Stable Model Semantics for Partial vs. Total Intensional Functions

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Abstract

Recently, several versions of the stable model semantics that incorporate the concept of intensional functions have emerged in order to express defaults involving non-Herbrand functions. One group defines a stable model in terms of minimality on the values of partial functions, and the other defines it in terms of uniqueness on the values of total functions. While it is known that these approaches coincide on some class of formulas, this paper goes further, and reveals a closer relationship between them. We show that the partial function based semantics can be fully embedded into the total function based semantics, which provides an alternative characterization of the former in terms of a more classical logic friendly way. Embedding in the other direction in full generality is still left open. In the special case of multi-valued formulas, we show that both semantics can be viewed as an abbreviation of the propositional stable model semantics, which allows us to easily relate the mathematical results established for propositional stable model semantics to multi-valued stable model semantics.

1 Introduction

The original stable model semantics (Gelfond and Lifschitz 1988) and many extensions have been restricted to Herbrand models, where the role of functions is quite limited. Several recent extensions of the stable model semantics are motivated by exploiting efficient computational methods involving non-Herbrand functions. In efforts to combine answer set programming (ASP) and constraint solving, such as (Gebser et al. 2009; Janhunen et al. 2011), constraint variables are essentially non-Herbrand functions whose values can be computed by CSP solvers. However, the loose coupling of ASP and CSP in this way is liable for the inability to express defaults involving functions (Bartholomew and Lee 2013a).

In order to overcome such limitations in the semantics, several extensions of the stable model semantics that allow *intensional functions* have emerged, such as (Cabalar 2011; Lifschitz 2012; Bartholomew and Lee 2012; Balduccini 2013). Bartholomew and Lee (2013b) showed that despite the different forms in which these semantics were defined (in terms of second-order logic, in terms of variants of the Logic of Here-and-There, or in terms of grounding and reduct), they can be essentially characterized into two groups: ensuring the stability of a model in terms of *minimality on the values of partial functions* (Cabalar 2011; Balduccini 2013), or in terms of *uniqueness on the valus of total functions* (Lifschitz 2012; Bartholomew and Lee 2012).

In (Bartholomew and Lee 2013b) it is shown that the two approaches coincide on the special case of so-called $\mathbf{c}$-*plain formulas* when we limit attention to *complete* interpretations only. However, it was left open which semantics is more desirable in the more general case. The partial function based approach may seem to be more expressive due to the fact that partial functions are a generalization of total functions, but its semantics is defined based on a non-classical notion of satisfaction, and looks complicated. For instance, $f = f$ is not a tautology under the partial function approach because it is false when f is undefined. On the other hand, $f \neq f$ is satisfiable because f can be undefined. Despite such a difference, in this paper, we show that the partial function based semantics can be fully embedded into the total function based semantics. On the other hand, the other direction is not obvious due to the difference between minimality vs. uniqueness checking, and is left open.

Another contribution of this paper is to show that those functional stable model semantics, when they are applied to the special class of "multi-valued formulas," can be viewed as abbreviations of propositional formulas under the stable model semantics with a slight difference to each other. This finding reveals the close relationship between the functional stable model semantics and their relationships to the propositional stable model semantics, and allows us to easily relate the mathematical results established for propositional formulas to multi-valued formulas, such as the theorem on strong equivalence (Lifschitz et al. 2001) and the splitting theorem (Ferraris et al. 2009). In (Lee et al. 2013), a new action language called $\mathcal{BC}$ was defined by a translation to multi-valued formulas and by a translation to logic programs. The equivalence between the two translations follows from our finding.

This paper is organized as follows. We show how multi-valued formulas under the partial and total function based stable model semantics can be turned into standard propositional formulas under the stable model semantics, and then show how the two functional stable model semantics can be reduced to each other. In the general first-order case, we show how the partial function based approach can be fully embedded in the total function based approach.

2 Multi-Valued Formulas under the Stable Model Semantics

We first discuss our main results in terms of multi-valued formulas, a simple extension of standard propositional formulas that allows atoms to express functions over finite domains. The convenience of multi-valued formulas for knowledge representation is demonstrated in the context of nonmonotonic causal theories and action language $\mathcal{C}$+ (Giunchiglia et al. 2004). For this paper, multi-valued propositional formulas serve as a simple but useful special case of first-order formulas to compare different extensions of the functional stable model semantics.

2.1 Review: Stable Models of Multi-Valued Formulas

We first review stable models of multi-valued formulas from (Bartholomew and Lee 2012), and present a variant based on the concept of partial functions, which is essentially a special case of the semantics from (Cabalar 2011) and a generalization of the semantics from (Balduccini 2013) to more complex formulas.

A multi-valued signature is a set σ of symbols called *constants*, along with a finite set $Dom(c)$ of symbols that is disjoint from σ and contains at least two elements, assigned to each constant c. We call $Dom(c)$ the *domain* of c. A *multi-valued atom* of σ is $\bot$, or an expression of the form $c{=}v$ ("the value of c is v") where $c \in \sigma$ and $v \in Dom(c)$. A *multi-valued formula* of σ is a propositional combination of multi-valued atoms.

A *multi-valued interpretation* of σ is a function that maps every element of σ to an element in its domain. We often identify an interpretation with the set of atoms of σ that are satisfied by I. A multi-valued interpretation I *satisfies* an atom $c{=}v$ (symbolically, $I \models c{=}v$) if $I(c) = v$. The satisfaction relation is extended from atoms to arbitrary formulas according to the usual truth tables for the propositional connectives. We say that I is a *model* of F if it satisfies F.

An expression of the form $c=d$, where both c and d are constants, will be understood as an abbreviation for the formula

$$\bigvee_{v\in Dom(c)\cap Dom(d)} (c=v \wedge d=v). \tag{1}$$

Let F be a multi-valued formaula of signature σ, and let I be a multi-valued interpretation of σ. The *reduct* of F relative to I (denoted $F^{\underline{I}}$) is the formula obtained from F by replacing each (maximal) subformula that is not satisfied by I with $\bot$. I is a *multi-valued stable model* of F if I is the *only* multi-valued interpretation of σ that satisfies $F^{\underline{I}}$.

