7. Logic of Here-and-There and Strong Equivalence

1 Natural Deduction

In the natural deduction system **N**, the derivable objects are *sequents*—expressions of the form $\Gamma \Rightarrow F$ ("F under the assumptions Γ"), where F is a formula and Γ is a finite set of formulas. Notationally, we will identify the set of assumption in a sequent with the list of its elements. For instance, we will write

$$\begin{array}{rcl} F, G \Rightarrow H & \text{for} & \{F, G\} \Rightarrow H, \\ \Gamma, F \Rightarrow G & \text{for} & \Gamma \cup \{F\} \Rightarrow G, \\ \Gamma, \Delta \Rightarrow F & \text{for} & \Gamma \cup \Delta \Rightarrow G. \end{array}$$

The axiom schemas of **N** are

$$F \Rightarrow F \tag{1}$$

and

$$\Rightarrow F \vee \neg F. \tag{2}$$

The latter is called *the law of excluded middle.* The inference rules of **N** are:

$$(\wedge I)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow F} \quad \frac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow G}$$

$$(\vee I)\ \frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow F \vee G} \quad \frac{\Gamma \Rightarrow G}{\Gamma \Rightarrow F \vee G} \qquad (\vee E)\ \frac{\Gamma \Rightarrow F \vee G \quad \Delta_1, F \Rightarrow H \quad \Delta_2, G \Rightarrow H}{\Gamma, \Delta_1, \Delta_2 \Rightarrow H}$$

$$(\to I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G} \qquad (\to E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$$

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F}$$

$$(W)\ \frac{\Gamma \Rightarrow F}{\Gamma' \Rightarrow F}$$

if $\Gamma \subseteq \Gamma'$

Among the first six inference rules, the rules in the left column are *introduction rules*, and the rules in the right column are *elimination rules*. Rule (C) is the *contradiction rule*, and (W) is *weakening*.

Since we defined $\neg F$ as an abbreviation for $F \to \bot$ (Handout 3), "negation introduction"

$$\frac{\Gamma, F \Rightarrow \bot}{\Gamma \Rightarrow \neg F}$$

is a special case of $(\to I)$, and "negation elimination"

$$\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow \neg F}{\Gamma, \Delta \Rightarrow \bot}$$

is a special case of $(\to E)$. Similarly, the introduction and elimination rules for equivalence

$$\frac{\Gamma \Rightarrow F \to G \quad \Delta \Rightarrow G \to F}{\Gamma, \Delta \Rightarrow F \leftrightarrow G} \qquad \frac{\Gamma \Rightarrow F \leftrightarrow G}{\Gamma \Rightarrow F \to G} \qquad \frac{\Gamma \Rightarrow F \leftrightarrow G}{\Gamma \Rightarrow G \to F}$$

1.	$p \to (q \to r) \Rightarrow p \to (q \to r)$	— axiom	Assume $p \to (q \to r)$. Now our goal is to prove $(p \wedge q) \to r$.
2.	$p \wedge q \Rightarrow p \wedge q$	— axiom	Assume $p \wedge q$. Now our goal is to prove r.
3.	$p \wedge q \Rightarrow p$	— by $(\wedge E)$ from 2	From the second assumption, p.
4.	$p \to (q \to r),\ p \wedge q \Rightarrow q \to r$	— by $(\to E)$ from 3, 1	Consequently, by the first assumption, $q \to r$.
5.	$p \wedge q \Rightarrow q$	— by $(\wedge E)$ from 2	On the other hand, from the second assumption, q.
6.	$p \to (q \to r),\ p \wedge q \Rightarrow r$	— by $(\to E)$ from 5, 4	Consequently r.
7.	$p \to (q \to r) \Rightarrow (p \wedge q) \to r$	— by $(\to I)$ from 6	Thus we proved $(p \wedge q) \to r$
8.	$\Rightarrow (p \to (q \to r)) \to ((p \wedge q) \to r)$	— by $(\to I)$ from 7	and consequently we are done.

