Functional Stable Model Semantics and General Theory of Stable Models

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Abstract

Recently, several versions of functional stable model semantics have emerged in order to express defaults involving non-Herbrand functions. We present a new perspective on these formalisms by reducing them to the General Theory of Stable Models, which allows us to understand intensional functions in terms of intensional predicates. This perspective not only allows us to transfer theoretical results established for the General Theory of Stable Models, such as safety, splitting theorem, and strong equivalence, to functional stable model semantics, but also tells us how to make further extensions, such as allowing aggregates and generalized quantifiers. In addition, we show that the functional stable model semantics that allows partial functions is no more expressive than the one that does not, in the sense that partial functions can be eliminated in favor of total functions. Based on these results, we present an implementation of different versions of functional stable model semantics by using existing ASP solvers.

Introduction

The original stable model semantics (Gelfond and Lifschitz 1988) and many extensions have been restricted to Herbrand models, where the role of functions is quite limited. Several recent extensions of the stable model semantics are related to enriching the role of functions. In efforts to combine answer set programming and constraint solving, such as (Gebser, Ostrowski, and Schaub 2009; Janhunen, Liu, and Niemel 2011), constraint variables are essentially non-Herbrand functions, but these functions are not characterized by logic programs, and thus do not allow us to express defaults involving them.

In order to overcome such limitations in the semantics, several extensions of the stable model semantics that allow *intensional functions* have emerged, such as (Cabalar 2011; Lifschitz 2012; Bartholomew and Lee 2012; Balduccini 2013). Bartholomew and Lee showed that despite the different forms in which these semantics were defined (in terms of second-order logic, in terms of variants of the Logic of Here-and-There, or in terms of grounding and reduct), they can be essentially characterized in two groups: ensuring the stability of a model by checking the "minimality of partially defined functions" (Cabalar 2011; Balduccini 2013) vs. by checking the "uniqueness of totally defined functions" (Lifschitz 2012; Bartholomew and Lee 2012). While the relationship between some of them has been studied in (Bartholomew and Lee 2013b), less known is how traditional mathematical results established for answer set programming that have been developed in the absence of intensional functions would carry over to these extensions. Also, while the semantics relying on the concept of partial functions may seem more expressive, it is not clear whether this is indeed the case.

In this paper, we present a new perspective on those functional stable model semantics by relating them to the General Theory of Stable Models (Ferraris, Lee, and Lifschitz 2011)—an extension of the stable model semantics to first-order formulas containing "intensional predicates." In fact, the functional stable model semantics from (Bartholomew and Lee 2012) is a further extension of this formalism to incorporate "intensional functions" as well. What we establish in this paper is a simple view that functions can be represented in terms of predicates, which is well known in classical logic, but is not so obvious under the stable model semantics. In other words, we show how both versions of the functional stable model semantics, one relying on the concept of partial functions and the other relying on the concept of total functions, can be similarly reduced to the General Theory of Stable Models. This fact tells us that many mathematical results that were established for the General Theory of Stable Models, such as safety (Lee, Lifschitz, and Palla 2008), splitting theorem (Ferraris et al. 2009), and strong equivalence (Lifschitz, Pearce, and Valverde 2007), can be easily carried over to the functional stable model semantics. In addition, it easily allows for further extensions which have not been well-addressed in the context of the functional stable model semantics, such as aggregates and generalized quantifiers (Lee and Meng 2012).

Given that both versions of the functional stable model semantics can be expressed in terms of the General Theory of Stable Models, one may wonder about the relationship between the two versions of the functional stable model semantics. Interestingly, we show that the functional stable model semantics that is based on partial functions can be fully embedded into the one that is based on total functions.

These results provide a way to implement the functional stable model semantics using existing ASP solvers. We present system MVSM based on this idea. The system is essentially a preprocessor to F2LP, which in turn is a preprocessor to the ASP grounder GRINGO.

This paper is organized as follows. We first present our main results in the context of multi-valued propositional formulas, which can be viewed as a special case of first-order formulas containing intensional functions. We show how multi-valued propositional formulas under the partial and total function based stable model semantics can be turned into standard propositional formulas under the stable model semantics; we also show how the partal function based stable model semantics can be translated into the total function based stable model semantics. Further, we note that the use of strong negation is closely related to Boolean functions. Next, these results are generalized to the first-order case. Then we present an implementation of multi-valued propositional formulas using existing ASP solvers. A longer version with the complete proofs is available from `http://reasoning.eas.asu.edu/papers/newfsm-long.pdf`.

Multi-Valued Propositional Formulas under the Stable Model Semantics

We first discuss our main results in terms of multi-valued propositional formulas, a simple extension of standard propositional formulas that allows atoms to express functions over finite domains. The convenience of using multi-valued propositional formulas for knowledge representation is demonstrated in the context of nonmonotonic causal theories and action language $\mathcal{C}$+ (Giunchiglia et al. 2004). For this paper, multi-valued propositional formulas serve as a simple but useful special case of first-order formulas to compare different extensions of the functional stable model semantics.

We present two stable model semantics for multi-valued propositional formulas, which we call the BL-semantics, and the CB-semantics. The BL-semantics is from (Bartholomew and Lee 2012). The CB-semantics is a variant that is based on the concept of partial functions. It is essentially a special case of the semantics from (Cabalar 2011) and a generalization of the semantics from (Balduccini 2013).

Syntax and Models of Multi-Valued Propositional Formulas

A *(multi-valued propositional) signature (or, simply MVP-signature)* is a set σ of symbols called *constants*, along with a finite set *Dom*(c) of symbols that is disjoint from σ and contains at least two elements, assigned to each constant c. We call *Dom*(c) the *domain* of c. An *MVP-atom* of a signature σ is $\bot$, $\top$, or an expression of the form $c\!=\!v$ ("the value of c is v") where $c \in \sigma$ and $v \in Dom(c)$. A *multi-valued propositional formula (MVP-formula)* of σ is a propositional combination of atoms.

A *multi-valued propositional interpretation (MVP-interpretation)* of σ is a function that maps every element of σ to an element in its domain. An MVP-interpretation I *satisfies* an atom $c\!=\!v$ (symbolically, $I \models c\!=\!v$) if $I(c) = v$. The satisfaction relation is extended from atoms to arbitrary formulas according to the usual truth tables for the propositional connectives. We say that I is a *model* of F if it satisfies F.

A *Boolean* constant is one whose domain is the set $\{\text{FALSE}, \text{TRUE}\}$ of truth values. A *Boolean* signature is one all of whose constants are Boolean. For Boolean constant c, $c = \text{FALSE}$ is equivalent to $\neg(c = \text{TRUE})$, but as we will see below, this is not the case under the stable model semantics.

An expression of the form $c = d$, where both c and d are constants, will be understood as an abbreviation for the disjunction

$$\bigvee_{v \in Dom(c) \cap Dom(d)} (c\!=\!v \wedge d\!=\!v). \qquad (1)$$

BL-Stable Models of Multi-Valued Propositional Formulas

Let F be a formaula of an MVP-signature σ, and let I be an MVP-interpretation of σ. The *reduct* of F relative to I (denoted $F^{\underline{I}}$) is the MVP-formula obtained from F by replacing each (maximal) subformula that is not satisfied by I with $\bot$. I is a *BL-stable model* of F if I is the *only* MVP-interpretation of σ that satisfies $F^{\underline{I}}$.

Example 1 *Take $\sigma = \{c\}$ and Dom$(c) = \{1, 2, 3\}$, and let F be*

$$c\!=\!1 \vee \neg(c\!=\!1), \qquad (2)$$

and let I_i $(i = 1, 2, 3)$ be the interpretation that maps c to i. All three interpretations satisfy (2)*, but I_1 is the only stable model of F: the reduct $F^{\underline{I_1}}$ is $c\!=\!1 \vee \bot$, and I_1 is the only model of the reduct; the reduct of F_1 relative to other interpretations is $\bot \vee \neg\bot$, which does not have a unique model.*

If we conjoin $c = 2$ with (2)*, we can check that the only stable model is $c\!=\!2$.*

Under the BL-semantics, formula (2) represents that by default c is mapped to 1. Since this form of formulas is useful for representing defaults involving functions, we abbreviate formulas of the form $F \vee \neg F$ as $\{F\}$.

Example 2 *Formula*

$$\begin{array}{l}(p_0\!=\!\text{TRUE} \vee p_0\!=\!\text{FALSE})\\ \wedge\ (p_0\!=\!\text{TRUE} \rightarrow \{p_1\!=\!\text{TRUE}\})\\ \wedge\ (p_0\!=\!\text{FALSE} \rightarrow \{p_1\!=\!\text{FALSE}\})\end{array} \qquad (3)$$

has two BL-stable models: $\{p_0 = \text{TRUE}, p_1 = \text{TRUE}\}$ and $\{p_0\!=\!\text{FALSE}, p_1\!=\!\text{FALSE}\}$. The formula represents that the Boolean fluent p is exogenous at time 0 and the truth value is preserved by inertia. The last two conjunctive terms succinctly express the commonsense law of inertia.

Let σ be an MVP-signature, and let σ^{prop} be the propositional signature consisting of all propositional atoms $c = v$ where $c \in \sigma$ and $v \in Dom(c)$. We identify an MVP-interpretation of σ with the corresponding set of propositional atoms from σ^{prop}. For example, for σ in Example 1, σ^{prop} is the set $\{c = 1, c = 2, c = 3\}$, where each element is understood as a propositional atom; the interpretation I_1 is identified with a subset $\{c\!=\!1\}$. It is clear that an MVP-interpretation I of signature σ satisfies an MVP-formula F iff I satisfies F viewed as a propositional formula of signature σ^{prop}.

It is not difficult to show that multi-valued propositional formulas can be turned into standard propositional formulas while preserving models. Less obvious is whether such a relationship would hold while preserving stable models. Theorem 1 below shows that MVP-formulas under the BL-stable model semantics can be reduced to standard propositional formulas under the stable model semantics (Ferraris 2005).

Given a multi-valued propositional signature σ, we define UEC_σ ("Uniqueness and Existence Constraints") as the conjunction of formulas of σ^{prop}:

$$\neg\neg \bigvee_{v \in Dom(c)} c = v\ , \tag{4}$$

$$\bigwedge_{v \neq w \ \mid \ v,w \in Dom(c)} \neg(c = v \wedge c = w) \tag{5}$$

for each $c \in \sigma$.

Theorem 1 *Let F be an MVP-formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.*

(a) Every BL-stable model of F viewed as an MVP-formula of signature σ is a stable model of $F \wedge UEC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005).

(b) Every stable model of $F \wedge UEC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005) is a BL-stable model of F viewed as an MVP-formula of σ.

Example 1 Continued *$F \wedge UEC_\sigma$ is*

$$\begin{array}{l}(c{=}1 \vee \neg(c{=}1)) \wedge \neg\neg(c{=}1 \vee c{=}2 \vee c{=}3) \\ \wedge\ \neg(c{=}1 \wedge c{=}2) \wedge \neg(c{=}2 \wedge c{=}3) \wedge \neg(c{=}1 \wedge c{=}3)\ .\end{array} \tag{6}$$

Viewing this as a standard propositional formula of signature σ^{prop}, any model satisfies exactly one of atoms $c = 1$, $c{=}2$, and $c{=}3$ in the signature. The reduct of (6) relative to $\{c{=}1\}$ is

$$\begin{array}{l}(c{=}1 \vee \bot) \wedge \neg\bot \\ \wedge\ \neg\bot \wedge \neg\bot \wedge \neg\bot\ .\end{array}$$

Note that the presence of $\neg\neg$ in (4) is essential for Theorem 1 to be valid. Indeed, if we drop the double negations in (4), then the reduct of $F \wedge UEC_\sigma$ relative to $\{c{=}i\}$ ($i = 1, 2, 3$) is equivalent to $c{=}i$, so there would be three stable models of $F \wedge UEC_\sigma$, two of which do not correspond to the stable model I_1 of F.

CB-Stable Models of MVP-Formulas

In this section we introduce a variant of the stable model semantics in the previous section, which we call the CB-stable model semantics. This definition of a stable model follows the approach by Cabalar (2011), and later by Balduccini (2013), which allow functions to be partially defined. In other words, interpretations may not map some constants to values in their domains, but instead to some *undefined value* u, which does not belong to the domain of any constant. By *complete* interpretations, we mean a special case of partial interpretations where all constants are defined (i.e., no constants are mapped to u). Complete partial interpretations can be identified with usual "total" interpretations.

We consider the same syntax of a multi-valued propositional formula as in the previous section, which implies that no formula contains the undefined value u.

A partial interpretation I *satisfies* an atom $c = v$ if $c = v \in I$ (i.e., $I(c) = v$). This implies that an interpretation that maps c to u does not satisfy any atom $c{=}v$. As before, the satisfaction relation is extended from atoms to arbitrary formulas according to the usual truth tables for the propositional connectives. We call I a *model* of F if it satisfies F.

It is convenient to identify a partial interpretation I with the set of atoms of σ that are satisfied by this interpretation, that is, the set of atoms $c = I(c)$ such that $I(c) \neq$ u. For instance, an interpretation of $\sigma = \{c\}$ which maps c to u is partial, and can be identified with the empty set.

The reduct $F^{\underline{I}}$ is defined the same as before. We say that a partial interpretation I is a *CB-stable model* of F if I satisfies F and no subset J of I satisfies $F^{\underline{I}}$.

Example 1 Continued *Under the CB-semantics, $\{c = 1\}$ does not mean that c is mapped to 1 by default. Instead, it means that c can be mapped to 1 or the undefined value u. As before, the reduct $F^{\underline{I_1}}$ relative to $I_1 := \{c = 1\}$ is $c = 1 \vee \bot$, and I_1 is the minimal model of the reduct.*[1] *Further, the reduct $F^{\underline{I_0}}$ relative to $I_0 := \emptyset$ (i.e., $I_0(c) =$ u) is $\bot \vee \neg\bot$, and I_0 is the minimal model of the reduct.*

Example 2 Continued *Formula (3) has four CB-stable models. The two BL-stable models of (3) are also CB-stable models that are complete, and there are two other CB-stable models, $\{p_0 = \text{FALSE}\}$ and $\{p_1 = \text{FALSE}\}$, that are not complete.*

In the examples above, the complete CB-stable models coincide with the BL-stable models. Indeed, the following theorem tells us that this is the case in general.

