Online appendix for the paper

Module Theorem for the General Theory of Stable Models

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Appendix A Proofs

A.1 Splitting Lemma

We use the splitting lemma (Ferraris et al. 2009) to prove a few theorems below.

Splitting Lemma
Let F be a first-order sentence, and let $\mathbf{p}$, $\mathbf{q}$ be lists of distinct predicate constants. If each strongly connected component of $\mathrm{DG}[F;\mathbf{pq}]$ is a subset of $\mathbf{p}$ or a subset of $\mathbf{q}$ then

$$\mathrm{SM}[F;\mathbf{pq}] \quad \text{is equivalent to} \quad \mathrm{SM}[F;\mathbf{p}] \wedge \mathrm{SM}[F;\mathbf{q}]\ .$$

The statement is slightly more general than the one from (Ferraris et al. 2009) in that $\mathbf{p}$ and $\mathbf{q}$ are not required to be disjoint. The proof of this enhancement follows from the Version 3 of the Splitting Lemma from (Ferraris et al. 2009).

A.2 Proof of Lemma 1

Lemma 1
X is a module answer set of $(\Pi, \mathcal{I}, \mathcal{O})$ iff X is an answer set of $\Pi \cup \{\{p\} \leftarrow \mid p \in \mathcal{I}\}$.

Proof

$$X \text{ is an answer set of } \Pi \cup \{p \leftarrow \mid p \in (\mathcal{I} \cap X)\}$$

iff

$$X \text{ is an answer set of } \Pi \cup \{p \leftarrow \mathit{not}\ \mathit{not}\ p \mid p \in \mathcal{I}\}$$

iff

$$X \text{ is an answer set of } \Pi \cup \{\{p\} \leftarrow \mid p \in \mathcal{I}\}\ .$$

The equivalence between the first and the second follows from the equivalence between the reducts of each program relative to X.

The equivalence between the second and third is because the transformation preserves strong equivalence. ∎

A.3 Proof of Theorem 3

Theorem 3
Let F, G, H be first-order sentences, and let $\mathbf{p}$, $\mathbf{q}$ be finite lists of distinct predicate constants. If

(a) each strongly connected component of $\mathrm{DG}[F \wedge G \wedge H;\ \mathbf{pq}]$ is a subset of $\mathbf{p}$ or a subset of $\mathbf{q}$,
(b) F is negative on $\mathbf{q}$, and
(c) G is negative on $\mathbf{p}$

then

$$\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{pq}] \quad \text{is equivalent to} \quad \mathrm{SM}[F \wedge H;\ \mathbf{p}] \wedge \mathrm{SM}[G \wedge H;\ \mathbf{q}]\ .$$

Proof
By the Splitting Lemma above, $\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{pq}]$ is equivalent to

$$\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{p}] \wedge \mathrm{SM}[F \wedge G \wedge H;\ \mathbf{q}]\ .$$

Since G is negative on $\mathbf{p}$, the first conjunctive term can be rewritten as

$$\mathrm{SM}[F \wedge H;\ \mathbf{p}] \wedge G\ . \tag{A1}$$

Similarly, the second conjunctive term can be rewritten as

$$\mathrm{SM}[G \wedge H;\ \mathbf{q}] \wedge F\ . \tag{A2}$$

It remains to observe that the second conjunctive term of each of the formulas (A1) and (A2) is entailed by the first conjunctive term of the other.
∎

A.4 Proof of Proposition 1

Proposition 1
For any first-order modules $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, the following properties hold:

- $\mathbf{F}_1 \sqcup \mathbf{F}_2$ is defined iff $\mathbf{F}_2 \sqcup \mathbf{F}_1$ is defined.
- $\mathrm{SM}[\mathbf{F}_1 \sqcup \mathbf{F}_2]$ is equivalent to $\mathrm{SM}[\mathbf{F}_2 \sqcup \mathbf{F}_1]$.
- $(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3$ is defined iff $\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)$ is defined.
- $\mathrm{SM}[(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3]$ is equivalent to $\mathrm{SM}[\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)]$.

