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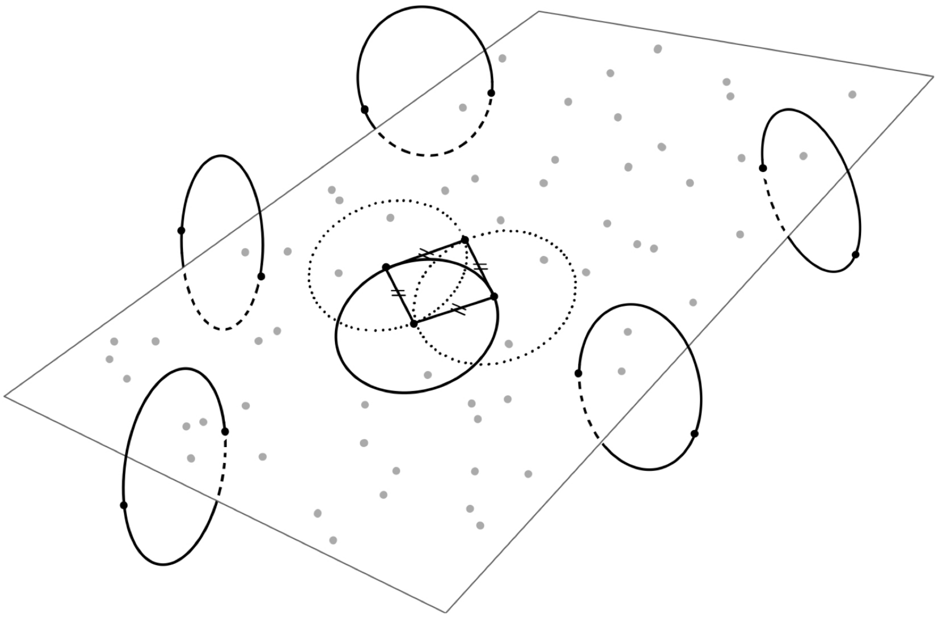
Paradoxical sets and the Axiom of Choice



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Mathematics

Paradoxical sets and the Axiom of Choice

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Contents

Chapter 1

Introduction

The present thesis is divided thematically into two parts: *Algebraic closures of certain subfields of the reals* (Part I) and *Paradoxical sets of reals* (Part II). Part I investigates a folklore result about the forcing extension by one Cohen real: the transcendence degree of the reals over the set of reals in the ground model is of cardinality $\mathfrak{c}$ (in the extension). We extend this to the case in which more Cohen reals are added obtaining the following result:

Theorem A (cf. Theorem 3.14)
Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then, in $V[X]$ the transcendence degree of $\mathbb{R}$ over F is continuum.

This result requires that V is a model of the Axiom of Choice so that we can talk about the transcendence degree in its extension. Nevertheless, this result is going to be used in Chapter 7 to prove the existence of a specific paradoxical set in some choiceless models.

In Part II, we consider some paradoxical sets of reals and study their interaction with the Axiom of Choice. Informally, paradoxical sets are subsets of $\mathbb{R}^n$ that can be constructed using the Axiom of Choice. Their existence can be counter-intuitive at first sight: for example, the well-known examples of a non-measurable set by Vitali [59] and the partition given by the Banach–Tarski paradox [3]. Although there are many examples of paradoxical sets (see, for example, [37]), much remains unknown about these objects.

In this work we will focus on the following examples of paradoxical sets:

Example 1. A **Hamel basis** is a basis of $\mathbb{R}$ as $\mathbb{Q}$-vector space.

Example 2. A **two-point set**, also called a *Mazurkiewicz set*, is a subset A of $\mathbb{R}^2$ such that, for every line l on the plane, $|A \cap l| = 2$.

Example 3. A partition of $\mathbb{R}^3$ into unit circles (**PUC**).

The known proofs of existence of these objects (Examples 1–3) rely on a transfinite induction on a well-order of the reals [45, 17].

For any notion of paradoxical set there are natural questions to ask about its properties. For example, whether we can have a paradoxical set that is Borel, measurable, meager, etc. We will give a literature review of the objects in Examples 1, 2, and 3 in the Sections 5.1, 6.1 and 7.1 respectively. Nevertheless, many questions remain unanswered, such as the following.

Question 1. *Can a two-point set or a* PUC *be Borel?* [43, 26]

After long efforts in trying to answer whether a two-point set can be Borel [6, 43, 46], another approach to this question was needed. Recent work has shifted focus to studying these objects from a set-theoretical perspective [8, 12, 42, 52]. In that direction, the main question considered through this work is the following.

Question 2. *Can we recover some weakening of the Axiom of Choice from the existence of a particular paradoxical set?*

There have been a number of recent results on this topic. Larson and Zapletal [42] developed a broad technique that deals with a similar type of problems, under some large cardinals assumptions. There has also been some progress using symmetric extensions of models of set theory. In particular, Schilhan [52] gives a partial negative answer to Question 2 for sets of reals that can be defined as *maximal independent sets*.

However, the approach that is taken here is different from the aforementioned lines of work. First, we do not have any large cardinal assumption. Second, the objects of Examples 2 and 3 above cannot be defined as maximal independent sets. This is usually the case for partitions. Instead, the direction of this work follows the lines of other authors (for example, [8, 9, 12, 33, 54]).

The contribution of this thesis is giving negative answers for different versions of Question 2, changing the particular paradoxical set and the weakening of the Axiom of Choice considered. The choice principles we will examine are: the existence of a well-ordering of the reals ($\mathsf{WO}(\mathbb{R})$), the Principle of Dependent Choices (DC), the existence of a non-principal ultrafilter on ω ($\mathsf{Ul}(\omega)$), and Countable Choice (AC_ω). For their definitions and the relations between them the reader is invited to see Section 2.2.

For the notion of paradoxical set given by a Hamel basis of $\mathbb{R}$, it was known that there is a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$ where there is a Hamel basis of $\mathbb{R}$ [54, Theorem 1.1]. We have proven the following strengthening:

Theorem B (cf. Theorem 5.2)
There is a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{UI}(\omega)$ *with a Hamel basis.*

For the case of Mazurkiewicz sets, our methods will recover a known negative answer to Question 2, namely, that there is model of $\mathsf{ZF}+\mathsf{DC}+\neg\mathsf{WO}(\mathbb{R})$ with a Mazurkiewicz set [7]. See Theorem 6.6.

For the paradoxical set of Example 3, we give a negative answer to Question 2 in the following theorems.

Theorem C (cf. Theorem 7.8)
There is a model of

$$\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \textit{there is partition of } \mathbb{R}^3 \textit{ in unit circles.}$$

Theorem D (cf. Theorem 7.9)
There is a model of

$$\mathsf{ZF} + \neg\mathsf{AC}_\omega + \textit{there is partition of } \mathbb{R}^3 \textit{ in unit circles.}$$

The model of Theorem 7.9 is a well-known model called *Cohen model* or *Cohen-Halpern-Lévy model*. It was already known that in this model there are many examples of paradoxical sets while the Axiom of Choice fails dramatically (see Theorem 2.31).

Furthermore, the main contribution of the present work is to develop a framework that addresses most of these results. That is the main goal of Chapter 4. We then apply the theorems of Chapter 4 to the following paradoxical sets: Hamel bases in Chapter 5, Mazurkiewicz sets in Chapter 6 and PUCs in Chapter 7.

Chapter 2

Preliminaries

We start by listing several results that will be used throughout this text. If the reader is familiar with basic Set theory, they can probably skip all of Section 2.1. Section 2.2 contains the choice principles that will be considered in Part II, and some more for context. Finally, we end this chapter with Sections 2.3 and 2.4 that carry some basic results in Algebra and Analysis, respectively. These will be used mainly through chapters 3, 6, and 7.

2.1 General Set Theory

We include here several results of Set Theory, most of which are well-known and included here for completeness. The reader used to Set Theory can skip this section, at most taking a look at the statements of Lemma 2.12, Theorem 2.17 and Theorem 2.18, which will be used very often.

Definition 2.1. An **inner model** is a transitive class that contains all the ordinals and satisfies all the axioms of ZF. We also say that M is an *inner model of* N if it contains the same ordinals than N and M is a transitive class of N.

