Euler-Lagrange Equation

1 Introduction

The Euler-Lagrange equation is an equation that is satisfied by a critical point of a functional. If $I[y] = \int_a^b f(x, y, y')dx$ is the functional, its critical point $y(\cdot) \in X$ satisfies the Euler-Lagrange equation,

$$\partial_y f = \frac{d}{dx}\left[\partial_{y'} f\right]$$

2 Proof

Recall that the first variation of a functional is defined by $\delta I[y]\eta = \frac{d}{ds} I[y]|_{s=0}$

$$\implies \delta I[y]\eta = \int_a^b \frac{\partial}{\partial s} f(x, y + s\eta, y' + s\eta')dx|_{s=0}$$

$$\implies \delta I[y]\eta = \int_a^b \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' dx$$

We can now integrate the second term by parts. $u = \frac{\partial f}{\partial y'}$, $v' = \eta'$

$$\implies \delta I[y]\eta = \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x=a}^{b} + \int_a^b \eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right)dx$$

Since $\eta \in H$, the first term is 0.

$$\implies \delta I[y]\eta = \int_a^b \eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right)dx$$

Recall that, if $y(\cdot)$ is a critical point of I if $\delta I[y]\eta = 0$ for all $\eta(\cdot) \in H$. If y is a critical point of I, we therefore have:

$$\implies 0 = \int_a^b \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right] \right) dx$$

Since $\eta(\cdot)$ is an arbitrary element of the Space of Variations, H, we can apply the Fundamental Lemma of the Calculus of Variations,

$$\implies 0 = \frac{\partial f}{\partial y} - \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right]$$
$$\implies \frac{\partial f}{\partial y} = \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right]$$

This is the Euler-Lagrange equation. The solution to this PDE that also satisfies the boundary conditions of the initial problem, is a minimiser of the problem.