Proof
Claims (a) and (b) follow immediately from the definitions.

We prove Claim (c). Let $\mathbf{F}_i = (F_i, \mathcal{I}_i, \mathcal{O}_i)$ for each $i \in \{1, 2, 3\}$ and without loss of generality assume that each F_i is a conjunction of the form $F_{i,1} \wedge \cdots \wedge F_{i,k_i}$.

From left to right: Assume that $(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3$ is defined. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are joinable,

(i) $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$;
(ii) each conjunctive term of F_1 is negative on $\mathcal{O}_2$, or is one of the conjunctive terms of F_2;
(iii) each conjunctive term of F_2 is negative on $\mathcal{O}_1$, or is one of the conjunctive terms of F_1;
(iv) each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$ is a subset of $\mathcal{O}_1$ or a subset of $\mathcal{O}_2$.

Also, since $(\mathbf{F}_1 \sqcup \mathbf{F}_2)$ and $\mathbf{F}_3$ are joinable,

(v) $(\mathcal{O}_1 \cup \mathcal{O}_2) \cap \mathcal{O}_3 = \emptyset$;
(vi) each conjunctive term of $F_1 \wedge F_2$ is negative on $\mathcal{O}_3$, or is one of the conjunctive terms of F_3;
(vii) each conjunctive term of F_3 is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$, or is one of the conjunctive terms of $F_1 \wedge F_2$;
(viii) each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is a subset of $\mathcal{O}_1 \cup \mathcal{O}_2$ or a subset of $\mathcal{O}_3$.

We first prove that $\mathbf{F}_2 \sqcup \mathbf{F}_3$ is defined.

(ix) From (v), it follows that $\mathcal{O}_2 \cap \mathcal{O}_3 = \emptyset$.
(x) From (vi), it follows that each conjunctive term of F_2 is negative on $\mathcal{O}_3$ or is one of the conjunctive terms of F_3.
(xi) We prove that each conjunctive term of F_3 is negative on $\mathcal{O}_2$ or is one of the conjunctive terms of F_2.
Consider any conjunctive term C of F_3. By (vii), C is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$, or is one of the conjunctive terms of $F_1 \wedge F_2$.
  - Case 1: C is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$. Clearly, it is negative on $\mathcal{O}_2$ as well.
  - Case 2: C is one of the conjunctive terms of $F_1 \wedge F_2$. If C is one of the conjunctive terms of F_2, the claim trivially follows. If C is one of the conjunctive terms of F_1, by (ii), it is either negative on $\mathcal{O}_2$ or is one of the conjunctive terms of F_2. In either case, the claim follows.

(xii) We first prove that each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is contained in only one of $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$, from which the fact that each strongly connected component of $\mathrm{DG}[F_2 \wedge F_3; \mathcal{O}_2\mathcal{O}_3]$ is contained in $\mathcal{O}_2$ or $\mathcal{O}_3$ follows, as $\mathrm{DG}[F_2 \wedge F_3; \mathcal{O}_2\mathcal{O}_3]$ is a subgraph of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$. By (i) and (v), $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are pairwise disjoint. Consider any strongly connected component S of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$. By (viii) S is a subset of $\mathcal{O}_1 \cup O_2$ or a subset of $\mathcal{O}_3$. Assume that S is a subset of $\mathcal{O}_1 \cup \mathcal{O}_2$. Clearly, S is also a strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2]$. In view of (vii), $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2]$ is the same as $\mathrm{DG}[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$, so that S is a strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$ as well. By (iv) S is contained in $\mathcal{O}_1$ or $\mathcal{O}_2$.

We now prove that $\mathbf{F_1} \sqcup (\mathbf{F_2} \sqcup \mathbf{F_3})$ is defined.