Definition 2.2. Let $\mathrm{def}(M)$ denote the family of definable sets over a model $(M, \in)$ with parameters in M. We define $L(A)$ as follows. Let $T = TC(\{A\})$

be the transitive closure of A, and let

$$\begin{aligned} L_0(A) &= T, \\ L_{\alpha+1}(A) &= def(L_\alpha(A)), \\ L_\alpha(A) &= \bigcup_{\beta<\alpha} L_\beta(A) \text{ if } \alpha \text{ is a limit ordinal, and} \\ L(A) &= \bigcup_{\alpha\in\mathrm{OR}} L_\alpha(A). \end{aligned}$$

The transitive class $L(A)$ is an inner model of ZF, contains A as an element, and is the smallest such inner model. If $A = \emptyset$, we write L_α instead of $L_\alpha(\emptyset)$ and L instead of $L(\emptyset)$. This model is usually called *Gödel's constructible universe*.

The most important property of the hierarchy described in Definition 2.2 is that the function $\alpha \mapsto L_\alpha$ is absolute between transitive models of ZF [40, Lemma II.6.13]. Similarly, $\alpha \mapsto L_\alpha(A)$ is absolute between transitive models of ZF containing A. In particular, if M is an inner model of N then $L^M = L^N$, because M and N have the same ordinals.

Definition 2.3. If M is an inner model of N and $a \in N$, we let $M[a]$ denote the smallest inner model of N which includes M and contains a as an element.

Let V be a model of ZFC, L the constructive universe in V, and let $a \in V\setminus L$. Notice that then $L[a]$ as in Definition 2.3 coincides with $L(a)$ as in Definition 2.2 because they are the smallest inner model containing a. Also, if $M[g]$ is a generic extension of M and $g \in N$, where M is an inner model of N, then $M[g]$ is also an inner model of N and it is the least such model that contains $M \cup \{g\}$. So the notation $M[g]$ as forcing extension coincides in this case with the notation of Definition 2.3.

We will introduce next another way of constructing an inner model of a given model, the class HOD_A.

Definition 2.4. A set x is **ordinal-definable** over A if there is a formula ϕ such that

$$X = \{u \mid \phi(u, \alpha_0, \ldots, \alpha_{n-1}, x_0, \ldots, x_{k-1}, A)\}$$

for some ordinal numbers $\alpha_0, \ldots, \alpha_{n-1}$ and elements $x_0, \ldots, x_{k-1}$ of A.

The class HOD_A of all hereditarily ordinal-definable sets over A is a transitive model of ZF.

Definition 2.5. Let $\mathcal{A}^2(\mathsf{a})$ (with $\mathsf{a} \in {}^\omega\omega$) be the structure $\langle \omega, {}^\omega\omega, \mathrm{ap}, +, \times, \exp, <, 0, 1, \mathsf{a}\rangle$ with $+, \times, \exp, <, 0, 1$ as the usual structure on ω, $\mathrm{ap}\colon {}^\omega\omega \times \omega \to \omega$ given by $\mathrm{ap}(x, m) = x(m)$, and where a is a binary relation.

Quantifiers over ${}^\omega\omega$ are called **real quantifiers** and denoted by $\forall^1$, $\exists^1$.

Definition 2.6. Let $A \subseteq {}^r\omega \times {}^k({}^\omega\omega)$ and $a \in {}^\omega\omega$. A is **arithmetical** in a iff A is definable in $\mathcal{A}^2(a)$ by a formula without real quantifiers.

For $n > 0$, we say

$$\begin{aligned} &A \text{ is } \Sigma^1_n(a) \text{ iff } \forall w(A(w) \iff \exists^1 x_1 \forall^1 x_2 \dots Qx_n R(w, x_1, \dots, x_n)), \textit{and} \\ &A \text{ is } \Pi^1_n(a) \text{ iff } \forall w(A(w) \iff \forall^1 x_1 \exists^1 x_2 \dots Qx_n R(w, x_1, \dots, x_n)) \end{aligned}$$

where $R \subseteq {}^r\omega \times {}^{k+n}({}^\omega\omega)$ is some arithmetical relation in a, and Q is $\exists^1$ or $\forall^1$ depending on whether n is odd or even to preserve the alternation of quantifiers.

This hierarchy of sets of reals given by Definition 2.6 is called the **analytical hierarchy**. There is another well-known hierarchy called the *projective hierarchy* which is defined by projections and complements of open sets. Here we will define it by its relation with the analytical hierarchy instead.

Definition 2.7. Suppose that $A \subseteq {}^k({}^\omega\omega)$ and $n < \omega$. Then A is $\boldsymbol{\Sigma^1_n}$ iff A is $\Sigma^1_n(a)$ for some $a \in {}^\omega\omega$, and A is $\boldsymbol{\Pi^1_n}$ iff A is $\Pi^1_n(a)$ for some $a \in {}^\omega\omega$.

We say $A \subseteq {}^\omega\omega$ is **Borel** iff A is both $\boldsymbol{\Sigma^1_1}$ and $\boldsymbol{\Pi^1_1}$. We say $A \subseteq {}^\omega\omega$ is **analytic** iff A is $\boldsymbol{\Sigma^1_1}$. This is equivalent to A being the projection of a closed set in ${}^\omega\omega \times {}^\omega\omega$.

One of the important properties of the hierarchy defined in 2.7 is that sets in the lower levels are absolute:

Definition 2.8. If M is a transitive model and ϕ is a formula, we say that the formula ϕ is **absolute** for M iff for all $x_0, \dots, x_{n-1}$

$$\phi^M(x_0, \dots, x_{n-1}) \iff \phi(x_0, \dots, x_{n-1}).$$

Theorem 2.9 (Mostowski's Absoluteness [29, Theorem 25.4])
If P is a $\boldsymbol{\Sigma^1_1}$ property then P is absolute for every transitive model of ZF *that contains the parameter of the definition of P.*

Theorem 2.10 (Shoenfield's Absoluteness Theorem [29, Theorem 25.20])
Every $\Sigma^1_2(a)$ and $\Pi^1_2(a)$ relation is absolute for all inner models M of $\mathsf{ZF} + \mathsf{DC}$ *such that $a \in M$. In particular $\boldsymbol{\Sigma^1_2}$ and $\boldsymbol{\Pi^1_2}$ relations are absolute for L.*

We will assume the reader is familiar with the basics of forcing, but we will explicitly state some definitions and properties that will be used often. Here, a forcing notion $\mathbf{P} = (\mathbf{P}, \leq_{\mathbf{P}}, \mathbb{1}_{\mathbf{P}})$ is a partially ordered set with a largest element $\mathbb{1}_{\mathbf{P}}$.

Definition 2.11. Let $\mathbf{C} = {}^{<\omega}\omega$ the forcing given by

$$\mathbf{C} = \{p\colon \omega \to \omega \mid p \text{ is a partial function with } \operatorname{dom}(p) < \omega\},$$

ordered by reverse inclusion and $\mathbb{1}_{\mathbf{C}} = \emptyset$. We call this forcing **Cohen forcing**[1].

For a set of ordinals X, we write $\mathbf{C}(X)$ for the finite support product of X-many copies of $\mathbf{C}$. Namely,

$$\mathbf{C}(X) = \{p \in \Pi_{\alpha\in X}\mathbf{C} : |\{\alpha \in X : p(\alpha) \neq \emptyset\}| < \omega\},$$

ordered coordinatewise.

If $g \subseteq \mathbf{C}$ is a generic filter over a model V, $\cup g \in {}^{\omega}\omega \cap V[g]$ is a real usually called the **Cohen real** added by g. If $g \subseteq \mathbf{C}(X)$ is a generic filter over V, then $\cup(g \restriction \{\alpha\})$ is also a real for each $\alpha \in X$. We will often mix up the generics with the reals added by them for these forcings.

We will use several times a nice fact of $\mathbf{C}(\omega_1)$, that establishes that any real in a forcing extension of this forcing is in the model produced by some initial segment of the generic.

