Plan for remaining classes...

10. Quantization and sampling **(11/5, 8.30-10.10)**
11. Data theorems and their effect on control **(12/5, 8.00-9.40)**
12. Quantized feedback control **(25/5, 8.30-10.10)**
13. Hierarchical control/Reviews **(26/5, 8.00-9.40)**

14-15. Paper presentation **(~2/6)**

16. Final exam **(~8/6)**

EE185524

Data Theorems in Control

Yurid E. Nugraha
TSP DTE FTEIC ITS

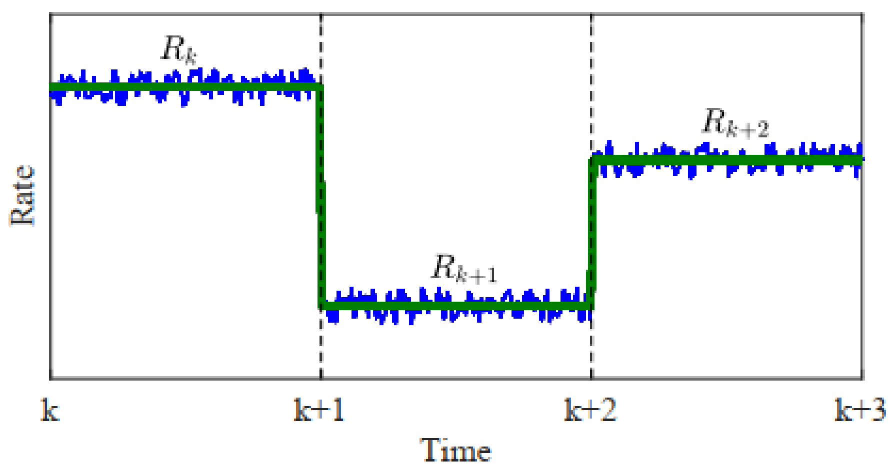
Modeling the channel

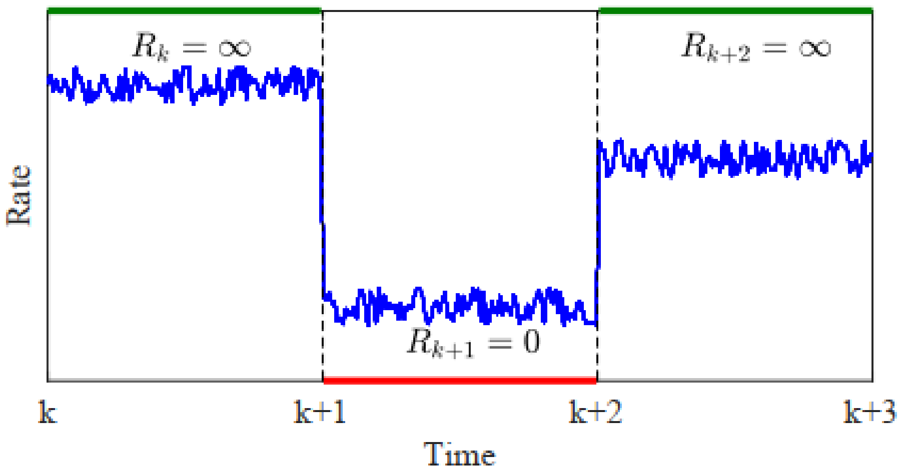
- Information-theoretic approach:
- Channels are bit pipes which can transmit R_k bits/time
- By deriving **data-rate theorems**, the rate needed for constructing a stabilizing controller/quantizer pair is quantified

- Network-theoretic approach:
- Packets are transmitted over the channel. These packets contain enough information such that they can represent real numbers.
- By deriving **critical packet loss** probability, the system cannot be stabilized by any control scheme over this critical value

Modeling the channel

- Two main approaches:

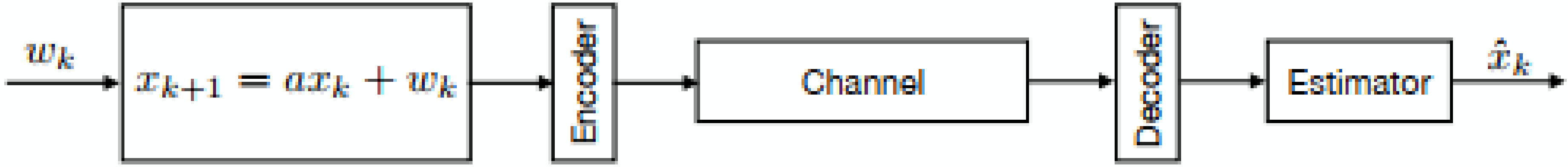




Data in control

(from previous slides..)

