Stable Models of Fuzzy Propositional Formulas

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Abstract. We introduce the stable model semantics for fuzzy propositional formulas, which generalizes both fuzzy propositional logic and the stable model semantics of classical propositional formulas. Combining the advantages of both formalisms, the introduced language allows highly configurable default reasoning involving fuzzy truth values. We show that several properties of Boolean stable models are naturally extended to this formalism, and discuss how it is related to other approaches to combining fuzzy logic and the stable model semantics.

1 Introduction

Answer set programming (ASP) [1] is a widely applied declarative programming paradigm for the design and implementation of knowledge intensive applications. One of the attractive features of ASP is its capability to model the nonmonotonic aspect of knowledge. However, as its mathematical basis, the stable model semantics, is restricted to Boolean values, it is too rigid to represent imprecise and vague information. Fuzzy logic [2], as a form of many-valued logic, can handle vague information by interpreting propositions with a truth degree in the interval of real numbers $[0, 1]$. The availability of various fuzzy operators gives the user great flexibility in combining truth degrees. However, the semantics of fuzzy logic is monotonic, and is not flexible enough to handle default reasoning allowed in answer set programming.

Both the stable model semantics and fuzzy logic are generalizations of classical propositional logic in different ways. While they do not subsume each other, it is clear that many real-world problems require both their strengths. This led to the body of work on combining fuzzy logic and the stable model semantics, known as fuzzy answer set programming (e.g., [3, 4]). However, their syntax is restricted to rules, and does not allow connectives nested arbitrarily as in fuzzy logic.

Unlike existing work on fuzzy answer set semantics, in this paper, we extend the general stable model semantics from [5] to many-valued propositional formulas. The syntax of this language is the same as the syntax of fuzzy propositional logic. The semantics, on the other hand, defines *stable* models instead of models. The language is a proper generalization of both fuzzy propositional logic and the stable model semantics for Boolean propositional formulas. This generalization is not simply a pure theoretical pursuit, but has practical use in conveniently modeling defaults involving fuzzy truth values in dynamic domains. For example, consider modeling dynamics of *trust* in social network. People trust each other in different degrees under some normal assumptions. If person A trusts another person B, then A tends to trust person C whom B trusts to a degree which is positively correlated to the degree to which A trusts B and the degree to which B trusts C. If nothing happens, the trust degrees would not change. But there may be less trust between two people when a conflict arises between them. Modeling such a domain requires expressing defaults involving fuzzy truth values. We demonstrate that such examples can be conveniently modelled in our proposed language by taking advantage of its generality over the existing approaches to fuzzy ASP.

The paper is organized as follows. Section 2 reviews the syntax and the semantics of fuzzy propositional logic we discuss in the paper, as well as the stable model semantics of classical propositional formulas. Section 3 presents the stable model semantics of fuzzy propositional formulas along with examples, followed by Section 4 that formalizes the trust example above in the proposed language. Section 5 shows how the fuzzy stable model semantics is related to the Boolean stable model semantics, and Section 6 shows how our fuzzy stable model semantics is related to other approaches to fuzzy ASP. Section 7 shows that several well-known properties of the Boolean stable model semantics can be easily extended to our fuzzy stable model semantics.

2 Preliminaries

2.1 Review: Stable Models of Classical Propositional Formulas

We review the definition of a stable model from [5] by limiting attention to the syntax of propositional formulas. Instead of defining stable models in terms of second-order logic as in [5] , we express the same concept using auxiliary atoms that do not belong to the original signature. This slight reformulation will simplify our efforts in extending the stable model semantics to fuzzy propositional formulas without resorting to "second-order fuzzy logic."

Let σ be a classical propositional signature, $\mathbf{p} = (p_1, \dots, p_n)$ be a list of distinct atoms belonging to σ, and let $\mathbf{q} = (q_1, \dots, q_n)$ be a list of new propositional atoms not in σ.

For two interpretations I and J of σ, $I \cup J^{\mathbf{p}}_{\mathbf{q}}$ denotes the interpretation of $\sigma \cup \mathbf{q}$ that agrees with I and J on all atoms not in $\mathbf{p} \cup \mathbf{q}$, and

- for each $p \in \mathbf{p}$, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})(p) = I(p)$;
- for each $q \in \mathbf{q}$, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})(q) = J(p)$.[1]

For any classical propositional formula F of signature σ, $F^*(\mathbf{q})$ is a classical propositional formula of signature $\sigma \cup \mathbf{q}$ that is defined recursively as follows:

- $p_i^* = q_i$ for each $p_i \in \mathbf{p}$;
- $F^* = F$ for any atom $F \notin \mathbf{p}$;
- $\bot^* = \bot$; $\quad \top^* = \top$;
- $(\neg F)^* = \neg F$;
- $(F \wedge G)^* = F^* \wedge G^*$; $\qquad (F \vee G)^* = F^* \vee G^*$;
- $(F \rightarrow G)^* = (F^* \rightarrow G^*) \wedge (F \rightarrow G)$.

[1] $I(p)$ denotes the truth value of p under I. We identify a list with a set if there is no confusion.

Let I and J be two interpretations of σ, and let $\mathbf{p}$ be a subset of σ. We say $J \leq^{\mathbf{p}} I$ if

- J and I agree on all atoms not in $\mathbf{p}$, and
- for all $p \in \mathbf{p}$, if $J \models p$, then $I \models p$.

We say $J <^{\mathbf{p}} I$ if $J \leq^{\mathbf{p}} I$ and $J \neq I$.

Definition 1. *An interpretation I is a* stable model *of F relative to $\mathbf{p}$ (denoted $I \models \mathrm{SM}[F; \mathbf{p}]$)*

- *if $I \models F$, and*
- *there is no interpretation J such that $J <^{\mathbf{p}} I$ and $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models F^*(\mathbf{q})$.*

Example 1. Consider a logic program

$$p \leftarrow \textit{not } q, \quad q \leftarrow \textit{not } p$$

which is understood as an alternative notation for propositional formula

$$F_1 = (\neg q \rightarrow p) \wedge (\neg p \rightarrow q).$$

$F_1^*(u, v)$ is

$$(\neg q \rightarrow u) \wedge (\neg q \rightarrow p) \wedge (\neg p \rightarrow v) \wedge (\neg p \rightarrow q).$$

We check that $I_1 = \{p\}$ (that is, p is TRUE and q is FALSE) [2] is a stable model of F_1 (relative to $\{p, q\}$): I_1 satisfies F_1, and $\emptyset$ is the only interpretation J such that $J <^{pq} I_1$. However, $I_1 \cup J^{pq}_{uv} = \{p\}$ does not satisfy $F_1^*(u, v)$ because it does not satisfy the first conjunctive term of $F_1^*(u, v)$.

Similarly, we can check that $\{q\}$ is another stable model of F_1.

2.2 Review: Fuzzy Logic

Let σ be a fuzzy propositional signature, which is a set of symbols called *fuzzy atoms*. In addition, we assume the presence of a set $\mathbb{C}$ of fuzzy conjunction symbols, a set $\mathbb{D}$ of fuzzy disjunction symbols, a set $\mathbb{N}$ of fuzzy negation symbols, and a set $\mathbb{I}$ of fuzzy implication symbols.

A *fuzzy (propositional) formula* of σ is defined recursively as follows.

- every fuzzy atom $p \in \sigma$ is a fuzzy formula;
- every numeric constant $\overline{c}$ where c is a real number in $[0, 1]$ is a formula;
- if F is a formula, then $\neg F$ is a formula, where $\neg \in \mathbb{N}$;
- if F and G are formulas, then $F \otimes G$, $F \oplus G$ and $F \rightarrow G$ are formulas, where $\otimes \in \mathbb{C}$, $\oplus \in \mathbb{D}$, and $\rightarrow \in \mathbb{I}$.

[2] We identify a propositional interpretation with the set of atoms that are true in it.

The models of a fuzzy formula are defined as follows [2]. The *fuzzy truth values* are the real numbers in the range $[0, 1]$. A *fuzzy interpretation* I of σ is a mapping from σ into $[0, 1]$.

The fuzzy operators are functions mapping one or two truth values into a truth value. Among the operators, $\neg$ denotes a function from $[0, 1]$ into $[0, 1]$; $\otimes$, $\oplus$, and $\rightarrow$ denote functions from $[0, 1] \times [0, 1]$ into $[0, 1]$. The actual mapping performed by each operator can be defined in many different ways, but all of them satisfy the following conditions, which imply that they are generalizations of the corresponding classical propositional connectives:[3]

- a fuzzy negation $\neg$ is decreasing, and satisfies $\neg(0) = 1$ and $\neg(1) = 0$;
- a fuzzy conjunction $\otimes$ is increasing, commutative, associative, and $\otimes(1, x) = x$ for all $x \in [0, 1]$;
- a fuzzy disjunction $\oplus$ is increasing, commutative, associative, and $\oplus(0, x) = x$ for all $x \in [0, 1]$;
- a fuzzy implication $\rightarrow$ is decreasing in its first argument and increasing in its second argument; and $\rightarrow(1, x) = x$ and $\rightarrow(0, 0) = 1$ for all $x \in [0, 1]$.

Figure 1 lists some specific fuzzy operators that we use in this paper.

Symbol	Name	Definition
$\otimes_l$	Łukasiewicz t-norm	$\otimes_l(x, y) = max\,(x + y - 1, 0)$
$\oplus_l$	Łukasiewicz t-conorm	$\oplus_l(x, y) = min\,(x + y, 1)$
$\otimes_m$	minimum t-norm	$\otimes_m(x, y) = min\,(x, y)$
$\oplus_m$	maximum t-conorm	$\oplus_m(x, y) = max\,(x, y)$
$\otimes_p$	product t-norm	$\otimes_p(x, y) = x \cdot y$
$\oplus_p$	product t-conorm	$\oplus_p(x, y) = x + y - x \cdot y$
$\neg_s$	standard negator	$\neg_s(x) = 1 - x$
$\rightarrow_r$	the residual implicator of $\oplus_m$	$\rightarrow_r(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
$\rightarrow_s$	the S-implicator induced by $\neg_s$ and $\oplus_m$	$\rightarrow_s(x, y) = max\,(1 - x, y)$

Fig. 1. Some t-norms, t-conorms, negator, and implicators

The *truth value* of a formula F under I, denoted F^I, is defined recursively as follows:

- for any atom $p \in \sigma$, $p^I = I(p)$;
- for any numeric constant $\overline{c}$, $\overline{c}^I = c$;
- $(\neg F)^I = \neg(F^I)$;
- $(F \otimes G)^I = \otimes(F^I, G^I)$; $(F \oplus G)^I = \oplus(F^I, G^I)$; $(F \rightarrow G)^I = \rightarrow(F^I, G^I)$.

[3] We say that a function f of arity n is *increasing in its i-th argument* ($1 \leq i \leq n$) if $f(arg_1, \ldots, arg_i, \ldots, arg_n) \leq f(arg_1, \ldots, arg'_i, \ldots, arg_n)$ for all arguments such that $arg_i \leq arg'_i$; f is said to be *increasing* if it is increasing in all its arguments. The definition of *decreasing* is similar.

(For simplicity, we identify the symbols for the fuzzy operators with the truth value functions represented by them.)

Definition 2. *We say that a fuzzy interpretation I* satisfies *a fuzzy formula F w.r.t. a threshold $y \in [0,1]$ if $F^I \geq y$, and denote it by $I \models_y F$. We call I a* fuzzy y-model *of F.*

We often omit the threshold y when it is 1.

3 Definition and Examples

We extend the notion of $J <^{\mathbf{p}} I$ in Section 2.1 as follows. For any two fuzzy interpretations J and I of the same signature σ and any subset $\mathbf{p}$ of σ, we say $J \leq^{\mathbf{p}} I$ if

- J and I agree on all fuzzy atoms not in $\mathbf{p}$, and
- for all $p \in \mathbf{p}$, $p^J \leq p^I$.

We say $J <^{\mathbf{p}} I$ if $J \leq^{\mathbf{p}} I$ and $J \neq I$.

As before, we assume a list $\mathbf{q}$ of new, distinct fuzzy atoms, and define $I \cup J^{\mathbf{p}}_{\mathbf{q}}$ in the same way. That is, $I \cup J^{\mathbf{p}}_{\mathbf{q}}$ denotes the interpretation of $\sigma \cup \mathbf{q}$ that agrees with I and J on all atoms not in $\mathbf{p} \cup \mathbf{q}$, and

- for each $p \in \mathbf{p}$, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})(p) = I(p)$;
- for each $q \in \mathbf{q}$, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})(q) = J(p)$.

The definition of F^* is also extended in a straightforward way: For any fuzzy formula F of signature σ, $F^*(\mathbf{q})$ is defined as follows.