Lemma 2.12. *Let g be $\mathbf{C}(\omega_1)$-generic over V and let $r \in {}^{\omega}\omega \cap V[g]$. Then there is $\alpha < \omega_1$ such that $r \in V[g \restriction \alpha]$.*

Proof: Identify r as an element of ${}^{\omega}2$. Let τ be a name for r. We can assume τ is of the form $\{(\check{n}, p) \mid p \in A_n\}$, where A_n is an antichain for every $n < \omega$. It is a well-known fact that $\mathbf{C}(\omega_1)$ is ccc, namely, every antichain on $\mathbf{C}(\omega_1)$ is countable. Therefore each A_n is countable. For each n, let α_n the supremum of the supports of conditions in A_n. Since A_n is countable, $\alpha_n < \omega_1$. Let α be the supremum of $\{\alpha_n\}_{n<\omega}$. Again, $\alpha < \omega_1$. Then τ is also a name in $V^{\mathbf{C}(\alpha)}$ and therefore $r = \tau_g = \tau_{g\restriction\alpha} \in V[g \restriction \alpha]$. $\square$

Notice that $g \restriction \alpha$ is $\mathbf{C}(\alpha)$-generic over V; so the notation $V[g \restriction \alpha]$ makes sense. Also, observe that $\mathbf{C}(\omega_1)$ is essentially the same (there is a natural isomorphism of partial orders) as $\mathbf{C}(\alpha) \times \mathbf{C}(\omega_1 \backslash \alpha)$ for each $\alpha < \omega_1$. Moreover, if X has cardinality $\aleph_1$ in V, then $\mathbf{C}(X)$ is isomorphic to $\mathbf{C}(\omega_1)$.

[1]Notice that here we denote this forcing by $\mathbf{C}$ and not $\mathbb{C}$ as frequently seen in the literature. This is to avoid confusion with the complex plane $\mathbb{C}$.

There is another way in which two forcing relations can be related, which is called *forcing equivalence*.

Definition 2.13. Let $\mathbf{P}$, $\mathbf{Q}$ be two forcing posets. We say that $\mathbf{P}$ and $\mathbf{Q}$ are **forcing equivalent** ($\mathbf{P} \cong \mathbf{Q}$) iff there is a third poset $\mathbf{R}$ and there are i and j such that $i\colon \mathbf{P} \to \mathbf{R}$ and $j\colon \mathbf{Q} \to \mathbf{R}$ are dense embeddings.

The following lemma illustrates a useful property of forcing equivalence.

Lemma 2.14. *Let M be a transitive model of ZFC and let $\mathbf{P}, \mathbf{Q}, i \in M$, where $i\colon \mathbf{P} \to \mathbf{Q}$ is a dense embedding. If h is $\mathbf{Q}$-generic over M and $g = i^{-1}(h)$, then g is $\mathbf{P}$-generic over M and $M[g] = M[h]$.*

The following theorem gives us a nice characterization of Cohen forcing up to forcing equivalence.

Theorem 2.15 ([24, Theorem 1 Section 4.5])
Let $\mathbf{P}$ be a separative, countable and atomless poset. Then $\mathbf{P}$ contains a dense subset isomorphic to $\mathbf{C}$. In other words, Cohen forcing $\mathbf{C}$ is the only countable atomless forcing (modulo forcing equivalence).

Remark.
If X is a set of ordinals that is countable in V, then $\mathbf{C} \cong \mathbf{C}(X)$.

It is a well known fact that any forcing is forcing equivalent to a separative forcing [29, Lemma 14.11], so we do not need to check whether $\mathbf{P}$ is separative in Theorem 2.15.

Notation: If M and N are two models of ZF, we write $N \hookrightarrow M$ to denote that M is a forcing extension of N. We write $N \overset{\mathbf{P}}{\hookrightarrow} M$ if N is a **$\mathbf{P}$-ground** of M, i.e., $M = N[g]$ for some $\mathbf{P}$-generic filter g over N.

The following is a result that we will use multiple times.

Theorem 2.16 (The Solovay basis result [24, Theorem 2 Section 2.14])
Let M be a model of ZF, $\mathbf{P} \in M$ be a forcing notion and let g be a $\mathbf{P}$- generic filter over M. If $a \in M[g]$ and $a \subseteq M$, then

$$M \hookrightarrow M[a] \hookrightarrow M[g].$$

Moreover, the first forcing is given by a complete subalgebra of the completion boolean algebra of $\mathbf{P}$ and the second is a forcing given by the quotient $\mathcal{B}/H$ where H is a generic filter of the first forcing.

Now we will apply Theorem 2.16 to our favorite forcings $\mathbf{C}$ and $\mathbf{C}(\omega_1)$.

Theorem 2.17
Let g be a $\mathbf{C}$-generic filter over V and let $r \in {}^{\omega}\omega \cap V[g]$. Then

$$V \overset{a}{\hookrightarrow} V[r] \overset{b}{\hookrightarrow} V[g],$$

where $a \cong \mathbf{C}$ if $V[r] \neq V$, and $b \cong \mathbf{C}$ if $V[r] \neq V[g]$.

Theorem 2.18
Let g be a $\mathbf{C}(\omega_1)$-generic filter over V and let $r \in {}^{\omega}\omega \cap V[g]$. Then

$$V \overset{a}{\hookrightarrow} V[r] \overset{b}{\hookrightarrow} V[g],$$

where $a \cong \mathbf{C}$ if $V[r] \neq V$ and $b \cong \mathbf{C}(\omega_1)$.

Proof: Let α be such that $r \in V[g \restriction \alpha]$ (see Lemma 2.12). Let x be the Cohen real such that $V[x] = V[g \restriction \alpha]$ (see the discussion below Theorem 2.15). By Theorem 2.16,

$$V \overset{a}{\hookrightarrow} V[r] \overset{c}{\hookrightarrow} V[x] = V[g \restriction \alpha] \xhookrightarrow{\mathbf{C}(\omega_1 \backslash \alpha)} V[g].$$

Moreover, applying Theorem 2.17, a and c are each forcing equivalent to Cohen forcing or a trivial forcing. Then $b = \mathbf{C}(\omega_1 \backslash \alpha)$ or $b \cong \mathbf{C} \times \mathbf{C}(\omega_1 \backslash \alpha)$. In any case, $b \cong \mathbf{C}(\omega_1)$. □

Lemma 2.19 (Product Lemma [53, Lemma 6.65])**.** *Let M be a transitive model of* ZFC *and let $\mathbf{P}$ and $\mathbf{Q}$ be partial orders in M. If g is $\mathbf{P}$-generic over M and h is $\mathbf{Q}$-generic over $M[g]$, then $g \times h$ is $(\mathbf{P} \times \mathbf{Q})$-generic over M. Conversely, if $k \subseteq \mathbf{P} \times \mathbf{Q}$ is $(\mathbf{P} \times \mathbf{Q})$-generic over M, then*

$$\begin{aligned} g &= \{p \in \mathbf{P} \mid \exists q \in \mathbf{Q}\, (p, q) \in K\}, \text{ and} \\ h &= \{q \in \mathbf{Q} \mid \exists p \in \mathbf{P}\, (p, q) \in K\} \end{aligned}$$

are $\mathbf{P}$ generic over M and $\mathbf{Q}$-generic over $M[g]$, respectively.

Definition 2.20. In the situation of Lemma 2.19, we say that g and h are **mutually generic**.

For example, if g is $\mathbf{C}(\omega_1)$-generic over M, then $g \restriction \alpha \subseteq \mathbf{C}(\alpha)$ and $g \restriction (\omega_1 \backslash \alpha) \subseteq \mathbf{C}(\omega_1 \backslash \alpha)$ are mutually generic for any $\alpha < \omega_1$.

One property of mutual genericity that we will use often is the following: if g and h are mutually generic over a model M, then $M[g] \cap M[h] = M$.

Definition 2.21. Let x be a function $x\colon \omega \to \omega$. We can **split** x in two reals x_0, x_1 such that $x = x_0 \oplus x_1$, where $\oplus$ is the operation of alternating digits from each of the reals, namely, $x(2n) = x_0(n)$ and $x(2n+1) = x_1(n)$ for all $n < \omega$. If s is a finite initial segment of x, we say we **split** x **according to** s in two reals x_0 and x_1 iff s is an initial segment of both x_0 and x_1 and $x \backslash s = (x_0 \backslash s) \oplus (x_1 \backslash s)$.

Remark.
Let x be a Cohen real over a model M. In $M[x]$, let x_0, x_1 be the split of x according to s. Then x_0 and x_1 are mutually generic Cohen reals over M.

Definition 2.22. Let $\mathbf{P}$ be a poset. We call $\mathbf{P}$ **homogeneous** iff for all $p, q \in \mathbf{P}$, there is a dense homomorphism π from $\mathbf{P}$ to itself such that $\pi(p) = q$.