Figure 1: A proof of $(p \to (q \to r)) \to ((p \wedge q) \to r)$ in propositional logic

1.	$\neg p \to q \Rightarrow \neg p \to q$	— axiom	Assume $\neg p \to q$. Now our goal is to prove $\neg q \to p$.
2.	$\neg q \Rightarrow \neg q$	— axiom	Assume $\neg q$. Now our goal is to prove p.
3.	$\Rightarrow p \vee \neg p$	— axiom	Consider two cases.
4.	$p \Rightarrow p$	— axiom	Case 1: p. This case is trivial.
5.	$\neg p \Rightarrow \neg p$	— axiom	Case 2: $\neg p$.
6.	$\neg p \to q,\ \neg p \Rightarrow q$	— by $(\to E)$ from 5, 1	Then, by the first assumption, q.
7.	$\neg p \to q,\ \neg q,\ \neg p \Rightarrow \bot$	— by $(\neg E)$ from 6, 2	This contradicts the second assumption,
8.	$\neg p \to q,\ \neg q,\ \neg p \Rightarrow p$	— by (T) from 7	so that we can conclude p also.
9.	$\neg p \to q,\ \neg q \Rightarrow p$	— by $(\vee E)$ from 3, 4, 8	Thus, in either case, p.
10.	$\neg p \to q \Rightarrow \neg q \to p$	— by $(\to I)$ from 9	We have proved $\neg q \to p$
11.	$\Rightarrow (\neg p \to q) \to (\neg q \to p)$	— by $(\to I)$ from 10	and consequently we are done.

Figure 2: A proof of $(\neg p \to q) \to (\neg q \to p)$ in propositional logic

are special cases of $(\wedge I)$ and $(\wedge E)$.

To prove a formula F in system **N** means to prove the sequent $\Rightarrow F$.

Figure 1 shows a proof of the formula

$$(p \to (q \to r)) \to ((p \wedge q) \to r)$$

written in this manner, with the corresponding "informal proof" to the right of the bar.

Figure 2 is a proof of

$$(\neg p \to q) \to (\neg q \to p)$$

in **N**, again with an "English translation."

7.1 Prove $\neg\neg F \leftrightarrow F$ in **N**.

Soundness and Completeness Theorem for Propositional Logic *A formula F is provable in* **N** *iff F is a tautology.*

2 Intuitionistic Provability

A sequent or a formula is *intuitionistically provable* if it can be proved in system **N** without axiom schema (2). A formula F is *intuitionistically equivalent* to a formula G if $F \leftrightarrow G$ is intuitionistically provable.

For instance, $F \rightarrow \neg\neg F$ is intuitionistically provable, while $\neg\neg F \rightarrow F$ is known to be *not* intuitionistically provable. The implication cannot be proved without the law of excluded middle, in general.

It's easy to check that the formulas

$$\neg(F \wedge G), \quad F \rightarrow \neg G, \quad G \rightarrow \neg F \tag{3}$$

are intuitionistically equivalent to each other. But it is known that the formulas

$$F \vee G, \quad \neg F \rightarrow G, \quad \neg G \rightarrow F \tag{4}$$

are not intuitionistically equivalent to each other if, for instance, F and G are two different atoms.

Intuitionistic provability, which plays an important role in the theory of answer sets, was introduced in [Heyting, 1930], long before the invention of logic programming.

In Problems 7.2–7.6, prove the given formulas in intuitionistic logic.

7.2 $F \wedge G \leftrightarrow G \wedge F$, $F \vee G \leftrightarrow G \vee F$.

7.3 $F \wedge \top \leftrightarrow F$, $F \vee \bot \leftrightarrow F$.

7.4 $(F \rightarrow \neg G) \leftrightarrow \neg(F \wedge G)$.

7.5 $\neg(F \vee G) \leftrightarrow \neg F \wedge \neg G$.