Remote state estimation:



$$x[k+1] = ax[k] + w[k]$$

$w[k]$ Gaussian w/ zero mean and variance σ_w^2:

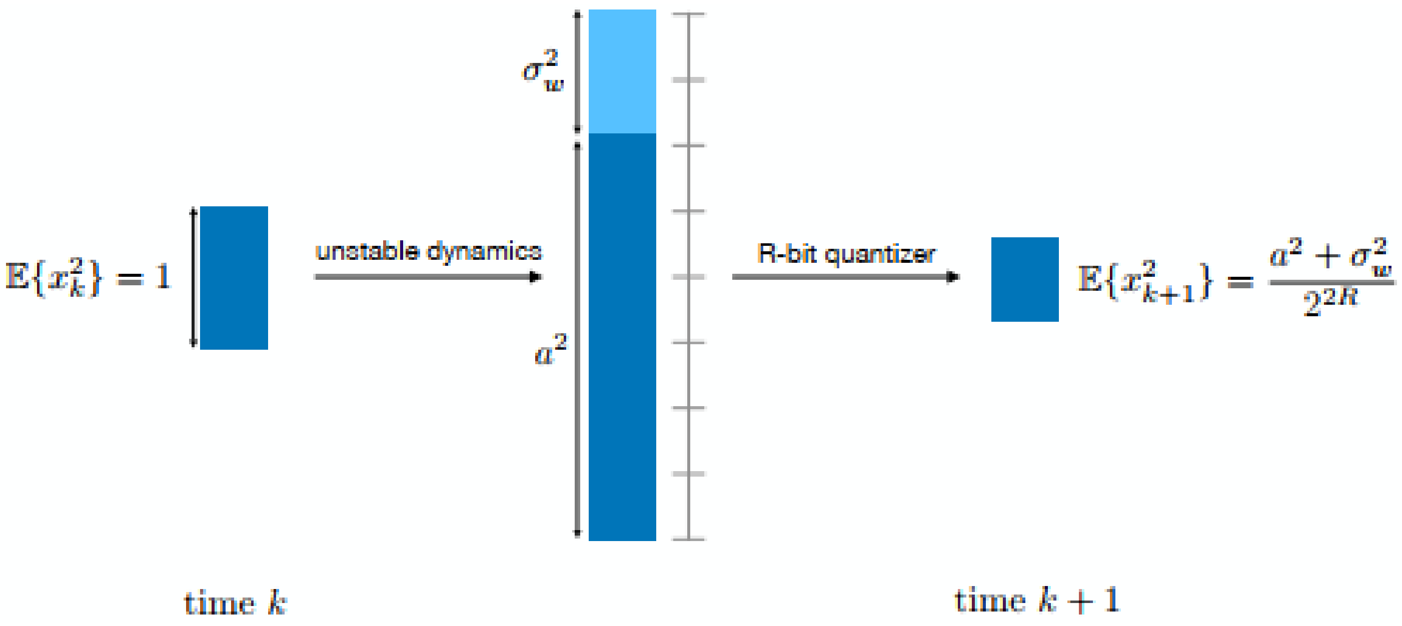
$$\mathbb{E}\{x_{k+1}^2\} = a^2\mathbb{E}\{x_k^2\} + \sigma_w^2$$

Unstable if $|a| > 1$

Data rate theorem

(from previous slides..)

