Comparison Test For Series

1 Introduction

The Comparison Test for series is a method of determining whether a series converges by comparing it to a series that you already know converges / diverges. The Comparison Test states two things.

- If for some $N \in \mathbb{N}$, $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges
- If for some $N \in \mathbb{N}$, $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges

2 Proof

Note that the first statement is the contra-positive of the second, and thus it is sufficient to prove just one statement. In this resource, the first will be proved.

Let B denote the value of the value of $\sum_{n=1}^{\infty} b_n$ with B_n denoting the nth partial sum, and let A_n be the nth partial sum of $\sum_{n=1}^{\infty} a_n$.

Note that, since term $a_n \geq 0$, A_n is increasing ($A_n \leq A_{n+1}$ for all n), meaning that A_n is a monotonic sequence. The same is true for B_n. Since B_n is increasing, clearly $B_n \leq B$ for all n. Further, since $a_n \leq b_n$ for all n, it must also be true that $A_n \leq B$.

This means that the sequence A_n is bounded. Since A_n is both bounded and monotonic, by the Monotone Convergence Theorem, A_n must converge and, as A_n is the sequence representing the nth partial sum of $\sum_{n=1}^{\infty} a_n$, the sum must also converge.