ЕВРОПЕЙСКИЙ УНИВЕРСИТЕТ В САНКТ-ПЕТЕРБУРГЕ
EUROPEAN UNIVERSITY AT ST PETERSBURG

Department of Economics

Факультет экономики

Working Paper Series

Concavity of the Consumption Function and Economic Growth

Kirill Borissov and Anastasia Zotova

Working paper # 2003/01

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Abstract
A model of economic growth based on the assumption that the consumption function is concave is proposed. In this model there can be two types of steady-state equilibria, dividing and non-dividing. In a dividing steady-state equilibrium, the population is divided into savers and spenders; in a non-dividing steady-state equilibrium, all consumers are in the same position. Dividing steady-state equilibria are indeterminate and all indeterminate steady-state equilibria, no matter whether they are dividing or non-dividing, are efficient. Several numerical examples indicate that possible patterns of dynamic behavior can vary substantially with both the starting point and the form of the production function.

Keywords: economic growth, consumption function, indeterminacy

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CONCAVITY OF THE CONSUMPTION FUNCTION AND ECONOMIC GROWTH

Kirill Borissov and Anastassia Zotova

1. Introduction

The aim of this paper is to introduce into a model of economic growth the explicit assumption that the consumption function is concave, and to derive some of its implications for the theory of economic growth. More specifically, we assume that consumption is a concave function of wealth (it is not difficult to show that the result will be the same if consumption is a concave function of income).

The assumption that the consumption function is concave dates back to Keynes who wrote that "...with the growth in wealth [comes] the diminishing marginal propensity to consume..." (Keynes, 1936, p.349). The authors of an up-to-date famous graduate textbook on macroeconomics write in the context of economic growth theory that "we usually think it is the rich who are more likely to be patient" (Blanchard and Fischer, 1989, p. 73). Recently, Carroll and Kimball (1996) have provided a solid analytical explanation for concavity of the consumption rule under uncertainty. Empirical evidence (see, e.g., Lusardi (1996)) shows that the marginal propensity to consume is substantially higher for consumers with low wealth or low income than for consumers with high wealth or income.

The idea that the saving rate tends to rise with income or wealth is akin to the assumptions underlying the Cambridge model of growth and distribution. Whereas Kaldor (1956) assumed that the saving rate from profit income is higher than that from wage income, Pasinetti (1962) made the assumption that the saving rate of capitalists is higher than that of workers. The Cambridge model provoked a wide scholarly debate (see, e.g., Samuelson and Modigliani, 1966). It is noteworthy that after some hiatus, interest in macroeconomic models with heterogeneous consumers has been rekindled (see, e.g., Foley and Michl, 2000; Mankiw, 2000; Michel and Pestieau, 1998; Smetters, 1999).

A distinctive feature of our model is that, unlike most previous models, the division of the population into different groups of consumers occurs endogenously. A similar approach was developed for a modification to the Barro-Ramsey model in (Borissov, 2002).

Our main emphasis is on steady states. In our model, all consumers are identical in their exogenous parameters, but in steady-state equilibria they can have different wealth and different saving rates. More precisely, there can be two types of steady-state equilibria: dividing and non-dividing. In a dividing steady-state equilibrium, the population is divided into savers and spenders. In a non-dividing steady-state equilibrium, all consumers are in the same position.

We show that dividing steady-state equilibria are indeterminate in the sense that the set of equilibrium interest and wage rates is a continuum. The indeterminacy arises because the variables that show the fractions of consumers with different saving rates are endogenous and hence, roughly speaking, the number of endogenous variables is larger than the number of conditions determining steady-state equilibria. It should be noted that non-dividing steady-state equilibria can also be indeterminate, though this possibility is non-generic. We show that all indeterminate steady-state equilibria, no matter whether they are dividing or non-dividing, are efficient; as for determinate steady-state equilibria, they can be inefficient like in the Solow model. The dynamic properties of our model are not studied in detail, but several numerical examples indicate that

possible patterns of dynamic behavior can be relatively diverse. Also we propose an endogenous-growth modification to our model and show that dividing steady-state equilibria are indeterminate in the sense that the set of equilibrium growth rates is a continuum.

The rest of the paper is organized as follows. Section 2 introduces the model. In Section 3, steady-state equilibria are introduced and necessary and sufficient conditions for a state to be a steady-state equilibrium are proposed. We study dividing equilibria in Section 4; in particular, we propose some sufficient conditions for their existence and non-existence and show that they are efficient and indeterminate. In Section 5 we prove that all indeterminate equilibria are efficient. Section 6 contains several numerical examples illustrating different possible types of dynamic behavior in the model. Section 7 is devoted to the endogenous-growth version of our model. Section 8 summarizes our concluding remarks.

2. The model

Production sector

Consider a world with two factors and a single good. Firms borrow capital K and hire labor L to produce a homogeneous output that can be consumed directly or used as capital in the production process. We assume that the state of technology can be described by an aggregate production function

$$Y=F(K,L),$$

where Y is aggregate output. We also assume that $F(\cdot)$ is a continuous and concave function exhibiting constant returns to scale and that both capital and labor are essential for production:

$$F(K,0)=F(0,L)=0. \quad (1)$$

Capital does not depreciate, the production function gives net output, not including the nondepreciating capital.

The production function can be written in intensive form as:

$$Y/L=f(k)\equiv F(k,1)=F(K/L,1)=F(K,L)/L,$$

where:

$$k=K/L.$$

It is assumed that:

$$r_0\equiv\lim_{k\to 0}f(k)/k>n,$$

where $n>0$ is the rate of population growth. It follows from (1) that:

$$\lim_{k\to\infty}f(k)/k=0.$$

Also we assume that for any $r\in[n,r_0)$, the maximization problem:

$$\max_{k\geq 0}\{f(k)-rk\}$$

has a unique solution; this solution will be denoted by $k(r)$. Note that $k(\cdot)$ is a continuous function.

Consumers

There is a large number (a continuum) of families in the economy. They are identical in their exogenous parameters.

