Beltrami Identity

1 Introduction

The Beltrami Identity is a reduction of the Euler-Lagrange Equation where the function, F, does not explicitly depend on x ($\frac{\partial F}{\partial x} = 0$) in the functional

$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$

The Beltrami Identity states that, if F is autonomous, then the solution satisfies

$$F - y'\frac{\partial F}{\partial y'} = C, \text{ where } C \text{ is a constant}$$

2 Proof

First, we will state the Euler-Lagrange Equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\implies y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ (Equation A)}$$

Now, we will take the total derivative of F with respect to x (noting that y and y' are both functions of x) by the Chain Rule:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

$$\implies y'\frac{\partial F}{\partial y} = \frac{dF}{dx} - \frac{\partial F}{\partial x} - y''\frac{\partial F}{\partial y'} \text{ (Equation B)}$$

Substituting Equation B into Equation A gives:

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - y'' \frac{\partial F}{\partial y'} - y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{dF}{dx} - \frac{\partial F}{\partial x} - \left[y'' \frac{\partial F}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = 0$$

The expression in the square brackets can be simplified by applying the Inverse Chain Rule

$$\implies \frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{dF}{dx} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

Given that F does not explicitly depend on x, $\frac{\partial F}{\partial x} = 0$

$$\implies \frac{dF}{dx} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{d}{dx}\left(F - y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\implies F - y' \frac{\partial F}{\partial y'} = C$$

Which is the Beltrami Identity.