Answers:
Separation Of Variables

1 Questions

1. Solve the following differential questions for $y(x)$:

   (a) $\frac{dy}{dx} = 2\left(\frac{y}{x}\right)$

   (b) $\frac{dy}{dx} = (1-x)y$

   (c) $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$

   (d) $\cos x \frac{dy}{dx} = y^2$

   (e) $\frac{dy}{dx} = xy + x + y + 1$

   (f) $\frac{dy}{dx} = x^2 \sec 2y$

   (g) $\frac{dy}{dx} = \cos^2 y \sin x$

   (h) $\frac{dy}{dx} = \sec y \csc y \tan x$

2. Solve the following initial value problems:

   (a) $\frac{dy}{dx} = \frac{1-2x}{y}$, $y(1) = -2$

   (b) $\frac{dy}{dx} = \frac{2x+\sec^2 x}{2y}$, $y(0) = -5$

   (c) $\frac{dy}{dx} = \cos x e^{y+\sin x}$, $y(0) = 0$

   (d) $\frac{dy}{dx} = \frac{\arcsin x}{y^2\sqrt{1-x^2}}$, $y(0) = 0$

2 Answers

1. (a) Separating the variables gives $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$. Integrating both sides with respect to x results in $\int \frac{1}{y}dy = 2\int \frac{1}{x}dx$. Therefore, $\ln|y| = 2\ln|x| + C$. Raising e to the power of both sides gives $|y| = Ae^{2\ln|x|}$. Note that $e^{2\ln|x|} = |x|^2$. This can further be simplified to x^2. This gives $|y| = Ax^2$. Since $x^2 \geq 0$, the sign of y is determined entirely by the sign of A. Because of this, we can drop the absolute value, giving the final answer of $y(x) = Ax^2$, where A is a constant.

   We can check this by taking the derivative with respect to x, which gives $\frac{dy}{dx} = 2Ax$. Note that $x = \frac{x^2}{x}$, meaning that $\frac{dy}{dx} = 2\left(\frac{Ax^2}{x}\right)$. Finally, substituting out Ax^2 for y, we get $\frac{dy}{dx} = 2\left(\frac{y}{x}\right)$, and thus $y(x) = Ax^2$ satisfies the differential equation.

   (b) Separating the variables gives $\frac{1}{y}\frac{dy}{dx} = 1 - x$. Integrating both sides with respect to x gives $\int \frac{1}{y}dy = \int(1-x)dx$. Therefore $\ln|y| = x - \frac{1}{2}x^2 + C$. Raising e to the power of both sides results in $|y| = Ae^{x-\frac{1}{2}x^2}$. $e^k > 0$, and so the sign of y is constant (determined by A) and thus the absolute value can be dropped. This gives a final answer of $y(x) = Ae^{x-\frac{1}{2}x^2}$.

   We can check this by taking the derivative with respect to x, which can be done via the Chain Rule. Let $u = x - \frac{1}{2}x^2$, meaning that $\frac{du}{dx} = 1 - x$. Writing y as a function of u gives $y(u) = Ae^u$, meaning that $\frac{dy}{du} = Ae^u = y(u)$. Applying the Chain Rule gives $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$. Substituting in our values of $\frac{du}{dx}$ and $\frac{dy}{du}$ gives $\frac{dy}{dx} = (1-x)Ae^u$. Substituting $x - \frac{1}{2}x^2 = u$ leads to $\frac{dy}{dx} = (1-x)Ae^{x-\frac{1}{2}x^2}$. Substituting in our solution for $y(x) = Ae^{x-\frac{1}{2}x^2}$ gives $\frac{dy}{dx} = (1-x)y$, and thus $y = Ae^{x-\frac{1}{2}x^2}$ satisfies the differential equation.

   (c) Separating the variables gives $y\frac{dy}{dx} = \frac{x^2}{1+x^3}$. Integrating both sides with respect to x gives $\int ydy = \int \frac{x^2}{1+x^3}dx$. To find the right side we can apply the substitution $u = 1+x^3$, meaning that $\frac{1}{3}du = x^2dx$. Making this substitution gives $\int ydy = \frac{1}{3}\int \frac{1}{u}du$. That gives $\frac{1}{2}y^2 = \frac{1}{3}\ln|u| + C$.Substituting back in $u = 1 + x^3$ gives $\frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + C$. Rearranging for y gives $y(x) = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + A}$.

