Class: Sistem Pengaturan Formasi dan Kolaboraasi (EE185523)
Lecturer: Yurid Nugraha
Deadline: 2023/02/22

Assignment 1

(You can answer with either English or Indonesian. You can choose which problems to solve; however, the maximum grade given will still be 100%.)

1. (**Weight: 20%**) Show that any graph on n vertices with more than $(n-1)(n-2)/2$ edges is connected.

2. (**Weight: 30%**) For any non-negative $A \in \mathbb{R}^{n\times n}$ and any numbers k and $m \in \mathbb{N}$, show that:
(i) if the ith row of A^k is positive, then the ith row of $A^{k+m} > 0$,
(ii) if the jth column of A^k is positive, then the jth column of $A^{k+m} > 0$,
(iii) if $A^k > 0$, then $A^{k+m} > 0$.

3. (**Weight: 25%**) Consider a digraph $G = (V, E)$ with at least two nodes. Prove that the following statements are equivalent: (i) G has a *globally reachable node*, and (ii) for every pair S_1, S_2 of non-empty disjoint subsets of V , there exists a node that is an out-neighbor of S_1 or S_2.

4. (**Weight: 30%**) Let G be a weighted digraph with Laplacian matrix L. Prove the following statements are *equivalent*:
(i) G is weight-balanced,
(ii) $L + L^T$ is the Laplacian matrix of the undirected digraph associated to the adjacency matrix $A + A^T$.

5. (**Weight: 20%**) How many 2×2 matrices do there exist that are simultaneously doubly stochastic, irreducible, and not primitive?

6. (**Weight: 25%**) Assume the square matrix A is non-negative and irreducible.
(i) Show that if A has a positive diagonal element, then A is primitive.
(ii) Show that if A is primitive, then it is false that A must have a positive diagonal element.
(iii) Discuss whether having a positive diagonal element is necessary to decide that A is primitive.

7. (**Weight: 20 %**) Consider the row-stochastic matrices

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad A_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad A_4 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(i) Draw the digraphs G_1, G_2, and G_3 associated with these three matrices.
(ii) Show that the matrices A_1, A_2, and A_3 are irreducible and primitive,
(iii) Show that the digraphs G_1, G_2 and G_3 are strongly connected.

8. (**Weight: 20 %**) Let A be the binary adjacency matrix for an undirected graph G without self-loops. Recall that the trace of A is $\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$.
(i) Show that $\text{trace}(A) = 0$.
(ii) Show that $\text{trace}(A_2) = 2|E|$, where $|E|$ is the number of edges of G.

(iii) Show that $\text{trace}(A_3) = 6|T|$, where $|T|$ is the number of triangles of G. (A *triangle* is a complete subgraph with three nodes.)
(iv) Verify results (i)–(iii) on the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$