- $p_i^* = q_i$ for each $p_i \in \mathbf{p}$;
- $F^* = F$ for any atom $F \notin \mathbf{p}$;
- $\overline{c}^* = \overline{c}$ for any numeric constant $\overline{c}$;
- $(\neg F)^* = \neg F$;
- $(F \otimes G)^* = F^* \otimes G^*$; $(F \oplus G)^* = F^* \oplus G^*$;
- $(F \to G)^* = (F^* \to G^*) \otimes_m (F \to G)$.

Definition 3. *An interpretation I is a y-stable model of F relative to $\mathbf{p}$ (denoted $I \models_y$ SM$[F; \mathbf{p}]$) if*

- *$I \models_y F$, and*
- *there is no interpretation J such that $J <^{\mathbf{p}} I$ and $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y F^*(\mathbf{q})$.*

We often omit the threshold y when it is 1, and omit $\mathbf{p}$ if it contains all atoms in σ.

Clearly, when $\mathbf{p}$ is empty, Definition 3 reduces to the definition of a fuzzy model in Definition 2 because there is no J such that $J <^{\emptyset} I$.

Also, Definition 3 is very similar to the definition of a stable model for classical propositional formulas in Definition 1. The main difference is that simply in the latter, atoms may have various degrees of truth, and accordingly the notion of $J <^{\mathbf{p}} I$ is more general. The precise relationship between the definitions is discussed in Section 5.

Example 2. Consider the formula $F = \neg_s p \rightarrow_r q$ and the interpretation $I = \{(p, 0), (q, 0.6)\}$. $F^*(u, v)$ is

$$((\neg_s p)^* \rightarrow_r q^*) \otimes_m (\neg_s p \rightarrow_r q) \;=\; (\neg_s p \rightarrow_r v) \otimes_m (\neg_s p \rightarrow_r q).$$

$I \models_{0.6}$ SM$[F;\ p, q]$. First, it is easy to see that $I \models_{0.6} F$, as

$$F^I = \rightarrow_r ((\neg_s p)^I, q^I) = \rightarrow_r (1 - p^I, q^I) = \rightarrow_r (1, 0.6) = 0.6.$$

Suppose there exists $J <^{pq} I$ such that $I \cup J^{pq}_{uv} \models_{0.6} F$, i.e.,

$$\begin{aligned} F^*(u,v)^{I \cup J^{pq}_{uv}} &= min\left(\rightarrow_r (\neg_s(p^I), v^{J^{pq}_{uv}}), \rightarrow_r (\neg_s(p^I), q^I)\right) \\ &= min\left(\rightarrow_r (1, v^{J^{pq}_{uv}}), 0.6\right) \\ &= min\left(v^{J^{pq}_{uv}}, 0.6\right) \geq 0.6. \end{aligned}$$

So $v^{J^{pq}_{uv}} \geq 0.6$. Since $v^{J^{pq}_{uv}} \leq q^I = 0.6$, we conclude that $v^{J^{pq}_{uv}} = 0.6$. However, this contradicts the assumption that $J <^{pq} I$. Therefore, such J does not exist, and I is a 0.6-stable model of F.

Example 3. p and $\neg_s\neg_s p$ have the same fuzzy models, but their stable models are different. This is similar to the fact that p and $\neg\neg p$ have different stable models according to the semantics from [5].

Clearly, any interpretation $I = \{(p, y)\}$, where y is any positive real number in $[0, 1]$, is a y-stable model of p relative to $\{p\}$. On the other hand, $I = \{(p, y)\}$ is not a y-stable model of $F = \neg_s\neg_s p$ relative to $\{p\}$. Formula $F^*(q)$ is $\neg_s\neg_s F$, and although $I \models_y F$, we have $I \cup J^p_q \models_y F^*(q)$ regardless of any J.

Example 4. Let $F_1 = p \rightarrow_s p$ and $F_2 = \neg_s p \oplus_m p$. Their fuzzy models are the same, but their stable models are not. This is similar to the relation between $p \rightarrow p$ and $\neg p \vee p$ in the Boolean stable model semantics. Indeed, observe that $F_1^*(q) = (p \rightarrow_s p) \otimes_m (q \rightarrow_s q)$ and $F_2^*(q) = \neg_s p \oplus_m q$.

The interpretation $I = \{(p, 1)\}$ is not a 1-stable model of F_1 relative to p, as witnessed by $J = \{(p, 0)\}$. However, I is a 1-stable model of F_2 relative to p: for any J^p_q,

$$F_2^*(q)^{I \cup J^p_q} = max\left(1 - p^I, q^{J^p_q}\right) = max\left(0, q^{J^p_q}\right) = q^{J^p_q}.$$

So, for $I \cup J^p_q$ to satisfy $F_2^*(q)$ to degree 1, $q^{J^p_q}$ should be 1, or equivalently, p^J should be 1. Consequently, it is not possible to have $J <^p I$.

The following example illustrates how the commonsense law of inertia involving fuzzy truth values can be represented.

Example 5. Let σ be $\{p, {\sim}p, q, {\sim}q\}$ [4] and let F be $F_1 \otimes_m F_2$, where F_1 represents that p and ${\sim}p$ are complementary, i.e., the sum of their truth values is 1:

$$F_1 = \neg_s(p \otimes_l {\sim}p) \otimes_m \neg_s\neg_s(p \oplus_l {\sim}p).$$

[4] Note that $\sim$ is not a connective; it is just a part of the symbol representing an atom.

F_2 represents that by default p has the truth value of q, and $\sim p$ has the truth value of $\sim q$:

$$F_2 = ((q \otimes_m \neg_s\neg_s p) \rightarrow_r p) \otimes_m ((\sim q \otimes_m \neg_s\neg_s \sim p) \rightarrow_r \sim p).$$

Let $\mathbf{p} = \{p, \sim p\}$ and $\mathbf{u} = \{u, \sim u\}$. $F^*(\mathbf{u})$ is

$$\begin{array}{l} \neg_s(p \otimes_l \sim p) \otimes_m \neg_s\neg_s(p \oplus_l \sim p) \\ \otimes_m((q \otimes_m \neg_s\neg_s p) \rightarrow_r u) \otimes_m ((q \otimes_m \neg_s\neg_s p) \rightarrow_r p) \\ \otimes_m((\sim q \otimes_m \neg_s\neg_s \sim p) \rightarrow_r \sim u) \otimes_m ((\sim q \otimes_m \neg_s\neg_s \sim p) \rightarrow_r \sim p). \end{array}$$

One can check that the interpretation $I_1 = \{(p, x), (\sim p, 1 - x), (q, x), (\sim q, 1 - x)\}$ (x is any value in $[0, 1]$) is a 1-stable model of F relative to $(p, \sim p)$; The interpretation $I_2 = \{(p, y), (\sim p, 1 - y), (q, x), (\sim q, 1 - x)\}$, where $y > x$, is not. Similarly, if $y < x$, I_2 is not a 1-stable model of F relative to $(p, \sim p)$.

On the other hand, if we conjoin F with $\overline{y} \rightarrow_r p$ to yield $F \otimes_m (\overline{y} \rightarrow_r p)$, then the default behavior is overridden, and I_2 is a 1-stable model of $F \otimes_m (\overline{y} \rightarrow_r p)$ relative to $(p, \sim p)$.

This behavior is useful in expressing the commonsense law of inertia involving fuzzy values. Suppose q represents some fluent at time t, and p represents the fluent at time $t+1$. Then F states that, "by default, the fluent retains the previous value." The default value is overridden if there is an action that sets p to a different value.

4 Further Examples

The trust example in the introduction can be formalized in the fuzzy stable model semantics as follows. Below x, y, z are schematic variables ranging over people, and t is a schematic variable ranging over time steps. *Trust*(x, y, t) is a fuzzy atom representing that "x trusts y at time t." Similarly, *Distrust*(x, y, t) is a fuzzy atom representing that "x distrusts y at time t."

The trust relation is reflexive:

$$F_1 = \mathit{Trust}(x, x, t).$$

The trust and distrust degrees are complementary, i.e., their sum is 1 (similar to Example 5):

$$\begin{array}{l} F_2 = \neg_s(\mathit{Trust}(x, y, t) \otimes_l \mathit{Distrust}(x, y, t)), \\ F_3 = \neg_s\neg_s(\mathit{Trust}(x, y, t) \oplus_l \mathit{Distrust}(x, y, t)). \end{array}$$

Initially, if x trusts y to degree d_1 and y trusts z to degree d_2, then x trusts z to degree $d_1 \times d_2$; further the initial distrust degree is 1 minus the initial trust degree.

$$\begin{array}{l} F_4 = \mathit{Trust}(x, y, 0) \otimes_p \mathit{Trust}(y, z, 0) \rightarrow_r \mathit{Trust}(x, z, 0), \\ F_5 = \neg_s \mathit{Trust}(x, y, 0) \rightarrow_r \mathit{Distrust}(x, y, 0). \end{array}$$

The inertia assumption (similar to Example 5):

$$\begin{array}{l} F_6 = \mathit{Trust}(x, y, t) \otimes_m \neg_s\neg_s \mathit{Trust}(x, y, t{+}1) \rightarrow_r \mathit{Trust}(x, y, t{+}1), \\ F_7 = \mathit{Distrust}(x, y, t) \otimes_m \neg_s\neg_s \mathit{Distrust}(x, y, t{+}1) \rightarrow_r \mathit{Distrust}(x, y, t{+}1). \end{array}$$

A conflict increases the distrust degree by the conflict degree:

$$F_8 = \mathit{Conflict}(x,y,t) \oplus_l \mathit{Distrust}(x,y,t) \rightarrow_r \mathit{Distrust}(x,y,t+1),$$
$$F_9 = \neg_s(\mathit{Conflict}(x,y,t) \oplus_l \mathit{Distrust}(x,y,t)) \rightarrow_r \mathit{Trust}(x,y,t+1).$$

Let F_{TW} be $F_1 \otimes_m F_2 \otimes_m \cdots \otimes_m F_9$. Suppose we have the formula $F_{Fact} = \mathit{Fact}_1 \otimes_m \mathit{Fact}_2$ that gives the initial trust degree.

$$\mathit{Fact}_1 = \overline{0.8} \rightarrow_r \mathit{Trust}(\mathit{Alice}, \mathit{Bob}, 0),$$
$$\mathit{Fact}_2 = \overline{0.7} \rightarrow_r \mathit{Trust}(\mathit{Bob}, \mathit{Carol}, 0).$$

Although there is no fact about how much *Alice* trusts *Carol*, any 1-stable model of $F_{TW} \otimes_m F_{Fact}$ assigns value 0.56 to the atom $\mathit{Trust}(\mathit{Alice}, \mathit{Carol}, 0)$. On the other hand, the 1-stable model assigns value 0 to $\mathit{Trust}(\mathit{Alice}, \mathit{David}, 0)$ due to the closed world assumption under the stable model semantics.

When we conjoin $F_{TW} \otimes F_{Fact}$ with $\overline{0.2} \rightarrow \mathit{Conflict}(\mathit{Alice}, \mathit{Carol}, 0)$, the 1-stable model of $F_{TW} \otimes_m F_{Fact} \otimes_m (\overline{0.2} \rightarrow \mathit{Conflict}(\mathit{Alice}, \mathit{Carol}, 0))$, manifests that the trust degree between *Alice* and *Carol* decreases to 0.36 at time 1. More generally, if we have more actions that change the trust degree in various ways, by specifying the entire history of actions, we can determine the evolution of the trust distribution among all the participants. Useful decisions can be made based on this information. For example, *Alice* may decide not to share her personal pictures to those whom she trusts less than degree 0.48.

Note that this example, like Example 5, uses nested connectives, such as $\neg_s\neg_s$, that are not available in previous fuzzy ASP semantics, such as [3, 4].

5 Relation to Boolean-Valued Stable Models

The Boolean stable model semantics in Section 2.1 can be embedded into the fuzzy stable model semantics as follows:

For any classical propositional formula F, define F^{fuzzy} to be the fuzzy propositional formula obtained from F by replacing $\bot$ with $\overline{0}$, $\top$ with $\overline{1}$, $\neg$ with $\neg_s$, $\wedge$ with $\otimes_m$, $\vee$ with $\oplus_m$, and $\rightarrow$ with $\rightarrow_s$. We identify the signature of F^{fuzzy} with the signature of F. Also, for any interpretation I, we define the corresponding fuzzy interpretation I^{fuzzy} as

- $I^{\mathit{fuzzy}}(p) = 1$ if $I(p) = \text{TRUE}$;
- $I^{\mathit{fuzzy}}(p) = 0$ otherwise.