Lemma 2.23 ([53, Lemma 6.53])**.** $\mathbf{C}$ *is homogeneous. If α is an ordinal, then $\mathbf{C}(\alpha)$ is homogeneous.*

Lemma 2.24 ([53, Lemma 6.61])**.** *Let M be a transitive model of* ZFC*, let $\mathbf{P} \in M$ be a homogeneous forcing notion. Let ϕ be a formula and $x_0, \ldots, x_{n-1} \in M$. Then*

$$\mathbb{1} \Vdash^{\mathbf{P}}_{M} \phi(\check{x}_0, \ldots, \check{x}_{n-1}), \ \textit{or} \ \mathbb{1} \Vdash^{\mathbf{P}}_{M} \neg\phi(\check{x}_0, \ldots, \check{x}_{n-1}).$$

Definition 2.25. A partial order $\mathbf{P}$ is σ-closed iff whenever $\langle p_n \mid n < \omega \rangle$ is a decreasing sequence of elements of $\mathbf{P}$, then there is $q \in \mathbf{P}$ such that $q \leq p_n$ for all $n \in \omega$.

Lemma 2.26 ([40, Lemma IV.7.15])**.** *Assume that ($\mathbf{P}$ is σ-closed)^M, M a transitive model of* ZFC*, and fix $f\colon \omega \to \omega$ with $f \in M[g]$. Then $f \in M$. In other words, σ-closed forcings do not add reals.*

Definition 2.27. Let g be a $\mathbf{C}(\omega)$-generic filter over L. Let us write $A = \{c_n : n < \omega\}$ for the set of Cohen reals added by g, i.e., $c_n = \cup(g \restriction \{n\})$ for $n < \omega$. The model

$$H = \mathsf{HOD}_A^{L[g]}$$

of all sets which are hereditarily ordinal definable inside $L[g]$ from parameters in $A \cup \{A\}$ is called **the Cohen–Halpern–Lévy model**.

We will use this model in Section 7.4 so we will describe some of its properties. It was introduced by Cohen [16, pp. 136–141], and explored later in a different presentation by Halpern and Lévy [25].

Theorem 2.28 ([16, pp. 136–141])
In H,

- $\mathbb{R} = \bigcup_{a\in[A]^{<\omega}}(\mathbb{R}\cap L[a])$,
- *there is no well-ordering of the reals, and*
- *A has no countable subset.*

Lemma 2.29 ([9])**.** *Consider the model H. Fix an enumeration of the rudimentary functions, and for any $a \in [A]^{<\omega}$ consider the natural order on a as a finite subset of reals. Then this fixes a global order $<_a$ in $L[a]$. In other words, the relation consisting on triples (a, x, y) such that $x <_a y$ is definable over H.*

Theorem 2.30 ([25])
The Prime Ideal Theorem holds in H.

From the Prime Ideal Theorem we get that there is a non-principal ultrafilter on ω.

Theorem 2.31
In H, there is

1. *a Luzin set [48, Section II],*
2. no *Sierpiński set [9, Theorem 1.6],*
3. *a Bernstein set [9, Theorem 1.7],*
4. *a Vitali set [48, II.3]* [2],
5. *a Hamel basis [9, Theorem 2.1], and*
6. *a Mazurkiewicz set. [8, Corollary 0.3].*

Lemma 2.32. *In H there is no countable subset of A.*

[2]Pincus and Prikry attributed it to Feferman.

2.2 Axiom of Choice

In this section we will introduce all the *choice principles* that will be mentioned throughout the text, and some more for context. They do not follow from ZF, and most of them are strictly weaker than the Axiom of Choice. We mostly use here the notation and definitions given by Jech [30].

AXIOM OF CHOICE (AC): For every family $\mathcal{F}$ of nonempty sets, there is a choice function, namely a function f such that $f(S) \in S$ for each S in the family $\mathcal{F}$.

WELL-ORDERING PRINCIPLE (WO): Every set can be well ordered.

WELL-ORDERING OF THE REALS (WO($\mathbb{R}$)): There is a well order of $\mathbb{R}$.

ZORN'S LEMMA: Let $(P, <)$ be a nonempty partially ordered set, and let every chain in P have an upper bound. Then P has a maximal element.

BASES FOR VECTOR SPACES (VS): Every vector space has a basis.

PRIME IDEAL THEOREM (PIT): Every Boolean algebra has a prime ideal.

ULTRAFILTER THEOREM: Every filter over a set S can be extended to an ultrafilter.

NON-PRINCIPAL ULTRAFILTER (UI): For each infinite set S, there is a non-trivial ultrafilter over S, namely, one that is not given by all the sets that contain a fixed element.

NON-PRINCIPAL ULTRAFILTER ON ω (UI(ω)): There is a non principal ultrafilter on ω.

COUNTABLE AXIOM OF CHOICE (AC_ω): Every countable family of nonempty sets has a choice function.

COUNTABLE AXIOM OF CHOICE FOR SETS OF REALS ($\mathsf{AC}_\omega(\mathbb{R})$): Every countable family of nonempty sets of reals has a choice function.

PRINCIPLE OF DEPENDENT CHOICES (DC): If R is a relation on a nonempty set A such that for every $x \in A$ there is $y \in A$ with xRy, then there is a sequence $\{x_n\}_{n<\omega}$ such that $x_n R x_{n+1}$ for all $n < \omega$.

Under ZF, there are some relations between these principles. For example, the Ultrafilter Theorem is equivalent to the Prime Ideal Theorem [27, Theorem 2.16.1]. Figure 2.1 shows all the relevant known implications between these notions, and each implication cannot be reversed.

$$
\begin{tikzcd}
\mathsf{AC} \arrow[r, leftrightarrow, "\text{Thm 2.2}"] \arrow[d, "\text{Thm 2.12.1}"] & \text{Zorn's Lemma} \arrow[r, leftrightarrow, "\text{Thm 1.4}"] \arrow[d] & \mathsf{WO} \arrow[r, leftrightarrow, "\text{Thm 4.4}"] \arrow[d] & \mathsf{VS} \arrow[dd] \\
\mathsf{DC} \arrow[d, "\text{Thm 2.12.2}"] & \mathsf{PIT} \arrow[d, "\text{Thm 2.12}"] & \mathsf{WO}(\mathbb{R}) \arrow[ddll] \arrow[ddl, "\text{Ex. 2.2.E10}"] \arrow[dr] & \\
\mathsf{AC}_\omega \arrow[d] & \mathsf{UI} \arrow[d] & & \exists \text{ Hamel basis} \\
\mathsf{AC}_\omega(\mathbb{R}) & \mathsf{UI}(\omega) & &
\end{tikzcd}
$$

Figure 2.1: Implications between some relevant choice principles. The references are all from Herrlich's book [27]. The arrows without reference follow directly from the definitions, except for a well order of the reals yielding a Hamel basis, for which we refer the reader to Chapter 5.

2.3 Algebra

In this section we will introduce all the concepts of algebra needed in Part I.

Definition 2.33. A field F is an **extension** of a field K if K is a subfield of F, namely, $K \subseteq F$ and the operations on K are the ones on F restricted to K.

If F is a field and $X \subseteq F$, then the **subfield** (resp. **subring**) **generated by** X is the intersection of all subfields (resp. subrings) of F that contain X. If F is an extension of K and $X \subseteq F$, then the subfield (resp. subring) generated by $K \cup X$ is denoted by $K(X)$ (resp. $K[X]$).

Notation: Given a field K and $n \in \omega$, we denote by $K[x_0, \ldots, x_{n-1}]$ the ring of polynomials in n variables over K.

Theorem 2.34 ([28, Theorem 1.3 Chapter V])
If F is an extension field of a field K and $X \subseteq F$, then the subfield $K(X)$ consists of all elements of the form

$$\frac{f(u_0, \ldots, u_{n-1})}{g(u_0, \ldots, u_{n-1})} = f(u_0, \ldots, u_{n-1})g(u_0, \ldots, u_{n-1})^{-1},$$

where $n \in \omega$, $f, g \in K[x_0, \ldots, x_{n-1}]$, $u_0, \ldots, u_{n-1} \in X$ and $g(u_0, \ldots, u_{n-1}) \neq 0$.

Definition 2.35 ([28, Theorem 1.4 Chapter V])**.** Let F be an extension field of K. An element u of F is said to be **algebraic** over K iff u is a root of some nonzero polynomial $f \in K[x]$. If u is not a root of any nonzero $f \in K[x]$, u is **transcendental** over K.

Definition 2.36. A field F is **algebraically closed** if every non-constant polynomial $f \in F[x]$ has a root in F. If F is an extension field of K, F is algebraically closed, and every element of F is algebraic over K, we say that F is an **algebraic closure** of K.

