Shift Rule For Series

1 Introduction

The Shift Rule is a simple rule for the convergence of series. It states that, letting $N \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_{N+n}$ converges.

2 Proof

First, we will prove that $\sum_{n=1}^{\infty} a_{N+n}$ converges if $\sum_{n=1}^{\infty} a_n$ converges.

Suppose $\sum_{n=1}^{\infty} a_n$ converges, and let A denote the value to which $\sum_{n=1}^{\infty} a_n$ converges.

Note that the series can be rewritten as

$$\sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{N+n}$$

This means that the sum we know converges can be expressed as

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

Since the sum from 1 to N is the sum of a finite number of finite terms, the sum itself must be finite. Let C denote the value of this sum. Since the left side of the equality converges, we can replace it with its value to get

$$A = C + \sum_{n=N+1}^{\infty} a_n$$

$$\implies \sum_{n=N+1}^{\infty} a_n = A - C$$

Since the left side is also $\sum_{n=1}^{\infty} a_{N+n}$,

$$\sum_{n=1}^{\infty} a_{N+n} = A - C$$

$A - C$ is finite, and thus $\sum_{n=1}^{\infty} a_{N+n}$ converges if $\sum_{n=1}^{\infty} a_n$ converges.

Now to prove that $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} a_{N+n}$ converges.

Suppose $\sum_{n=1}^{\infty} a_{N+n}$ converges, and let B denote the value to which it converges.

As shown before, the two sums can be expressed as

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$\implies \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=1}^{\infty} a_{N+n}$$

As before, let C denote the value of $\sum_{n=1}^{N} a_n$ which is finite.

$$\implies \sum_{n=1}^{\infty} a_n = C + B$$

Therefore, if $\sum_{n=1}^{\infty} a_{N+n}$ converges, then so does $\sum_{n=1}^{\infty} a_n$.