Example 1
Consider describing the capacity of a container of water that has a leak but that can be refilled to the maximum amount, say 10000 liter, with the action *FillUp*. Let σ be a multi-valued signature $\{Amount_0,\ Amount_1,\ FillUp\}$. The example can be described by multi-valued formula F_1 of σ which is the conjunction of

$$\begin{array}{rcll} Amount_1=x \vee \neg(Amount_1=x) & \leftarrow & Amount_0=y & \qquad (y=x+1) \\ Amount_1=10000 & \leftarrow & FillUp=\text{TRUE} & \\ Amount_0=z \vee \neg(Amount_0=z) & & & \\ FillUp=b \vee \neg(FillUp=b) & & & \end{array} \tag{2}$$

(Notation: $x \in \{0,\dots,9999\}$, $y \in \{1,\dots,10000\}$, $z \in \{0,\dots,10000\}$, $b \in \{\text{TRUE},\text{FALSE}\}$. We understand $G \leftarrow F$ as alternative notation of $F \rightarrow G$.)

Under the stable model semantics, the first formula in (2) represents that the amount is decreased by 1 by default, and the second formula provides an exception to this default behavior. Indeed, $I_1 = \{Amount_0=6,\ FillUp=\text{FALSE},\ Amount_1=5\}$ is a stable model of F_1: the reduct $F_1^{\underline{I_1}}$ is equivalent to the conjunction of

$$\begin{array}{rcl} Amount_1=5 & \leftarrow & Amount_0=6 \\ Amount_0=6 & & \\ FillUp=\text{FALSE} & & \end{array} \tag{3}$$

and I_1 is the unique model of the reduct. For $I_2 = \{Amount_0=6,\ FillUp=\text{FALSE},\ Amount_1=8\}$, the reduct $F_1^{\underline{I_2}}$ is equivalent to $Amount_1=8$, which has more than one model. So I_2 is not a stable model. For $I_3 = \{Amount_0=6,\ Amount_1=10000,\ FillUp=\text{TRUE}\}$, the reduct $F_1^{\underline{I_3}}$ is equivalent to

$$\begin{array}{rcl} Amount_1=10000 & \leftarrow & FillUp=\text{TRUE} \\ Amount_0=6 & & \\ FillUp=\text{TRUE} & & \end{array}$$

and I_3 is the unique model of the reduct, so I_3 is a stable model.

As shown in Example 1, formulas of the form $F \vee \neg F$ under the stable model semantics are useful for representing defaults involving functions. We abbreviate $F \vee \neg F$ as $\langle F\rangle$. For example, the first formula in (2) can be written as $\langle Amount_1 = x\rangle \leftarrow Amount_0 = y$, and can be read as "by default $Amount_1$ is x if $Amount_0$ is y.

2.2 Reducing Multi-Valued SM to Propositional SM

In this section we show that multi-valued stable model semantics can be viewed as a special case of propositional stable model semantics. Let σ be a multi-valued signature, and let σ^{prop}

be the propositional signature consisting of all propositional atoms $c = v$ where $c \in \sigma$ and $v \in Dom(c)$. For example, for σ in Example 1, σ^{prop} is the set $\{Amount_0 = 0, \ldots, Amount_0 = 10000,\ Amount_0 = 1, \ldots, Amount_1 = 10000,\ FillUp = \text{TRUE},\ FillUp = \text{FALSE}\}$, where each element is understood as a propositional atom.[1] We identify a multi-valued interpretation of σ with the corresponding set of propositional atoms from σ^{prop}. It is clear that a multi-valued interpretation I of signature σ satisfies a multi-valued formula F iff I satisfies F viewed as a propositional formula of signature σ^{prop}. Also, it is not difficult to show that multi-valued formulas can be turned into standard propositional formulas while preserving *models*. Less obvious is whether such a relationship would hold while preserving *stable models*. Theorem 1 below shows such a translation.

Given a multi-valued signature σ, by UC_σ ("Uniqueness Constraint") we denote the conjunction of

$$\bigwedge_{v \neq w \ \mid \ v,w \in Dom(c)} \neg(c = v \wedge c = w) \tag{4}$$

for all $c \in \sigma$, and by EC_σ ("Existence Constraint") we denote the conjunction of

$$\neg\neg \bigvee_{v \in Dom(c)} c = v \ , \tag{5}$$

for all $c \in \sigma$. By UEC_σ we denote the conjunction of (4) and (5) for all $c \in \sigma$.

Theorem 1

[thm:mvpf-sm-total] Let F be a multi-valued formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.

(a) If an interpretation I of σ is a multi-valued stable model of F, then I can be viewed as an interpretation of σ^{prop} that is a propositional stable model of $F \wedge UEC_\sigma$ in the sense of (Ferraris 2005).

(b) If an interpretation I of σ^{prop} is a propositional stable model of $F \wedge UEC_\sigma$ in the sense of (Ferraris 2005), then I can be viewed as an interpretation of σ that is a multi-valued stable model of F.

Example 1 Continued UEC_{Amount_0} is the conjunction of

$$\bigwedge_{v \neq w \ \mid \ v,w \in \{0,\ldots,10000\}} \neg(Amount_0 = v \wedge Amount_0 = w) \ ,$$

$$\neg\neg \bigvee_{v \in \{0,\ldots,10000\}} Amount_0 = v$$

Note that the presence of $\neg\neg$ in (5) is essential for Theorem 1 to be valid. For instance, consider a signature containing only one constant c whose domain is $\{1, 2\}$ and F to be $\top$. F has no multi-valued stable models, [2] but $F \wedge \neg(c = 1 \wedge c = 2) \wedge (c = 1 \vee c = 2)$ has two propositional stable models: $\{c = 1\}$ and $\{c = 2\}$.

Though Theorem 1 tells us that semantically multi-valued stable models are a special class of

[1] We could have included in σ^{prop} different expressions such as $c(v)$ in place of $c = v$. Viewing $c = v$ as both multi-valued atoms and propositional atoms under different signatures simplifies the formal statements.

