Metric Spaces

1 Metrics And Metrics

If a Norm is considered a way of measuring the size of a vector, then a Metric is a way of measuring the distance between two points.

A Metric, d, on a set X is a map $X \times X \to \mathbb{R}^+$ satisfying the following three conditions:

1. $d(x,y) = 0 \iff x = y$, for all $x, y \in X$
2. $d(x,y) = d(y,x)$, for all $x, y \in X$
3. $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in X$

Note that the third condition is the Triangle Inequality.

Similar to Normed Spaces, (X,d) denotes a Metric Space.

2 Norms And Metric Spaces

Lemma: If X is a vector space and $||\cdot|| : X \to \mathbb{R}$ is a Norm on X then $d(x,y) := ||x-y||$ is a metric on X.

First, to prove the first property, if $x = y$, then $x - y = 0$. As seen in the Introducing Norms section, a Norm is zero when its input is zero, and thus $||x-y|| = 0$. If $||x-y|| = 0$, by the same fact about Norms, $x - y = 0$, and thus $x = y$, proving the first condition.

The second property is trivial to prove since $||x-y|| = ||y-x||$.

Finally, $||x-z|| \leq ||x-y|| + ||y-z||$, by the Triangle Inequality of Norms, and thus all three conditions are satisfied, proving the lemma.

3 Examples of Metrics

The simplest examples of Metrics are those defined by the examples of the ℓ^p Norms (on $\mathbb{R}^n$),

$$d_p(x, y) := ||x - y||_{\ell^p}$$

The Discrete Metric is the metric defined on any non-empty set, X, by

$$d(x, y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

The Sunflower Metric on $\mathbb{R}^2$,

$$d(x, y) := \begin{cases} ||x - y||, & \text{if } x \text{ and } y \text{ lie on the same straight line that passes through the origin} \\ ||x|| + ||y||, & \text{otherwise} \end{cases}$$