p-Test For Series

1 Introduction

The p-Test For Series is a condition for the convergence of series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. If $p > 1$, the series converges; if $p \leq 1$, the series diverges.

2 Proof

The method by which this will be proven is the Integral Test.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if the following integral converges (and diverges if the integral diverges),

$$\int_1^\infty \frac{dx}{x^p}$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p}$$
$$\implies \int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_1^t$$
$$\implies \int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \frac{1}{1-p} t^{1-p} - \frac{1}{1-p}$$

The right side converges if $p > 1$ since the exponent of t will be less than zero (and therefore $\lim_{t\to\infty} t^{1-p} converges$). The right side diverges if $p < 1$ since the exponent of t will be greater than 0 (the limit diverges by similar logic).

For $p = 1$, the integral instead evaluates to $\lim_{t\to\infty} \ln |t|$ which diverges, and thus the integral diverges if $p \leq 1$ (the divergence of this particular series is expanded upon in the section on the Harmonic Series).

Since the integral diverges if $p \leq 1$ and converges if $p > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ by the Integral Test.