7.6 $\neg\neg\neg F \leftrightarrow \neg F$.

Even if F is equivalent to G in propositional logic, their answer sets may not be the same. (Remember examples?) However, if they are equivalent in intuitionistic logic, they have the same answer sets.

Theorem on Intuitionistic Equivalence. *If F is intuitionistically equivalent to G, then F and G have the same answer sets.*

We will now state several theorems that relate intuitionistic provability to the concept of a tautology.

Replacement Theorem for Intuitionistic Logic. *Let F, G, F', G' be formulas such that G' is obtained from F' by replacing an occurrence of F*

with G. (a) The sequent

$$F \leftrightarrow G \Rightarrow F' \leftrightarrow G'$$

is intuitionistically provable. (b) If F is intuitionistically equivalent to G then F' is intuitionistically equivalent to G'.

The replacement theorem shows, for instance, that replacing a subformula of one of the forms (3) in any formula with another expression from that list will produce an intuitionistically equivalent result.

Glivenko's Theorem. If $\bigwedge\Gamma \rightarrow \neg F$ is a tautology then $\Gamma \rightarrow \neg F$ is provable in intuitionistic logic.

For instance, from Glivenko's theorem we can conclude that, for any tautology F, $\neg\neg F$ is provable intuitionistically. As another example of the use of Glivenko's theorem, we can establish that the formula from Problem 7.4 is intuitionistically provable by observing that it can be derived intuitionistically from the sequents

$$F \rightarrow \neg G \Rightarrow \neg(F \wedge G)$$

and

$$F, \neg(F \wedge G) \Rightarrow \neg G.$$

Each of these sequents is intuitionistically provable by Glivenko's theorem.

7.7 Apply Glivenko's theorem to Problems 7.5 and 7.6.

Theorem on Excluded Middle. *Let $A_1, \ldots, A_n$ be all atoms occurring in a formula F. The sequent*

$$\{A_1, \ldots, A_n\}^c \Rightarrow F$$

is intuitionistically provable iff F is a tautology.

For instance, the sequent

$$p \vee \neg p \Rightarrow \neg\neg p \leftrightarrow p$$

is intuitionistically provable because $\neg\neg p \leftrightarrow p$ is a tautology.

The following fact can be sometimes used to show that a formula is *not* provable intuitionistically,

Disjunction Property. If $\Gamma \Rightarrow F \vee G$ is intuitionistically provable and Γ does not contain $\vee$, then at least one of the sequents $\Gamma \Rightarrow F$, $\Gamma \Rightarrow G$ is intuitionistically provable also.

For instance, the disjunction property shows that $F \vee G$ can be intuitionistically provable only if at least one of the formulas F, G is a tautology. In particular, $p \vee \neg p$ (where p is an atom) is not intuitionistically provable.

7.8 Show that the formulas

$$\begin{gathered}\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q \\ p \rightarrow q \leftrightarrow \neg p \vee q \\ (p \leftrightarrow q) \leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)\end{gathered}$$

are not provable intuitionistically.

7.9 Show that $\neg\neg p \rightarrow p$ is not provable intuitionistically.

3 Logic of Here-and-There

The logic of here-and-there was originally proposed by the inventor of intuitionistic logic Arend Heyting as a technical tool for the purpose of proving that intuitionistic logic is weaker than classical logic [Heyting, 1930].

Natural Deduction for the Logic of Here-and-There

The logic of here-and-there can be obtained from system **N** by replacing the law of the excluded middle

$$\Rightarrow F \vee \neg F$$

with the axiom schema

$$\Rightarrow F \vee (F \rightarrow G) \vee \neg G.$$

7.10 (a) Prove this sequent in classical logic. (b) Show that some instances of this schema are not provable intuitionistically.

In the following two problems, prove the given formula in the logic of here-and-there.

7.11 $\neg F \vee \neg\neg F$.

7.12 $\neg(F \wedge G) \leftrightarrow \neg F \vee \neg G$.