Suppose that at some time t (time t is the end of time period $[t-1,t]$ and, at the same time, the beginning of time period $[t,t+1]$) the size of a family is $L_t>0$ and that the gross savings of this family are $Z_t\geq 0$. Then the gross wealth of the family at time $t+1$ is $(1+r_t)Z_t+w_tL_t$, where r_t is the interest rate in period $[t,t+1]$ and w_t is the wage rate. At time $t+1$, the division of the gross wealth into consumption $C_t\geq 0$ and savings $Z_{t+1}\geq 0$ is made by means of the consumption function $c(\cdot)$:

$$C_t=L_tc\left(\frac{(1+r_t)Z_t+w_tL_t}{L_t}\right),$$

$$Z_{t+1}=(1+r_t)Z_t+w_tL_t-C_t.$$

These equations can be rewritten as:

$$c_t=c((1+r_t)z_t+w_t),$$

$$(1+n)z_{t+1}=(1+r_t)z_t+w_t-c_t,$$

where $c_t=C_t/L_t$ is *per capita* consumption and $z_t=Z_t/L_t$ ($z_{t+1}=Z_{t+1}/L_{t+1}$) are *per capita* savings. The consumption function $c:\Re_+\to\Re_+$ is continuous, strictly concave and satisfies the following condition:

$$0\leq c(y)\leq y \ (y\geq 0).$$

It is clear that the consumption function is strictly increasing and that the saving function $s:\Re_+\to\Re_+$ defined by:

$$s(y)=y-c(y)$$

is continuous, strictly convex, and strictly increasing.

Dynamics

To specify the dynamics of the model, suppose that at time $t=0$ the population is divided into a finite number of different groups of families in such a way that the *per capita* savings of the families that belong to the same group are equal. Let J be the set of such groups and $\alpha_j>0$ be the fraction of group j in the population (clearly, $\sum_{j\in J}\alpha_j=1$) and let $\bar{L}_t=(1+n)^t$ be the total population at time t. Then the population of group j at time t is $\alpha_j\bar{L}_t$.

The stock of capital at each time t, K_t, is equal to the gross savings in the economy:

$$K_t=\sum_{j\in J}\alpha_j\bar{L}_tz_{jt},$$

where z_{jt} is the *per capita* savings of group j at time t, and capital per worker, $k_t = K_t / \overline{L}_t$, is given by:

$$k_t = \sum_{j \in J} \alpha_j z_{jt}. \tag{2}$$

Recall that the production sector maximizes its profit. Therefore, the interest and wage rates, r_t and w_t, are given by

$$w_t = f(k_t) - r_t k_t = \max_{k \geq 0} \{f(k) - r_t k\}.$$

In particular, if it is possible to differentiate the production function, then:

$$r_t = f'(k_t)$$

and:

$$w_t = f(k_t) - f'(k_t) k_t.$$

The consumption c_{jt} of each individual from group j is determined by:

$$c_{jt} = c((1+r_t) z_{jt} + w_t).$$

Thus:

$$z_{jt+1} = s((1+r_t) z_{jt} + w_t)/(1+n)$$

and, therefore:

$$k_{t+1} = \sum_{j \in J} \alpha_j s((1+r_t) z_{jt} + w_t)/(1+n).$$

3. Steady-State Equilibria

We define a steady-state equilibrium as a state where the level of capital per worker and the *per capita* savings and consumption of each family do not change over time.

Prior to defining steady-state equilibria formally, several points should be clarified. Suppose that we are given steady-state equilibrium real wage and interest rates, r^* and w^*. Then

the *per capita* savings of a family, z, must satisfy the following equality

$$s((1+r^*) z + w^*) = (1+n) z. \tag{3}$$

Consider (3) as an equation in z. Since $s(\cdot)$ is a strictly convex function, this equation has at most two solutions. Denote the smaller solution by z_l^* and the larger solution by z_h^*. Of course, degenerate cases where $z_l^* = z_h^*$ are possible.

Thus, each family can find itself at two steady-state positions depending on whether its *per capita* savings are z_l^* or z_h^*, and the population can split at most into two groups. Those whose *per capita* savings are z_l^* will be called *spenders* and those whose *per capita* savings are z_h^* *savers*. If $z_l^* < z_h^*$, then the former are less wealthy than the latter, and therefore spenders may be called the poor and savers the rich.

Let σ equal the proportion of savers in the population and therefore $1-\sigma$ equals the proportion of spenders in the economy. These proportions are determined endogenously in the following definition.

An array (r^*, w^*, k^*, z_l^*, z_h^*, σ^*), where $r^* \geq 0$, $w^* > 0$, $k^* > 0$, $z_l^* > 0$, $z_h^* > 0$ and $\sigma^* \in [0,1]$, is called a *steady-state equilibrium* if:

E.1) $w^* = f(k^*) - r^* k^* = \max_{k \geq 0} \{f(k) - r^* k\} > 0$;

E.2) z_l^* is the smaller solution to (3);

E.3) z_h^* is the larger solution to (3);

E.4) $\sigma^* z_h^* + (1-\sigma^*) z_l^* = k^*$.

Note that r^* in this definition is an *equilibrium interest rate*.

In this definition, condition E.4) is a steady-state version of (2); it says that the stock of capital must be equal to gross savings in the economy.

A steady-state equilibrium (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) will be called *dividing* if $z_l^*<z_h^*$ and $0<\sigma^*<1$, otherwise it will be called *non-dividing*.

In a dividing steady-state equilibrium, the population is in fact divided into savers and spenders which differ in their *per capita* wealth and savings. In a non-dividing steady-state equilibrium, savers and spenders do not differ, or the proportion of one of these groups in the population is nil.

Note that the strict concavity of $c(y)$ implies that $c(y)/y$ is a strictly decreasing function of y and analogously $s(y)/y$ is a strictly increasing function of y.

We are now in a position to discuss what specific values of the rate of interest can be equilibrium values. To answer this question, suppose we are given some values of interest and wage rates (r^* and w^* respectively) and a value of capital per worker (k^*) satisfying E.1). Looking now at equation (3), it is clear that if this equation has no solution (see Fig. 1), then r^* cannot be an equilibrium interest rate.

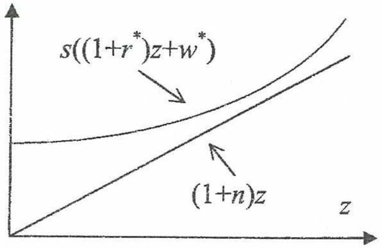

Figure 1.

Consider the case where (3) has two solutions (see Fig. 2). Let z_l^* be the smaller solution and z_h^* be the larger solution.

