IOAA Notes

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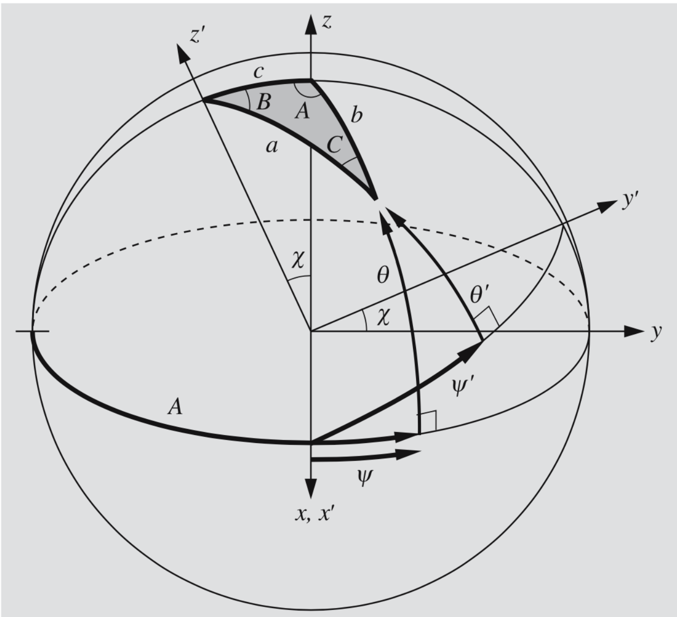
August 2022

1 Spherical Trig

Readings: Roy and Clarke Chapter 9, Kartunnen Chapter 2
(the carroll ostlie chapter is honestly not very good)

1.1 Triangle Formulas



rea:

$$E = A + B + C - \pi$$
$$\text{Area} = E \cdot r^2$$

Spherical law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Spherical law of cosines:

$$\cos a = \cos A \sin b \sin c + \cos b \cos c$$

$$\cos A = \cos a \sin B \sin C - \cos B \cos C$$

Spherical law of cosines (alternate):

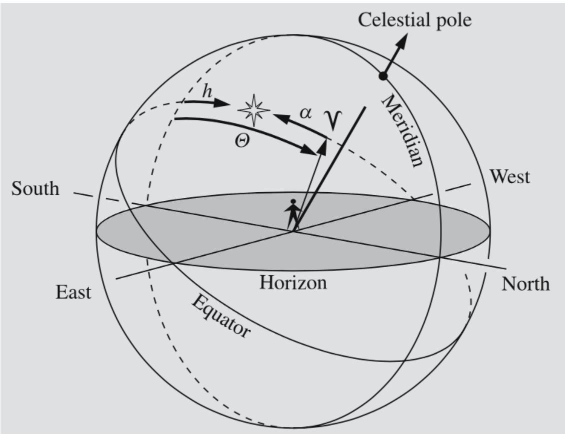
$$\cos B \sin a = -\cos A \sin b \cos c + \cos b \sin c$$

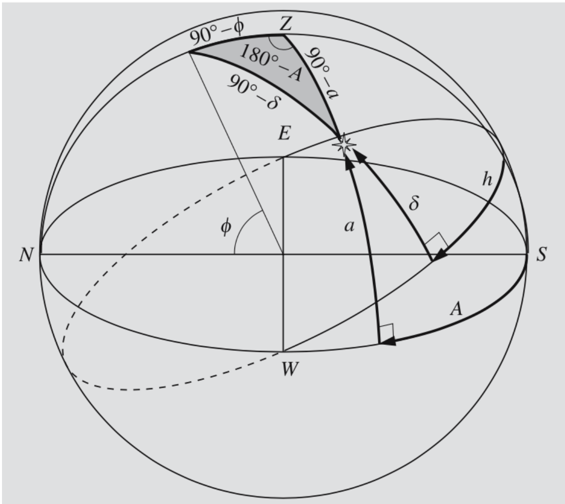
Four parts formula:

$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

$$\cos A \cos c = \sin A \cot B - \sin c \cot b$$

1.2 Coordinate Systems





Equatorial: (Right Ascension, Declination) = (α, δ)
Horizontal: (Azimuth, Altitude) = (A, a)
Ecliptic: (Longitude, Latitude) = (λ, β)
Local Sidereal Time = $\Theta = h + \alpha$. (AKA hour angle of Υ)

- Azimuth A measured clockwise from either the northernmost or southernmost point on observor's horizon (usually whichever is more convenient, but stick to one notation to be consistent).
- RA α measured counterclockwise from the vernal equinox (Υ)
- Longitude λ measured counterclockwise from the vernal equinox (Υ)
- Hour angle h measured clockwise from the southern meridian. Measures the time that has elapsed after an object has culminated (in the northern hemisphere).

