Given a CFG G, and an input string w, does G generate w?

$A_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$

Thm : $A_{CFG}$ is decidable

Bad idea : Design a TM M to try all derivations.

Q Why does not work?

That TM is a recognizer, not a decider.

We need a different (smart) idea.

Hint : P 2.26

Good idea : Convert G into Chomsky Normal Form.

- Need only consider a finite number of derivations for the given string.

"On input $\langle G, w\rangle$

1. Convert G into Chomsky Normal Form
2. List all derivations with $2|w|-1$ steps

The first undecidable problem

Given a TM M and a string w, does M accept w?

$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$

$A_{TM}$ is undecidable, but recognizable.

The following machine U recognizes $A_{TM}$.

U = "On input $\langle M, w\rangle$ where M is a TM and w is a string":

1. Simulate M on input w.
2. If M accepts, accept
   If M rejects, reject.

What if M does not halt?

U is an example of Universal Turing machine.
(Stored-program computers)

## The first undecidable problem

$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$ is undecidable. That is, there is no TM that decides $A_{TM}$.

Proof:
Assume for the sake of contradiction that there is such a TM $H$.

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{otherwise} \end{cases}$$

Consider a new TM $D$ that uses $H$ as a subroutine

$D$ = "On input $\langle M\rangle$ where $M$ is a TM,
1. Run $H$ on input $\langle M, \langle M\rangle\rangle$.
2. Output the opposite of what $H$ outputs

That is,

$$D(\langle M\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \text{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

What if $\langle M\rangle$ is $\langle D\rangle$?

## The first undecidable problem (Another proof)

Given a TM $M$ and a string $w$, does $M$ accept $w$?

$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$ is undecidable. That is, there is no TM that decides $A_{TM}$.

proof: Assume that $A_{TM}$ is decidable.
Then $D = \{\langle M\rangle : M \text{ does not accept } \langle M\rangle\}$ is decidable also, because

$$D = \{\langle M\rangle \mid \langle M, \langle M\rangle\rangle \notin A_{TM}\}$$

Let $M_D$ be a TM that decides $D$:
For any TM $M$,

$M_D$ accepts $\langle M\rangle$ iff $M$ does not accept $\langle M\rangle$.

In particular,

$M_D$ accepts $\langle M_D\rangle$ iff $M_D$ does not accept $\langle M_D\rangle$