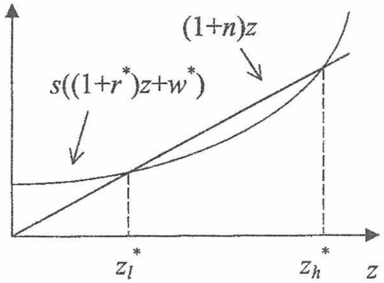

Figure 2.

Further, let σ^* be a solution to the equation

$$\sigma z_h^*+(1-\sigma)z_l^*=k^* \tag{4}$$

in σ. The array (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) satisfies conditions E.1)-E.4) but this array represents a steady-state equilibrium if, and only if, $0\leq\sigma^*\leq 1$, that is,

$$z_l^*\leq k^*\leq z_h^*.$$

Suppose now that equation (3) has just one solution (see Fig.3). Setting both z_l^* and z_h^* equal to this solution and noting

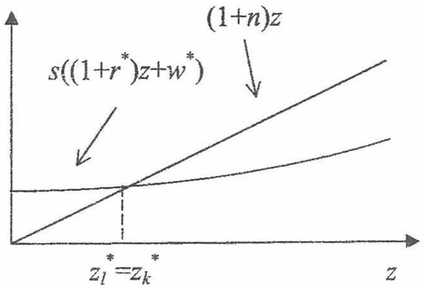

Figure 3.

that (4) reduces to:

$z_h{}^*=z_l{}^*=k^*$

implies that, in this case, if k^* is a solution to (3), then for any $\sigma^*\in[0,1]$, the array (r^*, w^*, k^*, $z_l{}^*$, $z_h{}^*$, σ^*) is a non-dividing steady-state equilibrium; otherwise r^* is not an equilibrium interest rate.

Now we can preliminarily summarize.

Proposition 1. *Let $r^*<r_0$ be given and let w^* and k^* be determined by* E.1). *The interest rate r^* is an equilibrium interest rate if, and only if, either*

- *equation* (3) *has just two solutions, $z_l{}^*$ and $z_h{}^*$, and*

$$z_l{}^*<k^*<z_h{}^* \tag{5}$$

or

- *k^* is a solution to* (3).

In the first case the steady-state equilibrium corresponding to r^* is dividing, and in the second case it is non-dividing.

If we denote

$$a=\lim_{y\to\infty}\frac{s(y)}{y}=\sup_{y>0}\frac{s(y)}{y},$$

then the following lemma will help us to clarify Proposition 1.

Lemma 1. *Let $r^*<r_0$ be given and let w^* and k^* be determined by* E.1). *The following conditions are equivalent:*

i) *the following inequalities hold:*

$$s(f(k^*)+k^*)<(1+n)k^* \tag{6}$$

and

$$1+r^*>\frac{1+n}{a}. \tag{7}$$

ii) *equation* (3) *has just two solutions, $z_l{}^*$ and $z_h{}^*$, and these solutions satisfy* (5).

Proof. To simplify the notation, introduce the function $\varphi(\cdot)$ by:

$\varphi(z)=s((1+r^*)z+w^*)$.

The function $\varphi(\cdot)$ is a strictly convex and strictly increasing function and:

$$\lim_{z\to\infty}\frac{\varphi(z)}{z}=\lim_{z\to\infty}\frac{s((1+r^*)z+w^*)}{z}$$

$$=\lim_{z\to\infty}\left(\frac{s((1+r^*)z+w^*)}{((1+r^*)z+w^*)}\times\frac{((1+r^*)z+w^*)}{z}\right)=(1+r^*)a. \tag{8}$$

i)⇒ii) Suppose that inequalities (6) and (7) hold. Since, $w^*>0$, we have $s(w^*)>0$. Therefore, there is a solution to (3) that belongs to the interval $(0, k^*)$. At the same time, taking into account (7), we have

$$\lim_{z\to\infty}\frac{s((1+r^*)z+w^*)}{(1+n)z}=\lim_{z\to\infty}\frac{\varphi(z)}{(1+n)z}=a\frac{1+r^*}{1+n}>1.$$

Therefore, there exists a solution to (3) that is larger than k^*.

ii)⇒i) Suppose that equation (3) has just two solutions, $z_l{}^*$ and $z_h{}^*$, and these solutions satisfy (5). First, note that

$$\varphi(k^*)=s((1+r^*)k^*+w^*)=s(f(k^*)+k^*)<(1+n)k^*, \tag{9}$$

otherwise one of the following inequalities would hold:

$k^*\le z_l{}^*$

or

$k^*\ge z_h{}^*$,

which contradicts (5).

Assume now that (9) holds and:

$$1+r^*\le\frac{1+n}{a}. \tag{10}$$

Show that this is impossible. Let z' be the solution to (3) that is larger than k^*. We have:

$$\varphi(z')=(1+n)z'. \tag{11}$$

Denote

$$\gamma=\frac{\varphi(z')-\varphi(k^*)}{z'-k^*}.$$

and note that by (9) and (11),

$$\gamma>1+n.$$

Since the function $\varphi(\cdot)$ is strictly convex, $\frac{\varphi(z)-\varphi(k^*)}{z-k^*}$ is an increasing function of z. Therefore,

$$\lim_{z\to\infty}\frac{\varphi(z)-\varphi(k^*)}{z-k^*}>\frac{\varphi(z')-\varphi(k^*)}{z'-k^*}=\gamma>1+n.$$

On the other hand, taking into account (8),

$$\lim_{z\to\infty}\frac{\varphi(z)-\varphi(k^*)}{z-k^*}=\lim_{z\to\infty}\left(\frac{\varphi(z)}{z}\times\frac{z}{z-k^*}-\frac{\varphi(k^*)}{z-k^*}\right)=$$

$$=\lim_{z\to\infty}\frac{\varphi(z)}{z}=(1+r^*)a.$$

Thus,

$$(1+r^*)a>1+n,$$

which contradicts (10). This contradiction shows that (7) holds.□

The following theorem follows directly from Proposition 1 and Lemma 1.