- The noise and bandwidth limitations of the channels are captured by modeling channels capable of transmitting only R bits in each time slot
- By transmitting enough bits at each time step, we can ensure the uncertainty decreases



Data rate theorem

Second moment stability is possible iff

$$|a^2|2^{-2R} < 1 \leftrightarrow R > \log_2 |a|$$

(Rate greater than topological entropy $\log_2 |a|$ (or $\log_2 \lambda_i$ for vector cases))

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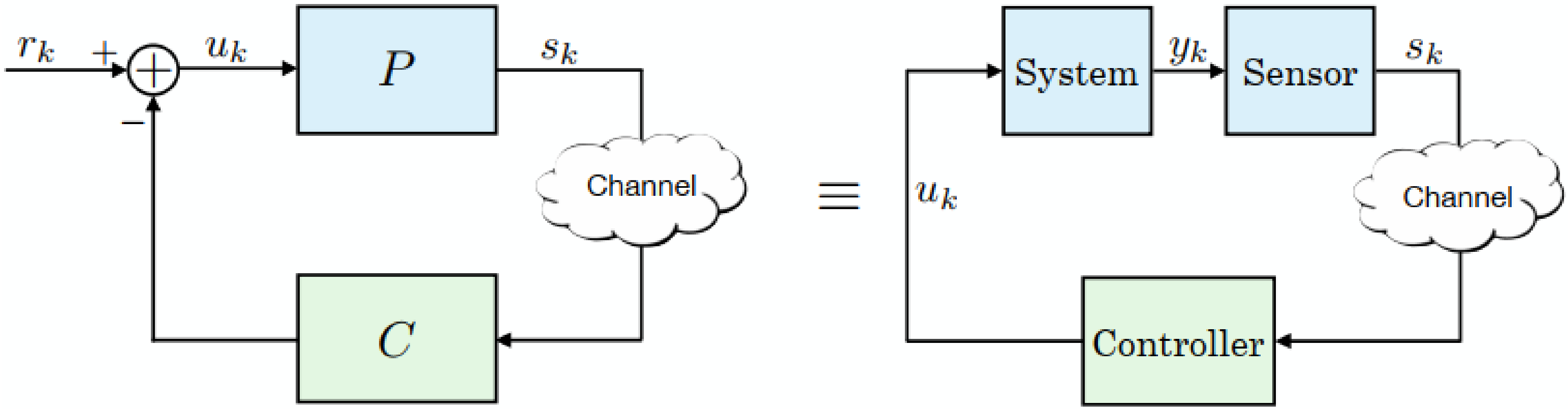
Data Theorems in Control

Stabilization over noiseless channels

Data in control



- Sensor encodes the output and sends a symbol s_k to the controller over a noiseless digital channel
- If R is too small, s_k carries limited information about the output: the controller will not have adequate information for stabilizing the system.

Data-rate theorem for channels with a fixed rate

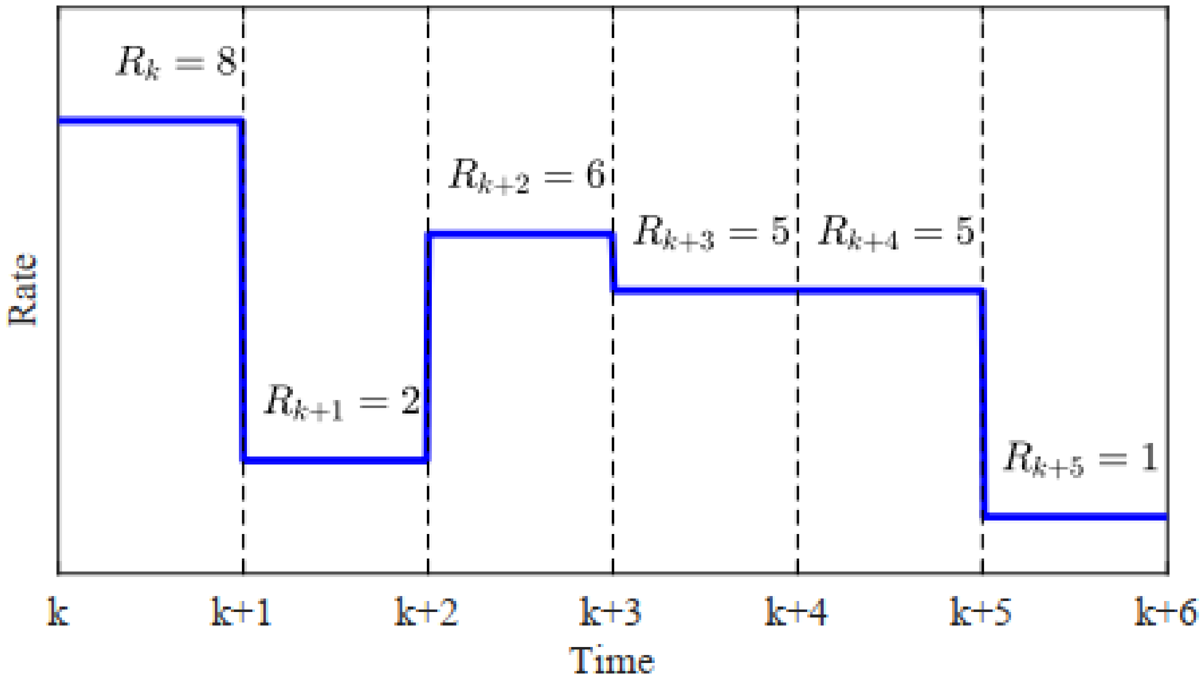
Theorem:

Consider networked control system where the output sensor is connected to the controller via a noiseless digital channel. Then, a necessary and sufficient condition for the asymptotic stabilization of the system is that

$$R > \sum_{|\lambda_i|>1} \log_2|\lambda_i| := R_{inf} = H_T(A)$$

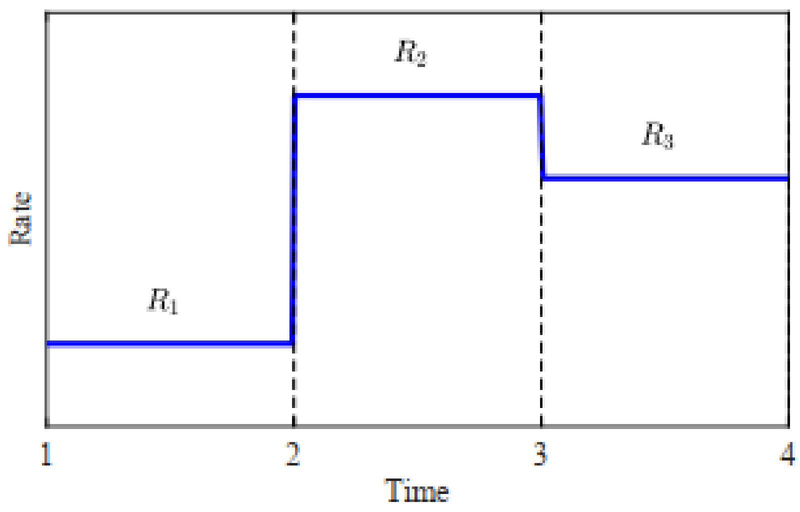
Topological entropy

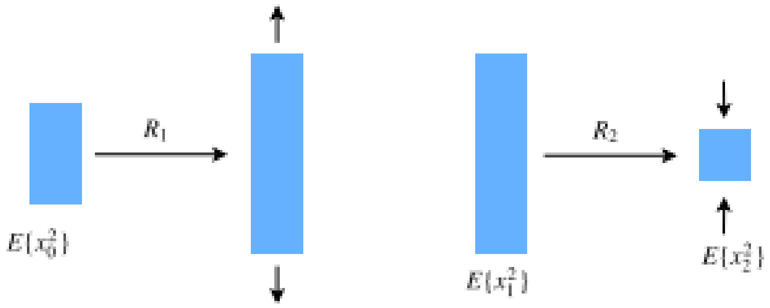
Stochastic time-varying channels



- The variations of rate $\{R_k\}$ are independently, identically distributed (i.i.d) random in time
- There is causal knowledge of the rate process $\{R_i\}_{i=0}^{k}$ at encoder and decoder

