Class: Sistem Pengaturan Formasi dan Kolaborasi (EE185523)
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Date and Time: 2023/04/06, 10.00-13.30
Rule: Take home

Midterm Exam (2022 Genap)

Rule: Each problem is worth the same. **You are allowed to only attempt at most four problems total: one problem in each block**. Do not attempt more than those.

Block 1:

1. Assume the square matrix A is non-negative and irreducible. Discuss whether having a positive diagonal element is **necessary**, **sufficient**, or **necessary and sufficient** to decide that A is primitive.

2. Let A be the binary adjacency matrix for an undirected graph G without self-loops. Recall that the trace of A is $\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$.
   (i) Show that $\text{trace}(A) = 0$.
   (ii) Show that $\text{trace}(A^2) = 2|E|$, where $|E|$ is the number of edges of G.
   (iii) Verify results (i)–(ii) on the matrix
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

3. For the following matrices, use Gersgorin disk/circle theorem to characterize their eigenvalues. Show the steps and sketch the circles. Then, verify the results by obtaining the exact values of the eigenvalues (you can obtain the exact values using matlab, for example).
$$\text{a. } A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \text{b. } A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 5 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 7 & 1 \end{bmatrix}$$

Block 2:

1. Consider a group of five moving small robots that interact to each other according to a graph with adjacency matrix A. Their position $x[k]$ is modeled as discrete consensus model $x[k+1] = Ax[k]$, where
$$A = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.4 \\ 0 & 0.6 & 0 & 0 & 0.4 \\ 0.3 & 0.05 & 0.05 & 0 & 0.6 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}.$$

For $x_0 = [4\ 6\ 1\ 9\ 30]^\top$, compute the final position. Do they finally converge to the same position $\lim_{k\to\infty} x[k]$? Is the final position the average of initial position? Support your answer with an analysis.

2. What is the second eigenvalue of Laplacian matrix for disconnected graph and how it may change the agents' states at infinite time? Explain your answer.

3. By now, we know that the consensus dynamics can be written as $\dot{x} = -Lx$, with the corresponding agent-level dynamics expressed as $\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))$. If, instead, we have $\dot{x} = -L^2x$, what is the corresponding agent-level dynamics and where approximately the agents will be at infinite time?

Block 3:

1. Recall the definition of exponential matrix $\exp(A) = \sum_{k=0}^{\infty}\frac{1}{k!}A^k$ for any square matrix A. Complete the following tasks:

   (a) Show that $A = \mathrm{diag}(a_1, \ldots, a_n)$ implies $\exp(A) = \mathrm{diag}(e^{a_1}, \ldots, e^{a_n})$ and

   (b) Show that $\frac{d}{dt}\exp(At) = A\exp(At) = \exp(At)A$.

2. Given a strongly-connected unweighted digraph G, design weights along the edges of G (and possibly add self-loops) so that the weighted adjacency matrix is doubly-stochastic. Also, discuss which class/type of graph is the easiest to design such weights. Explain your steps.

3. Consider a so-called leader-follower network with two leaders and two followers, as shown below. Assume that the leaders and followers all live on the real line and that the network topology is a line graph, with the peripheral nodes being the static leaders. Moreover, let the dynamics be given by

$$\begin{aligned}
\dot{x}_1 &= \alpha_1((x_3 - x_1) + (x_2 - x_1)) \\
\dot{x}_2 &= \alpha_2((x_1 - x_2) + (x_4 - x_2)) \\
\dot{x}_3 &= 0 \\
\dot{x}_4 &= 0
\end{aligned}$$

   where $\alpha_1; \alpha_2 > 0$. Where do x_1 and x_2 end up as $t \to \infty$, if $x_3 = \beta$ and $x_4 = \gamma$?

Block 4:

1. (a) Explain why and how incidence matrix is used for formation control. What is the advantage of using incidence matrix compared to other graph matrices?

   (b) In the relative-state based formation control, the dynamic is defined as

$$\dot{x}(t) = kL(\tilde{G})x(t) + kB(G)z_{ref},$$

   where B being incidence matrix and $\tilde{G}$ being the undirected version of G. Discuss why undirected version of the graph is needed here.

2. Suppose agents with line graph as formation graph G_f follow a modified consensus protocol with desired distance $d_{ij} = d_{ji} = ||\zeta_j - \zeta_i|| > 0$ for $j \in \mathcal{N}_i$ as

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_{f_i}} (x_j(t) - x_i(t)) - ||\zeta_j - \zeta_i||.$$

Discuss whether the protocol above will always achieve a line formation or not.

3. Design a simple **dynamics** of a three-agent formation starting from the same initial position $x_1(0) = x_2(0) = x_3(0)$ (supposing agents are integrators) which accomplish letter "v". You can assume that both the interaction graph and the formation graph is a complete graph. Provide a reason why your proposed dynamics achieve letter "v" at infinite time.