1.2: Multiplication Principle: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example: If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

Sol: For the chair position, there are 22 total possibilities. For each of those 22 possibilities, there are 21 possibilities to elect the treasurer. Using the multiplication rule, we obtain $n_1 \times n_2 = 22 \times 21 = 462$ different ways.

Example: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Sol: Since, $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are $n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$ different ways to order the parts.

Permutation: A permutation is an arrangement of all or part of a set of objects.

Eg. Consider the three letters a, b, and c. The possible permutations are $abc, acb, bac, bca, cab,$ and cba. Thus, we see that there are 6 distinct arrangements. There are $n_1 = 3$ choices for the first position. No matter which letter is chosen, there are always $n_2 = 2$ choices for the second position. No matter which two letters are chosen for the first two positions, there is only $n_3 = 1$ choice for the last position, giving a total of $n_1 \times n_2 \times n_3 = (3)(2)(1) = 6$ permutations.

Definition: For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1) \cdots (2)(1), \text{ with special case } 0! = 1.$$

Theorem: The number of permutations of n distinct objects taken r at a time is $n_{P_r} = \frac{n!}{(n-r)!}$.

Eg. In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Sol: Since the awards are distinguishable, it is a permutation problem. The total number of sample points is $25_{P_3} = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800$.

Theorem: The number of permutations of n objects arranged in a circle is $(n-1)!$

Theorem: The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, …, n_k of a k-th kind is $\frac{n!}{n_1!\, n_2! \ldots n_k!}$.

Eg. How many different letter arrangements can be made from the letters in the word *STATISTICS*?

Sol: Here we have 10 total letters, with 2 letters (S, T) appearing 3 times each, letter I appearing twice, and letters A and C appearing once each. So, the different arrangement of the letters are $\frac{10!}{3!\,3!\,2!1!1!} = 50,400$.

Combinations: In many problems, we are interested in the number of ways of selecting $\boldsymbol{r}$ objects from $\boldsymbol{n}$ without regard to order. These selections are called **combinations**. A combination is actually a partition with two cells, the one cell containing the $\boldsymbol{r}$ objects selected and the other cell containing the $(\boldsymbol{n}-\boldsymbol{r})$ objects that are left. In combinations, you can select the items in any order.

Theorem: The number of combinations of n distinct objects taken r at a time is $\boldsymbol{n}_{\boldsymbol{C}_{\boldsymbol{r}}} = \frac{\boldsymbol{n}!}{\boldsymbol{r}!\,(\boldsymbol{n}-\boldsymbol{r})!}$.

Eg. A boy asks his mother to select $\mathbf{5}$ balls from his collection of $\mathbf{10}$ red and $\mathbf{5}$ blue balls. How many ways are there that his mother can select $\mathbf{3}$ red and $\mathbf{2}$ blue balls?

Sol: The number of ways of selecting $\mathbf{3}$ red balls from $\mathbf{10}$ is $\mathbf{10}_{\boldsymbol{C}_{\mathbf{3}}} = \frac{\mathbf{10}!}{\mathbf{3}!\,(\mathbf{10}-\mathbf{3})!} = \mathbf{120}$.

The number of ways of selecting $\mathbf{2}$ red balls from $\mathbf{5}$ is $\mathbf{5}_{\boldsymbol{C}_{\mathbf{2}}} = \frac{\mathbf{5}!}{\mathbf{2}!\,(\mathbf{5}-\mathbf{2})!} = \mathbf{10}$.

Using the multiplication rule with $\boldsymbol{n}_{\mathbf{1}} = \mathbf{120}$ and $\boldsymbol{n}_{\mathbf{2}} = \mathbf{10}$, we have $(\mathbf{120})(\mathbf{10}) = \mathbf{1200}$ ways.

Eg. A bag contains $\mathbf{10}$ white, $\mathbf{6}$ red, $\mathbf{4}$ black & $\mathbf{7}$ blue balls. $\mathbf{5}$ balls are drawn at random. What is the probability that $\mathbf{2}$ of them are red and one is black?