Stochastic time-varying channels





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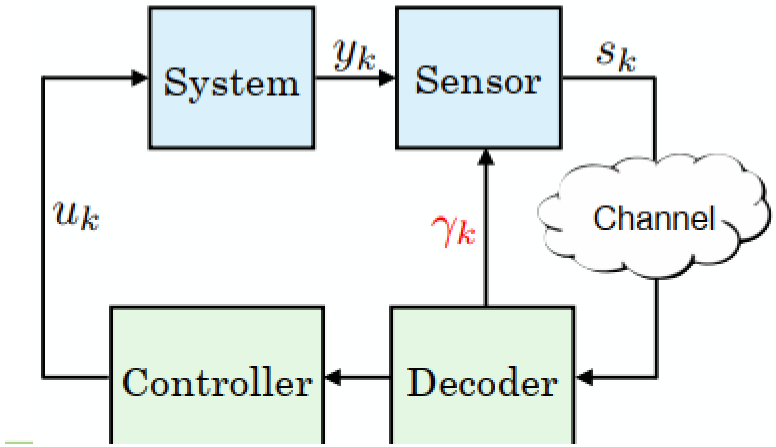
Data Theorems in Control

Memoryless erasure channel

Memoryless erasure channel

- A **special case** of stochastic rate channel: memoryless erasure channel: (erasure channel: channel in which some data are erased)
  - Packet reception is represented by a random variable γ_k ($\gamma_k = 1$ indicating that the packet is successfully delivered)
  - Packet loss process $\{\gamma_k\}_{k\geq 0}$ is assumed to be i.i.d. process w/ probability distribution:

$$\mathbb{P}(\gamma_k = 1) = 1 - p$$
$$\mathbb{P}(\gamma_k = 0) = p$$



Memoryless erasure channel

$\mathbb{E}[x_k^2] < \infty$

- Second moment stability of a scalar system

$$x_{k+1} = a x_k + u_k + w_k$$

can be ensured iff

$= \sum x \cdot p(x)$

$$|a|^2 \mathbb{E}\{2^{-2R}\} < 1$$

$$|a|^2 (2^{-2R}(1-p) + p) < 1$$

$$\text{If R} \to \infty, p < p_c = \frac{1}{a^2}$$

- Despite the infinite channel capacity, the system may be unstable when the erasure probability is high

Memoryless erasure channel (vector case)

Theorem:

- The system

$$x_{k+1} = Ax_k + Bu_k$$

asymptotically stabilizable in the mean square sense via quantization transmissions iff

- Packet loss rate is small enough:

$$p < \frac{1}{M(A)^2}$$

- Data rate R satisfies

$$R > H_T(A) + \frac{1}{2}\log_2[\frac{(1-p)}{1-(pM)(A)^2}]$$

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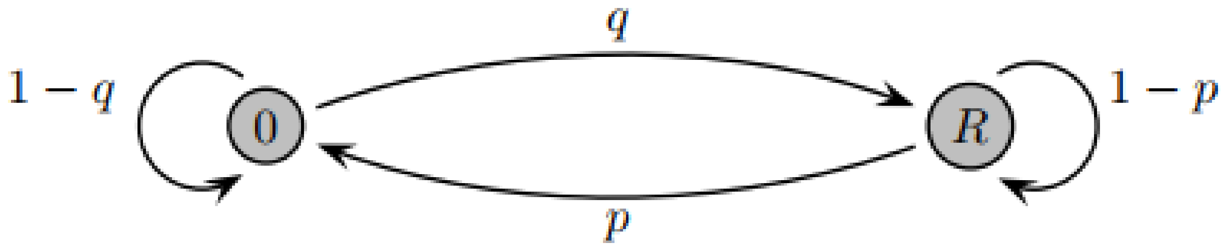
Data Theorems in Control

Erasure channel with memory

Erasure channel with memory

- If the noise in the channel is correlated over time, a time-homogeneous Markov chain can be used:

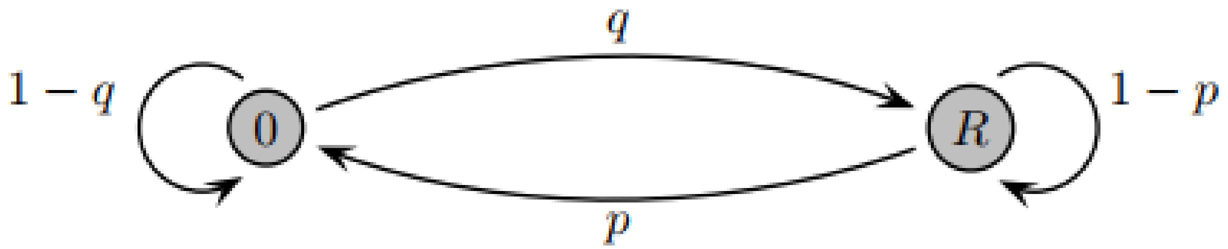


- $\gamma_k = 1$ if packets successfully received, $\gamma_k = 0$ if packets lost

