Introducing Norms

1 Definition Of Norms

A norm of a vector space can be thought of as defining the length of any vector in that vector space.

A norm on a vector space X is a map, $||\cdot|| : X \to \mathbb{R}$ satisfying:

1. $||x|| = 0 \iff x = 0$
2. $||\lambda x|| = |\lambda|||x||$, for all $\lambda \in \mathbb{R}$ and $x \in X$
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$

The second criterion is commonly referred to as "Homogeneity", and the third is the Triangle Inequality.

It is usual for a vector space to have a typical (standard) norm.

2 Euclidean Norm

The Euclidean Norm is the standard norm of $\mathbb{R}^n$. It is usually denoted in one of three ways: $||\cdot||, ||\cdot||_2, ||\cdot||_{\ell^2}$. The Euclidean Norm, $||\cdot||_2 : \mathbb{R}^n \to \mathbb{R}$, is defined as

$$||x||_2 := \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}}$$

Proving that it is a norm is fairly simple. For the first criteria, this is true since all $|x_j| \geq 0$ and $|x_j| = 0 \implies x_j = 0$, and thus the Euclidean Norm is zero if and only if $x = 0$.

To prove Homogeneity, suppose $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. We have that

$$||\lambda x||_2 = \left(\sum_{j=1}^{n} |\lambda x_j|^2 \right)^{\frac{1}{2}}$$

$$\implies ||\lambda x||_2 = \left(\sum_{j=1}^{n} |\lambda|^2 |x_j|^2 \right)^{\frac{1}{2}}$$

$$\implies ||\lambda x||_2 = |\lambda| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} = |\lambda| ||x||_2$$

Proving the Triangle Inequality is typically the hardest condition to prove, but it is not too much trouble for the Euclidean Norm. Note that for $x, y \in \mathbb{R}^n$, $||x+y||_2^2 = (x+y) \cdot (x+y)$ and thus

$$||x+y||_2^2 = ||x||_2^2 + 2(x \cdot y) + ||y||_2^2$$

$$\implies ||x+y||_2^2 \leq ||x||_2^2 + 2||x||_2||y||_2 + ||y||_2^2$$

$$\implies ||x+y||_2^2 \leq (||x||_2 + ||y||_2)^2$$

Since the Euclidean Norm satisfies all criteria, it is a norm as expected.

3 Other Norms On $\mathbb{R}^n$

The Euclidean Norm, whilst the standard norm on $\mathbb{R}^n$, is not the only norm on $\mathbb{R}^n$. As will be explored in the section on ℓ^p Norms, there exist norms denoted by $|| \cdot ||_{\ell^p}$ on $\mathbb{R}^n$.

One instance of these ℓ^p Norms is

$$||x||_{\ell^1} := \sum_{j=1}^{n} |x_j|$$