Ratio Test For Series

1 Introduction

The Ratio Test is a useful tool for determining the convergence or divergence of a series. The test states that if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Then the following is true:

- The series converges absolutely, if $L < 1$
- The series diverges, if $L > 1$
- The test is inconclusive, if $L = 1$

2 Proof

First, we will prove that, if $L > 1$, then the series diverges.

By the definition of convergence, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon$.

Select some ϵ such that $L - \epsilon > 1$. We know that this always exists since $L > 1$, and thus for any $\epsilon < L - 1$, the inequality holds.

This means that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$1 < L - \epsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon$$

$$\implies |a_n| < (L-\epsilon)|a_n| < |a_{n+1}| < (L+\epsilon)|a_n|$$

Clearly, $|a_{N+1}| > (L-\epsilon)|a_N|$, and thus $|a_{N+2}| > (L-\epsilon)|a_{N+1}| > (L-\epsilon)^2|a_N|$. More generally, for any $M \in \mathbb{N}$,

$$|a_{N+M}| > (L-\epsilon)^M|a_N|$$

$$\implies \lim_{M\to\infty} |a_{N+M}| \geq \lim_{M\to\infty} (L-\epsilon)^M|a_N|$$

Note that $\lim_{M\to\infty}(L-\epsilon)^M|a_N| = \infty$, since $L-\epsilon > 1$, and thus $\lim_{M\to\infty}|a_{N+M}| = \infty$. Note that this also means that $\lim_{n\to\infty}|a_n| = \infty$.

Since the sequence (a_n) does not converge to 0, by the Divergence Test, the series $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$.

Next, we will prove that, if $L < 1$, the series converges.

Choose some $\epsilon > 0$ such that $L + \epsilon < 1$. Values for ϵ exist since $L < 1$ and thus any choice $\epsilon < 1 - L$ works.

This means that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$L - \epsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon < 1$$

$$\implies (L-\epsilon)|a_n| < |a_{n+1}| < (L+\epsilon)|a_n| < |a_n|$$

Since $|a_{N+1}| < (L+\epsilon)|a_N|$, $|a_{N+2}| < (L+\epsilon)|a_{N+1}| < (L+\epsilon)^2|a_N|$. More generally, for any $M \in \mathbb{N}$,

$$0 < |a_{N+M}| < (L+\epsilon)^M|a_M|$$

$$\implies |a_N| + |a_{N+1}| + |a_{N+2}| + ... < |a_N| + (L+\epsilon)|a_N| + (L+\epsilon)^2|a_N| + ...$$

$$\implies |a_N| + |a_{N+1}| + |a_{N+2}| + ... < |a_N|(1 + (L+\epsilon) + (L+\epsilon)^2 + ...)$$

$$\implies \sum_{n=N}^{\infty} |a_n| < |a_N| \sum_{n=0}^{\infty} (L+\epsilon)^n$$

The right sum converges since $(L+\epsilon) < 1$ (the proof of this in the Geometric Series section uses the Ratio Test, however this is not the only method of proving it).

$$\implies \implies \sum_{n=N}^{\infty} |a_n| < |a_N| \sum_{n=0}^{\infty} (L+\epsilon)^n < \infty$$

Therefore, by the Comparison Test, the left sum converges. Since, $\sum_{n=0}^{N-1}$ is finite, the sum of it and the left sum must also converge.Therefore $\sum_{n=0}^{\infty} |a_n|$ converges (or $\sum_{n=0}^{\infty} a_n$ converges absolutely) if $L < 1$.

Finally, to show that the test is inconclusive for $L = 1$, it is sufficient to show an example for which $L = 1$ and the series converges absolutely and another for which $L = 1$ and the series diverges.

An example of a convergent series is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n\left(\frac{1}{n}\right)}{(n+1)\left(\frac{1}{n}\right)}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right) = 1$

For a divergent series, the Harmonic Series is perhaps the simplest example. As proven in the section dedicated to the Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Further, the $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

Since there are cases in which $L = 1$ and the series converges, and cases where $L = 1$ and the series diverges, the Ratio Test cannot be used to guarantee convergence in the case that $L = 1$.