Theorem 2 *For any MVP-formula F and any complete interpretation I, we have that I is a BL-stable model of F iff I is a CB-stable model of F.*

Similar to Theorem 1, the following theorem tells us that the CB-stable models of an MVP-formula can be identified with the stable models of a standard propositional formula. By UC_σ we denote the conjunction of formulas (5) for each $c \in \sigma$.

Theorem 3 *Let F be an MVP-formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.*

(a) Every CB-stable model of F viewed as an MVP-formula of signature σ is a stable model of $F \wedge UC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005).

(b) Every stable model of $F \wedge UC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005) is a CB-stable model of F viewed as an MVP-formula of σ.

Comparing Theorem 1 and Theorem 3, the only difference between BL-stable models and CB-stable models can be explained in terms of propositional formulas under the stable model semantics—whether we enforce both the uniqueness and existence of total function values vs. we enforce only the uniqueness of partial function values.

The definition of CB-stable models is essentially a simplification of the semantics in (Cabalar 2011), and a generalization of (Balduccini 2013). In the latter paper, $c = d$ where

[1] Minimality is understood in terms of set inclusion.

both c and d are constants were called *t-literals*, and the satisfaction was defined for t-literals directly: I satisfies $c = d$ if I satisfies both $c = v$ and $d = v$ for some value v other than u. This means that I does not satisfy $c = c$, but instead satisfies $\neg(c = c)$ when I maps c to u. This turns out to be equivalent to the way we understand $c = d$ as shorthand for (1): I satisfies (1) iff I satisfies both $c = v$ and $d = v$ for some value v other than u.

Representing Partial Stable Models for MVP-Formulas by Total Stable Models

The CB-stable model semantics may seem more expressive as it allows interpretations to be partially defined: any BL-stable model of F is a CB-stable model, but not vice versa because an incomplete partial interpretation has no counterpart in total interpretations. However, in this section we show that it is possible to embed the CB-stable model semantics into the BL-stable model semantics.

Let σ be an MVP-signature, and let σ^{none} be the signature that is the same as σ except that the domain of each constant has an additional new value NONE. Given a partial interpretation I of σ, by I^{none} we denote an interpretation that agrees with I on all defined constants, and maps to NONE on all undefined constants. That is,

$$I^{none}(c) = \begin{cases} I(c) & \text{if } I(c) \neq \mathrm{u}; \\ \text{NONE} & \text{otherwise.} \end{cases}$$

For example, for the interpretation I_0 in Example 1, I_0^{none} maps c to NONE. Same as I_1, I_1^{none} maps c to 1.

Recall that expression $\{F\}$ stands for the formula $F \vee \neg F$.

Theorem 4 *Let F be an MVP-formula of signature σ.*

(a) An interpretation I of σ is a CB-stable model of F iff I^{none} is a BL-stable model of

$$F \wedge \bigwedge_{c \in \sigma} \{c = \text{NONE}\}$$

of σ^{none}.

(b) An interpretation J of σ^{none} is a BL-stable model of

$$F \wedge \bigwedge_{c \in \sigma} \{c = \text{NONE}\}$$

iff $J = I^{none}$ for some CB-stable model I of F.

Example 2 Continued *The formula* (3) *conjoined with* $\{p_0 = \text{NONE}\} \wedge \{p_1 = \text{NONE}\}$ *has four BL-stable models:* $\{p_0 = \text{TRUE}, p_1 = \text{TRUE}\}$, $\{p_0 = \text{FALSE}, p_1 = \text{FALSE}\}$, $\{p_0 = \text{TRUE}, p_1 = \text{NONE}\}$, *and* $\{p_0 = \text{FALSE}, p_1 = \text{NONE}\}$. *Each of them corresponds to the four CB-stable models of* (3).

Strong Negation in Functional View

Interestingly, the translation $F \wedge UC_\sigma$ in Theorem 3 is similar to the way that strong negation (denoted $\sim$) is understood in terms of auxiliary atoms. In (Ferraris, Lee, and Lifschitz 2011), the stable model semantics of formulas involving strong negation is defined as follows. For any propositional signature σ, σ^{sneg} is the set of propositional atoms consisting of c and $\sim c$ for all $c \in \sigma$. We consider *coherent* interpretations of σ^{sneg} only, which do not satisfy both c and $\sim c$. For any formula F of signature σ^{sneg}, an interpretation I of σ^{sneg} is a stable model of F if I is a minimal model of the reduct F relative to I.

Given a propositional signature σ, by σ^{bool} we denote the MVP-signature where each $c \in \sigma$ is understood as a Boolean constant. We identify an interpretation of σ^{sneg} with an interpretation of σ^{bool} as follows:

- the MVP-interpretation of σ^{bool} maps c to TRUE if the propositional interpretation of σ^{sneg} maps c to TRUE and $\sim c$ to FALSE;
- the MVP-interpretation of σ^{bool} maps c to FALSE if the propositional interpretation of σ^{sneg} maps $\sim c$ to TRUE and c to FALSE;
- the MVP-interpretation of σ^{bool} maps c to u if the propositional interpretation of σ^{sneg} maps both c and $\sim c$ to FALSE.

We identify a propositional formula F of signature σ^{sneg} with an MVP-formula of σ^{bool} by identifying the occurrence of c in F with $c = \text{TRUE}$ and $\sim c$ with $c = \text{FALSE}$. The following corollary, which follows from Theorem 3, justifies such identification.

Corollary 1 *Let F be a formula of signature σ^{sneg}, which can be also viewed as an MVP-formula of signature σ^{bool}. We have that I is a coherent stable model of F viewed as a propositional formula of σ^{sneg} iff I is a CB-stable model of F viewed as an MVP-formula of signature σ^{bool}.*

Syntactically strong negation is not a connective; it occurs only in front of propositional atoms. This aligns with the view that $\sim c$ is identified with $c = \text{FALSE}$. Also, replacing $\sim c$ in a formula with $\neg c$, or vice versa, affect stable models in general. Likewise, replacing $c = \text{FALSE}$ in a formula with $\neg(c = \text{TRUE})$, or vice versa, affects its CB-stable models.

Example 3 *The formula*

$$(p_0 \vee \sim p_0) \wedge (p_0 \wedge \neg \sim p_1 \rightarrow p_1) \wedge (\sim p_0 \wedge \neg p_1 \rightarrow \sim p_1) \quad (7)$$

expresses that fact p is either true or false initially, and is inertial—both $\{p_0, p_1\}$ and $\{\sim p_0, \sim p_1\}$ are the stable models. In accordance with Corollary 1, the corresponding formula of σ^{bool}

$$\begin{aligned} &(p_0 = \text{TRUE} \vee p_0 = \text{FALSE}) \\ &\wedge (p_0 = \text{TRUE} \wedge \neg(p_1 = \text{FALSE}) \rightarrow p_1 = \text{TRUE}) \\ &\wedge (p_0 = \text{FALSE} \wedge \neg(p_1 = \text{TRUE}) \rightarrow p_1 = \text{FALSE}) \end{aligned} \quad (8)$$

has two CB-stable models: $\{p_0 = \text{TRUE}, p_1 = \text{TRUE}\}$ *and* $\{p_0 = \text{FALSE}, p_1 = \text{FALSE}\}$.

A similar result holds with the BL-stable model semantics. The main difference is that the statement considers total interpretations only.

Corollary 2 *Let F be a formula of signature σ^{sneg}, which can be also viewed as a multi-valued formula of signature σ^{bool}. For any total interpretation I of σ^{bool}, we have that I is a stable model of F viewed as a propositional formula of σ^{sneg} iff I is a BL-stable model of F viewed as an MVP-formula of signature σ^{bool}.*

Example 3 Continued *Formula* (8) *has two BL-stable models, which correspond to the two stable models of* (7). *Under the BL-semantics,* (8) *is strongly equivalent to*

$$
\begin{array}{l}
(p_0 = \text{TRUE} \vee p_0 = \text{FALSE}) \\
\wedge\ (p_0 = \text{TRUE} \wedge \neg\neg(p_1 = \text{TRUE}) \rightarrow p_1 = \text{TRUE}) \\
\wedge\ (p_0 = \text{FALSE} \wedge \neg\neg(p_1 = \text{FALSE}) \rightarrow p_1 = \text{FALSE})
\end{array} \tag{9}
$$

which is further strongly equivalent to (3). *On the other hand, under the CB-semantics,* (9) *has two other stable models than* (8), *which are* $\{p_0 = \text{TRUE}\}$ *and* $\{p_0 = \text{FALSE}\}$.

Stable Model Semantics of Intensional Functions

In this section we generalize the results in the previous section to the first-order level. We first define functional stable model semantics to first-order formulas. This definition is a slight modification to the definition in (Bartholomew and Lee 2012) and turns out to be equivalent to the definition in (Cabalar 2011) when we consider total stable models only. It is simpler than the latter definition because it does not use a rather complex notion of satisfaction for partial functions. Further, it is closely related to the General Theory of Stable Models.

Review: Stable Models of c-Plain Formulas

c-Plain formulas Let f be a function constant. A first-order formula is called *f-plain* (Lifschitz and Yang 2011) if each atomic formula

- does not contain f, or
- is of the form $f(\mathbf{t}) = t$ where $\mathbf{t}$ is a tuple of terms not containing f, and t is a term not containing f.

For example, $f = 1$ is f-plain, but each of $p(f)$, $g(f) = 1$, and $1 = f$ is not f-plain.

For any list $\mathbf{c}$ of predicate and function constants, we say that F is **c**-*plain* if F is f-plain for each function constant f of $\mathbf{c}$.

It is known that the BL semantics and the CB semantics coincide on $\mathbf{c}$-plain formulas when we consider complete interpretations only (Bartholomew and Lee 2013b). Below we review the BL-semantics in term of grounding and reduct.

Infinitary Ground Formulas Since the universe can be infinite, grounding a quantified sentence introduces infinite conjunctions and disjunctions over the elements in the universe. Here we rely on the concept of grounding *relative to an interpretation* from (Truszczynski 2012). The following is the definition of an *infinitary ground formula*, which is adapted from (Truszczynski 2012). A main difference is that we allow atomic formulas to be any ground atomic formulas in the sense of first-order logic, rather than limiting attention to propositional atoms as in (Truszczynski 2012).

For each element ξ in the universe $|I|$ of I, we introduce a new symbol $\xi^\diamond$, called an *object name*. By σ^I we denote the signature obtained from σ by adding all object names $\xi^\diamond$ as additional object constants. We will identify an interpretation I of signature σ with its extension to σ^I defined by $I(\xi^\diamond) = \xi$.[2]

[2] For details, see (Lifschitz, Morgenstern, and Plaisted 2008).

We assume the primary connectives to be $\bot$, $\{\}^\wedge$, $\{\}^\vee$, and $\rightarrow$. Propositional connectives $\wedge, \vee, \neg, \top$ are considered as shorthands: $F \wedge G$ as $\{F, G\}^\wedge$; $F \vee G$ as $\{F, G\}^\vee$. $\neg$ and $\top$ are defined as before.

Let A be the set of all ground atomic formulas of signature σ^I. The sets $\mathcal{F}_0, \mathcal{F}_1, \dots$ are defined recursively as follows:

- $\mathcal{F}_0 = A \cup \{\bot\}$;
- $\mathcal{F}_{i+1} (i \geq 0)$ consists of expressions $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$, for all subsets $\mathcal{H}$ of $\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_i$, and of the expressions $F \rightarrow G$, where $F, G \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_i$.

We define $\mathcal{L}_A^{inf} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$, and call elements of $\mathcal{L}_A^{inf}$ *infinitary ground formulas* of σ w.r.t. I.

For any interpretation I of σ and any infinitary ground formula F w.r.t. I, the definition of satisfaction, $I \models F$, is as follows:

- For atomic formulas, the definition of satisfaction is the same as in the standard first-order logic;
- $I \models \mathcal{H}^\wedge$ if, for every formula $G \in \mathcal{H}$, $I \models G$;
- $I \models \mathcal{H}^\vee$ if there is a formula $G \in \mathcal{H}$ such that $I \models G$;
- $I \models G \rightarrow H$ if $I \not\models G$ or $I \models H$.

Stable Models of c-Plain Formulas Let F be any first-order sentence of a signature σ, and let I be an interpretation of σ. By $gr_I[F]$ we denote the infinitary ground formula w.r.t. I that is obtained from F by the following process:

- If F is an atomic formula, $gr_I[F]$ is F;
- $gr_I[G \odot H] = gr_I[G] \odot gr_I[H] \quad (\odot \in \{\wedge, \vee, \rightarrow\})$;
- $gr_I[\forall x G(x)] = \{gr_I[G(\xi^\diamond)] \mid \xi \in |I|\}^\wedge$;
- $gr_I[\exists x G(x)] = \{gr_I[G(\xi^\diamond)] \mid \xi \in |I|\}^\vee$.

For any two interpretations I, J of the same signature and any list $\mathbf{c}$ of distinct predicate and function constants, we write $J <^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants p in $\mathbf{c}$; and
- J and I do not agree on $\mathbf{c}$.

The *reduct* $F^{\underline{I}}$ of an infinitary ground formula F relative to an interpretation I is defined as follows:

- For each atomic formula F, $F^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models F; \\ F & \text{otherwise}; \end{cases}$
- $(\mathcal{H}^\wedge)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models \mathcal{H}^\wedge; \\ \{G^{\underline{I}} \mid G \in \mathcal{H}\}^\wedge & \text{otherwise}; \end{cases}$
- $(\mathcal{H}^\vee)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models \mathcal{H}^\vee; \\ \{G^{\underline{I}} \mid G \in \mathcal{H}\}^\vee & \text{otherwise}; \end{cases}$
- $(G \rightarrow H)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models G \rightarrow H; \\ G^{\underline{I}} \rightarrow H^{\underline{I}} & \text{otherwise}. \end{cases}$

Definition 1 *let* $\mathbf{c}$ *be a list of predicate and function constants, and let* F *be a* $\mathbf{c}$*-plain first-order sentence of signature* σ. *An interpretation* I *of* σ *is a* stable model *of* F *relative to* $\mathbf{c}$ *if*

- *I satisfies F, and*
- *every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[F])^{\underline{I}}$.*

Relation to MVP-Formulas MVP-formulas can be viewed as a special case of $\mathbf{c}$-plain formulas, for which we limit attention to some special case of first-order interpretations.