The following theorem tells us that the Boolean-valued stable model semantics can be viewed as a special case of the fuzzy stable model semantics.

Theorem 1 *For any classical propositional formula F and any classical propositional interpretation I, I is a stable model of F relative to* $\mathbf{p}$ *iff I^{fuzzy} is a 1-stable model of F^{fuzzy} relative to* $\mathbf{p}$.

Example 6. Let F be the classical propositional formula $\neg p \rightarrow q$. F has only one stable model $I = \{q\}$. Clearly $I^{\mathit{fuzzy}} = \{(p, 0), (q, 1)\}$ is a 1-stable model of $F^{\mathit{fuzzy}} = \neg_s p \rightarrow_s q$.

Theorem 1 does not hold for an arbitrary choice of operators, as illustrated by the following example.

Example 7. Let F be the classical propositional formula $p \vee p$. Classical interpretation $I = \{p\}$ is a stable model of F. However, $I^{fuzzy} = \{(p, 1)\}$ is not a stable model of $F' = p \oplus_l p$ because there is $J = \{(p, 0.5)\}$ such that $I \cup J^p_q \models_1 q \oplus_l q$.

However, one direction of Theorem 1 holds for arbitrary choice of fuzzy operators.

Theorem 2 *For any classical propositional formula F, let F^{fuzzy}_1 be the formula obtained from F by replacing $\bot$ with $\overline{0}$, $\top$ with $\overline{1}$, $\neg$ with any fuzzy negation symbol, $\wedge$ with any fuzzy conjunction symbol, $\vee$ with any fuzzy disjunction symbol, and $\rightarrow$ with any fuzzy implication symbol. For any classical propositional interpretation I, if I^{fuzzy} is a 1-stable model of F^{fuzzy}_1 relative to $\mathbf{p}$, then I is a stable model of F relative to $\mathbf{p}$.*

6 Relation to Other Approaches to Fuzzy ASP

6.1 Relation to Stable Models of Normal FASP Programs

A normal FASP program is a finite set of rules of the form

$$a \ \leftarrow \ b_1 \otimes \ldots \otimes b_m \otimes \neg b_{m+1} \otimes \ldots \otimes \neg b_n,$$

where $n \geq m \geq 0$, $a, b_1, \ldots, b_n$ are fuzzy atoms or numeric constants in $[0, 1]$, and $\otimes$ is any fuzzy conjunction. We identify the rule with the fuzzy implication

$$b_1 \otimes \ldots \otimes b_m \otimes \neg_s b_{m+1} \otimes \ldots \otimes \neg_s b_n \ \rightarrow_r \ a.$$

We say that a fuzzy interpretation I of signature σ *satisfies* a rule R if $R^I = 1$. I *satisfies* an FASP program Π if $I \models R$ for every rule R in Π. According to [3], an interpretation I is a *fuzzy answer set* of a normal FASP program Π if I satisfies Π, and no interpretation J such that $J <^\sigma I$ satisfies the reduct of Π w.r.t. I, which is the program obtained from Π by replacing each negative literal $\neg b$ with the constant for $1 - b^I$.

Theorem 3 *For any normal FASP program $\Pi = \{r_1, \ldots, r_n\}$, let F be the formula $r_1 \otimes_m \ldots \otimes_m r_n$. An interpretation I is a fuzzy answer set of Π in the sense of [3] if and only if I is a 1-stable model of F.*

Example 8. Let Π be the following program

$$p \leftarrow \neg q, \quad q \leftarrow \neg p.$$

The answer sets of Π according to [3] are $\{(p, x), (q, 1 - x)\}$, where x is any value in $[0, 1]$: the corresponding fuzzy formula F is $(\neg_s q \rightarrow_r p) \otimes_m (\neg_s p \rightarrow_r q)$; $F^*(u, v)$ is

$$F \otimes_m ((\neg_s q \rightarrow_r u) \otimes_m (\neg_s p \rightarrow_r v)).$$

One can check that the 1-stable models of F are also $\{(p, x), (q, 1 - x)\}$, where $x \in [0, 1]$.

6.2 Relation to Fuzzy Equilibrium Logic

Like our fuzzy stable model semantics, fuzzy equilibrium logic [6] generalizes fuzzy ASP programs to arbitrary propositional formulas, but its definition is highly complex. Nonetheless we show that if we disregard strong negation considered there, fuzzy equilibrium logic is essentially equivalent to the fuzzy stable model semantics where the threshold is set to 1 and all atoms are subject to minimization.[5]

We review the definition of fuzzy equilibrium logic in the absence of strong negation. For any fuzzy propositional signature σ, a (fuzzy N5) *valuation* is a mapping from $\{h,t\} \times \sigma$ to subintervals of $[0,1]$ such that $V(t,a) \subseteq V(h,a)$ for each atom $a \in \sigma$. For $V(w,a) = [u,v]$, where $w \in \{h,t\}$, we write $V^-(w,a)$ to denote the lower bound u and $V^+(w,a)$ to denote the upper bound v. The truth value of a formula under V is defined as follows.

- $V(w,\overline{c}) = [c,\ c]$ for any numeric constant $\overline{c}$;
- $V(w, F \otimes G) = [V^-(w,F) \otimes V^-(w,G),\ \ V^+(w,F) \otimes V^+(w,G)]$; [6]
- $V(w, F \oplus G) = [V^-(w,F) \oplus V^-(w,G),\ \ V^+(w,F) \oplus V^+(w,G)]$;
- $V(h, \neg F) = [1 - V^-(t,F),\ \ 1 - V^-(h,F)]$;
- $V(t, \neg F) = [1 - V^-(t,F),\ \ 1 - V^-(t,F)]$;
- $V(h, F \rightarrow G) = [min(V^-(h,F) \rightarrow V^-(h,G), V^-(t,F) \rightarrow V^-(t,G)),$ $V^-(h,F) \rightarrow V^+(h,G)]$;
- $V(t, F \rightarrow G) = [V^-(t,F) \rightarrow V^-(t,G),\ \ V^-(t,F) \rightarrow V^+(t,G)]$.

A valuation V is a (fuzzy N5) model of a formula F if $V^-(h,F) = 1$, which implies $V^+(h,F) = V^-(t,F) = V^+(t,F) = 1$. For two valuations V and V', we say $V' \preceq V$ if $V'(t,a) = V(t,a)$ and $V(h,a) \subseteq V'(h,a)$ for all atoms a. We say $V' \prec V$ if $V' \preceq V$ and $V' \neq V$. We say that a model V of F is *h-minimal* if there is no model V' of F such that $V' \prec V$. An h-minimal fuzzy N5 model V of F is a *fuzzy equilibrium model* of F if $V(h,a) = V(t,a)$ for all atoms a.

For two fuzzy interpretations I, J of signature σ such that $J \leq^\sigma I$, define the N5 fuzzy valuation $V_{J,I}$ as $V_{J,I}(h,a) = \left[a^J, 1\right]$, $V_{J,I}(t,a) = \left[a^I, 1\right]$ for all atoms a in σ. Since $J \leq^\sigma I$, we have $a^J \leq a^I$, and $V_{J,I}(t,a) \subseteq V_{J,I}(h,a)$ for all atoms a.

As in [6], we assume that the fuzzy negation $\neg$ is $\neg_s$.

The following proposition relates the notions used in the fuzzy equilibrium models and the fuzzy stable models.

Proposition 1 *(a) $I \models_1 F$ if and only if $V_{I,I}$ is a model of F.*
(b) For $\mathbf{p} = \sigma$, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$ if and only if $V_{J,I}$ is a model of F.
(c) For two interpretations I and J, we have $V_{J,I} \prec V_{I,I}$ if and only if $J < I$.

The theorem below shows that the fuzzy stable model semantics can be reduced to fuzzy equilibrium logic semantics.

Theorem 4 *For any fuzzy formula F (that contains no strong negation) and any fuzzy interpretation I, I is a 1-stable model of F if and only if $V_{I,I}$ is a fuzzy equilibrium model of F.*

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[5] Strong negation can be simulated in our semantics using new atoms as illustrated in Example 5.

[6] For readability, we write the infix notation $(x \odot y)$ in place of $\odot(x,y)$.

Next we show the other direction, i.e., reducing fuzzy equilibrium logic to the fuzzy stable model semantics. For any valuation V, define the fuzzy interpretation I_V as $a^{I_V} = V^-(h,a)$ for all atoms a.

Theorem 5 *For any fuzzy formula F (that contains no strong negation) and any valuation V, we have that V is an equilibrium model of F if and only if*

(i) $V^+(h,a) = V^+(t,a) = 1$ *for all atoms a, and*
(ii) I_V *is a 1-stable model of F relative to σ.*

Theorem 5 tells us that in the absence of strong negation, the upper bounds of both worlds in any equilibrium model are always 1.

7 Properties of Fuzzy Stable Models

In this section, we show that several well-known properties of the Boolean stable model semantics can be naturally extended to the fuzzy stable model semantics.

7.1 Alternative Definition of F^*

Proposition 2 *For any fuzzy formula F and any fuzzy interpretations I, J with $J \leq^{\mathbf{p}} I$,*

- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y \neg F^*(\mathbf{q}) \otimes_m \neg F$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y \neg F$.
- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \otimes G^*)(\mathbf{q}) \otimes_m (F \otimes G)$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \otimes G^*)(\mathbf{q})$.
- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \oplus G^*)(\mathbf{q}) \otimes_m (F \oplus G)$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \oplus G^*)(\mathbf{q})$.

This proposition tells us that F^* in Section 3 can be equivalently defined by treating the fuzzy operators in the uniform way:

- $(\neg F)^* = \neg F^* \otimes_m \neg F$;
- $(F \odot G)^* = (F^* \odot G^*) \otimes_m (F \odot G)$ for any binary operator $\odot$.

7.2 Theorem on Constraints

In answer set programming, constraints—rules with $\bot$ in the head—play an important role in view of the fact that adding a constraint eliminates the stable models that "violate" the constraint. The following theorem is the counterpart of Theorem 3 from [5] for fuzzy propositional formulas.

Theorem 6 *For any fuzzy formulas F and G, I is a 1-stable model of $F \otimes \neg G$ (relative to $\mathbf{p}$) if and only if I is a 1-stable model of F (relative to $\mathbf{p}$) and $I \models_1 \neg G$.*

Example 9. Consider $F = (\neg_s p \rightarrow_r q) \otimes_m (\neg_s q \rightarrow_r p) \otimes_m \neg_s p$. Formula F has only one 1-stable model $I = \{(p,0),(q,1)\}$, which is the only 1-stable model of $(\neg_s p \rightarrow_r q) \otimes_m (\neg_s q \rightarrow_r p)$ that satisfies $\neg_s p$ to degree 1.

If we consider a more general y-stable model, then only one direction holds.

Theorem 7 *For any fuzzy formulas F and G, if I is a y-stable model of $F \otimes \neg G$ (relative to $\mathbf{p}$), then I is a y-stable model of F (relative to $\mathbf{p}$) and $I \models_y \neg G$.*

Example 10. The other direction, that is, "if I is a y-stable model of F and $I \models_y \neg G$, then I is a y-stable model of $F \otimes \neg G$," does not hold in general. For example, consider $F = G = p$ and $\otimes$ to be $\otimes_l$, and interpretation $I = \{(p, 0.4)\}$. Clearly I is a 0.4-stable model of p and $I \models_{0.4} \neg p$, but I is not a 0.4-stable model of $p \otimes_l \neg p$. In fact, I is not even a 0.4-model of the formula.

7.3 Theorem on Choice Formulas

In the Boolean stable model semantics, formulas of the form $p \vee \neg p$ are called *choice formulas*, and adding them to the program makes atoms p exempt from minimization. Choice formulas have been shown to be useful in composing a program in the "Generate-and-Test" method. This section shows their counterpart in the fuzzy stable model semantics.

For any fuzzy atom p, *Choice*(p) stands for $p \oplus_l \neg_s p$. For any list $\mathbf{p} = (p_1, \dots p_n)$ of fuzzy atoms, *Choice*$(\mathbf{p})$ stands for

$$\mathit{Choice}(p_1) \otimes \ldots \otimes \mathit{Choice}(p_n),$$

where $\otimes$ is any fuzzy conjunction.

The following proposition tells that choice formulas are tautological.

Proposition 3 *For any fuzzy interpretation I and any list $\mathbf{p}$ of fuzzy atoms, $I \models_1 \mathit{Choice}(\mathbf{p})$.*

Theorem 8 is an extension of Theorem 2 from [5].