Remark.
In Part I, we will consider several subfields of $\mathbb{R}$. If K is a subfield of $\mathbb{R}$, we will often talk about *the* algebraic closure of K. We will always mean the algebraic closure of K that is a subfield of $\mathbb{C}$. Moreover, we will be interested in the algebraic closure of K relative to $\mathbb{R}$, which is the algebraic closure of K ($\subseteq \mathbb{C}$) intersected with $\mathbb{R}$. We will denote the algebraic closure of K relative to $\mathbb{R}$ simply by $\overline{K}$.

Definition 2.37. Let F be an extension field of K and S a subset of F. S is **algebraically dependent** over K if for some $n < \omega$ there is a nonzero polynomial $f \in K[x_0, \ldots, x_{n-1}]$ such that $f(s_0, \ldots, s_{n-1}) = 0$ for some distinct $s_0, \ldots, s_{n-1} \in S$. S is **algebraically independent** over K if S is not algebraically dependent over K.

Definition 2.38. Let F be an extension field of K. A **transcendence base** of F over K is a subset S of F which is algebraically independent over K and is a maximal (with respect to $\subseteq$) set with this property.

Theorem 2.39 ([28, Theorem 1.5 Chapter VI])
Let F be an extension field of K, S a subset of F algebraically independent over K and $u \in F \backslash K(S)$. Then $S \cup \{u\}$ is algebraically independent over K if and only if u is transcendental over $K(S)$.

Theorem 2.40 ([28, Theorems 1.8-1.9 Chapter VI])
Let F be an extension field of K. Let S be a transcendence base of F over K. Then every transcendence base of F over K has the same cardinality as S.

Definition 2.41. Let F be an extension field of K. The **transcendence degree** of F over K is the cardinal given by $|S|$, where S is any transcendence base of F over K.

Remark.
Let F be an extension field of K and S a subset of F of cardinality strictly less than the transcendence degree of F over K. Take C an algebraic closure of $K(S)$. Then $F \backslash C \neq \emptyset$, by a combination of Theorem 2.39 and Definition 2.41.

2.4 Analysis

In this section we will introduce some basic theorems of analysis that will be used later on. We start by recalling the definition of *uniformly continuous*.

Definition 2.42. Let $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that f is **uniformly continuous** on A if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in A\ (\|x - y\| < \delta \implies |f(x) - f(y)| < \varepsilon),$$

where $\|\cdot\|$ denotes the norm of a vector in $\mathbb{R}^n$, and $|\cdot|$ denotes the absolute value of a real number.

Theorem 2.43 ([50, Theorem 4.19])
Let $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function on a compact set A. Then f is uniformly continuous on A.

Recall that $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded (see for example [50, Theorem 2.41]). So a continuous function on the closure of a ball in $\mathbb{R}^n$ is uniformly continuous.

Theorem 2.44 (Implicit Function Theorem [50, Theorem 9.28])
Let $f\colon A \to \mathbb{R}$ be a function on an open set $A \subseteq \mathbb{R}^n \times \mathbb{R}$ such that $f(\mathsf{a}, \mathsf{z}) = 0$ for some point $(\mathsf{a}, \mathsf{z}) \in A$ and $\frac{\partial f}{\partial z}(\mathsf{a}, \mathsf{z}) \neq 0$. Then there are open sets $U \subseteq \mathbb{R}^{n+1}$ and $W \subseteq \mathbb{R}^n$ with $(\mathsf{a}, \mathsf{z}) \in U$ and $\mathsf{a} \in W$ having the following property:

To every $a \in W$ corresponds a unique z such that $(a, z) \in U$ and $f(a, z) = 0$. If this z is defined to be $g(a)$ then $g\colon W \to \mathbb{R}$ is continuously differentiable, $g(\mathsf{a}) = \mathsf{z}$ and $f(g(a), a) = 0$ for all $a \in W$.

Remark.
Let $f, A, (\mathsf{a}, \mathsf{z})$ as in Theorem 2.44. Suppose there is another point $(\mathsf{a}_0, \mathsf{z}_0) \in U$, $\mathsf{a}_0 \in W$ such that $f(\mathsf{a}_0, \mathsf{z}_0) = 0$ and $\frac{\partial f}{\partial z}(\mathsf{a}_0, \mathsf{z}_0) \neq 0$. Using the Implicit Function Theorem for the point $(\mathsf{a}_0, \mathsf{z}_0)$, we get two open sets $U_0 \subseteq \mathbb{R}^{n+1}$ and $W_0 \subseteq \mathbb{R}^n$ with $(\mathsf{a}_0, \mathsf{z}_0) \in U_0$ and $\mathsf{a}_0 \in W_0$, and a continuously differentiable function $g_0\colon W_0 \to \mathbb{R}$ such that $g_0(\mathsf{a}_0) = \mathsf{z}_0$ and $f(g_0(a), a) = 0$ for all $a \in W$.

By uniqueness, g and g_0 coincide in $W \cap W_0$, which is an open set. Then $\tilde{g} = g \cup g_0 \colon W \cup W_0 \to \mathbb{R}$ is continuously differentiable and $f(\tilde{g}(a), a) = 0$ for all $a \in W \cup W_0$.

Definition 2.45. Let $f \colon U \to \mathbb{C}$ be a function defined in an open set U of the complex plane $\mathbb{C}$. We say that f is **analytic** in U iff it is representable by power series, namely, for every $a \in U$ there are $r \in \mathbb{R}_{>0}$ and coefficients $\{c_n\}_{n<\omega}$ such that the following series converges

$$\sum_{n=0}^{\infty} c_n (z-a)^n,$$

and it is equal to $f(z)$ for all z in the disk $D(a, r) = \{z \colon |z-a| < r\}$.

Remark.
Sums, products and compositions of analytic functions are analytic. The exponential function and polynomials are analytic in the whole plane, $\frac{1}{z}$ is analytic in $\mathbb{C} \backslash \{0\}$, each branch of $\log z$ is analytic in $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$.

Theorem 2.46 ([51, Theorem 10.6])
If f is analytic then it has derivatives of all orders. Moreover, let $\{c_n\}_{n<\omega}$ be as in Definition 2.45. Then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

for every $n < \omega$.

We will use the following property of analytic functions.

Theorem 2.47 ([51, Theorem 10.18])
Let U be an open connected set, and let $f \colon U \to \mathbb{C}$ be an analytic function. If f vanishes in a set that has a limit point in U, then $f \equiv 0$ in U.

Part I

Algebraic closure of certain subfields of $\mathbb{R}$

Chapter 3

Algebraic closures

In this section we will approach some algebraic questions related to different subfields of $\mathbb{R}$, which are actually $\mathbb{R}$ in some inner model.

These results will be useful for the applications in Part II and originally that was the motivation for these results, but they can be written totally independently from Part II and are of interest by themselves.

The Lemma 3.1 is a well-known fact. It states that the transcendence degree of the reals after adding one Cohen real is continuum with respect to the ground model reals. This chapter deals with generalizations of this statement in two directions: taking into account other type of reals added (see Corollary 3.5) and adding more than one Cohen real (see Theorem 3.14).

Lemma 3.1 (Folklore)**.** *Let M be a model of ZFC and $\tilde{g}$ be a $\mathbb{C}$-generic filter over M. In $M[\tilde{g}]$, the transcendence degree of $\mathbb{R}$ over the algebraic closure relative to $\mathbb{R}$ of $\mathbb{R} \cap M$ is maximal (i.e. the cardinality of $\mathbb{R}$).*

In particular, if y is a Cohen real over $V[x]$, then in $V[x, y]$ the transcendence degree of $\mathbb{R}$ over $\overline{\mathbb{R} \cap V[x]}$ is $\mathfrak{c}$.

It is natural to ask the following question.

Question 3. *For which forcings that add reals does Lemma 3.1 hold?*

It is known by private (independent) conversations with Ben de Bondt and Elliot Glazer that Lemma 3.1 is true for any type of reals added to V, giving a complete answer to Question 3. This result will follow from Proposition 3.4. But before that we need some definitions. We will give the proofs of Proposition 3.4 and Corollary 3.5 by Ben De Bondt.

Definition 3.2. Let $P\colon \mathbb{R}_{>0} \to \mathbb{R}$. We say P is a **generalized polynomial with coefficients** $\{a_i, r_i\}_{i<k}$ for some $k < \omega$ iff $r_i \neq r_j$ for $i \neq j$, $a_i \neq 0$ for

$i < k$, and

$$P(x) = \sum_{i<k} a_i x^{r_i}.$$

Definition 3.3. Let $R \subseteq \mathbb{R}$ be a subfield of the reals. We say R is **really closed** if it is closed under roots of generalized polynomials with coefficients in R. Namely, if P is a generalized polynomial with coefficients $\{a_i, r_i\}_{i<k} \subseteq R$ and $x \in \mathbb{R}_{>0}$ is such that $P(x) = 0$, then $x \in R$.