Erasure channel with memory

➢ Transition probability matrix:

$$\left(\mathbb{P}\{\gamma_{k+1} = j | \gamma_k = i\}\right)_{i,j\in\{0,R\}} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}$$



Scalar noise-free case

Theorem:

Consider $x_{k+1} = ax_k + bu_k$ where $|a| > 1$ and x_0 is random variable w/ a known bounded support. The system is **asymptotically mean square stabilizable (asymptotic mean square stability:** $\lim_{k\to\infty} \mathbb{E}\{\|x_k^2\|\} = 0$**)** iff

- The recovery rate of the channel is large enough:

$$q > q_c = 1 - \frac{1}{a^2}$$

$q = 0 \longrightarrow R$

- Data rate R satisfies

$$R > \frac{1}{2}\log_2 \boxed{\mathbb{E}(|a|^{2T})} = \log_2 |a| + \frac{1}{2}\log_2[1 + \frac{p(|a|^2 - 1)}{1 - (1 - q)|a|^2}]$$

T: sojourn time

Stochastic scalar case

Theorem:

Consider $x_{k+1} = ax_k + bu_k + w_k$ where $|a| > 1$. The system is **mean square stabilizable** iff

- The recovery rate of the channel is large enough:

$$q > q_c = 1 - \frac{1}{a^2}$$

- Data rate R satisfies

$$R > \frac{1}{2} \log_2 \mathbb{E}(|a|^{2T})$$

Vector case

- System model:

$$\boldsymbol{x}_{k+1} = J\boldsymbol{x}_k + B\boldsymbol{u}_k$$

where $J = TAT^{-1}$, $J = \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_d \end{bmatrix}$, $J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 0 & & \cdots & \lambda_i \end{bmatrix}$

- Quantifies the joint effect of **packet losses** and **finite communication data rate** on the mean square stabilization of linear scalar systems.

Vector case

Theorem (necessity):

A necessary condition for asymptotic mean square stabilization of the networked system is that for any $s_i \in \{d_{i1}, \dots, d_{in}\}$ and $s = \sum_{i=1}^{d} s_i$, *the following hold:*

- The probability for the channel recovering from packet loss is

$$q > 1 - \frac{1}{\left(\Pi_{i=1}^{d} |\lambda|^{2s_i}\right)^{1/s}}$$

- Data rate R satisfies

$$R > \frac{s}{2} \log_2 \mathbb{E}\left\{\left(\Pi_{i=1}^{d} |\lambda|^{2s_i}\right)^{\frac{T}{s}}\right\}$$

Vector case

Theorem (sufficient):

A sufficient condition for asymptotic mean square stabilization of the networked system is that the following hold:

- The probability for the channel recovering from packet loss is

$$q > 1 - \frac{1}{\max_{i\in\{1,\ldots,d\}} |\lambda_i|}$$

- Data rate R satisfies

$$R > \frac{s_i}{2a_i(R)} \log_2 \mathbb{E}(|\lambda_i|^{2T}),\ \forall i \in \{1, \ldots, d\}$$

Remark

Theorem (sufficient, cont'd):

- $\boldsymbol{a}(R) = [a_1(R) \quad \dots \quad a_d(R)]$ *being rate allocation vector and satisfy*