   We can check this by taking the derivative with respect to x, which can be done via the Chain Rule. Let $u = \frac{2}{3}\ln|1+x^3| + A$, meaning that $\frac{du}{dx} = \frac{2}{3}\frac{d}{dx}\left[\ln|1+x^3|\right]$. This derivative also has to be computed with Chain Rule. Let $v = 1 + x^3$ and therefore $\frac{dv}{dx} = 3x^2$ and $w = \ln|v|$, meaning that $\frac{dw}{dv} = \frac{1}{v}$. Applying the Chain Rule, $\frac{dw}{dx} = \frac{dv}{dx}\frac{dw}{dv}$ and so $\frac{dw}{dx} = \frac{3x^2}{1+x^3}$. Substituting this into the equation for $\frac{du}{dx}$ gives $\frac{du}{dx} = \frac{2x^2}{(1+x^3)}$. $y = \pm\sqrt{u}$ which implies that $\frac{dy}{du} = \pm\frac{1}{2\sqrt{u}}$. Applying the Chain Rule ($\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$) gives $\frac{dy}{dx} = \frac{x^2}{(1+x^3)(\pm\sqrt{\frac{2}{3}\ln|1+x^3|+A})}$. The second bracket of the denominator is the equation $y(x)$ and thus $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$, meaning that $y(x) = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + A}$ satisfies the differential equation.

   (d) Separating the variables gives $\frac{1}{y^2}\frac{dy}{dx} = \sec x$. Integrating with respect to x gives $\int \frac{1}{y^2}dy = \int \sec x dx$. The right side is a difficult integral. To solve it, multiply $\sec x$ by $\frac{\sec x + tan x}{\sec x + \tan x}$ $(= 1)$. This gives $\int \frac{1}{y^2}dy = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$, which simplifies to $\int \frac{1}{y^2}dy = \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x}dx$. Let $u = \sec x + \tan x$, $du = (\tan x \sec x + \sec^2 x)dx$. Substituting this into the right integral gives $\int \frac{1}{y^2}dy = \int \frac{1}{u}du$. Evaluating the integral gives $\frac{-1}{y} = \ln|u| + C$, or $\frac{-1}{y} = \ln|\sec x + \tan x| + C$. Rearranging for $y(x)$

gives $y(x) = \frac{-1}{\ln|\sec x + \tan x| + C}$.

This solution can be checked by differentiating with respect to x with the Quotient Rule. Recall that the Quotient Rule states that $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. Letting $u = -1$ and $v = \ln|\sec x + \tan x| + C$, we get $\frac{dy}{dx} = \frac{v\frac{d}{dx}(-1) - (-1)\frac{d}{dx}(\ln|\sec x + \tan x| + C)}{(\ln|\sec x + \tan x| + C)^2}$. Note that $\frac{d}{dx}(-1) = 0$ and so the left term on the numerator is zero. To calculate the remaining derivative on the top, we use the Chain Rule. Let $m = \sec x + \tan x$, $\frac{dm}{dx} = \tan x \sec x + \sec^2 x$. Let $w = \ln|m| + C$, so $\frac{dw}{dm} = \frac{1}{m}$. Combining these via the Chain Rule gives $\frac{d}{dx}(\ln|\sec x + \tan x| + C) = \frac{\tan x \sec x + \sec^2 x}{\sec x + \tan x}$ which simplifies to $\frac{d}{dx}(\ln|\sec x + \tan x| + C) = \sec x$. Substituting this into $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \frac{\sec x}{(\ln|\sec x + \tan x| + C)^2}$. Noting that $y(x)^2 = \frac{1}{(\ln|\sec x + \tan x| + C)^2}$, $\frac{dy}{dx} = \sec x y^2$. Alternatively, to fit the format of the original differential equation, $\cos x \frac{dy}{dx} = y^2$, and thus $y(x) = \frac{-1}{\ln|\sec x + \tan x| + C}$ satisfies the differential equation.