Theorem 1. *Let $r^*<r_0$ be given and let w^* and k^* be determined by* E.1). *Then*

- *z_l^*, z_h^*, and σ^* such that the array (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is a dividing steady-state equilibrium exist if, and only if,*

$$s(f(k^*)+k^*)<(1+n)k^*$$

and

$$1+r^*>\frac{1+n}{a}. \tag{7}$$

- *z_l^*, z_h^*, and σ^* such that the array (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is a non-dividing steady-state equilibrium exist if, and only if,*

  $$s(f(k^*)+k^*)=(1+n)k^*.$$

Remark. *Let* $\tilde{k}=k\left(\frac{1+n}{a}-1\right)$. *If $k(\cdot)$ is a strictly decreasing function, then in Lemma* 1 *and Theorem* 1, (7) *can be rewritten as:*

$$k^*<\tilde{k}. \tag{12}$$

Note that for the Leontieff production function, (7) is not equivalent to (12).

4. Dividing steady-state equilibria

The most interesting feature of our approach is the notion of dividing steady-state equilibrium. In this section we study equilibria of this kind. Namely, we provide some sufficient conditions for their existence and non-existence and show that they are efficient and indeterminate.

The following trivial proposition provides some simple sufficient conditions for the existence of dividing steady-state equilibria.

Proposition 2. *If:*

$$\lim_{k\to0}\frac{s(f(k)+k)}{k}<(1+n)$$

and:

$$a>\frac{1+n}{1+r_0},$$

then dividing steady-state equilibria exist.□

Example. Consider the following consumption function:

$$c(y)=\ln(y+1).$$

If the production function is such that $f'(0)<+\infty$, then dividing steady-state equilibria exist. If the production function is Cobb-Douglas:

$$f(k)=Ak^{\alpha},\quad A>0,\quad 0<\alpha<1,$$

then dividing steady-state equilibria exist if $\alpha>1/2$. □

There exists no dividing steady-state equilibria in the Solow model. Our model also has no dividing steady-state equilibrium if the consumption (and therefore the saving function is linear) that is, if the marginal propensity to consume is constant and is equal to the average propensity to consume.

More precisely, if the saving function is linear, that is:

$$s(y)=\bar{s}y,\quad 0<\bar{s}<1,$$

then there is no k^* satisfying both (6) and (12). Indeed, in this case $\tilde{k}$ is a solution to:

$$\max_{k\geq 0}\{\bar{s}(f(k)+k)-(1+n)k\} \tag{13}$$

and if (6) holds then $k^*>\tilde{\tilde{k}}$, where $\tilde{\tilde{k}}$ is the larger solution (of at most two solutions) to the equation:

$$\bar{s}(f(k)+k)=(1+n)k. \tag{14}$$

Since $\tilde{\tilde{k}}\geq\tilde{k}$, $k^*>\tilde{k}$.

It is natural to conjecture that if the saving function is close to a linear one, then there is no dividing steady-state equilibrium in our model. In a sense, this conjecture is generically true. To show this, suppose that the production function is given and also that we are given a continuos function

$S(\xi,y)$ defined on $[\underline{\xi},1]\times[0,y_1]$, where $0<\underline{\xi}<1$, $y_1=f(k_1)+k_1$, and k_1, in its turn, is a positive solution to the equation:

$$(f(k)+k)=(1+n)k.$$

Suppose further that for any $\xi\in(\underline{\xi},1)$, the function:

$$s^{\xi}(y)\equiv S(\xi,y)$$

is strictly convex and strictly increasing and satisfies:

$$0\leq s^{\xi}(y)\leq y\quad (y\geq 0)$$

and that for some $\bar{s}\in(0,1)$,

$$S(1,y)=\bar{s}y\quad (y\geq 0).$$

Proposition 3. *If* $\bar{s}\neq\frac{1+n}{1+r_0}$, *then there is* $\bar{\xi}\in[\underline{\xi},1)$ *such that for any* $\xi\in(\bar{\xi},1)$, *the model with the saving function* $s^{\xi}(y)$ *has no dividing steady-state equilibrium.*

Proof. Assume the converse and suppose that $\bar{s}>\frac{1+n}{1+r_0}$. Then there is a sequence $(\xi_i)_{i=1,2,\ldots}$ such that $\xi_i\to 1$ and for each i, the model with the saving function $s_i(y)\equiv S(\xi_i,y)$ has a dividing steady-state equilibrium and therefore there exists k_i such that:

$$s_i(f(k_i)+k_i)<(1+n)k_i$$

and:

$$k_i<\tilde{k}_i,$$

where $\tilde{k}_i$ is a solution to:

$$\max_{k\geq 0}\{a_i(f(k)+k)-(1+n)k\}$$

for:

$$a_i=\lim_{y\to\infty}\frac{s_i(y)}{y}.$$

Without any loss of generality we can suppose that the sequences $(k_i)_{i=1,2,\dots}$ and $(\tilde{k}_i)_{i=1,2,\dots}$ are convergent. Let:

$$k^*=\lim_{i\to\infty} k_i$$

and:

$$\tilde{k}=\lim_{i\to\infty}\tilde{k}_i.$$

It is clear that $k^*\leq\tilde{k}$ and that $\tilde{k}$ is a solution to (13). Moreover, $\bar{s}(f(k^*)+k^*)\leq(1+n)k^*$ and hence $k^*\geq\tilde{\tilde{k}}$, where $\tilde{\tilde{k}}$ is the larger solution to (14). Therefore, $\tilde{k}\geq\tilde{\tilde{k}}$. This inequality can hold only for $\bar{s}\leq\frac{1+n}{1+r_0}$. This contradiction proves Proposition 3 for $\bar{s}>\frac{1+n}{1+r_0}$.

To prove Proposition 3 for $\bar{s}<\frac{1+n}{1+r_0}$ it is sufficient to note that if ξ is sufficiently close to 1, then for all $k>0$,

$$s^{\xi}(f(k)+k))<(1+n)k.$$

This implies that the model with the saving function $s^{\xi}(y)$ has no steady-state equilibrium at all.□

We turn now to the indeterminacy and efficiency of dividing steady-state equilibria.

A steady-state equilibrium (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is called *indeterminate* if there exist $r_1<r_0$ and $r_2<r_1$ such that $r_2\leq r^*\leq r_1$ and any $r\in[r_2,r_1]$ is an equilibrium interest rate.

