Divergence Test For Series

1 Introduction

The Divergence Test for series is a simple method of checking to see if a series diverges. It can be stated in two ways. The first states that a series, $\sum_{n=1}^{\infty} a_n$, diverges if $\lim_{k\to\infty} a_k \neq 0$. Note that the inverse is not true (the divergence of a series does not imply that the sequence does not converge to zero).

Alternatively, if the series converges then $\lim_{k\to\infty} a_k = 0$. This statement is the contra-positive statement of the first statement, and thus proving one proves the other.

2 Proof

We will prove the second statement which ultimately proves both.

Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series. Let S denote the value to which the series converges ($\sum_{n=1}^{\infty} a_n := S$).

Let s_n denote the nth partial sum of the series, meaning that:

$$s_n = \sum_{k=1}^{n} a_n$$

$$\implies s_{n-1} = \sum_{k=1}^{n-1} a_n$$

Note that the difference between s_n and s_{n-1} is a_n since $s_n - s_{n-1} = (a_1 + a_2 + ... + a_n) - (a_1 + a_2 + ... + a_{n-1}) = a_n$.

Since s_n and s_{n-1} are the nth partial sum of the infinite series,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = S$$

$$\implies \lim_{n\to\infty} [s_n - s_{n-1}] = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$

$$\implies \lim_{n\to\infty} [s_n - s_{n-1}] = S - S = 0$$

As we had previously established, $s_n - s_{n-1} = a_n$, and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} [s_n - s_{n-1}] = 0$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$.