[2] Recall that the doamin contains at least two elements.

propositional stable models, it does not mean that the multi-valued stable models should be computed in the same way as propositional stable models. While computing the propositional stable models of $F \wedge \mathit{UEC}_\sigma$ may involve a large number of ground rules, the explicit representation of functions in multi-valued formulas allows us to take the advantage of efficient computation methods involving functions directly. For instance, according to (Bartholomew and Lee 2013a), the multi-valued stable models of F_1 correspond to the models of

$$\big((\mathit{Amount}_0 = \mathit{Amount}_1 + 1) \vee (\mathit{Amount}_1 = 10000 \wedge \mathit{FillUp})\big) \wedge (\mathit{FillUp} \rightarrow \mathit{Amount}_1 = 10)\ , \quad (6)$$

which can be computed by Satisfiability Modulo Theory (SMT) solvers without grounding value variables for Amount_0 and Amount_1.

2.3 CB-Stable Models of Multi-Valued Formulas

In this section we introduce a variant of the stable model semantics in the previous section, which we call the CB-stable model semantics. This definition of a stable model follows the approach by Cabalar (2011), and later by Balduccini (2013), which allows functions to be partially defined. In other words, interpretations are allowed to leave some constants undefined. By *complete* interpretations, we mean a special case of partial interpretations where all constants are defined. Complete partial interpretations can be identified with the classical ("total") interpretations.

We consider the same syntax of a multi-valued formula as in the previous section. As with total interpretations, a partial interpretation I *satisfies* an atom $c = v$ if $I(c)$ is defined and is mapped to v. This implies that an interpretation that is undefined on c does not satisfy any atom of the form $c = w$ where $w \in \mathit{Dom}(c)$. As before, it is convenient to identify a partial interpretation I with the set of atoms of σ that are satisfied by this interpretation. For instance, an interpretation of $\sigma = \{c\}$ which is undefined on c is identified with the empty set. As before, the satisfaction relation is extended from atoms to arbitrary formulas according to the usual truth tables for the propositional connectives. We call I a *model* of F if it satisfies F.

The reduct $F^{\underline{I}}$ is defined to be the same as before. We say that a partial interpretation I is a *CB-stable model* of F if I satisfies F and no subset J of I satisfies $F^{\underline{I}}$.

Example 2

Every stable model of F_1 is a CB-stable model as well. In addition, F_1 has other CB-stable models, which are not complete. For instance, $\emptyset$ (i.e., every constant is undefined) is a CB-stable model: the reduct $F_1^{\underline{\emptyset}}$ is equivalent to $\top$, and $\emptyset$ is the minimal model of the reduct. There are other CB-stable models, such as $\{\mathit{Amount}_0 = 6,\ \mathit{FillUp} = \text{FALSE}\}$ ("Amount becomes unknown even though no action was executed"); $\{\mathit{FillUp} = \text{TRUE},\ \mathit{Amount}_1 = 10000\}$ ("Initial amount is unknown but once *FillUp* is executed, the resulting amount is set to 10000"); and $\{\mathit{FillUp} = \text{FALSE}\}$ ("the action is not executed, and the amounts at both times are unknown"). Some CB stable models may seem unnatural.

The first formula in (2) under the CB-stable model semantics expresses the choice between either Amount_1 has a value x or is undefined. This is in contrast with the stable model semantics in the previous section, where the formula expresses the concept of default.

Similar to Theorem 1, the following theorem tells us that the CB-stable models of a multi-valued formula can be identified with the stable models of a propositional formula. The only difference is that we impose UC_σ in place of UEC_σ.

Theorem 2
[thm:mvpf-sm-partial] Let F be a multi-valued formula of signature σ, which can be also viewed as a propositional formula of signature σ^{prop}.

(a) If a partial interpretation I of σ is a CB-stable model of F, then I can be viewed as an interpretation of σ^{prop} that is a propositional stable model of $F \wedge UC_\sigma$ (in the sense of (Ferraris 2005)).
(b) If an interpretation I of σ^{prop} is a propositional stable model of $F \wedge UC_\sigma$ (in the sense of (Ferraris 2005)), then I can be viewed as a partial interpretation of σ that is a CB-stable model of F.

The following corollary immediately follows from Theorems 1 and 2. It tells us that the stable model semantics can be fully embedded into the CB-semantics.

Corollary 1
[cor:sm-cbsm] For any multi-valued formula F of signature σ and any partial interpretation I, we have that I is a multi-valued stable model of F iff I is a CB-stable model of $F \wedge EC_\sigma$.

The definition of a CB-stable model is essentially a simplification of the semantics from (Cabalar 2011) and a generalization of the one from (Balduccini 2013). Unlike the way we treat $c = d$ as an abbreviation of (1), in the latter paper, it was called a *t-atom*, for which the notion of satisfaction was defined directly: I *satisfies* $c = d$ if I is defined on both c and d, and maps them to the same value. This is essentially equivalent to the way we understand $c = d$ as shorthand for formula (1).[3]

Since I satisfies $c = c$ iff I is defined on c, the assertion in Corollary 1 remains valid when we replace EC_σ in the statement with $\neg\neg\left(\bigwedge_{c\in\sigma} c = c\right)$.

2.4 Reducing CB semantics to Multi-Valued Stable Models

The CB-stable model semantics may seem more expressive as it allows interpretations to be partially defined: any multi-valued stable model of F is a CB-stable model, but not vice versa because an incomplete partial interpretation has no direct counterpart as a total interpretation. Nevertheless, we show that it is possible to embed the CB-stable model semantics (based on partial functions) into the multi-valued stable model semantics (based on total functions).

Let σ be a multi-valued signature, and let σ^{none} be the signature that is the same as σ except that the domain of each constant has an additional new value NONE. Given a partial interpretation I of σ, by I^{none} we denote an interpretation of σ^{none} that agrees with I on all defined constants, and maps undefined constants to NONE. Recall that expression $\langle F\rangle$ stands for the formula $F \vee \neg F$.

Theorem 3
[thm:mvpf-partial-total] Let F be a multi-valued formula of signature σ.

(a) If an interpretation I of σ is a CB-stable model of F, then I^{none} is a stable model of $F \wedge \bigwedge_{c\in\sigma}\langle c = \text{NONE}\rangle$.
(b) If an interpretation J of σ^{none} is a stable model of $F \wedge \bigwedge_{c\in\sigma}\langle c = \text{NONE}\rangle$ then $J = I^{none}$ for some CB-stable model I of F.

[3] In a sense, our treatment is more general because it allows the domains of c and d to be different.