- From (i) and (v), it follows that $\mathcal{O}_1 \cap (\mathcal{O}_2 \cup O_3) = \emptyset$;
- From (ii) and (vi), it follows that each conjunctive term of F_1 is negative on $\mathcal{O}_2 \cup \mathcal{O}_3$ or is one of the conjunctive terms of $F_2 \wedge F_3$;
- From (iii) and (vii), it follows that each conjunctive term of $F_2 \wedge F_3$ is negative on $\mathcal{O}_1$ or is one of the conjunctive terms of F_1;
- From the claim proven in (viii), it follows that each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is contained in $\mathcal{O}_1$ or $\mathcal{O}_2 \cup \mathcal{O}_3$.

From right to left: Assume that $\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)$ is defined. By Claim (a), $(\mathbf{F}_2 \sqcup \mathbf{F}_3) \sqcup \mathbf{F}_1$ is defined, and then $(\mathbf{F}_3 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_1$ is defined. By the first part of Claim (c) that was proven, $\mathbf{F}_3 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_1)$ is defined, and then by applying Claim (a) twice, we have that $(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3$ is defined.

We now prove Claim (d). Using Theorem 4 and Claim (c),

$$\begin{aligned}
\mathrm{SM}[(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3] &\Leftrightarrow \mathrm{SM}[\mathbf{F}_1 \sqcup \mathbf{F}_2] \wedge \mathrm{SM}[\mathbf{F}_3] \\
&\Leftrightarrow \mathrm{SM}[\mathbf{F}_1] \wedge \mathrm{SM}[\mathbf{F}_2] \wedge \mathrm{SM}[\mathbf{F}_3] \\
&\Leftrightarrow \mathrm{SM}[\mathbf{F}_1] \wedge \mathrm{SM}[\mathbf{F}_2 \sqcup \mathbf{F}_3] \\
&\Leftrightarrow \mathrm{SM}[\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)] \ .
\end{aligned}$$

∎

A.5 Proof of Theorem 4

Theorem 4
Let $\mathbf{F}_1 = (F_1, \mathcal{I}_1, \mathcal{O}_1)$ and $\mathbf{F}_2 = (F_2, \mathcal{I}_2, \mathcal{O}_2)$ be first-order modules of a signature σ that are joinable, and, for $i = 0, 1$, let $\mathbf{c}_i$ be a subset of σ that contains $\mathbf{c}(F_i) \cup \mathcal{O}_i$, and let I_i be a $\mathbf{c}_i$-partial interpretation of σ. If I_1 and I_2 are compatible with each other, then

$$I_1 \cup I_2 \models \mathrm{SM}[\mathbf{F}_1 \sqcup \mathbf{F}_2] \quad \text{iff} \qquad I_1 \models \mathrm{SM}[\mathbf{F}_1] \ \text{ and } \ I_2 \models \mathrm{SM}[\mathbf{F}_2] \ .$$

Proof
Let us identify $\mathbf{F}_1$ with $(F_1' \wedge H, \mathcal{I}_1, \mathcal{O}_1)$ and $\mathbf{F}_2$ with $(F_2' \wedge H, \mathcal{I}_2, \mathcal{O}_2)$ as in the definition of join (Definition 8).

By definition $\mathrm{SM}[\mathbf{F}_1 \sqcup \mathbf{F}_2]$ is $\mathrm{SM}[F_1' \wedge F_2' \wedge H;\ \mathcal{O}_1 \cup \mathcal{O}_2]$. By Theorem 3,

$$\begin{aligned}
&I_1 \cup I_2 \models \mathrm{SM}[F_1' \wedge F_2' \wedge H;\ \mathcal{O}_1 \cup \mathcal{O}_2] \text{ iff} \\
&\qquad I_1 \cup I_2 \models \mathrm{SM}[F_1' \wedge H;\ \mathcal{O}_1] \text{ and } I_1 \cup I_2 \models \mathrm{SM}[F_2' \wedge H;\ \mathcal{O}_2]
\end{aligned}$$