Theorem 8 *(a) If I is a y-stable model of F relative to $\mathbf{p} \cup \mathbf{q}$, then I is a y-stable model of F relative to $\mathbf{p}$.*
(b) I is a 1-stable model of F relative to $\mathbf{p}$ iff I is a 1-stable model of $F \otimes \mathit{Choice}(\mathbf{q})$ relative to $\mathbf{p} \cup \mathbf{q}$.

Theorem 8 (b) does not hold for arbitrary threshold y (i.e., if "$1-$" is replaced with "$y-$"). For example, consider $F = \neg_s \neg_s q$ and $I = \{(q, 0.5)\}$. Clearly I is a 0.5-model of F, and thus I is a 0.5-stable model of F relative to $\emptyset$. However, I is not a 0.5-stable model of $F \otimes_m \mathit{Choice}(q) = \neg_s \neg_s q \otimes_m (q \oplus_l \neg_s q)$ relative to $\emptyset \cup \{q\}$, as witnessed by $J = \{(q, 0)\}$.

Since the 1-stable models of F relative to $\emptyset$ are the models of F, it follows from Theorem 8 (b) that the 1-*stable models* of $F \otimes \mathit{Choice}(\sigma)$ relative to σ are exactly the 1-*models* of F.

Corollary 1 *Let F be a formula of a finite signature σ. I is a 1-model of F relative to σ iff I is a 1-stable model of $F \otimes \mathit{Choice}(\sigma)$.*

Example 11. Consider the formula $F = \neg_s p \rightarrow_r q$. Although any interpretation I that satisfies $1 - p^I \leq q^I$ is a 1-model of F, among them only $\{(p, 0), (q, 1)\}$ is a 1-stable model of F. However, we check that all 1-models of F are exactly the 1-stable models of $G = F \otimes_m \mathit{Choice}(p) \otimes_m \mathit{Choice}(q)$: $G^*(u,v)$ is

$$(\neg_s p \rightarrow_r q) \otimes_m (\neg_s p \rightarrow_r v) \otimes_m (u \oplus_l \neg_s p) \otimes_m (v \oplus_l \neg_s q)$$

and for $K = I \cup J^{pq}_{uv}$,

$$G^*(u,v)^K = 1 \otimes_m ((1-p^K) \rightarrow_r v^K) \otimes_m (u^K \oplus_l (1-p^K)) \otimes_m (v^K \oplus_l (1-q^K)).$$

So, for K to satisfy $G^*(u,v)$ to degree 1, u^K should be at least p^K and v^K should be at least q^K. So there does not exist $J <^{pq} I$ such that $I \cup J^{pq}_{uv} \models_1 G^*(u,v)$, from which it follows that I is a 1-stable model of G.

8 Conclusion

We introduced a general stable model semantics for fuzzy propositional formulas, which generalizes both the Boolean stable model semantics and fuzzy propositional logic. The syntax is the same as the syntax of fuzzy propositional logic, but the semantics defines *stable models* instead of *models*. The formalism allows highly configurable default reasoning involving fuzzy truth values. Our semantics, when we restrict threshold to be 1 and assume all atoms to be subject to minimization, is equivalent to fuzzy equilibrium logic in the absence of strong negation, but is much more simpler. To the best of our knowledge, our representation of commonsense law of inertia involving fuzzy values is new. The representation uses nested fuzzy operators, which are not available in earlier fuzzy ASP semantics for a restricted syntax.

We showed that several traditional results in answer set programming can be naturally extended to this formalism, and expect that more results can be carried over. Future work includes implementing this language using mixed integer programming solvers or bilevel programming solvers [7].

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A Proof of Theorem 1

Lemma 1. *For any fuzzy conjunction $\otimes$, if $\otimes(x,y) \geq z$, then $x \geq z$ and $y \geq z$.*

Proof. From the definition of fuzzy conjunction, for any $x, y \in [0,1]$, $\otimes(x,y) \leq \otimes(x,1) \leq x$ and $\otimes(x,y) \leq \otimes(1,y) \leq y$. If $\otimes(x,y) \geq z$, then clearly $x \geq z$ and $y \geq z$. ∎

The following is a corollary to Lemma 1.

Corollary 2 *For any fuzzy conjunction $\otimes$, $\otimes(x,y) = 1$ if and only if $x = 1$ and $y = 1$.*

Define the mapping *defuz*$(\cdot)$ from a fuzzy interpretation $I = \{(p_1, x_1), \dots, (p_n, x_n)\}$ to a classical interpretation as *defuz*$(I) = \{p_i | (p_i, x_i) \in I \; and \; x_i = 1\}$.

Lemma 2. *for any fuzzy interpretation I and any classical propositional formula F,*

(i) if $I \models_1 F^{fuzzy}$, then defuz$(I) \models F$, and
(ii) if $(F^{fuzzy})^I = 0$, then defuz$(I) \not\models F$.

Proof. We prove by induction on F.

- F is an atom p. (i) Suppose $I \models_1 F^{fuzzy}$. Then $p^I = 1$, and thus $p \in$ *defuz*(I). So *defuz*$(I) \models F$. (ii) Suppose $(F^{fuzzy})^I = 0$. Then $p^I = 0$, and thus $p \notin$ *defuz*(I). So *defuz*$(I) \not\models F$.
- F is $\bot$. (i) There is no interpretation that satisfies $F^{fuzzy} = \bar{0}$ to the degree 1. So the claim is trivially true. (ii) Since no interpretation satisfies F, *defuz*$(I) \not\models F$.
- F is $\top$. (i) Then all interpretations satisfy F. So *defuz*$(I) \models F$. (ii) There is no interpretation I such that $(F^{fuzzy})^I = 0$. So (ii) is trivially true.
- F is $G \wedge H$. Then F^{fuzzy} is $G^{fuzzy} \otimes_m H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. By Corollary 2, $(G^{fuzzy})^I = (H^{fuzzy})^I = 1$, i.e., $I \models_1 G^{fuzzy}$ and $I \models_1 H^{fuzzy}$. By I.H., *defuz*$(I) \models G$ and *defuz*$(I) \models H$. It follows that *defuz*$(I) \models G \wedge H = F$. (ii) Suppose $(F^{fuzzy})^I = min((G^{fuzzy})^I, (H^{fuzzy})^I) = 0$. Then $(G^{fuzzy})^I = 0$ or $(H^{fuzzy})^I = 0$. By I.H., *defuz*$(I) \not\models G$ or *defuz*$(I) \not\models H$. It follows that *defuz*$(I) \not\models G \wedge H = F$.
- F is $G \vee H$. Then F^{fuzzy} is $G^{fuzzy} \oplus_m H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$, and as the disjunction is defined as $\oplus_m(x,y) = max(x,y)$, $(G^{fuzzy})^I = 1$ or $(H^{fuzzy})^I = 1$, i.e., $I \models_1 G^{fuzzy}$ or $I \models_1 H^{fuzzy}$. By I.H., *defuz*$(I) \models G$ or *defuz*$(I) \models H$. It follows that *defuz*$(I) \models G \vee H = F$. (ii) Suppose $(F^{fuzzy})^I = max((G^{fuzzy})^I, (H^{fuzzy})^I) = 0$. Then $(G^{fuzzy})^I = 0$ and $(H^{fuzzy})^I = 0$. By I.H., *defuz*$(I) \not\models G$ and *defuz*$(I) \not\models H$. It follows that *defuz*$(I) \not\models G \vee H = F$.
- F is $G \rightarrow H$. Then F^{fuzzy} is $G^{fuzzy} \rightarrow_s H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. $(F^{fuzzy})^I = max(1 - (G^{fuzzy})^I, (H^{fuzzy})^I) = 1$, and thus $(G^{fuzzy})^I = 0$ or $(H^{fuzzy})^I = 1$[7]. By I.H., *defuz*$(I) \not\models G$ or *defuz*$(I) \models H$. It follows that *defuz*$(I) \models$

[7] This does not hold for arbitrary choice of implication. For example, consider $\rightarrow_l (x,y) = min(1 - x + y, 1)$, then from $I \models_1 G \rightarrow_l H$, we can only conclude $H^I \geq G^I$. Furthermore, under this choice of implication, the modified statement of the lemma does not hold. A counterexample is $F = \neg_s p \rightarrow_l q$, $I = \{(p, 0.5), (q, 0.6)\}$. Clearly $I \models_1 F^{fuzzy}$ but *defuz*$(I) = \emptyset \not\models F$

$G \rightarrow H = F$. (ii) Suppose $(F^{fuzzy})^I = max\left\{1 - (G^{fuzzy})^I, (H^{fuzzy})^I\right\} = 0$. Then $(G^{fuzzy})^I = 1$ and $(H^{fuzzy})^I = 0$. By I.H., *defuz*$(I) \models G$ and *defuz*$(I) \not\models H$. Therefore *defuz*$(I) \not\models G \rightarrow H$.
- F is $\neg G$. Then F^{fuzzy} is $\neg_s G^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. Then $(F^{fuzzy})^I = 1 - (G^{fuzzy})^I = 1$, so $(G^{fuzzy})^I = 0$. By I.H., *defuz*$(I) \not\models G$, and thus *defuz*$(I) \models F$. (ii) Suppose $(F^{fuzzy})^I = 0$, Then $1 - (G^{fuzzy})^I = 0$, $(G^{fuzzy})^I = 1$, i.e., $I \models_1 (G^{fuzzy})^I$. By I.H., *defuz*$(I) \models G$, and thus *defuz*$(I) \not\models F$.

∎

Lemma 3. *For any fuzzy interpretation I where all atoms are assigned only 0 or 1 and any fuzzy formula F where every numeric constant in it is either $\bar{0}$ or $\bar{1}$, F^I belongs to $\{0, 1\}$.*

Proof. We prove by induction on F.

- F is an atom p. Clearly $F^I = p^I \in \{0, 1\}$.
- F is numeric constant $\bar{0}$ or $\bar{1}$. Clearly $F^I \in \{0, 1\}$.
- F is $G \odot H$, where $\odot$ is $\otimes$, $\oplus$ or $\rightarrow$. By I.H., $G^I, H^I \in \{0, 1\}$. As any function associated with fuzzy operators $\otimes$, $\oplus$ or $\rightarrow$ must be a proper generalization of their classical counterpart, $F^I = (G \odot H)^I \in \{0, 1\}$.
- F is $\neg G$. By I.H., $G^I \in \{0, 1\}$. As any function associated with fuzzy negation must be a proper generalization of classical negation, $F^I = (\neg G)^I \in \{0, 1\}$.

∎

Lemma 4. *For any classical interpretation I and any classical propositional formula F, $I \models F$ if and only if $I^{fuzzy} \models_1 F^{fuzzy}$.*

Proof. We prove by induction on F. Note that the way by which I^{fuzzy} is constructed guarantees that atoms in F^{fuzzy} can only be assigned 0 or 1 in I^{fuzzy}.

- if F is an atom p, according to the construction of I^{fuzzy}, $F^I = p^I = \mathbf{t} \iff p^{I^{fuzzy}} = (F^{fuzzy})^{I^{fuzzy}} = 1$;
- if F is $\bot$, the two directions are both trivially true; if F is $\top$, $F^I = \textbf{true}$ and $(F^{fuzzy})^{I^{fuzzy}} = 1$;
- if F is $\neg G$, then $F^{fuzzy} = \neg G^{fuzzy}$. By I.H., $G^I = \textbf{true} \iff (G^{fuzzy})^{I^{fuzzy}} = 1$, which says the same as $G^I = \textbf{false} \iff (G^{fuzzy})^{I^{fuzzy}} \neq 1$. Since atoms can only be assigned 0 or 1 in I^{fuzzy}, by Lemma 3, $(G^{fuzzy})^{I^{fuzzy}} \in \{0, 1\}$. It follows that $G^I = \textbf{false} \iff (G^{fuzzy})^{I^{fuzzy}} = 0$. So $F^I = \textbf{true} \iff G^I = \textbf{false} \iff (G^{fuzzy})^{I^{fuzzy}} = 0 \iff (F^{fuzzy})^{I^{fuzzy}} = 1$;
- if F is $G \odot H$, then $F^{fuzzy} = G^{fuzzy} \odot' H^{fuzzy}$, where $\odot'$ is the corresponding fuzzy operator of $\odot$. By I.H., $G^I = \textbf{true} \iff (G^{fuzzy})^{I^{fuzzy}} = 1$ and $H^I = \textbf{true} \iff (H^{fuzzy})^{I^{fuzzy}} = 1$. Consequently, $F^I = \textbf{true} \iff \odot(G^I, H^I) = \textbf{true}$. As atoms can only be assigned 0 or 1 in I^{fuzzy}, by Lemma 3, $(G^{fuzzy})^{I^{fuzzy}}, (H^{fuzzy})^{I^{fuzzy}} \in \{0, 1\}$. As any valid fuzzy operator must be proper generalization of its corresponding classical operator, $\odot(G^I, H^I) = \textbf{true} \iff \odot'((G^{fuzzy})^{I^{fuzzy}}, (H^{fuzzy})^{I^{fuzzy}}) = 1$.

