The ℓ^p Norms

1 The ℓ^p Norms

The ℓ^p Norms are a set of norms on $\mathbb{R}^n$. For $1 \leq p < \infty$, the ℓ^p Norms are defined as:

$$||x||_{\ell^p} := \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}}$$

The Norm ℓ^∞ is also defined, being

$$||x||_{\ell^\infty} := \max_{j=1,\ldots,n} |x_j|$$

This definition comes from the fact that, by the above definitions, $\lim_{p\to\infty} ||x||_{\ell^p} = ||x||_{\ell^\infty}$.

To prove that these are indeed norms, they must be tested against the three conditions (discussed in the Introducing Norms section).

For the first criterion, for $1 \leq p < \infty$, the ℓ^p norm can only be zero if x is zero since the sum $\sum_{j=1}^{n} |x_j|$ is increasing. For $p = \infty$, $||x||_{\ell^\infty}$ is only zero if $\max_{j=1,\ldots,n} |x_j| = 0$. Since $|x_j| \geq 0$, if $||x||_{\ell^\infty} = 0$ then all $|x_j| = 0$, and thus $x = 0$. Therefore, for all $1 \leq p \leq \infty$, $||x||_{\ell^p} = 0 \iff x = 0$.

For the second, the proof is very similar to the proof for the Euclidean Norm (Introducing Norms) which is natural as the Euclidean Norm is the ℓ^2 Norm. First, for $1 \leq p < \infty$. Suppose that $\lambda \in \mathbb{R}$. For all $x \in \mathbb{R}^n$,

$$||\lambda x||_{\ell^p} = \left(\sum_{j=1}^{n} |\lambda x_j|^p \right)^{\frac{1}{p}}$$

$$\implies ||\lambda x||_{\ell^p} = \left(\sum_{j=1}^{n} |\lambda|^p |x_j|^p\right)^{\frac{1}{p}}$$

$$\implies ||\lambda x||_{\ell^p} = |\lambda| \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$$

$$\implies ||\lambda x||_{\ell^p} = |\lambda| ||x||_{\ell^p}$$

Therefore, for $1 \leq p < \infty$, the ℓ^p Norms satisfy Homogeneity.

For $p = \infty$, the fact is more direct.

$$||\lambda x||_{\ell^\infty} = \max_{j=1,\ldots,n} |\lambda x_j|$$

$$\implies ||\lambda x||_{\ell^\infty} = |\lambda| \max_{j=1,\ldots,n} |x_j| = |\lambda| ||x||_{\ell^\infty}$$

Therefore, for all $1 \leq p \leq \infty$, the ℓ^p Norms satisfy the first two criteria.

2 Minkowski's Inequality In $\mathbb{R}^n$

Minkowski's Inequality in $\mathbb{R}^n$ states that for all $1 \leq p \leq \infty$, if $x, y \in \mathbb{R}^n$ then $||x + y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$

Note that Minkowski's Inequality deals with the third criterion for the ℓ^p norms.

First, we will consider the case of $p = \infty$. For every $x, y \in \mathbb{R}^n$,

$$||x + y||_{\ell^\infty} = \max_{j=1,\ldots,n} (x_j + y_j)$$

$$\implies ||x + y||_{\ell^\infty} \leq \max_{j=1,\ldots,n} x_j + \max_{j=1,\ldots,n} y_j$$

$$\implies ||x + y||_{\ell^\infty} \leq ||x||_{\ell^\infty} + ||y||_{\ell^\infty}$$

Thus Minkowski's Inequality holds for $p = \infty$.

When considering the case of $1 \leq p \infty$, let $x, y \in \mathcal{B}$.

$$||\lambda x + (1-\lambda)y||_{\ell^p}^p = \sum_{j=1}^{n} |\lambda x_j + (1-\lambda)y_j|^p$$

Applying the Triangle Inequality (for absolute values), we can see that

$$\implies ||\lambda x + (1-\lambda)y||_{\ell^p}^p \leq \sum_{j=1}^{n} (\lambda |x_j|^p + (1-\lambda)|y_j|^p)$$

Since $x, y \in \mathcal{B}$, $\sum_{j=1}^{n} |x_j|^p \leq 1, \sum_{j=1}^{n} |y_j|^p \leq 1$.

$$\implies ||\lambda x + (1-\lambda)y||_{\ell^p}^p \leq \lambda + (1-\lambda) = 1$$

Therefore, the Closed Unit Ball, $\mathcal{B}$, is convex. By applying the lemma introduced in Normed Spaces, since for $1 \leq p < \infty$ the Closed Unit Ball is convex and the ℓ^p Norm meets the first two criteria and thus satisfies the Triangle Identity.

Finally, we can conclude that for all $1 \leq p \leq \infty$, for all $x, y \in \mathbb{R}^n$, $||x + y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$ as stated in Minkowski's Inequality.

3 Conclusion

Since Minkowski's Inequality holds for all $1 \leq p \leq \infty$ (meaning $||x + y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$), and for all $1 \leq p \leq \infty$ the first two criteria holds. Therefore, for all $1 \leq p \leq \infty$, $|| \cdot ||_{\ell^p}$ is a norm.