# Normed Spaces

## 1 Definition Of A Normed Space

If $X$ is a vector space and $||\cdot||$ is a norm on $X$ then the pair $(X, ||\cdot||)$ is said to be a Normed Space.

It is common to simply refer to the "normed space of $X$" when referring to the normed space $(X, ||\cdot||)$ where $||\cdot||$ is the typical norm on $X$. For instance, when dealing with $\mathbb{R}^n$, it is usual to refer to $(\mathbb{R}^n, ||\cdot||_2)$ as "the normed space of $\mathbb{R}^n$.

## 2 Convex Sets

Let $X$ be a vector space. A subset of $X$, $K \subset X$, is said to be convex if when $x, y \in K$ and $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)y \in K$.

Less formally, $K$ is convex if the line segment connecting any two points in $K$ is fully contained within $K$.

## 3 Closed (Unit) Ball In $(X, ||\cdot||)$

The Closed (Unit) Ball in $(X, ||\cdot||)$ is defined as

$$\mathcal{B}_X := \{x \in X : ||x|| \leq 1\}$$

Lemma: In any normed space $(X, ||\cdot||)$, the Closed Unit Ball $\mathcal{B}_X$ is convex.

To prove this, suppose that $(X, ||\cdot||)$ is a normed space with $\mathcal{B}_X$ the Closed Unit Ball in the normed space.

Let $x, y \in \mathcal{B}_X$.By definition, $||x|| \leq 1, ||y|| \leq 1$. For any $0 \leq \lambda \leq 1$, we have

$$||\lambda x + (1-\lambda)y|| \leq |\lambda| ||x|| + |1-\lambda| ||y|| \leq \lambda + (1-\lambda) = 1$$

Since $||\lambda x + (1-\lambda)y|| \leq 1$, it must be in $\mathcal{B}_X$, and thus $\mathcal{B}_X$ is convex.

# 4 Balls Prove Norms

Lemma: Suppose that a function, $N : X \to \mathbb{R}^+$, satisfies the first two criteria to be a norm (discussed in the Introducing Norms section), and the set $\mathcal{B} := \{x \in X : N(x) \leq 1\}$ is convex. Then $N$ satisfies the Triangle Inequality and is thus a norm.

Proof:

Suppose that the function $N : X \to \mathbb{R}^+$ satisfies the above conditions, and $\mathcal{B} := \{x \in X : N(x) \leq 1\}$.

Let $x, y \in X$ such that $N(x) > 0, N(y) > 0$. The points $\frac{x}{N(x)}, \frac{y}{N(y)} \in \mathcal{B}$. Since $\mathcal{B}$ is convex, we know that

$$\frac{N(x)}{N(x)+N(y)} \left( \frac{x}{N(x)} \right) + \frac{N(y)}{N(x)+N(y)} \left( \frac{y}{N(y)} \right) \in \mathcal{B}$$

By Homogeneity, we know that $N\left(\frac{1}{N(x)+N(y)}k\right) = \frac{1}{N(x)+N(y)}N(k)$ for all $k \in X$.

$$\implies N\left(\frac{x+y}{N(x)+N(y)}\right) = \frac{N(x+y)}{N(x)+N(y)} \leq 1$$
$$\implies N(x+y) \leq N(x) + N(y)$$

And thus $N$ satisfies the Triangle Inequality. Further, since $N$ satisfies all three conditions to be a norm, $N$ is a norm of $X$.

Note that when trying to apply this lemma, it is often necessary to show that a real-valued function is convex. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for all $x, y \in [a, b]$, $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$. A sufficient condition for $f \in C^2((a, b)) \cap C^1([a, b])$ is that $f''(c) \geq 0$ for all $x \in (a, b)$.