Boundedness Condition

1 Introduction

The Boundedness Condition for series states that for any $a_n \geq 0$, $\sum_{j=1}^{\infty} a_j$ converges if and only if the sequence of partial sums is bounded ($s_n = \sum_{j=1}^{n} a_j$ is bounded).

The condition also applies for $a_n \leq 0$ since a sequence, $b_n = -a_n$, can be constructed such that $\sum_{j=1}^{\infty} b_n = -\sum_{i=1}^{\infty} a_n$ (this fact will not be proved).

2 Proof

Suppose $a_n \geq 0$, and let s_n be the nth partial sum of the series where s_n is bounded.

Since $a_n \geq 0$, and $s_{n+1} = s_n + a_n$, s_n must be increasing ($s_{n+1} \geq s_n$). Since s_n is monotonic and bounded, it is a convergent sequence by the Monotone Convergence Theorem (note that this only depends on the existence of an upper bound, although since $a_n \geq 0$ the lower bound exists and is greater than or equal to 0).

As a result of the convergence of the partial sums, the series must converge.