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Introduction to Formation Control

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Formation control?

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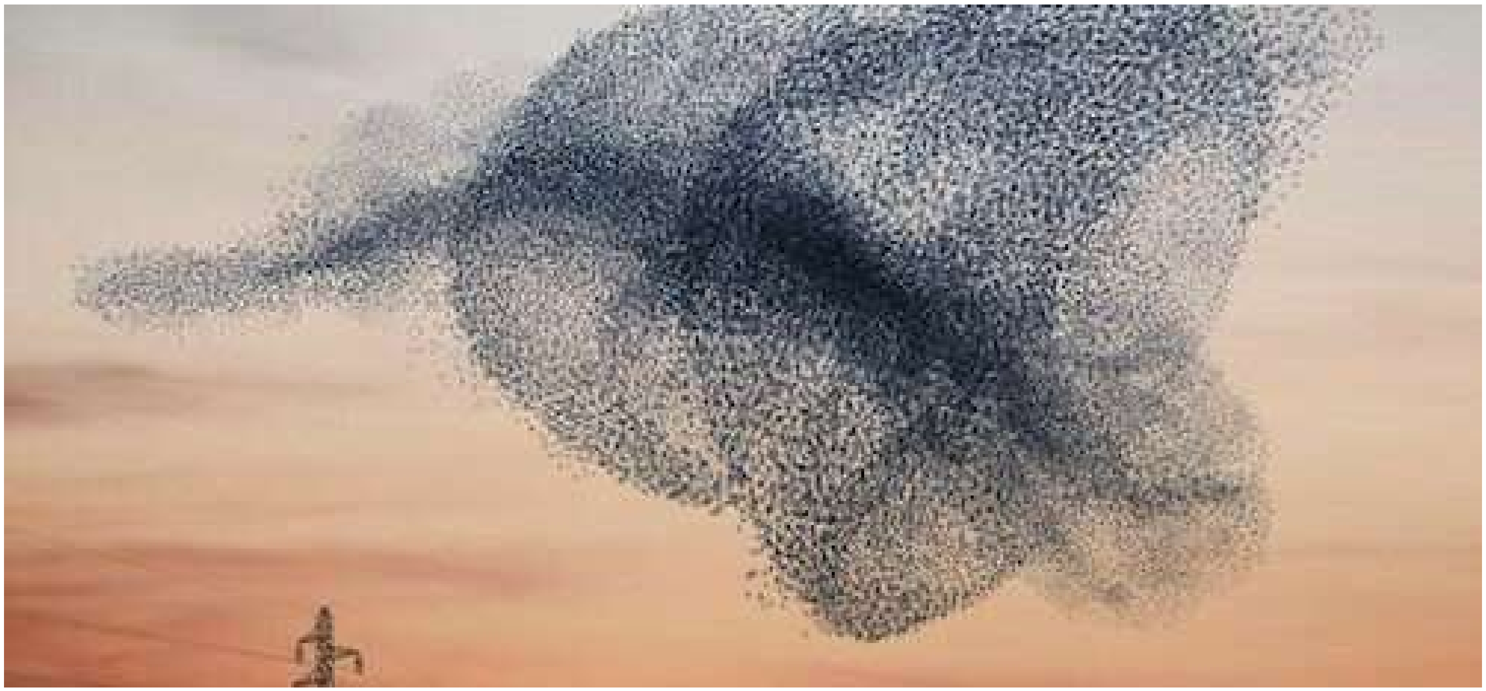


Formation control

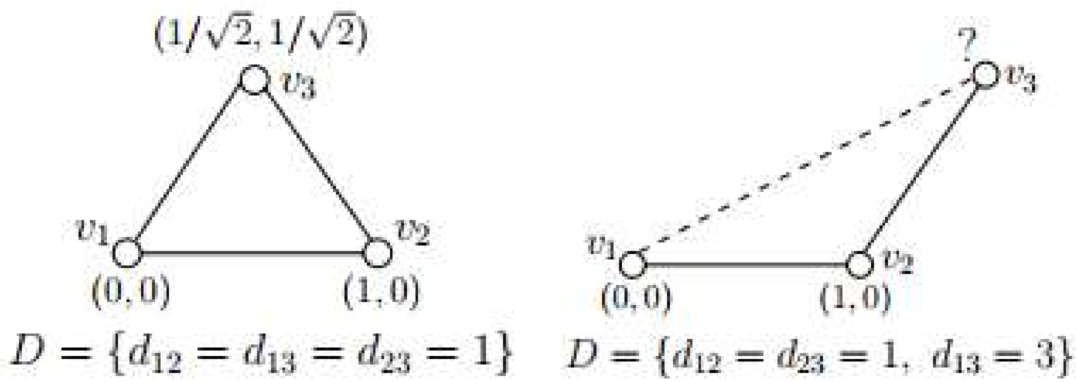
- Involving moving the agents in such a way that they satisfy a particular ***shape* or *relative state*** and a certain aspect of assigning roles (targets in the shape or the relative state) to individual agents

