EE185523

Consensus Protocol (Discrete)

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Multiagent states and communication protocols

Consider $x[k+1] = Ax[k], \;\; x[0] = x_0, \;\; x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$

or..

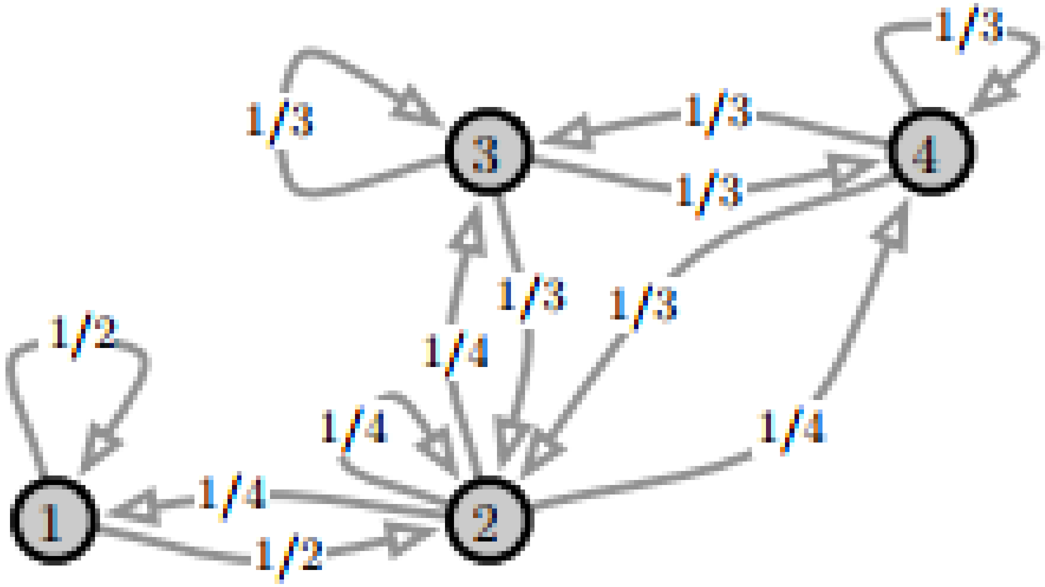
$$x[k] = A^k x_0$$

where..

A row-stochastic.

what will x be?

Graph with row-stochastic matrix



$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Theorem:

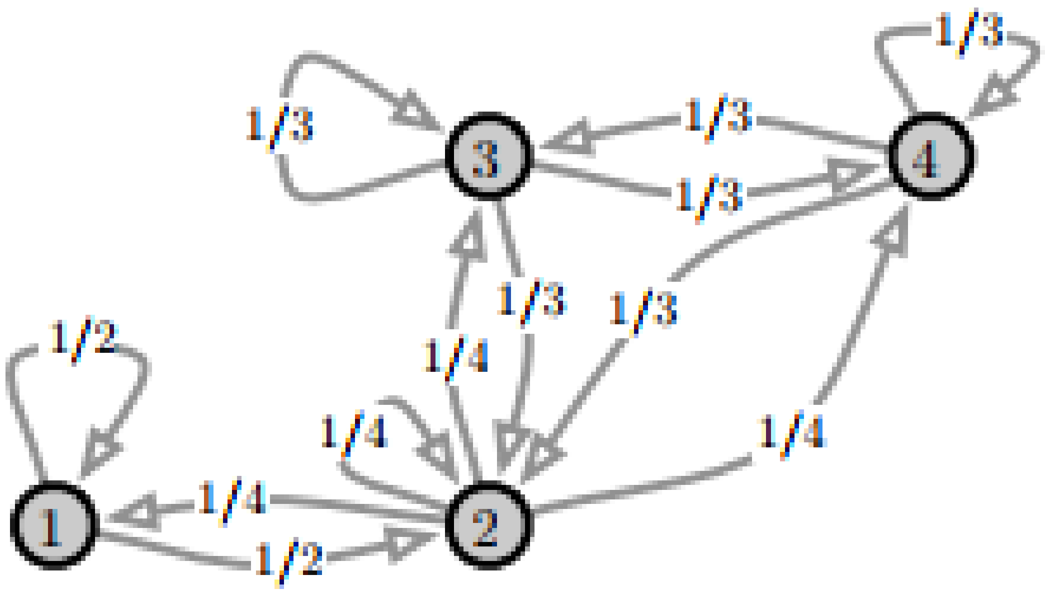
For non-negative matrix A :