Let σ be a multi-valued signature and let σ^{fo} be a first-order signature consisting of all object constants c in σ, and all object constants v for $v \in Dom(c)$ for every $c \in \sigma$. Given a multi-valued interpretation I, we define the first-order interpretation I^{fo} as follows:

- $|I^{fo}| = \{v \mid v \in Dom(c)$ for some $c \in \sigma\}$;
- $I^{fo}(v) = v$ for each $v \in Dom(c)$ for each $c \in \sigma$;
- $I^{fo}(c) = I(c)$ for each multi-valued constant $c \in \sigma$.

Proposition 1 *For any MVP-formula F of signature σ, and any MVP-interpretation I of σ whose multi-valued constants are $\mathbf{c}$, I is a BL-stable model of F iff I^{fo} is a stable model of F relative to $\mathbf{c}$ viewed as a first-order formula of signature σ^{fo}.*

Stable Models of Arbitrary Formulas

We extend Definition 1 to non-$\mathbf{c}$-plain formulas by first "unfolding" them to eliminate nested intensional function constants.

The process of *unfolding F w.r.t.* $\mathbf{c}$, denoted by $UF_{\mathbf{c}}(F)$, is formally defined as follows.

- If F is of the form $p(t_1, \dots, t_n)$ $(n \geq 0)$ such that $t_{k_1}, \dots, t_{k_j}$ are all the terms in $t_1, \dots, t_n$ that contain some members of $\mathbf{c}$, then $UF_{\mathbf{c}}(p(t_1, \dots, t_n))$ is
$$\exists x_1 \dots x_j \Big(p(t_1, \dots, t_n)'' \wedge \bigwedge_{1 \leq i \leq j} UF_{\mathbf{c}}(t_{k_i} = x_i) \Big)$$
where $p(t_1, \dots, t_n)''$ is obtained from $p(t_1, \dots, t_n)$ by replacing each t_{k_i} with the variable x_i.
- If F is of the form $f(t_1, \dots, t_n) = t_0$ $(n \geq 0)$ such that $t_{k_1}, \dots, t_{k_j}$ are all the terms in $t_0, \dots, t_n$ that contain some members of $\mathbf{c}$, then $UF_{\mathbf{c}}(f(t_1, \dots, t_n) = t_0)$ is
$$\exists x_1 \dots x_j \Big((f(t_1, \dots, t_n) = t_0)'' \wedge \bigwedge_{0 \leq i \leq j} UF_{\mathbf{c}}(t_{k_i} = x_i) \Big)$$
where $(f(t_1, \dots, t_n) = t_0)''$ is obtained from $f(t_1, \dots, t_n) = t_0$ by replacing each t_{k_i} with the variable x_i.
- $UF_{\mathbf{c}}(F \odot G)$ is $UF_{\mathbf{c}}(F) \odot UF_{\mathbf{c}}(G)$ where $\odot \in \{\wedge, \vee, \rightarrow\}$.
- $UF_{\mathbf{c}}(QxF)$ is $Qx\ UF_{\mathbf{c}}(F(x))$ where $Q \in \{\forall, \exists\}$.

For example, $UF_f(p(f))$ is $\exists x(p(x) \wedge f = x)$; $UF_{(f,g)}(f = g)$ is $\exists x(f = x \wedge g = x)$.

Definition 2 *let $\mathbf{c}$ be a list of predicate and function constants, and let F be any first-order sentence of signature σ. An interpretation I of σ is a* stable model *of F relative to $\mathbf{c}$ if*

- *I satisfies F, and*
- *every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[UF_{\mathbf{c}}(F)])^{\underline{I}}$.*

Relation to General Stable Models

Corollary 1 from (Bartholomew and Lee 2012) shows how to eliminate intensional predicates in favor of intensional functions, which tells us that the General Theory of Stable Models can be embedded into the BL-stable model semantics. The other direction is also possible: just as MVP-formulas can be identified with standard propositional formulas with additional constraints representing the uniqueness and the existence of function values, the stable model semantics in this section can be identified with formulas under the General Theory of Stable Models (Ferraris, Lee, and Lifschitz 2011) as follows.

Let σ be a signature as in first-order logic. $\sigma^{pred[\mathbf{c}]}$ is the signature obtained from σ by replacing each function constant f in $\mathbf{c}$ with a new predicate constant p of arity $n+1$, where n is the arity of f. For any $\mathbf{c}$-plain formula G of signature σ, $G^{pred[\mathbf{c}]}$ is the formula of signature $\sigma^{pred[\mathbf{c}]}$ that is obtained from G by replacing each subformula $f(\mathbf{t}) = t$ where $f \in \mathbf{c}$ with $p(\mathbf{t}, t)$.

For any interpretation I of signature σ, by $I^{pred[\mathbf{c}]}$ we denote the interpretation of signature $\sigma^{pred[\mathbf{c}]}$ that is obtained from I by replacing each function f^I with the set p^I that consists of the tuples
$$\langle \xi_1, \dots, \xi_n, f^I(\xi_1, \dots, \xi_n) \rangle$$
for all $\xi_1, \dots, \xi_n$ from the universe of I.

By UEC_f we denote the following formulas that enforce the functional image on the predicates, which generalizes (4) and (5) to first-order formulas:
$$\begin{array}{c} \forall \mathbf{x} y z (y \neq z \wedge f(\mathbf{x}) = y \wedge f(\mathbf{x}) = z \rightarrow \bot), \\ \neg\neg \forall \mathbf{x} \exists y\, (f(\mathbf{x}) = y), \end{array} \tag{10}$$
where $\mathbf{x}$ is a n-tuple of variables, and all variables in $\mathbf{x}$, y, and z are pairwise distinct. Note that each formula is a constraint. By $UEC_{\mathbf{c}}$ we denote the conjunction of UEC_f for all function constants f in $\mathbf{c}$. $\mathbf{c}^{pred}$ denotes the list of the predicate constants in $\sigma^{pred[\mathbf{c}]}$ that corresponds to $\mathbf{c}$, where all function constants in $\mathbf{c}$ are replaced with predicate constants as described above.

The following theorem is a generalization of Theorem 1. It is similar to Corollary 2 of (Bartholomew and Lee 2012) except that it applies to non-$\mathbf{c}$-plain formulas as well.

Theorem 5 *Let $\mathbf{c}$ be a list of predicate and function constants, and let F be a first-order sentence of signature σ.*

(a) An interpretation I of σ that satisfies $\exists xy(x \neq y)$ is a stable model of F relative to $\mathbf{c}$ in our sense iff $I^{pred[\mathbf{c}]}$ is a stable model of $(UF_{\mathbf{c}}(F) \wedge UEC_{\mathbf{c}})^{pred[\mathbf{c}]}$ relative to $\mathbf{c}^{pred}$ in the sense of (Ferraris, Lee, and Lifschitz 2011).

(b) An interpretation J of $\sigma^{pred[\mathbf{c}]}$ that satisfies $\exists xy(x \neq y)$ is a stable model of $(UF_{\mathbf{c}}(F) \wedge UEC_{\mathbf{c}})^{pred[\mathbf{c}]}$ relative to $\mathbf{c}^{pred}$ in the sense of (Ferraris, Lee, and Lifschitz 2011) iff $J = I^{pred[\mathbf{c}]}$ for some stable model I of F relative to $\mathbf{c}$ in our sense.

Example 4 *To see why we need the condition that there should be at least two elements in the universe (i.e., $\exists xy(x \neq y)$), consider the formula $\top$ with signature $\sigma = \{c\}$ and universe $\{1\}$. There is only one interpretation and it maps c to*

1. *This is a stable model of* $\top$. *On the other hand considering the same universe but signature* $\sigma^{pred[\mathbf{c}]}$, *the formula* $(UF_{\mathbf{c}}(\top) \wedge UEC_{\mathbf{c}})^{pred[\mathbf{c}]}$ *is equivalent to* $\top \wedge \neg\neg p(1)$, *which has no stable models.*

Theorem 5 tells us that a formula under the functional stable model semantics defined in this section can be viewed as a formula under the General Theory of Stable Models.

One advantage of this characterization is that mathematical results established for the General Theory of Stable Models can be easily carried over to the functional stable model semantics. For instance, the splitting theorem in (Ferraris et al. 2009) and the theorem on loop formulas (Lee and Meng 2011) can be applied to the functional stable model semantics. The first-order version of the Logic of Here-and-There in (Lifschitz, Pearce, and Valverde 2007), which characterizes strong equivalence for the General Theory of Stable Models, can be used to prove strong equivalence for the functional stable model semantics.

Also, the characterization in Theorem 5 tells us how to further extend the functional stable model semantics to allow aggregates and generalized quantifiers, which was already well studied in the context of the General Theory of Stable Models. For instance,

$$\#count\langle x \ : \ Loc(x)\!=\!b\rangle > 3 \rightarrow TooMany(b)$$

would express that $TooMany(b)$ is true if there are more than 3 blocks on top of block b. The semantics of such a language can be defined by first eliminating intensional functions in favor of intensional predicates as in Theorem 5, and then referring to the stable model semantics of generalized quantifiers in (Lee and Meng 2012).

Relation to the Stable Model Semantics from (Bartholomew and Lee 2012)

Definition 2 is almost the same as the reformulation of the BL-semantics given in (Bartholomew and Lee 2013b) except that instead of $(gr_I[UF_{\mathbf{c}}(F)])^{\underline{I}}$, the reduct there is defined to be $(gr_I[F])^{\underline{I}}$. Obviously this difference does not affect stable models of $\mathbf{c}$-plain formulas, but may affect stable models of non-$\mathbf{c}$-plain formulas. For instance, consider F to be $p(f)$ where p is a predicate constant and f is an object constant, and take an interpretation I such that $p^I = \{1,2\}$ and $f^I = 1$. I is not a stable model of F relative to f, but is a stable model of $UF_f(F)$, which is $\exists x(p(x) \wedge f = x)$. For another instance, for the formula

$$f(1) = 1 \wedge f(2) = 1 \wedge (f(g) = 1 \rightarrow g = 1), \tag{11}$$

and an interpretation I such that the universe $|I|$ is $\{1,2\}$, and $1^I = 1$, $2^I = 2$, $f^I(1) = 1$, $f^I(2) = 1$, $g^I = 1$, one can check that I is a stable model of (11) relative to (f,g) according to (Bartholomew and Lee 2012). One may argue that this involves a cyclic dependency because the fact that the stable model maps g to 1 is "supported" by the fact that g is mapped to 1. On the other hand, in this paper we identify (11) with $UF_{(f,g)}((11))$,

$$f(1) = 1 \wedge f(2) = 1 \wedge (\exists x(f(x) = 1 \wedge g = x) \rightarrow g = 1),$$

which has no stable models. Indeed, this formula, when viewed as an abbreviation of a formula under the General Theory of Stable Models as in Theorem 5, obviously has a cyclic dependency involving $g = 1$.

Below we present a class of non-$\mathbf{c}$-plain formulas for which the semantics in this paper coincides with the semantics from (Bartholomew and Lee 2012). An occurrence of a symbol or a subformula in a formula F is called *strictly positive* in F if that occurrence is not in the antecedent of any implication in F (Ferraris, Lee, and Lifschitz 2011). We say that a formula is *head-$\mathbf{c}$-plain* if every strictly positively occurring atomic formula is $\mathbf{c}$-plain. For instance, $f(g) = 1 \rightarrow h = 1$ is head-(f,g,h)-plain, though it is not (f,g,h)-plain.

Theorem 6 *For any head-$\mathbf{c}$-plain sentence F of signature σ that is tight on $\mathbf{c}$, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, I is a stable model of F relative to $\mathbf{c}$ in the sense of (Bartholomew and Lee 2012) iff I is a stable model of F relative to $\mathbf{c}$ in the sense of Definition 2.* [3]

Both examples above do not satisfy the condition of Theorem 6: formula $p(f)$ is not head-f-plain, and formula (11) is not tight on (f,g).

del Cerro, Pearce, and Valverde (2013) present a variant of the Logic-of-Here-and-There, called system FHTG, which can be used to prove strong equivalence for the semantics in (Bartholomew and Lee 2012). However, contrary to their claim, their result does not apply to non-$\mathbf{c}$-plain formulas. According to (del Cerro, Pearce, and Valverde 2013),

$$p(f) \leftrightarrow \exists x(p(x) \wedge f = x) \tag{12}$$

is a theorem of the deductive system FHTG, but as we saw above, their stable models are different. We conjecture that this mismatch does not arise when their theorem is limited to $\mathbf{c}$-plain formulas.

Relation to the Semantics from (Cabalar 2011)

Review: Cabalar Semantics We first define the notion of a *partial* interpretation, which is a generalization of a partial interpretation for MVP-formulas. Given a first-order signature σ comprised of function and predicate constants, a *partial* interpretation I of σ consists of

- a non-empty set $|I|$, called the *universe* of I;
- for every function constant f of arity n, a function f^I from $(|I| \cup \{\mathrm{u}\})^n$ to $|I| \cup \{\mathrm{u}\}$, where u is not in $|I|$ ("u" stands for *undefined*);
- for every predicate constant p of arity n, a function p^I from $(|I| \cup \{\mathrm{u}\})^n$ to $\{\text{TRUE}, \text{FALSE}\}$.

For each term $f(t_1, \dots, t_n)$, we define

$$f(t_1,\dots,t_n)^I = \begin{cases} \mathrm{u} & \text{if } t_i^I = \mathrm{u} \text{ for some } i; \\ f^I(t_1^I,\dots,t_n^I) & \text{otherwise.} \end{cases}$$

The satisfaction relation $\models_{\mathrm{p}}$ between a partial interpretation I and a first-order formula F is the same as the one for first-order logic except for the following base cases:

- For each atomic formula $p(t_1, \dots, t_n)$,

$$p(t_1,\dots,t_n)^I = \begin{cases} \text{FALSE} & \text{if } t_i^I = \mathrm{u} \text{ for some } i; \\ p^I(t_1^I,\dots,t_n^I) & \text{otherwise.} \end{cases}$$

[3] Tight formulas are defined in (Bartholomew and Lee 2013a).