HT-interpretation

The logic of here-and-there can be explained by a 3-valued logic. Heyting remarks that the truth values in his truth tables "can be interpreted as follows: 0 denotes a correct proposition, 1 denotes a false proposition, and 2 denotes a proposition that cannot be false but whose correctness is not proved."

An *HT-interpretation* is an ordered pair $\langle Y, X\rangle$ of sets of atoms such that $Y \subseteq X$. The *satisfaction* relation $\models$ between an HT-interpretation $\langle Y, X\rangle$ and a formula F is defined recursively:

- for an atom A, $\langle Y, X\rangle \models A$ if $A \in Y$;
- $\langle Y, X\rangle \not\models \bot$;
- $\langle Y, X\rangle \models F \wedge G$ if $\langle Y, X\rangle \models F$ and $\langle Y, X\rangle \models G$;
- $\langle Y, X\rangle \models F \vee G$ if $\langle Y, X\rangle \models F$ or $\langle Y, X\rangle \models G$;
- $\langle Y, X\rangle \models F \rightarrow G$ if
  (i) $\langle Y, X\rangle \not\models F$ or $\langle Y, X\rangle \models G$, and
  (ii) $X \models F \rightarrow G$.

(The symbol $\models$ in the last line refers to the satisfaction relation of classical logic defined in Handout 1.)

A formula is *valid in the logic of here-and-there* if it is satisfied by every HT-interpretation. A formula F is *equivalent* to a formula G *in the logic of here-and-there* if $F \leftrightarrow G$ is valid in the logic of here-and-there (or, equivalently, if F and G are satisfied by the same HT-interpretations).

The following facts relate the satisfaction relation of the logic of here-and-there to the satisfaction relation of classical logic:

- $$\langle X, X\rangle \models F \text{ iff } X \models F. \tag{5}$$
- $$\text{If } \langle Y, X\rangle \models F \text{ then } X \models F.$$
- $$\langle Y, X\rangle \models \neg F \text{ iff } X \models \neg F. \tag{6}$$

From property (5) we see that a formula can be valid in the logic of here-and-there only if it is a tautology. It follows that two formulas can be equivalent to each other in the logic of here-and-there only if they are classically equivalent. To see where the two equivalence relations differ from each other, note that $\neg\neg p$ is not equivalent to p in the logic of here-and-there. Indeed, by (6), the HT-interpretation $\langle\emptyset, \{p\}\rangle$ satisfies $\neg\neg p$, but it clearly does not satisfy p.

Soundness and Completeness Theorem for the Logic of Here-and-There. *A formula F is provable in the logic of here-and-there iff F is satisfied by every HT-interpretation.*

7.13 Verify the assertion of the theorem for the formulas

$$(p \wedge q) \rightarrow p,$$
$$(p \wedge \neg p) \rightarrow q.$$

7.14 Use the theorem to determine, for each of the formulas

$$p \vee \neg p,$$
$$\neg p \vee \neg\neg p,$$

whether it is provable in the logic of here-and-there.

Answer sets can be characterized in terms of HT-interpretation as follows.

Theorem on Answer Sets and HT-interpretation For any formula F, a set X of atoms is an answer set of F iff

$$\langle X, X\rangle \models F, \text{ and, for all proper subsets } Y \text{ of } X, \langle Y, X\rangle \not\models F. \qquad (7)$$

The fact that the logic of here-and-there is related to the concept of an answer set was discovered by David Pearce [1997]. He calls the HT-interpretation $\langle X, X\rangle$ that satisfies (7) *equilibrium model*. The theorem above is due to Paolo Ferraris [2005], and it follows from the following lemma.

7.15 For any formula F, and for any sets X, Y of atoms such that $Y \subseteq X$,

$$Y \models F^X \text{ iff } \langle Y, X\rangle \models F\ .$$

4 Strong Equivalence

We say that a formula F is *strongly equivalent* to a formula G if any formula H that contains an occurrence of F has the same stable models as the formula obtained from H by replacing that occurrence with G. The theorem on intuitionistic equivalence and the replacement theorem for intuitionistic logic show that intuitionistically equivalent formulas are strongly equivalent. The other direction does not hold in general.

