Informed Search

- Uninformed search is systematic but inefficient.

- Would like to use additional knowledge such that better nodes are considered for expansion and exploration first.
  - In terms of pseudo-code, we want to order the open queue.

- We will introduce an **evaluation function** f(n) that indicates the desirability of considering node n next for exploration and expansion.
  - Nodes with a better f(n) are always considered first.

- How should we compute f(n) ???

Recall Uniform Cost Search

- UCS orders the open queue according to the path cost $g(n)$.
  - Path cost is distance from root to state n.
- So, UCS uses an evaluation function $f(n) = g(n)$.
- The path cost $g(n)$ only accounts for cost to reach n.
  - UCS is NOT goal directed/oriented.
- Would also like to consider the cost from state n to the goal.

Being Goal Oriented

- Would like $f(n)$ to include a measure of the cost to goal.
- Use what's called a heuristic function $h(n)$.
  - an estimate of the cost from the current state to goal.

Two ideas/approaches; always try to expand nodes that are:

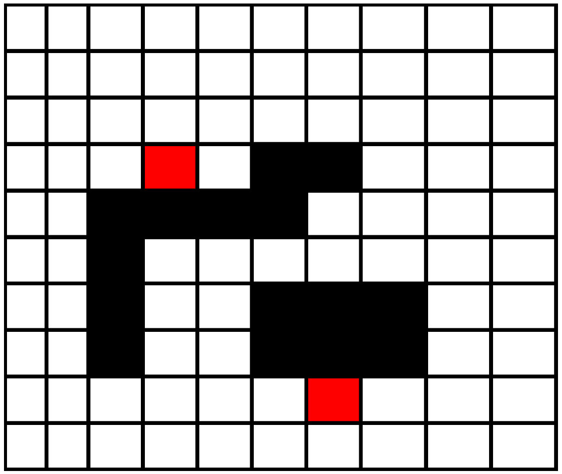
- Estimated to be closest to goal.
  - $f(n) = h(n)$ (Greedy Best First Search)
- On least cost path from root, through current state, to the goal.
  - $f(n) = g(n) + h(n)$ (A* Search)

- Heuristic function $h(n)$ is:
  - Only an estimate (whereas path cost $g(n)$ is exact).
  - Equal to zero when at the goal.

Greedy Best First Search

- Always explores and expands the node judged to be closest to goal.
- So, uses $f(n) = h(n)$ and ignores the path cost $g(n)$ entirely.
- Consider it to be the complement of UCS.

Illustration of Greedy Best First Search



6	5	4	3	4	5	6	7	8	9
5	4	3	2	3	4	5	6	7	8
4	3	2	1	2	3	4	5	6	7
3	2	1	0	1			4	5	6
4	3					4	5	6	7
5	4		2	3	4	5	6	7	8
6	5		3	4				8	9
7	6		4	5				9	10
8	7	6	5	6	7	8	9	10	11
9	8	7	6	7	8	9	10	11	12

Distances to goal h(n)

- Consider wanting to find a path from (6,1) to (3,6), but we have obstacles.
- Let the estimate of distance to goal be the manhattan distance between the current position (x,y) and the goal (dstx,dsty); i.e., h(n) = |x-dstx|+|y-dsty|.

- **NOTE:** our heuristic does not look at obstacles (problem relaxation, more Friday)

Illustration of Greedy Best First Search

-	-	-	-	-	-	-	-	-
-	-	-	18	17	16	15	14	-
-	-	18	17	16	15	14	13	14
-	-	-	18	17			12	13
-	-					10	11	12
-	-		6	7	8	9	10	-
-	-		5	6				-
-	-		4	3				-
-	-	4	3	2	1	0	1	-
-	-	-	4	3	2	1	-	-

-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-
-	-	-	19	18	17	16	15	-
-	-	-	20	-			14	-
-	-					12	13	-
-	-		6	7	9	11	-	-
-	-		5	8				-
-	-		4	10				-
-	-	-	3	2	1	0	-	-
-	-	-	-	-	-	-	-	-

- Left figure shows nodes expanded and explored and those nodes on the fringe of the search.
- Right figure shows the order (time) at which a square was explored.
- Final path returned is:
  - (6,1)->(5,1)->(4,1)->(3,1)->(3,2)->(3,3)->(3,4)->(4,4)->(5,4)->(6,4)->(6,5)->(7,5)->(7,6)->(7,7)->(6,7)->(5,7)->(4,7)->(3,7)->(3,6)
  - Which is not optimal.
  - **NOTE:** Only one path was followed until the goal was found. (similar to DFS)

Queue Contents

- Can illustrate the contents of the open and closed queue. Note that:
  - Goal appears at the head of the open queue just prior to termination.
  - Open queue contains the fringe nodes upon termination.

1	open	[(6,1,8.0)]
	closed	
2	open	[(5,1,7.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)]
3	open	[(4,1,6.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)]
4	open	[(3,1,5.0)][(4,2,5.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)]
5	open	[(3,2,4.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)]
6	open	[(3,3,3.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)]
7	open	[(3,4,2.0)][(4,3,4.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)]
8	open	[(4,4,3.0)][(4,3,4.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)]
9	open	[(4,3,4.0)][(5,4,4.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)]
10	open	[(5,4,4.0)][(4,2,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)]
11	open	[(4,2,5.0)][(6,4,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]

Queue Contents cont'd

12	open	[(6,4,5.0)][(2,1,6.0)][(3,0,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)]
13	open	[(6,5,4.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)]
14	open	[(7,5,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)]
15	open	[(7,6,4.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)]
16	open	[(7,7,5.0)][(8,6,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)]
17	open	[(6,7,4.0)][(8,6,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)][(7,8,6.0)][(8,7,6.0)][(4,0,7.0)][(5,0,8.0)]
		[(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)][(7,7,5.0)]
18	open	[(5,7,3.0)][(8,6,5.0)][(6,8,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)][(7,8,6.0)][(8,7,6.0)][(4,0,7.0)]
		[(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)][(7,7,5.0)][(6,7,4.0)]
19	open	[(4,7,2.0)][(5,8,4.0)][(8,6,5.0)][(6,8,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)][(7,8,6.0)][(8,7,6.0)]
		[(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)][(7,7,5.0)][(6,7,4.0)][(5,7,3.0)]
20	open	[(3,7,1.0)][(4,6,1.0)][(4,8,3.0)][(5,8,4.0)][(8,6,5.0)][(6,8,5.0)][(2,1,6.0)][(3,0,6.0)][(7,4,6.0)][(8,5,6.0)]
		[(7,8,6.0)][(8,7,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)][(7,7,5.0)][(6,7,4.0)][(5,7,3.0)][(4,7,2.0)]
21	open	[(3,6,0.0)][(4,6,1.0)][(3,8,2.0)][(2,7,2.0)][(4,8,3.0)][(5,8,4.0)][(6,8,5.0)][(8,6,5.0)][(2,1,6.0)][(3,0,6.0)]
		[(7,4,6.0)][(8,5,6.0)][(7,8,6.0)][(8,7,6.0)][(4,0,7.0)][(5,0,8.0)][(6,0,9.0)][(7,1,9.0)]
	closed	[(6,1,8.0)][(5,1,7.0)][(4,1,6.0)][(3,1,5.0)][(3,2,4.0)][(3,3,3.0)][(3,4,2.0)][(4,4,3.0)][(4,3,4.0)][(5,4,4.0)]
		[(4,2,5.0)][(6,4,5.0)][(6,5,4.0)][(7,5,5.0)][(7,6,4.0)][(7,7,5.0)][(6,7,4.0)][(5,7,3.0)][(4,7,2.0)][(3,7,1.0)]