If P is a generalized polynomial then the set of roots $S = \{x \mid P(x) = 0\}$ is finite. Clearly S is definable over $(\mathbb{R}, +, \cdot, <, \exp)$ with parameters $\{a_i, r_i\}_{i<k}$. By o-minimality [61], S is a finite union of singletons and intervals. Since P is an analytic function over $\mathbb{R}_{>0}$, it cannot vanish in an interval, otherwise it would be constantly 0 (see Theorem 2.47). Therefore S is a finite union of singletons, i.e., a finite set.

Proposition 3.4. *If $R \subsetneq \mathbb{R}$ is a really closed subfield of $\mathbb{R}$, the transcendence degree of $\mathbb{R}$ over R is continuum.*

Proof: If $|R| < \mathfrak{c}$, the transcendence degree of $\mathbb{R}$ over R is $\mathfrak{c}$ by cardinality. Namely, if B is algebraically independent over R, and $S = \{x \mid \exists P$ with coefficients in R such that $P(x, b) = 0$ for some $b \in B^{<\omega}\}$, then $|S| = \max\{|B|, |R|\}$. If B is a transcendence base, then $S = \mathbb{R}$ and therefore we have $|B| = \mathfrak{c}$.

If $|R| = \mathfrak{c}$, pick $x \in \mathbb{R} \backslash R$ such that $x > 0$, and take $T \subseteq R$ linearly independent over $\mathbb{Q}$ of cardinality $\mathfrak{c}$. Let $B = \{x^t\}_{t \in T}$. We claim that B is algebraically independent over R. Suppose not, then there are $k \in \omega$, $\{t_i\}_{i<k} \subseteq T$ and P a non-zero polynomial over R with k variables such that

$$P\left(x^{t_0}, \ldots, x^{t_{k-1}}\right) = 0.$$

Specifically, there are $J \subseteq \omega^k$ and non-zero coefficients $\{a_j\}_{j \in J} \subseteq R$, such that

$$\sum_{j \in J} a_j x^{t_0 j_0} \cdots x^{t_{k-1} j_{k-1}} = 0,$$

where for each $j \in J \subseteq \omega^k$ we denote its coordinates by $(j_0, \ldots, j_{k-1})$. We can rewrite this by grouping the exponents, namely,

$$\sum_{i \in J} a_j x^{s_j} = 0,$$

where $s_j = \sum_{i<k} j_i t_i$. Notice that all of these $\{s_j\}_{j \in J}$ are different since $\{t_i\}_{i<k} \subseteq T$ and T is linearly independent over $\mathbb{Q}$. So x is the root of a generalized polynomial with coefficients in R. Since R is a really closed

subfield of $\mathbb{R}$, we get that $x \in R$, which is a contradiction. Therefore, B is algebraically independent over R and the transcendence degree of $\mathbb{R}$ over R is $\mathfrak{c}$. $\square$

Corollary 3.5. *If M and N are models of set theory such that $M \subseteq N$ and $\mathbb{R}^M \subsetneq \mathbb{R}^N$, then in N the transcendence degree of $\mathbb{R}^N$ over $\mathbb{R}^M$ is $(2^{\aleph_0})^N$. In particular, if N is a forcing extension of M via a forcing that adds reals, the same conclusion holds.*

Proof: In N, $\mathbb{R}^M$ is a subfield of $\mathbb{R}^N$. Recall that a generalized polynomial has always finitely many roots. If a generalized polynomial P with coefficients $\{a_i, r_i\}_{i<k} \subseteq \mathbb{R}^M$ has n roots in N, this is expressed by the formula

$$\exists z_0, \ldots, z_{n-1} \in \mathbb{R}_{>0}\ P(z_0) = \cdots = P(z_{n-1}) = 0,$$

which is $\Sigma^1_1(c)$ for $c = \bigoplus_{i<k}(a_i \oplus r_i)$, and therefore absolute between M and N. So the roots of P belong to M and $\mathbb{R}^M$ is a really closed subfield of $\mathbb{R}^N$. By applying Proposition 3.4 in N, we get that the transcendence degree of $\mathbb{R}^N$ over $\mathbb{R}^M$ is continuum. $\square$

Notation: Let $S \subseteq \mathbb{R}$. We write $\overline{S}^{\text{field}}$ to denote the minimal subfield of $\mathbb{R}$ containing S. We write $\overline{S}^{\text{gp}}$ to denote the (real) closure under roots of generalized polynomials with coefficients in S. Finally, we write $\overline{S}^{\exp}$ for the set defined as follows:

$$\overline{S}^{\exp} := \bigcup_{n<\omega} S_n,$$

where $S_0 := S$, and $S_{n+1} := \overline{\overline{S_n}^{\text{gp}}}^{\text{field}}$.

Let S be a subset of $\mathbb{R}$. Then $\overline{S}^{\exp}$ is the smallest really closed subfield of $\mathbb{R}$ containing S. Clearly, $0, 1 \in \overline{S}^{\exp}$. If $a, b, c \in \overline{S}^{\exp}$ then there is $n \in \omega$ such that $a, b, c \in S_{n+1}$. Since S_{n+1} is a field, $ab^{-1} - c \in S_{n+1} \subseteq \overline{S}^{\exp}$. Thus, $\overline{S}^{\exp}$ is a field. Let P be a generalized polynomial with coefficients on $\overline{S}^{\exp}$ and let $z \in \mathbb{R}_{>0}$ be such that $P(z) = 0$. Let $\{a_i, r_i\}_{i \in k} \subseteq \overline{S}^{\exp}$ be the coefficients of P. There is n such that $\{a_i, r_i\}_{i \in k} \subseteq S_n$. Then $z \in \overline{S_n}^{\text{gp}} \subseteq S_{n+1} \subseteq \overline{S}^{\exp}$, and therefore $\overline{S}^{\exp}$ is closed under roots of generalized polynomials.

Lemma 3.6. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ a continuously differentiable function in V. Let $V[g]$ be a forcing extension of V and $\bar{f}$ the version of f in $V[g]$. Suppose that in $V[g]$ there is $\mathsf{a} \in \mathbb{R}^n$ and $\mathsf{z} \in \mathbb{R}$ such that $\bar{f}(\mathsf{a}, \mathsf{z}) = 0$ and $\frac{\partial \bar{f}}{\partial z}|_{(\mathsf{a},\mathsf{z})} \neq 0$.*

Then in V there is an open set $U \subseteq \mathbb{R}^n$ and a continuously differentiable function $h : U \to \mathbb{R}$ such that its version $\bar{h}$ in $V[g]$ satisfies $\bar{h}(\mathsf{a}) = \mathsf{z}$ and $\bar{f}(a, \bar{h}(a)) = 0$ for all $a \in \bar{U}$, where $\bar{U}$ is the version of U in $V[g]$, and $\mathsf{a} \in \bar{U}$.

This is just an application of the Implicit Function Theorem. But for our purposes we need to check precisely that in this case the implicit function comes from the ground model V.

Remark.
If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuously differentiable function in V, then its version $\bar{f}$ in $V[g]$ is also continuously differentiable.

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function in V, we can represent it by a real $r \in \mathbb{R}^V$ coding $f \restriction \mathbb{Q}^{n+1}$ (which is a sequence of countably many reals). Notice that "r codes a continuous function f_r" is equivalent to

$$\varphi(r)\colon \forall (q_m)_{m\in\omega} \subseteq \mathbb{Q}^n \text{ Cauchy } \exists y \in \mathbb{R} \forall \varepsilon \in \mathbb{Q}_{>0} \exists N \in \omega\, \forall n \geq N\, |f_r(q_n) - y| < \varepsilon.$$

This is a $\mathbf{\Pi^1_2}$ formula so absolute between V and $V[g]$ (Theorem 2.10) and $\bar{f}$ is well defined and continuous.

Let $\phi(f, f')$ be the following formula stating that f' is the derivative of f.

$$\begin{aligned}\phi(f, f')\colon & \forall x_0 \in \mathbb{R}^n \forall \varepsilon \in \mathbb{Q}_{>0} \exists \delta \in \mathbb{Q}_{>0} \forall x \in \mathbb{R}^n \\ & \qquad \|x - x_0\| < \delta \to \left| \frac{f(x) - f(x_0) - \langle f'(x_0), x - x_0\rangle}{\|x - x_0\|} \right| < \varepsilon\end{aligned}$$

which is equivalent to a $\mathbf{\Pi^1_1}$ formula.