Theorem on Strong Equivalence For any formulas F and G, the following conditions are equivalent to each other.

(i) F is strongly equivalent to G;

(ii) F is equivalent to G in the logic of here-and-there;

(iii) For all sets X of atoms, F^X is equivalent to G^X in propositional logic.

7.16 The program

$$\begin{array}{l} \{p\}^c \\ \leftarrow \textit{not } p \end{array}$$

is strongly equivalent to

$$\begin{array}{l} \{p\}^c \\ p. \end{array}$$

7.17 Determine whether the program

$$\begin{array}{l} p \leftarrow q \\ p \leftarrow \textit{not } q \end{array}$$

is strongly equivalent to p.

7.18 The program

$$\begin{array}{l} F; G \\ \leftarrow F, G \end{array}$$

is strongly equivalent to

$$\begin{array}{l} F \leftarrow \textit{not } G \\ G \leftarrow \textit{not } F \\ \leftarrow F, G. \end{array}$$

7.19 Any two constraints $\leftarrow F$ and $\leftarrow G$ are strongly equivalent to each other if and only if F is equivalent to G in classical logic. True or false?

To simplify a formula F means to find a formula that is "simpler" than F but is strongly equivalent to F.

7.20 Simplify the program

$$\begin{array}{l} \{p, q, r\}^c \\ \leftarrow \textit{not } p. \end{array}$$

7.21 Replacing the formula

$$\{p, q, r\} \leq 2$$

in any program with

$$1 \leq \{\textit{not } p, \textit{not } q, \textit{not } r\}$$

does not affect the program's answer sets. True or false?

7.22 Simplify the program

$$\begin{array}{l} s \leftarrow 2 \leq \{p, q, r\} \\ p. \end{array}$$

7.23 The program

$$l \leq \{F_1, \dots, F_n\}^c \leq u \leftarrow G$$

is strongly equivalent to

$$\begin{array}{l} \{F_1, \dots, F_n\}^c \leftarrow G \\ \leftarrow u+1 \leq \{F_1, \dots, F_n\},\, G \\ \leftarrow \{F_1, \dots, F_n\} \leq l-1,\, G. \end{array}$$

The next two problems are about the solution to the n-queens problem given in Handout 5, which consists of the rules

$$1 \leq \{q(1, j), \dots, q(n, j)\}^c \leq 1 \qquad (1 \leq j \leq n), \tag{8}$$

$$\leftarrow q(i, j), q(i, j') \qquad (1 \leq i, j, j' \leq n;\ j < j') \tag{9}$$

and

$$\leftarrow q(i, j), q(i', j') \qquad (1 \leq i, i' j, j' \leq n;\ j < j';\ |i - i'| = |j - j'|). \tag{10}$$

7.24 Show that adding the rule

$$q(1, 1)$$

to program (8)–(10) changes its answer sets in the same way as adding the constraint

$$\leftarrow \mathit{not}\ q(1,1).$$

7.25 Show that removing ≤ 1 from each of the rules (8) does not change the answer sets of program (8)–(10).

References

[Ferraris, 2005] Paolo Ferraris. Answer sets for propositional theories. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, pages 119–131, 2005.

[Heyting, 1930] Arend Heyting. Die formalen Regeln der intuitionistischen Logik. *Sitzunsberichte der Preussischen Akademie von Wissenschaften. Physikalisch-mathematische Klasse*, pages 42–56, 1930.

[Pearce, 1997] David Pearce. A new logical characterization of stable models and answer sets. In Jürgen Dix, Luis Pereira, and Teodor Przymusinski, editors, *Non-Monotonic Extensions of Logic Programming (Lecture Notes in Artificial Intelligence 1216)*, pages 57–70. Springer-Verlag, 1997.