- For each atomic formula $t_1 = t_2$,

$$(t_1 = t_2)^I = \begin{cases} \text{TRUE} & \text{if } t_1^I \neq \text{u}, t_2^I \neq \text{u}, \text{ and } t_1^I = t_2^I; \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We say that $I \models_{\text{p}} F$ if $F^I = \text{TRUE}$.

Observe that under a partial interpretation, $t = t$ is not necessarily true: $I \not\models_{\text{p}} t = t$ iff $t^I = \text{u}$. On the other hand, $\neg(t_1 = t_2)$, also denoted by $t_1 \neq t_2$, is true under I even when both t_1^I and t_2^I are mapped to the same u.

Given any two partial interpretations J and I of the same signature σ, and a set of constants $\mathbf{c}$, we write $J \preceq^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants in $\mathbf{c}$; and
- $f^J(\boldsymbol{\xi}) = \text{u}$ or $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ for all function constants in $\mathbf{c}$ and all lists $\boldsymbol{\xi}$ of elements in the universe.

We write $J \prec^{\mathbf{c}} I$ if $J \preceq^{\mathbf{c}} I$ but not $I \preceq^{\mathbf{c}} J$. Note that $J \prec^{\mathbf{c}} I$ is defined similar to $J <^{\mathbf{c}} I$ except for the treatment of functions.

Definition 3 *Let F be a first-order sentence of signature σ and let $\mathbf{c}$ be a list of intensional constants. A partial interpretation I of σ is a Cabalar stable model of F relative to $\mathbf{c}$ if*

- *$I \models_{\text{p}} F$, and*
- *for every partial interpretation J of σ such that $J \prec^{\mathbf{c}} I$, we have $J \not\models_{\text{p}} gr_I[F]^{\underline{I}}$.*

Unlike Definition 1, Definition 3 remains equivalent even if $gr_I[F]^{\underline{I}}$ is replaced with $(gr_I[UF_{\mathbf{c}}(F)])^{\underline{I}}$ as shown in (Bartholomew and Lee 2013b, Theorem 7).

The following theorem is a generalization of Theorem 2.

Theorem 7 *For any formula F and any total interpretation I, we have that I is a stable model of F relative to $\mathbf{c}$ (in the sense of Definition 2) iff I is a Cabalar stable model of F relative to $\mathbf{c}$.*

Relation to ASPMT

Given that the functional stable model semantics is reducible to the General Theory of Stable Models, and vice versa, one might wonder what is the advantage of the distinction. As in classical logic, the functional stable model semantics allows a natural representation involving functions. Further, it leads to computational advantages by facilitating the connection to Constraint Processing (CP) and Satisfiability Modulo Theories (SMT), where functions are primary constructs. Bartholomew and Lee (2013a) define the language of Answer Set Programming Modulo Theories (ASPMT) as a special case of the functional stable model semantics with a fixed background interpretation for designated constants, and showed that a fragment of ASPMT instances can be turned into the input language of SMT solvers, effectively handling large numeric domains or even real number domains because variables representing values of functions do not have to be grounded.

Eliminating Partial Functions

We show how partial functions can be eliminated in favor of total functions. The work tells us that instead of relying on the rather complex notion of satisfaction involving partial functions, one can achieve the same effect in terms of the standard satisfaction using total functions.

Let F be a first-order sentence of a signature σ. F^{none} is the formula of signature $\sigma \cup \{\text{NONE}\}$ (where $\text{NONE} \notin \sigma$) that is obtained from F as follows.

- for any atomic formula F, $F^{none} = F$;
- $(G \odot H)^{none} = (G^{none} \odot H^{none})$ where $\odot \in \{\wedge, \vee, \rightarrow\}$;
- $\forall x G(x)^{none}$ is $\forall x(x \neq \text{NONE} \rightarrow G(x)^{none})$;
- $\exists x G(x)^{none}$ is $\exists x(G(x)^{none} \wedge x \neq \text{NONE})$.

Let $\text{UNDEF}_{\mathbf{c}}$ stand for the conjunction of

$$\forall \mathbf{x}(f(\mathbf{x}) = \text{NONE} \vee \neg(f(\mathbf{x}) = \text{NONE}))$$

for each function constant f in $\mathbf{c}$.

Given a partial interpretation I, we define the total interpretation I^{none} as

- $|I^{none}| = |I| \cup \{\text{NONE}\}$;
- $\text{NONE}^{I^{none}} = \text{NONE}$;
- for every function constant $f \in \sigma$ and $\boldsymbol{\xi} \in |I^{none}|^n$ where n is the arity of f,

$$f^{I^{none}}(\boldsymbol{\xi}) = \begin{cases} f^I(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \text{ is in } |I|^n \text{ and } f^I(\boldsymbol{\xi}) \neq \text{u}; \\ \text{NONE} & \text{otherwise;} \end{cases}$$

- For every predicate $p \in \sigma$ and $\boldsymbol{\xi} \in |I^{none}|^n$ where n is the arity of p,

$$p^{I^{none}}(\boldsymbol{\xi}) = \begin{cases} p^I(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \text{ is in } |I|^n; \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

The following theorem is a generalization of Theorem 4.

Theorem 8 *For any sentence F of signature σ,*

(a) I is a Cabalar stable model of F relative to $\mathbf{c}$ iff I^{none} is a stable model of $(UF_\sigma(F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ relative to $\mathbf{c}$.

(b) An interpretation J such that $\text{NONE}^J = \text{NONE}$ is a stable model of $(UF_\sigma(F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ relative to $\mathbf{c}$ iff $J = I^{none}$ for some Cabalar stable model I of F relative to $\mathbf{c}$.

Example 5 *Let F be $f = f$ and let $\mathbf{c}$ be f, let I be such that the universe is $\{1, 2\}$. There are two Cabalar stable models of this formula: $\{f = 1\}$ and $\{f = 2\}$. On the other hand, $(UF_\sigma(F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ is $\exists x(f = x \wedge x \neq \text{NONE}) \wedge (f = \text{NONE} \vee \neg(f = \text{NONE}))$, whose stable models are the same as the Cabalar stable models.*

For $\neg F$, $\{f = \text{u}\}$ is the only Cabalar stable model. $(UF_\sigma(\neg F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ is $\neg(\exists x(f = x \wedge x \neq \text{NONE})) \wedge (f = \text{NONE} \vee \neg(f = \text{NONE}))$, and $\{f = \text{NONE}\}$ is the only stable model.

In the special case when $\mathbf{c}$ is empty, Theorem 8 tells us how to eliminate partial functions while preserving models.

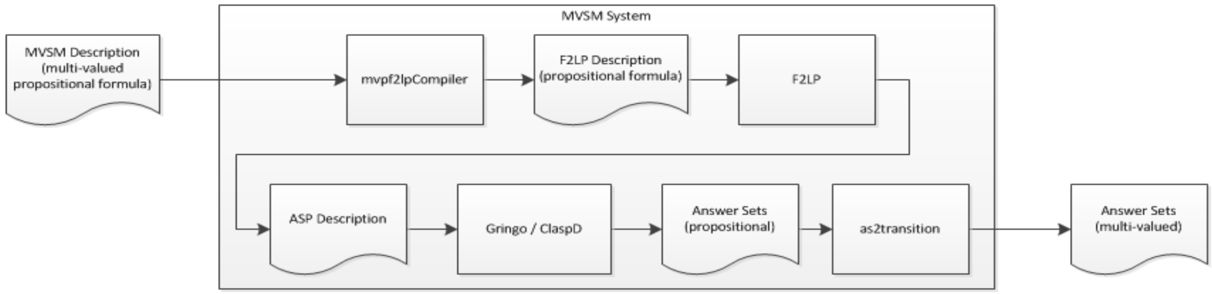

Figure 1: Architecture of MVSM

System MVSM

System `mvsm`[4] is a prototype implementation of multi-valued propositional formulas under the stable model semantics. In fact, it is a script that invokes several software, such as MVPF2LPCOMPILER, F2LP, GRINGO, CLASPD, and AS2TRANSITION. MVPF2LPCOMPILER is an implementation of the translations in Theorem 1 and Theorem 3, which translates multi-valued propositional formulas into standard propositional formulas under the BL-semantics and CB-semantics. As the theorems show, the translations are very similar, and the user can choose which translation to use. F2LP then transforms the propositional formula into an ASP program in the input language of GRINGO. AS2TRANSITION takes the output of CLASPD and outputs propositional atoms in the form of multi-valued atoms. The composition of these software is depicted in Figure 1.

Shown below is a description of the blocks world domain in the language of MVSM assuming the BL-semantics. The syntax of declarations follows the one in the input language of the Causal Calculator V2 [5]. Compared to the usual ASP encoding, explicit declaration of sorts and type checking help reduce the programmer's mistakes.[6] The inertia and exogeneity assumptions in the last three rules have simple reading, once we understand $\{F\}$ as representing defaults. There is no need to use strong negation.

```
% File 'bw': The blocks world

:- sorts
  step; astep;
  location >> block.

:- objects
  0..maxstep              :: step;
  0..maxstep-1            :: astep;
  1..6                    :: block;
  table                   :: location.

:- variables
  ST                      :: step;
  T                       :: astep;
  Bool                    :: boolean;
  B,B1                    :: block;
  L                       :: location.

:- constants
  loc(block,step)               :: location;
  move(block,location,astep)    :: boolean.

% two blocks can't be on the same block at the same time
<- loc(B1,ST)=B & loc(B2,ST)=B & B1!=B2.

% effect of moving a block
loc(B,T+1)=L <- move(B,L,T).

% a block can be moved only when it is clear
<- move(B,L,T) & loc(B1,T)=B.

% a block can't be moved onto a block that is being
% moved also
<- move(B,B1,T) & move(B1,L,T).

% initial location is exogenous
{loc(B,0)=L}.

% actions are exogenous
{move(B,L,T)=Bool}.

% fluents are inertial
{loc(B,T+1)=L} <- loc(B,T)=L.
```

[4] `http://sourceforge.net/projects/aspmt/`

[5] `http://www.cs.utexas.edu/~tag/cc/`

[6] This was observed in the course taught by the second author where MVSM was used to teach answer set programming.

Conclusion

We showed how to embed functional stable model semantics into the general theory of stable models. This has the advantage that many mathematical results that were established for the general theory of stable models, such as safety, splitting theorem, and strong equivalence, to be carried over to the functional stable model semantics. What we established is a simple view that functions are a special case of relations, which is well known in classical logic, but it was not obvious under the stable model semantics. We also showed that partial functions can be eliminated in favor of total functions, which provides a method of computing theories with partial functions using standard SMT solvers, which do not have a notion of partial functions.

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Appendix

Relating Cabalar Semantics to MVP-Formulas

Given a multi-valued interpretation I, we define the partial first-order interpretation I^{pfo} as follows:

- $|I^{pfo}| = \{v \mid v \in \mathit{Dom}(c)$ for some $c \in \sigma\}$;
- $I^{pfo}(v) = v$ for each $v \in \mathit{Dom}(c)$ for each $c \in \sigma$;
- $I^{pfo}(c) = I(c)$ for each multi-valued constant $c \in \sigma$.

Proposition 2 *For any MVP-formula F of signature σ, and any partial MVP-interpretation I of σ whose multi-valued constants are* $\mathbf{c}$*, I is a CB-stable model of F iff I^{pfo} is a partial stable model of F with respect to* $\mathbf{c}$ *viewed as a first-order formula of signature σ^{pfo}.*

Proof. ($\Rightarrow$) Consider any CB-stable model I of F. This means that I satisfies F and no subset K of I satisfies $F^{\underline{I}}$. It is clear by induction that $I^{pfo} \models_p F$; the base case is when F is an atomic formula $c = v$ and clearly by definition of I^{pfo}, we have $I \models c = v$ iff $I^{pfo} \models c = v$.

Thus, we must show that there is no $J \prec^{bfc} I^{pfo}$ such that $J \models_p F^{\underline{I^{pfo}}}$. To do so, we will show that if there is such a J, then we can create a partial MVP-interpreation K such that $K \subset I$ and $K \models_p F^{\underline{I}}$.

Assume there is some J such that $J \prec^{bfc} I^{pfo}$ and $J \models_p F^{\underline{I^{pfo}}}$. We create K from J as follows. For each $c \in \sigma$

$$K(c) = \begin{cases} I(c) & \text{if } c^J = I(c) \\ u & \text{otherwise} \end{cases}$$

We first show that $K \subset I$. Since $J \prec^{bfc} I^{pfo}$, there must be some constant $c \in \sigma$ such that $c^J = \mathrm{u}$ and $c^{I^{pfo}} \neq \mathrm{u}$. However, since $c^{I^{pfo}} = I(c)$ by definition of I^{pfo}, we have that $c^J \neq I(c)$ and so $K(c) = \mathrm{u}$ but $I(c) \neq \mathrm{u}$. Thus, $K \subset I$.

We now show that $K \models_p F^{\underline{I}}$ iff $J \models_p F^{\underline{I^{pfo}}}$ by induction on F. From this, we will conclude that since we assume I is a CB-stable model, then no such K exists and so it follows that no such J exists, which means I^{pfo} is a partial stable model of F with respect to $\mathbf{c}$.

- Case 1: F is an MVP atom $c = v$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $c = v$. Then, by definition of J, we have $K \models_p F^{\underline{I}}$ iff $J \models_p F^{\underline{I^{pfo}}}$.

  On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.
- Case 2: F is $G \wedge H$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \wedge H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \wedge H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.
- Case 3: F is $G \vee H$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \vee H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \vee H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.
- Case 4: F is $G \rightarrow H$. If $I \not\models_p G$ then by definition, $I^{pfo} \not\models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $\bot \rightarrow H^{\underline{I^{pfo}}}$ and $\bot \rightarrow H^{\underline{I}}$. Then we have $K \models_p F^{\underline{I}}$ and $J \models_p F^{\underline{I^{pfo}}}$.

