

Separation Of Variables

1 Method

Separation of Variables is a method of solving both ODEs and PDEs (this article will focus on ODEs). The method states that for differential equations of the form $\frac{dy}{dx} = f(x)h(y)$, the following is true:

$$\int \frac{1}{h(y)} dy = \int f(x) dx$$

2 Proof

To prove this, we will first rewrite the original equation as $\frac{dy}{dx} = f(x)h(y(x))$. Clearly, as long as $h(y) \neq 0$, we can divide both sides by $h(y(x))$ (this is why the original equation is sometimes written as $\frac{dy}{dx} = \frac{f(x)}{h(y)}$) giving $\frac{1}{h(y(x))} \frac{dy}{dx} = f(x)$. Both sides can be integrated with respect to x , giving $\int \frac{1}{h(y(x))} \frac{dy}{dx} dx = \int f(x) dx$. Intuitively, we can see that the dx s on the left side cancel however this is not rigorous as $\frac{dy}{dx}$ is not a fraction. Instead, consider the substitution $y = y(x)$. Then, by differentiating with respect to x , $dy = y'(x)dx$. Since this is with respect to x , we can rewrite this as $dy = \frac{dy}{dx} dx$ (this result is a standard use of the inverse chain rule). Finishing the substitution, we get our desired result of $\int \frac{1}{h(y)} dy = \int f(x) dx$.