1.3 Timekeeping

Solar vs. Sidereal: T = Local Solar Time (acronym LST not to be confused with Θ = Local Sidereal Time). `12:00` local solar time corresponds to the time when the sun culminates for a particular location. On March 20th, the sun aligns with the vernal equinox and thus culminates at the same time as the vernal equinox, meaning that $T =$ `12:00` and $\Theta =$ `0:00` align on this date.

Every solar day takes 24 hours; the time between consecutive culminations of the sun takes 24 hours. Every sidereal day takes 23 hours 56 minutes 4 seconds; for any background star, the time between consecutive culminations takes 23 hours 56 minutes 4 seconds. The exact conversion is

$$\begin{aligned} 24 \text{ sidereal hours} &= 24 \cdot \left(\frac{365.25}{366.25}\right) \text{ solar hours} \\ 24 \text{ solar hours} &= 24 \cdot \left(\frac{366.25}{365.25}\right) \text{ sidereal hours.} \end{aligned}$$

To derive this, consider that sidereal time accounts for one additional rotation of the entire earth's orbit around the sun per year, so it necessary runs faster, i.e., a sidereal hour is less than a solar hour, and any time period measured in solar time will be numerically smaller than any period measured in sidereal time.

To compute solar to sidereal times at specific dates, approximate

$$\Theta = \begin{cases} T + 12 \text{ hr} & \text{Vernal Equinox} \\ T + 18 \text{ hr} & \text{Summer Solstice} \\ T + 0 \text{ hr} & \text{Autumnal Equinox} \\ T + 12 \text{ hr} & \text{Winter Solstice.} \end{cases}$$

Then, for example, if you want to compute sidereal time on March 31st, this corresponds to a period of 10 solar days that passed, which is equal to $10 \cdot (366.25/365.25)$ sidereal days that passed. The difference is about 40 minutes, as we expect, since a solar day is about 4 minutes longer than a sidereal day (in solar time). So, on March 31st, we have $\Theta \approx T + 12 \text{ hr} + 40 \text{ min}$.

Zero Meridian: Greenwich, London is the 0° meridian. Greenwich Mean Time (GMT), also called Universal Time (UT) is the local solar time in Greenwich and is used as the "universal" time. Greenwich Mean Sidereal Time (GMST) is the hour angle of Υ ignoring the effects of nutational variation (since it is measured against the background stars). Greenwich Apparent Sidireal Time (GAST) is the actual position of Υ. Equation of the Equinoxes:

$$\mathcal{E}_\Upsilon = \text{GAST} - \text{GMST}.$$

Convert between Greenwich and any other location with longitude λ with:

$$GHA\star = HA \star + \lambda,$$

where west λ is positive and east λ is negative, so that "Longitude east, Greenwich least; Longitude west, Greenwich best".

Equation of Time ($\mathcal{E}$): The mean sun (MS) is a fictitious sun that moves along the celestial equator at the rate of $(360°)/365.25$ days. The true sun ($\odot$) moves along the ecliptic at the same rate. We introduce MS for timekeeping, since the true sun does not technically have a constant angular velocity (eccentricity of orbit) and also suffers from the 23.5° obliquity. Now, equation of Time ($\mathcal{E}$):

$$\mathcal{E} = \text{RAMS} - \text{RA}\odot$$

(Note that this is backwards from the Equation of the Equinoxes). Some other notes on the equation of time that are important to remember (from that one Singapore problem):

1. The equation of time can also be written as $\mathcal{E} = \text{HA}\odot - \text{HAMS}$.
2. Recall that timekeeping is based on the mean sun. So, whenever the equation of time is not equal to zero, $HA\odot = 0$ does not necessarily correspond to local solar time $LST = 12^h$.
3. Instead, use this: $LST = HAMS \pm 12^h = HA\odot - \mathcal{E} \pm 12^h$. In particular, this is one reason why the rising time will become earlier between the vernal and summer solstice, since the equation of time becomes more and more positive. At the summer solstice, the equation of time is equal to zero, so the only reason why the sun still rises earlier at this stage is due to the fact that the sun's declination is 23.5°.

