Fourth Problem Set

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Problem 1

$B \models C$ just in case $\models (B \to C)$
just in case $\bar{v}(B \to C) = T$
just in case
$\bar{v}(B) = T$ and $\bar{v}(C) = T$
or
$\bar{v}(B) = F$ and $\bar{v}(C) = T$
or
$\bar{v}(B) = F$ and $\bar{v}(C) = F$
In the second two cases, $\bar{v}(B) = F$ which means $\bar{v}(\neg B) = T$
Hence $\models \neg B$
In the first two cases $\bar{v}(C) = T$ which means $\models C$
So in all three cases $\models \neg b$ or $\models C$.

Problem 2

(a)

$(x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge \neg z)$

(b)

$\neg(x \wedge y \wedge z) \wedge \neg(x \wedge \neg y \wedge \neg z) \wedge \neg(\neg x \wedge \neg y \wedge z)$
$\Rightarrow (\neg x \vee \neg y \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z)$

Problem 3

(a)

Let $C = \{\text{customers}\}$ $J = \{\text{Jeans}\}$ $J_x = \{\text{Jeans that fit } x \in C\}$. We know that $\forall x_1, \ldots, x_m \; \exists j_1, \ldots, j_m$ such that $j_i \in J_{x_i}$ where $i = 1, \ldots, m$

Assume J_x is finite $\forall x \in C$

Need to show that there is a way of assigning jeans to customers so that every $x \in C$ buys a $j \in J$ that fits and no two customers but the same pair of jeans.
(i.e. $\exists p : C \to J$ such that $p(x) \in J_x$ and p is injective)

Let $\Sigma \subseteq \bar{\mathcal{S}}$ such that $\exists v : \mathcal{S} \to \{T, F\}$ satisfying Σ iff There exists such a function p

We will use sentence letters: b_{xj} to build the wffs in Σ

Let $\Sigma = A \cup B \cup C$ where
$A = \{\bigvee_{j \in J_x} b_{xj}, \forall x \in C\}$
$B = \{b_{xj} \to \neg b_{xl}, \forall x \in C, \forall j, l \in J, j \neq l\}$
$C = \{b_{xj} \to \neg b_{yj}, \forall x, y \in C, \forall j \in J\}$

It is useful to note that by A every customer $x \in C$ buys a pair of jeans (i.e. $p(x) \in J_x$), by B every customer only buys one pair of jeans (i.e. p is well defined) and by C every pair of jeans is only bought by one customer (i.e. p is injective).

Now we can show that Σ is satisfiable using the compactness theorem.

Let $\Delta \subseteq \Sigma$ be finite.

Lemma: $\Delta \subseteq \Delta'$ and Δ' is satisfiable then Δ is satisfiable.

Let $C_0 = \{x \in C | b_{xj}$ shows up in some wff in $\Delta\}$ and $J_0 = \{j \in J | b_{xj}$ shows up in some wff in $\Delta\}$, these sets are both finite because Δ is finite.
Now let $J_1 = J_0 \cup \bigcup_{x \in C_0} J_x$ and take Δ' to be the set:

$$\Delta' = \{\text{all wffs in } \Sigma \text{ built from } b_{xj} \text{ with } x \in C_0 \text{ and } j \in J_1\}$$

Now to show that Δ' is satisfiable, we have a finite set of customers C_0 and we can define

$$p' : C_0 \to J_1$$

We can show that $p'(x) \in J_{1_x}$ because the definition of J_1 says that it contains all of the sets of fitting jeans for each customer. We know that p' is also well defined and injective. Therefore Δ' is satisfiable. Meaning Δ is satisfiable and by the compactness theorem Σ is satisfiable.

(b)

We can show that the conclusion will not hold if we drop the extra hypothesis by providing a coutnerexample.

Let $C = \{c_0, c_1, c_2, \ldots\}$ and $J = \{j_0, j_1, j_2, \ldots\}$
There is at least one customer that fits infinitely many jeans because we dropped the extra hypothesis (i.e. say c_0 fits $j_0, j_1, j_2, \ldots$)
Say
c_1 only fits j_0
c_2 only fits j_1
$\vdots$
This satisfies the conditions of the problem because given a finite set $D = \{x_0, x_1, \ldots, x_m\} \subseteq C$ we can say c_i fits j_{i-1} for $i = 1, \ldots, m$ then c_0 fits j_m.
Now if we set up a function $p : C \to J$ as:

$$p(c_i) = j_{i-1}, \forall i > 0$$

Then there is no pair of jeans for c_0 to buy.