  If $I \models_p H$ and $I \models_p G$ then by definition, $I^{pfo} \models_p H^{pfo}$ and $I^{pfo} \models_p G^{pfo}$. Then we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \rightarrow H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \rightarrow H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  If $I \not\models_p H$ and $I \models_p G$ then by definition, $I^{pfo} \not\models_p H^{pfo}$ and $I^{pfo} \models G^{pfo}$. Then we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.
- Case 5: F is $\neg G$. If $I \models_p G$ then by definition, $I^{pfo} \models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

  On the other hand, if $I \not\models_p G$ then by definition, $I^{pfo} \not\models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\neg\bot$. Then we have $K \models_p F^{\underline{I}}$ and $J \models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

($\Leftarrow$) Consider any partial stable model I^{pfo} of F. This means that $I^{pfo} \models F$ and there is no interpretation J such that $J \prec^{\mathbf{c}} I^{pfo}$ and $J \models F^{\underline{I^{pfo}}}$. It is clear by induction that $I \models_p F$; the base case is when F is an atomic formula $c = v$ and clearly by definition of I^{pfo}, we have $I \models c = v$ iff $I^{pfo} \models c = v$.

Then it only remains to be shown no partial MVP-interpretation K that is a subset of I satisfies $F^{\underline{I}}$. To show this, we will show that if there is such a K, then we can create an interpretation J such that $J \prec^{bfc} I^{pfo}$ and $J \models F^{\underline{I^{pfo}}}$.

Assume such a K exists and let $J = K^{pfo}$.

We first show that $J \prec^{bfc} I^{pfo}$. Since K is a subset of I, there must be some constant $c \in \sigma$ such that $I(c) \neq \mathrm{u}$ but $K(c) = \mathrm{u}$. Then, by definition of $J = K^{pfo}$, we have that $c^J = \mathrm{u}$ but $c^{I^{pfo}} \neq \mathrm{u}$. Thus, $J \prec^{bfc} I^{pfo}$.

We now show that $K \models_p F^{\underline{I}}$ iff $J \models_p F^{\underline{I^{pfo}}}$ by induction on F. From this, we will conclude that since we assume I^{pfo} is a partial stable model with respect to $\mathbf{c}$, then no such J exists and so it follows that no such K exists, which means I is a CB-stable model of F.

- Case 1: F is an MVP atom $c = v$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $c = v$. Then, by definition of J, we have $K \models_p F^{\underline{I}}$ iff $J \models_p F^{\underline{I^{pfo}}}$.

On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

- Case 2: F is $G \wedge H$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \wedge H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \wedge H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

- Case 3: F is $G \vee H$. If $I \models_p F$ then by definition, $I^{pfo} \models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \vee H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \vee H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models_p F$ then by definition, $I^{pfo} \not\models_p F^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

- Case 4: F is $G \rightarrow H$. If $I \not\models_p G$ then by definition, $I^{pfo} \not\models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $\bot \rightarrow H^{\underline{I^{pfo}}}$ and $\bot \rightarrow H^{\underline{I}}$. Then we have $K \models_p F^{\underline{I}}$ and $J \models_p F^{\underline{I^{pfo}}}$.

  If $I \models_p H$ and $I \models_p G$ then by definition, $I^{pfo} \models_p H^{pfo}$ and $I^{pfo} \models_p G^{pfo}$. Then we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{pfo}}} \rightarrow H^{\underline{I^{pfo}}}$ and $G^{\underline{I}} \rightarrow H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{pfo}}}$ and $H^{\underline{I^{pfo}}}$, $H^{\underline{I}}$.

  If $I \not\models_p H$ and $I \models_p G$ then by definition, $I^{pfo} \not\models_p H^{pfo}$ and $I^{pfo} \models G^{pfo}$. Then we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

- Case 5: F is $\neg G$. If $I \models_p G$ then by definition, $I^{pfo} \models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models_p F^{\underline{I}}$ and $J \not\models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

  On the other hand, if $I \not\models_p G$ then by definition, $I^{pfo} \not\models_p G^{pfo}$ and so we have $F^{\underline{I^{pfo}}}$ and $F^{\underline{I}}$ are both $\neg\bot$. Then we have $K \models_p F^{\underline{I}}$ and $J \models_p F^{\underline{I^{pfo}}}$ so in this case, the claim holds.

Proofs

Proof of Theorem 1

Lemma 1 *Assume that K and X are multi-valued interpretations of σ and Y is a propositional interpretation of σ^{prop} which is a subset of X such that*

$$K(c) = X(c) \text{ iff } c = X(c) \in Y.$$

We have that $K \models F^X$ (when we view F as a multi-valued formula of σ) iff $Y \models F^X$ (when we view F as a propositional formula of σ^{prop}).

Proof. By induction on F. We show only the case of atoms. The other cases are straightforward.

Let F be an atom $c = v$. If $X \models c = v$, then F^X is F. The claim follows from the assumption since $K \models c = v$ iff $Y \models c = v$. If $X \not\models c = v$, then F^X is $\bot$, which neither K nor Y satisfies. ∎

Theorem 1 *Let F be a MVP-formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.*

(a) Any BL-stable model of F viewed as a MVP-formula of signature σ is a stable model of $F \wedge UEC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005).

(b) Any stable model of $F \wedge UEC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005) is a BL-stable model of F viewed as a MVP-formula of σ.

Proof. (a) Assume X of signature σ is a stable model of F. This means $X \models F$ and no multi-valued interpretation K different from X satisfies F^X. Now since X is a multi-valued intepretation, $X \models UEC_\sigma$. Then clearly $X \models F$ when viewed as a propositional formula of signature σ^{prop}.

So, we wish to show that there is no interpretation Y of signature σ^{prop} such that $Y \subset X$ when X is viewed as a set of propositional atoms and $Y \models (F \wedge UEC_\sigma)^X$ when viewed as a propositional formula of signature σ^{prop}. To do so, we prove the contrapositive. We will show that if there is an interpretation Y of signature σ^{prop} such that $Y \subset X$ when X is viewed as a set of propositional atoms and $Y \models (F \wedge UEC_\sigma)^X$ when viewed as a propositional formula of signature σ^{prop}, then there is an interpretation K different from X that satisfies F^X when viewed as a multi-valued formula of signature σ.

Given such an interpretation Y, we create K as follows. For each $c \in \sigma$,

$$K(c) = \begin{cases} v & \text{if } c = v \in Y \\ m_c(v) & : \text{if } c = v \in X \text{ and } c = v \notin Y \end{cases}$$

where m_c is any mapping from $m : Dom(c) \rightarrow Dom(c)$ such that $m(x) \neq x$. Note that this requires that every $Dom(c)$ have at least two elements. Note that since $Y \subset X$, there is at least one $c \in \sigma$ and $v \in Dom(c)$ such that $c = v \in X$ but $c = v \notin Y$. For this c, $K(c) = m(X(c)) \neq X(c)$ so K and X are different.

In addition, we have that $K(c) = X(c)$ iff $c = X(c) \in Y$. Now, since $Y \models (F \wedge UEC_\sigma)^X$, it follows that $Y \models F^X$. Thus, from Lemma 1 it follows that since $Y \models F^X$, then $K \models F^X$.

(b) Assume X of signature σ^{prop} is a stable model of $F \wedge UEC_\sigma$. This means that $X \models F \wedge UEC_\sigma$ and no interpretation Y such that $Y \subset X$ satisfies $(F \wedge UEC_\sigma)^X$. Since $X \models UEC_\sigma$, then X can be viewed as a multi-valued interpretation. Then clearly, $X \models F$.

Now, we wish to show that there is no interpretation K of signature σ that is different from X satisfying F^X. To do so, we prove the contrapositive. We will show that if there is an interpretation K of signature σ different from X and

$K \models F^X$, then there is an interpretation Y such that $Y \subset X$ that satisfies $(F \wedge UEC_\sigma)^X$. Now since we already have seen that $X \models UEC_\sigma$, then $(UEC_\sigma)^X$ is equivalent to $\top$ so we need only show that there is an interpretation Y such that $Y \subset X$ that satisfies F^X.

Given such an interpretation K, we create Y as follows. Let us view K as a set of propositional atoms. We will take $Y = X \cap K$. Clearly $Y \subset X$. In addition, we have that $K(c) = X(c)$ iff $c = X(c) \in Y$. Thus, from Lemma 1 it follows that since $Y \models F^X$, then $K \models F^X$. ∎

Proof of Theorem 2

Theorem 2 *For any multi-valued propositional formula F and any total interpretation I, I is a BL-stable model of F iff I is a CB-stable model of F.*

Proof. This is a consequence of Theorem 5 from (Bartholomew and Lee 2013b) when we note that the stable models of a formula F under the semantics presented here are the stable models of $UF_{\mathbf{c}}(F)$.

Proof of Theorem 3

Lemma 2 *Assume that K and X are partial multi-valued interpretations of σ and Y is a propositional interpretation of σ^{prop} which is a subset of X such that*

$$K(c) = X(c) \textit{ iff either } c = X(c) \in Y \textit{ or } X(c) = \text{UNDEF}.$$

We have that $K \models F^X$ (when we view F as a multi-valued formula of σ) iff $Y \models F^X$ (when we view F as a propositional formula of σ^{prop}).

Proof. By induction on F. We show only the case of atoms. The other cases are straightforward.

Let F be an atom $c = v$. If $X \models c = v$, then F^X is F and it cannot be that $X(c) = \text{UNDEF}$. The claim follows from the assumption since $K \models c = v$ iff $Y \models c = v$. If $X \not\models c = v$, then F^X is $\bot$, which neither K nor Y satisfies. ∎

Theorem 3 *Let F be a multi-valued formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.*

(a) Any CB-stable model of F viewed as a MVP-formula of signature σ is a stable model of $F \wedge UC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005).

(b) Any stable model of $F \wedge UC_\sigma$ viewed as a propositional formula of signature σ^{prop} in the sense of (Ferraris 2005) is a CB-stable model of F viewed as a MVP-formula of σ.

Proof. (a) Assume X of signature σ is a CB-stable model of F. This means $X \models F$ and no partial multi-valued interpretation K that is a subset of X satisfies F^X. Now since X is a partial multi-valued intepretation, $X \models UC_\sigma$. Then clearly $X \models F$ when viewed as a propositional formula of signature σ^{prop}.

So, we wish to show that there is no interpretation Y of signature σ^{prop} such that $Y \subset X$ when X is viewed as a set of propositional atoms and $Y \models (F \wedge UC_\sigma)^X$ when viewed as a propositional formula of signature σ^{prop}. To do so, we prove the contrapositive. We will show that if there is an interpretation Y of signature σ^{prop} such that $Y \subset X$ when X is viewed as a set of propositional atoms and $Y \models (F \wedge UC_\sigma)^X$ when viewed as a propositional formula of signature σ^{prop}, then there is a partial interpretation K that is a subset of X that satisfies F^X when viewed as a multi-valued formula of signature σ.

Given such an interpretation Y, we create K as follows. For each $c \in \sigma$,

$$K(c) = \begin{cases} v & \text{if } c = v \in Y \\ \text{UNDEF} & : \text{if } c = v \notin Y \text{ for any } v \end{cases}$$

Note that this no longer requires there to be two explicit elements in $Dom(c)$

Note that since $Y \subset X$, there is at least one $c \in \sigma$ and $v \in Dom(c)$ such that $c = v \in X$ but $c = v \notin Y$. For this c, $K(c) = \text{UNDEF} \neq X(c)$ so K and X are different. Further, when $c = v \in Y$, then $c = v \in X$ and so $X(c) = K(c)$, thus K is a subset of X.

In addition, we have that $K(c) = X(c)$ iff $c = X(c) \in Y$ or $X(c) = \text{UNDEF}$. Now, since $Y \models (F \wedge UC_\sigma)^X$, it follows that $Y \models F^X$. Thus, from Lemma 2 it follows that since $Y \models F^X$, then $K \models F^X$.

(b) Assume X of signature σ^{prop} is a stable model of $F \wedge UC_\sigma$. This means that $X \models F \wedge UC_\sigma$ and no interpretation Y such that $Y \subset X$ satisfies $(F \wedge UC_\sigma)^X$. Since $X \models UC_\sigma$, then X can be viewed as a partial multi-valued interpretation. Then clearly, $X \models F$.

Now, we wish to show that there is no partial interpretation K of signature σ that is a subset of X satisfying F^X. To do so, we prove the contrapositive. We will show that if there is a partial interpretation K of signature σ that is a subset of X and $K \models F^X$, then there is an interpretation Y such that $Y \subset X$ that satisfies $(F \wedge UC_\sigma)^X$. Now since we already have seen that $X \models UC_\sigma$, then $(UC_\sigma)^X$ is equivalent to $\top$ so we need only show that there is an interpretation Y such that $Y \subset X$ that satisfies F^X.

Given such an interpretation K, we create Y as follows. Let us view K as a set of propositional atoms. We will take $Y = X \cap K$. Clearly $Y \subset X$. In addition, we have that $K(c) = X(c)$ iff $c = X(c) \in Y$ or $X(c) = \text{UNDEF}$. Thus, from Lemma 1 it follows that since $Y \models F^X$, then $K \models F^X$. ∎

Proof of Theorem 4

Theorem 4

(a) An interpretation I of σ is a CB-stable model of F iff I^{none} is a stable model of

$$F \wedge \bigwedge_{c \in \sigma} \Big(c = \text{NONE} \vee \neg(c = \text{NONE})\Big)$$

of σ^{none}.

(b) An interpretation J such that $none^J = none$ of σ^{none} is a stable model of

$$F \wedge \bigwedge_{c \in \sigma} \Big(c = \text{NONE} \vee \neg(c = \text{NONE})\Big)$$

iff $J = I^{none}$ for some CB-stable model of F.

Proof. We first note that by Proposition 1, we can view $F \wedge \bigwedge_{c\in\sigma} \Big(c = \text{NONE} \vee \neg(c = \text{NONE})\Big)$ of signature σ^{none} as a first-order formula under the functional stable model semantics. Similarly by Proposition 2, we can view F as a first-order formula under the cabalar semantics. Then, by Theorem 8 the claim follows.