Analemma: plot with equation of time on the horizontal axis and declination on the vertical axis. See this link for a nice visual of the projection of an Analemma: https://medium.com/sentinel-hub/the-shadow-of-a-celestial-dance-90968f1f42fb

Time Zones:

zone	longitude	zone	longitude
±0 (UTC±0)	0^h30^mW—0^h30^mE		
+1 (UTC-1)	1^h30^mW—0^h30^mW	-1 (UTC+1)	1^h30^mE—0^h30^mE
+2 (UTC-2)	2^h30^mW—1^h30^mW	-2 (UTC+2)	2^h30^mE—1^h30^mE
⋮	⋮	⋮	⋮
+12 (UTC-12)	12^hW—11^h30^mW	-12 (UTC+12)	12^hE—11^h30^mE

The international date line is defined by 12^h E/W. If you're travelling from east to west (in the direction from China to America) across the international date line, then you're moving from Zone −12 (UTC+12) to Zone +12 (UTC-12). To change times from Zone −12 to UTC, you subtract 12 hours; to change times from UTC to Zone +12, you subtract 12 hours again. In total, you lose 24 hours. If you're travelling from west to east, you gain 24 hours.

2 Cosmology

Readings: Carroll Ostlie Chapter 29

Cosmological Principle: the assumption that the expanding universe is both isotropic and homogenous.

Hubble's Law:

$$v = H_0 r.$$

v is recessional velocity. H_0 is Hubble's constant. r is distance. Works best for $r > 10$ Mpc, since smaller r are obscured by *peculiar velocities* (i.e. "movement" that is not caused by the expansion of spacetime, such as gravity between interacting galaxies).

Co-moving coordinate and scale factor:

$$r(t) = R(t) \cdot \varpi$$

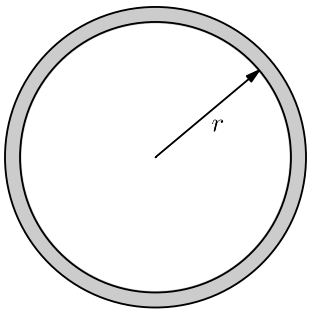
$r(t)$ is the coordinate distance from the observer, $R(t)$ is the scale factor, and ϖ is the *co-moving coordinate.*

From cosmological time dilation and cosmological redshift, we have

$$R(z) = \frac{1}{1+z} = \frac{\Delta t_{emit}}{\Delta t_{obs}} = \frac{\lambda_{emit}}{\lambda_{obs}}.$$

Deriving Friedmann:

(this only works for a one-component universe of pressureless dust, so this technically isn't a fully complete proof)



Consider an expanding ring of dust with mass m radially expanding with velocity v (by the cosmological principle). Let M_r be the total mass inside of the shell. Now, by conservation of energy,

$$\frac{1}{2}mv^2(t) - G\frac{M_r m}{r(t)} = -\frac{1}{2}mkc^2\varpi^2. \tag{1}$$

Also,

$$H(t) = \frac{v(t)}{r(t)} = \frac{v(t)}{R(t)\varpi} = \frac{d/dt(R(t))\varpi}{R(t)\varpi} = \frac{\dot{R}}{R}.$$

Plug stuff into Eq 1 to eventually obtain:

$$R^2\left(H^2 - \frac{8}{3}\pi G\rho\right) = -kc^2.$$

This is equivalent to

$$\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho R^2 = -kc^2,$$

which is also equivalent to

$$\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\frac{\rho_{m,0}}{R} + \frac{\rho_{rel,0}}{R^2} + \rho_{\Lambda,0}R^2\right) = -kc^2.$$

Density:

$\rho = \rho_m + \rho_{rel} + \rho_\Lambda$ accounts for matter (dark + baryonic = protons, neutrons), relativistic particles (photons and neutrinos), and dark energy. Relativistic density and dark energy density are taken as equivalent mass densities; that is, we let

$$\rho_{rel} = \frac{u_{rel}}{c^2} = \frac{g^* a T^4}{2c^2} = \frac{2g^*\sigma T^4}{c^3},$$

where u_{rel} is the energy density of relativistic particles, both photons and neutrinos. This differs from the normal energy density of blackbody radiation ($u = aT^4$, which only accounts for photons) by a factor of $g^*/2 \approx 1.68$. g^* itself is the "effective number of degrees of freedom for relativistic particles". We also let

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G},$$

so that the Friedmann equation looks consistent $(8/3\pi G(\rho_m + \rho_{rel}) + 1/3\Lambda c^2)$ when fully expanded.