Proposition 4. *Any dividing steady-state equilibrium* (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) *is indeterminate.*

Proof. It follows directly from Theorem 1.□

Suppose we are given a series of *per capita* capital stocks $(k_t)_{t=0,1,\dots}$. Then the total consumption in the economy at time $t+1$ is:

$$F(K_t,\bar{L}_t)+K_t-K_{t+1},$$

where $\bar{L}_t$ is the total population and $K_t=k_t\bar{L}_t$ is the gross stock of capital at time t, and average *per capita* consumption is

$$f(k_t)+k_t-(1+n)k_{t+1}\ (=[F(K_t,\bar{L}_t)+K_t-K_{t+1}]/\bar{L}_t).$$

A series of (*per capita*) capital stocks $(k_t)_{t=0,1,\dots}$ is called *feasible* if, for all t=0,1…, total consumption at time t+1 is non-negative:

$$f(k_t)+k_t\geq(1+n)k_{t+1}.$$

A feasible sequence of capital stocks $(k_t^*)_{t=0,1,\dots}$ is called *inefficient* if there exists another feasible sequence of capital stocks $(k_t)_{t=0,1,\dots}$ such that:

- $k_0=k_0^*$;
- $f(k_t)+k_t-(1+n)k_{t+1}\geq f(k_t^*)+k_t^*-(1+n)k_{t+1}^*$ for all t=0,1…;
- there exists at least one t=0,1,… such that:

$$f(k_t)+k_t-(1+n)k_{t+1}>f(k_t^*)+k_t^*-(1+n)k_{t+1}^*.$$

A feasible sequence of capital stocks $(k_t^*)_{t=0,1,\dots}$ is called *efficient* if it is not inefficient.

We say that a steady-state equilibrium (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is *efficient* if the sequence of capital stocks $(k_t)_{t=0,1,\dots}$ given by:

$$k_t=k^*\ (t=0,1,\dots) \tag{15}$$

is efficient.

Efficiency means that it is impossible to increase total consumption at one date without decreasing total consumption at another date.

It is well known that the sequence given by (15) is efficient if $k^*\leq k^G$, where:

$$k^G\equiv k(n)$$

is the golden rule capital stock, and inefficient if $k^*>k^G$.

Proposition 5. *Any dividing steady-state equilibrium* (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) *is efficient.*

Proof. Since $a \le 1$, we have

$$\left(\frac{1+n}{a}-1\right) \ge n.$$

Therefore, taking into account (7),

$$k^* < \tilde{k} \le k^G. \square$$

5. Efficiency of indeterminate steady-state equilibria

In the previous section we noted that any dividing steady-state equilibrium is efficient and indeterminate. At the same time, there can exist non-dividing indeterminate steady-state equilibria (though the existence of non-dividing indeterminate steady-state equilibria is not generic), and also there can exist non-dividing inefficient steady-state equilibria. The question of whether all indeterminate steady-state equilibria are efficient arises. The answer to this question is positive.

Theorem 2. *Any indeterminate steady-state equilibrium* (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) *is efficient.*

Proof. Let (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) be an indeterminate steady-state equilibrium. Assume that:

$$k^* > k^G.$$

The indeterminacy of (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) implies that there is another steady-state equilibrium, (r^{**}, w^{**}, k^{**}, z_l^{**}, z_h^{**}, σ^{**}) such that $k^{**} \ne k^*$ and $k^{**} > k^G$.

It follows from proposition 5 that both steady-state equilibria, (r^*, w^*, k^*, z_l^*, z_h^*, σ^*) and (r^{**}, w^{**}, k^{**}, z_l^{**}, z_h^{**}, σ^{**}), are non-dividing. Therefore:

$$s(f(k^*)+k^*)=(1+n)k^* \tag{16}$$

and:

$$s(f(k^{**})+k^{**})=(1+n)k^{**}. \tag{17}$$

Consider the case where:

$$k^{**} > k^* > k^G. \tag{18}$$

Taking into account (16) and (17) and the inequality:

$$c(f(k^{**})+k^{**}) > c(f(k^*)+k^*),$$

we get:

$$\begin{aligned}(1+n)k^{**}-(1+n)k^* &= s(f(k^{**})+k^{**})-s(f(k^*)+k^*)\\ &= (f(k^{**})+k^{**})-(f(k^*)+k^*)-[c(f(k^{**})+k^{**})-c(f(k^*)+k^*)]\\ &< (f(k^{**})+k^{**})-(f(k^*)+k^*).\end{aligned}$$

Therefore:

$$f(k^{**})-nk^{**} > f(k^*)-nk^*,$$

which contradicts (18) since $f(k)-nk$ decreases as k goes up for $k>k^G$. This contradiction shows that $k^* \le k^G$.

The case where:

$$k^* > k^{**} > k^G$$

can be considered in a similar way.$\square$

6. Numerical Examples

In the Solow model a steady state is unique and stable. Several numerical examples presented in this section show that, in contrast to the Solow model, in our model, short-run as well as long-run behavior of the economy depends on the starting point.

In all of the examples, the rate of population growth is set at n=0.1. The saving function is specified as follows:

$$s(y) = y - \ln(1+y).$$

We will consider two cases, distinguished by the shape of the production function.

Case A. The production function is Cobb-Douglas:

$$Y = F(K,L) = K^{\alpha} L^{1-\alpha},\ 0<\alpha<1,$$

that is, $y = f(k) = k^{\alpha}$. We set α=0.4. For the set of different groups of families, J, the share of each group j in the population, α_j, and the *per capita* savings of each group j at t=0, z_{j0}, are given. This implies that the rest of the dynamic path is fully determined as described in Section 2.

Example A1.1. Suppose there are two groups of families in the economy (J={1,2}). We set the share of the wealthier of the two groups in the population, α_1, equal to 0.1. The *per capita* savings of this group at t=0 are set at z_{10}=0.2. Clearly, the share of the other group, α_2, is equal to 0.9. Their *per capita* savings at t=0 are set at z_{20}=0.1.

The time paths for the *per capita* savings of each group (z_{1t} and z_{2t}) and for capital per worker (k_t) are depicted in Figure 4 which shows that the economy converges to a non-dividing steady-state equilibrium. In the steady-state, the *per capita* savings of all families are equal.

Example A1.2. Let us slightly change the initial conditions. The values of α_1, α_2, and z_{20} remain unchanged but the *per capita* savings of the wealthier group at t=0 are now set at z_{10}=1.

