Equivalent Norms And ℓ^p Spaces

1 Equivalent Norms

Two norms, $||\cdot||_1, ||\cdot||_2$, are said to be equivalent if and only if there exist constants $0 < c_1 \leq c_2$ such that for all $x \in X$

$$c_1||x||_1 \leq ||x||_2 \leq c_2||x||_1$$

Alternatively, this can also be written as two norms, $||\cdot||_1, ||\cdot||_2$, are said to be equivalent if and only if there exist constants $0 < c_1 \leq c_2$ such that

$$c_1\mathcal{B}_{(X,||\cdot||_2)} \subset \mathcal{B}_{(X,||\cdot||_1)} \subset c_2\mathcal{B}_{(X,||\cdot||_2)}$$

2 The ℓ^p Sequence Space

For $1 \leq p < \infty$, the ℓ^p Sequence Space is defined as the space of all sequences $x = (x_j)_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

The ℓ^∞ Sequence Space is defined as the space of bounded sequences equipped with the ℓ^∞ Norm.

Proof that the ℓ^p Sequence Space is closed under addition is provided later. A full proof of ℓ^p as a vector space is not provided.

3 Minkowski's Inequality In ℓ^p

Minkowski's Inequality In ℓ^p states that for all $1 \leq p \leq \infty$, if $x, y \in \ell^p$, $x + y \in \ell^p$ and $||x + y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$.

Proving the case for ℓ^∞ is simple. For $x, y \in \ell^\infty$,

$$||x+y||_{\ell^\infty} = \max_{j\in[1,\infty)} |x_j + y_j|$$
$$\implies ||x+y||_{\ell^\infty} = \max_{j\in[1,\infty)} |x_j| + \max_{j\in[1,\infty)} |y_j||$$
$$\implies ||x+y||_{\ell^\infty} = ||x||_{\ell^\infty} + ||y||_{\ell^\infty}$$

Therefore Minkowski's Inequality holds for ℓ^∞.

For the cases of $1 \leq p < \infty$, consider Minkowski's Inequality In $\mathbb{R}^n$ (seen in The ℓ^p Norms). For $x, y \in \mathbb{R}^n$, and $1 \leq p < \infty$,

$$||x+y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$$
$$\implies \left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{\frac{1}{p}} = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

Since ℓ^p infinite dimensional and the series of elements of ℓ^p are finite, we can take the limit as $n \to \infty$, meaning that

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty} \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \lim_{n\to\infty} \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$
$$\implies ||x+y||_{\ell^p} \leq ||x||_{\ell^p} + ||y||_{\ell^p}$$

For all $x, y \in \ell^p$, and thus Minkowski's Inequality holds for $1 \leq p < \infty$. Finally, we can see that for all $1 \leq p \leq \infty$, Minkowski's Inequality In ℓ^p holds.

4 Proof That ℓ^p Is Closed Under Addition

To prove that ℓ^p is closed under addition, for $1 \leq p < \infty$ and $x, y \in \ell^p$, we have that

$$\sum_{j=1}^{n} |x_j + y_j|^p \leq \sum_{j=1}^{n} |2\max\{x_j, y_j\}|^p$$
$$\implies \sum_{j=1}^{n} |x_j + y_j|^p \leq 2^p \left(\sum_{j=1}^{n} |\max\{x_j^p, y_j^p\}| \right)$$
$$\implies \sum_{j=1}^{n} |x_j + y_j|^p \leq 2^p \left(\sum_{j=1}^{n} |x_j^p| + \sum_{j=1}^{n} |y_j^p| \right)$$

Taking the limit as $n \to \infty$, we see that

$$\implies \lim_{n\to\infty} \sum_{j=1}^{n} |x_j + y_j|^p \leq \lim_{n\to\infty} 2^p \left(\sum_{j=1}^{n} |x_j^p| + \sum_{j=1}^{n} |y_j^p| \right)$$

Since the two sums on the right are finite (as $x, y \in \ell^p$), the left side must also be finite, and thus $\sum_{j=1}^{\infty} |x_j + y_j|^p < \infty$ and thus $x + y \in \ell^p$ if $x, y \in \ell^p$ for $1 \leq p < \infty$.

All other conditions are trivial to prove (Additive Commutativity, Associativity, Inverse, and Identity, Multiplicative Closure, Distributivity, and Identity), including the additive closure in the case of ℓ^∞.