Proof of Corollary 1

Corollary 1 *Let F be a formula of signature σ^{sneg}, which can be also viewed as a multi-valued formula of signature σ^{bool}. We have that I is a coherent stable model of F viewed as a propositional formula of σ^{sneg} iff I is a CB-stable model of F of signature σ^{bool}.*

Proof. From Theorem 3, we first note that I is a CB-stable model of F of signature σ^{bool} iff I^{prop} is a stable model of $F \wedge UC_\sigma$ (when we identify c with $c = \text{TRUE}$ and $\sim c$ with $c = \text{FALSE}$). We also consider the standard elimination of strong negation in favor of new predicates; let F' be obtained from F by replacing c and $\sim c$ with c and c' where c' is a new predicate and adding the constraint $\neg(c \wedge c')$. Now, we can see if we identify c with $c = \text{TRUE}$ and c' with $c = \text{FALSE}$, these are precisely the same formula so the claim follows.

Proof of Corollary 2

Corollary 2 *Let F be a formula of signature σ^{sneg}, which can be also viewed as a multi-valued formula of signature σ^{bool}. For any total interpretation I of σ^{bool}, we have that I is a stable model of F viewed as a propositional formula of σ^{sneg} iff I is a BL-stable model of F of signature σ^{bool}.*

Proof. From Theorem 1, we first note that I is a BL-stable model of F of signature σ^{bool} iff I^{prop} is a stable model of $F \wedge UEC_\sigma$ (when we identify c with $c = \text{TRUE}$ and $\sim c$ with $c = \text{FALSE}$. We also consider the standard elimination of strong negation in favor of new predicates while enforcing complete interpretations; let F' be obtained from F by replacing c and $\sim c$ with c and c' where c' is a new predicate and adding the constraints $\neg(c \wedge c') \wedge \neg\neg(c \vee c')$. Now, we can see if we identify c with $c = \text{TRUE}$ and c' with $c = \text{FALSE}$, these are precisely the same formula so the claim follows.

Proof of Proposition 1

Proposition 1 *For any MVP-formula F of signature σ, and any MVP-interpretation I of σ whose multi-valued constants are* **c**, *I is a BL-stable model of F iff I^{fo} is a stable model of F with respect to* **c** *viewed as a first-order formula of signature σ^{fo}.*

Proof. ($\Rightarrow$) Consider any BL-stable model I of F. This means that I is the only MVP-interpretation that satisfies $F^{\underline{I}}$. Clearly $I \models F$ since if not, $F^{\underline{I}}$ would be $\bot$, which would contradict that I satisfies $F^{\underline{I}}$. It is clear by induction that $I^{fo} \models F$; the base case is when F is an atomic formula $c = v$ and clearly by definition of I^{fo}, we have $I \models c = v$ iff $I^{fo} \models c = v$.

Thus, we must show that there is no $J <^{bfc} I^{fo}$ such that $J \models F^{\underline{I^{fo}}}$. To do so, we will show that if there is such a J, then we can create a MVP-interpreation K such that $K \neq I$ and $K \models F^{\underline{I}}$.

Assume there is some J such that $J <^{bfc} I^{fo}$ and $J \models F^{\underline{I^{fo}}}$. We create K from J as follows. For each $c \in \sigma$

$$K(c) = \begin{cases} I(c) & \text{if } c^J = I(c) \\ m_c(I(c)) & \text{otherwise} \end{cases}$$

where m_c is a mapping from $Dom(c)$ to $Dom(c)$ so that for every $x \in Dom(c)$, we have $m_c(x) \neq x$.

We first show that K is different from I. Since $J <^{bfc} I^{fo}$, there must be some constant $c \in \sigma$ such that $c^J \neq c^{I^{fo}}$. However, since $c^{I^{fo}} = I(c)$ by definition of I^{fo}, we have that $c^J \neq I(c)$ and so $K(c) = m_c(I(c)) \neq I(c)$. Thus, K is different from I.

We now show that $K \models F^{\underline{I}}$ iff $J \models F^{\underline{I^{fo}}}$ by induction on F. From this, we will conclude that since we assume I is a BL-stable model, then no such K exists and so it follows that no such J exists, which means I^{fo} is a stable model of F with respect to **c**.

- Case 1: F is an MVP atom $c = v$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $c = v$. Then, by definition of J, we have $K \models F^{\underline{I}}$ iff $J \models F^{\underline{I^{fo}}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 2: F is $G \wedge H$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \wedge H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \wedge H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 3: F is $G \vee H$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \vee H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \vee H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 4: F is $G \to H$. If $I \not\models G$ then by definition, $I^{fo} \not\models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $\bot \to H^{\underline{I^{fo}}}$ and $\bot \to H^{\underline{I}}$. Then we have $K \models F^{\underline{I}}$ and $J \models F^{\underline{I^{fo}}}$.

  If $I \models H$ and $I \models G$ then by definition, $I^{fo} \models H^{fo}$ and $I^{fo} \models G^{fo}$. Then we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \to H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \to H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  If $I \not\models H$ and $I \models G$ then by definition, $I^{fo} \not\models H^{fo}$ and $I^{fo} \models G^{fo}$. Then we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 5: F is $\neg G$. If $I \models G$ then by definition, $I^{fo} \models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

  On the other hand, if $I \not\models G$ then by definition, $I^{fo} \not\models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\neg\bot$. Then we have $K \models F^{\underline{I}}$ and $J \models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

($\Leftarrow$) Consider any stable model I^{fo} of F. This means that $I^{fo} \models F$ and there is no interpretation J such that $J <^{\mathbf{c}} I$ and $J \models F^{\underline{I^{fo}}}$. It is easy to see that $I^{fo} \models F^{\underline{I^{fo}}}$ and by definition of I^{fo}, it follows that $I \models F^{\underline{I}}$.

Then it only remains to be shown no other MVP-interpretation K satisfies $F^{\underline{I}}$. To show this, we will show that if there is such a K, then we can create an interpretation J such that $J <^{bfc} I^{fo}$ and $J \models F^{\underline{I^{fo}}}$.

Assume such a K exists and let $J = K^{fo}$.

We first show that $J <^{bfc} I^{fo}$. Since K is different from I, there must be some constant $c \in \sigma$ such that $I(c) \neq K(c)$. However, since we have $c^{I^{fo}} = I(c)$ by definition of I^{fo} and $c^J = K(c)$ by definition of $J = K^{fo}$, we have that $c^J \neq c^{I^{fo}}$). Thus, $J <^{bfc} I^{fo}$.

We now show that $K \models F^{\underline{I}}$ iff $J \models F^{\underline{I^{fo}}}$ by induction on F. From this, we will conclude that since we assume I^{fo} is a stable model with respect to $\mathbf{c}$, then no such J exists and so it follows that no such K exists, which means I is a BL-stable model of F.

- Case 1: F is an MVP atom $c = v$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $c = v$. Then, by definition of J, we have $K \models F^{\underline{I}}$ iff $J \models F^{\underline{I^{fo}}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 2: F is $G \wedge H$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \wedge H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \wedge H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 3: F is $G \vee H$. If $I \models F$ then by definition, $I^{fo} \models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \vee H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \vee H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  On the other hand, if $I \not\models F$ then by definition, $I^{fo} \not\models F^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 4: F is $G \rightarrow H$. If $I \not\models G$ then by definition, $I^{fo} \not\models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $\bot \rightarrow H^{\underline{I^{fo}}}$ and $\bot \rightarrow H^{\underline{I}}$. Then we have $K \models F^{\underline{I}}$ and $J \models F^{\underline{I^{fo}}}$.

  If $I \models H$ and $I \models G$ then by definition, $I^{fo} \models H^{fo}$ and $I^{fo} \models G^{fo}$. Then we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are $G^{\underline{I^{fo}}} \rightarrow H^{\underline{I^{fo}}}$ and $G^{\underline{I}} \rightarrow H^{\underline{I}}$ so the claim follows by induction on $G^{\underline{I}}$, $G^{\underline{I^{fo}}}$ and $H^{\underline{I^{fo}}}$, $H^{\underline{I}}$.

  If $I \not\models H$ and $I \models G$ then by definition, $I^{fo} \not\models H^{fo}$ and $I^{fo} \models G^{fo}$. Then we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

- Case 5: F is $\neg G$. If $I \models G$ then by definition, $I^{fo} \models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\bot$. Then we have $K \not\models F^{\underline{I}}$ and $J \not\models F^{\underline{I^{fo}}}$ so in this case, the claim holds.

  On the other hand, if $I \not\models G$ then by definition, $I^{fo} \not\models G^{fo}$ and so we have $F^{\underline{I^{fo}}}$ and $F^{\underline{I}}$ are both $\neg\bot$. Then we have $K \models F^{\underline{I}}$ and $J \models F^{\underline{I^{fo}}}$ so in this case, the claim holds. ∎

Proof of Theorem 5

Corollary 2 from (Bartholomew and Lee 2012) is presented below as a lemma using notation from this paper.

Lemma 3 *Let F be an f-plain sentence of signature σ.*

An interpretation I of σ satisfying $\exists xy(x \neq y)$ is a stable model of F relative to $\mathbf{c}$ iff $I^{pred[f]}$ is a stable model of $(F \wedge UEC_f)^{pred[f]}$ relative to $(\mathbf{c} \setminus \{f\}) \cup pred[f]$.

(b) An interpretation J of $\sigma^{pred[f]}$ that satisfies $\exists xy(x \neq y)$ is a stable model of $(F \wedge UEC_f)^{pred[f]}$ relative to $(\mathbf{c} \setminus \{f\}) \cup pred[f]$ in the sense of (Ferraris, Lee, and Lifschitz 2011) iff $J = I^{pred[f]}$ for some stable model I of F relative to $\mathbf{c}$ in our sense.

Theorem 5

(a) An interpretation I of σ is a stable model of F relative to $\mathbf{c}$ iff $I^{pred[\mathbf{c}]}$ is a stable model of $(UF_{\mathbf{c}}(F) \wedge UEC_{\mathbf{c}})^{pred[\mathbf{c}]}$ relative to $pred[\mathbf{c}]$.

(b) An interpretation J of $\sigma^{pred[\mathbf{c}]}$ that satisfies $\exists xy(x \neq y)$ is a stable model of $(UF_{\mathbf{c}}(F) \wedge UEC_{\mathbf{c}})^{pred[\mathbf{c}]}$ relative to $\mathbf{c}^{pred}$ in the sense of (Ferraris, Lee, and Lifschitz 2011) iff $J = I^{pred[\mathbf{c}]}$ for some stable model I of F relative to $\mathbf{c}$ in our sense.

Proof. Immediate from Lemma 3 when we note that the stable models of a formula F under the semantics presented here are the stable models of $UF_{\mathbf{c}}(F)$.

Proof of Theorem 6

We present Theorem 6 from (Bartholomew and Lee 2013b) here as a lemma in the notation in this paper.

Lemma 4 *For any head-$\mathbf{c}$-plain sentence F of signature σ that is tight on $\mathbf{c}$, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, I is a stable model of F relative to $\mathbf{c}$ in the sense of (Bartholomew and Lee 2012) iff I is a Cabalar stable model of F relative to $\mathbf{c}$.*

Theorem 6 *For any head-$\mathbf{c}$-plain sentence F of signature σ that is tight on $\mathbf{c}$, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, I is a stable model of F relative to $\mathbf{c}$ in the sense of (Bartholomew and Lee 2012) iff I is a stable model of F relative to $\mathbf{c}$ in the sense of Definition 2.*

Proof. This an immediate consequence of Theorem 2 and Lemma 4.

Proof of Theorem 7

We present Corollary 1 from (Bartholomew and Lee 2013b) here as a lemma in the notation in this paper.

Lemma 5 *For any sentence F, any list $\mathbf{c}$ of constants, and any total interpretation I satisfying $\exists xy(x \neq y)$, we have I is a Cabalar stable model of F iff I is a Cabalar stable model of $UF_{\mathbf{c}}(F)$ iff I is a stable model of $UF_{\mathbf{c}}(F)$.*

Theorem 7 *For any formula F and any total interpretation I, I is a stable model of F relative to $\mathbf{c}$ iff I is a Cabalar stable model of F relative to $\mathbf{c}$.*

Proof. We first note that by Lemma 5, the Cabalar stable models of F are precisely the stable models of $UF_{\mathbf{c}}(F)$. Then, by noting that the stable models of a formula F under the semantics presented here are the stable models of $UF_{\mathbf{c}}(F)$ presented in (Bartholomew and Lee 2013b), the claim follows.

Proof of Theorem 8

Lemma 6 *Given a σ-plain formula G of signature σ, a partial interpretation I satisfies G iff I^{none} satisfies G^{none}.*

Proof. By induction on G.

- Case 1: G is a (σ-plain) ground atomic formula of signature σ^* which is σ extended with object names from $|I|$ (not including NONE). G^{none} is the same as G in this case.
  - G is $p(\boldsymbol{\xi}^*)$. Note that since $\boldsymbol{\xi}^*$ are object names, I does not map any of these to u. Thus, by definition of I^{none}, $p(\boldsymbol{\xi}^*)^{I^{none}} = p(\boldsymbol{\xi}^*)^I$ so certainly the claim holds.
  - G is $\xi_1^* = \xi_2^*$. The claim follows immediately from the fact that $(\xi_1^*)^I = (\xi_1^*)^{I^{none}} = \xi_1$ and $(\xi_2^*)^I = (\xi_2^*)^{I^{none}} = \xi_2$.
  - G is $f(\boldsymbol{\xi}^*) = \xi^*$. Note that since $\boldsymbol{\xi}^*$ and ξ^* are object names, I does not map any of these to u. Now if $f(\boldsymbol{\xi}^*)^I = $ u, then by definition of I^{none}, $f(\boldsymbol{\xi}^*)^{I^{none}} =$ NONE. In this case, neither I nor I^{none} satisfy G. On the other hand, if $f(\boldsymbol{\xi}^*)^I \neq$ u, then by definition of I^{none}, $f(\boldsymbol{\xi}^*)^{I^{none}} = f(\boldsymbol{\xi}^*)^I$ so certainly the claim follows.
- Case 2: G is $H_1 \odot H_2$ where $\odot \in \{\wedge, \vee, \rightarrow\}$. G^{none} is $(H_1)^{none} \odot (H_2)^{none}$. By I.H. on H_1 and H_2, the claim follows.
- Case 3: G is $\exists x H(x)$. G^{none} is $\exists x(H(x)^{none} \wedge x \neq \text{NONE})$.
  ($\Rightarrow$) Assume $I \models_{\rm p} G$. That means there is some $\xi \in |I|$ such that $I \models_{\rm p} H(\xi)$. By I.H. on $H(\xi^\diamond)$ for every $\xi \in |I|$, we then have that there is some $\xi \in |I|$ such that $I^{none} \models H(\xi)^{none}$. Since $\xi \neq$ NONE for all $\xi \in |I|$, we have that there is some $\xi \in |I|$ such that $I^{none} \models H(\xi)^{none} \wedge \xi \neq$ NONE. Finally, since $|I| \subseteq |I^{none}|$, we further have that there is some $\xi \in |I^{none}|$ such that $I^{none} \models H(\xi)^{none} \wedge \xi \neq$ NONE, which is the definition of $I^{none} \models G^{none}$.

