Formulas for Quantum Optics

Displacement Operator

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a} = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a} \quad D(\alpha)\,|0\rangle = |\alpha\rangle$$

$$D(\alpha) = D(-\alpha)$$

$$D(\alpha)D(\beta) = \mathrm{e}^{\mathrm{i}\Im(\alpha\beta^*)}D(\alpha+\beta)$$

Squeeze Operator

$$S(\zeta) = \mathrm{e}^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)} \qquad \zeta = r\mathrm{e}^{i\theta}$$

$$\theta = 0 \implies \Delta X_1^2 = \frac{1}{4}e^{-2r} \qquad \Delta X_2^2 = \frac{1}{4}e^{2r}$$

$$\left\langle \zeta \,\middle|\, a^\dagger a \,\middle|\, \zeta \right\rangle = \sinh^2 r$$

$$\langle \zeta \,|\, a \,|\, \zeta \rangle = 0$$

$$\langle \zeta \,|\, a^2 \,|\, \zeta \rangle = -e^{\mathrm{i}\varphi}\sinh r \cosh r$$

$$\left\langle \zeta \,\middle|\, (a^\dagger a)^2 \,\middle|\, \zeta \right\rangle = 2\cosh^2 r \sinh^2 r + \sinh^4 r$$

Properties of the Commutator

$$[A, BC] = [A,B]C + B[A,C]$$

$$[BC, A] = [B,A]C + B[C,A]$$

$$[A,B] = -[B,A]$$

Beam Splitter phase shift operator

$$P(\phi) = \mathrm{e}^{-\mathrm{i}\phi\hat{n}}$$

$$a_0 = \frac{a_2 - \mathrm{i}a_3}{\sqrt{2}} \qquad a_1 = \frac{a_3 - \mathrm{i}a_2}{\sqrt{2}}$$

$$D_0(\alpha) = D_2\left(\frac{\alpha}{\sqrt{2}}\right)D_3\left(\frac{-\mathrm{i}\alpha}{\sqrt{2}}\right)$$

$$D_1(\alpha) = D_3\left(\frac{\alpha}{\sqrt{2}}\right)D_2\left(\frac{\mathrm{i}\alpha}{\sqrt{2}}\right)$$

Baker-Campbell Hausdorff Theorem

$$\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B\mathrm{e}^{\frac{1}{2}[B,A]} = \mathrm{e}^B\mathrm{e}^A\mathrm{e}^{\frac{1}{2}[A,B]}$$

falls $[A,[A,B]] = [B,[A,B]] = 0$

Baker-Campbell Hausdorff Lemma

Number Operator

Quantum Harmonic Oscillator

Spread of Ground wave function $\sqrt{\hbar/(m\omega)}$

$$E_x = E_0(a + a^\dagger)\sin kz \quad B_y = -\mathrm{i}B_0(a - a^\dagger)\cos kz$$

$$H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{p}{m} + m\omega^2 q^2\right)$$

$$a, a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} \pm \mathrm{i}p)$$

$$\hat{n} = a^\dagger a \qquad \hat{n}\,|n\rangle = n\,|n\rangle \qquad |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}\,|0\rangle$$

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$[a^m, a^\dagger] = ma^{m-1} \qquad [a, a^{\dagger m}] = m(a^\dagger)^{m-1}$$

$$[n, a^m] = -ma^m \qquad [n, a^{\dagger m}] = ma^{\dagger m}$$

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$$

Quadratures

$$X_1 = \frac{1}{2}\left(a + a^\dagger\right) \propto q \quad X_2 = \frac{1}{2\mathrm{i}}\left(a - a^\dagger\right) \propto p$$

Phase State

$$P(\varphi) = \frac{1}{2\pi}\,|\langle \varphi \,|\, \psi \rangle|^2 \qquad |\varphi\rangle = \sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n\varphi}\,|n\rangle$$

Coherent States

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \qquad \bar{n} = |\alpha|^2$$

$$\langle \alpha \,|\, \beta \rangle = \mathrm{e}^{\frac{1}{2}(\beta^*\alpha - \beta\alpha^*)}\mathrm{e}^{-\frac{1}{2}|\beta-\alpha|^2} \qquad a\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

Characteristic Functions

$$W(\alpha, \alpha^*) = \frac{2}{\pi^2}\mathrm{e}^{2|\alpha|^2}\int d^2\eta\,\langle -\eta \,|\, \rho \,|\, \eta \rangle\,\mathrm{e}^{-2(\eta\alpha^* - \eta^*\alpha)}$$

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\,\langle \alpha \,|\, \rho \,|\, \alpha \rangle$$

Correlation Functions

$$\hat{E}^+(\vec{r}, t) = \mathrm{i}\epsilon_0/2a\mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\vec{r})} \qquad \hat{E}^- = \hat{E^+}^\dagger$$

$$g^{(1)}(\tau) = \frac{\langle E^-(t)E^+(t+\tau)\rangle}{\langle E^-(t)E^+(t)\rangle}$$

$$g^{(2)}(\tau) = \frac{\langle E^-(t)E^-(t+\tau)E^+(t+\tau)E^+(\tau)\rangle}{\langle E^-(t)E^+(t)\rangle^2}$$

$$Q = \langle \hat{n} \rangle\,(g^{(2)}(0) - 1)$$

Bloch Equations

$$\sigma_+ = |e\rangle\,\langle g| \quad \sigma_- = |g\rangle\,\langle e| \quad \sigma_3 = |e\rangle\,\langle e| - |g\rangle\,\langle g|$$

$$u = 2\Re(\rho_{12}) \qquad v = 2\Im(\rho_{12}) \qquad w = \rho_{11} - \rho_{22}$$

$W = \Omega\vec{e}_1 + \Delta\vec{e}_3$ Precession around this vector

$$\Delta = \omega_L - \omega_0$$