Other important equations:

Fluid equation (for a pressureless universe, $P = 0$ confirms M_r constant in our simplified dust model):

$$\frac{d}{dt}(R^3\rho) = -\frac{P}{c^2}\frac{d(R^3)}{dt}.$$

Acceleration equation:

$$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi G\left(\rho + \frac{3P}{c^2}\right)R.$$

Equation of state:

$$P = wu = w\rho c^2,$$

which along with the fluid equation implies

$$R^{3(1+w)}\rho_x = \text{constant} = \rho_{0,x},$$

where $w = 0$ when x is matter, $w = 1/3$ when x is radiation, and $w = -1$ when x is dark energy. (for example, we know $P_{\text{rad}} = u_{\text{rad}}/3$).

We also have

$$RT = T_0,$$

which comes from Wien's law or from the relationship above for $x =$ radiation.

Critical Density:
When $k = 0$ in the friedmann equation:

$$\rho_c(t) = \frac{3H(t)^2}{8\pi G}.$$

Density Paramters:

$$\Omega_x(t) = \frac{\rho_x(t)}{\rho_c(t)} = \frac{8\pi G \rho_x(t)}{3H(t)^2}.$$

Try to remember approximately what the current density parameter values are:

$$\begin{cases} \Omega_{m,0} = \dfrac{8\pi G \rho_{m,0}}{3H_0^2} \approx 0.27 \\ \Omega_{rel,0} = \dfrac{16\pi G g^* \sigma T_0^4}{3H_0^2 c^3} \approx 8.24 \cdot 10^{-5} \\ \Omega_{\Lambda,0} = \dfrac{\Lambda c^2}{3H_0^2} \approx 0.73. \end{cases}$$

(the current universe is dominated by dark energy, and relativistic particles are almost negligible so that $P/c^2 << \rho$).

Some very useful reformulations of the friedmann equation:

$$H^2(1-\Omega)R^2 = -kc^2 = H_0^2(1-\Omega_0),$$

where $\Omega = \Omega_m + \Omega_{rel} + \Omega_\Lambda$. Also equivalent to

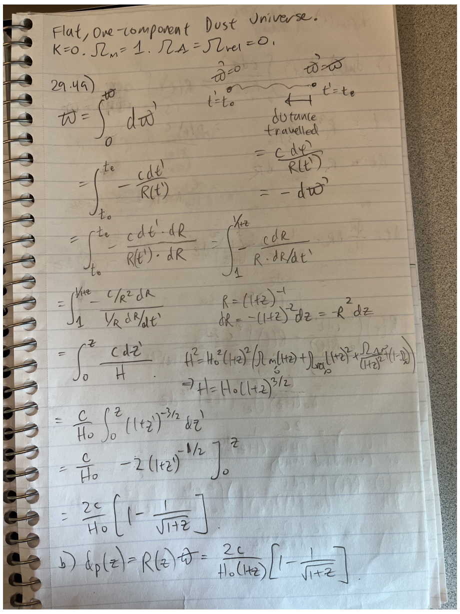
$$H = H_0(1+z)\left[(1+z)\Omega_{m,0} + (1+z)^2\Omega_{rel,0} + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right].$$

Distances:

- co-moving distance (ϖ): equal to current proper distance $d_{p,0}$. this definition is somewhat tautological but just accept it for now.
- proper distance (d_p): distance between two events that occur simultaneously. In a flat universe, the proper distance is just equal to the coordinate distance, i.e., $d_p = \varpi/(1+z)$.
- horizon distance (d_h): proper distance to the farthest observable point in the universe (the particle horizon).
- luminosity distance (d_L): measured flux $F = L/(4\pi d_L^2) = L/(4\pi\varpi^2(1+z)^2)$, so $d_L = \varpi(1+z)$.

- angular distance (d_A): $d_A = D/\theta = \varpi/(1+z)$.

Example calculation of ϖ, d_p, and d_h for a flat, one-component universe of pressureless dust:



Flat, One-component Dust Universe.