Clearly, $I_1 \cup I_2$ is compatible with I_1. Since $\mathbf{c}_1$ contains $\mathbf{c}(F_1' \wedge H) \cup \mathcal{O}_1$, it follows that $I_1 \cup I_2 \models \mathrm{SM}[F_1' \wedge H;\ \mathcal{O}_1]$ iff $I_1 \models \mathrm{SM}[F_1' \wedge H;\ \mathcal{O}_1]$. Similarly, $I_1 \cup I_2 \models \mathrm{SM}[F_2' \wedge H;\ \mathcal{O}_2]$ iff $I_2 \models \mathrm{SM}[F_2' \wedge H;\ \mathcal{O}_2]$. Consequently, the claim follows. ∎

A.6 Proof of Proposition 3

Lemma 1
Let $\langle B, P[t], Q[t]\rangle$ be an incremental first-order theory, and let $\mathbf{P}_i$ and $\mathbf{R}_k$ be as in Proposition 3. It holds that

$$\begin{aligned} Out(\mathbf{P}_i) &= pr(B \wedge P[1] \wedge \cdots \wedge P[i]),\\ Out(\mathbf{R}_k) &= pr(B \wedge P[1] \wedge \cdots \wedge P[k] \wedge Q[k]). \end{aligned}$$

Proof
We show the first clause by induction. The second clause is similar.

- Base case: $\mathbf{P}_0 = FM(B, \emptyset) = (B^\omega,\ \emptyset,\ pr(B))$.
- Inductive step: Assume that $Out(\mathbf{P}_{i-1}) = pr(B \wedge P[1] \wedge \cdots \wedge P[i-1])$. The module $FM(P[i], Out(\mathbf{P}_{i-1}))$ is

  $$(P[i]^\omega,\ Out(\mathbf{P}_{i-1}),\ pr(P[i]) \setminus Out(\mathbf{P}_{i-1}))\ .$$

  Thus

  $$Out(\mathbf{P}_i) = Out(\mathbf{P}_{i-1}) \cup \big(pr(P[i]) \setminus Out(\mathbf{P}_{i-1})\big) = Out(\mathbf{P}_{i-1}) \cup pr(P[i])$$

  and by the I.H., this is then $pr(B \wedge P[1] \wedge \cdots \wedge P[i])$.

∎

Lemma 2
Given any two first-order formulas F_1, F_2 and disjoint sets of predicate constants $\mathbf{p}_1, \mathbf{p}_2$ such that $pr(F_1) \subseteq \mathbf{p}_1$, and F_2 is negative on $\mathbf{p}_1$. Every strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathbf{p}_1\mathbf{p}_2]$ is contained in $\mathbf{p}_1$ or $\mathbf{p}_2$.

Proof
Since F_2 is negative on $\mathbf{p}_1$, we have that $head(F_2) \cap \mathbf{p}_1 = \emptyset$. Thus every outgoing edge in the dependency graph from a predicate constant in $\mathbf{p}_1$ must be obtained from F_1. Since $pr(F_1) \subseteq \mathbf{p}_1$, such outgoing edge always leads to a vertex in $\mathbf{p}_1$. Consequently, every strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathbf{p}_1\mathbf{p}_2]$ containing a predicate constant from $head(F_1)$ is contained in $\mathbf{p}_1$, so the claim follows. ∎

Proposition 3
If an incremental first-order theory $\langle B, P[t], Q[t]\rangle$ is acyclic, then the following modules are defined for all $k \geq 0$.

$$\begin{aligned} \mathbf{P}_0 &= FM(B, \emptyset),\\ \mathbf{P}_i &= \mathbf{P}_{i-1} \sqcup FM(P[i], Out(\mathbf{P}_{i-1})), && (1 \leq i \leq k)\\ \mathbf{R}_k &= \mathbf{P}_k \sqcup FM(Q[k], Out(\mathbf{P}_k))\ . \end{aligned}$$

Proof
We first prove by induction that $\mathbf{P}_i$ is defined.