Sol: Total no. of balls $= \mathbf{10} + \mathbf{6} + \mathbf{4} + \mathbf{7} = \mathbf{27}$

$\mathbf{5}$ balls can be drawn from these $\mathbf{27}$ balls $= \mathbf{27}_{\boldsymbol{C}_{\mathbf{5}}}$ways $= \mathbf{80730}$ ways,

$\mathbf{2}$ red balls can be drawn from $\mathbf{6}$ red balls $= \mathbf{6}_{\boldsymbol{C}_{\mathbf{2}}}$ ways $= \mathbf{15}$ ways

and, $\mathbf{1}$ black balls can be drawn from $\mathbf{4}$ black balls $= \mathbf{4}_{\boldsymbol{C}_{\mathbf{1}}}$ways $= \mathbf{4}$ ways

$\therefore$ No. of favourable cases $= \mathbf{15} \times \mathbf{4} = \mathbf{60}$

So, probability $= \frac{\textbf{Number of favourable cases}}{\textbf{Total number of exhaustive cases}} = \frac{\mathbf{60}}{\mathbf{80730}} = \frac{\mathbf{6}}{\mathbf{8073}}$

Exercise:

1. Find the probability that a hand at bridge will consist of $\mathbf{3}$ spades, $\mathbf{5}$ hearts, $\mathbf{2}$ diamonds & $\mathbf{3}$ clubs? **Ans:** $\frac{\mathbf{13}_{\boldsymbol{C}_{\mathbf{3}}}\times\mathbf{13}_{\boldsymbol{C}_{\mathbf{5}}}\times\mathbf{13}_{\boldsymbol{C}_{\mathbf{2}}}\times\mathbf{13}_{\boldsymbol{C}_{\mathbf{3}}}}{\mathbf{52}_{\boldsymbol{C}_{\mathbf{13}}}}$
2. In a committee of $\mathbf{4}$ persons from a group of $\mathbf{10}$ persons, what is the probability that a particular person is on the committee? **Ans:** $\frac{\mathbf{9}_{\boldsymbol{C}_{\mathbf{3}}}}{\mathbf{10}_{\boldsymbol{C}_{\mathbf{4}}}}$, For not committee **Ans:** $\frac{\mathbf{9}_{\boldsymbol{C}_{\mathbf{4}}}}{\mathbf{10}_{\boldsymbol{C}_{\mathbf{4}}}}$

1.3: CONDITIONAL PROBABILITY:

The conditional probability of an event $\boldsymbol{A}$, given that event $\boldsymbol{B}$ has occurred, is defined by

$$\boldsymbol{P}(\boldsymbol{A}|\boldsymbol{B}) = \frac{\boldsymbol{P}(\boldsymbol{A}\cap\boldsymbol{B})}{\boldsymbol{P}(\boldsymbol{B})};\quad \text{provided that } \boldsymbol{P}(\boldsymbol{B}) > \mathbf{0}$$

The probability that two events, $\boldsymbol{A}$ and $\boldsymbol{B}$, both occur is given by the multiplication rule,

$$\boldsymbol{P}(\boldsymbol{A}\cap\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{B})\boldsymbol{P}(\boldsymbol{A}|\boldsymbol{B});\quad \text{provided that } \boldsymbol{P}(\boldsymbol{B}) > \mathbf{0}$$

Or,

$$\boldsymbol{P}(\boldsymbol{A}\cap\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A})\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A});\quad \text{provided that } \boldsymbol{P}(\boldsymbol{A}) > \mathbf{0}$$

Sometimes, after considering the nature of the random experiment, one can make reasonable assumptions so that it is easier to assign $\boldsymbol{P}(\boldsymbol{B})$ and $\boldsymbol{P}(\boldsymbol{A}|\boldsymbol{B})$ rather than $\boldsymbol{P}(\boldsymbol{A}\cap\boldsymbol{B})$. Then $\boldsymbol{P}(\boldsymbol{A}\cap\boldsymbol{B})$ can be computed with these assignments.

Problems:

1. A bag contains **3** red & **4** white balls. Two draws are made **without replacement**. What is the probability that both the balls are **red**?

Solution: Here, total no. of balls $= \mathbf{3} + \mathbf{4} = \mathbf{7}$

$\boldsymbol{P}(\text{drawing a red ball in the first draw}) = \boldsymbol{P}(\boldsymbol{A}) = \frac{\mathbf{3}}{\mathbf{7}}$

$\boldsymbol{P}(\text{drawing a red ball in the second draw given that first ball drawn is red}) = \boldsymbol{P}(\boldsymbol{B}\backslash\boldsymbol{A}) = \frac{\mathbf{2}}{\mathbf{6}} = \frac{\mathbf{1}}{\mathbf{3}}$

So, the probability that both the balls are red, $\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A})\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}) = \frac{\mathbf{3}}{\mathbf{7}} \times \frac{\mathbf{1}}{\mathbf{3}} = \frac{\mathbf{1}}{\mathbf{7}}$

2. A bag contains **3** red & **4** white balls. Two draws are made **without replacement**. What is the probability that both the balls are **different color**?