$$0 \le a_i(R) \le 1, \forall i$$

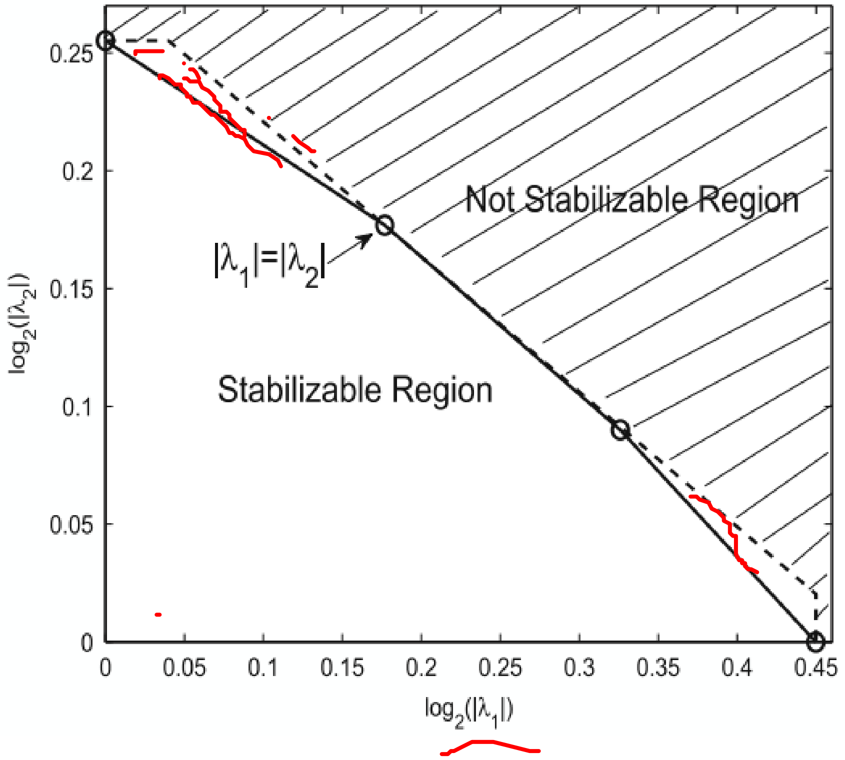
$$\sum_{i=1}^{d} a_i(R) \le 1$$

$$\frac{R a_i(R)}{s_i} \in \mathbb{N}$$

- If $|\lambda_1| = \cdots = |\lambda_d|$, *then the sufficient condition is also necessary*

Example

- Suppose MJLS w/ $p = 1/2$, $q = 2/3$, $R = 1$
- Consider an unstable system w/ distinct eigenvalues $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{C}$ and $s_1 = 1$, $s_2 = 2$.
- It is shown that necessary condition **is almost sufficient**