Example 2 Continued The CB-stable models of F_1 are in a 1-1 correspondence with the stable models of $F_1 \wedge \langle Amount_1 = \text{NONE}\rangle \wedge \langle Amount_0 = \text{NONE}\rangle \wedge \langle FillUp = \text{NONE}\rangle$.

However, if we want to capture the partial stable models that match the intended behavior of the domain while omitting the unintuitive ones, we can modify the first rule in F_1 to be

$$\langle Amount_1 = x\rangle \wedge Amount_1 = Amount_1 \;\leftarrow\; Amount_0 = y \qquad (y = x + 1)\ .$$

to obtain F_1' and now $\{FillUp = \text{TRUE},\ Amount_1 = 10000\}$ ("Initial amount is unknown but once *FillUp* is executed, the resulting amount is set to 10000") and $\{FillUp = \text{FALSE}\}$ ("the action is not executed, and the amounts at both times are unknown") are both still CB-stable models while the unintended $\{Amount_0 = 6, FillUp = \text{FALSE}\}$ ("Amount becomes unknown though no action was executed") is no longer a CB-stable model. We can use Theorem 3 to obtain that $F_1' \wedge \langle Amount_1 = \text{NONE}\rangle \wedge \langle Amount_0 = \text{NONE}\rangle \wedge \langle FillUp = \text{NONE}\rangle$ exhibits this same behavior. Alternatively, we can represent this by mentioning the constant NONE directly. F_1'' is obtained from F_1 by rewriting the first rule as

$$\langle Amount_1 = x\rangle \;\leftarrow\; Amount_0 = y \wedge \neg(Amount_0 = \text{NONE}) \qquad (y = x + 1)\ .$$

$F_1'' \wedge \langle Amount_1 = \text{NONE}\rangle \wedge \langle Amount_0 = \text{NONE}\rangle \wedge \langle FillUp = \text{NONE}\rangle$ exhibits this same behavior.

2.5 Strong Negation in Functional View

Interestingly, the translation $F \wedge UC_\sigma$ in Theorem 2 is similar to the way that strong negation (denoted $\sim$) is understood in terms of auxiliary atoms. According to (Ferraris et al. 2011), the stable model semantics of propositional formulas involving strong negation is defined as follows. For any propositional signature σ, signature σ^{sneg} is the set of propositional atoms consisting of c and $\sim c$ for all $c \in \sigma$. We consider *coherent* interpretations of σ^{sneg} only, which do not satisfy both c and $\sim c$. For any formula F of signature σ^{sneg}, an interpretation I of σ^{sneg} is a stable model of F if I is a minimal model of the reduct F relative to I.

Given a propositional signature σ, by σ^{bool} we denote the multi-valued signature where each $c \in \sigma$ is understood as a Boolean constant. We identify an interpretation I of σ^{sneg} with an interpretation I^{bool} of σ^{bool} as follows:

- I^{bool} maps c to TRUE and $\sim c$ to FALSE if I maps c to TRUE;
- I^{bool} maps c to FALSE and $\sim c$ to TRUE if I maps c to FALSE;
- I^{bool} is undefined on both c and $\sim c$ if I is undefined on c.

We identify a propositional formula F of signature σ^{sneg} with a multi-valued formula of σ^{bool} by identifying the occurrence of c in F with $c = \text{TRUE}$ and $\sim c$ with $c = \text{FALSE}$. The following corollary, which follows from Theorem 2, justifies such identification.

Corollary 2
[cor:sneg-cbsm-mvpf] Let F be a formula of signature σ^{sneg}, which can be also viewed as a multi-valued formula of signature σ^{bool}. We have that I is a coherent propositional stable model of F iff I^{bool} is a CB-stable model of F.

Syntactically strong negation is not a connective; it occurs only in front of propositional atoms. This aligns with the view that $\sim c$ is identified with $c = \text{FALSE}$. Just like interchaning $\sim c$ and $\neg c$ in a formula may result in different stable models, interchanging $c = \text{FALSE}$ and $\neg(c = \text{TRUE})$ may result in different CB-stable models.

Example 3
The formula

$$(p_0 \vee \sim p_0) \wedge (p_0 \wedge \neg \sim p_1 \rightarrow p_1) \wedge (\sim p_0 \wedge \neg p_1 \rightarrow \sim p_1) \tag{7}$$

expresses that fact p is either true or false initially, and is inertial—both $\{p_0, p_1\}$ and $\{\sim p_0, \sim p_1\}$ are the stable models. In accordance with Corollary 2, the corresponding formula of σ^{bool}

$$\begin{aligned} &(p_0 = \text{TRUE} \vee p_0 = \text{FALSE}) \\ &\wedge (p_0 = \text{TRUE} \wedge \neg(p_1 = \text{FALSE}) \rightarrow p_1 = \text{TRUE}) \\ &\wedge (p_0 = \text{FALSE} \wedge \neg(p_1 = \text{TRUE}) \rightarrow p_1 = \text{FALSE}) \end{aligned} \tag{8}$$

has two CB-stable models: $\{p_0 = \text{TRUE}, p_1 = \text{TRUE}\}$ and $\{p_0 = \text{FALSE}, p_1 = \text{FALSE}\}$.

A similar result holds with the multi-valued stable model semantics. The only difference is that the statement is restricted to total interpretations only.

Corollary 3
[cor:sneg-fsm-mvpf] Let F be a formula of signature σ^{sneg}, which can be also viewed as a multi-valued formula of signature σ^{bool}. We have that I is a coherent propositional stable model of $F \wedge \bigwedge_{p \in \sigma} \neg(\neg p \wedge \neg \sim p)$ iff I^{bool} is a multi-valued stable model of F.

3 Stable Model Semantics of Intensional Functions

In this section we generalize the results in the previous section to the first-order level.

3.1 Review: Stable Models from Bartholomew and Lee (2012)

We review the definition of a stable model from (Bartholomew and Lee 2012), reformulated in terms of the grounding and reduct as in (Bartholomew and Lee 2013b).

Let F be any first-order sentence of signature σ, and let I be an interpretation of σ. By $gr_I[F]$ we denote the infinitary ground formula w.r.t. I that is obtained from F by the following process:[4]

- If F is an atomic formula, $gr_I[F]$ is F;
- $gr_I[G \odot H] = gr_I[G] \odot gr_I[H] \quad (\odot \in \{\wedge, \vee, \rightarrow\})$;
- $gr_I[\forall x G(x)] = \{gr_I[G(\xi^\diamond)] \mid \xi \in |I|\}^\wedge$; $\quad gr_I[\exists x G(x)] = \{gr_I[G(\xi^\diamond)] \mid \xi \in |I|\}^\vee$.