Performance of Greedy Best First Search

- Resembles DFS in that it follows a single path until it hits a dead end (and backs up … false starts).
- Optimal? NO
  - Can get stuck going down an incorrect path.
- Complete? NO (in theory)
  - Can get stuck going down an infinite path.
- Time Complexity? $O(b^m)$
  - Worst case: must expand all nodes for a tree of depth m.
- Space Complexity? $O(b^m)$
  - Worst case: must maintain all nodes for a tree of depth m (since it back-tracks and jumps around* in the search tree).
- However, like DFS, in the best case (with a good h(n)) it can be efficient.

A* Search

- Uniform cost search orders the queue according to the **path cost** g(n).
  - Optimal, complete, but inefficient in time and space.
- Greedy best first search orders the queue using the **heuristic cost** h(n).
  - Not optimal, not complete (in theory) but efficient and directed (with good heuristic … what makes a good heuristic?).

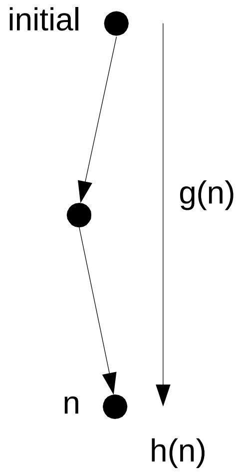
- Idea behind A* is to combine the two strategies; Use an **evaluation function** f(n) = g(n) + h(n) to order the nodes to be explored.
  - f(n) measures the **cheapest total estimated cost** from the initial state to the goal state **passing through** the current state n.*

- The resulting search is both optimal and complete assuming certain conditions on the heuristic cost h(n).
  - Since we are goal oriented, we find that, in practice, the time and space complexity is reduced (but not necessarily).

Issues In Using f(n) = g(n)+h(n) [Case 1]

- When expanding a new node, we need to consider queue management.

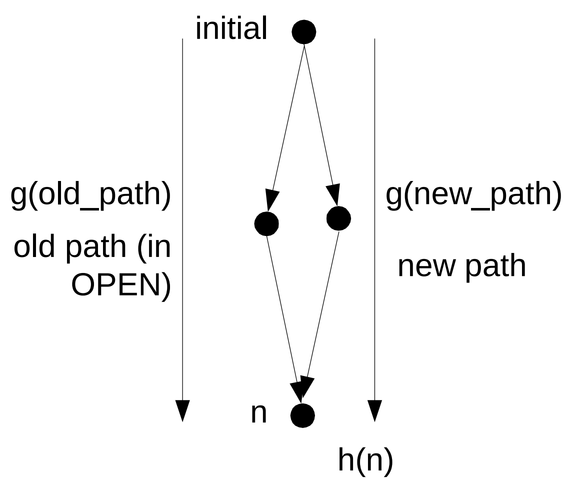
- Case 1: new node, n, expanded is not currently in the open or closed queue.
  - This is the only/first path found to node n.
  - Compute g(n), h(n) and f(n) and insert node n into the open queue.



Issues In Using f(n) = g(n)+h(n) [Case 2]

- When expanding a new node, we need to consider queue management.

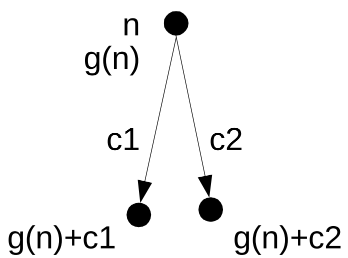
- CASE 2: expanded node already in open queue.
  - We must have found a new path to node n.
  - It might be possible that g(new_path) < g(old_path). Of course, h(n) is the same.
  - This means we have a better overall path going through node n, so we replace the entry on the open queue with the new path if it is better.



Issues In Using f(n) = g(n)+h(n) [Case 3]

- When expanding a new node, we need to consider queue management.

- CASE 3: expanded node already in closed queue.
  - We must have found a new path to node n.
  - It might be possible that g(new_path) < g(old_path). Of course, h(n) is the same.
  - If g(new_path) < g(old_path) need to reconsider node n by moving it from the closed queue to the open queue.

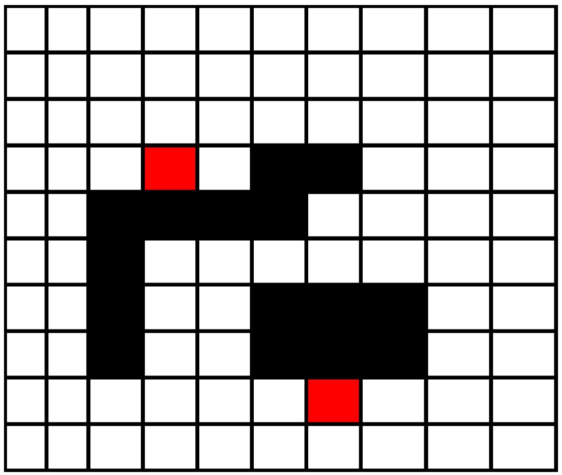


- Will also need to reconsider children of node n too but they will get reconsidered when node n is revisited.

Pseudo-code for A* Search

```
open_queue.insert(init_state); closed_queue.clear();
while (open_queue.size() != 0) {
  curr_state = open_queue.remove_front();
  if (is_goal(curr_state)) { return success; } // and solution
  closed_queue.insert(curr_state);
  child = expand(curr_state);
  for (i = 1 ; i <= child.size() ; i++) {
    compute_h(child[i]);
    compute_g(child[i]) = g(curr_state) + action_cost;
    if (open_queue.find(child[i])) {
      if (g(child_in_open) > g(child[i])) {
        replace_child_on_open;
      } else ; // discard new path.
    } else if (closed_queue.find(child[i])) {
      if (g(Child_in_closed) > g(child[i])) {
        closed_queue.remove(child[i]);
        open_queue.enqueue(child[i]);
      } else ; // discard new path.
    } else {  open_queue.enqueue(Child[i]); } // first visit.
  }
  sort(open_queue); // sort using f(n)=g(b)+h(n)
}
return failure; // no solution found.
```

Illustration of A* Search



6	5	4	3	4	5	6	7	8	9
5	4	3	2	3	4	5	6	7	8
4	3	2	1	2	3	4	5	6	7
3	2	1	0	1			4	5	6
4	3					4	5	6	7
5	4		2	3	4	5	6	7	8
6	5		3	4				8	9
7	6		4	5				9	10
8	7	6	5	6	7	8	9	10	11
9	8	7	6	7	8	9	10	11	12

Distances to goal h(n)

- Consider wanting to find a path from (6,1) to (3,6), but we have obstacles.
- Let h(n) be the manhattan distance from current position (x,y) to the goal .
  - Clearly admissible, since it is the straight path w/o obstacles (more on admissibility later).
- Let g(n) be the distance traveled from initial position to (x,y).