$$\lim_{k\to\infty} \frac{A^k}{\lambda^k} = vw^\top$$

w : left dominant eigenvector

v : right dominant eigenvector

Graph with row-stochastic matrix



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For row-stochastic matrix,

$$\lim_{k\to\infty} A^k = \mathbf{1}_n w^{\mathrm{T}}$$

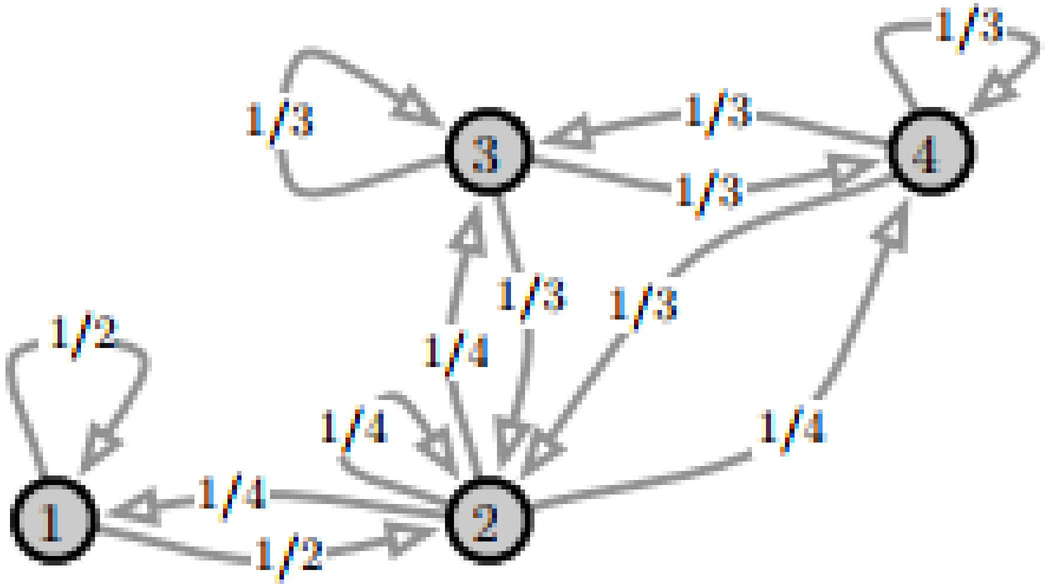
w : left dominant eigenvector with sum 1

