6. Natural Deduction

Next we turn to the study of deductive systems for propositional logic. A deductive system is a collection of axioms and inference rules. A "derivation" in a deductive system shows how a formula can be derived from a set of hypotheses using the postulates (axioms and inference rules) of the system.

Deductive System N

In the natural deduction system **N**, the derivable objects are *sequents*—expressions of the form

$$\Gamma \Rightarrow F \tag{1}$$

("F under assumptions Γ") where F is a formula and Γ is a finite set of formulas. If Γ is written as $\{G_1, \ldots, G_n\}$, we will usually drop the braces and write (1) as

$$G_1, \ldots, G_n \Rightarrow F\ . \tag{2}$$

Intuitively, a sequent (2) is understood as the formula

$$(G_1 \wedge \cdots \wedge G_n) \rightarrow F \tag{3}$$

if $n > 0$, and as F if $n = 0$.

The axioms of **N** are sequents of the forms

$$\Rightarrow \top\ ,$$

$$F \Rightarrow F$$

and

$$\Rightarrow F \vee \neg F.$$

The last one is called *the law of excluded middle*.

In the list of inference rules below, Γ, Δ, Δ_1, Δ_2 are finite sets of formulas. All inference rules of **N** except for the two rules at the end are classified into *introduction rules* (the left column) and *elimination rules* (the right column); the exceptions are the *contradiction rule* (C) and the *weakening rule* (W):

$$(\wedge I)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow F} \quad \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow G}$$

$$(\vee I)\ \frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow F \vee G} \quad \frac{\Gamma \Rightarrow G}{\Gamma \Rightarrow F \vee G} \qquad (\vee E)\ \frac{\Gamma \Rightarrow F \vee G \quad \Delta_1, F \Rightarrow H \quad \Delta_2, G \Rightarrow H}{\Gamma, \Delta_1, \Delta_2 \Rightarrow H}$$

$$(\rightarrow I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad (\rightarrow E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \rightarrow G}{\Gamma, \Delta \Rightarrow G}$$

$$(\neg I)\ \frac{\Gamma, F \Rightarrow \bot}{\Gamma \Rightarrow \neg F} \qquad (\neg E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow \neg F}{\Gamma, \Delta \Rightarrow \bot}$$

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F}$$

$$(W)\ \frac{\Gamma \Rightarrow F}{\Gamma' \Rightarrow F} \quad \text{if } \Gamma \subseteq \Gamma'$$

We regard the introduction and elimination rules for equivalence

$$\frac{\Gamma \Rightarrow F \rightarrow G \quad \Delta \Rightarrow G \rightarrow F}{\Gamma, \Delta \Rightarrow F \leftrightarrow G} \qquad \frac{\Gamma \Rightarrow F \leftrightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad \frac{\Gamma \Rightarrow F \leftrightarrow G}{\Gamma \Rightarrow G \rightarrow F}$$

as special cases of $(\wedge I)$ and $(\wedge E)$.

System N is sound and complete:

Theorem 1 *a sequent $\Gamma \Rightarrow F$ is provable in* **N** *iff Γ entails F.*

To prove a formula F in System **N** means to prove the sequent $\Rightarrow F$. The theorem asserts that a formula is provable in **N** iff it is a tautology.

Figure 1 shows a proof of the formula

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$$

with the corresponding "informal proof" to the right of the bar. Figure 2 is a proof of

$$(\neg p \rightarrow q) \rightarrow (\neg q \rightarrow p)$$

in **N**, again with an "English translation."

1.	$p \to (q \to r) \Rightarrow p \to (q \to r)$	— axiom	Assume $p \to (q \to r)$. Now our goal is to prove $(p \wedge q) \to r$.
2.	$p \wedge q \Rightarrow p \wedge q$	— axiom	Assume $p \wedge q$. Now our goal is to prove r.
3.	$p \wedge q \Rightarrow p$	— by $(\wedge E)$ from 2	From the second assumption, p.
4.	$p \to (q \to r),\ p \wedge q \Rightarrow q \to r$	— by $(\to E)$ from 3, 1	Consequently, by the first assumption, $q \to r$.
5.	$p \wedge q \Rightarrow q$	— by $(\wedge E)$ from 2	On the other hand, from the second assumption, q.
6.	$p \to (q \to r),\ p \wedge q \Rightarrow r$	— by $(\to E)$ from 5, 4	Consequently r.
7.	$p \to (q \to r) \Rightarrow (p \wedge q) \to r$	— by $(\to I)$ from 6	Thus we proved $(p \wedge q) \to r$
8.	$\Rightarrow (p \to (q \to r)) \to ((p \wedge q) \to r)$	— by $(\to I)$ from 7	and consequently we are done.

Figure 1: A proof of $(p \to (q \to r)) \to ((p \wedge q) \to r)$

In each of the following problems, find a proof of the given formula in System **N**, and show also the corresponding informal proof.

6.1 $((p \wedge q) \to r) \to (p \to (q \to r))$.

6.2 $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

6.3 $((p \wedge q) \vee r) \to (p \vee r)$.

6.4 $(p \to q) \to (\neg q \to \neg p)$.

6.5 $(p \to q) \vee (q \to p)$.

In the following F and G are formulas. Find a proof in System **N**.

6.6 $F \wedge G \leftrightarrow G \wedge F$, $F \vee G \leftrightarrow G \vee F$.

6.7 $F \wedge \top \leftrightarrow F$, $F \vee \bot \leftrightarrow F$.

1.	$\neg p \to q \Rightarrow \neg p \to q$	— axiom	Assume $\neg p \to q$. Now our goal is to prove $\neg q \to p$.
2.	$\neg q \Rightarrow \neg q$	— axiom	Assume $\neg q$. Now our goal is to prove p.
3.	$\Rightarrow p \vee \neg p$	— axiom	Consider two cases.
4.	$p \Rightarrow p$	— axiom	Case 1: p. This case is trivial.
5.	$\neg p \Rightarrow \neg p$	— axiom	Case 2: $\neg p$.
6.	$\neg p \to q,\ \neg p \Rightarrow q$	— by $(\to E)$ from 5, 1	Then, by the first assumption, q.
7.	$\neg p \to q,\ \neg q,\ \neg p \Rightarrow \bot$	— by $(\neg E)$ from 6, 2	This contradicts the second assumption,
8.	$\neg p \to q,\ \neg q,\ \neg p \Rightarrow p$	— by (C) from 7	so that we can conclude p also.
9.	$\neg p \to q,\ \neg q \Rightarrow p$	— by $(\vee E)$ from 3, 4, 8	Thus, in either case, p.
10.	$\neg p \to q \Rightarrow \neg q \to p$	— by $(\to I)$ from 9	We have proved $\neg q \to p$
11.	$\Rightarrow (\neg p \to q) \to (\neg q \to p)$	— by $(\to I)$ from 10	and consequently we are done.

Figure 2: A proof of $(\neg p \to q) \to (\neg q \to p)$

6.8 $\neg F \leftrightarrow (F \to \bot)$, $\top \leftrightarrow (\bot \to \bot)$.

6.9 $(F \to \neg G) \leftrightarrow \neg(F \wedge G)$.

6.10 $\neg(F \vee G) \leftrightarrow \neg F \wedge \neg G$.

6.11 $\neg(F \wedge G) \leftrightarrow \neg F \vee \neg G$.

6.12 $\neg\neg\neg F \leftrightarrow \neg F$.