- In fact, formation control problems can be defined by a shape or a relative state, as well as an ***assignment*** component
  - First component as dictating what the formation should "look" like,
  - The second component as codifying which agent should take on what role in the formation.

Shapes

Feasible formation:
$D = \{d_{ij} \in \mathbb{R} \mid d_{ij} > 0\}$

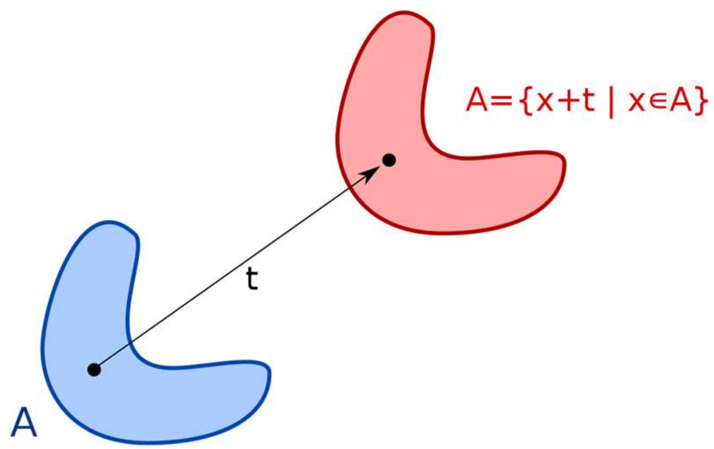
There exist points $\zeta_1, \zeta_2, \ldots, \zeta_n \in \mathbb{R}^p$, $p = 2$ or 3



Formation types

What's allowed and what's not?

- Scale invariant: agents are free to scale
- Rigid: agents are **not** free to scale
- (Only) translational invariant: agents cannot rotate freely



Formation types

What's allowed and what's not?

Interagent distances

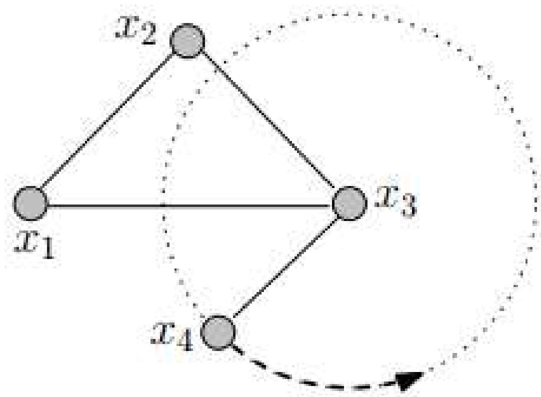
$$D = \{d_{ij} = d_{ji} \geq 0,\ i, j = 1, \ldots, n,\ i \neq j\}$$

formation	specification	interpretation
scale invariant	D	$\|x_i - x_j\| = \alpha d_{ij}$ for some $\alpha > 0$
rigid	D	$\|x_i - x_j\| = d_{ij}$
translational invariant	Ξ	$x_i = \xi_i + \tau$ for some $\tau \in \mathbf{R}^p$

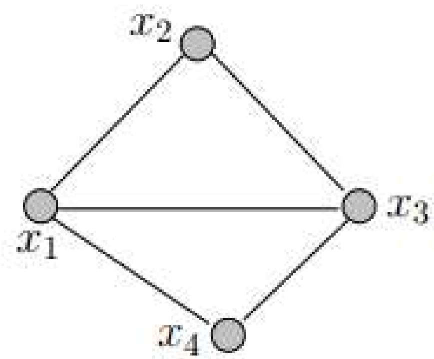
Rigidity

- Only permissible motions, while maintaining proper edge distances, are rigid motions
- **Definition:** *A framework is rigid if and only if all edge-consistent trajectories of the framework are rigid trajectories*
- **Rigid formation:** formation whose shape can be maintained rigidly while **only maintaining** the desired interagent distances
- Formation graph $G(\Xi) = (\Xi,\ G)$
- Framework Ξ: feasible points
- Trajectory: set of continuous states