For example, for formula

$$\forall x \big((\mathit{Amount}_1 = x) \vee \neg(\mathit{Amount}_1 = x) \quad \leftarrow \quad \mathit{Amount}_0 = x + 1 \big)$$

and an interpretation I whose universe is the set of nonnegative integers, $gr_I[F_1]$ is the infinite conjunction of the following formulas.

$$\begin{aligned} &(\mathit{Amount}_1 = 0) \vee \neg(\mathit{Amount}_1 = 0) \ \leftarrow \ \mathit{Amount}_0 = 0 + 1 \\ &(\mathit{Amount}_1 = 1) \vee \neg(\mathit{Amount}_1 = 1) \ \leftarrow \ \mathit{Amount}_0 = 1 + 1 \\ &\qquad \cdots \end{aligned}$$

For any two interpretations I, J of the same signature and any list $\mathbf{c}$ of distinct predicate and function constants (called "intensional constants"), we write $J <^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants p in $\mathbf{c}$; and

[4] The definition of infinitary ground formulas is reviewed in Appendix A.

- J and I do not agree on $\mathbf{c}$.

The *reduct* $F^{\underline{I}}$ of an infinitary ground formula F relative to an interpretation I is defined as follows:

- For each atomic formula F, $F^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models F; \\ F & \text{otherwise;} \end{cases}$
- $(\mathcal{H}^\wedge)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models \mathcal{H}^\wedge; \\ \{G^{\underline{I}} \mid G \in \mathcal{H}\}^\wedge & \text{otherwise;} \end{cases}$
- $(\mathcal{H}^\vee)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models \mathcal{H}^\vee; \\ \{G^{\underline{I}} \mid G \in \mathcal{H}\}^\vee & \text{otherwise;} \end{cases}$
- $(G \to H)^{\underline{I}} = \begin{cases} \bot & \text{if } I \not\models G \to H; \\ G^{\underline{I}} \to H^{\underline{I}} & \text{otherwise.} \end{cases}$

Definition 1
[def:fsm-reduct] An interpretation I is a *stable model* of F relative to $\mathbf{c}$ if

- I satisfies F, and
- every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[F])^{\underline{I}}$.

In the absence of intensional function constants, this definition reduces to the definition from (Ferraris et al. 2011). In other words, this definition is a proper extension of the semantics from (Ferraris et al. 2011).

3.2 Multi-Valued Formulas as a Special Case

Multi-valued stable model semantics can be viewed as a special case of the stable model semantics in the previous section when the formulas are restricted to be $\mathbf{c}$-plain.

Let σ be a multi-valued signature and let σ^{fo} be a first-order signature consisting of all object constants c in σ, and all object constants v for $v \in \mathit{Dom}(c)$ for every $c \in \sigma$. Given a multi-valued interpretation I, we define the first-order interpretation I^{fo} as follows:

- $|I^{fo}| = \{v \mid v \in \mathit{Dom}(c) \text{ for some } c \in \sigma\}$;
- $I^{fo}(v) = v$ for each $v \in \mathit{Dom}(c)$ for each $c \in \sigma$;
- $I^{fo}(c) = I(c)$ for each multi-valued constant $c \in \sigma$.

Proposition 1
[prop:mvpf-fsm] For any multi-valued formula F of signature σ, and any multi-valued interpretation I of σ whose multi-valued constants are $\mathbf{c}$, I is a stable model of F iff I^{fo} is a stable model of F relative to $\mathbf{c}$ when F is viewed as a first-order formula of signature σ^{fo}.

3.3 Reducing Cabalar Semantics to SM Under Complete Interpretations

We show how to embed the Cabalar semantics in terms of the stable model semantics when we restrict attention to complete interpretations. Due to lack of space we refer the reader to Appendix B for the definition of a Cabalar stable model, which is a reformulation of the original semantics from (Cabalar 2011) in terms of grounding and reduct (Bartholomew and Lee 2013b).

The process of "unfolding" (Bartholomew and Lee 2013b) turns any first-order formula into a $\mathbf{c}$-plain formula. The *unfolding of F w.r.t.* $\mathbf{c}$, denoted by $\mathit{UF}_{\mathbf{c}}(F)$, is defined as follows.

- If F is $\bot$, $\mathit{UF}_{\mathbf{c}}(\bot)$ is $\bot$;

- If F is of the form $p(t_1,\dots,t_n)$ ($n\ge 0$) such that $t_{k_1},\dots,t_{k_j}$ are all the terms in $t_1,\dots,t_n$ that contain some members of $\mathbf{c}$, then $\mathit{UF}_{\mathbf{c}}(p(t_1,\dots,t_n))$ is
$$\exists x_1\dots x_j\Big(p(t_1,\dots,t_n)''\land\bigwedge_{1\le i\le j}\mathit{UF}_{\mathbf{c}}(t_{k_i}=x_i)\Big)$$
where $p(t_1,\dots,t_n)''$ is obtained from $p(t_1,\dots,t_n)$ by replacing each t_{k_i} with the variable x_i.
- If F is of the form $f(t_1,\dots,t_n)=t_0$ ($n\ge 0$) such that $t_{k_1},\dots,t_{k_j}$ are all the terms in $t_0,\dots,t_n$ that contain some members of $\mathbf{c}$, then $\mathit{UF}_{\mathbf{c}}(f(t_1,\dots,t_n)=t_0)$ is
$$\exists x_1\dots x_j\Big((f(t_1,\dots,t_n)=t_0)''\land\bigwedge_{0\le i\le j}\mathit{UF}_{\mathbf{c}}(t_{k_i}=x_i)\Big)$$
where $(f(t_1,\dots,t_n)=t_0)''$ is obtained from $f(t_1,\dots,t_n)=t_0$ by replacing each t_{k_i} with the variable x_i.
- $\mathit{UF}_{\mathbf{c}}(F\odot G)$ is $\mathit{UF}_{\mathbf{c}}(F)\odot\mathit{UF}_{\mathbf{c}}(G)$ where $\odot\in\{\land,\lor,\rightarrow\}$.[5]
- $\mathit{UF}_{\mathbf{c}}(QxF)$ is $Qx\;\mathit{UF}_{\mathbf{c}}(F(x))$ where $Q\in\{\forall,\exists\}$.