- Solution returned by A* is: (6,1)->(5,1)->(4,1)->(3,1)->(2,1)->(1,1)->(1,2)->(1,3)->(1,4)->(1,5)->(1,6)->(2,6)->(3,6) which is optimal!

Illustration of A* Search

-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-
-	11	12	-	-	-	-	-	8	-
11	10	11	12	-			8	7	8
10	9					8	9	6	7
9	8		6	5	6	7	8	5	6
8	7		5	4				4	5
7	6		4	3				3	4
6	5	4	3	2	1	0	1	2	3
-	6	5	4	3	2	1	2	3	-

Distances from initial g(n)

-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-
-	3	2	-	-	-	-	-	6	-
3	2	1	0	-			4	5	6
4	3					4	5	6	7
5	4		2	3	4	5	6	7	8
6	5		3	4				8	9
7	6		4	5				9	10
8	7	6	5	6	7	0	9	10	11
-	8	7	6	7	8	9	10	11	-

Distances to goal h(n)

- Left figure shows g(n); Right figure shows h(n) (only for nodes expanded).
  - Can add values together to determine f(n).

Illustration of A* Search

-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-
-	11	12	-	-	-	-	-	8	-
11	10	11	12	-			8	7	8
10	9					8	9	6	7
9	8		6	5	6	7	8	5	6
8	7		5	4				4	5
7	6		4	3				3	4
6	5	4	3	2	1	0	1	2	3
-	6	5	4	3	2	1	2	3	-

-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-
-	32	33	34	-			-	31	-
-	30					24	-	29	-
-	28		9	8	16	21	-	27	-
-	26		7	6				25	-
-	23		5	4				22	-
-	19	14	3	2	1	0	11	18	-
-	-	20	15	13	12	10	17	-	-

- Left figure shows nodes expanded and explored and those nodes on the fringe of the search.
- Right figure shows the order (time) at which a square was explored.

- Searches more than greedy best first search, but is optimal.
  - At end of search: open queue has 20 nodes, closed queue has 34 nodes.
  - cf. Greedy search: 17 nodes in open queue, 20 in closed (faster, not optimal result!)

Illustration of A* Search (Compare to breadth first search)

- Can compare to breadth first search (also optimal).
  - Illustration of nodes expanded, explored and on fringe on left, time visited on right.
  - At end of search, open contains 7 nodes, closed contains 69 nodes.

- A* search: returns optimal solution, uses less memory and less time (in this example) compared to breadth first search.

-	13	-	-	-	-	12	11	10	11
13	12	13	-	-	12	11	10	9	10
12	11	12	13	12	11	10	9	8	9
11	10	11	12	-			8	7	8
10	9					8	7	6	7
9	8		6	5	6	7	6	5	6
8	7		5	4				4	5
7	6		4	3				3	4
6	5	4	3	2	1	0	1	2	3
7	6	5	4	3	2	1	2	3	4

-	-	-	-	-	-	-	64	57	65
-	67	-	-	-	-	63	56	51	58
66	60	68	-	-	62	55	50	46	52
59	54	61	69	-			45	40	47
53	49					44	39	33	41
48	43		30	24	31	38	32	25	34
42	37		23	17				19	26
36	29		16	10				12	20
27	21	14	8	4	1	0	3	7	13
35	28	22	15	9	5	2	6	11	18

Illustration of A* Search (Compare to depth first search)

- Can compare to depth first search (not optimal).
  - Illustration of nodes expanded, explored and on fringe on left, time visited on right.
  - At end of search, open contains 29 nodes, closed contains 37 nodes.
  - Returned solution has length 35!

- A* search: returns optimal solution, uses less memory and less time (in this example) compared to depth first search.

-	-	-	-	-	-	14	13	12	11
-	-	-	-	-	-	15	14	11	10
-	35	36	-	-	-	16	15	10	9
35	34	35	36	-			16	9	8
34	33					18	17	8	7
33	32		22	21	20	19	18	7	6
32	31		23	22				6	5
31	30		24	23				3	4
30	29	28	25	24	1	0	1	2	3
-	28	27	26	25	26	1	2	3	4

-	-	-	-	-	-	-	13	12	11
-	-	-	-	-	-	-	14	-	10
-	-	-	-	-	-	-	15	-	9
-	35	36	37	-			16	-	8
-	34					-	17	-	7
-	33		-	21	20	19	18	-	6
-	32		-	22				-	5
-	31		-	23				-	4
-	30	29	-	24	-	0	1	2	3
-	-	28	27	25	26	-	-	-	-

Optimality of A* Search - Heuristic Function Admissibility

- Optimality depends on h(n).

- The heuristic cost h(n) is said to be **admissible** if it **never overestimates** the actual cost from node n to the goal.
  - If h(n) never overestimates, then f(n) never overestimates true cost to goal through node n.

Admissibility guarantees optimality! Sketch of proof:

- Let f* be optimal cost to reach the goal, and G2 be a discovered goal with f(G2) > f*.
- Consider a fringe node n along the path to the optimal solution. If h(n) admissible, then f(n) = g(n)+h(n) <= f*.
- Also, f(n) <= f* < f(G2).
- A* expands lowest cost nodes first, so reaching G2 implies f(G2)<= f(n) and we have a contradiction.

- With h(n) admissible, we must expand any node n along the optimal path first, and this includes the optimal goal node itself.

- Even without admissibility, we might still get a good search, it just won't necessarily be optimal.

Completeness of A* Search

- A* expands nodes with $f(n) < f^*$, and possibly some nodes with $f(n)=f^*$ on the fringe before reaching the goal.
  - Therefore, we will eventually find the goal (complete).

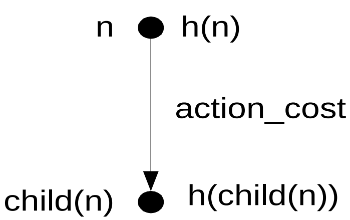
- Two reasons for failure:
  - infinite branching factor,
  - path with finite path cost but infinitely long.

Monotonicity of Heuristic Functions

- Relates to queue management (from Wed.) and the need to re-consider nodes already in the open and closed queue when rediscovered.

- **Monotonicity** (or consistency) means that for every node n and each of its children, the estimated cost h(n) is never greater than the cost h(child(n)) plus the action of getting to child(n).

