CSE 591 Knowledge Representation and Reasoning, Fall 2009

1. Review: Classical Logic

Reading: The Handbook, Chapter 1.

In Problems 1.1-1.5, we consider propositional logic.

1.1 (a) Find a formula F of the signature $\{p, q, r\}$ such that (1) is the only interpretation satisfying F.

p	q	r
f	f	t

(1)

(b) Find a formula F of the signature $\{p, q, r\}$ whose truth table is

p	q	r	F
f	f	f	f
f	f	t	t
f	t	f	f
f	t	t	f
t	f	f	t
t	f	t	f
t	t	f	f
t	t	t	f

In the following two problems, we assume that the underlying signature is finite: $\sigma = \{p_1, \ldots, p_n\}$.

1.2^e Prove this: For any interpretation I, there exists a formula F such that I is the only interpretation satisfying F.

1.3^e Prove this: For any function α from interpretations to truth values, there exists a formula F such that, for all interpretations I, $F^I = \alpha(I)$.

1.4 Prove this: A set $\{F_1, \ldots, F_n\}$ is unsatisfiable iff $\neg(F_1 \wedge \cdots \wedge F_n)$ is a tautology.

1.5 Prove this: For any set Γ of formulas and any formula F, $\Gamma \models F$ iff the set $\Gamma \cup \{\neg F\}$ is unsatisfiable.

In Problems 1.6–1.9 we consider first-order logic.

Let σ be the signature $\{a, P, Q\}$ where a is an object constant, P is a unary predicate constant, and Q is a binary predicate constant. Let I be the following interpretation:

$$
\begin{aligned}
&|I| = \mathbf{N}, \\
&a^I = 10, \\
&P^I(n) = \begin{cases} \mathsf{t}, & \text{if } n \text{ is prime}, \\ \mathsf{f}, & \text{otherwise}, \end{cases} \\
&Q^I(m,n) = \begin{cases} \mathsf{t}, & \text{if } m < n, \\ \mathsf{f}, & \text{otherwise}. \end{cases}
\end{aligned}
\tag{2}
$$

1.6 Represent the following English sentence by a first-order formula of signature σ and prove that I satisfies it.

> There are infinitely many prime numbers.

1.7 Determine whether the following set of sentences is satisfiable.

(a) $P(a)$, $\exists x \neg P(x)$.

(b) $P(a)$, $\forall x \neg P(x)$.

(c) $\forall x \exists y P(x,y)$, $\forall x \neg P(x,x)$, $\forall xyz((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$.

Recall that the *transitive closure* of Q of a binary relation P on a set X is the smallest *transitive relation* on X that contains P. In the following we will prove that transitive closure cannot be expressed in first-order logic, even using infinitely many formulas. We will assume the following fact, known as the *compactness theorem* for first-order logic, without proving it.

Compactness Theorem. *For any set Γ of first-order formulas, if every finite subset of Γ is satisfiable then Γ is satisfiable also.*

In the following two problems, we assume that Γ is a set of sentences of the signature consisting of binary predicate constants P, Q, such that an interpretation I of that signature is a model of T iff Q^I is the transitive closure of P^I. Let Δ be the following set of formulas:

1. Γ
2. $\forall xy(P(x,y) \leftrightarrow y = f(x))$,

3. $\forall x(x = a \vee Q(a, x))$,
4. $\neg(b = a), \neg(b = f(a)), \neg(b = f(f(a))), \ldots$

1.8 Show that Δ is unsatisfiable.

1.9 Show that every finite subset of Δ is satisfiable.

These two statements are in conflict with the compactness theorem. Consequently, Γ does not exist.