For example, $\mathit{UF}_f(p(f))$ is $\exists x(p(x)\land f=x)$; $\mathit{UF}_{(f,g)}(f=g)$ is $\exists x(f=x\land g=x)$. For $\mathbf{c}$-plain formula F, $\mathit{UF}_{\mathbf{c}}(F)$ is exactly F.

The following theorem generalizes Theorem 3 to arbitrary first-order sentences. It tells us that, instead of relying on the rather complex notion of satisfaction involving partial functions, one can define Cabalar stable models in terms of the standard satisfaction using total functions.

Theorem 4

[thm:fsm-cbl-total] For any sentence F and any complete interpretation I that satisfies $\exists xy(x\ne y)$, I is a Cabalar stable model of F relative to $\mathbf{c}$ iff I is a stable model of $\mathit{UF}_{\mathbf{c}}(F)$ relative to $\mathbf{c}$.

Example 4

To see why we need the condition that there should be at least two elements in the universe (i.e., $\exists xy(x\ne y)$), let F be the formula $\top$ with signature $\sigma=\{c\}$. When the universe is $\{1\}$, the only interpretation that maps c to 1 is a stable model of $\mathit{UF}_\sigma(\top)$ relative to c, but it is not a Cabalar stable model of F relative to c.

Example 5

The theorem becomes incorrect if $\mathit{UF}_{\mathbf{c}}(F)$ in the statement is replaced with F. For example, if F is $f=g$ where f and g are object constants, and the universe is $\{1,2\}$, then each of $\{f=1,\ g=1\}$ and $\{f=2,\ g=2\}$ is a Cabalar stable model of $f=g$ relative to (f,g), as well as a stable model of $\mathit{UF}_{(f,g)}(F)$, but neither of them is a stable model of F relative to (f,g). For another example, consider F to be $p(f)$ where p is a non-intensional predicate constant and f is an object constant, and take an interpretation I such that $p^I=\{1,2\}$ and $f^I=1$. I is a Cabalar stable model of F relative to f, as well as a stable model of $\mathit{UF}_f(F)$, but not a stable model of F relative to f.

[5] We understand $\neg F$ as shorthand for $F\rightarrow\bot$.

3.4 Reducing Cabalar Semantics to SM Under Partial Interpretations

The restriction in Theorem 4 that I be complete can be removed as follows.

Let F be a first-order sentence of signature σ. F^{none} is the formula of signature $\sigma \cup \{\text{NONE}\}$ (where NONE is a new object constant) that is obtained from F as follows.

- for any atomic formula F, $F^{none} = F$;
- $(G \odot H)^{none} = (G^{none} \odot H^{none})$ where $\odot \in \{\wedge, \vee, \rightarrow\}$;
- $\forall x G(x)^{none}$ is $\forall x(x \neq \text{NONE} \rightarrow G(x)^{none})$;
- $\exists x G(x)^{none}$ is $\exists x(G(x)^{none} \wedge x \neq \text{NONE})$.

Given a partial interpretation I, we define the total interpretation I^{none} as

- $|I^{none}| = |I| \cup \{\text{NONE}\}$;
- $\text{NONE}^{I^{none}} = \text{NONE}$;
- for every function constant $f \in \sigma$ and $\boldsymbol{\xi} \in |I^{none}|^n$ where n is the arity of f,

$$f^{I^{none}}(\boldsymbol{\xi}) = \begin{cases} f^I(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \text{ is in } |I|^n \text{ and } f^I(\boldsymbol{\xi}) \text{ is defined;} \\ \text{NONE} & \text{otherwise;} \end{cases}$$

- For every predicate $p \in \sigma$ and $\boldsymbol{\xi} \in |I^{none}|^n$ where n is the arity of p,

$$p^{I^{none}}(\boldsymbol{\xi}) = \begin{cases} p^I(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \text{ is in } |I|^n; \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

Theorem 5
[thm:fsm-cbl-partial] For any sentence F of signature σ,

(a) If I is a Cabalar stable model of F relative to $\mathbf{c}$, then I^{none} is a stable model of

$$(UF_\sigma(F))^{none} \wedge \bigwedge_{f \in \mathbf{c}} \forall \mathbf{x} \langle f(\mathbf{x}) = \text{NONE} \rangle \tag{9}$$

relative to $\mathbf{c}$.

(b) If an interpretation J such that $\text{NONE}^J = \text{NONE}$ is a stable model of (9) relative to $\mathbf{c}$, then $J = I^{none}$ for some Cabalar stable model I of F relative to $\mathbf{c}$.

Example 6
Let F be $f = f$, and let $\mathbf{c}$ be f. Assuming that the universe is $\{1, 2\}$, F has two Cabalar stable models: $\{f = 1\}$ and $\{f = 2\}$. The translation (9) yields the formula

$$\exists x(f = x \wedge x \neq \text{NONE}) \wedge (f = \text{NONE} \vee \neg(f = \text{NONE})),$$

and, in accordance with Theorem 5, its stable models are the same as the Cabalar stable models.

For $\neg F$, set $\emptyset$ is the only Cabalar stable model. Accordingly, $(UF_\sigma(\neg F))^{none} \wedge \langle f = \text{NONE} \rangle$ has only one stable model: $\{f = \text{NONE}\}$.

Theorem 5 becomes incorrect if we do not apply unfolding, i.e., if we replace $UF_\sigma(F)$ in the statement with F. Indeed, for formula $f = f$ above, the modification of (9) yields $f = f \wedge \langle f = \text{NONE} \rangle$, which has $\{f = \text{NONE}\}$ as the only stable model.

Also, Theorem 5 becomes incorrect if the unfolding is restricted to $\mathbf{c}$ only rather than to the whole σ, i.e., if we replace $UF_\sigma(F)$ with $UF_{\mathbf{c}}(F)$. Indeed, consider F to be $a = b$ where neither a nor b is intensional (i.e., $a, b \notin \mathbf{c}$). Formula (9) is still $a = b$. $I = \emptyset$ is not a Cabalar stable model of $a = b$ relative to $\emptyset$, but $I^{none} = \{a = \text{NONE}, b = \text{NONE}\}$ is a stable model of $a = b$ relative to $\emptyset$.