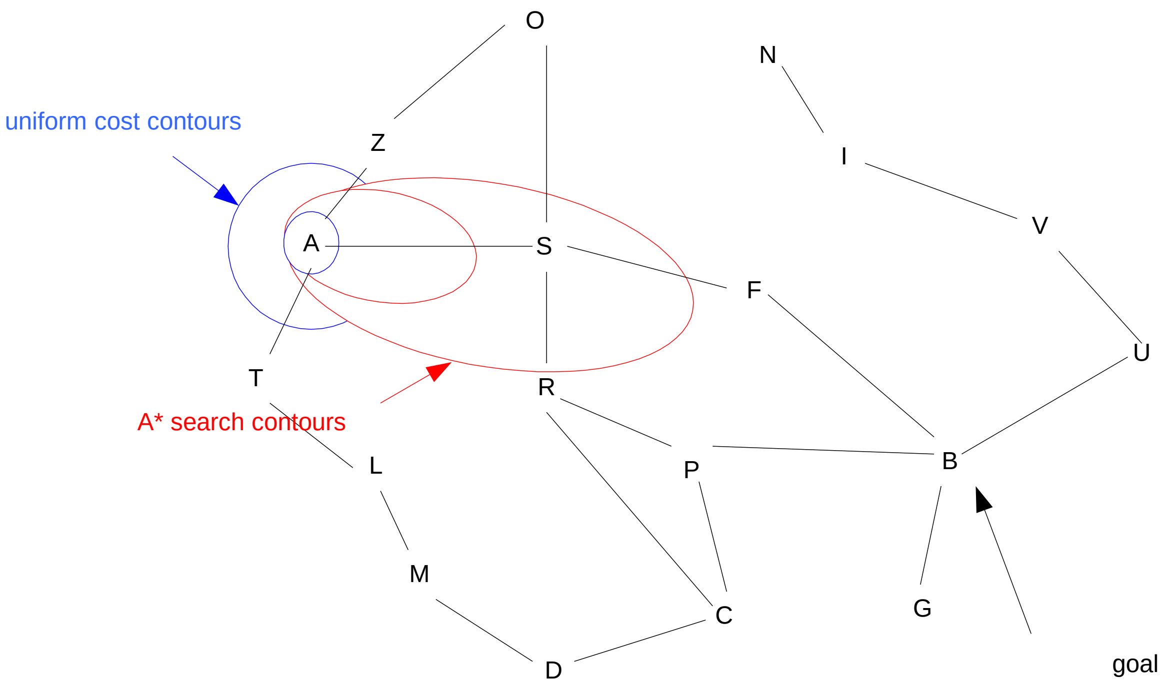
$$h(n) \leq h(\text{ child}(n)\ ) + \text{cost}(\ n \rightarrow \text{child}(n)\ )$$



- With some mathematical arrangements:
  - $$\begin{aligned} f(n) &= h(n) + g(n) \\ &\leq h(\text{ child}(n)\ ) + \text{cost}(\ n \rightarrow \text{child}(n)\ ) + g(n) \\ &= h(\text{ child}(n)\ ) + g(\text{ child}(n)\ ) \\ &= f(\text{ child}(n)\ ) \end{aligned}$$
- So, f(n) never decreases as we approach the goal.

- Monotonicity guarantees that states are always visited by the cheapest path first; no need to check if subsequent paths are better than first.

Visualization of A* Search



- Uniform cost expands in "circular" cost contours (h(n) = 0).
- A* search elongates and rotates contours towards the goal. More narrow and elongated the better h(n) is.
  - More directed!

Informedness of Heuristic Functions

- One heuristic function might be better than another for a given problem!
- Informedness: For two admissible heuristic functions, h1 & h2:
  - if h2(n) >= h1(n), then h2(n) is more informed than h1(n). (alternatively say that h2(n) *dominates* h1(n)).
  - More informedness implies fewer expanded states (as in previous slide).

Informedness of Heuristic Functions

A* will consider all nodes with $f(n) < f^*$, and possibly some on the contour of $f(n) = f^*$ before finding the goal state.

- Since $h2(n) >= h1(n)$, nodes expanded by $h2(n)$ will be expanded by $h1(n)$. The opposite is not true
- => not all nodes expanded by $h1(n)$ will be expanded by $h2(n)$.*
- => $h2(n)$ will expand fewer nodes!

Can also think of it as follows:

- Assume n expanded by $h2(n)$ but not $h1(n)$ … implies that $f2(n) < f1(n)$.
- But, $f2(n) = h2(n)+g(n)$ and $f1(n) = h1(n) + g(n)$ … implies that $f2(n) \geq f1(n)$ since $h2(n) \geq h1(n)$.
  - $\rightarrow$ contradiction*

- Always best to pick $h(n)$ large (but admissible).

Creation of Heuristic Functions

- How to choose a heuristic function for a problem?
  - Sometimes obvious, other times not due to constraints of the problem.

Can invent a heuristic function using a **problem relaxation**.

- Leave out constraints and get an easier problem.
- Solution to original problem solves relaxation, but not visa-versa.*
- Admissible heuristic function for relaxation is admissible for original.
  - Solution to original problem solves the relaxed problem.
  - => solution must be at least as expensive and the relaxed solution.*
  - => h(n) for relaxed problem is <= cost in original problem.

- Can also have different heuristics and always choose the best one:

$$h(n) = \max\{h_1(n), h_2(n), \cdots, h_m(n)\}$$

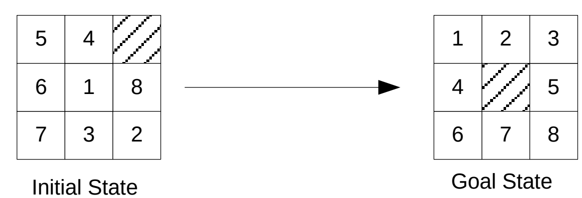
Example of Heuristic Functions

- Consider the 8-puzzle and a verbal description of a move:
  - “Tile can move from location A to location B if A,B are adjacent and B is blank.

Consider several relaxations of this verbal description:

- “I’m confused”
  - => h1(n) = 0 (BFS)
- “Tile can move from location A to location B”
  - => h2(n) = number of tiles out of place.
- “Tile can move from location A to location B if A,B adjacent”
  - => h3(n) = sum of distance of tiles from goal locations.

- Note: h1(n) <= h2(n) <= h3(n) <= h*(n).
  - h1(init_state) = 0.
  - h2(init_state) = 1+1+1+1+1+1+1+1=8
  - h3(init_state) = 2+3+3+2+3+1+1+1=16



Iterative Deepening A* Search

- A* Search can still generate a lot of nodes (won't consider exact complexity).
- Iterative Deepening A* (IDA*):
  - Use iterative deepening (each iteration is limited DFS), but use f(n) costs rather than depth to limit search.
  - Examines all nodes within a certain cost contour using DFS.
  - If no solution, increase cost cutoff to the next smallest f(n), where b is a node on the fringe.
- Complete and optimal like A*, but memory requirements of DFS.
- Can perform poorly if small action costs (small steps each iteration).

Game playing

- One of the earliest AI problems tackled as games seemed to require intelligence.
  - At least 4 attempts at chess alone by 1950 (Zuse, Shannon, Wiener & Turing)
  - Computers better than humans at checkers, Othello.

- Games are (usually) well defined; e.g.,
  - Board games have well-defined board configurations.
  - Legal movements are well-defined.