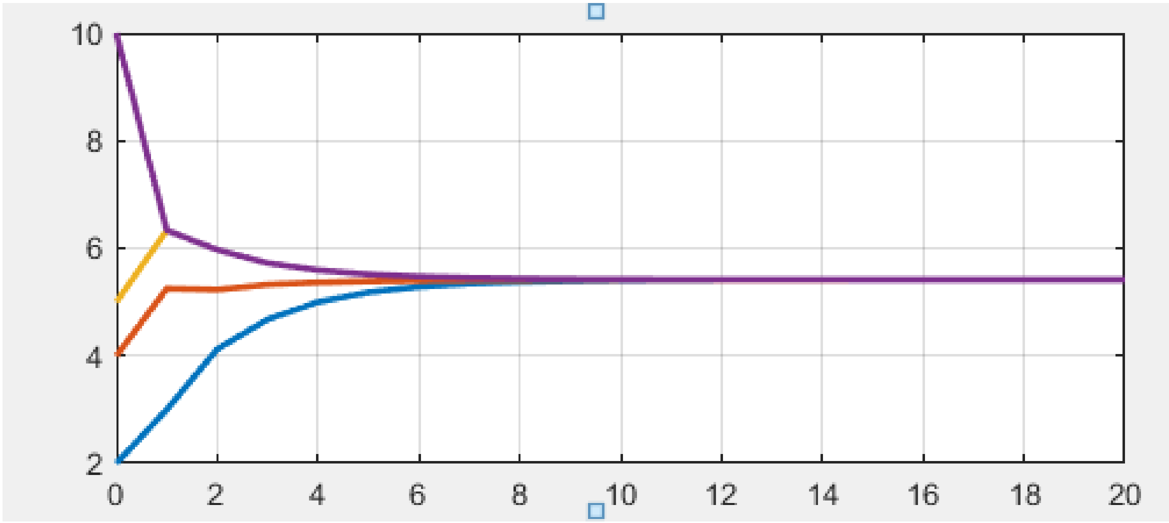
$w : [1/6/1/3/1/4/1/4]^\top$

$$\lim_{k\to\infty} x(k) = (1/6)x_1(0) + (1/3)x_2(0) + (1/4)x_3(0) + (1/4)x_4(0)$$

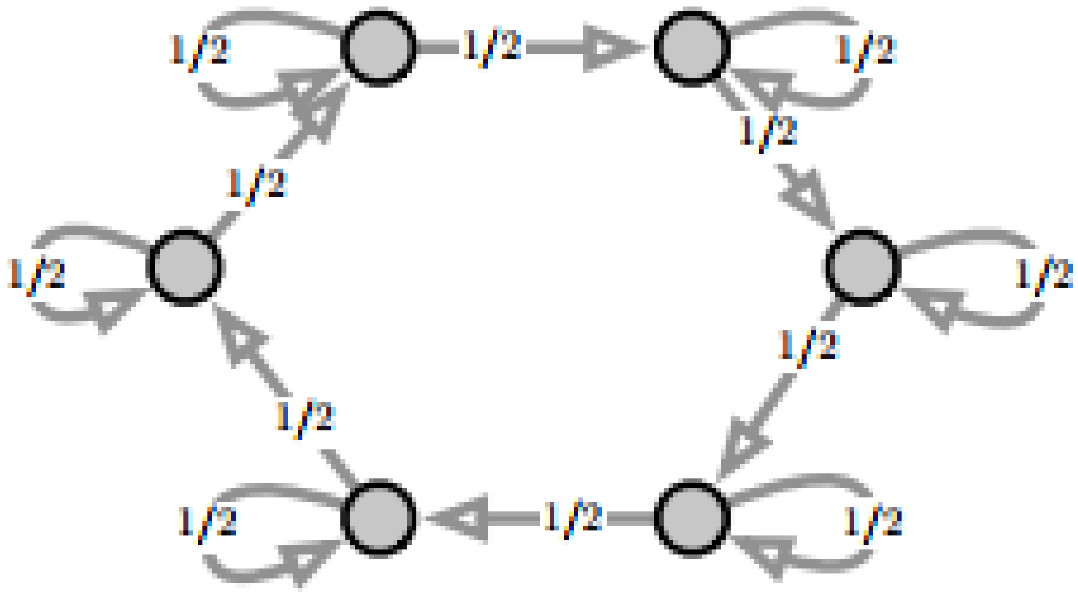
Where do they converge?

