Mass-Spring System

1 Introduction

In this article, we will derive, solve, and non-dimensionalise the equations governing the behaviour of the mass-spring system. The mass-spring system is comprised of a mass attached to the end of a spring. The movement of this spring is dependent on the initial displacement and velocity of the spring. The second-order linear ODE governing this system's behaviour is:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

2 Derivation

There are three forces that will be acting on the system, the first of which is the spring force, F_s, which is determined by Hooke's Law. Hooke's Law states that $F_s = -kx$, where k is the spring constant and x is the spring's compression/stretch. The second force is friction, which can be modeled by $F_f = -cv$, where c is a constant. Note that these two forces are negative since friction opposes motion and the spring pulls when the spring is stretched (positively displaced). Since velocity is the derivative of displacement (x), we can express friction as $F_f = -c\dot{x}$. Finally, we have external forces not dependent on the displacement, which we shall simply denote by $f(t)$.

Since forces are additive, the total force acting on the system, F, is $F = -kx - c\dot{x} + f(t)$. Recalling Newton's second law, $F = ma$, we can write this as $ma = -kx - c\dot{x} + f(t)$. Rewriting a as the second derivative of displacement, we can see that $m\ddot{x} + c\dot{x} + kx = f(t)$.

The function $f(t)$ can be anything, but it is often modeled as $F_0 \cos(\omega t)$. For a vertical system where the mass is hanging from the spring, we may wish to have $f(t) = mg$ to represent the force of friction. Note that the particular solution of this ODE would be constant ($x_p = \frac{mg}{k}$), meaning that we can set this to zero by a co-ordinates change (equivalent to changing the point of equilibrium in the system).

Often, friction is negligible in the model, and so the differential equation is written as $\ddot{x} = \frac{k}{m}x$.

3 Solving The System

The system is a linear second-order ODE, so the homogeneous solution is simple to find. The auxiliary (characteristic) equation is $mr^2 + cr + k = 0$. Solving this, we get $r = \frac{-c \pm \sqrt{(c^2 - 4mk)}}{2m}$. If $c^2 > 4mk$, we can express the homogeneous solution as $x_c = Ae^{\frac{-c+\sqrt{c^2-4mk}}{2m}t} + Be^{\frac{-c-\sqrt{(c^2-4mk)}}{2m}t}$, where A and B are constants that can be solved for with initial conditions. If $c^2 < 4mk$, we can instead write the homogeneous solution as $x_c = e^{\frac{-c}{2m}t}(A\cos\frac{\sqrt{4mk-c^2}}{2m}t + B\sin\frac{\sqrt{4mk-c^2}}{2m}t)$. The first case is called overdamped, the second is underdamped. In addition to those two cases, there is also the critically damped, which is when $c^2 = 4mk$. This case can be expressed like the first, however, it has different properties. Note how in both underdamped and overdamped systems, there is a factor of $e^{-\frac{c}{2m}}$. As t increases, this factor decreases, meaning that the overall displacement is dampened as time increases. This is dependent on c which should be expected as it was the coefficient of friction. The non-homogeneous case can be dealt with by applying standard methods to get a particular solution.

Suppose we have initial conditions $x(0) = x_0$ and $\dot{x}(0) = u_0$. Let us first consider the underdamped case. Since $x = e^{\frac{-c}{2m}t}(A\cos\frac{\sqrt{4mk-c^2}}{2m}t + B\sin\frac{\sqrt{4mk-c^2}}{2m}t)$, we can write the derivative of x as

$$\dot{x} = -\frac{c}{2m}x + e^{\frac{-c}{2m}t}\left(-\frac{\sqrt{4mk-c^2}}{2m}A\sin\frac{\sqrt{4mk-c^2}}{2m}t + \frac{\sqrt{4mk-c^2}}{2m}B\cos\frac{\sqrt{4mk-c^2}}{2m}t\right)$$

Evaluating both of these at $t = 0$ gives $x(0) = x_0 = A$, and $\dot{x}(0) = u_0 = -\frac{c}{2m}x_0 + \frac{\sqrt{4mk-c^2}}{2m}B$. Simplifying these, we get that $A = x_0$ and $B = \frac{cx_0}{\sqrt{4mk-c^2}}$.

For the overdamped case, we have a system of linear equations.

1. $x_0 = A + B$
2. $u_0 = \frac{-c+\sqrt{c^2-4mk}}{2m}A + \frac{-c-\sqrt{c^2-4mk}}{2m}B$

Solving this is straightforward forward so it is left as an exercise.

4 Non-Dimensionalisation

In this section, we will non-dimensionalise two cases. The first is the homogeneous case, We wish to convert the variables x, t to new variables $\tilde{x}, \tilde{t}$. To do so, we need to find x_r, x_s, t_r, t_s in terms of m, k, c, F_0, and ω such that they satisfy the relationship $\tilde{x} = \frac{x-x_r}{x_s}$, $\tilde{t} = \frac{t-t_r}{t_s}$. For the purposes of this section, we do not wish to shift the variables, only scale them, so we will set x_r and t_r to zero. We are left with the following relationships: $x_s\tilde{x} = x$ and $t_s\tilde{t} = t$. Substituting these into our original ODE, we get

$$m\frac{d}{d(\tilde{t}t_s)}\left(\frac{d(\tilde{x}x_s)}{d(\tilde{t}t_s)}\right) + c\frac{d(\tilde{x}x_s)}{d(\tilde{t}t_s)} + k(\tilde{x}x_s) = 0$$

$$\implies \frac{mx_s}{t_s^2}\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{cx_s}{t_s}\frac{d\tilde{x}}{d\tilde{t}} + k(\tilde{x}x_s) = 0$$

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{ct_s}{m}\frac{d\tilde{x}}{d\tilde{t}} + \frac{kt_s^2}{m}\tilde{x} = 0$$

To reduce the number of coefficients, we can set one to 1. Choosing the latter, we get $\frac{kt_s^2}{m} = 1$, meaning that $t_s = \sqrt{\frac{m}{k}}$. Since there is no x_s in the final equation, we can let $x = \tilde{x}$ (for the purposes of the non-homogeneous case, we shall leave it as $\tilde{x}$). Substituting back into the equation gives us the ODE in non-dimensionalised form:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = 0$$

Let us now consider the non-homogenous case, $f(t) = F_0 \cos\omega t$. Following through the same steps from before, we get

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \frac{F_0t_s^2}{mx_s}\cos\omega t_s\tilde{t}$$

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \frac{F_0}{kx_s}\cos\omega\sqrt{\frac{m}{k}}\tilde{t}$$

Taking the coefficient of $\cos\omega\sqrt{\frac{m}{k}}t$ and setting it to 1, we get $\frac{F_0}{kx_s} = 1$, meaning that $\frac{F_0}{k} = x_s$. Substituting this back in results in

$$\implies \frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{c}{\sqrt{mk}}\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \cos\omega\sqrt{\frac{m}{k}}\tilde{t}$$

Finally, we let $\alpha := \frac{c}{\sqrt{mk}}$ and $\beta := \omega\sqrt{\frac{m}{k}}$ (with $x = \frac{F_0}{k}$ and $t = \sqrt{\frac{m}{k}}$). This leaves us with only two parameters in the equation compared with the original's five. The non-dimensionalised differential equation is then:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \alpha\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = \cos\beta\tilde{t}$$

Doing this gives us a better understanding of the relationship between parameters (in a dimensionless group) and how they affect the behaviour of the system.