- Games definitely require heuristics; usually <u>too hard</u> to solve such problems using an exhaustive approach

E.g., chess:

- Well defined: 8x8 board, pieces move in specific ways.
- Has an average branching factor of ~35,
- Average game lasts for ~50 moves per player,
- Implies a search tree with $\mathbf{35}^{100}$ nodes!

Adversarial search

- Regular search heuristics are not appropriate for games due to involvement of an opponent (adversary).
  - We don't have control over all of the movements (e.g., in a two-player game, we only have control over half of the moves)
  - Can't really search for an "optimal solution"

- E.g. chess:
  - Goal is any state with checkmate
  - Don't know which of these states (if any) we will reach, since we don't know what moves the other player will make.

- Opponent introduces concept of uncertainty.
  - We need to make assumptions about behavior of opponent; better able to predict their movements and reach goal state.
  - Normally, assume "perfect play" behavior.

Types of games

- Will focus on games that can be solved with search concepts.
- Such games can be categorized:
  - Whether or not chance can affect the outcome of actions,
  - Whether or not all state information is available to all players.

	deterministic	chance
perfect information	**chess, checkers, go, othello**	**backgammon, monopoly**
imperfect information	**battleship**	**poker, scabble**

Two player strategy games

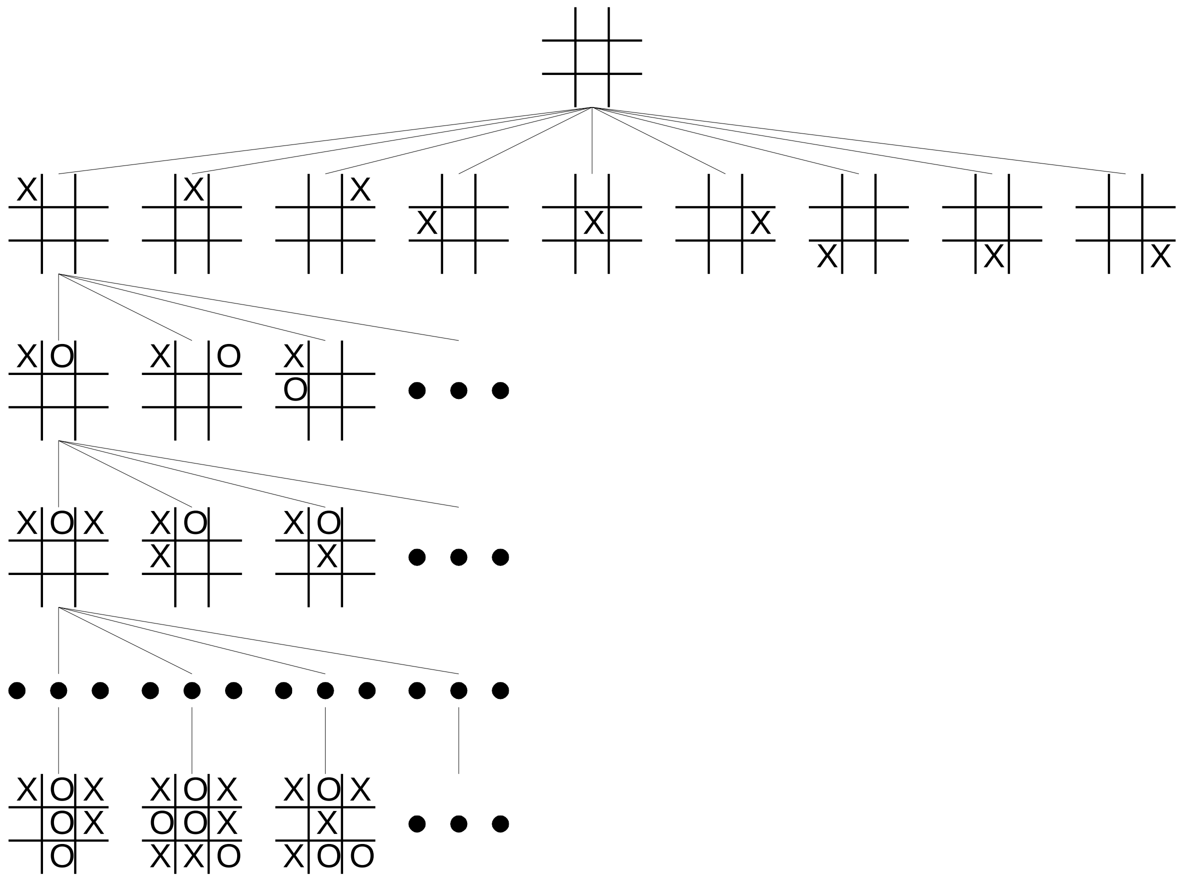
- Will consider two player games in which players (i) make alternating sequences of discrete moves (turn based) and (ii) each player wants to win ("I win, you lose").

- Will consider games that are (i) deterministic (no chance involved), (ii) have perfect information (fully observable) and (iii) have well-defined rules and goals.

- Can decide upon a move by making a "game tree":
  - Root of tree is the current game configuration;
  - Other nodes are game configurations reached via a sequence of moves by each player;
  - Each level is associated with one of the two players.

- We will refer to each level of the tree as a "ply";
  - A "ply" corresponds to a "half-move" (move by one player),
  - For most two-player games with alternating moves, a "full move" corresponds to two plies (levels) in the game tree.