∎

Theorem 1 *For any classical propositional formula F and any classical propositional interpretation I, I is a stable model of F relative to $\mathbf{p}$ iff I^{fuzzy} is a 1-stable model of F^{fuzzy} relative to $\mathbf{p}$.*

Proof. ($\Rightarrow$) Suppose I is a stable model of F relative to $\mathbf{p}$. From the fact that $I \models F$, by Lemma 4, $I^{fuzzy} \models_1 F^{fuzzy}$. Next we show that there does not exist fuzzy interpretation $J <^{\mathbf{p}} I^{fuzzy}$ such that $I^{fuzzy} \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 (F^{fuzzy})^*(\mathbf{q})$. Suppose, to the contrary, there exists such J. Since $I^{fuzzy} \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 (F^{fuzzy})^*(\mathbf{q}) = F^*(\mathbf{q})^{fuzzy}$, by Lemma 2, *defuz*$(I^{fuzzy} \cup J^{\mathbf{p}}_{\mathbf{q}}) \models F^*(\mathbf{q})$. It is easy to see that *defuz*$(I^{fuzzy} \cup J^{\mathbf{p}}_{\mathbf{q}}) = I \cup$ *defuz*$(J^{\mathbf{p}}_{\mathbf{q}}) = I \cup$ *defuz*$(J)^{\mathbf{p}}_{\mathbf{q}}$, because in I^{fuzzy} all atoms are only assigned 0 or 1. So we have $I \cup$ *defuz*$(J)^{\mathbf{p}}_{\mathbf{q}} \models F^*(\mathbf{q})$. Since $J <^{\mathbf{p}} I^{fuzzy}$, J and I^{fuzzy} agree on all atoms not in $\mathbf{p}$. Since in I^{fuzzy} atoms are only assigned 0 or 1, for all $p \notin \mathbf{p}$, $p^J \in \{0,1\}$, so *defuz*(J) and J agree on all atoms not in $\mathbf{p}$. So the construction of *defuz*(J) guarantees that *defuz*$(J)^{fuzzy} \leq^{\mathbf{p}} J <^{\mathbf{p}} I^{fuzzy}$. Since in both *defuz*$(J)^{fuzzy}$ and I^{fuzzy}, atoms are only assigned 0 or 1, there is at least one atom $p \in \mathbf{p}$ such that $p^{defuz(J)^{fuzzy}} = 0$ and $p^{I^{fuzzy}} = 1$, and consequently $p^{defuz(J)} =$ **false** and $p^I =$ **true**. So *defuz*$(J) <^{\mathbf{p}} I$, and thus I cannot be a stable model of F, which is a contradiction. Thus, there does not exist such J. Therefore, I^{fuzzy} is a 1-stable model of F^{fuzzy} relative to $\mathbf{p}$.[8]

($\Leftarrow$) Suppose I^{fuzzy} is a 1-stable model of a fuzzy formula F^{fuzzy} relative to $\mathbf{p}$. Then $I^{fuzzy} \models_1 F^{fuzzy}$. By Lemma 4, $I \models F$. Next we show there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models F^*(\mathbf{q})$. Suppose, to the contrary, that there exists such J. Then by Lemma 4, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})^{fuzzy} = I^{fuzzy} \cup (J^{\mathbf{p}}_{\mathbf{q}})^{fuzzy} \models_1 F^*(\mathbf{q})^{fuzzy}$, and obviously $J^{fuzzy} <^{\mathbf{p}} I^{fuzzy}$. It follows that I^{fuzzy} cannot be a 1-stable model of F^{fuzzy}, which is a contradiction. So there does not exist such J. Therefore, I is a stable model of F relative to $\mathbf{p}$. ∎

B Proof of Theorem 2

Theorem 2 *For any classical propositional formula F, let F_1^{fuzzy} be the formula obtained from F by replacing $\bot$ with $\overline{0}$, $\top$ with $\overline{1}$, $\neg$ with any fuzzy negation symbol, $\wedge$ with any fuzzy conjunction symbol, $\vee$ with any fuzzy disjunction symbol, and $\rightarrow$ with any fuzzy implication symbol. For any classical propositional interpretation I, if I^{fuzzy} is a 1-stable model of F_1^{fuzzy} relative to $\mathbf{p}$, then I is a stable model of F relative to $\mathbf{p}$.*

Proof. (Exactly the same as ($\Leftarrow$) in the proof of Theorem 1) Suppose I^{fuzzy} is a 1-stable model of a fuzzy formula F^{fuzzy} relative to $\mathbf{p}$. Then $I^{fuzzy} \models_1 F^{fuzzy}$. By Lemma 4, $I \models F$. Next we show there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models F^*(\mathbf{q})$. Suppose, to the contrary, that there exists such J. Then by Lemma 4, $(I \cup J^{\mathbf{p}}_{\mathbf{q}})^{fuzzy} = I^{fuzzy} \cup (J^{\mathbf{p}}_{\mathbf{q}})^{fuzzy} \models_1 F^*(\mathbf{q})^{fuzzy}$, and obviously $J^{fuzzy} <^{\mathbf{p}} I^{fuzzy}$. It follows that

[8] This direction does not hold for arbitrary choice of operators, because Lemma 2 does not hold for arbitrary choice of operators.

I^{fuzzy} cannot be a 1-stable model of F^{fuzzy}, which is a contradiction. So there does not exist such J. Therefore, I is a stable model of F relative to $\mathbf{p}$. ∎

C Proof of Theorem 3

Theorem 3 *For any normal FASP program $\Pi = \{r_1, \ldots, r_n\}$, let F be the formula $r_1 \otimes_m \ldots \otimes_m r_n$[9]. An interpretation I is an answer set of Π in the sense of [3] if and only if I is a 1-stable model of F.*

Proof. ($\Rightarrow$) Suppose I is an answer set of Π. By definition, $I \models \Pi$, and thus $r_i^I = 1$ for all $r_i \in \Pi$. So $F^I = (\underset{r_i \in \Pi}{\otimes_m} r_i)^I = 1$. So $I \models_1 F$. Next we show that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$ for $\mathbf{p} = \sigma$. Suppose $\mathbf{q} = (a', b'_1, \ldots, b'_n)$, corresponding to $\mathbf{p} = (a, b_1, \ldots, b_n)$, respectively. For each r_i,

$$r_i^*(\mathbf{q}) = r_i \otimes_m (a' \leftarrow_r b'_1 \otimes \cdots \otimes b'_m \otimes \neg_s b_{m+1} \otimes \cdots \otimes \neg_s b_n).$$

Suppose, to the contrary, that there exists an interpretation $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$. Then for all $r_i \in \Pi$, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 r_i^*(\mathbf{q})$, i.e.,

$$J \models_1 \overline{(r_i)^I} \otimes_m (a \leftarrow_r b_1 \otimes \cdots \otimes b_m \otimes \overline{(\neg_s b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg_s b_n)^I}).$$

It follows that

$$J \models_1 a \leftarrow_r b_1 \otimes \cdots \otimes b_m \otimes \overline{(\neg_s b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg_s b_n)^I}.$$

So $J \models \Pi^{\underline{I}}$. Since $J <^{\mathbf{p}} I$, I cannot be an answer set of Π, which is a contradiction. So there does not exist such J. Therefore, I is a 1-stable model of F.

($\Leftarrow$) Suppose I is a 1-stable model of F. Then $I \models_1 F$. By Corollary 2, for all $r_i \in \Pi$, $r_i^I = 1$, i.e, $I \models \Pi$. Next we show that there does not exist $J < I$ such that $J \models \Pi^{\underline{I}}$. The reduct $\Pi^{\underline{I}}$ contains the following rule for each original rule $r_i \in \Pi$:

$$a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \overline{(\neg b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg b_n)^I}.$$

Suppose, to the contrary, that there exists $J < I$ such that $J \models \Pi^{\underline{I}}$. Then for each rule $r_i \in \Pi$,

$$J \models_1 a \leftarrow_r b_1 \otimes \cdots \otimes b_m \otimes \overline{(\neg_s b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg_s b_n)^I}.$$

As $I \models \Pi$, for all $r_i \in \Pi$, $r_i^I = 1$, so

$$J \models_1 \overline{(r_i)^I} \otimes_m (a \leftarrow_r b_1 \otimes \cdots \otimes b_m \otimes \overline{(\neg_s b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg_s b_n)^I})$$

for each $r_i \in \Pi$. Suppose $\mathbf{q} = (a', b'_1, \ldots, b'_n)$, corresponding to $\mathbf{p} = (a, b_1, \ldots, b_n)$, respectively. Replacing each atom p with p', it follows that

$$J^{\mathbf{p}}_{\mathbf{q}} \models_1 \overline{(r_i)^I} \otimes (a' \leftarrow_r b'_1 \otimes \cdots \otimes b'_m \otimes \overline{(\neg_s b_{m+1})^I} \otimes \cdots \otimes \overline{(\neg_s b_n)^I})$$

[9] It is not necessary to have $\otimes = \otimes_m$ for this theorem to hold. $\otimes$ can be any fuzzy conjunction.

for each $r_i \in \Pi$. So

$$I \cup J^{\mathbf{p}}_{\mathbf{q}} \vDash_1 r_i \otimes (a' \leftarrow b'_1 \otimes \cdots \otimes b'_m \otimes \neg b_{m+1} \otimes \cdots \otimes \neg b_n) = r^*_i(\mathbf{q})$$

for each $r_i \in \Pi$. Therefore, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \vDash_1 \underset{r_i \in \Pi}{\otimes_m} r^*_i(\mathbf{q}) = F^*(\mathbf{q})$. Since $J < I$, I cannot be a 1-stable model of F, which is a contradiction. So there does not exist such J. So I is an answer set of Π. ∎

D Proof of Theorem 4

Lemma 5. *For* $\mathbf{p} = \sigma$, $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = V^-_{J,I}(t, F)$.

Proof. We prove by induction on F.

- F is an atom p. Then $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = p^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = p^I = V^-_{J,I}(t, p) = V^-_{J,I}(t, F)$.
- F is a numeric constant $\bar{c}$. Then $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = c = V^-_{J,I}(t, \bar{c}) = V^-_{J,I}(t, F)$.
- F is $G \odot H$, where $\odot$ is $\otimes$, $\oplus$ or $\rightarrow$. Then $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \odot(G^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}, H^{I \cup J^{\mathbf{p}}_{\mathbf{q}}})$. By I.H., $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \odot(V^-_{J,I}(t, G), V^-_{J,I}(t, H)) = V^-_{J,I}(t, F)$.
- F is $\neg G$. Then $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \neg(G^{I \cup J^{\mathbf{p}}_{\mathbf{q}}})$. By I.H., $F^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \neg(V^-_{J,I}(t, G)) = V^-_{J,I}(t, F)$.

∎

The following is a corollary to Lemma 5.

Corollary 3 *(Proposition 1(a)) $I \vDash_1 F$ if and only if $V_{I,I}$ is a model of F.*

Proof. By Lemma 5, $F^{I \cup I^{\mathbf{p}}_{\mathbf{q}}} = F^I = V^-_{I,I}(t, F) = V^-_{I,I}(h, F)$.

Lemma 6. $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = V^-_{J,I}(h, F)$.

Proof. We prove by induction on F.

- F is an atom p. Then $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = p^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = p^J = V^-_{J,I}(h, p) = V^-_{J,I}(h, F)$.
- F is a numeric constant $\bar{c}$. Then $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = c = V^-_{J,I}(h, \bar{c}) = V^-_{J,I}(h, F)$.
- F is $G \odot H$, where $\odot$ is $\otimes$ or $\oplus$. Then $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = (G^*(\mathbf{q}) \odot H^*(\mathbf{q}))^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \odot(G^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}, H^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}})$. By I.H., $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \odot(V^-_{J,I}(h, G), V^-_{J,I}(h, H)) = V^-_{J,I}(h, G \odot H) = V^-_{J,I}(h, F)$.
- F is $G \rightarrow H$. Then

$$\begin{aligned} F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} &= ((G^*(\mathbf{q}) \rightarrow H^*(\mathbf{q})) \otimes_m (G \rightarrow H))^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} \\ &= min((G^*(\mathbf{q}) \rightarrow H^*(\mathbf{q}))^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}, (G \rightarrow H)^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}) \\ &= min(\rightarrow (G^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}, H^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}), \rightarrow (G^{I \cup J^{\mathbf{p}}_{\mathbf{q}}}, H^{I \cup J^{\mathbf{p}}_{\mathbf{q}}})). \end{aligned}$$

  By I.H. and Lemma 5,

$$\begin{aligned} F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} &= min(\rightarrow (V^-_{J,I}(h, G), V^-_{J,I}(h, H)), \rightarrow (V^-_{J,I}(t, G), V^-_{J,I}(t, H))) \\ &= V^-_{J,I}(h, G \rightarrow H) = V^-_{J,I}(h, F). \end{aligned}$$

- F is $\neg G$. Then $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \neg(G^{I \cup J^{\mathbf{p}}_{\mathbf{q}}})$. By Lemma 5, $F^*(\mathbf{q})^{I \cup J^{\mathbf{p}}_{\mathbf{q}}} = \neg(V^-_{J,I}(t, G)) = V^-_{J,I}(t, F)$.