The time paths for z_{1t}, z_{2t} and k_t are displayed in Figure 5. In this case the economy converges to a steady-state equilibrium in which the population is divided into two groups. In this

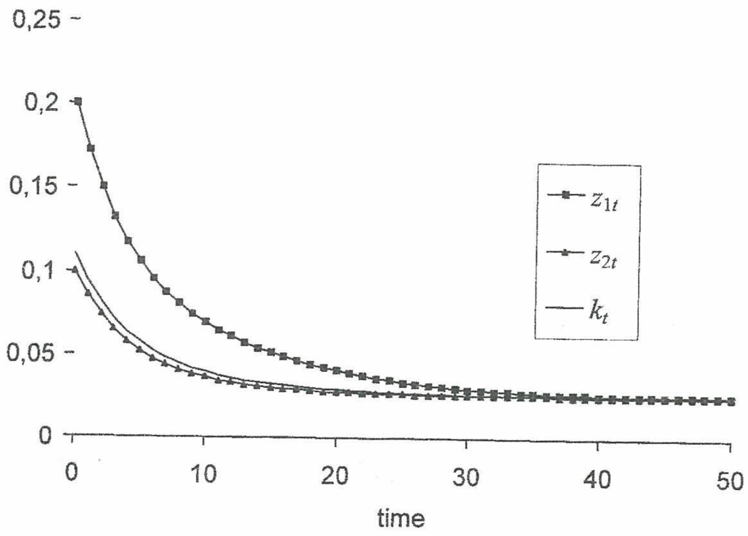

Figure 4.

steady state, the shares of both savers and spenders in the population are positive. Therefore this steady-state equilibrium is dividing.

A comparison of examples A1.1 and A1.2 reveals that, depending on the initial conditions, the economy may converge to fundamentally different steady states.

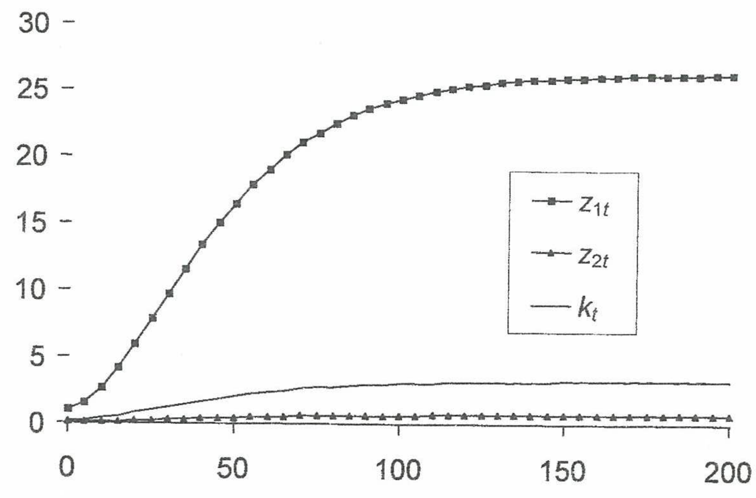

Figure 5

Examples A1.2 and A2.2 are characterized by different values of capital per worker and hence the wage and interest rates at t=0 are also different. Let us now consider two examples where the initial values of these variables are set equal.

Example A2.1. We set k_0=0.5. As in the previous examples, at t=0 the population is divided into two groups. The *per capita* savings of the wealthier group at t=0 are set at z_{10}=0.85. The *per capita* savings of the poorer group at t=0 are taken to be z_{20}=0.45. The values of z_{10}, z_{20} and k_0 uniquely determine the shares of both groups in the population. We have:

$$\alpha_1 = \frac{k_0 - z_{20}}{z_{10} - z_{20}} = 0.125 \text{ and } \alpha_2 = 1 - \alpha_1 = 0.875.$$

The time paths for z_{1t}, z_{2t} and k_t are illustrated in Figure 6. The economy converges to the same steady-state equilibrium as in example A1.1. In the steady state, the capital stock is evenly

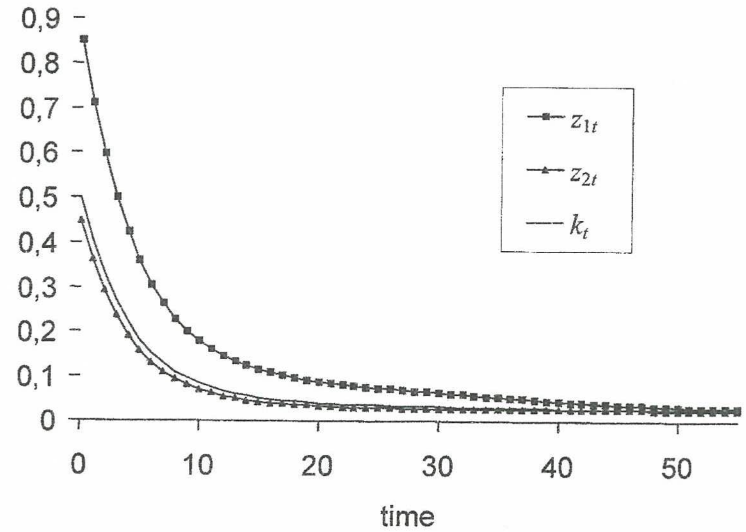

Figure 6.

distributed among the consumers.

Example A2.2. Now we set k_0=0.5, z_{10}=2 and z_{20}=0.3. It can easily be checked that this implies that α_1=2/17 and α_2=15/17. The time paths for z_{1t}, z_{2t} and k_t are shown in Figure 7. It is clear that the dynamic behavior of the economy is similar to that in example A1.2.