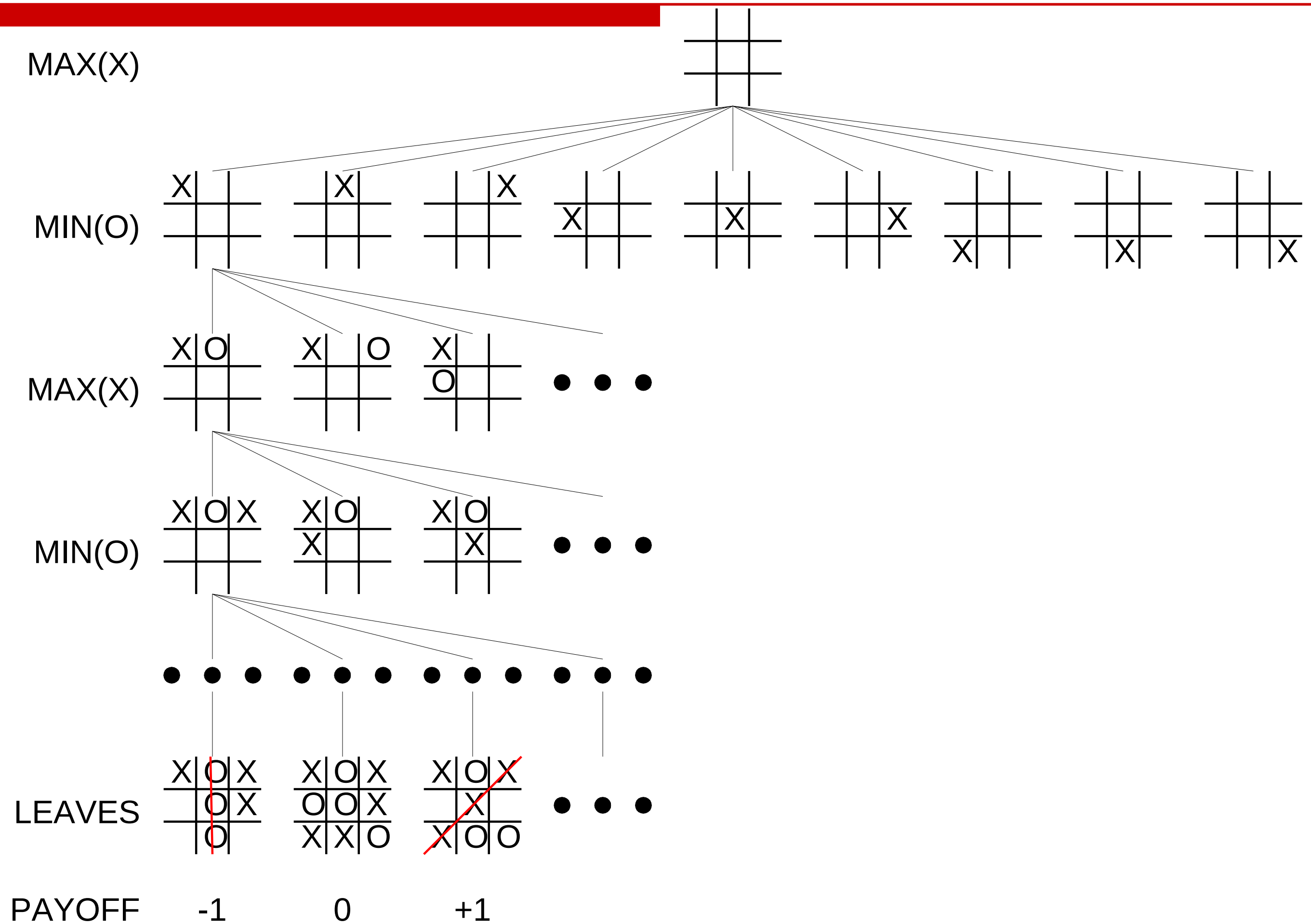
Game tree for tic-tac-toe



Minimax search

- Talked about game trees, but haven't talked about the searching of game trees and/or the selection of a move! Use <u>minimax algorithm</u>
- Call one player MAX and the other player MIN; MAX is the player making the current move (at the root of the tree).
- Use a **PAYOFF FUNCTION** (UTILITY FUNCTION) that assigns a numerical value to each leaf node in the game tree. The payoff function should be:
  - From the point of view of MAX.
  - Be larger for better game configurations for MAX.
    - I.e., MAX wants to maximize payoff, MIN wants to minimize payoff.
- Note that the payoff function might be simple (+1 win/0 draw/-1 lose), but could be more complex
  - ** goal states such as win/draw/lose might not appear at the leaves of the tree, depending on number of plies; only appear at **terminal states**.

Game tree for tic-tac-toe (players/payoffs labeled)



- **NOTE:** leaves are not at the same level

Minimax search

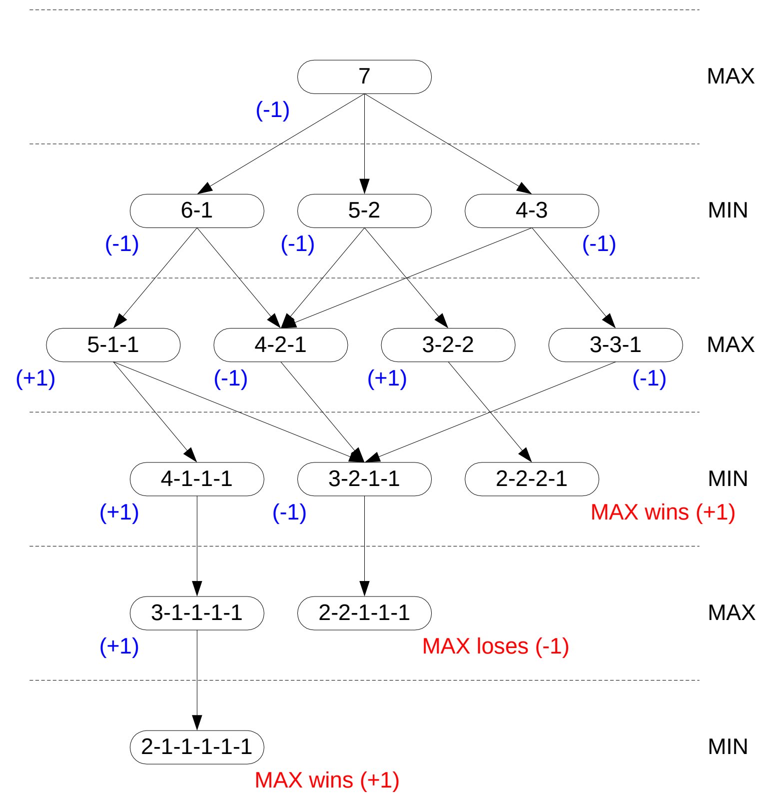
- In deciding upon the move to make, we assume:
  - MAX wants the payoff function to be as large as possible,
  - MIN wants the payoff function to be as small as possible.

- We make the assumption that each player plays perfectly; i.e., in their own best interest at each step of the algorithm.

- Minimax algorithm:
  - Generate game tree to some number of plies,
  - Compute payoffs at leaf nodes,
  - Propagate payoffs up the tree toward the root (done differently at each level);
    - MAX nodes choose child with maximum value;
    - MIN nodes choose child with minimum value;
  - At root, MAX chooses a move to make.

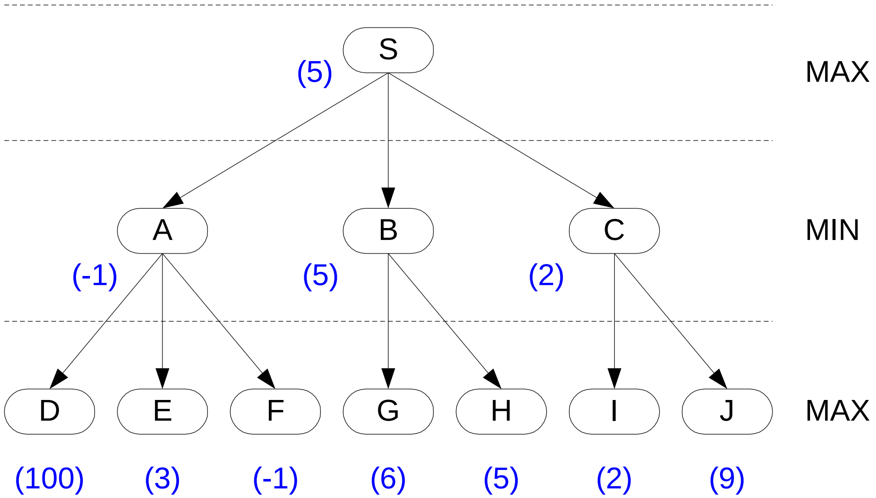
Minimax example (full game tree)

- Game of NIM; 7 matches placed in a pile; each player divides a pile of matches into 2 non-empty piles with a different number of matches. Player who cannot make a move is the loser.
- Let payoff be +1 (MAX WIN) or -1 (MAX LOSE).
- Values at each node n represents the best value of the best terminal state the current player (MAX or MIN) can hope to achieve.
- Turns out here that all first moves for MAX are bad!



