Absolutely Convergent Series

1 Introduction

A series, $\sum a_n$, is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$. The most important property of absolutely convergent series is that if a series is absolutely convergent, it is also convergent (ie: $\sum_{n=1}^{\infty} |a_n|$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges).

Note that the opposite does not hold.

2 Proof

Suppose that the infinite sum of a_n is absolutely convergent.

Note that for all $n \in \mathbb{N}$, $0 \leq |a_n + |a_n|| \leq 2|a_n|$. Because of this,

$$0 \leq \left| \sum_{k=1}^{n} (a_k + |a_k|) \right| \leq 2 \sum_{k=1}^{n} |a_k|$$

$$\implies 0 \leq \left| \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} |a_k| \right| \leq 2 \sum_{k=1}^{n} |a_k|$$

Applying the Triangle Inequality gives,

$$\implies 0 \leq \left| \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} |a_k| \right| \leq \left| \sum_{k=1}^{n} a_k \right| + \sum_{k=1}^{n} |a_k| \leq 2 \sum_{k=1}^{n} |a_k|$$

$$\implies 0 \leq \left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k|$$

$$\implies 0 \leq \lim_{n\to\infty} \left| \sum_{k=1}^{n} a_k \right| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$$

We know that the right side of the inequality converges and thus $|\sum a_n|$ converges. Since $\sum a_n = \pm |\sum a_k|$, it must also converge.