Mass-Spring System

1   Introduction

In this article, we will derive, solve, and non-dimensionalise the equations
governing the behaviour of the mass-spring system. The mass-spring system
is comprised of a mass attached to the end of a spring. The movement of this
spring is dependent on the initial displacement and velocity of the spring.
The second-order linear ODE governing this system's behaviour is:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

2   Derivation

There are three forces that will be acting on the system, the first of which
is the spring force, F_s, which is determined by Hooke's Law. Hooke's Law
states that $F_s = -kx$, where k is the spring constant and x is the spring's
compression/stretch. The second force is friction, which can be modeled
by $F_f = -cv$, where c is a constant. Note that these two forces are nega-
tive since friction opposes motion and the spring pulls when the spring is
stretched (positively displaced). Since velocity is the derivative of displace-
ment (x), we can express friction as $F_f = -c\dot{x}$. Finally, we have external
forces not dependent on the displacement, which we shall simply denote by
$f(t)$.

Since forces are additive, the total force acting on the system, F, is $F =
-kx - c\dot{x} + f(t)$. Recalling Newton's second law, $F = ma$, we can write this
as $ma = -kx - c\dot{x} + f(t)$. Rewriting a as the second derivative of displace-
ment, we can see that $m\ddot{x} + c\dot{x} + kx = f(t)$.

The function $f(t)$ can be anything, but it is often modeled as $F_0 \cos(\omega t)$.
For a vertical system where the mass is hanging from the spring, we may
wish to have $f(t) = mg$ to represent the force of friction. Note that the

particular solution of this ODE would be constant ($x_p = \frac{mg}{k}$), meaning that we can set this to zero by a co-ordinates change (equivalent to changing the point of equilibrium in the system).

Often, friction is negligible in the model. If so the differential equation is written as $\ddot{x} = \frac{k}{m}x$.

3   Solving The System

The system is a linear second-order ODE, so the homogeneous solution is simple to find. The auxiliary (characteristic) equation is $mr^2 + cr + k = 0$. Solving this, we get $r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$. If $c^2 > 4mk$, we can express the homogeneous solution as $x_c = Ae^{\frac{-c+\sqrt{c^2-4mk}}{2m}t} + Be^{\frac{-c-\sqrt{c^2-4mk}}{2m}t}$, where A and B are constants that can be solved for with initial conditions. If $c^2 < 4mk$, we can instead write the homogeneous solution as $x_c = e^{\frac{-c}{2m}t}(A\cos\frac{\sqrt{4mk-c^2}}{2m}t + B\sin\frac{\sqrt{4mk-c^2}}{2m}t)$. The first case is called overdamped, the second is underdamped. In addition to those two cases, there is also the critically damped, which is when $c^2 = 4mk$. This case can be expressed like the first, however, it has different properties. Note how in both underdamped and overdamped systems, there is a factor of $e^{-\frac{c}{2m}}$. As t increases, this factor decreases, meaning that the overall displacement is dampened as time increases. This is dependent on c which should be expected as it was the coefficient of friction. The non-homogeneous case can be dealt with by applying standard methods to get a particular solution.

Suppose we have initial conditions $x(0) = x_0$ and $\dot{x}(0) = u_0$. Let us first consider the underdamped case. Since $x = e^{\frac{-c}{2m}t}(A\cos\frac{\sqrt{4mk-c^2}}{2m}t + B\sin\frac{\sqrt{4mk-c^2}}{2m}t)$, we can write the derivative of x as

$$\dot{x} = -\frac{c}{2m}x + e^{\frac{-c}{2m}t}\left(-\frac{\sqrt{4mk-c^2}}{2m}A\sin\frac{\sqrt{4mk-c^2}}{2m}t + \frac{\sqrt{4mk-c^2}}{2m}B\cos\frac{\sqrt{4mk-c^2}}{2m}t\right)$$

Evaluating both of these at $t = 0$ gives $x(0) = x_0 = A$, and $\dot{x}(0) = u_0 = -\frac{c}{2m}x_0 + \frac{\sqrt{4mk-c^2}}{2m}B$. Simplifying these, we get that $A = x_0$ and $B = \frac{cx_0}{\sqrt{4mk-c^2}}$.

For the overdamped case, we have a system of linear equations.

2

1. $x_0 = A + B$

2. $u_0 = \frac{-c+\sqrt{c^2-4mk}}{2m}A + \frac{-c-\sqrt{c^2-4mk}}{2m}B$

Solving this is straightforward forward so it is left as an exercise.

4 Non-Dimensionalisation

In this section, we will non-dimensionalise two cases. The first is the homogeneous case, We wish to convert the variables x, t to new variables $\tilde{x}, \tilde{t}$. To do so, we need to find x_r, x_s, t_r, t_s in terms of m, k, c, F_0, and ω such that they satisfy the relationship $\tilde{x} = \frac{x-x_r}{x_s}$, $\tilde{t} = \frac{t-t_r}{t_s}$. For the purposes of this section, we do not wish to shift the variables, only scale them, so we will set x_r and t_r to zero. We are left with the following relationships: $x_s\tilde{x} = x$ and $t_s\tilde{t} = t$. Substituting these into our original ODE, we get

$$m\frac{d}{d(\tilde{t}t_s)}\left(\frac{d(\tilde{x}x_s)}{d(\tilde{t}t_s)}\right) + c\frac{d(\tilde{x}x_s)}{d(\tilde{t}t_s)} + k(\tilde{x}x_s) = 0$$

$$\implies \frac{mx_s}{t_s^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{cx_s}{t_s}\frac{d\tilde{x}}{d\tilde{t}} + k(\tilde{x}x_s) = 0$$

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{ct_s}{m}\frac{d\tilde{x}}{d\tilde{t}} + \frac{kt_s^2}{m}\tilde{x} = 0$$

To reduce the number of coefficients, we can set one to 1. Choosing the latter, we get $\frac{kt_s^2}{m} = 1$, meaning that $t_s = \sqrt{\frac{m}{k}}$. Since there is no x_s in the final equation, we can let $x = \tilde{x}$ (for the purposes of the non-homogeneous case, we shall leave it as $\tilde{x}$). Substituting back into the equation gives us the ODE in non-dimensionalised form:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = 0$$

Let us now consider the non-homogenous case, $f(t) = F_0\cos\omega t$. Following through the same steps from before, we get

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \frac{F_0t_s^2}{mx_s}\cos\omega t_s\tilde{t}$$

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \frac{F_0}{kx_s}\cos\omega\sqrt{\frac{m}{k}}\tilde{t}$$

Taking the coefficient of $\cos\omega\sqrt{\frac{m}{k}}t$ and setting it to 1, we get $\frac{F_0}{kx_s} = 1$, meaning that $\frac{F_0}{k} = x_s$. Substituting this back in results in

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \cos\omega\sqrt{\frac{m}{k}}\tilde{t}$$

Finally, we let $\alpha := \frac{c}{\sqrt{mk}}$ and $\beta := \omega\sqrt{\frac{m}{k}}$ (with $x = \frac{F_0}{k}$ and $t = \sqrt{\frac{m}{k}}$). This leaves us with only two parameters in the equation compared with the original's five. The non-dimensionalised differential equation is then:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \alpha\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \cos\beta\tilde{t}$$

Doing this gives us a better understanding of the relationship between parameters (in a dimensionless group) and how they affect the behaviour of the system.