Minimax example (partial game tree)

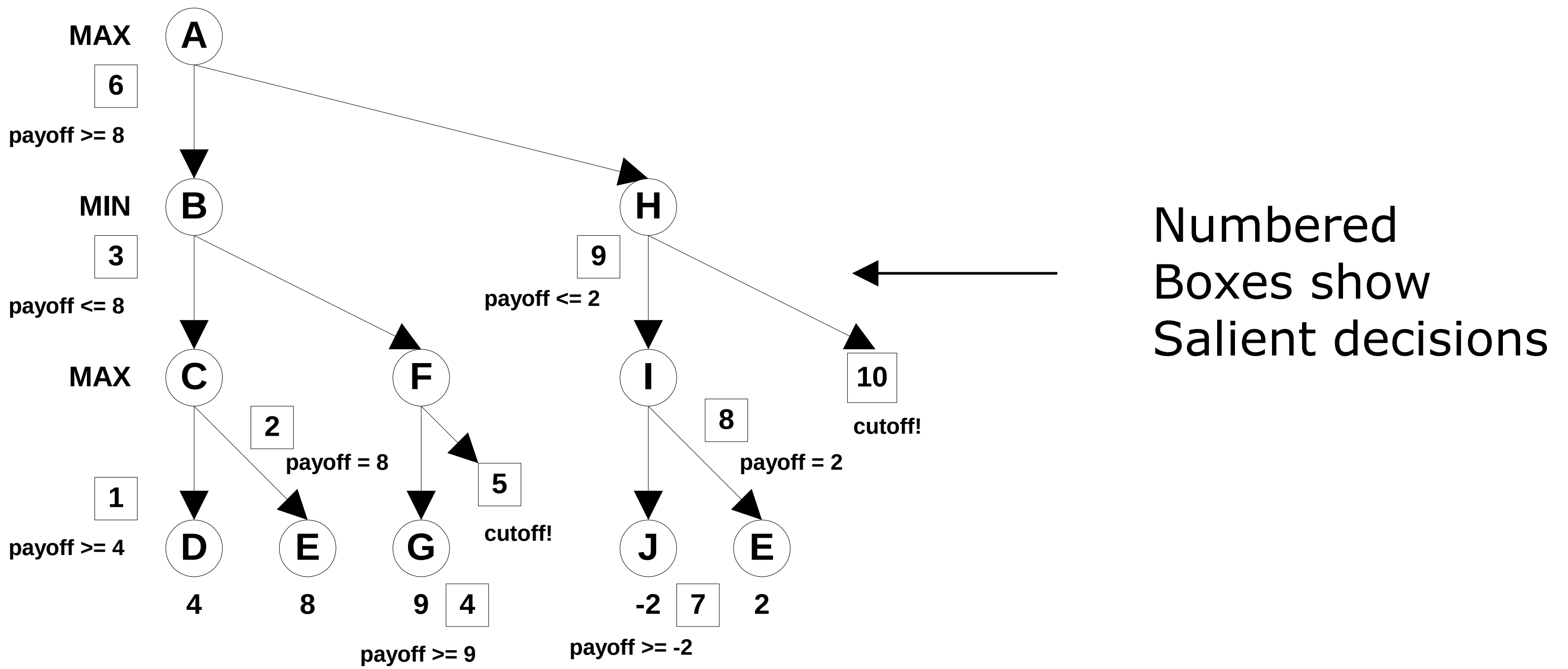
- Tree not deep enough for leaves to indicate win/lose; only "goodness" of game configuration (for MAX).
- MAX would like to get to game configuration D, but MIN would not allow it.
- MAX will choose to go to game configuration B.
- Again, minimax assumes MIN plays perfectly using the same payoff function as MAX (i.e., plays optimally from the same point of view as MAX).



Problems with minimax

- Minimax effectively uses only depth limiting (e.g., a limited number of plies).
- Large branching factors will cause minimax to be very inefficient; can't search too deep in a given amount of time.
- Need to introduce the concept of pruning; Want a modified algorithm that
  1. Returns the exact same result as minimax (for given number of plies), but
  2. Does not need to search as much of the game tree (faster).
- I.e., we want the same decision, but with less work.
- The ability to search a tree with a given number of plies faster means we can actually go deeper!
- Modification of minimax to use pruning is called alpha-beta pruning.

Pruning illustrated



- Prune at F after exploring G; F has payoff >= 9 (MAX); there exists a MIN ancestor with payoff <= 8; hence, MIN would prevent us from ever getting to F.
- Prune at H after exploring I; H has payoff <= 2 (MIN); there exists a MAX ancestor with payoff >= 8; hence, MAX would prevent us from ever getting to H.

Alpha-beta pruning

- The algorithm maintains two values α and β.
  - Both computed using payoffs propagated up the tree.
  - The α represents the minimum score that the maximizing player is assured of at any point during the search.
  - The β represents the maximum score that the minimizing player is assured of at any point during the search.
- As the search of the game tree progresses the "window" of values decreases.
  - Fewer possible values for α and β.
- Whenever β becomes less than or equal to α ($\beta \leq \alpha$)
  - ⇨ Current game configuration cannot be the result of the best play by both players and does not need to be explored further.
  - ⇨ More later …