Graph with row-stochastic matrix





Graph with row-stochastic matrix

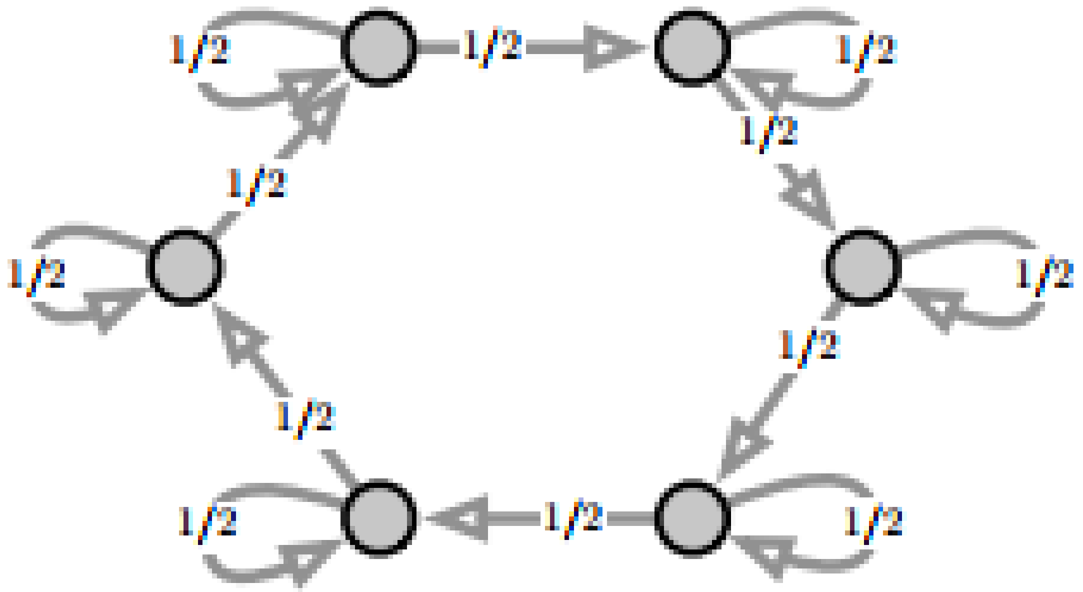


$$A = \begin{bmatrix} 1/2 & 1/2 & \dots & 0 \\ 0 & 1/2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/2 \end{bmatrix}$$

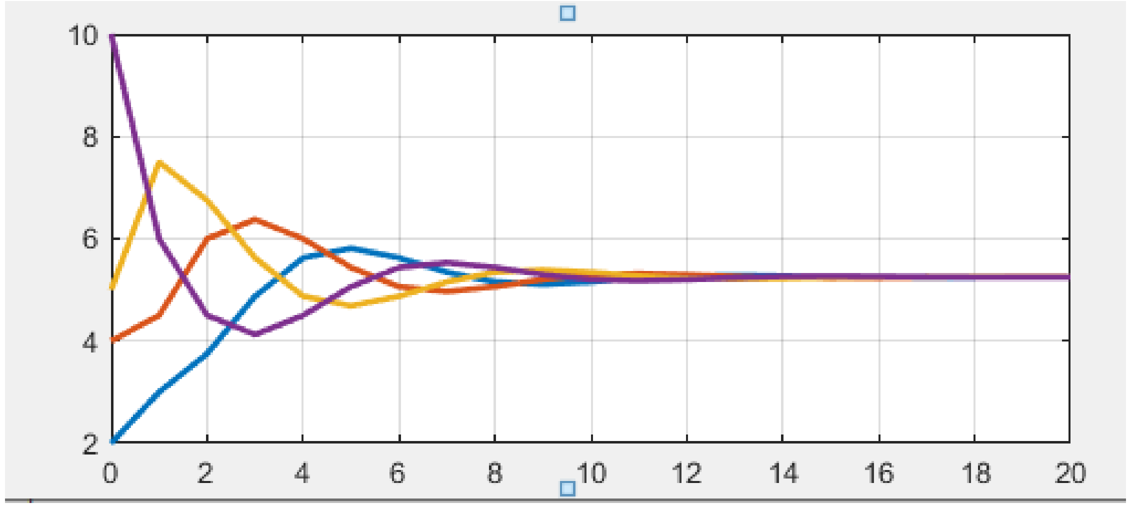
Difference from previous graph?

Row-stochastic?