Note that in examples A2.1 and A2.2 the initial values of capital per worker are equal. Therefore, in the Solow model the values of k_t, (and hence of r_t and w_t) in the two economies would be equal for each t because if the average propensity to consume is assumed to be constant, the distribution of income and wealth does not affect the time paths of capital per worker, wage rates or interest rates. As examples A2.1 and A2.2 demonstrate, if the assumption that the consumption function is concave is made, the distribution of wealth at t=0 is one of the main factors that determines the dynamics of the economy. We have seen that the steady-state equilibria in examples A2.1 and A2.2 differ both in

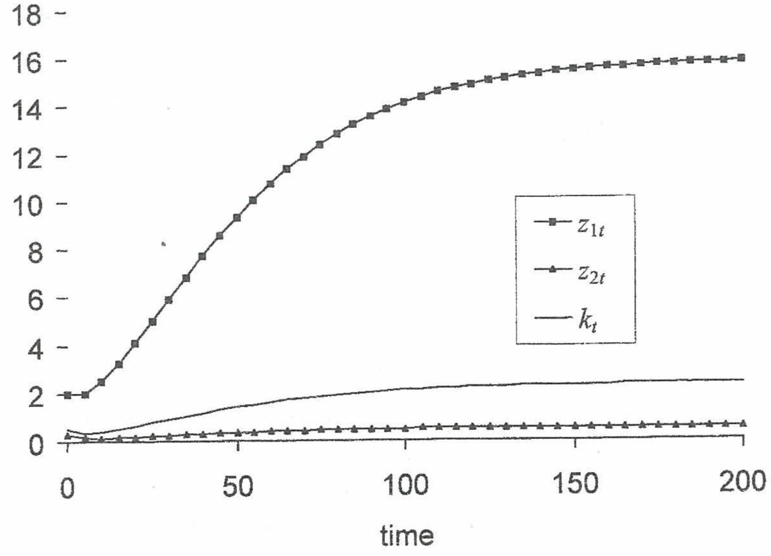

Figure 7.

the distribution of wealth and in the value of capital per worker.

Example A3. We now investigate the case where at t=0 there are three groups of families in the economy (J={1,2,3}). Let z_{1t}, z_{2t} and z_{3t} denote the *per capita* savings at time t of the wealthiest, the second wealthiest and the poorest group, respectively. In a steady state, the maximum number of different groups of families is two, which implies that, if the economy converges to a steady state, then $\lim_{t\to\infty}(z_{1t}-z_{2t})=0$, or $\lim_{t\to\infty}(z_{2t}-z_{3t})=0$, or both. In order to illustrate this case, for α_j (j=1, 2, 3) and z_{j0} (j=1, 2, 3), the following values are chosen: α_1=0.01, α_2=0.01, α_3=0.98; z_{10}=11, z_{20}=10.9999, z_{30}=0.5.

Figure 8 depicts the time paths for z_{1t}, z_{2t} and z_{3t}. As in

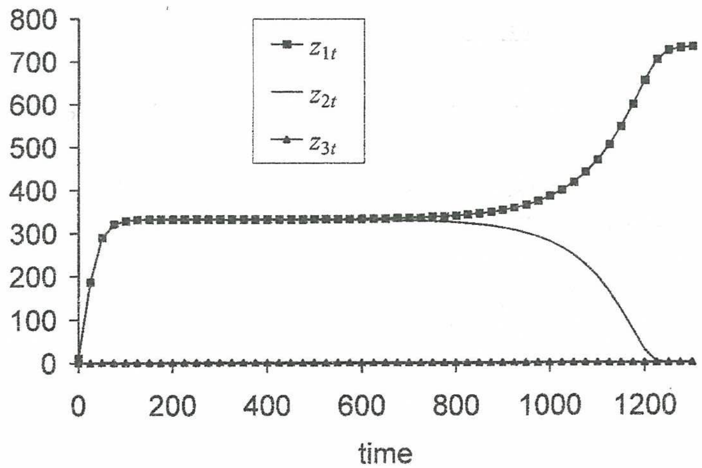

Figure 8.

example A2, the economy converges to the steady-state equilibrium in which the population is divided into two groups. It is interesting that over a relatively long period the *per capita* savings of the wealthiest and the second wealthiest groups are almost equal, but $(z_{2t}-z_{3t})\to 0$ as $t\to\infty$.

Table 1 reports the steady-state values for r^*, w^*, k^*, z_l^*, z_h^*, σ^* for examples A1, A2 and A3. The values are rounded to the fourth decimal point.

Table 1.

	r^*	w^*	k^*	z_l^*	z_h^*	σ^*
A1.1	3.6088	0.1384	0.0256	0.0256	X	0
A1.2	0.1970	0.9621	3.2563	0.7012	26.2518	0.1
A2.1	3.6088	0.1384	0.0256	0.0256	X	0
A2.2	0.2400	0.8434	2.3427	0.5330	15.9155	0.1176
A3	0.1072	1.4437	8.9804	1.6650	733.2076	0.01

Case B. A CES production function has the form $F(K,L)=(aK^{\rho}+bL^{\rho})^{1/\rho}$. We will consider the case where $\rho<0$. The parameter values for a and b are set to a=0.4 and b=0.6. We suppose that at t=0 there are two different groups of families in the economy (J={1,2}). For α_1, α_2, z_{10} and z_{20}, we chose the following values: α_1=0.1, α_2=0.9, z_{10}=3, z_{20}=1. The examples that follow are distinguished only by different values of ρ.

Example B1. Taking $\rho=-40$ Figure 9 displays the time paths for z_{1t}, z_{2t} and k_t.

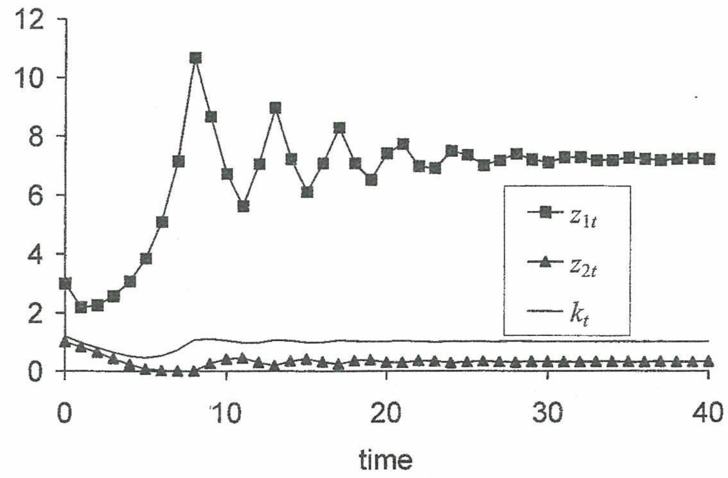

Figure 9.