∎

The following is a corollary to Lemma 6.

Corollary 4 *(Proposition 1(b)) For $\mathbf{p} = \sigma$, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$ if and only if $V_{J,I}$ is a model of F.*

Lemma 7. *(Proposition 1(c)) For two interpretations I, J of the same signature, $V_{J,I} \prec V_{I,I}$ if and only if $J < I$.*

Proof. ($\Rightarrow$) Suppose $V_{J,I} \prec V_{I,I}$. Then for every atom a, $V_{I,I}(h, a) \subseteq V_{J,I}(h, a)$, which means $V^-_{J,I}(h, a) \leq V^-_{I,I}(h, a)$. So $J \leq I$. Furthermore, there is at least one atom p satisfying $V_{I,I}(h, p) \subset V_{J,I}(h, p)$. By the definition of $V_{I,I}$ and $V_{J,I}$, $V^+_{I,I}(h, a) = V^+_{J,I}(h, a) = 1$ for all a. So $V^-_{I,I}(h, p) > V^-_{J,I}(h, p)$, which means $p^J < p^I$. So $J < I$.

($\Leftarrow$) Suppose $J < I$. Then for every atom a, $a^J \leq a^I$. It follows that $V^-_{J,I}(h, a) \leq V^-_{I,I}(h, a)$ for all a, and as $V^+_{I,I}(h, a) = V^+_{J,I}(h, a) = 1$ for all a, $V_{I,I}(h, a) \subseteq V_{J,I}(h, a)$. Furthermore there is at least one atom p s.t $p^J < p^I$, i.e., $V^-_{J,I}(h, a) < V^-_{I,I}(h, a)$. So $V_{J,I} \prec V_{I,I}$. ∎

Theorem 4 *For any fuzzy formula F (that contains no strong negation) and any interpretation I, I is a 1-stable model of F if and only if $V_{I,I}$ is a fuzzy equilibrium model of F.*

Proof. ($\Rightarrow$) Suppose I is a 1-stable model of F. As I is a model of F, by Corollary 3, $V_{I,I}$ is a model of F. Next we show that there does not exist any other model V' of F such that $V' \prec V_{I,I}$. Suppose, to the contrary, that there exists such V'. Then we define the interpretation J as $a^J = V'^-(h, a)$ for all atoms a. Obviously $V' = V_{J,I}$. As $V' = V_{J,I}$ is a model of F, by Corollary 4, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$, for $\mathbf{p} = \sigma$. Furthermore, as $V' = V_{J,I} \prec V_{I,I}$, by Lemma 7, $J < I$. So I cannot be a 1-stable model of F, which is a contradiction. So there does not exist such V'. So $V_{I,I}$ is an h-minimal model of F and clearly $V_{I,I}(h, a) = V_{I,I}(t, a)$ for all atoms a. So $V_{I,I}$ is an equilibrium model of F.

($\Leftarrow$) Suppose $V_{I,I}$ is an equilibrium model of F. As $V_{I,I}$ is a model of F, by Corollary 3, $I \models_1 F$. Next we show that there does not exist any $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$, for $\mathbf{p} = \sigma$. Suppose, to the contrary, that there exists such J. Then by Corollary 4, the valuation $V_{J,I}$ is a model of F. Furthermore, by Lemma 7, $V_{J,I} \prec V_{I,I}$. So $V_{I,I}$ is not an h-minimal model of F, and thus $V_{I,I}$ cannot be an equilibrium model of F, which is a contradiction. So there does not exist such J. Therefore, I is a 1-stable model of F. ∎

E Proof of Theorem 5

For a valuation V, define valuation V' as $V'^-(w, a) = V^-(w, a)$ and $V'^+(w, a) = 1$ for all atom a, where $w \in \{h, t\}$.

Lemma 8. *For any fuzzy formula F with no strong negation, a valuation V is a model of F iff V' is a model of F.*

Proof. We show by induction that $V^-(w, F) = V'^-(w, F)$, where $w \in \{h, t\}$.

- F is an atom p. $V^-(w, F) = V^-(w, p) = V'^-(w, p) = V'^-(w, F)$.
- F is a numeric constant $\overline{c}$. Clearly $V^-(w, F) = c = V'^-(w, F)$.
- F is $G \odot H$ where $\odot$ is $\otimes$ or $\oplus$. By I.H, $V^-(w, G) = V'^-(w, G)$ and $V^-(w, H) = V'^-(w, H)$. So

$$\begin{aligned} V^-(w, F) &= \odot(V^-(w, G), V^-(w, H)) \\ &= \odot(V'^-(w, G), V'^-(w, H)) \\ &= V'^-(w, F). \end{aligned}$$

- F is $G \rightarrow H$. By I.H, $V^-(w, G) = V'^-(w, G)$ and $V^-(w, H) = V'^-(w, H)$. So

$$\begin{aligned} V^-(h, F) &= min(\odot(V^-(h, G), V^-(h, H)), \odot(V^-(t, G), V^-(t, H))) \\ &= min(\odot(V'^-(h, G), V'^-(h, H)), \odot(V'^-(t, G), V'^-(t, H))) \\ &= V'^-(h, F). \end{aligned}$$

  And

$$\begin{aligned} V^-(t, F) &= \rightarrow (V^-(t, G), V^-(t, H)) \\ &= \rightarrow (V'^-(t, G), V'^-(t, H)) \\ &= V'^-(t, F). \end{aligned}$$

So $V^-(h, F) = V'^-(h, F)$ and thus $V^-(h, F) = 1$ if and only if $V'^-(h, F) = 1$, i.e, V is a model of F if and only if V' is a model of F.

The following is a corollary to Lemma 8.

Corollary 5 *For any two valuations V_1 and V_2, such that $V_1^-(w, a) = V_2^-(w, a)(w \in \{h, t\})$ for all atoms a, and any formula F with no strong negation, V_1 is a model of F iff V_2 is a model of F.*

Proof. By Lemma 8, V_1 is a model of F iff V_1' is a model of F, and V_2 is a model of F iff V_2' is a model of F. Since $V_1^-(w, a) = V_2^-(w, a)$, $V_1' = V_2'$. So V_1 is a model of F iff V_2 is a model of F.

Lemma 9. *Given a formula F with no strong negation, any equilibrium model V of F satisfies $V^+(h, a) = V^+(t, a) = 1$ for all atom a.*

Proof. Assume that V is an equilibrium model of F. It follows that $V^+(h, a) = V^+(t, a)$. Further, for the sake of contradiction, assume that $V^+(h, a) = V^+(t, a) = v < 1$. Define V'' as $V''^-(w, a) = V^-(w, a)$, $V''^+(t, a) = V^+(t, a)$ and $V''^+(h, a) = v'$ where $v' \in (v, 1]$. Clearly $V'' \prec V$ and by Corollary 5, V'' is a model of F. So V is not an h-minimal model of F, and thus V cannot be an equilibrium model of F, which is a contradiction. Therefore, there does not exist such V.

∎

Theorem 5 *For any fuzzy formula F (that contains no strong negation) and any valuation V, we have that V is an equilibrium model of F if and only if*

(i) $V^+(h,a) = V^+(t,a) = 1$ *for all atoms* a*, and and*
(ii) I_V *is a* 1*-stable model of* F *relative to* σ*.*

Proof. ($\Rightarrow$) Suppose V is an equilibrium model of F. By Lemma 9, V satisfies $V^+(h,a) = V^+(t,a) = 1$ for all atoms a. Furthermore, it can be seen that $V = V_{I_V,I_V}$. So V_{I_V,I_V} is an equilibrium model of F. By Theorem 4, I_V is a 1-stable model of F.

($\Leftarrow$) Suppose V satisfies $V^+(h,a) = V^+(t,a) = 1$ for all atom a and I_V is a 1-stable model of F. By Theorem 4, V_{I_V,I_V} is an equilibrium model of F. As $V^+(h,a) = V^+(t,a) = 1$ for all atom a, $V = V_{I_V,I_V}$, so V is an equilibrium model of F. ∎

F Proof of Proposition 2

Lemma 10. *For any fuzzy propositional formula* F *and any two interpretations* I, J *of the underlying signature with* $J \leq^{\mathbf{p}} I$*,* $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} \leq F^I$*.*

Proof. We prove by induction on F. Suppose $J \leq^{\mathbf{p}} I$.

- F is a numerical constant $\bar{c}$. Then $F^*(\mathbf{q}) = F = \bar{c}$. $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = F^I = c$;
- F is an atom $r \notin \mathbf{p}$. Then $F^*(\mathbf{q}) = F = r$. $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = r^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = r^I = F^I$;
- F is an atom $p \in \mathbf{p}$. Then $F^*(\mathbf{q}) = q$, where q is the corresponding atom of p in $\mathbf{q}$. As $J \leq^{\mathbf{p}} I$, $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = q^{J^{\mathbf{p}}_{\mathbf{q}}} = p^J \leq p^I = F^I$;
- $F = \neg G$. Then $F^*(\mathbf{q}) = \neg G = F$. $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (\neg G)^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (\neg G)^I = F^I$;
- $F = G \odot H$, where $\odot$ is $\otimes$ or $\oplus$. Then $F^*(\mathbf{q}) = (G^* \odot H^*)(\mathbf{q})$. $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (G^* \odot H^*)(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (G^*(\mathbf{q}) \odot H^*(\mathbf{q}))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$. By I.H., $G^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} \leq G^I$ and $H^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} \leq H^I$. Since both $\otimes$ and $\oplus$ are increasing, $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = \odot(G^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, H^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}) \leq \odot(G^I, H^I) = F^I$;
- $F = G \rightarrow H$. Then $F^*(\mathbf{q}) = (G^*(\mathbf{q}) \rightarrow H^*(\mathbf{q})) \otimes_m (G \rightarrow H)$. $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = min((G^* \rightarrow H^*)^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, (G \rightarrow H)^I) = min((G^*(\mathbf{q}) \rightarrow H^*(\mathbf{q}))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, F^I) \leq F^I$.

Proposition 2 *For any fuzzy formula* F *and any fuzzy interpretations* I*,* J *with* $J \leq^{\mathbf{p}} I$*,*

- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y \neg F^*(\mathbf{q}) \otimes_m (\neg F)$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (\neg F)$*;*
- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \otimes G^*)(\mathbf{q}) \otimes_m (F \otimes G)$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \otimes G^*)(\mathbf{q})$*;*
- $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \oplus G^*)(\mathbf{q}) \otimes_m (F \oplus G)$ *iff* $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \oplus G^*)(\mathbf{q})$*.*

Proof. For any interpretations I, J with $J \leq^{\mathbf{p}} I$,

- $(\neg F^*(\mathbf{q}) \otimes_m (\neg F))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = min(\neg(F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}), \neg(F^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}))$. By Lemma 10, $F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} \leq F^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$. Since the fuzzy negation $\neg$ is always decreasing, $\neg(F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}) \geq \neg(F^{I\cup J^{\mathbf{p}}_{\mathbf{q}}})$. Therefore, $min(\neg(F^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}), \neg(F^{I\cup J^{\mathbf{p}}_{\mathbf{q}}})) = \neg(F^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}) = \neg(F^I) = (\neg F)^I = (\neg F)^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$. Since $(\neg F^*(\mathbf{q}) \otimes_m (\neg F))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (\neg F)^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y \neg F^*(\mathbf{q}) \otimes_m (\neg F)$ iff $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (\neg F)$.
- For $\odot$ being $\otimes$ or $\oplus$, $((F^* \odot G^*)(\mathbf{q}) \otimes_m (F \odot G))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = min((F^* \odot G^*)(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, (F \odot G)^I) = min((F \odot G)^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, (F \odot G)^I)$. By Lemma 10, $(F \odot G)^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} \leq (F \odot G)^I)$. So $min((F \odot G)^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}, (F \odot G)^I) = (F \odot G)^*(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (F^* \odot G^*)(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$. Since $((F^* \odot G^*)(\mathbf{q}) \odot_m (F \odot G))^{I\cup J^{\mathbf{p}}_{\mathbf{q}}} = (F^* \odot G^*)(\mathbf{q})^{I\cup J^{\mathbf{p}}_{\mathbf{q}}}$, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \odot G^*)(\mathbf{q}) \odot_m (F \odot G)$ iff $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F^* \odot G^*)(\mathbf{q})$.

