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Laplacian Matrix and Consensus Protocol in Continuous Time

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TSP DTE FTEIC ITS

Laplacian matrix

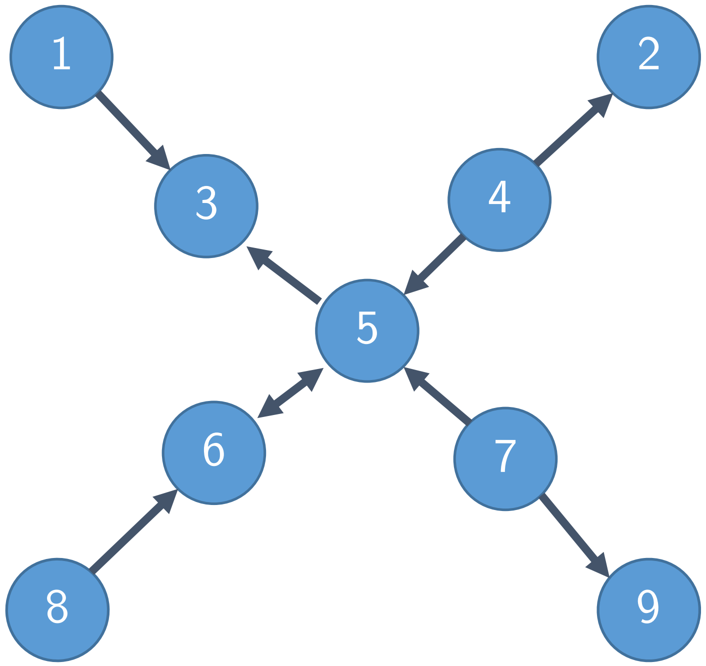
$$L := D - A \qquad \text{(undirected)}$$

$$L := D_{\text{out}} - A \quad \text{(directed)}$$

Properties:

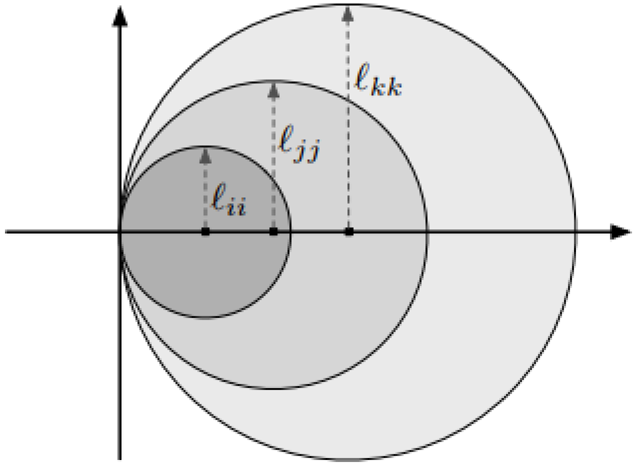
- row-sums are zero,
- non-diagonal entries are non-positive,
- diagonal entries are non-negative.

Applications?



Some other properties of Laplacian matrix

- $L\mathbf{1}_n = \mathbf{0}_n$
- For a digraph, $\mathbf{1}_n^{\mathrm{T}} L = \mathbf{0}_n^{\mathrm{T}}$ implies that the graph is weight-balanced
- Real values of nonzero eigenvalues are all positive $(0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n)$



https://www.geogebra.org/m/wDEj3Xg9

Laplacian and agents' states

Consensus on discrete-time case:

$$x[k+1] = Ax[k] \longrightarrow x_i[k+1] = \sum_{j \in \mathcal{N}_i} A_{ij} x_j[k]$$

Consensus on continuous-time case:

$$\dot{x}(t) = ?? \longrightarrow \dot{x}_i(t) = ??$$

Laplacian and agents' states

From consensus on discrete-time case:

$$x[k+1] = Ax[k], \quad x \in \mathbb{R}^n$$

$$\Downarrow$$

$$x[k+1] - x[k] = (A - I_n)x[k]$$

$$\Big\Downarrow \text{ for some } t,$$

$$\dot{x}(t) = (A - I_n)x(t)$$

$$\Downarrow$$

$$\dot{x}(t) = -Lx(t)$$

Laplacian and agents' states

$$\dot{x}(t) = -Lx(t)$$

$$\Downarrow$$

$$\dot{x}_i(t) = \text{ ??}$$

Solution of $\dot{x}(t) = -Lx(t)$?

Matrix exponential

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Generalize usual exponential function

Formal definition [edit]

Main article: Characterizations of the exponential function

The real exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ can be characterized in a variety of equivalent ways. It is commonly defined by the following power series:[1][7]

$$\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Solution of $\dot{x}(t) = -Lx(t)$

For undirected graph, with orthonormal eigenvectors:

$$x(t) = e^{-\lambda_1 t}(v_1^\top x(0))v_1 + e^{-\lambda_2 t}(v_2^\top x(0))v_2 + \ldots + e^{-\lambda_n t}(v_n^\top x(0))v_n$$

Recall that.. $(0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n)$

$$\Downarrow$$

$$\boxed{x(t) = \text{average}(x(0))\mathbf{1}_n + e^{-\lambda_2 t}(v_2^\top x(0))v_2 + \ldots + e^{-\lambda_n t}(v_n^\top x(0))v_n}$$

Solution of $\dot{x}(t) = -Lx(t)$

$$x(t) = \text{average}(x(0))\mathbf{1}_n + e^{-\lambda_2 t}(v_2^\top x(0))v_2 + \ldots + e^{-\lambda_n t}(v_n^\top x(0))v_n$$

$$\lim_{t\to\infty} x(t) = ??$$

Eigenvalues and convergence

$$x(t) = \text{average}(x(0))\mathbf{1}_n + e^{-\lambda_2 t}(v_2^\top x(0))v_2 + \ldots + e^{-\lambda_n t}(v_n^\top x(0))v_n$$

Which eigenvalue is the most important?

Eigenvalues and convergence

$$x(t) = \mathrm{average}(x(0))\mathbf{1}_n + e^{-\lambda_2 t}(v_2^\top x(0))v_2 + \ldots + e^{-\lambda_n t}(v_n^\top x(0))v_n$$

Which eigenvalue is the most important?

λ_2 : Fiedler eigenvalue/algebraic connectivity

Consensus with a globally-reachable aperiodic strongly-connected component (continuous)

Theorem:

Let L be a Laplacian matrix associated with a digraph G.
If L is semi-convergent and $\lim_{t\to\infty} \exp(-Lt) = \mathbf{1}_n w^\top$ where $\mathbf{1}_n^\top w = 1,\ w^\top A = \mathbf{0}^\top$, then G contains a **globally reachable node.**

Furthermore,
(i) The solution of the model $\dot{x} = -Lx(t)$ satisfies $\lim_{k\to\infty} x(t) = (w^\top x(0))\mathbf{1}_n$

(ii) If additionally G is weight-balanced, then G is strongly connected and

$$\lim_{t\to\infty} x(k) = \frac{\mathbf{1}_n^{\mathrm{T}} x(0)}{n} = \mathrm{average}(x(0))\mathbf{1}_n$$