  ($\Leftarrow$) Assume $I^{none} \models G^{none}$. That means there is some $\xi \in |I^{none}|$ such that $I^{none} \models H(\xi)^{none} \wedge \xi \neq$ NONE. It then follows that there is some $\xi \in |I|$ such that $I^{none} \models H(\xi)^{none}$. By I.H. on $H(\xi^\diamond)$ for every $\xi \in |I|$, it then follows that there is some $\xi \in |I|$ such that $I \models_{\rm p} H(\xi)$, which is the definition of $I \models_{\rm p} G$.
- Case 4: G is $\forall x H(x)$. G^{none} is $\forall x(x \neq \text{NONE} \rightarrow H(x)^{none})$.
  ($\Rightarrow$) Assume $I \models_{\rm p} G$. That means for every $\xi \in |I|$, we have $I \models_{\rm p} H(\xi)$. By I.H. on $H(\xi^\diamond)$ for every $\xi \in |I|$, it follows that for every $\xi \in |I|$ we have $I^{none} \models H(\xi)^{none}$. Since $\xi \neq$ NONE for all $\xi \in |I|$, we have that there is some $\xi \in |I|$ such that $I^{none} \models \xi \neq \text{NONE} \rightarrow H(\xi)^{none}$. Finally, since $|I| \subseteq |I^{none}|$ and since the implication $\xi \neq \text{NONE} \rightarrow H(\xi)^{none}$ is trivially satisfied when $\xi =$ NONE, it further follows that for every $\xi \in |I^{none}|$ we have $I^{none} \models \xi \neq \text{NONE} \rightarrow H(\xi)^{none}$, which is the definition of $I^{none} \models G^{none}$.

  ($\Leftarrow$) Assume $I^{none} \models G^{none}$. That means for every $\xi \in |I^{none}|$ we have $I^{none} \models \xi \neq \text{NONE} \rightarrow H(\xi)^{none}$. Since $|I| \subseteq |I^{none}|$, it certainly follows that for every $\xi \in |I|$ we have $I^{none} \models \xi \neq \text{NONE} \rightarrow H(\xi)^{none}$. Then, since $\xi \neq$ NONE is true for every $\xi \in |I|$, it follows that for every $\xi \in |I|$ we have $I^{none} \models H(\xi)^{none}$. Then by I.H. on $H(\xi^\diamond)$ for every $\xi \in |I|$ it follows that for every $\xi \in |I|$ we have $I \models_{\rm p} H(\xi)$, which is the definition of $I \models_{\rm p} G$

Note: the σ-plain assumption is only used for the atomic formulas $t_1 = t_2$ not $p(t)$.

Theorem 8 *For any sentence F of signature σ,*

(a) I is a Cabalar stable model of F relative to $\mathbf{c}$ iff I^{none} is a stable model of

$$(UF_\sigma(F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$$

relative to $\mathbf{c}$.

(b) An interpretation J is a stable model of

$$(UF_\sigma(F))^{none} \wedge \text{UNDEF}_{\mathbf{c}}$$

relative to $\mathbf{c}$ iff $J = I^{none}$ for some Cabalar stable model I of F relative to $\mathbf{c}$.

We first note that by Theorem 7 in (Bartholomew and Lee 2013b), I is a Cabalar stable model of F relative to $\mathbf{c}$ iff I is a Cabalar stable model of $UF_\sigma(F)$ relative to $\mathbf{c}$ (the theorem is about $UF_{\mathbf{c}}(F)$ but the same proof should hold for $UF_\sigma(F)$).

For notational simplicity, let $G = UF_\sigma(F)$. We will prove the theorem in terms of G. That is, we will show

(a) I is a Cabalar stable model of G relative to $\mathbf{c}$ iff I^{none} is a stable model of

$$G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$$

relative to $\mathbf{c}$.

(b) An interpretation J is a stable model of

$$G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$$

relative to $\mathbf{c}$ iff $J = I^{none}$ for some Cabalar stable model I of G relative to $\mathbf{c}$.

Proof. (a $\Rightarrow$) Assume that I is a Cabalar stable model of G relative to $\mathbf{c}$. That is, $I \models_{\rm p} G$ and for every partial interpretation K such that $K \prec^{\mathbf{c}} I$, we have $K \not\models_{\rm p} gr_I[G]^{\underline{I}}$. We wish to show that I^{none} is a stable model of $G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ relative to $\mathbf{c}$. That is, we wish to show that $I^{none} \models G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ and for every interpretation L such that $L <^{\mathbf{c}} I^{none}$, we have $L \not\models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$.

By Lemma 6, we have that since we assume $I \models_{\text{p}} G$, we conclude that $I^{none} \models G^{none}$. Then, since $\text{UNDEF}_{\mathbf{c}}$ is a tautology in classical logic, we have $I^{none} \models G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$.

We now show that if for every partial interpretation K such that $K \prec^{\mathbf{c}} I$, we have $K \not\models_{\text{p}} gr_I[G]^{\underline{I}}$ then for any L such that $L <^{\mathbf{c}} I^{none}$, we have $L \not\models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$. To do so, we prove the contrapositive; if there is some L such that $L <^{\mathbf{c}} I^{none}$ and $L \models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$, then there is some partial interpretation K such that $K \prec^{\mathbf{c}} I$ and $K \not\models_{\text{p}} gr_I[G]^{\underline{I}}$. Given such an L, we construct such a K as follows. First, let $|K| = |I|$. For every predicate $p \in \sigma$, we define $p^K = p^L$. For every function $f \in \sigma$ of arity n and every tuple of objects $\boldsymbol{\xi}$ from $(|I| \cup \{\text{u}\})^n$, we define

$$f^K(\boldsymbol{\xi}) = \begin{cases} f^L(\boldsymbol{\xi}) & \text{if } f^L(\boldsymbol{\xi}) \neq \text{NONE and } f^L(\boldsymbol{\xi}) = f^{I^{none}}(\boldsymbol{\xi}) \text{ and } \boldsymbol{\xi} \in |I|^n; \\ \text{u} & \text{otherwise.} \end{cases}$$

Assuming $L <^{\mathbf{c}} I^{none}$, we show that $K \prec^{\mathbf{c}} I$. We first show that $K \preceq^{\mathbf{c}} I$.

- By definition, K and I both have the same universe.
- Since L and I^{none} agree on all constants not in $\mathbf{c}$, it is easy to see by definition of K and I^{none} that K and I agree on all constants not in $\mathbf{c}$.
- Consider any predicate constant $p \in \mathbf{c}$ and any tuple $\boldsymbol{\xi}$ from $|I|$. If $p(\boldsymbol{\xi})^K = \text{TRUE}$, then by definition of K, it must be that $p(\boldsymbol{\xi})^L = \text{TRUE}$. Then, since $p^L \subseteq p^{I^{none}}$, it must be that $p(\boldsymbol{\xi})^{I^{none}} = \text{TRUE}$. Finally, by definition of I^{none}, it follows that $p(\boldsymbol{\xi})^I = \text{TRUE}$. Thus it holds that $p^K \subseteq p^I$.
- Consider any function constant $f \in \mathbf{c}$ and any tuple $\boldsymbol{\xi}$ from $|I|$. We have three cases and wish to show that $f^k(\boldsymbol{\xi}) = \text{u}$ or $f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$.
  - If $f^L(\boldsymbol{\xi}) = \text{NONE}$, then by definition of K, $f^K(\boldsymbol{\xi}) = \text{u}$ so in this case, the claim follows.
  - If $f^L(\boldsymbol{\xi}) \neq \text{NONE}$ and $f^L(\boldsymbol{\xi}) = f^{I^{none}}$. Then by definition of K, $f^K(\boldsymbol{\xi}) = f^L(\boldsymbol{\xi}) = f^{I^{none}}(\boldsymbol{\xi})$. Then, by definition of I^{none}, we have $f^{I^{none}}(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ and so $f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ so in this case, the claim follows.
  - If $f^L(\boldsymbol{\xi}) \neq \text{NONE}$ and $f^L(\boldsymbol{\xi}) \neq f^{I^{none}}$. Then by definition of K, $f^K(\boldsymbol{\xi}) = \text{u}$ so in this case, the claim follows.

  If $f^L(\boldsymbol{\xi}) = f^{I^{none}}(\boldsymbol{\xi})$, then by definition of K, $f^K(\boldsymbol{\xi}) = f^L(\boldsymbol{\xi}) = f^{I^{none}}(\boldsymbol{\xi})$. Then, by definition of I^{none}, we have $f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$. On the other hand, if $f^L(\boldsymbol{\xi}) \neq f^{I^{none}}(\boldsymbol{\xi})$, then by definition of K, we have $f^K(\boldsymbol{\xi}) = u$. Therefore, it holds for every function constant $f \in \sigma$ and every list $\boldsymbol{\xi}$ of elements from $|I|$ that $f^k(\boldsymbol{\xi}) = u$ or $f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$.

Thus, we have $K \preceq^{\mathbf{c}} I$.

We now show $(I \preceq^{\mathbf{c}} K)$ does not hold and conclude that $K \prec^{\mathbf{c}} I$. Since $L <^{\mathbf{c}} I^{\text{NONE}}$, we consider two cases:

- Case 1: There is some predicate $p \in \mathbf{c}$ of arity n and tuple $\boldsymbol{\xi}$ of objects from $|I^{none}|$ such that $p(\boldsymbol{\xi})^L = \text{FALSE}$ but $p(\boldsymbol{\xi})^{I^{none}} = \text{TRUE}$. We first note that by definition of I^{none} that if $\boldsymbol{\xi}$ is not in $|I|^n$, then $p(\boldsymbol{\xi})^{I^{none}} = \text{FALSE}$ so it must be that $\boldsymbol{\xi}$ is in $|I|^n$. Then by definition of K, we have that $p(\boldsymbol{\xi})^K = \text{FALSE}$ and by definition of I^{none}, it follows that $p(\boldsymbol{\xi})^I = \text{TRUE}$ so in this case $(I \preceq^{\mathbf{c}} K)$ does not hold.
- Case 2: There is some function $f \in \mathbf{c}$ of arity n and tuple $\boldsymbol{\xi}$ of objects from $|I^{none}|$ such that $f(\boldsymbol{\xi})^L \neq f(\boldsymbol{\xi})^{I^{none}}$. We need to show that $f(\boldsymbol{\xi})^I \neq \text{u}$ and $f(\boldsymbol{\xi})^K \neq f(\boldsymbol{\xi})^I$.

  We show that for this to be the case, it must be that $f(\boldsymbol{\xi})^{I^{none}} \neq \text{NONE}$. Assume to the contrary that $f(\boldsymbol{\xi})^{I^{none}} = \text{NONE}$, then since $L \models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$, and in particular $L \models gr_{I^{none}}(\text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$, which contains a conjunctive term equivalent to $f(\boldsymbol{\xi}) = \text{NONE} \vee \bot$ and so it must be that $f(\boldsymbol{\xi})^L = \text{NONE}$ which contradicts the assumption that $f(\boldsymbol{\xi})^L \neq f(\boldsymbol{\xi})^{I^{none}}$.

  Thus it must be that $f(\boldsymbol{\xi})^{I^{none}} \neq \text{NONE}$. By definition of I^{none}, this means that $\boldsymbol{\xi}$ is in $|I|^n$ and $f^I \neq u$. However, by definition of K, since $f^L(\boldsymbol{\xi}) = f^{I^{none}}(\boldsymbol{\xi})$ does not hold, $f^K(\boldsymbol{\xi}) = \text{u}$ and so we have $f^K(\boldsymbol{\xi}) \neq f^I(\boldsymbol{\xi})$. Thus in this case $(I \preceq^{\mathbf{c}} K)$ does not hold.

We now show by induction on G that $K \models_{\text{p}} gr_I[G]^{\underline{I}}$ iff $L \models gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ and then since we assume $L \models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$ then certainly $L \models gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$, and then we will conclude that $K \models_{\text{p}} gr_I[G]^{\underline{I}}$.

- Case 1: G is a (σ-plain) ground atomic formula of extended signature σ^* which is σ extended with object names from $|I|$ (not including NONE).

  If $I \not\models_{\text{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  If instead $I \models_{\text{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both G, so by definition of K, $K \models_{\text{p}} G$ iff $L \models G$.
- Case 2: G is $H_1 \odot H_2$ where $\odot \in \{\wedge, \vee, \rightarrow\}$ and so $gr_I[G]^{\underline{I}}$ is $gr_I[H_1]^{\underline{I}} \odot gr_I[H_2]^{\underline{I}}$. G^{none} is $(H_1)^{none} \odot (H_2)^{none}$.

  If $I \not\models_{\text{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  On the other hand, if $I \models_{\text{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_{(I^{none})}[G^{none}]^{\underline{I^{none}}}$ is

  $$gr_{(I^{none})}[(H_1)^{none}]^{\underline{I^{none}}} \odot gr_{(I^{none})}[(H_2)^{none}]^{\underline{I^{none}}}.$$

  By I.H. on $gr_I[H_1]^{\underline{I}}$ and $gr_{(I^{none})}[(H_1)^{none}]^{\underline{I^{none}}}$ and $gr_I[H_2]^{\underline{I}}$ and $gr_{(I^{none})}[(H_2)^{none}]^{\underline{I^{none}}}$.
- Case 3: G is $\exists x H(x)$. G^{none} is $\exists x (H(x)^{none} \wedge x \neq \text{NONE})$.

