3. Model development

3.1. Model assumptions

Prior to developing a mathematical model, we made the following assumptions:

1. Unloading/loading time at each node is negligible.
2. The driver has no break.
3. There is no possibility of the vehicle breaking down during the pickup and delivery process.
4. The client's appointment time and pickup time are known in advance.
5. Client inconvenience is a linear function of the total excess riding time, early delivery time, late delivery time, and late pickup time.
6. Travel time and distance between two nodes are symmetric.

3.2. Model formulation

The model has a multiple objective framework because its soft time window requirements can only be satisfied at the expense of a longer route. Thus, it necessitates a bi-objective model that is formulated as follows:

k index for vehicles; $k \in K$, $K = \{1, \ldots, m\}$
i index for requests; $i \in R$, $R = \{1, \ldots, n\}$
Request i is identified by four nodes belonging to four different sets as follows:

i pickup of request i before service; $P_B = \{i | i \in R\}$
$n + i$ delivery of request i before service; $D_B = \{n + i | i \in R\}$
$2n + i$ pickup of request i after service; $P_A = \{2n + i | i \in R\}$
$3n + i$ delivery of request i after service; $D_A = \{3n + i | i \in R\}$
Vehicle k is identified by two nodes belonging to two different sets as follows:

$4n + k$ origination depot of vehicle k; $O_K = \{4n + k | k \in K\}$
$4n + m + k$ destination depot of vehicle k; $D_K = \{4n + m + k | k \in K\}$
N $P_B \cup P_A \cup D_B \cup D_A$; set of nodes identifying all requests
A $N \cup O_k \cup D_K$; set of all nodes, including vehicle origination and destination depots.

3.2.1. Model parameters

d_i transported units from node i; $l_i = d_i$, $l_{n+i} = -d_i$, $i \in P_B \cup P_A$
C_k capacity of vehicle k
t_{ij} travel time between distinct nodes $i, j \in A$
c_{ij} transportation cost between distinct nodes $i, j \in A$
β_1 unit cost of vehicle waiting time

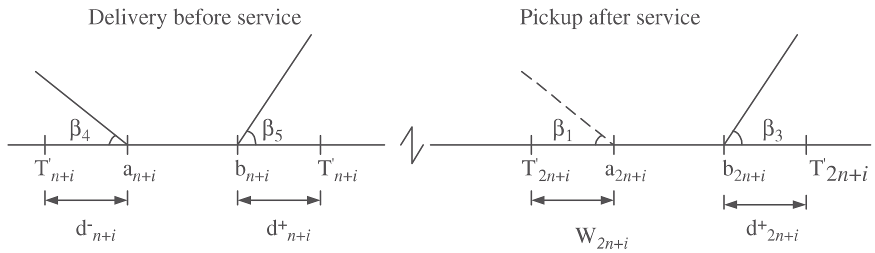

Fig. 1. Deviations, time windows and associated weight coefficients.

$\beta_2, \beta_3, \beta_4, \beta_5$ weight coefficients of extra riding time, late pickup time after service, early delivery time before service and late delivery time before service, respectively

a_i, b_i time window limits of node $i \in D_B \cup P_A$ where $a_{n+i} \leqslant b_{n+i} \leqslant a_{2n+i} \leqslant b_{2n+i}, i \in R$.

3.2.2. Variables

X_{ijk} $\begin{cases} 1 & \text{if vehicle } k \text{ traverses arc } (i, j) \in N \times N \\ 0 & \text{otherwise} \end{cases}$

T_{ik} time at which vehicle k departs from node $i \in N$

L_{ik} load of vehicle k immediately after it departs from node i

w_i vehicle waiting time at node $i \in P_A$

d_i^+ positive deviation due to late pickup at node $i \in P_A$, or due to late delivery at node $i \in D_B$

d_i^- negative deviation due to early delivery at node $i \in D_B$

(see Fig. 1 for the relationship between time window limits and deviational variables for a given request i, where T'_{n+i} is the client delivery time before service and T'_{2n+i} is the arrival time of the vehicle to pick up a client after service).