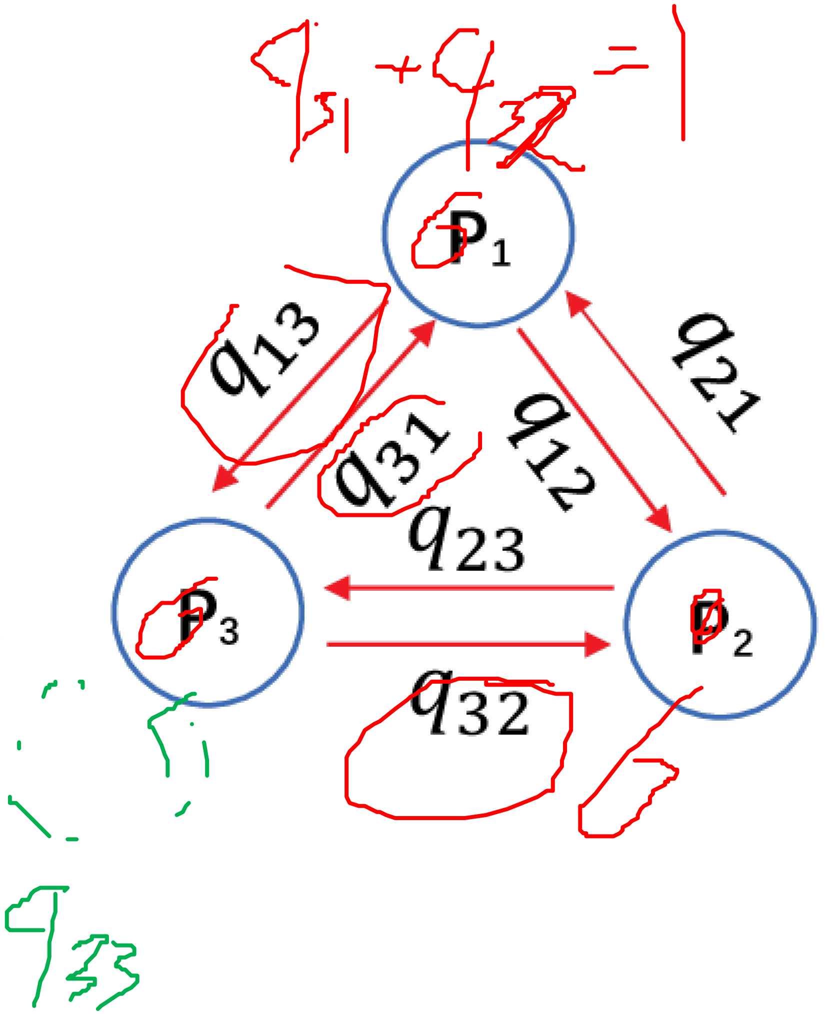


Markov jump linear systems

➢ Consider a dynamical system that is, in a certain moment, described by a model G_1.

➢ Suppose that this system is subject to abrupt changes that cause it to be described, after a certain amount of time, by G_2.

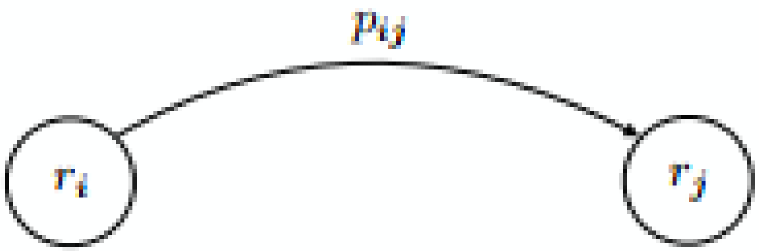
➢ When a system is subject to a series of possible changes that make it switch among a countable set of models, for example, $\{G_1, G_2, \ldots, G_N\}$, we can say system *jumps* from one mode to another.



Markov jump linear systems

- Stability of MJLS are used to characterize the stability of linear dynamical systems where the estimated state is sent to the controller **whose state is described by a Markov chain**
- It is considered that state observer is connected to the actuator through a noiseless digital communication link that at each time k allows R_k bits transmission
- The rate process is given by $\{R_k\}_{k\geq 0}$ is modeled as a Markov chain on the finite set, $R = \{r_1, \dots, r_n\}$

Markov jump linear systems

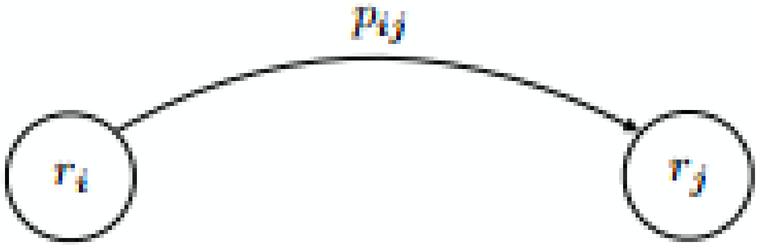


- Arbitrary positive recurrent time-invariant Markov chain of n states

$$p_{ij} = \mathbb{P}\{R_{k+1} = r_j | R_k = r_i\}$$

- For dynamical system $z_{k+1} = \frac{|\lambda|}{2^{2R_k}} z_k + c$ is *mean square stable* iff $\rho(H) < 1$
- H being the matrix defined by **transition probability matrix**

Markov jump linear systems



- Stabilization in mean square sense over Markov time-varying channels is possible iff the corresponding MJLS is mean square stable, that is:

$$|\lambda|^2 \rho(H) < 1$$

Summary

- Stabilization in mean square sense over Markov time-varying channels is possible iff the corresponding MJLS is mean square stable:

$$|a|^2\rho(H) < 1$$

- **Erasure channel w/ memory:**

$$R > \frac{1}{2}\log_2 \mathbb{E}(|a|^{2T})$$

- **Memoryless erasure channel:**

$$R > \log_2|a| + \frac{1}{2}\log_2 \frac{1-p}{1-p|a|^2}$$

Take home message

- Rate of the link should be larger than topological entropy
- Possible to find the data rate theorems for scalar systems for both **noiseless** and **noisy** channels without/with memory
- Still limited to scalar systems