Solution: Here, total no. of balls $= \mathbf{3} + \mathbf{4} = \mathbf{7}$

$\boldsymbol{P}(\text{drawing a red ball in the first draw}) = \boldsymbol{P}(\boldsymbol{A}) = \frac{\mathbf{3}}{\mathbf{7}}$

$\boldsymbol{P}(\text{drawing a white ball in the second draw given that first ball drawn is red}) = \boldsymbol{P}(\boldsymbol{B}\backslash\boldsymbol{A}) = \frac{\mathbf{4}}{\mathbf{6}} = \frac{\mathbf{2}}{\mathbf{3}}$

So, the probability that first ball is red and second ball is white,

$$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A})\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}) = \frac{\mathbf{3}}{\mathbf{7}} \times \frac{\mathbf{2}}{\mathbf{3}} = \frac{\mathbf{2}}{\mathbf{7}}$$

Similarly,

$\boldsymbol{P}(\text{drawing a white ball in the first draw}) = \boldsymbol{P}(\boldsymbol{C}) = \frac{\mathbf{4}}{\mathbf{7}}$

$\boldsymbol{P}(\text{drawing a red ball in the second draw given that first ball drawn is white}) = \boldsymbol{P}(\boldsymbol{D}\backslash\boldsymbol{C}) = \frac{\mathbf{3}}{\mathbf{6}} = \frac{\mathbf{1}}{\mathbf{2}}$

So, the probability that first ball is white and second ball is red,

$$\boldsymbol{P}(\boldsymbol{C} \cap \boldsymbol{D}) = \boldsymbol{P}(\boldsymbol{C})\boldsymbol{P}(\boldsymbol{D}|\boldsymbol{C}) = \frac{\mathbf{4}}{\mathbf{7}} \times \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{2}}{\mathbf{7}}$$

Thus, the probability that both the balls are different color, $\frac{\mathbf{2}}{\mathbf{7}} + \frac{\mathbf{2}}{\mathbf{7}} = \frac{\mathbf{4}}{\mathbf{7}}$.

3. Find the probability of drawing a queen and a king from a pack of cards in two consecutive draws, the cards drawn **not being replaced**.

Solution: $\boldsymbol{P}(\text{drawing a queen card}) = \boldsymbol{P}(\boldsymbol{A}) = \frac{\mathbf{4}}{\mathbf{52}}$

$\boldsymbol{P}(\text{drawing a king after a queen has been drawn}) = \boldsymbol{P}(\boldsymbol{B}\backslash\boldsymbol{A}) = \frac{\mathbf{4}}{\mathbf{51}}$

So, the probability of drawing a queen and a king from a pack of cards in two consecutive draws,

$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A})\boldsymbol{P}(\boldsymbol{B}|\boldsymbol{A}) = \frac{\mathbf{4}}{\mathbf{52}} \times \frac{\mathbf{4}}{\mathbf{51}} = \frac{\mathbf{4}}{\mathbf{663}}$

4. In a box, there are $\mathbf{100}$ resistors having resistance and tolerance as shown in the following table. Let a resistor be selected from the box and assume each resistor has the same likelihood of being chosen. Define three events $\boldsymbol{A}$ as draw a $\mathbf{47}\boldsymbol{\Omega}$ resistor, $\boldsymbol{B}$ as draw a resistor with $\mathbf{5\%}$ tolerance and $\boldsymbol{C}$ as draw a $\mathbf{100}\boldsymbol{\Omega}$ resistor. Find $\boldsymbol{P}(\boldsymbol{A}\backslash\boldsymbol{B})$, $\boldsymbol{P}(\boldsymbol{A}\backslash\boldsymbol{C})$ and $\boldsymbol{P}(\boldsymbol{B}\backslash\boldsymbol{C})$.

Resistance Ω	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Solution: $\boldsymbol{P(A) = P(47\Omega) = \frac{44}{100}, P(B) = P(5\%) = \frac{62}{100}}$ and $\boldsymbol{P(C) = P(100\Omega) = \frac{32}{100}}$.