Base case: It is clear that $\mathbf{P}_0 = FM(B, \emptyset)$ is defined.

Inductive step: Assume that $\mathbf{P}_{i-1} = (F_{i-1}, \mathcal{I}_{i-1}, \mathcal{O}_{i-1})$ is defined for any $i > 0$. Also,

$$FM(P[i], \mathcal{O}_{i-1}) = (P[i]^\omega, \mathcal{O}_{i-1}, pr(P[i]) \backslash \mathcal{O}_{i-1})$$

is trivially defined. To show that they are joinable, we will check the following:

(i) $head(F_{i-1}) \cap (pr(P[i]) \backslash \mathcal{O}_{i-1}) = \emptyset$;
(ii) $head(P[i]^\omega) \cap \mathcal{O}_{i-1} = \emptyset$;
(iii) every strongly connected component of

$$\mathrm{DG}[F_{i-1} \wedge P[i]^\omega;\ \mathcal{O}_{i-1} \cup (pr(P[i]) \backslash \mathcal{O}_{i-1})]$$

is a subset of $\mathcal{O}_{i-1}$ or $pr(P[i]) \backslash \mathcal{O}_{i-1}$.

Note that

$$pr(F_{i-1}) \subseteq pr(B \wedge P[1] \wedge \dots P[i-1]) \tag{A3}$$

and

$$head(P[i]^\omega) \subseteq head(P[i]) \ . \tag{A4}$$

Proof of Claim (i): By Lemma 1, $\mathcal{O}_{i-1}$ is $pr(B \wedge P[1] \wedge \dots \wedge P[i-1])$, and Claim (i) trivially follows in view of (A3) and the fact that $head(F_{i-1}) \subseteq pr(F_{i-1})$.

Proof of Claim (ii): Since the theory is acyclic,

$$head(P[i]) \cap pr(B \wedge P[1] \wedge \dots \wedge P[i-1]) = \emptyset \ ,$$

and from (A4) and Lemma 1, we have that

$$head(P[i]^\omega) \cap \mathcal{O}_{i-1} = \emptyset \ . \tag{A5}$$

Proof of Claim (iii): The claim follows from (A5) and Lemma 2.

We next show that $\mathbf{R}_k$ is defined. By our previous result, $\mathbf{P}_k = (F_k, \mathcal{I}_k, \mathcal{O}_k)$ is defined. It also holds that

$$FM(Q[k], \mathcal{O}_k) = (Q[k]^\omega, \mathcal{O}_k, pr(Q[k]) \setminus \mathcal{O}_k)$$

is defined trivially. The rest of the reasoning is similar to the previous one.

A.7 Proof of Proposition 4

Proposition 4
Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental first-order theory and let $\mathbf{R}_k$ be the

module as defined in the statement of Proposition 3. For any nonnegative integer k,

$$\begin{aligned} I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[\mathbf{R}_k] & \\ \text{iff} \quad & I_B \models \mathrm{SM}[\mathit{FM}(B, \emptyset)] \\ & \text{and} \;\; I_{P[1]} \models \mathrm{SM}[\mathit{FM}(P[1], \mathit{Out}(\mathbf{P}_0))] \\ & \text{and} \;\; \ldots \\ & \text{and} \;\; I_{P[k]} \models \mathrm{SM}[\mathit{FM}(P[k], \mathit{Out}(\mathbf{P}_{k-1}))] \\ & \text{and} \;\; I_{Q[k]} \models \mathrm{SM}[\mathit{FM}(Q[k], \mathit{Out}(\mathbf{P}_k))] \; . \end{aligned}$$

where I_B ($I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$, respectively) is a $\mathbf{c}(B)$-partial interpretation ($\mathbf{c}(P[1])$, $\ldots, \mathbf{c}(P[k]), \mathbf{c}(Q[k])$-partial interpretation, respectively) such that $I_B, I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$ are pairwise compatible.

