MATH 135 Fall 2020: Assignment 1
Due at 11:55 PM EDT on Friday, September 18th, 2020
Covers the contents of Lessons from 1.1 to 1.5

Q01. Determine if the following statements are *true* or *false*. No justification is required.

(a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x^2 \leq y^4$

(b) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x^2 \leq y^4$

(c) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x^2 \leq y^4$

(d) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x^2 \leq y^4$

(e) $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x^2 \leq y^4$

(f) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x^2 \leq y^4$

Q02. An integer $p > 1$ is called *prime* if its only positive divisors are 1 and p. For example, 2, 3 and 5 are prime, but 4 is not prime because, apart from being divisible by 1 and 4, it is also divisible by 2.

Consider the following statement:

For any prime p the equation $x^2 - py^2 = 1$ has a solution in positive integers x and y.

(a) Let $\mathbb{P} = \{2, 3, 5, \ldots\}$ denote the set of all prime numbers. Express the above statement symbolically, without using words.

(b) State the negation of the above mathematical statement without using words or the $\neg$ symbol.

(c) The above mathematical statement was proved in XVIII century by the French mathematician Joseph Louis Lagrange. Since this statement has been proved, is it *true* or *false*?

Q03. Consider the following universally quantified statement:

For every integer x such that $x > 1$, it is the case that $2x^2 + 7x + 4 > 25$.

(a) Below you will find three proofs of this statement. One of them is erroneous, another one is correct and well-written, and the remaining one, though correct, needs to be re-written. Which one is which?

- **Proof A.** $\forall x \in \mathbb{Z}$ with $x > 1$ we have that the smallest value of x is 2. When $x = 2$ $P(2)$ is true, thus the statement is true.

- **Proof B.** Let x be an integer such that $x > 1$. Then $x \geq 2$, so $2x^2 \geq 8$ and $7x \geq 14$. Therefore,
$$2x^2 + 7x + 4 \geq 8 + 14 + 4 = 26 > 25.$$
- **Proof C.** $2x^2 + 7x + 4 > 25$ is the same as $2x^2 + 7x - 21 > 0$. Since $x \geq 2$,
$$2x^2 + 7x - 21 \geq 1 > 0.$$

(b) Write a reflection about the proofs given in Part (a). Make sure to include the following details:

- For the erroneous proof, what do you think the problems are? Think not only about the validity of the arguments, but also about the presentation.
- For the proof that is correct, but which requires reformulation, why do you think it *is* correct? What sort of reformulations would you introduce? What kind of details would you add?
- For the proof that is correct and well-written, what do you think makes it a good proof? Is it a particular structure? A particular flow of the argument? Was it easy for you to follow the proof?

Q04. Choose the appropriate domain, S, and the appropriate open sentence, $P(a, b)$, to make each statement true. You can use each domain and each open sentence exactly once. **Note that there is only one solution!**

Domain, S	Open sentence, $P(a, b)$
$\left\{-\frac{1}{3}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right\}$	$a \leq b$
$\mathbb{N}$	$\frac{a}{b^2+1} \in S$
$\mathbb{Q}$	$a = 7b^3 + 10$
$\mathbb{R}$	$\sin\left(\frac{\pi a}{2}\right) + \cos\left(\frac{\pi b}{2}\right) = \frac{\sqrt{2}+\sqrt{3}}{2}$

(a) $\forall a \in S, \forall b \in S, P(a, b)$, where $S =$ ____________ and $P(a, b) =$ ______________________________.

(b) $\forall a \in S, \exists b \in S, P(a, b)$, where $S =$ ____________ and $P(a, b) =$ ______________________________.

(c) $\exists a \in S, \forall b \in S, P(a, b)$, where $S =$ ____________ and $P(a, b) =$ ______________________________.

(d) $\exists a \in S, \exists b \in S, P(a, b)$, where $S =$ ____________ and $P(a, b) =$ ______________________________.

Q05. Let $P(x, n)$ denote the open sentence "$\exists y \in \mathbb{Z}, x^2 - xy + y^2 = n$".

(a) Determine the truth values of mathematical statements $P(3, 19)$ and $P(-1, 11)$.

(b) Is the sentence $\exists x \in \mathbb{Z}, P(x, n)$ a mathematical statement or an open sentence? If it is a mathematical statement, determine its truth value, justifying your answer. If it is an open sentence, explain what variable(s) it depends on.

(c) Is the sentence $\forall n \in \mathbb{N}, \exists x \in \mathbb{Z}, P(x, n)$ a mathematical statement or an open sentence? If it is a mathematical statement, determine its truth value, justifying your answer. If it is an open sentence, explain what variable(s) it depends on.

Hint: Note that $x^2 - xy + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2$.