The joint probabilities are,

$\boldsymbol{P(A \cap B) = P(47\Omega \cap 5\%) = \frac{28}{100}, P(A \cap C) = P(47\Omega \cap 100\Omega) = 0}$ and

$\boldsymbol{P(B \cap C) = P(5\% \cap 100\Omega) = \frac{24}{100}}$.

$\therefore\ \boldsymbol{P(A\backslash B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{28}{100}}{\frac{62}{100}} = \frac{28}{62} = \frac{14}{31}, P(A\backslash C) = \frac{P(A\cap C)}{P(C)} = \frac{0}{\frac{32}{100}} = 0}$ and,

$\boldsymbol{P(B\backslash C) = \frac{P(B\cap C)}{P(C)} = \frac{\frac{24}{100}}{\frac{32}{100}} = \frac{24}{32}} = \frac{3}{4}$

5. The Hindu newspaper publishes three columns entitled politics **(A)**, books **(B)**, cinema **(C)**. Reading habits of a randomly selected reader with respect to three columns are,

Read Regularly	A	B	C	A ∩ B	A ∩ C	B ∩ C	A ∩ B ∩ C
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05

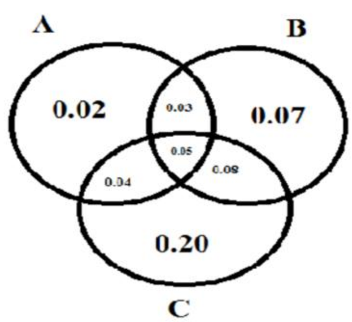
Find $\boldsymbol{P(A\backslash B), P\ (A\backslash B \cup C), P(A\backslash}\text{reads at least one}\boldsymbol{), P(A \cup B\backslash C)}$.

Solution: Here, $\boldsymbol{P(A\backslash B) = \frac{P(A\cap B)}{P(B)} = \frac{0.08}{0.23} = \frac{8}{23}}$

$$\boldsymbol{P\ (A\backslash B \cup C) = \frac{P(A\cap(B\cup C))}{P(B\cup C)} = \frac{0.04+0.05+0.03}{0.04+0.05+0.03+0.08+0.07+0.20} = \frac{11}{47}}$$

$$\boldsymbol{P(A\backslash}\text{reads at least one}\boldsymbol{) = P\ (A\backslash(A \cup B \cup C)) = \frac{P(A\cap(A\cup B\cup C))}{P(A\cup B\cup C)}}$$

$$\boldsymbol{= \frac{P(A)}{P(A\cup B\cup C)} = \frac{0.14}{0.49} = \frac{14}{49}}$$



and, $\boldsymbol{P(A \cup B\backslash C) = \frac{P((A\cup B)\cap C)}{P(C)} = \frac{0.04+0.05+0.08}{0.37} = \frac{17}{37}}$

Formula: $\boldsymbol{P(A'|B) = 1 - P(A|B)}$

Examples: 1.3-1 to 1.3-12 (See yourself)

The multiplication rule for the events $\boldsymbol{A}$, $\boldsymbol{B}$ & $\boldsymbol{C}$ is

$$\boldsymbol{P(A \cap B \cap C) = P[(A \cap B) \cap C] = P(A \cap B)P(C|A \cap B)}$$

Since, $\boldsymbol{P(A \cap B) = P(A)P(B|A)}$, so $\boldsymbol{P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)}$

Example: 1.3-10: A grade school boy has ***five*** blue and ***four*** white marbles in his left pocket and ***four*** blue and ***five*** white marbles in his right pocket. If he transfers ***one*** marble at random from his **left** to his **right** pocket, what is the probability of his then drawing a blue marble from his **right** pocket?

Solution: Let **BL**, **BR**, and **WL** denote drawing blue from left pocket, blue from right pocket, and white from left pocket, respectively. Then,

$$\boldsymbol{P(BR) = P(BL \cap BR) + P(WL \cap BR) = P(BL)P(BR \mid BL) + P(WL)P(BR \mid WL)}$$

$$\boldsymbol{= \frac{5}{9} \times \frac{5}{10} + \frac{4}{9} \times \frac{4}{10} = \frac{41}{90}}$$

Exercises: 1.3-1 to 1.3-4 & 1.3-9. (Try yourself)