Proof

Via repeated applications of Theorem 4 on $\mathbf{R}_k$ as indicated by Proposition 3. ∎

A.8 Proof of Proposition 5

Lemma 3

Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental first-order theory, let k be a nonnegative integer, let $H_k = B \wedge P[1] \wedge \cdots \wedge P[k]$, and let R_k be the k-expansion of the incremental theory. It holds that $I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[R_k]$ iff

$$\begin{aligned} & I_B \models \mathrm{SM}[B;\; pr(B)] \\ & \quad \text{and} \;\; I_{P[1]} \models \mathrm{SM}[P[1];\; pr(P[1]) \setminus pr(H_0)] \\ & \quad \text{and} \;\; \ldots \\ & \quad \text{and} \;\; I_{P[k]} \models \mathrm{SM}[P[k];\; pr(P[k]) \setminus pr(H_{k-1})] \\ & \quad \text{and} \;\; I_{Q[k]} \models \mathrm{SM}[Q[k];\; pr(Q[k]) \setminus pr(H_k)] \end{aligned} \tag{A6}$$

where I_B ($I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$, respectively) is a $\mathbf{c}(B)$-partial interpretation ($\mathbf{c}(P[1])$, $\ldots, \mathbf{c}(P[k]), \mathbf{c}(Q[k])$-partial interpretation, respectively) such that $I_B, I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$ are pairwise compatible.

Proof

Formula H_k is trivially negative on $pr(Q[k]) \setminus pr(H_k)$, and since the theory is acyclic, $Q[k]$ is negative on $pr(H_k)$. Also, by Lemma 2, every strongly connected component of $\mathrm{DG}[H_k \wedge Q[k];\; pr(H_k) \cup pr(Q[k])]$ is a subset of $pr(H_k)$ or $pr(Q[k]) \setminus pr(H_k)$. By Theorem 4, it then holds that

$$I_{H_k} \cup I_{Q[k]} \models \mathrm{SM}[R_k] \quad \text{iff} \quad I_{H_k} \models \mathrm{SM}[H_k] \text{ and } I_{Q[k]} \models \mathrm{SM}[Q[k];\; pr(Q[k]) \setminus pr(H_k)]$$

where I_{H_k} is a $\mathbf{c}(H_k)$-partial interpretation that is compatible with $I_{Q[k]}$.

Next we check by induction that $I_{H_k} \models \mathrm{SM}[H_k]$ is equivalent to

$$\begin{aligned}
&I_B \models \mathrm{SM}[B] \\
&\quad \text{and} \quad I_{P[1]} \models \mathrm{SM}[P[1];\ pr(P[1]) \setminus pr(H_0)] \\
&\quad \text{and} \quad \dots \\
&\quad \text{and} \quad I_{P[k]} \models \mathrm{SM}[P[k];\ pr(P[k]) \setminus pr(H_{k-1})]\ .
\end{aligned} \tag{A7}$$

Base case: when $k = 0$, $H_k = B$. Trivial.

Inductive step: Let the property hold for H_{k-1}. By definition, $H_k = H_{k-1} \wedge P[k]$. H_{k-1} is trivially negative on $pr(P[k]) \setminus pr(H_{k-1})$ and since the theory is acyclic, $P[k]$ is negative on $pr(H_{k-1})$. Also, by Lemma 2, every strongly connected component of $\mathrm{DG}[H_k; pr(H_k)]$ is a subset of $pr(H_{k-1})$ or $pr(P[k]) \setminus pr(H_{k-1})$. By Theorem 4, it then holds that

$$I_{H_k} \models \mathrm{SM}[H_k] \quad \text{iff} \quad I_{H_{k-1}} \models \mathrm{SM}[H_{k-1}] \text{ and } I_{P[k]} \models \mathrm{SM}[P[k];\ pr(P[k]) \setminus pr(H_{k-1})].$$

The property then holds by the I.H. ∎

Lemma 4

For any first-order formula F, $\mathrm{SM}[FM(F, \mathcal{I})]$ is equivalent to $\mathrm{SM}[F; pr(F) \setminus \mathcal{I}]$.