3.2.3. Mathematical formulation

$$Min\ z_1 = \sum_{k=1}^{m} \sum_{i=1}^{4n+2m} \sum_{j=1}^{4n+2m} c_{ij} X_{ijk} + \beta_1 \sum_{i=1}^{n} w_{2n+i}. \tag{1}$$

$$\begin{aligned} Min\ z_2 = \sum_{i=1}^{n} \Big\{ & \beta_2 \left[\left(T_{n+i,k} - T_{i,k} - t_{i,n+i} \right) + \left(T_{3n+i,k} + T_{2n+i,k} - t_{2n+i,3n+i} \right) \right] \\ & + \beta_3 d_{2n+i}^{+} + \beta_4 d_{n+i}^{-} + \beta_5 d_{n+i}^{+} \Big\}. \end{aligned} \tag{2}$$

Subject to

$$\sum_{k \in K} \sum_{j \in N} X_{ijk} = 1, \quad i \in P_B \bigcup P_A. \tag{3}$$

$$\sum_{j \in N} X_{ijk} - \sum_{j \in N} X_{j,n+i,k} = 0, \quad i \in P_B \bigcup P_{A,} \ \forall k \in K. \tag{4}$$

$$\sum_{j \in P_B \bigcup P_A} X_{4n+k,jk} \leqslant 1, \quad \forall k \in K. \tag{5}$$

$$\sum_{i \in D_B \cup D_A} X_{i,4n+m+k,k} \leqslant 1, \quad \forall k \in K. \tag{6}$$

$$\sum_{i \in N} X_{ijk} + X_{4n+k,jk} - \sum_{i \in N} X_{jik} + X_{j,4n+m+k,k} = 0, \quad j \in N, \ \forall k \in K. \tag{7}$$

$$X_{ijk}(T_{ik} + t_{ij} + w_j - T_{jk}) = 0, \quad i \in N, \ j \in P_A, \ \forall k \in K. \tag{8a}$$

$$X_{ijk}(T_{ik} + t_{ij} - T_{jk}) = 0, \quad i \in N, \ j \in P_B \cup D_B \cup D_A, \ \forall k \in K. \tag{8b}$$

$$a_i \leqslant T_{ik} - d_i^+ \leqslant b_i, \quad i \in P_A, \ \forall k \in K. \tag{9a}$$

$$a_i \leqslant T_{ik} - d_i^- - d_i^+ \leqslant b_i, \quad i \in D_B, \ \forall k \in K. \tag{9b}$$

$$T_{ik} + t_{i,n+i} \leqslant T_{n+i,k}, \quad i \in P_B \cup P_A, \ \forall k \in K. \tag{10}$$

$$X_{ijk}(L_{ik} + l_j - L_{jk}) = 0, \quad j \in N, \ i \in N \cup \{4n+k\}, \ \forall k \in K. \tag{11}$$

$$L_{n+i,k} \leqslant C_k - l_{n+i}, \quad i \in P_B \cup P_A, \ \forall k \in K. \tag{12}$$

$$L_{4n+k,k} = 0, \quad \forall k \in K. \tag{13}$$

$$X_{ijk} \in (0, 1), \quad \forall i \in A, \ \forall j \in A, \ \forall k \in K. \tag{14}$$

$$T_{ik}, L_{ik} \geqslant 0, \quad \forall i \in A, \ \forall j \in A, \ \forall k \in K. \tag{15}$$

$$d^-, d^+ > 0, \quad i \in D; \quad d^+ > 0, w > 0, \quad i \in P. \tag{16}$$

Objective function (1) minimizes total transportation cost. Objective function (2) minimizes total client inconvenience time comprised of vehicle riding time (excluding direct travel time) and the extent of earliness due to client delivery before service and lateness due to late delivery for service and late pickup after service. Constraint (3) assures that each client is served exactly once. Constraint (4) ensures that the same vehicle serves each client. Constraints (5) and (6) state that a vehicle should start traveling from its origination depot and terminate at its destination depot. Constraint (7) ensures a multi-commodity flow structure (balance equations). Constraints (8a) and (8b) impose compatibility requirements between routes and schedules. For example, constraint (8a) ensures that if vehicle k traverses arc (i, j), where j is a pickup node after service, then its departure time from node j is equal to the departure time from the previous node i plus travel time t_{ij} plus any waiting time at node j. Constraints (9a) and (9b) set the boundaries of the time windows. Time windows were set for client pickup schedules after service (9a) and delivery schedules before service (9b). When delivering clients before service, both lower and upper bounds of the time window are flexible. When picking up clients after service, the lower bound of the time window is rigid ("hard"), while its upper bound is flexible ("soft"). In other words, if a vehicle arrives early to pick up a client, it has to wait until the beginning of the time window (a client finishes service). This is consistent with constraint (8a). In all other cases a vehicle does not wait, but the client may wait. The client waiting time is computed in these two constraints as either a negative or positive deviation from the lower and upper bounds of the respective time window. This deviation is penalized in