If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

On the other hand, if $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_I[G]^{\underline{I}}$ is

$$\{gr_I[H(\xi^\diamond)]^{\underline{I}} : \xi \in |I|\}^\vee$$

and $gr_{(I^{none})}[G^{none}]^{\underline{I^{none}}}$ is

$$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{I^{none}}} \wedge \xi^\diamond \neq \text{NONE} : \xi \in |(I^{none})|\}^\vee.$$

Now we note that $|(I^{none})| = |I| \cup \{\text{NONE}\}$. Further, we note that since $\xi^\diamond \neq \text{NONE}$ is not satisfied when $\xi = \text{NONE}$, the latter reduct is equivalent to

$$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{(I^{none})}} \wedge \xi^\diamond \neq \text{NONE} : \xi \in |I|\}^\vee.$$

The further, we note that for all $\xi \in |I|$, $\xi^\diamond \neq \text{NONE}$ is satisfied so that this is further equivalent to

$$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{I^{none}} : \xi \in |I|\}^\vee.$$

Thus the claim follows by induction on $gr_I[H(\xi^\diamond)]$ and $gr_{(I^{none})}[H(\xi^\diamond)^{none}]$ for every $\xi \in |I|$.

- Case 4: G is $\forall x H(x)$. G^{none} is $\forall x(x \neq \text{NONE} \rightarrow H(x)^{none})$.

  If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  On the other hand, if $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_I[G]^{\underline{I}}$ is

$$\{gr_I[H(\xi^\diamond)]^{\underline{I}} : \xi \in |I|\}^\wedge$$

and $gr_{(I^{none})}[H^{none}]^{\underline{I^{none}}}$ is

$$\{\xi^\diamond \neq \text{NONE} \rightarrow gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{I^{none}}} : \xi \in |(I^{none})|\}^\wedge.$$

Now we note that $|(I^{none})| = |I| \cup \{\text{NONE}\}$. Further, we note that since $\xi^\diamond \neq \text{NONE}$ is not satisfied when $\xi = \text{NONE}$, the latter reduct is equivalent to

$$\{\xi^\diamond \neq \text{NONE} \rightarrow gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{(I^{none})}} : \xi \in |I|\}^\wedge.$$

The further, we note that for all $\xi \in |I|$, $\xi^\diamond \neq \text{NONE}$ is satisfied so that this is further equivalent to

$$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{I^{none}} : \xi \in |I|\}^\wedge.$$

Thus the claim follows by induction on $gr_I[H(\xi^\diamond)]$ and $gr_{(I^{none})}[H(\xi^\diamond)^{none}]$ for every $\xi \in |I|$.

(a $\Leftarrow$)

Assume that I^{none} is a stable model of $G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ relative to $\mathbf{c}$. That is, $I^{none} \models G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ and for every interpretation L such that $L <^{\mathbf{c}} I^{none}$, we have $L \not\models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$. We wish to show that I is a Cabalar stable model of G relative to $\mathbf{c}$. That is, we wish to show that $I \models_{\mathrm{p}} G$ and for every partial interpretation K such that $K \prec^{\mathbf{c}} I$, we have $K \not\models_{\mathrm{p}} gr_I[G]^{\underline{I}}$.

Since we assume $I^{none} \models G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ it follows that $I^{none} \models G^{none}$, then by Lemma 6, we conclude that $I \models_{\mathrm{p}} G$.

We now show that if for any L such that $L <^{\mathbf{c}} I^{none}$, we have $L \not\models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$, then for every partial interpretation K such that $K \prec^{\mathbf{c}} I$, we have $K \not\models_{\mathrm{p}} gr_I[G]^{\underline{I}}$.

To do so, we prove the contrapositive; if there is some partial interpretation K such that $K \prec^{\mathbf{c}} I$ and $K \not\models_{\mathrm{p}} gr_I[G]^{\underline{I}}$, then there is some L such that $L <^{\mathbf{c}} I^{none}$ and $L \models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$. Given such an K, we construct such a L as follows. First, let $|L| = |I^{none}|$. For every predicate $p \in \sigma$, we define $p^L = p^K$. For every function $f \in \sigma$ of arity n and every tuple of objects $\boldsymbol{\xi}$ from $(|I^{none}|)^n$, we define

$$f^L(\boldsymbol{\xi}) = \begin{cases} f^K(\boldsymbol{\xi}) & \text{if } f^K(\boldsymbol{\xi}) \neq \text{u and } \boldsymbol{\xi} \in |I|^n; \\ \text{NONE} & \text{otherwise.} \end{cases}$$

Assuming $K \prec^{\mathbf{c}} I$, we show that $L <^{\mathbf{c}} I^{none}$.

- By definition, L and I^{none} both have the same universe.
- Since K and I agree on all constants not in $\mathbf{c}$, it is easy to see by definition of L and I^{none} that L and I^{none} agree on all constants not in $\mathbf{c}$.
- Consider any predicate constant $p \in \mathbf{c}$ and any tuple $\boldsymbol{\xi}$ from $|I^{none}|$. We first note by definition of L that if $\boldsymbol{\xi}$ has at least one NONE, then $p^L(\boldsymbol{\xi}) = \text{FALSE}$ so there is nothing to be proven. Now we consider when $\boldsymbol{\xi}$ has no NONE. If $p^L(\boldsymbol{\xi}) = \text{TRUE}$, then by definition of L, it must be that $p^K(\boldsymbol{\xi}) = \text{TRUE}$. Then, since $p^K \subseteq p^I$, it must be that $p^I(\boldsymbol{\xi}) = \text{TRUE}$. Finally, by definition of I^{none}, it follows that $p^I(\boldsymbol{\xi}) = \text{TRUE}$ since $\boldsymbol{\xi}$ is from $|I|$. Thus it holds that $p^L \subseteq p^{I^{none}}$.
- We wish to show that L and I^{none} do not agree on $\mathbf{c}$. We consider two cases
  - There is some predicate $p \in \mathbf{c}$ and some list of objects $\boldsymbol{\xi}$ from $|I|$ such that $p^K(\boldsymbol{\xi}) = \text{FALSE}$ and $p^I(\boldsymbol{\xi}) = \text{TRUE}$. By definition of I^{none}, we have $p^{I^{none}}(\boldsymbol{\xi}) = \text{TRUE}$ and by definition of L, we have $p^L(\boldsymbol{\xi}) = \text{FALSE}$ so in this case, the claim holds.
  - There is some function $f \in \mathbf{c}$ and some list of objects $\boldsymbol{\xi}$ from $|I|$ such that $f^I(\boldsymbol{\xi}) \neq \text{u}$ and $f^K(\boldsymbol{\xi}) \neq f^I(\boldsymbol{\xi})$. In particular, since $K \preceq^{\mathbf{c}} I$, this means that $f^K = \text{u}$. Now since $f^I(\boldsymbol{\xi}) \neq \text{u}$, by definition of I^{none}, we have $f^I(\boldsymbol{\xi}) \neq \text{NONE}$ and by definition of L, we have $f^L(\boldsymbol{\xi}) = \text{NONE}$ so in this case, the claim holds.

Thus, we have $L <^{\mathbf{c}} I^{none}$.

To show that $L \models gr_{I^{none}}(\text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$ we first note that $gr_{I^{none}}(\text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$ is equivalent to the conjunction of $f(\boldsymbol{\xi}) = \text{NONE}$ for every $f \in \sigma$ and $\boldsymbol{\xi}$ from $|I^{none}|$ such that $f^{I^{none}}(\boldsymbol{\xi}) = \text{NONE}$. We see that by definition of I^{none} that it must be that $f^I(\boldsymbol{\xi}) = \text{u}$. Then, since we assume that $K \prec^{\mathbf{c}} I$, we have that $f^K(\boldsymbol{\xi}) = \text{u}$. Then, by definition of L, we have $f^L(\boldsymbol{\xi}) = \text{NONE}$. Thus we have $L \models gr_{I^{none}}(\text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$.

We now show by induction on G that $K \models_{\mathrm{p}} gr_I[G]^{\underline{I}}$ iff $L \models gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ and then since we assume $K \models_{\mathrm{p}} gr_I[G]^{\underline{I}}$, we will conclude that $L \models gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$. Finally, since we have already seen that $L \models gr_{I^{none}}(\text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$, we will concluded further that $L \models gr_{I^{none}}(G^{none} \wedge \text{UNDEF}_{\mathbf{c}})^{\underline{I^{none}}}$

- Case 1: G is a (σ-plain) ground atomic formula of extended signature σ^* which is σ extended with object names from $|I|$ (not including NONE).

  If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  If instead $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both G, so by definition of K, $K \models_{\mathrm{p}} G$ iff $L \models G$.

- Case 2: G is $H_1 \odot H_2$ where $\odot \in \{\wedge, \vee, \rightarrow\}$ and so $gr_I[G]^{\underline{I}}$ is $gr_I[H_1]^{\underline{I}} \odot gr_I[H_2]^{\underline{I}}$. G^{none} is $(H_1)^{none} \odot (H_2)^{none}$.

  If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  On the other hand, if $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_{(I^{none})}[G^{none}]^{\underline{I^{none}}}$ is

  $$gr_{(I^{none})}[(H_1)^{none}]^{\underline{I^{none}}} \odot gr_{(I^{none})}[(H_2)^{none}]^{\underline{I^{none}}}.$$

  By I.H. on $gr_I[H_1]^{\underline{I}}$ and $gr_{(I^{none})}[(H_1)^{none}]^{\underline{I^{none}}}$ and $gr_I[H_2]^{\underline{I}}$ and $gr_{(I^{none})}[(H_2)^{none}]^{\underline{I^{none}}}$.

- Case 3: G is $\exists x H(x)$. G^{none} is $\exists x(H(x)^{none} \wedge x \neq \text{NONE})$.

  If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  On the other hand, if $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_I[G]^{\underline{I}}$ is

  $$\{gr_I[H(\xi^\diamond)]^{\underline{I}} : \xi \in |I|\}^\vee$$

  and $gr_{(I^{none})}[G^{none}]^{\underline{I^{none}}}$ is

  $$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{I^{none}}} \wedge \xi^\diamond \neq \text{NONE} : \xi \in |(I^{none})|\}^\vee.$$

  Now we note that $|(I^{none})| = |I| \cup \{\text{NONE}\}$. Further, we note that since $\xi^\diamond \neq \text{NONE}$ is not satisfied when $\xi = \text{NONE}$, the latter reduct is equivalent to

  $$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{(I^{none})}} \wedge \xi^\diamond \neq \text{NONE} : \xi \in |I|\}^\vee.$$

  The further, we note that for all $\xi \in |I|$, $\xi^\diamond \neq \text{NONE}$ is satisfied so that this is further equivalent to

  $$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{I^{none}} : \xi \in |I|\}^\vee.$$

  Thus the claim follows by induction on $gr_I[H(\xi^\diamond)]$ and $gr_{(I^{none})}[H(\xi^\diamond)^{none}]$ for every $\xi \in |I|$.

- Case 4: G is $\forall x H(x)$. G^{none} is $\forall x(x \neq \text{NONE} \rightarrow H(x)^{none})$.

  If $I \not\models_{\mathrm{p}} G$, then $I^{none} \not\models G^{none}$ by Lemma 6 and so the reducts $gr_I[G]^{\underline{I}}$ and $gr_{I^{none}}(G^{none})^{\underline{I^{none}}}$ are both $\bot$, which neither L nor K satisfy so in this case, the claim holds.

  On the other hand, if $I \models_{\mathrm{p}} G$, then $I^{none} \models G^{none}$ by Lemma 6 and so $gr_I[G]^{\underline{I}}$ is

  $$\{gr_I[H(\xi^\diamond)]^{\underline{I}} : \xi \in |I|\}^\wedge$$

  and $gr_{(I^{none})}[H^{none}]^{\underline{I^{none}}}$ is

  $$\{\xi^\diamond \neq \text{NONE} \rightarrow gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{I^{none}}} : \xi \in |(I^{none})|\}^\wedge.$$

  Now we note that $|(I^{none})| = |I| \cup \{\text{NONE}\}$. Further, we note that since $\xi^\diamond \neq \text{NONE}$ is not satisfied when $\xi = \text{NONE}$, the latter reduct is equivalent to

  $$\{\xi^\diamond \neq \text{NONE} \rightarrow gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{\underline{(I^{none})}} : \xi \in |I|\}^\wedge.$$

  The further, we note that for all $\xi \in |I|$, $\xi^\diamond \neq \text{NONE}$ is satisfied so that this is further equivalent to

  $$\{gr_{(I^{none})}[H(\xi^\diamond)^{none}]^{I^{none}} : \xi \in |I|\}^\wedge.$$

  Thus the claim follows by induction on $gr_I[H(\xi^\diamond)]$ and $gr_{(I^{none})}[H(\xi^\diamond)^{none}]$ for every $\xi \in |I|$.

(b $\Rightarrow$) We assume that J is an interpretation such that $none^J = none$ and J is a stable model of $G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$. We wish to show that $J = I^{none}$ for some Cabalar stable model I of G relative to $\mathbf{c}$.

We will show this by constructing such an I from J. Let $I = J^{invnone}$ where $J^{invnone}$ is the partial interpretation obtained from J as follows:

- $|J^{invnone}| = |J| \setminus \{\text{NONE}\}$.
- for every function constant $f \in \sigma$ and $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f,

  $$f^{J^{invnone}}(\boldsymbol{\xi}) = \begin{cases} f^J(\boldsymbol{\xi}) & \text{if } f^J(\boldsymbol{\xi}) \neq \text{NONE}; \\ u & \text{otherwise;} \end{cases}$$

- For every predicate $p \in \sigma$ and $\boldsymbol{\xi} \in |I|^n$ where n is the arity of p, $p^{J^{invnone}}(\boldsymbol{\xi}) = p^J(\boldsymbol{\xi})$.

It is not difficult to see that $I^{none} = (J^{invnone})^{none} = J$ so then the claim follows from the $\Leftarrow$ of part (a).

(b $\Leftarrow$) We assume that $J = I^{none}$ for some Cabalar stable model I of G relative to $\mathbf{c}$. We wish to show that J is a stable model of $G^{none} \wedge \text{UNDEF}_{\mathbf{c}}$ and $none^J = none$ (which follows from the definition of $J = I^{none}$). However, this is the $\Rightarrow$ of part (a). ∎