2. Propositional Logic: Semantics

Semantics

The symbols f and t are called *truth values*. An *interpretation* of a propositional signature σ is a function from σ into $\{\mathsf{f}, \mathsf{t}\}$. If σ is finite then an interpretation can be defined by the table of its values, for instance:

p	q	r
f	f	t

(1)

The semantics of propositional formulas that we are going to introduce defines which truth value is assigned to a formula F by an interpretation I.

As a preliminary step, we need to associate functions with all unary and binary connectives: a function from $\{\mathsf{f}, \mathsf{t}\}$ into $\{\mathsf{f}, \mathsf{t}\}$ with the unary connective $\neg$, and a function from $\{\mathsf{f}, \mathsf{t}\} \times \{\mathsf{f}, \mathsf{t}\}$ into $\{\mathsf{f}, \mathsf{t}\}$ with each of the binary connectives. These functions are denoted by the same symbols as the corresponding connectives, and defined by the following tables:

x	$\neg(x)$
f	t
t	f

x	y	$\wedge(x, y)$	$\vee(x, y)$	$\rightarrow (x, y)$	$\leftrightarrow (x, y)$
f	f	f	f	t	t
f	t	f	t	t	f
t	f	f	t	f	f
t	t	t	t	t	t

For any formula F and any interpretation I, the truth value F^I that is *assigned* to F by I is defined recursively, as follows:

- for any atom F, $F^I = I(F)$,
- $\bot^I = \mathsf{f}$, $\top^I = \mathsf{t}$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$.

If $F^I = \mathsf{t}$ then we say that the interpretation I *satisfies* F (symbolically, $I \models F$).

2.1^e Find a formula F of the signature $\{p, q, r\}$ such that (1) is the only interpretation satisfying F.

2.2^e For any formulas $F_1, \dots, F_n$ $(n \geq 1)$ and any interpretation I,

$$(F_1 \wedge \cdots \wedge F_n)^I = \mathsf{t} \text{ iff } F_1^I = \cdots = F_n^I = \mathsf{t},$$
$$(F_1 \vee \cdots \vee F_n)^I = \mathsf{f} \text{ iff } F_1^I = \cdots = F_n^I = \mathsf{f}.$$

If the underlying signature is finite then the set of interpretations is finite also, and the values of F^I for all interpretations I can be represented by a finite table. This table is called the *truth table* of F. For instance, Problem 2.1 could be stated as follows: Find a formula F whose truth table is

p	q	r	F
f	f	f	f
f	f	t	t
f	t	f	f
f	t	t	f
t	f	f	f
t	f	t	f
t	t	f	f
t	t	t	f

In the following two problems, we assume that the underlying signature is finite: $\sigma = \{p_1, \dots, p_n\}$.

2.3 For any interpretation I, there exists a formula F such that I is the only interpretation satisfying F.

2.4 For any function α from interpretations to truth values, there exists a formula F such that, for all interpretations I, $F^I = \alpha(I)$.

Tautologies

A propositional formula F is a *tautology* if every interpretation satisfies F.

2.5^e Determine which of the following formulas are tautologies:

$$(p \to q) \vee (q \to p),$$
$$((p \to q) \to p) \to p,$$
$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

Equivalent Formulas

A formula F is *equivalent* to a formula G (symbolically, $F \Leftrightarrow G$) if, for every interpretation I, $F^I = G^I$. In other words, $F \Leftrightarrow G$ means that formula $F \leftrightarrow G$ is a tautology.

2.6 (a) Conjunction and disjunction are associative:

$$(F \wedge G) \wedge H \Leftrightarrow F \wedge (G \wedge H),$$
$$(F \vee G) \vee H \Leftrightarrow F \vee (G \vee H).$$

Does equivalence have a similar property?

(b) Conjunction distributes over disjunction:

$$F \wedge (G \vee H) \Leftrightarrow (F \wedge G) \vee (F \wedge H);$$

disjunction distributes over conjunction:

$$F \vee (G \wedge H) \Leftrightarrow (F \vee G) \wedge (F \vee H).$$

Do these connectives distribute over equivalence?