As in examples A2 and A3, the economy converges to a steady-state equilibrium in which savers coexist with spenders. The steady-state values of the model's key variables are: r^*=0.3462, w^*=0.6540, k^*=1.0056, z_l^*=0.3153, z_h^*=7.2191. However, the dynamic behavior in the short run is distinctly different from that of case A. All variables experience damped fluctuations before they eventually converge to the steady-state values.

Example B2. Now we fix $\rho=-70$ and report in Figure 10 the time paths for z_{1t}, z_{2t} and k_t. This example shows that under certain conditions convergence to a steady state does not take place at all. In our numerical example, the economy is in a state of perpetual fluctuation.

The numerical examples presented in this section suffice to

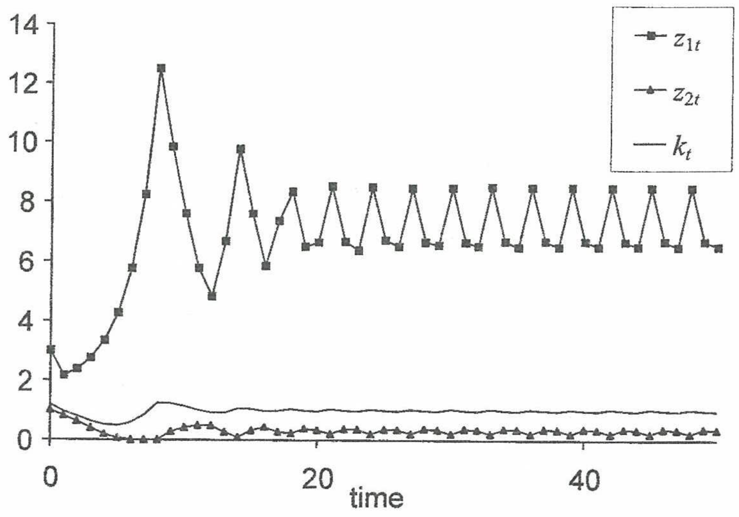

Figure10.

illustrate that the dynamic behavior of the economy may vary substantially with both the form of the production function and the initial conditions. While the Solow model of economic growth predicts that in the long run the behavior of the economy does not depend on the initial state, our model suggests that the past has a much stronger influence on the present and the future than it is widely believed.

7. Endogenous growth

Before concluding, we briefly outline an endogenous-growth modification to our model, introduce a notion of steady-state equilibrium, describe the set of steady-state growth rates

and show that dividing steady-state equilibria are indeterminate in the sense that the set of equilibrium growth rates is a continuum.

In this section we assume that the production function $F(K,L)$ is differentiable and interpret its second term, L, as the effective amount of labor, whereas the population, $\bar{L}$, is supposed to be constant over time. Economic growth is endogenized by assuming that technology grows because of aggregate spillovers coming from firms' production activities. Following Frankel (1962), we assume that firms face a fixed labor productivity that is proportional to the current economy-wide average of physical capital per worker. This implies that the effective amount of labor L is given by:

$$L=\tilde{K}\,\bar{L}/\bar{k}\text{, where } \tilde{K}=K/\bar{L} \text{ and } \bar{k}>0.$$

Therefore, irrespective of K, we have $K/L=\bar{k}$. Hence the aggregate output Y is determined by:

$$Y=F(K,L)=F(K,K/\bar{k})=[f(\bar{k})/\bar{k}]K.$$

In a competitive equilibrium, interest rate is equal to:

$$\bar{r}\equiv f'(\bar{k})$$

and wage rate is equal to:

$$\bar{w}\equiv f(\bar{k})-f'(\bar{k})\bar{k}.$$

It is clear that in this model the rate of growth is determined endogenously by the saving behavior of households.

Prior to defining steady-state equilibria in this newly introduced framework, we should make our interpretation of the consumption function $c(\cdot)$ more specific. In fact, our consumption function shows the consumption per effective unit of labor as a function of wealth per effective unit of labor.

An array (n^*, r^*, w^*, k^*, z_l^*, z_h^*, σ^*), where $n^*\geq 0$, $z_l^*>0$, $z_h^*>0$ and $\sigma^*\in[0,1]$, is called a *steady-state endogenous-growth equilibrium* if:

- $k^*=\bar{k}$;
- $r^*=\bar{r}$;
- $w^*=\bar{w}$;
- z_l^* is the smaller and z_h^* is the larger solution to the equation $s((1+r^*)z+w^*)=(1+n^*)z$ in z;
- $\sigma^* z_h^*+(1-\sigma^*)z_l^*=k^*$.

Note that in this definition n^* is an *equilibrium growth rate.*

A steady-state endogenous-growth equilibrium (n^*, r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is called *dividing* if $z_l^*<z_h^*$ and $0<\sigma^*<1$, otherwise it is called *non-dividing*.

The following theorem is a natural analogue to Theorem 1.

Theorem 3. $n^*>0$ *is an equilibrium growth rate if and only if either*:

$$s(f(\bar{k})+\bar{k})<(1+n^*)\bar{k}<a(1+\bar{r})$$

or:

$$s(f(\bar{k})+\bar{k})=(1+n^*)\bar{k}.$$

In the first case the steady-state endogenous-growth equilibrium corresponding to n^* *is dividing, and in the second case it is non-dividing.*

It follows from this theorem that any dividing steady-state endogenous-growth equilibrium (n^*, r^*, w^*, k^*, z_l^*, z_h^*, σ^*) is indeterminate in the sense that any n sufficiently close to n^* is an equilibrium growth rate.

7. Conclusion

Starting at least from Keynes (1936) onwards, many economists have intuitively assumed that the consumption function is concave, with marginal and average propensities to consume lower for the rich than for the poor. Nevertheless, much empirical and theoretical work, in particular in economic growth theory, has assumed, explicitly or implicitly, that the consumption function is linear.

This paper has demonstrated that assuming the concavity of the consumption function has important implications for the theory of economic growth. In particular, if the consumption function is concave, then, in steady states, the population that consists of identical families of consumers can split into groups with different saving rates leading to indeterminacy of steady-state equilibria. Several numerical examples have shown that the concavity of the consumption function results in a great diversity and different patterns of dynamic behavior.

Our model indicates that history matters much more than it is generally accepted. In particular, our model provides an explanation for persistence of *per capita* income differences across countries. Economies which are equal in all respects except for differences in their initial conditions may find themselves eventually in quite different positions.

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