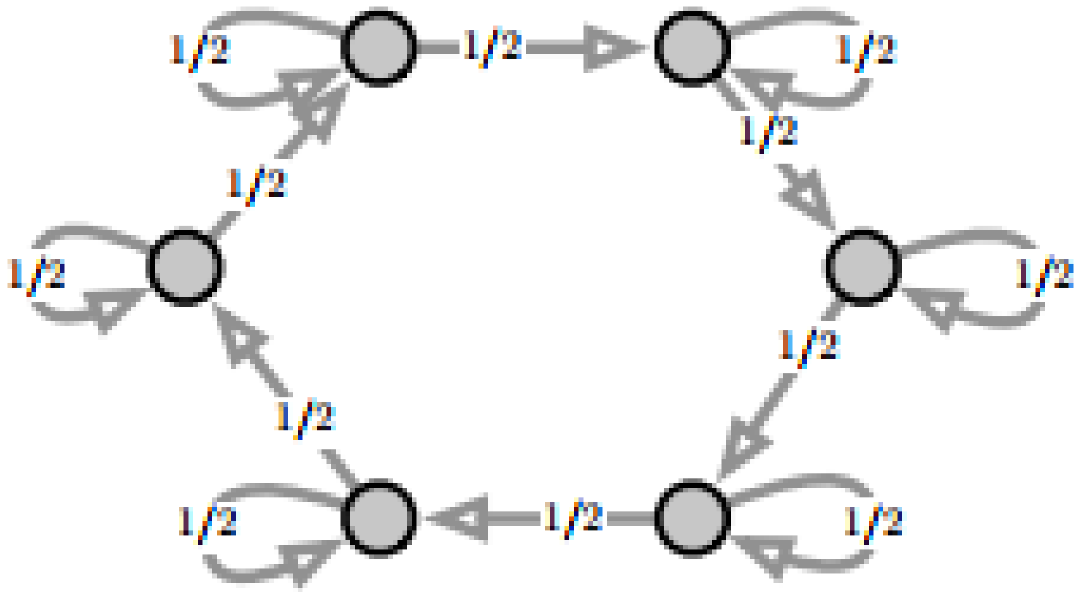
Graph with row-stochastic matrix



For $n = 4$:



Graph with row-stochastic matrix



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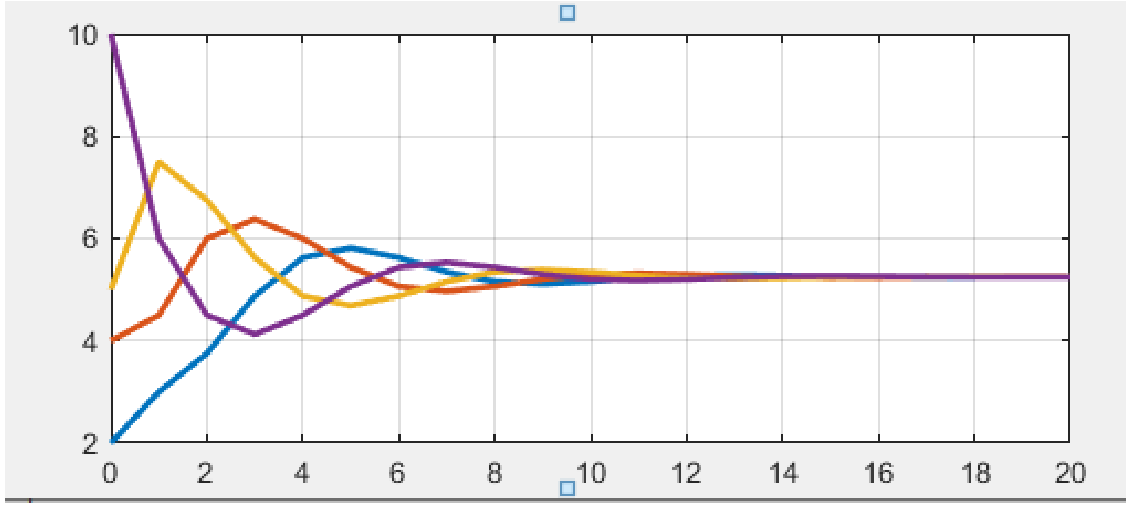
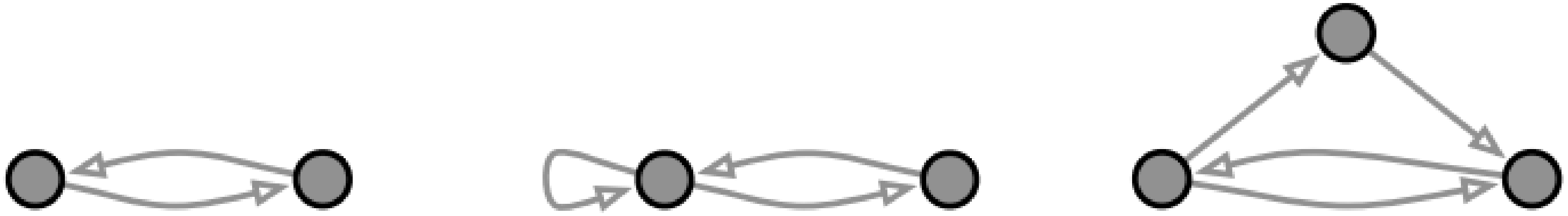