Flexible framework



Flexible



Rigid

Specification: relative states

- Translationally invariant formations can be directly specified by a set of **desired relative states** in the formation configuration space, as opposed to set of **relative distances**
- e.g., for agents moving in 3 dimension:

$$z(t) = [(x_1(t) - x_2(t))^\top, (x_2(t) - x_3(t))^\top]^\top, \; z \in \mathbb{R}^6$$

- Then, define **reference relative state** z_{ref} for desired formation

Specification: relative states

- ... specifying the formation inertially can be accomplished by letting x_0 be the coordinates of **a fictitious point mass** w.r.t. the inertial frame, and then specifying the formation of the three-point masses by defining

$$z(t) = [(x_0(t) - x_1(t))^\top, (x_1(t) - x_2(t))^\top, (x_2(t) - x_3(t))^\top]^\top,\ \mathbb{R}^9$$

- ... where x_0 is the origin of the inertial frame

Specification: relative states

➢ .. can be encoded using **incidence matrix**

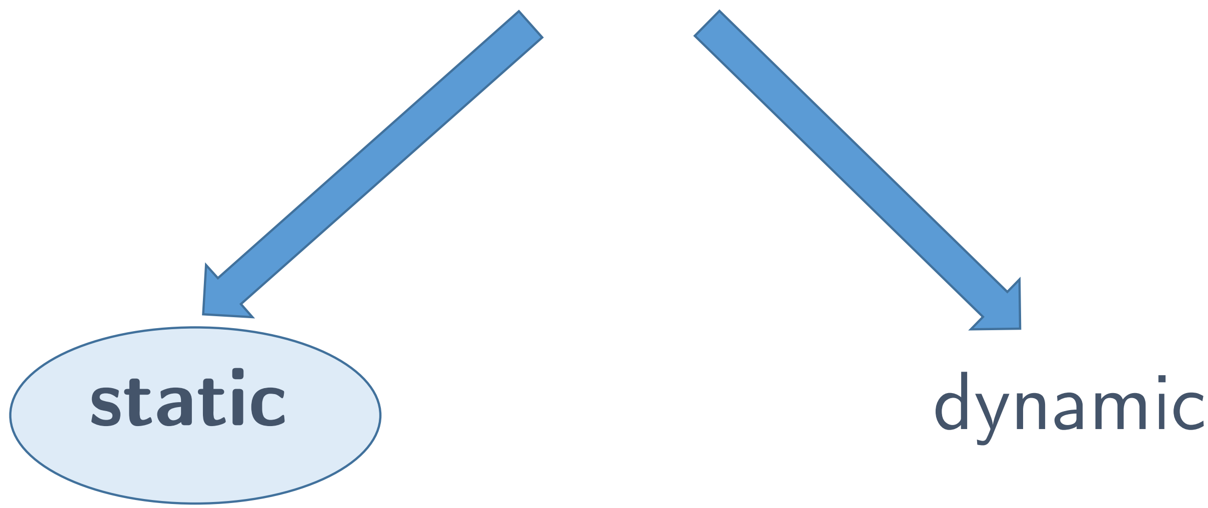
$$z(t) = [(x_1(t) - x_2(t))^\top, (x_2(t) - x_3(t))^\top]^\top, \;\; z \in \mathbb{R}^6$$

with Incidence matrix $\boldsymbol{B}(\boldsymbol{G})$: $\quad z(t) = B(G)^\top x(t)$

where $\quad B(G) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \otimes I$

Shape-based control

- Assumption: rotationally invariant formation
- **Consensus can be used**



Shape-based control

- Assumption: rotationally invariant formation
- **Consensus can be used**
- Notation: $x_i(t) \in \mathbb{R}^p$ denotes position of agent i
- Goals:
  - $\|x_i(t) - x_j(t)\|$ converges to d_{ij} $\forall\ i, j$ such that $\{i, j\} \in E_f$
  - If the interaction graph is dynamic, it will converge to a static graph in finite time (i.e., $E_f \in E(t)\ \forall t \geq T$ for some finite $T \geq 0$.)

