $R = \{\text{Reader applies for the job}\}$.

Here, $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}, P(C) = \frac{1}{6}, P(R|A) = 0.002, P(R|B) = 0.001$ and

$$P(R|C) = 0.005$$

So, by the law of total probability: $P(R) = P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C)$

$$= 0.002 \times \frac{1}{3} + 0.001 \times \frac{1}{2} + 0.005 \times \frac{1}{6} = 0.002$$

a) Now, $P(A|R) = \frac{P(R|A)P(A)}{P(R)} = \frac{0.002 \times \frac{1}{3}}{0.002} = \frac{1}{3}$

Similarly, $P(B|R) = \frac{1}{4}$ and $P(C|R) = \frac{5}{12}$.

b) Now, assuming that the replies and readerships are independent.

$P(\text{Both applicants read paper} A) = P(A|R) \times P(A|R) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

4. Consider a routine screening test for a disease. Suppose the frequency of the disease in the population (base rate) is 0.5%. The test is highly accurate with a 5% false positive rate and a 10% false negative rate. You take the test and it comes back positive. What is the probability that you have the disease?

Solution:

Let $D+=$ 'you have the disease', $D-=$ 'you do not have the disease, $T+=$ 'you tested positive' and $T-=$ 'you tested negative'.

Here, $P(D+) = 0.005$, $P(D-) = 0.995$, $P(\text{false positive}) = P(T+|D-) = 0.05$ and $P(\text{false negative}) = P(T-|D+) = 0.1$.

Now, $P(T-|D-) = 1 - P(T+|D-) = 0.95$ and $P(T+|D+) = 1 - P(T-|D+) = 0.9$

So, by the law of total probability: $P(T+) = P(T+|D-)P(D-) + P(T+|D+)P(D+)$

$$= 0.05 \times 0.995 + 0.9 \times 0.005 = 0.05425$$

Thus, $P(D+|T+) = \frac{P(T+|D+) \cdot P(D+)}{P(T+)} = \frac{0.9 \times 0.005}{0.05425} = 0.083$

Examples: 1.5-1 to 1.5-3 (See yourself)
Exercises: 1.5-1 to 1.5-4.& 1.5-9 to 1.5-11 (Try yourself)