For column-stochastic:

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) = \text{average}(x_0)\mathbf{1}_n$$

Periodic and aperiodic digraphs

- A strongly-connected directed graph is periodic if there exists a $k > 1$, called the *period*, that divides the length of every simple cycle of the graph.
- In other words: a digraph is periodic if the greatest common divisor of the lengths of all its simple cycles is larger than one.
- A digraph is aperiodic if it is not periodic.



(a) A periodic digraph with period 2

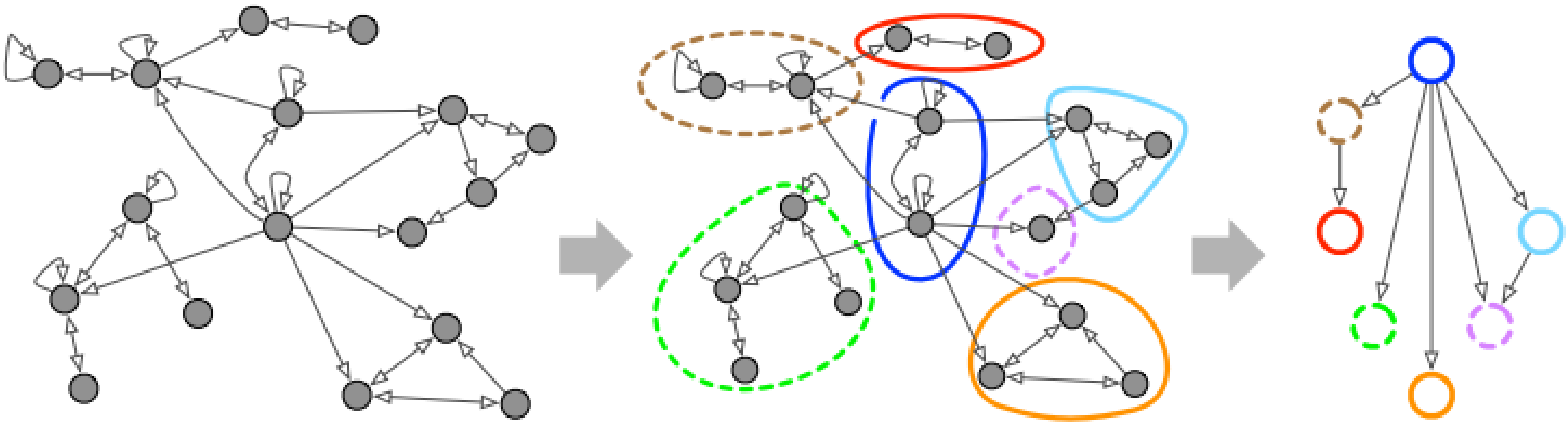
(b) An aperiodic digraph with simple cycles of length 1 and 2.

(c) An aperiodic digraph with simple cycles of length 2 and 3.

Condensation digraphs

- A subgraph H is a strongly connected component of G if H is strongly connected and any other subgraph of G strictly containing H is not strongly connected.
- The condensation digraph of a digraph G denoted by $C(G)$, is defined as follows: the nodes of $C(G)$ are the strongly connected components of G, and there exists a directed edge in $C(G)$ from node H_1 to node H_2 if and only if there exists a directed edge in G from a node of H_1 to a node of H_2.
- The condensation digraph has no self-loops.

