Divergence Of The Harmonic Series

1 Introduction

The Harmonic Series is a series defined as the sum of the sequence $a_n = \frac{1}{n}$,

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

The most important property of the Harmonic Series is that, despite $(a_n) \to 0$, the series diverges (an important fact when proving the inconclusiveness of the $L = 1$ case of the Ratio Test).

There are multiple methods that can be used to prove the divergence of the Harmonic Series, and only a couple will be shown here.

2 Method 1

Let s_n denote the nth partial sum of the Harmonic Series. That is to say that

$$s_n = \sum_{k=1}^{n} \frac{1}{k}$$

Note that $s_4 - s_2$ is greater than 1/2. This is clearly true since

$$s_4 - s_2 = \frac{1}{3} + \frac{1}{4} > 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

The same is also true for $s_8 - s_4$ as each term of a_n from n=5 to n=8 is greater than or equal to $\frac{1}{8}$, and there are four terms and thus

$$s_8 - s_4 > 4\left(\frac{1}{8}\right) = \frac{1}{2}$$

In general, it is simple to see that for any integer $m \geq 0$, $s_{2^{m+1}} - s_{2^m} > \frac{1}{2}$. Taking the sum of these terms which are greater than $\frac{1}{2}$ gives

$$s_1 + \sum_{k=0}^{m} s_{2^{k+1}} - s_{2^k} \geq s_1 + \sum_{k=0}^{m} \frac{1}{2} = \frac{m+3}{2}$$

Since the left series subtracts the previous partial sum to a power of two, the series can be simplified to

$$\sum_{k=1}^{2^{m+1}} \frac{1}{k} \geq \frac{m+3}{2}$$

Note that as $m \to \infty$, the left sum approaches the Harmonic Series and therefore

$$\lim_{m\to\infty} \sum_{k=1}^{2^{m+1}} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \geq \lim_{m\to\infty} \frac{m+3}{2}$$

Clearly, the right side of the inequality diverges, meaning that the left side must also diverge, and thus the Harmonic Series diverges.

3 Method 2: Integral Test

As seen in the section on the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges if the integral $\int_1^{\infty} \frac{dx}{x}$ diverges.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x}$$

$$\implies \int_1^{\infty} \frac{dx}{x} = \lim_{t\to\infty} [\ln|x|]_{x=1}^{x=t}$$

$$\implies \int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \ln|t| - \ln 1$$

$$\implies \int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \ln|t|$$

$\ln|t| \to \infty$ as $t \to \infty$, and thus the integral $\int_1^\infty \frac{dx}{x}$ diverges. Applying the Integral Test, this means that the Harmonic Series also diverges.