G Proof of Theorem 6

Theorem 6 *For any fuzzy formulas F and G, I is a 1-stable model of $F \otimes \neg G$ (relative to $\mathbf{p}$) if and only if I is a 1-stable model of F (relative to $\mathbf{p}$) and $I \models_1 \neg G$.*

Proof. $(F \otimes \neg G)^*(\mathbf{q}) = F^*(\mathbf{q}) \otimes \neg G$.

($\Rightarrow$) Suppose I is a 1-stable model of $F \otimes \neg G$ relative to $\mathbf{p}$. Then $I \models_1 F \otimes \neg G$ and there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q}) \otimes \neg G$. By Corollary 2, $I \models_1 F$ and $I \models_1 \neg G$. Further, it can be seen that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$, since otherwise this J must satisfy $J <^{\mathbf{p}} I$ and $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q}) \otimes \neg G$, which is a contradiction. So I is a 1-stable model of F relative to $\mathbf{p}$ and $I \models_1 \neg G$.

($\Leftarrow$) Suppose I is a 1-stable model of F relative to $\mathbf{p}$ and $I \models_1 \neg G$. Then $I \models_1 F$ and $I \models_1 \neg G$,. From the property of fuzzy conjunction, $I \models_1 F \otimes \neg G$[10]. Next we show that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 (F \otimes \neg G)^*(\mathbf{q}) = F^*(\mathbf{q}) \otimes \neg G$. Suppose, to the contrary, that there exists such J. Then by Corollary 2, $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_1 F^*(\mathbf{q})$. As $J <^{\mathbf{p}} I$, this means that I cannot be a 1-stable model of F relative to $\mathbf{p}$, which is a contradiction. So there does not exist such J, and thus I is a 1-stable model of $F \otimes \neg G$ relative to $\mathbf{p}$. ∎

H Proof of Theorem 7

Theorem 7 *For any fuzzy formulas F and G, if I is a y-stable model of $F \otimes \neg G$ (relative to $\mathbf{p}$), then I is a y-stable model of F (relative to $\mathbf{p}$) and $I \models_y \neg G$.*

Proof. $(F \otimes \neg G)^*(\mathbf{q}) = F^*(\mathbf{q}) \otimes \neg G$. Suppose I is a y-stable model of $F \otimes \neg G$ relative to $\mathbf{p}$. Then $I \models_y F \otimes \neg G$ and there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y (F \otimes \neg G)^*(\mathbf{p}) = F^*(\mathbf{p}) \otimes \neg G$. By Lemma 1, $I \models_y F$ and $I \models_y \neg G$. Further, it can be seen that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y F^*(\mathbf{q})$, since otherwise this J must satisfy $J <^{\mathbf{p}} I$ and $I \cup J^{\mathbf{p}}_{\mathbf{q}} \models_y F^*(\mathbf{q}) \otimes \neg G$, which is a contradiction. So I is a y-stable model of F and $I \models_y \neg G$. ∎

I Proof of Theorem 8

Proposition 3 *For any fuzzy interpretation I and any list $\mathbf{p}$ of fuzzy atoms, $I \models_1 Choice(\mathbf{p})$.*[11]

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[10] This does not hold if the threshold considered is not 1. For example, suppose $F^I = 0.5$ and $G^I = 0.5$, and consider $\otimes_l$ as the fuzzy conjunction. Clearly $I \models_{0.5} F$ and $I \models_{0.5} G$ but $I \not\models_{0.5} F \otimes_l G$.

[11] This lemma does not hold for arbitrary fuzzy disjunction. For example, consider $\oplus_m$ as the disjunction. Clearly the interpretation $I = \{(p, 0.5)\} \not\models_1 p \oplus_m \neg_s p$.

Proof. Suppose $\mathbf{p} = (p_1, \dots, p_n)$

$$\begin{aligned} Choice(\mathbf{p})^I &= (Choice(p_1) \otimes \dots \otimes Choice(p_n))^I \\ &= ((p_1 \oplus_l \neg_s p_1) \otimes \dots \otimes (p_n \oplus_l \neg_s p_n))^I \\ &= \overline{min(p_1^I + 1 - p_1^I, 1)} \otimes \dots \otimes \overline{min(p_n^I + 1 - p_n^I, 1)} \\ &= 1 \end{aligned}$$

∎

Theorem 8

(a) If I is a y-stable model of F relative to $\mathbf{p} \cup \mathbf{q}$, then I is a y-stable model of F relative to $\mathbf{p}$.
(b) I is a 1-stable model of F relative to $\mathbf{p}$ iff I is a 1-stable model of $F \otimes Choice(\mathbf{q})$ relative to $\mathbf{p} \cup \mathbf{q}$.[12]

Proof. (a) Suppose I is a y-stable model of F relative to $\mathbf{p} \cup \mathbf{q}$. Then clearly $I \models_y F$. Next we show that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{u}} \models_y F^*(\mathbf{u})$. Suppose, to the contrary, that there exists such J. Then since for all $q \in \mathbf{q}$, $q^J = q^I$, J must satisfy $J <^{\mathbf{pq}} I$ and $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_y F^*(\mathbf{u}, \mathbf{v})$, which is a contradiction. So such J cannot exist. So I is a y-stable model of F relative to $\mathbf{p}$.

(b) ($\Rightarrow$) Suppose I is a 1-stable model of F relative to $\mathbf{p}$. Clearly $I \models_1 F$. By Proposition 3, $I \models_1$ *Choice*$(\mathbf{q})$, so by the property of fuzzy conjunctions, $I \models_1 F \otimes$ *Choice*$(\mathbf{q})$.[13] Next we show that there does not exist $J <^{\mathbf{p} \cup \mathbf{q}} I$ such that $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 (F \otimes Choice(\mathbf{q}))^*(\mathbf{u}, \mathbf{v})$.

Suppose, to the contrary, there exists such J. Since $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 (F \otimes Choice(\mathbf{q}))^*(\mathbf{u}, \mathbf{v}) = F^*(\mathbf{u}, \mathbf{v}) \otimes (v_1 \oplus_l \neg_s q_1) \otimes \dots \otimes (v_n \oplus_l \neg_s q_n)$, where $v_1, \dots, v_n$ are atoms in $\mathbf{v}$ that correspond to $q_1, \dots, q_n$, respectively, for each $k = 1, \dots, n$, $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 (v_k \oplus_l \neg_s q_k)$, which means that $v_k^{I \cup J^{\mathbf{pq}}_{\mathbf{uv}}} \geq q_k^I$. By definition, $v_k^{I \cup J^{\mathbf{pq}}_{\mathbf{uv}}} = v_k^{J^{\mathbf{pq}}_{\mathbf{uv}}} = v_k^{J^{\mathbf{q}}_{\mathbf{v}}} = q_k^J$, so $q_k^J \geq q_k^I$. Since $J <^{\mathbf{pq}} I$, $q_k^J \leq q_k^I$. So $q_k^J = q_k^I$.[14] So there is at least one atom $p \in \mathbf{p}$ such that $p^J < p^I$, and since $q^J = q^I$ for all $q \in \mathbf{q}$, $J <^{\mathbf{p}} I$. Since $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 (F \otimes Choice(\mathbf{q})))^*(\mathbf{u}, \mathbf{v})$, $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} = I \cup J^{\mathbf{p}}_{\mathbf{u}} \models_1 F^*(\mathbf{u}, \mathbf{v}) = F^*(\mathbf{u})$, which contradicts that $I \models_1$ SM $[F; \mathbf{p}]$. So there does not exist such J. So $I \models_1$ SM $[F; \mathbf{p}, \mathbf{q}]$.

($\Leftarrow$) Suppose I is a 1-stable model of $F \otimes Choice(\mathbf{q})$ relative to $\mathbf{p} \cup \mathbf{q}$. Then $I \models_1 F \otimes Choice(\mathbf{q})$. By Corollary 2, $I \models_1 F$. Next we show that there does not exist $J <^{\mathbf{p}} I$ such that $I \cup J^{\mathbf{p}}_{\mathbf{u}} \models_1 F^*(\mathbf{u})$. Suppose to the contrary, there exists such J. Since $J <^{\mathbf{p}} I$, $J <^{\mathbf{pq}} I$. By Proposition 3, $I \cup J^{\mathbf{p}}_{\mathbf{u}} \models_1 Choice(\mathbf{q})$. Since $I \cup J^{\mathbf{p}}_{\mathbf{u}} \models_1$

[12] This does not hold for arbitrary threshold y. For example, consider $F = \neg_s \neg_s q$ and $I = \{(q, 0.5)\}$. Clearly I is a 0.5-model of F, and thus I is a 0.5-stable model of F relative to $\emptyset$. However, I is not a 0.5-stable model of $F \otimes_m$ *Choice*$(q) = \neg_s \neg_s q \otimes_m (q \oplus_l \neg_s q)$ relative to $\emptyset \cup \{q\}$, as witnessed by $J = \{(q, 0)\}((F \otimes_m Choice(q))^*(u) = \neg_s \neg_s q \otimes_m (u \oplus_l \neg_s q))$.

[13] It is not necessary to have the threshold $y = 1$ for this to hold. In general, suppose $I \models_y F$. Since $I \models_1 Choice(\mathbf{p})$, by the property of fuzzy conjunction($\otimes(x, 1) = x$), $I \models_y F \otimes Choice(\mathbf{p})$.

[14] This cannot be concluded if we have $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_y (v_k \oplus_l \neg_s q_k)$, instead of $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 (v_k \oplus_l \neg_s q_k)$. So this direction of the theorem does not hold for arbitrary threshold.

$F^*(\mathbf{u})$, by the property of fuzzy conjunctions, and since I and J agree on all atoms in $\mathbf{q}$, $I \cup J^{\mathbf{p}}_{\mathbf{u}} = I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 F^*(\mathbf{u}, \mathbf{v}) \otimes Choice(\mathbf{q})$. Since J and I agree on all atoms in $\mathbf{q}$, $Choice(\mathbf{q})^{I \cup J^{\mathbf{pq}}_{\mathbf{uv}}} = (Choice(\mathbf{q}))^*(\mathbf{u}, \mathbf{v})^{I \cup J^{\mathbf{pq}}_{\mathbf{uv}}}$. So $I \cup J^{\mathbf{pq}}_{\mathbf{uv}} \models_1 F^* \otimes (Choice(\mathbf{q}))^*(\mathbf{u}, \mathbf{v})$ which contradicts $I \models_1 \text{SM}[F \otimes Choice(\mathbf{q}); \mathbf{p}, \mathbf{q}]$. So there does not exist such J. So $I \models_1 \text{SM}[F; \mathbf{p}]$. ∎

J Representing Strong Negation

Let σ denotes the signature. For a formula F over σ, involving strong negation of atoms only, define F' over $\sigma \cup \{np | p \in \sigma\}$ as the formula obtained from F by replacing all strong negation of atom $\sim p$ with a new atom np. The transformation $nneg(F)$ is defined as $nneg(F) = F' \otimes_m \underset{p \in \sigma}{\otimes_m} \neg_s (p \otimes_l np)$.

For a valuation V of σ, define the valuation $nneg(V)$ of $\sigma \cup \{np | p \in \sigma\}$ as $nneg(V)(w, p) = [V^-(w, p), 1]$ and $nneg(V)(w, np) = [1 - V^+(w, p), 1]$ for all atoms $p \in \sigma$. Clearly for every valuation V of σ, there exists a corresponding interpretation $I_{nneg(V)}$ of $\sigma \cup \{np | p \in \sigma\}$. On the other hand, there exists interpretation I of $\sigma \cup \{np | p \in \sigma\}$ for which there is no corresponding valuation V of σ such that $I = I_{nneg(V)}$.

Example 12. Suppose $\sigma = \{p\}$. The interpretation I of $\sigma \cup \{np | p \in \sigma\}$, defined by $I = \{(p, 0.6), (np, 0.5)\}$, has no corresponding valuation V of σ such that $I = I_{nneg(V)}$. Those valuations V characterized by $V(h, p) = [0.6, 0.5]$ are not valid valuations since $[0.6, 0.5]$ is not a valid interval.

Lemma 11. *For F involving strong negation of atoms only, $V^-(w, F) = nneg(V)^-(w, nneg(F))$.*

Proof. First we show by induction that $V^-(w, F) = nneg(V)^-(w, F')$.