Formation control: static case

Definition: τ_i as displacement of x_i from the target location ζ_i

Thus, from consensus protocol:

$$\dot{\tau}_i(t) = \sum_{j \in \mathcal{N}_f(i)} (\tau_j(t) - \tau_i(t))$$

$$\Downarrow$$

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_f(i)} (x_j(t) - x_i(t)) - (\zeta_j(t) - \zeta_i(t))$$

Convergence has been proved in consensus protocol

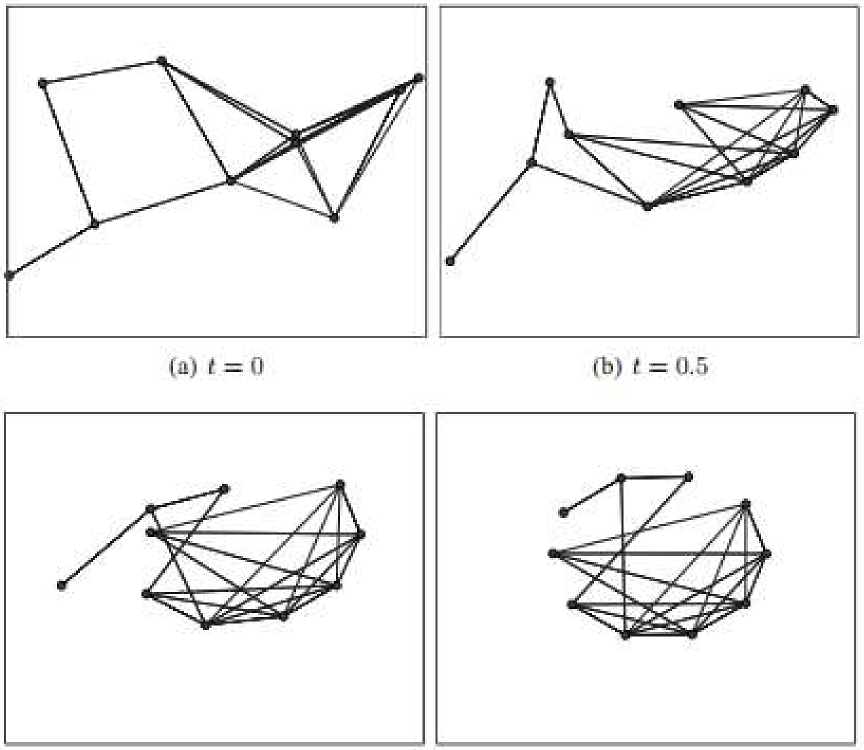
Formation control: static case

Theorem 6.12. *Consider the connected target formation graph $\mathcal{G}_f$ given by (V, E_f) and a set of target locations Ξ. If the static interaction graph $\mathcal{G} = (V, E)$ satisfies $E_f \subseteq E$, then the protocol (6.7) will asymptotically drive all agents to a constant displacement of the target positions, that is, for all i,*

$$x_i(t) - \xi_i \to \tau$$

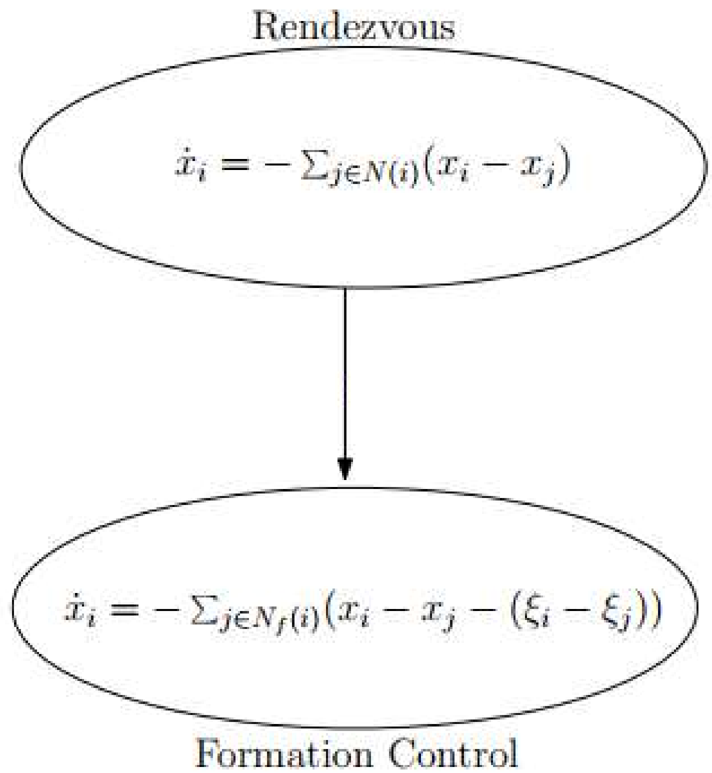
as $t \to \infty$.