Example 7
To see why we need the condition that $\text{NONE}^J = \text{NONE}$ in part (b), consider the formula $\top$ with signature $\sigma = \{c\}$ and the universe $\{1\}$. The only Cabalar stable model I is undefined on c. On the other hand, formula (9) yields $f = \text{NONE} \lor f \neq \text{NONE}$. Here, without the condition, we have a stable model J such that $\text{NONE}^J = 1$ and $f^J = 1$ but this does not correspond to the Cabalar stable model.

3.5 Reducing Stable Model Semantics to Cabalar Semantics

Theorem 5 from (Bartholomew and Lee 2013b) tells us that for any $\mathbf{c}$-plain sentence F, the the complete CB-stable models of F are precisely the stable models of F. The following corollary shows that the restriction to complete interpretations can instead be expressed in the sentence itself.

Corollary 4
For any $\mathbf{c}$-plain sentence F and any partial interpretation I that satisfies $\exists xy(x \neq y)$, I is a stable model of F relative to $\mathbf{c}$ iff I is a Cabalar stable model of $F \land \neg\neg \bigwedge_{f \in \mathbf{c}} \forall \mathbf{x}(f(\mathbf{x}) = f(\mathbf{x}))$ relative to $\mathbf{c}$.

However, the restriction that the sentence is $\mathbf{c}$-plain remains. We consider two examples of non-$\mathbf{c}$-plain sentences below.

Example 8
Consider the very simple problem of restricting the function f to a certain domain. To express that f is a member of dom_1 with the universe $\{1, 2, 3\}$, we can simply write $dom_1(f)$ where $\mathbf{c} = \{f\}$ (dom_1 is non-intensional) which alone has no stable models as long as dom_1 has more than one element. However, this has among its CB-stable models $\{dom_1(1), dom_1(2), f = 1\}$ and $\{dom_1(1), dom_1(2), f = 2\}$. We can try writing this as a constraint $\neg\neg dom_1(f)$ and no longer are there any CB-stable models. However, this does not work in general. Consider the extension to the previous example in which we know that f belongs to two different domains. To express that f is a member of dom_1 and a member of dom_2 with universe $\{1, 2, 3\}$, we can simply write $dom_1(f) \land dom_2(f)$ where $\mathbf{c} = \{f\}$ (dom_1 and dom_2 are non-intensional) which has a stable model in the case that the intersection of dom_1 and dom_2 is of size 1; e.g. $\{dom_1(1), dom_1(2), dom_2(2), dom_2(3), f = 2\}$ is a stable model. Now the approach to capture this in the Cabalar semantics in the previous example would write this $\neg\neg dom_1(f) \land \neg\neg dom_2(f)$ which has no CB-stable models.

It remains an open question whether this behavior can be captured in the Cabalar semantics.

4 Conclusion

This paper investigates the close relationship between the partial function based approach and the total function based approach for defining intensional functions. We showed that the partial function based approach can be fully embedded into the total function based approach, which provides an alternative characterization of the former in terms of a more classical logic friendly way. The other direction is open for future work. It appears that some behavior of non-$\mathbf{c}$-plain formulas under the total function based semantics is not easily captured in the partial function based semantics due to the difference between uniqueness vs. minimality checking.

In the special case of multi-valued formulas, we show that both semantics can be viewed as an abbreviation of the propositional stable model semantics.

References

BALDUCCINI, M. 2013. ASP with non-herbrand partial functions: a language and system for practical use. *TPLP 13,* 4-5, 547–561.

BARTHOLOMEW, M. AND LEE, J. 2012. Stable models of formulas with intensional functions. In *Proceedings of International Conference on Principles of Knowledge Representation and Reasoning (KR)*. 2–12.

BARTHOLOMEW, M. AND LEE, J. 2013a. Functional stable model semantics and answer set programming modulo theories. In *Proceedings of International Joint Conference on Artificial Intelligence (IJCAI)*.

BARTHOLOMEW, M. AND LEE, J. 2013b. On the stable model semantics for intensional functions. *TPLP 13,* 4-5, 863–876.

CABALAR, P. 2011. Functional answer set programming. *TPLP 11,* 2-3, 203–233.

FERRARIS, P. 2005. Answer sets for propositional theories. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*. 119–131.

FERRARIS, P., LEE, J., AND LIFSCHITZ, V. 2011. Stable models and circumscription. *Artificial Intelligence 175*, 236–263.

FERRARIS, P., LEE, J., LIFSCHITZ, V., AND PALLA, R. 2009. Symmetric splitting in the general theory of stable models. In *Proceedings of International Joint Conference on Artificial Intelligence (IJCAI)*. AAAI Press, 797–803.

GEBSER, M., OSTROWSKI, M., AND SCHAUB, T. 2009. Constraint answer set solving. In *Proceedings of International Conference on Logic Programming (ICLP)*. 235–249.

GELFOND, M. AND LIFSCHITZ, V. 1988. The stable model semantics for logic programming. In *Proceedings of International Logic Programming Conference and Symposium*, R. Kowalski and K. Bowen, Eds. MIT Press, 1070–1080.

GIUNCHIGLIA, E., LEE, J., LIFSCHITZ, V., MCCAIN, N., AND TURNER, H. 2004. Nonmonotonic causal theories. *Artificial Intelligence 153(1–2)*, 49–104.

JANHUNEN, T., LIU, G., AND NIEMELÄ, I. 2011. Tight integration of non-ground answer set programming and satisfiability modulo theories. In *Working notes of the 1st Workshop on Grounding and Transformations for Theories with Variables*.

LEE, J., LIFSCHITZ, V., AND YANG, F. 2013. Action language BC: Preliminary report. In *Proceedings of International Joint Conference on Artificial Intelligence (IJCAI)*.

LIFSCHITZ, V. 2012. Logic programs with intensional functions. In *Proceedings of International Conference on Principles of Knowledge Representation and Reasoning (KR)*. 24–31.

LIFSCHITZ, V., MORGENSTERN, L., AND PLAISTED, D. 2008. Knowledge representation and classical logic. In *Handbook of Knowledge Representation*, F. van Harmelen, V. Lifschitz, and B. Porter, Eds. Elsevier, 3–88.