Let f be as in Lemma 3.6. Let r be a real that codes this (continuous) function and let s be a real that codes the (continuous) function $g = Df : \mathbb{R}^n \to \mathbb{R}^n$.

Then "g is the derivative of f and g is continuous" is equivalent to

$$\varphi(r) \wedge \varphi(s) \wedge \phi(f_r, f_s),$$

for some reals r and s. This is $\mathbf{\Pi^1_2}$ and therefore absolute between V and $V[g]$. All of this is to conclude that $\bar{f}$ is also continuously differentiable in $V[g]$.

Proof: By the Remark above $\bar{f}$ is also continuously differentiable in $V[g]$. We can apply then the Implicit Function Theorem in $V[g]$. There is then an open set $\bar{U} \subseteq \mathbb{R}^n$ and $\bar{h} : \bar{U} \to \mathbb{R}$ such that $\bar{h}(\mathsf{a}) = \mathsf{z}$ and $\bar{f}(a, h(a)) = 0$ for all $a \in \bar{U}$. Without loss of generality, let us assume $\bar{U}$ is an open set in V.

Notice that

$$\exists(a,z) \ \in \bar{U}\times\mathbb{R}\,\bar{f}(a,z)=0\wedge\frac{\partial\bar{f}}{\partial z}(a,z)\neq 0$$

is true in $V[g]$ and it is $\mathbf{\Sigma^1_1}$ (considering $\bar{f}$ and $\frac{\partial\bar{f}}{\partial z}$ as coded by a real) thus absolute. Therefore, in V there is $(\mathsf{a}',\mathsf{z}')\in\bar{U}\times\mathbb{R}$ such that $f(\mathsf{a}',\mathsf{z}')=0$ and $\frac{\partial f}{\partial z}(\mathsf{a}',\mathsf{z}')\neq 0$.

By the Implicit Function Theorem (2.44) in V for the point $(\mathsf{a}',\mathsf{z}')$, there is an open set $U\subseteq\mathbb{R}^n$ and $h:U\to\mathbb{R}$ such that $\mathsf{a}'\in U$, $h(\mathsf{a}')=\mathsf{z}'$ and $f(a',h(a'))=0$ for all $a'\in U$. By the Remark after Theorem 2.44, we can assume h is defined in all of $\bar{U}$. Then by uniqueness of the implicit function $\bar{h}$ is the version of h in $V[g]$ and we can assume that $\bar{U}$ is the version of U in $V[g]$. □

Remark.

From now onwards, we will drop the notation $\bar{f}$ and we will just write f for a continuous function independently of the model in which the function is considered.

Definition 3.7. Working in $V[g]$ a forcing extension of V: Let $S\subseteq\mathbb{R}$ and $z\in\mathbb{R}$. We say that z **depends V-continuously on** S if there is $F\colon\mathbb{R}^k\to\mathbb{R}$ a continuous function in V with $k<\omega$ and $s\in S^k$ such that $F(s)=z$.

Notice that the interesting case is when $z\notin S$ and $z\notin\mathbb{R}^V$. Otherwise, $F=\mathrm{id}$ or $F\equiv z$ would trivially witness V-continuity.

Observe that V-continuously dependence on some set S is a local property. So the function F that witnesses this does not need to be defined in the full space $\mathbb{R}^k$, but rather in an open set $U\subseteq\mathbb{R}^k$ such that $s\in U$.

Using Definition 3.7 we can restate Lemma 3.6 as follows:

Lemma 3.8. *Working in $V[g]$ a forcing extension of V: Let $f\colon\mathbb{R}^{n+1}\to\mathbb{R}$ a continuously differentiable function in V. Let $(\mathsf{a},\mathsf{z})\in\mathbb{R}^n\times\mathbb{R}$ such that $f(\mathsf{a},\mathsf{z})=0$ and $\frac{\partial f}{\partial z}|_{(\mathsf{a},\mathsf{z})}\neq 0$. Then z depends V-continuously on a.*

Let $z\in\overline{F}$ (the real algebraic closure of a subfield F). By definition there is a polynomial P with coefficients in a field F such that $P(z)=0$. Then we can assume P has also the property that $P'(z)\neq 0$. This is because a non-zero polynomial cannot vanish in all its derivatives, so we can exchange P for the highest derivative of P which vanishes at z.

Remark.

Let $z\in\overline{F}^{\mathrm{gp}}$ (the closure of a field F by roots of generalized polynomials).

By definition there is a generalized polynomial P with coefficients in the field F such that $P(z) = 0$. Then we can assume P has also the property that $P'(z) \neq 0$.

A generalized polynomial is in particular an analytic function on its domain (Definition 2.45), which in this case includes $\mathbb{R}_{>0}$. If it vanished in all its derivatives at z, then the power series of P at z would have all coefficients equal to zero by Theorem 2.46. Then, there would be an open set U containing z for which $P \equiv 0$, then by Theorem 2.47 we get that $P \equiv 0$ in its domain. This is a contradiction. Therefore P has a non-zero derivative at z. The highest derivative of P that vanishes at z is itself a generalized polynomial with coefficients in F, has z as a root, and its derivative at z is non-zero.

Lemma 3.9. *Let $V[g]$ be a forcing extension of V. In $V[g]$, let $S \subseteq \mathbb{R}$ and $z \in \overline{S}^{\exp}$. Then z depends V-continuously on S.*

Proof: Recall that

$$\overline{S}^{\exp} := \bigcup_{n<\omega} S_n,$$

where $S_0 := S$, and $S_{n+1} := \overline{\overline{S_n}^{\mathrm{gp}}}^{\mathrm{field}}$.

Let $z \in \overline{S}^{\exp}$. Let us prove by induction on n that

$$z \in S_n \implies z \text{ depends } V\text{-continuously on } S.$$

For $n = 0$ and $z \in S_0 = S$, it is clear. Suppose the above statement is true for $n \in \omega$. Let $z \in S_{n+1} = \overline{\overline{S_n}^{\mathrm{gp}}}^{\mathrm{field}}$. Then there are polynomials P and Q with coefficients in $\mathbb{Q}$ and a finite set $t \subseteq \overline{S_n}^{\mathrm{gp}}$ such that $\frac{P(t)}{Q(t)} = z$ (See Theorem 2.34). Clearly the function $\frac{P}{Q}$ witnesses that z depends V-continuously on t. For each $u \in t$ there is $k < \omega$ and there is a generalized polynomial P_u with coefficients $\{a_u, r_u\} \subseteq S_n^k$ such that $P_u(u) = 0$. By the Remark above we can assume that $P_u'(u) \neq 0$. Consider the function $f\colon \mathbb{R}^{2k} \times \mathbb{R}_{>0} \to \mathbb{R}$ defined by

$$f(a, r, x) = \sum_{i=0}^{k-1} a_i x^{r_i}.$$

Then f is continuously differentiable, $f(a_u, r_u, u) = 0$ and $\frac{\partial f}{\partial x}(a_u, r_u, u) = P_u'(u) \neq 0$. Applying Lemma 3.8, u depends V-continuously on $a_u \cup r_u$.

We know that z depends V-continuously on t and each element u of t depends V-continuously on $a_u \cup r_u$. Then z depends V-continuously on the finite set $w := \{a_u \cup r_u \mid u \in t\} \subseteq S_n$, witnessed by the composition of the functions that witness each step. By inductive hypothesis, each element of

w depends V-continuously on S. Therefore z depends V-continuously on S. $\square$

Definition 3.10. Let $f\colon C \subseteq \mathbb{R}^k \to \mathbb{R}$ be a continuous function defined on a compact set C. We say that $\delta\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ **witnesses the uniform continuity of** f if

$$\forall \varepsilon \in \mathbb{Q}_{>0} \forall x, x' \in C \ \|x - x'\| < \delta(\varepsilon) \to |f(x) - f(x')| < \varepsilon.$$

Notice this is a $\mathbf{\Pi^1_1}$ formula therefore absolute between transitive models containing its parameters. Recall that f can be coded by a real.

Lemma 3.11. *Let x and y be mutually Cohen-generic reals over V. In $V[x,y]$, consider the set $S = \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Then there is $\mathsf{z} \in \mathbb{R}$ that does not depend V-continuously on S.*

Proof: We will define z from x and y. For this purpose we need a function $c\colon {}^\omega\omega \to [0,1]$ that will help us translate properties of the Baire space to the real line in a nice way [1].