$K=0.\ \Omega_M = 1.\ \Omega_\Lambda = \Omega_{rel} = 0.$

29.49)

$\varpi' = 0 \quad t' = t_0 \qquad \varpi' = \varpi \quad t' = t_e$

distance travelled $= \frac{c\, d\varphi'}{R(t')} = -d\varpi'$

$$\varpi = \int_0^{\varpi} d\varpi'$$

$$= \int_{t_0}^{t_e} -\frac{c\,dt'}{R(t')}$$

$$= \int_{t_0}^{t_e} -\frac{c\,dt' \cdot dR}{R(t') \cdot dR} = \int_1^{1/1+z} -\frac{c\,dR}{R \cdot dR/dt'}$$

$$= \int_1^{1/1+z} -\frac{c/R^2\,dR}{1/R\ dR/dt'} \qquad R = (1+z)^{-1},\quad dR = -(1+z)^{-2} dz = -R^2 dz$$

$$= \int_0^z \frac{c\,dz'}{H}. \qquad H^2 = H_0^2 (1+z)^2 \left(\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + (1-\Omega_0)\right)$$

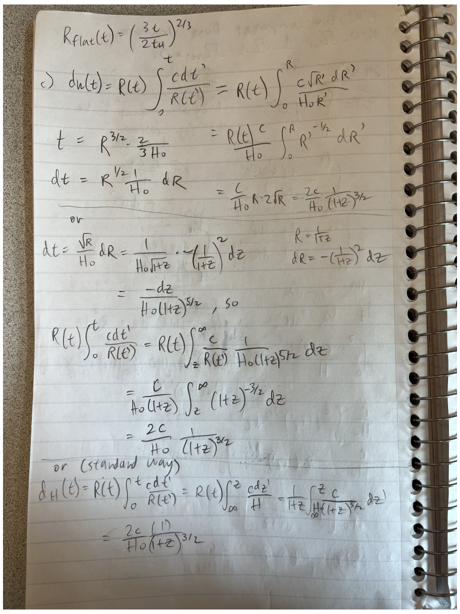
$$\Rightarrow H = H_0 (1+z)^{3/2}$$

$$= \frac{c}{H_0} \int_0^z (1+z')^{-3/2} dz'$$

$$= \frac{c}{H_0} \left. -2(1+z')^{-1/2} \right]_0^z$$

$$= \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right].$$

b) $d_p(z) = R(z)\varpi = \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right].$



$$R_{flat}(t) = \left(\frac{3t}{2t_H}\right)^{2/3}$$

c) $d_h(t) = R(t) \int_0^t \frac{c\,dt'}{R(t')} = R(t) \int_0^R \frac{c\sqrt{R'}\,dR'}{H_0 R'}$

$$t = R^{3/2} \cdot \frac{2}{3H_0} \qquad = \frac{R(t)\,c}{H_0} \int_0^R R'^{-1/2}\, dR'$$

$$dt = R^{1/2} \frac{1}{H_0} dR \qquad = \frac{c}{H_0} R \cdot 2\sqrt{R} = \frac{2c}{H_0} \frac{1}{(1+z)^{3/2}}$$

or

$$dt = \frac{\sqrt{R}}{H_0} dR = \frac{1}{H_0\sqrt{1+z}} \cdot -\left(\frac{1}{1+z}\right)^2 dz \qquad R = \frac{1}{1+z},\quad dR = -\left(\frac{1}{1+z}\right)^2 dz$$

$$= \frac{-dz}{H_0 (1+z)^{5/2}}, \text{ so}$$

$$R(t) \int_0^t \frac{c\,dt'}{R(t')} = R(t) \int_z^\infty \frac{c}{R(t')} \frac{1}{H_0 (1+z)^{5/2}} dz$$

$$= \frac{c}{H_0 (1+z)} \int_z^\infty (1+z)^{-3/2} dz$$

$$= \frac{2c}{H_0} \frac{1}{(1+z)^{3/2}}$$

or (standard way)

$$d_H(t) = R(t) \int_0^t \frac{c\,dt'}{R(t')} = R(t) \int_\infty^z \frac{c\,dz'}{H} = \frac{1}{1+z} \int_\infty^z \frac{c}{H_0 (1+z')^{3/2}} dz'$$

$$= \frac{2c}{H_0} \frac{(1)}{(1+z)^{3/2}}$$