Formation control: static case



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 2$ (d) $t = 5$

Consensus vs formation control



Relative-state based formation control

We begin from single integrator agents $\dot{x}_i = u_i$

- Let $z(t) = B(G)^\top x(t)$ and z_{ref} be the **constant reference relative position**
- Then, $e(t) = z_{ref} - z(t) \Rightarrow \dot{e}(t) = -B(G)^T u(t)$, where B is **incidence matrix**
- Suppose $u(t) = kB(G)e(t),\ k > 0$ ⇨ proportional control w/ gain $B(G)$

- **Resulting CL system:** $\dot{e}(t) = -kL_e(G)e(t)$, where $L_e(G) = B(G)^T B(G)$ is the edge-Laplacian of G

Edge-Laplacian

Graph Laplacian based on edges

$$L = \begin{bmatrix} B(G_1)^\top B(G_1) & 0 & \dots & 0 \\ 0 & B(G_2)^\top B(G_2) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & B(G_n)^\top B(G_n) \end{bmatrix}$$

Relative-state based formation control

We begin from single integrator agents $\dot{x}_i = u_i$

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- Then, $e(t) = z_{ref} - z(t) \Rightarrow \dot{e}(t) = -B(G)^T u(t)$
- Suppose $u(t) = kB(G)e(t),\ k > 0$ ➡ proportional control w/ gain $B(G)$
- **Resulting CL system:** $\dot{e}(t) = -kL_e(G)e(t)$, where $L_e(G) = B(G)^T B(G)$ is the edge-Laplacian of G
- Since L_e **positive definite for a graph with spanning tree/globally reachable node,** then $\lim_{t\to\infty} e(t) = 0$
- Thus, we have $\boxed{\dot{x}(t) = kL(\tilde{G})x(t) + kB(G)z_{ref}}$, with $\tilde{G}$ being undirected version of G

Relative-state based formation control

We continue to double integrator agents $\ddot{x}_i = u_i$

- Let desired (reference) velocity and position be $\left[z_{ref}(t)^T \; \dot{z}_{ref}(t)^T\right]^T$
- By setting $e(t) = z_{ref}(t) - B(G)^T x(t)$ and assuming that $\ddot{z}_{ref}(t) = 0$, it follows that $\ddot{e}(t) = -B(G)^T \ddot{x}(t) = -B(G)^T u(t)$
- PD controller (for $k > 0$): $u(t) = k[B(G) \quad B(G)] \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}$
- Guarantee that $\lim_{t\to\infty} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} = 0$

Relative-state based formation control

We continue to double integrator agents $\ddot{x}_i = u_i$

- Dynamics on agents become

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \tilde{L}(\tilde{G})\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \widetilde{D}(\tilde{G})\begin{bmatrix} z_{ref} \\ \dot{z}_{ref} \end{bmatrix}$$

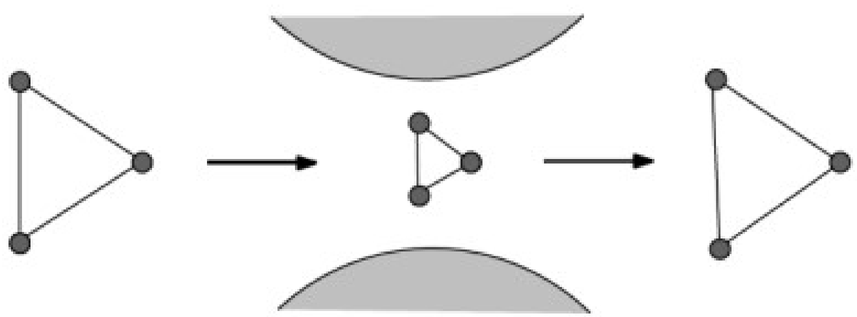
- where $\tilde{L}(\tilde{G}) = \begin{bmatrix} 0 & I \\ -kL(\tilde{G}) & -kL(\tilde{G}) \end{bmatrix}$ and $\widetilde{D}(\tilde{G}) = \begin{bmatrix} 0 & 0 \\ -kB(G) & -kB(G) \end{bmatrix}$,