Alpha-beta pruning

- Pseudo-code; called as `evaluate(root,-infinity,+infinity)`

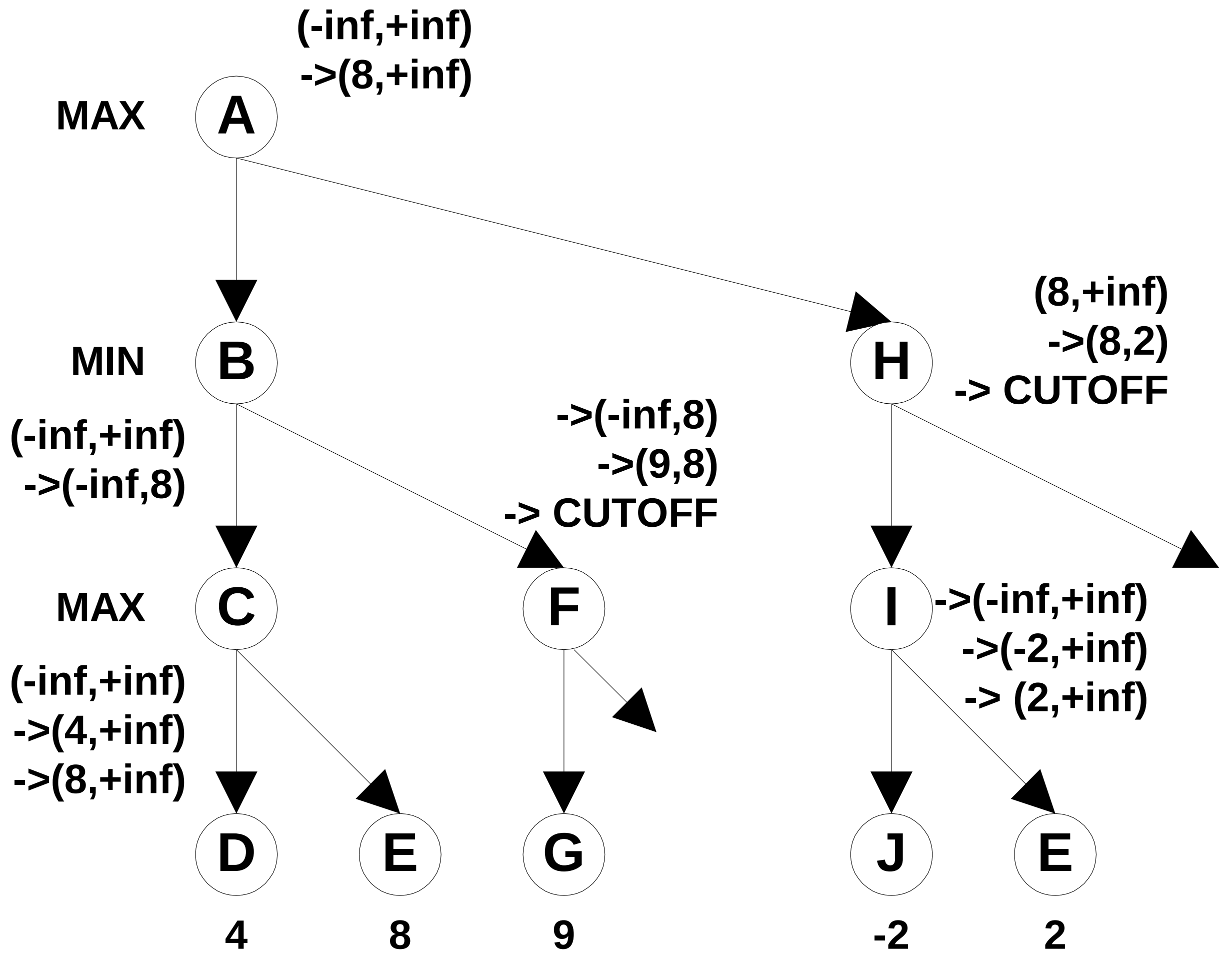
```
evaluate (node, alpha, beta)
    if node is a LEAF
        return the PAYOFF value of node
    if node is a MIN node
        beta = +inf
        for each child of node
            beta = min (beta, evaluate (child, alpha, beta))
            if beta <= alpha break
        return beta
    if node is a MAX node
        alpha = -inf
        for each child of node
            alpha = max (alpha, evaluate (child, alpha, beta))
            if beta <= alpha break
        return alpha
```

Alpha-beta pruning

- For MAX nodes,
  - β passed down equals best decision $\forall$ MIN node ancestors (fixed)
  - α calculated represents the best choice at the current node (variable)
  - Prune if $\alpha \geq \beta$
    - ⇨ $\exists$ MIN node ancestor that would prevent reaching this MAX node (MIN has a better choice somewhere else).

- For MIN nodes,
  - α passed down equals best decision $\forall$ MAX node ancestors (fixed)
  - β calculated represents the best choice at the current node (variable)
  - Prune if $\beta \leq \alpha$
    - ⇨ $\exists$ MAX node ancestor that would prevent reaching this MIN node (MAX can get a higher payoff somewhere else, so why would it choose this node?).

Pruning illustrated with (alpha,beta) values



Games of chance

- Game of chance is anything with a random factor; e.g., dice, cards, etc.

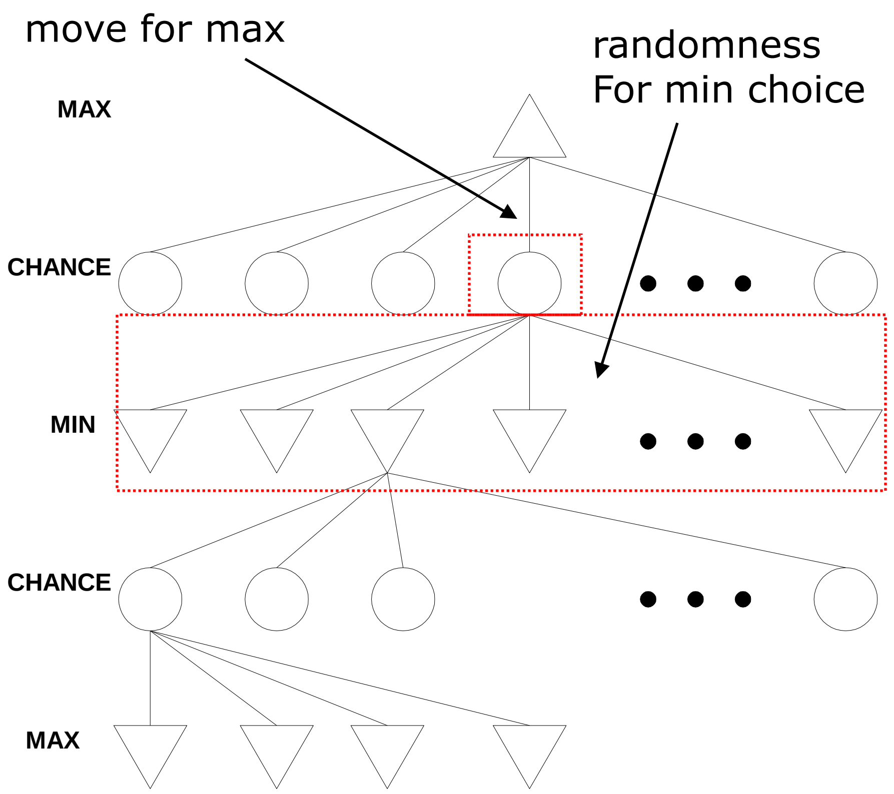
- Can extend the minimax method to handle games of chance.
  - This results in an **expectiminimax tree.**

- In an expectiminimax tree, levels of max and min nodes are **interleaved** with "chance" nodes.

- Rather than taking the max or min of the payoff values of their children, chance nodes take a weighted average (expected value) of their children.
  - Weights are the probability that the chance node is reached.
  - Each chance node gets an “expectiminimax value”.

Game tree with chance nodes

- Assume max have several possible moves.
- Before min decides, there is some element of chance involved (e.g., dice rolled; each outcome with some probability).
  - Chance nodes are added and the outgoing edges labeled with the probabilities.
- Same applies to max nodes further down the tree; i.e.,
  - For each move for min, there are chance nodes added before decisions for max.
- Tree can become large very fast!



Example of expectiminimax values

- Example below shows calculation of weighted expected values at chance nodes.

- Demonstrates potential problem when payoff function values are skewed;
  - Resulting values at chance nodes are an average (mean);
  - Large skewing in payoff values can cause MAX to make the wrong decision.