(e) The differential equation can be factorised to $\frac{dy}{dx} = (x+1)(y+1)$. Separating the variables gives $\frac{1}{y+1}\frac{dy}{dx} = (x+1)$. Integrating with respect to x gives $\int \frac{1}{y+1} dy = \int (x+1) dx$. This gives $\ln|y+1| = \frac{1}{2}x^2 + x + C$. Raising e to the power of both sides gives $|y+1| = Ae^{\frac{1}{2}x^2+x}$. Since $e^k > 0$, we can drop the absolute value, giving $y(x) = -1 + Ae^{\frac{1}{2}x^2+x}$.

To verify this, differentiate with respect to x. $\frac{dy}{dx} = A\frac{d}{dx}\left(e^{\frac{1}{2}x^2+x}\right)$. Let $u = \frac{1}{2}x^2 + x$, $\frac{du}{dx} = x + 1$. $z = e^u$ and so $\frac{dz}{du} = e^u$. Applying the Chain Rule gives $\frac{dz}{dx} = (x+1)e^{\frac{1}{2}x^2+x}$. This gives $\frac{dy}{dx} = (x+1)Ae^{\frac{1}{2}x^2+x}$, which can be rewritten as $\frac{dy}{dx} = (x+1)(y+1)$, and thus $y(x) = -1 + Ae^{\frac{1}{2}x^2+x}$ solves the differential equation.

(f) Separating the variables, $\cos 2y \frac{dy}{dx} = x^2$. Integrating with respect to x gives $\int \cos 2y dy = \int x^2 dx$. Therefore $\frac{1}{2}\sin 2y = \frac{1}{3}x^3 + \frac{1}{2}C$. Rearranging for y gives $y(x) = y = \frac{1}{2}\arcsin\left(\frac{2}{3}x^3 + C\right)$. Verify this by graphing the differential equation.

(g) Separating the variables, $\sec^2 y \frac{dy}{dx} = \sin x$. Integrating with respect to x results in $\int \sec^2 y dy = \int \sin x dx$. This means that $\tan x = C - \cos x$. Rearranging for y, $y(x) = \arctan(C - \cos x)$.

(h) Separating the variables, $\sin x \cos x \frac{dy}{dx} = \tan x$. Integrating with respect to x results in $\int \sin x \cos x dx = \int \tan x dx$. This implies that $\frac{1}{2}\int \sin 2y dx = \ln|\sec x| + C$ (by applying the double angle formula). Therefore, $-\frac{1}{4}\cos 2y = \ln|\sec x| + C$. Rearranging for y gives $y(x) = \frac{1}{2}\arccos(4\ln\cos x + C)$.

2. (a) Separating the variables, $y\frac{dy}{dx} = 1 - 2x$. Integrating with respect to x gives $\int y dy = \int 1 - 2x dx$, and thus $\frac{1}{2}y^2 = x - x^2 + \frac{1}{2}C$. Rearranging for y results in $y = \pm\sqrt{2x - 2x^2 + C}$.

To solve for C, we have to consider the initial condition. Letting $y(1) = -2$, $-2 = \pm\sqrt{C}$. Since square roots are greater than or equal to zero, the sign in front of the root is $-$. This means that $2 = \sqrt{C}$ and thus $C = 4$. This means that the equation satisfying both the initial condition and the differential equation is $y(x) = -\sqrt{2x - 2x^2 + 4}$. To check that it does fit the differential equation, take the derivative with respect to x. This gives $\frac{dy}{dx} = \frac{2-4x}{-2\sqrt{2x-2x^2+4}}$ which can be reduced to $\frac{dy}{dx} = \frac{1-2x}{y}$, as expected.

(b) Separating the variables, $2y\frac{dy}{dx} = 2x + \sec^2 x$. Integrating with respect to x, $\int 2ydy = \int 2x + \sec^2 xdx$. Therefore $y^2 = x^2 + \tan x + C$. Rearranging for y, $y = \pm\sqrt{x^2 + \tan x + C}$.