- We then have

$$\ddot{x}(t) = -kL(\tilde{G})x(t) - kL(\tilde{G})\dot{x}(t) + kB(G)z_{ref}(t) + kB(G)\dot{z}_{ref}(t)$$

Dynamic formation selection

- Centralized case vs decentralized case
- One can easily envision that the agents should be spread out when navigating and exploring free space, whereas a tighter formation is preferred when dealing with cluttered environments
- It could potentially be beneficial to let the agents switch between different formations in reaction to environmental changes.



Dynamic formation selection: Centralized

- "Formation error": w.r.t. each possible formation under consideration
- For example:

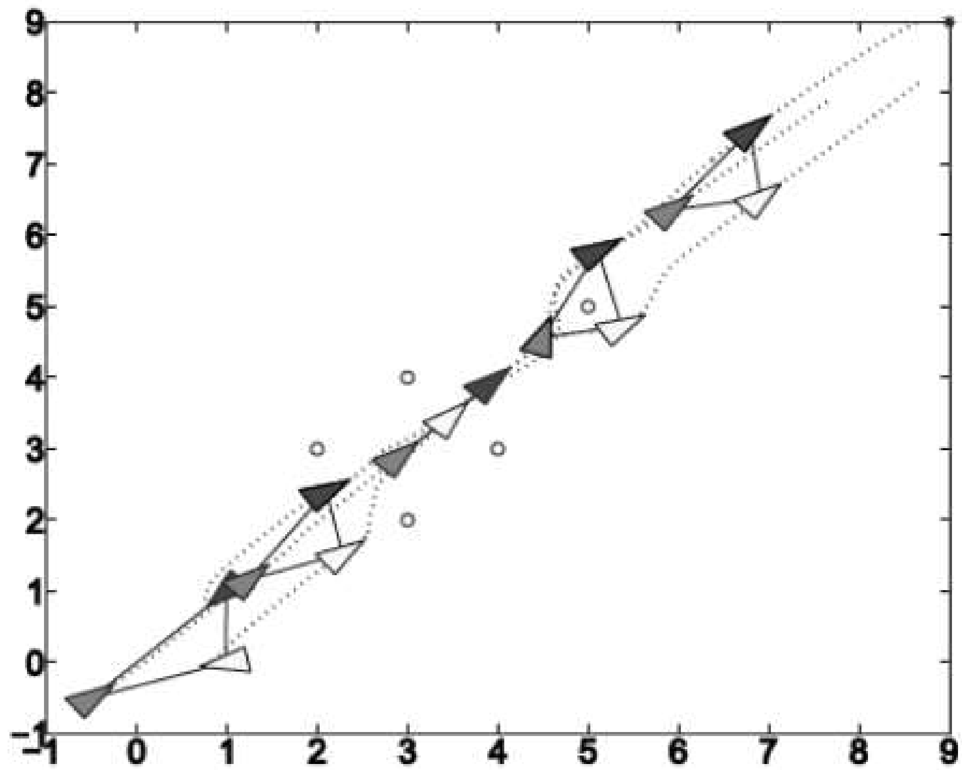
$$\mathcal{E}^k(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij}^k \left(\left\| x_i - x_j \right\|^2 - \left(d_{ij}^k \right)^2 \right)^2$$

Dynamic formation selection: Decentralized

- Global information unavailable to all agents: introduce $\mathcal{E}_i^k(t)$ as a measure of kth formation error as perceived locally by agent i
- For example,

$$\mathcal{E}_i^k = \sum_{j \in N_i} \omega_{ij}^k \left(\left\| x_i - x_j(t) \right\|^2 - \left(d_{ij}^k \right)^2 \right)^2$$

Dynamic formation selection: Decentralized



Assigning roles

- Assignment vs assignment free
- Hungarian method: algorithm used for computing best assignment