Proof

We introduce a notion that helps us prove. By $Simpl(F)$ we denote the least fixpoint of the sequence $F_0, F_1, \dots$: formula F_0 is defined as F, and F_{i+1} is defined as $F_i|_{head(F_i)}$.

Formula $\mathrm{SM}[FM(F, \mathcal{I})]$ is $\mathrm{SM}[(F^\omega, \mathcal{I}, pr(F) \setminus \mathcal{I})]$, which in turn is defined as $\mathrm{SM}[F^\omega;\ pr(F) \setminus \mathcal{I}]$. By Theorem 2 from (Ferraris et al. 2011), this is equivalent to $\mathrm{SM}[F^\omega \wedge Choice(\mathcal{I});\ pr(F)]$. From the definition of $Simpl$, the latter is equivalent to $\mathrm{SM}[Simpl(F \wedge Choice(\mathcal{I}));\ pr(F)]$, and, furthermore, by Theorem 4 from (Ferraris et al. 2011), is equivalent to $\mathrm{SM}[F \wedge Choice(\mathcal{I});\ pr(F)]$. ∎

Proposition 5

Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental theory, let k be a nonnegative integer, let R_k be the k-expansion of the incremental theory, and let $\mathbf{R}_k$ be the module as defined in Proposition 3. For any $\mathbf{c}$-partial interpretation I such that $\mathbf{c} \supseteq \mathbf{c}(R_k)$, we have that

$$I \models \mathrm{SM}[R_k] \ \text{ iff } \ I \models \mathrm{SM}[\mathbf{R}_k].$$

Proof

Without loss of generality, let $I = I_B \cup I_{P[1]} \cup \dots \cup I_{P[k]} \cup I_{Q[k]}$. By Lemma 3,

$I \models \mathrm{SM}[R_k]$ is equivalent to (A6), and by Lemma 1, this is further equivalent to

$$\begin{aligned} I_B &\models \mathrm{SM}[B;\ pr(B)] \\ &\text{and}\quad I_{P[1]} \models \mathrm{SM}[P[1];\ pr(P[1]) \setminus Out(\mathbf{P}_0)] \\ &\text{and}\quad \dots \\ &\text{and}\quad I_{P[k]} \models \mathrm{SM}[P[k];\ pr(P[k]) \setminus Out(\mathbf{P}_{k-1})] \\ &\text{and}\quad I_{Q[k]} \models \mathrm{SM}[Q[k];\ pr(Q[k]) \setminus Out(\mathbf{P}_k)]\ . \end{aligned}$$

We check the following:

- $I_B \models \mathrm{SM}[B;\ pr(B)]$ iff $I_B \models \mathrm{SM}[FM(B, \emptyset)]$;
- $I_{P[i]} \models \mathrm{SM}[P[i];\ pr(P[i]) \setminus Out(\mathbf{P}_{i-1})]$ iff $I_{P[i]} \models \mathrm{SM}[FM(P[i], Out(\mathbf{P}_{i-1}))]$;
- $I_{Q[k]} \models \mathrm{SM}[Q[k];\ pr(Q[k]) \setminus Out(\mathbf{P}_k)]$ iff $I_{Q[k]} \models \mathrm{SM}[FM(Q[k], Out(\mathbf{P}_k))]$.

The first clause is clear. The last two clauses follow from Lemma 4.
Therefore, by Proposition 4,

$$I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[\mathbf{R}_k]\ .$$

∎

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