- F is atom p.

$$\begin{aligned} V^-(w, F) &= V^-(w, p) \\ &= nneg(V)^-(w, p) \\ &= nneg(V)^-(w, F'). \end{aligned}$$

- F is $\sim p$, where p is an atom. Then F' is np.

$$\begin{aligned} V^-(w, F) &= V^-(w, \sim p) \\ &= 1 - V^+(w, p) \\ &= nneg(V)^-(w, np) \\ &= nneg(V)^-(w, F'). \end{aligned}$$

- F is a numerical constant $\overline{c}$. $V^-(w, F) = V^-(w, \overline{c}) = c = nneg(V)^-(w, \overline{c}) = nneg(V)^-(w, F')$.

– F is $\neg G$. By I.H., $V^-(w, G) = nneg(V)^-(w, G')$.

$$\begin{aligned} V^-(w, F) &= V^-(w, \neg G) \\ &= 1 - V^-(t, G) \\ &= 1 - nneg(V)^-(t, G') \\ &= nneg(V)^-(w, \neg G') \\ &= nneg(V)^-(w, F'). \end{aligned}$$

– F is $G \odot H$, where $\odot$ is $\otimes$ or $\oplus$. By I.H., $V^-(w, G) = nneg(V)^-(w, G')$ and $V^-(w, H) = nneg(V)^-(w, H')$.

$$\begin{aligned} V^-(w, F) &= V^-(w, G \odot H) \\ &= \odot(V^-(w, G), V^-(w, H)) \\ &= \odot(nneg(V)^-(w, G'), nneg(V)^-(w, H')) \\ &= nneg(V)^-(w, G' \odot H') \\ &= nneg(V)^-(w, F'). \end{aligned}$$

– F is $G \to H$. By I.H., $V^-(w, G) = nneg(V)^-(w, G')$ and $V^-(w, H) = nneg(V)^-(w, H')$.

$$\begin{aligned} V^-(h, F) &= V^-(h, G \to H) \\ &= min(\to (V^-(h, G), V^-(h, H)), \to (V^-(t, G), V^-(t, H))) \\ &= min(\to (nneg(V)^-(h, G), nneg(V)^-(h, H)), \to (nneg(V)^-(t, G), nneg(V)^-(t, H))) \\ &= nneg(V)^-(h, G' \to H') \\ &= nneg(V)^-(h, F'). \end{aligned}$$

And

$$\begin{aligned} V^-(t, F) &= V^-(t, G \to H) \\ &= \to (V^-(t, G), V^-(t, H)) \\ &= \to (nneg(V)^-(t, G'), nneg(V)^-(t, H')) \\ &= nneg(V)^-(t, F'). \end{aligned}$$

Now notice that, for any valuation V, it must be the case that for all atoms p, $V^-(w, p) \leq V^+(w, p)$, i.e, $V^-(w, p) + 1 - V^+(w, p) \leq 1$. It follows that $nneg(V)^-(w, p) + nneg(V)^-(w, np) \leq 1$. Therefore, we have that for all atoms p,

$$\begin{aligned} nneg(V)^-(w, \neg_s(p \otimes_l np)) &= 1 - nneg(V)^-(w, (p \otimes_l np)) \\ &= 1 - \otimes_m(nneg(V)^-(w, p), nneg(V)^-(w, np)) \\ &= 1 - max(nneg(V)^-(w, p) + nneg(V)^-(w, np) - 1, 0) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

It follows that

$$\begin{aligned}
V^{-}(w,F) &= nneg(V)^{-}(w,F') \\
&= \otimes_m(nneg(V)^{-}(w,F'),1) \\
&= \otimes_m(nneg(V)^{-}(w,F'), nneg(V)^{-}(w, \underset{p\in\sigma}{\otimes_m}\neg_s(p\otimes_l np))) \\
&= nneg(V)^{-}(w, F'\otimes_m \underset{p\in\sigma}{\otimes_m}\neg_s(p\otimes_l np)) \\
&= nneg(V)^{-}(w, nneg(F))
\end{aligned}$$

∎

Corollary 6 *For F involving strong negation of atoms only, a valuation V is a model of F iff $nneg(V)$ is a model of $nneg(F)$.*

Proof. By Lemma 11, $V^{-}(h,F) = nneg(V)^{-}(h, nneg(F))$. So $V^{-}(h,F) = 1$ iff $nneg(V)^{-}(h, nneg(F)) = 1$, i.e., V is a model of F iff $nneg(V)$ is a model of $nneg(F)$. ∎

Lemma 12. *for all atoms $a\in\sigma$, $V(h,a) = V(t,a)$ iff $nneg(V)(h,a) = nneg(V)(t,a)$ and $nneg(V)(h,na) = nneg(V)(t,na)$.*

Proof. For all atoms $a\in\sigma$, we have

$V(h,a) = V(t,a) \iff$
$V^{-}(h,a) = V^{-}(t,a)$ and $V^{+}(h,a) = V^{+}(t,a) \iff$
$nneg(V)(h,a) = nneg(V)(t,a)$ and $1 - V^{+}(h,a) = 1 - V^{+}(t,a) \iff$
$nneg(V)(h,a) = nneg(V)(t,a)$ and $nneg(V)(h,na) = nneg(V)(t,na)$.

∎

Lemma 13. *For two valuations V and V_1, $V_1 \prec V$ iff $nneg(V_1) \prec nneg(V)$*

Proof.

$V_1 \prec V \iff$
$(\forall p : V(t,p) = V_1(t,p) \wedge (V_1(h,p) \subseteq V(h,p)) \wedge$
$(\exists a : V_1(h,a) \subset V(h,a)) \iff$
$(\forall p : V^-(t,p) = V_1^-(t,p) \wedge V^+(t,p) = V_1^+(t,p) \wedge V_1^-(h,p) \leq V^-(h,p) \wedge V_1^+(h,p) \geq V^+(h,p))$
$\wedge (\exists a : V_1^-(h,a) < V^-(h,a) \vee V_1^+(h,a) > V^+(h,a)) \iff$
$(\forall p : V^-(t,p) = V_1^-(t,p) \wedge V^+(t,p) = V_1^+(t,p) \wedge V_1^-(h,p) \leq V^-(h,p) \wedge V_1^+(h,p) \geq V^+(h,p))$
$\wedge (\exists a : V_1^-(h,a) < V^-(h,a) \vee 1 - V_1^+(h,a) < 1 - V^+(h,a)) \iff$
$(\forall p : V^-(t,p) = V_1^-(t,p) \wedge V^+(t,p) = V_1^+(t,p) \wedge V_1^-(h,p) \leq V^-(h,p) \wedge V_1^+(h,p) \geq V^+(h,p))$
$\wedge (\exists a : V_1^-(h,a) < V^-(h,a) \vee nneg(V_1)^-(h,a) < nneg(V)^-(h,a)) \iff$
$(\forall p : nneg(V)(t,p) = nneg(V_1)(t,p) \wedge nneg(V)(t,np) = nneg(V_1)(t,np) \wedge$
$nneg(V)(h,p) \subseteq nneg(V_1)(h,p) \wedge$
$nneg(V)(h,np) \subseteq nneg(V_1)(h,np)) \wedge$
$(\exists a : nneg(V)(h,a) \subset nneg(V_1)(h,a) \vee nneg(V)(h,na) \subset nneg(V_1)(h,na)) \iff$
$nneg(V_1) \prec nneg(V)$.

∎

Lemma 14. *For F involving strong negation of atoms only, a valuation V is an equilibrium model of F iff $nneg(V)$ is an equilibrium model of $nneg(F)$.*

Proof. Let σ be the underlying signature of F, σ' be the extended signature $\sigma \cup \{np | p \in \sigma\}$. ($\Rightarrow$) Suppose V is an equilibrium model of F. Then V satisfies $V(h,a) = V(t,a)$ for all atoms $a \in \sigma'$ and V is a model of F. By Lemma 12 and Corollary 6, $nneg(V)$ satisfies $nneg(V)(h,a) = nneg(V)(t,a)$ for all atoms a and $nneg(V)$ is a model of $nneg(F)$. Next we show that there does not exists $V_1 \prec nneg(V)$ such that V_1 is a model of $nneg(F)$. Suppose, to the contrary, there exists such V_1. First note that by the definition of $nneg(V)$, for all atoms $a \in \sigma'$, $nneg(V)(w,a)^+ = 1$, which means $V_1(w,a)^+ = 1$. Second, since V_1 is a model of $nneg(F)$, V_1 is a model of $\bigotimes_{m\, p\in\sigma} \neg_s(p \otimes_l np)$, which guarantees that for all atoms $p \in \sigma$, $V_1(h,p)^- + V_1(h,np)^- \leq 1$; furthermore, by $V_1 \prec nneg(V)$, for all atoms $a \in \sigma'$, $V_1(t,a) = nneg(t,a)$, and thus $V_1(t,p)^- + V_1(t,np)^- \leq 1$. The above two conditions guarantee that there exists a valuation V' such that $V_1 = nneg(V')$. Since $V_1 \prec nneg(V)$, $nneg(V') \prec nneg(V)$. By Lemma 13, $V' \prec V$; by Corollary 6, V' is a model of F. Therefore, V cannot be an equilibrium model of F. So there does not exiss such V_1. So $nneg(V)$ is an equilibrium model of F'.

($\Leftarrow$) Suppose $nneg(V)$ is an equilibrium model of $nneg(F)$. Then $nneg(V)$ satisfies $nneg(V)(h,a) = nneg(V)(t,a)$ for all atoms $a \in \sigma'$ and $nneg(V)$ is a model of $nneg(F)$. By Lemma 12 and Corollary 6, V satisfies $V(h,a) = V(t,a)$ for all atoms $a \in \sigma'$ and V is a model of F. Next we show that there does not exists $V_1 \prec V$ such that V_1 is a model of F. Suppose, to the contrary, there exists such V_1. Then by Lemma

13 and Corollary 6, $nneg(V_1) \prec nneg(V)$ and $nneg(V_1)$ is a model of $nneg(F)$. So $nneg(V)$ cannot be an equilibrium model of $nneg(F)$, which is a contradiction. So there does not exist such V_1. So V is an equilibrium model of F. ∎

Theorem 9 *If V is an equilibrium model of F, then $I_{nneg(V)}$ is a 1-stable model of $nneg(F)$.*

Proof. Suppose V is an equilibrium model of F. By Lemma 14, $nneg(V)$ is an equilibrium model of $nneg(F)$. By Theorem 5 in the main body, $I_{nneg(V)}$ is a 1-stable model of $nneg(F)$. ∎

Theorem 10 *If I is a 1-stable model of $nneg(F)$, then there exists V such that $I = I_{nneg(V)}$ and V is an equilibrium model of F.*

Proof. Suppose I is a 1-stable model of $nneg(F)$. Then $I \models_1 \underset{p\in\sigma}{\otimes_m} \neg_s(p \otimes_l np)$, where σ is the underlying signature of F. It follows that for all atoms $p \in \sigma$, $p^I + np^I \leq 1$ and thus $p^I \leq 1 - np^I$. Construct the valuation V of σ by defining $V(w,p) = \left[p^I, 1 - np^I\right]$. Clearly $I = I_{nneg(V)}$. By Theorem 4 in the main body, $V_{I,I}$ is an equilibrium model of $nneg(F)$. From the definition of $V_{I,I}$, it can be seen that $V_{I,I} = nneg(V)$. So $nneg(V)$ is an equilibrium model of $nneg(F)$. By Lemma 14, V is an equilibrium model of F. ∎

From Theorem 9 and Theorem 10, we know that there is a 1-1 correspondence of the equilibrium models of a formula F and the 1-stable models of $nneg(F)$. The next example illustrates how to represent strong negation of atoms in fuzzy stable model semantics.

Example 13. Consider the formula

$$\begin{aligned} F = &(\overline{0.2} \rightarrow_r p) \otimes_m \\ &(\overline{0.3} \rightarrow_r \sim p). \end{aligned}$$

The formula $nneg(F)$ is

$$\begin{aligned} nneg(F) = &(\overline{0.2} \rightarrow_r p) \otimes_m \\ &(\overline{0.3} \rightarrow_r np) \otimes_m \\ &\neg_s(p \otimes_l np). \end{aligned}$$

One can check that the valuation V defined as $V(w,p) = [0.2, 0.7]$ is the only equilibrium model of F. By definition, $nneg(V)(w,p) = [0.2, 1]$ and $nneg(V)(w,np) = [0.3, 1]$. The interpretation $I_{nneg(V)} = \{(p, 0.2), (np, 0.3)\}$ is the only 1-stable model of $nneg(F)$.