Applying the initial condition, $y(0) = -5$ and so $-5 = \pm\sqrt{C}$. Since square roots are always greater than or equal to zero, the sign in front of the root must be $-$. This means that $25 = C$ and thus the equation satisfying both the differential equation and initial condition is $y(x) = -\sqrt{x^2 + \tan x + 25}$. To verify this, differentiate with respect to x, giving $\frac{dy}{dx} = \frac{2x+\sec^2 x}{-2\sqrt{x^2+\tan x+25}}$. The denominator can be replaced with $2y$, giving $\frac{dy}{dx} = \frac{2x+\sec^2 x}{2y}$, as expected.

(c) Separating the variables, $e^{-y}\frac{dy}{dx} = \cos xe^{\sin x}$. Integrating both sides with respect to x, $\int e^{-y}dy = \int \cos xe^{\sin x}dx$. Let $u = \sin x$, $du = \cos xdx$. Making this substitution into the right integral gives $\int e^{-y}dy = \int e^u du$ and thus $-e^{-y} = e^{\sin x} - C$. Rearranging for y, $y(x) = -\ln(C - e^{\sin x})$.

To solve for C, consider the initial condition, $y(0) = 0$. $0 = -\ln(C - 1)$, and thus $C = 2$. This gives a solution of $y(x) = -\ln 2 - e^{\sin x}$. This satisfies the initial condition. To check that it satisfies the differential equation, take the derivative with respect to x, $\frac{dy}{dx} = -\frac{d}{dx}[C - e^{\sin x}]\frac{d}{du}[\ln u]$ by the Chain Rule, where $u := C - e^{\sin x}$. For the first derivative, applying the Chain Rule gives $\frac{d}{dx}[C - e^{\sin x}] = -\cos xe^{\sin x}$. The second derivative is simply $\frac{1}{u}$ $(\frac{1}{C-e^{\sin x}})$. Substituting these into the equation for $\frac{dy}{dx} = -\frac{(-\cos xe^{\sin x})}{C-e^{\sin x}}$. Note that $e^y = \frac{1}{C-e^{\sin x}}$, and thus $\frac{dy}{dx} = \cos xe^{\sin x}e^y$. Combining the powers of e, $\frac{dy}{dx} = \cos xe^{y+\sin x}$, and thus $y(x) = -\ln(C - e^{\sin x})$ satisfies the differential equation.

(d) Separating the variables, $y^2\frac{dy}{dx} = \frac{\arcsin x}{\sqrt{1-x^2}}$. Integrating with respect to x, $\int y^2dy = \int \frac{\arcsin x}{\sqrt{1-x^2}}dx$. Let $u = \arcsin x$, and so $du = \frac{1}{\sqrt{1-x^2}}dx$. Substituting this into the right integral gives $\int y^2dy = \int udu$. Therefore $\frac{1}{3}y^3 = \frac{1}{2}(\arcsin x)^2 + \frac{1}{3}C$. Rearranging for y, $y(x) = \sqrt[3]{\frac{3}{2}(\arcsin x)^2 + C}$.

To check that the equation satisfies the differential equation, take the derivative with respect to x. $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{3}{2}(\arcsin x)^2 + C\right]\frac{d}{du}\left[\sqrt[3]{u}\right]$, where $u := \frac{3}{2}(\arcsin x)^2 + C$. Applying the Chain Rule to the left integral gives $\frac{d}{dx}\left[\frac{3}{2}(\arcsin x)^2 + C\right] = \frac{3\arcsin x}{\sqrt{1-x^2}}$. The right integral is $\frac{1}{3\sqrt[3]{u^2}}$ $(\frac{1}{3\sqrt{\frac{3}{2}(\arcsin x)^2+C}^2})$. Substituting these into the equation for $\frac{dy}{dx} = \frac{\arcsin x}{\sqrt{1-x^2}\sqrt[3]{\frac{3}{2}(\arcsin x)^2+C}^2}$. Substituting back in $y(x) = \sqrt[3]{\frac{3}{2}(\arcsin x)^2 + C}$, $\frac{dy}{dx} = \frac{\arcsin x}{y^2\sqrt{1-x^2}}$, as expected.

To solve for C, consider the initial condition, $y(0) = 0$. $0 = \sqrt[3]{C}$, and thus $C = 0$. This means the final solution is $y(x) = \sqrt[3]{\frac{3}{2}(\arcsin x)^2}$