Condensation digraphs



Digraph | Connected components | Condensation digraph

Consensus with a globally-reachable aperiodic strongly-connected component

Theorem:

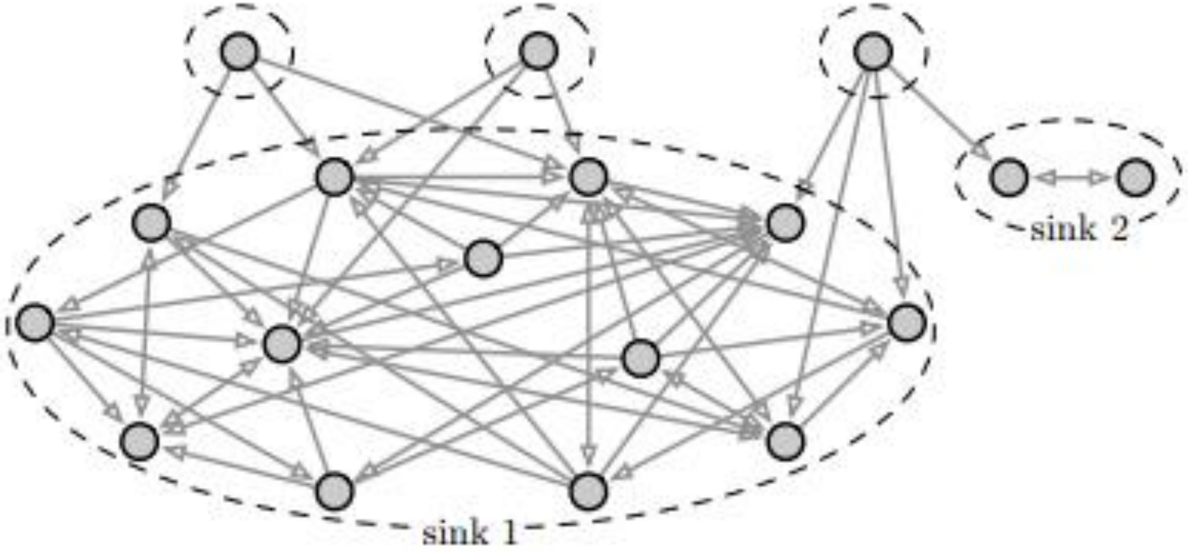
Let A be row-stochastic matrix associated with a digraph G.
If A is semi-convergent and $lim_{k\to\infty} A^k = \mathbf{1}_n w^{\mathrm{T}}$ where $\mathbf{1}_n^\top w = 1,\ w^\top A = w^\top$, then G contains a **globally reachable node.**

Furthermore, matrix A is said to be *indecomposable* and
(i) The solution of the model $x(k+1) = Ax(k)$ satisfies $\lim_{k\to\infty} x(k) = (w^\top x(0))\mathbf{1}_n$
(ii) If additionally A is doubly-stochastic, then

$$\lim_{k\to\infty} x(k) = \frac{\mathbf{1}_n^{\mathrm{T}} x(0)}{n} = \text{average}(x(0))\mathbf{1}_n$$

Averaging system reaching averaging disagreement

Digraph without globally reachable nodes:



Convergence for row-stochastic matrices with multiple aperiodic sinks

Theorem:

Let A be row-stochastic matrix associated with a digraph G and n_s be the number of sinks in the condensation digraph $C(G)$.

If A is semi-convergent, then the solution to $x(k+1) = Ax(k)$ satisfies

$$lim_{k\to\infty} x_i(k) = \begin{cases} (w^P)^{\mathrm{T}} x(0), & \text{if node } i \text{ belongs to sink } p, \\ \sum_{p=1}^{n_i} z_{i,p}((w^P)^{\mathrm{T}} x(0)), & \text{otherwise} \end{cases}$$

Where $z_{i,p},\ p \in \{1, \ldots, n_s\}$ are convex combination coefficient and $z_{i,p} > 0$ if and only if there exists a directed walk from node i to the sink p

Convergence for row-stochastic matrices with multiple aperiodic sinks

Message:

- Convergence does not occur to consensus (not all components of the state are equal)
- The final value of all nodes is independent of the initial values at nodes which are not in the sinks of the condensation digraph

Convergence for row-stochastic matrices with multiple aperiodic sinks