Dada la aplicación lineal $f : \mathbb{R}^3 \to \mathbb{R}^3$, tal que $f(x, y, z) = (3x - y, 2y - x - z, 3z - y)$ se pide:

a) Debe ser lineal y el espacio vectorial inicial = final

$f(a\bar{u}) = af(\bar{u})$ $\quad / \bar{u} = (u_1, u_2, u_3)$ y $\bar{e} = (e_1, e_2, e_3)$

$f(\bar{u}+\bar{e}) = f(\bar{u}) + f(\bar{e})$

$f(a\bar{u}) = (3au_1 - au_2, 2au_2 - au_1 - au_3, 3au_3 - au_2) = (a(3u_1 - u_2), a(2u_2 - u_1 - u_3), a(3u_3 - u_2))$

$= af(\bar{u})$

$f(\bar{u}+\bar{e}) = (3u_1 + 3e_1 - u_2 - e_2, 2u_2 + 2e_2 - u_1 - e_1 - u_3 - e_3, 3u_3 + 3e_3 - u_2 - e_2)$

$= f(\bar{u}) + f(\bar{e})$

$$(3x - y, 2y - x - z, 3z - y) = (3x, -x, 0) + (-y, 2y, -y) + (0, -z, 3z) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A_{B_c} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

b) $\bar{v} = (1, 2, 0)$

$$f(\bar{v}) = A\bar{v} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

c) $\bar{w} = (1, 1, 1)$ $\quad \exists \bar{u} \in \mathbb{R}^3 / f(\bar{u}) = \bar{w}$

$$\left.\begin{array}{r} 3x - y = 1 \\ -x + 2y - z = 1 \\ -y + 3z = 1 \end{array}\right\} \to \begin{array}{l} x = 0.75 \\ y = 1.25 \\ z = 0.75 \end{array} \to \bar{u} = (0.75, 1.25, 0.75)$$

d) $B' = \{(1,0,1), (1,-1,0), (1,0,0)\} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$

$$\left.\begin{array}{ll} f(\bar{u}_1) = (3, -2, 3) & [f(\bar{u}_1)]_{B'} = (3, 2, -2) \\ f(\bar{u}_2) = (4, -3, 1) & [f(\bar{u}_2)]_{B'} = (1, 3, 0) \\ f(\bar{u}_3) = (3, -1, 0) & [f(\bar{u}_3)]_{B'} = (0, 1, 2) \end{array}\right\} A_{B'} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 0 & 2 \end{pmatrix}$$

$\langle B' \rangle = (x + y + z, -y, x)$

e) $[\bar{z}]_{B'} = (0, 1, -1)$, $\quad [f(\bar{z})]_{B'}$?

Como tenemos la matriz de la aplicación respecto a B',

$\bar{z} = (0, -1, -1)$

$A_{B'} \bar{z} = (-1, -4, -2) = [f(\bar{z})]_{B'}$

f) Buscamos los autovalores:

$(A-\lambda I)\bar{v}=\bar{0}$

$\det(A-\lambda I)=0$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda)-(3-\lambda+3-\lambda) = (3-\lambda)^2(2-\lambda)-2(3-\lambda)$$

$$= (3-\lambda)((3-\lambda)(2-\lambda)-2)$$

$$= (3-\lambda)(4-5\lambda+\lambda^2)$$

$$= (3-\lambda)(\lambda-1)(\lambda-4)=0 \Rightarrow \lambda = 3,1,4$$

Como la m.a. de todos los autovalores es 1, su m.g. también y por tanto el endomorfismo es diagonalizable.

Buscamos los autovectores:

$$(A-\lambda_1 I)\bar{v} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-y, -x-y-z, -y) = \bar{0}$$

$$\left.\begin{array}{l} x+y+z=0 \\ y=0 \\ z=\lambda \end{array}\right\} H_1 = \{(-\lambda, 0, \lambda) / \lambda \in \mathbb{R}\} \Rightarrow \bar{v}_1 = (-1,0,1)$$

$$(A-\lambda_2 I) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x-y, -x+y-z, -y+2z) = \bar{0}$$

$$\left.\begin{array}{l} 2x-y=0 \\ \cancel{-x+y-z=0} \\ -y+2z=0 \end{array}\right\} \left.\begin{array}{l} y=\lambda \\ x=\frac{\lambda}{2} \\ z=\frac{\lambda}{2} \end{array}\right\} H_2 = \{(\tfrac{\lambda}{2}, \lambda, \tfrac{\lambda}{2}) / \lambda \in \mathbb{R}\} \Rightarrow \bar{v}_2 = (1,2,1)$$

$-(eq_1+eq_3) = 2eq_2$

$$(A-\lambda_3 I)\bar{v} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-x-y, -x-2y-z, -y-z) = \bar{0}$$

$$\left.\begin{array}{l} x+y=0 \\ \cancel{x+2y+z=0} \\ y+z=0 \end{array}\right\} \left.\begin{array}{l} y=\lambda \\ x=-\lambda \\ z=-\lambda \end{array}\right\} H_3 = \{(-\lambda, \lambda, -\lambda) / \lambda \in \mathbb{R}\} \Rightarrow \bar{v}_3 = (-1,1,-1)$$

$eq_1 + eq_3 = eq_2$

$B_0 = \{(-1,0,1), (1,2,1), (-1,1,-1)\}$

$$A_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

g) $\bar{v} = (1,2,0)$ $\quad [\bar{v}]_{B_0}$?

$< B_0 > = (-x+y-z, 2y+z, x+y-z)$

$$\left.\begin{array}{l} -x+y-z=1 \\ 2y+z=2 \\ x+y-z=0 \end{array}\right\} \Rightarrow \begin{array}{l} x=-0.5 \\ y=\frac{5}{6} \\ z=\frac{1}{3} \end{array}$$

h)

$[f(\bar{v})]_{B_0} = A_0 \bar{v} = (3\cdot 1, 1\cdot 2, 4\cdot 0) = (3, 2, 0)$