2.7 (a) De Morgan's laws

$$\neg(F \wedge G) \Leftrightarrow \neg F \vee \neg G,$$
$$\neg(F \vee G) \Leftrightarrow \neg F \wedge \neg G$$

show how to transform a formula of the form $\neg(F \odot G)$ when $\odot$ is conjunction or disjunction. Find similar transformations for the cases when $\odot$ is implication or equivalence.

(b) Implication distributes over conjunction:

$$F \to (G \wedge H) \Leftrightarrow (F \to G) \wedge (F \to H).$$

Find a similar transformation for $(F \vee G) \to H$.

(c) To simplify a formula means to find an equivalent formula that is shorter. Simplify the following formulas:

(i) $F \leftrightarrow \neg F$,

(ii) $F \vee (F \wedge G)$,

(iii) $F \wedge (F \vee G)$,

(iv) $F \vee (\neg F \wedge G)$.

(v) $F \wedge (\neg F \vee G)$.

2.8 (a) For each of the formulas

$$p \wedge q,\ p \vee q,\ p \leftrightarrow q,\ \neg p, \top$$

find an equivalent formula that contains no connectives other than $\rightarrow$ and $\bot$.
(b) For each of the formulas

$$p \rightarrow q,\ p \wedge q$$

find an equivalent formula that contains no connectives other than $\leftrightarrow$ and $\vee$.

Adequate Sets of Connectives

2.9 For any formula, there exists an equivalent formula that contains no connectives other than $\rightarrow$ and $\bot$.

In this sense, $\{\rightarrow, \bot\}$ is an "adequate" set of connectives.

2.10^e If the underlying signature is non-empty then each of the sets

$$\{\wedge, \neg\},\ \{\vee, \neg\},\ \{\rightarrow, \neg\}$$

is adequate.

2.11 Any propositional formula equivalent to $\bot$ contains at least one of the connectives $\bot$, $\neg$.

This fact shows that the set $\{\wedge, \vee, \rightarrow, \leftrightarrow, \top\}$ is not adequate.

Disjunctive and Conjunctive Normal Forms

A *literal* is an atom or the negation of an atom. A *simple conjunction* is a formula of the form $L_1 \wedge \cdots \wedge L_n$ $(n \geq 1)$, where $L_1, \ldots, L_n$ are literals. A formula is in *disjunctive normal form* if it has the form $C_1 \vee \cdots \vee C_m$ $(m \geq 1)$, where $C_1, \ldots, C_m$ are simple conjunctions.

A *simple disjunction* is a formula of the form $L_1 \vee \cdots \vee L_n$ $(n \geq 1)$, where $L_1, \ldots, L_n$ are literals. A formula is in *conjunctive normal form* if it has the form $D_1 \wedge \cdots \wedge D_m$ $(m \geq 1)$, where $D_1, \ldots, D_m$ are simple disjunctions.

Theorem on Disjunctive and Conjunctive Normal Form: If the underlying signature is non-empty, then

(a) any formula is equivalent to a formula in disjunctive normal form;

(b) any formula is equivalent to a formula in conjunctive normal form.

Satisfiability and Entailment

A set Γ of formulas is *satisfiable* if there exists an interpretation that satisfies all formulas in Γ, and *unsatisfiable* otherwise.

2.12 A set $\{F_1, \ldots, F_n\}$ is unsatisfiable iff $\neg(F_1 \wedge \cdots \wedge F_n)$ is a tautology.

For any atom A, the literals A, $\neg A$ are said to be *complementary* to each other.

A set Γ of formulas *entails* a formula F (symbolically, $\Gamma \models F$), if every interpretation that satisfies all formulas in Γ satisfies F also. Note that the notation for entailment uses the same symbol as the notation for satisfaction introduced earlier, the difference being that the expression on the left is an interpretation (I) in one case and a set of formulas (Γ) in the other. The formulas entailed by Γ are also called the *logical consequences* of Γ.

2.13^e $F_1, \ldots, F_n \models G$ iff $(F_1 \wedge \cdots \wedge F_n) \to G$ is a tautology.

(In the first of these expressions, we dropped the braces $\{\,\}$ around $F_1, \ldots, F_n$.)

2.14 For any set Γ of formulas and any formula F, $\Gamma \models F$ iff the set $\Gamma \cup \{\neg F\}$ is unsatisfiable.