LIFSCHITZ, V., PEARCE, D., AND VALVERDE, A. 2001. Strongly equivalent logic programs. *ACM Transactions on Computational Logic 2*, 526–541.

TRUSZCZYNSKI, M. 2012. Connecting first-order ASP and the logic FO(ID) through reducts. In *Correct Reasoning - Essays on Logic-Based AI in Honour of Vladimir Lifschitz*. 543–559.

Appendix A Infinitary Ground Formulas

Since the universe can be infinite, grounding a quantified sentence introduces infinite conjunctions and disjunctions over the elements in the universe. Here we rely on the concept of grounding *relative to an interpretation* from (Truszczynski 2012). The following is the definition of an *infinitary ground formula*, which is adapted from (Truszczynski 2012). A main difference is that we allow atomic formulas to be any ground atomic formulas in the sense of first-order logic, rather than limiting attention to propositional atoms as in (Truszczynski 2012).

For each element ξ in the universe $|I|$ of I, we introduce a new symbol $\xi^\diamond$, called an *object name*. By σ^I we denote the signature obtained from σ by adding all object names $\xi^\diamond$ as additional object constants. We will identify an interpretation I of signature σ with its extension to σ^I defined by $I(\xi^\diamond) = \xi$.[6]

We assume the primary connectives to be $\bot$, $\{\}^\wedge$, $\{\}^\vee$, and $\rightarrow$. Propositional connectives $\wedge, \vee, \neg, \top$ are considered as shorthands: $F \wedge G$ as $\{F, G\}^\wedge$; $F \vee G$ as $\{F, G\}^\vee$. We understand $\neg F$ as shorthand for $F \rightarrow \bot$ and $\top$ as shorthand for $\bot \rightarrow \bot$.

Let A be the set of all ground atomic formulas of signature σ^I. The sets $\mathcal{F}_0, \mathcal{F}_1, \dots$ are defined recursively as follows:

- $\mathcal{F}_0 = A \cup \{\bot\}$;
- $\mathcal{F}_{i+1} (i \geq 0)$ consists of expressions $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$, for all subsets $\mathcal{H}$ of $\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_i$, and of the expressions $F \rightarrow G$, where $F, G \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_i$.

We define $\mathcal{L}_A^{inf} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$, and call elements of $\mathcal{L}_A^{inf}$ *infinitary ground formulas* of σ w.r.t. I.

For any interpretation I of σ and any infinitary ground formula F w.r.t. I, the definition of satisfaction, $I \models F$, is as follows:

- For atomic formulas, the definition of satisfaction is the same as in the standard first-order logic;
- $I \models \mathcal{H}^\wedge$ if, for every formula $G \in \mathcal{H}$, $I \models G$;
- $I \models \mathcal{H}^\vee$ if there is a formula $G \in \mathcal{H}$ such that $I \models G$;
- $I \models G \rightarrow H$ if $I \not\models G$ or $I \models H$.

Appendix B Review: Cabalar Semantics

We first define the notion of a *partial* interpretation, which is a generalization of a partial interpretation for multi-valued formulas. Given a first-order signature σ comprised of function and predicate constants, a *partial* interpretation I of σ consists of

- a non-empty set $|I|$, called the *universe* of I;
- for every function constant f of arity n, a function f^I from $(|I| \cup \{\mathrm{u}\})^n$ to $|I| \cup \{\mathrm{u}\}$, where u is not in $|I|$ ("u" stands for *undefined*);
- for every predicate constant p of arity n, a function p^I from $(|I| \cup \{\mathrm{u}\})^n$ to $\{\text{TRUE}, \text{FALSE}\}$.

For each term $f(t_1, \dots, t_n)$, we define

$$f(t_1, \dots, t_n)^I = \begin{cases} \mathrm{u} & \text{if } t_i^I = \mathrm{u} \text{ for some } i; \\ f^I(t_1^I, \dots, t_n^I) & \text{otherwise.} \end{cases}$$

The satisfaction relation $\models_{\mathrm{p}}$ between a partial interpretation I and a first-order formula F is the same as the one for first-order logic except for the following base cases:

[6] For details, see (Lifschitz et al. 2008).

- For each atomic formula $p(t_1, \dots, t_n)$,

$$p(t_1, \dots, t_n)^I = \begin{cases} \text{FALSE} & \text{if } t_i^I = \text{u} \text{ for some } i; \\ p^I(t_1^I, \dots, t_n^I) & \text{otherwise.} \end{cases}$$

- For each atomic formula $t_1 = t_2$,

$$(t_1 = t_2)^I = \begin{cases} \text{TRUE} & \text{if } t_1^I \neq \text{u}, t_2^I \neq \text{u}, \text{ and } t_1^I = t_2^I; \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We say that $I \models_{\text{p}} F$ if $F^I = \text{TRUE}$.

Observe that under a partial interpretation, $t = t$ is not necessarily true: $I \not\models_{\text{p}} t = t$ iff $t^I = \text{u}$. On the other hand, $\neg(t_1 = t_2)$, also denoted by $t_1 \neq t_2$, is true under I even when both t_1^I and t_2^I are mapped to the same u.

Given any two partial interpretations J and I of the same signature σ, and a set of constants $\mathbf{c}$, we write $J \preceq^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants in $\mathbf{c}$; and
- $f^J(\boldsymbol{\xi}) = \text{u}$ or $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ for all function constants in $\mathbf{c}$ and all lists $\boldsymbol{\xi}$ of elements in the universe.

We write $J \prec^{\mathbf{c}} I$ if $J \preceq^{\mathbf{c}} I$ but not $I \preceq^{\mathbf{c}} J$. Note that $J \prec^{\mathbf{c}} I$ is defined similar to $J <^{\mathbf{c}} I$ except for the treatment of functions.

Definition 2

[def:cbl-reduct] Let F be a first-order sentence of signature σ and let $\mathbf{c}$ be a list of intensional constants. A partial interpretation I of σ is a Cabalar stable model of F relative to $\mathbf{c}$ if

- $I \models_{\text{p}} F$, and
- for every partial interpretation J of σ such that $J \prec^{\mathbf{c}} I$, we have $J \not\models_{\text{p}} gr_I[F]^{\underline{I}}$.