Let $a = (a_n)_{n<\omega}$ be an element of ${}^\omega\omega$. Then we define $c(a) \in [0,1]$ as

$$c(a) = \sum_{n<\omega} \frac{a_n \bmod 3}{3^{n+1}},$$

where $m \bmod 3$ denotes the unique number in $\{0,1,2\}$ that has the same residue as $m \in \omega$ in the division by 3.

Let s be a sequence of natural numbers of length n. Then we know that there is a closed interval I_s of length $\frac{1}{3^n}$ such that for any $a \in {}^\omega\omega$: $c(a) \in \operatorname{int}(I_s)$ implies s is an initial segment of a; and if s is an initial segment of a, then $c(a) \in I_s$.

Think of x and y as elements of the Baire space. Consider $y \circ x \in {}^\omega\omega$ (composition of functions) and $\mathsf{z} = c(y \circ x)$. By definition of c, $\mathsf{z} \in \mathbb{R}^{V[x,y]}$.

Suppose on the contrary that z depends V-continuously on S. Then there are $k, l \in \omega$, U an open set in $\mathbb{R}^{k+l}$, $F\colon U \to \mathbb{R}$ a continuous function in V, and sets $\mathsf{r} \subseteq \mathbb{R}^{V[x]}$, $\mathsf{s} \subseteq \mathbb{R}^{V[y]}$ of sizes k and l such that $F(\mathsf{r}, \mathsf{s}) = \mathsf{z}$. Restrict F to a closed ball C in V such that (r, s) belongs to (the version in $V[g]$ of) C. Let $\delta\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be a witness for the uniform continuity of F restricted to

[1] At a first glance, it seems that is not so relevant which is the c we choose. But, for example, a representation of reals as binary sequences does not work for the last part of the proof.

C (see Theorem 2.43).

Given a name $\sigma \in V^{\mathbf{C}\times\mathbf{C}}$, we define σ^* and ${}^*\sigma$ in $V^{\mathbf{C}\times\mathbf{C}}$ recursively:

$$\sigma^* = \{(\pi^*, (p,q)) \mid (p,q) \in \mathbf{C}\times\mathbf{C} \text{ and } \exists q'(\pi,(p,q')) \in \sigma\}$$
$${}^*\sigma = \{({}^*\pi, (p,q)) \mid (p,q) \in \mathbf{C}\times\mathbf{C} \text{ and } \exists p'(\pi,(p',q)) \in \sigma\}$$

Let τ be a name for z and $\dot{\mathsf{r}}$ and $\dot{\mathsf{s}}$ nice names for r and s that only depend on the first and second coordinate respectively, namely, $\dot{\mathsf{r}} = \pi^*$ and $\dot{\mathsf{s}} = {}^*\sigma$ for some π, σ names for r and s respectively.

Let (p,q) be a condition in $\mathbf{C}\times\mathbf{C}$ such that

$$(p,q) \Vdash^{\mathbf{C}\times\mathbf{C}}_{V} (\dot{\mathsf{r}}, \dot{\mathsf{s}}) \in \check{C} \text{ and } \check{F}(\dot{\mathsf{r}}, \dot{\mathsf{s}}) = \tau.$$

We will find a condition $(\tilde{p}, \tilde{q})$ below (p,q) that forces the opposite statement, reaching a contradiction.

First let us assume $\mathrm{lh}(p) = \mathrm{lh}(q) = m$. We will extend (p,q) in several steps until getting to the desired $(\tilde{p}, \tilde{q})$, as Figure 3.1 shows.

1. Extend (p,q) to (p,q') so that it decides $(y \circ x) \restriction m$ by setting $\mathrm{lh}(q') = \max(\mathrm{im}\, p) + 1$. Notice that (p,q') does not decide $(y \circ x)(m)$.

   This implies that (p,q') forces that $\tau \in \check{I}$, where I is some interval given by the code c. More precisely, $I = I_t$ where t is the sequence given by $(q'_{p_0}, \dots, q'_{p_{m-1}})$.

   Let $\varepsilon = \frac{\mathrm{lh}(I)}{6}$ and $\delta = \delta(\varepsilon)$ (this is computed in V).

2. Extend (p,q') to (p,q'') so that it decides $\dot{\mathsf{s}} \in \check{J}_\delta$ (some ball in $\mathbb{R}^l$ with a rational center and of radius less than $\delta/4$).

3. Extend (p,q'') to (p',q'') so that $p'(m) = \mathrm{dom}(q'')$ and $\mathrm{dom}(p') = m+1$ (see Figure 3.1). This implies $q''(p'(m))$ is not defined and therefore (p',q'') does not decide the value of $(y \circ x)(m)$.

4. Extend (p',q'') to (p'',q'') so that p' decides $\dot{\mathsf{r}} \in \check{I}_\delta$ (another ball in $\mathbb{R}^k$ of rational center and radius less than $\delta/4$).

   In V, take $(r,s) \in I_\delta \times J_\delta$. Notice that

   $$(p'',q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} (\dot{\mathsf{r}}, \dot{\mathsf{s}}), (\check{r}, \check{s}) \in \check{I}_\delta \times \check{J}_\delta \subseteq \check{C}.$$

   If (r, s) and (r,s) are elements of $I_\delta \times J_\delta$, then $\|r - \mathsf{r}\| < 2\delta/4 = \delta/2$ and $\|s - \mathsf{s}\| < \delta/2$. Then, $\|(r,s) - (\mathsf{r}, \mathsf{s})\| < \delta$. So we obtain

   $$(p'',q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \|(\dot{\mathsf{r}}, \dot{\mathsf{s}}) - (\check{r}, \check{s})\| < \delta.$$

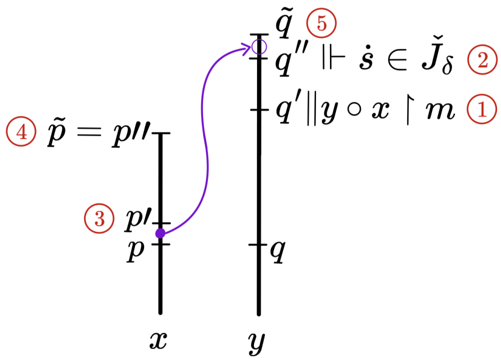

Figure 3.1: Steps 1–5. Here the arrow represents the fact that $p'(m) = \mathrm{dom}(q'')$, where $m = \mathrm{lh}(p)$.

Let $z = F(r, s) \in \mathbb{R}^V$. Notice that

$$\mathbb{1} \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \check{\delta} \text{ witnesses uniform continuity of } \check{F} \text{ on } \check{C}.$$

Then we get that

$$\begin{aligned} (p'', q'') &\Vdash^{\mathbb{C}\times\mathbb{C}}_{V} |\check{F}(\dot{\mathsf{r}}, \dot{\mathsf{s}}) - \check{F}(\check{r}, \check{s})| < \varepsilon, \text{ i.e.,} \\ (p'', q'') &\Vdash^{\mathbb{C}\times\mathbb{C}}_{V} |\tau - \check{z}| < \varepsilon. \end{aligned} \tag{3.1}$$

5. Finally, extend (p'', q'') to $(\tilde{p}, \tilde{q})$ so that $\tilde{p} = p''$ and $\mathrm{lh}(\tilde{q}) = \mathrm{lh}(q'') + 1$ as follows. Let $I' = (z - \varepsilon, z + \varepsilon)$. We know that $\mathrm{lh}(I') = 2\varepsilon = \frac{\mathrm{lh}\, I}{3}$, so $I' \cap I \subsetneq I$. Notice that I and I' are intervals in V, in the sense of their endpoints being reals in V. Recall that I is an interval I_t for some finite sequence t and $I_t = I_{t^\frown 0} \cup I_{t^\frown 1} \cup I_{t^\frown 2}$ where each of the intervals $I_{t^\frown i}$ is of length $\frac{\mathrm{lh}(I)}{3}$. Since $\mathrm{lh}(I') = \frac{\mathrm{lh}\, I}{3}$ as well, there is a number $j \in \{0, 1, 2\}$ such that $I_{t^\frown j} \cap I' = \emptyset$. Let us assign to $\tilde{q}(\tilde{p}(m))$ the number j, so that z avoids being in I'.

   Then we get

   $$(\tilde{p}, \tilde{q}) \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \tau \notin \check{I}',$$

   which contradicts Equation 3.1.

Thus, z as we defined it does not depend V-continuously on $\mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. □

Remark.
The composition of two mutually generic Cohen reals is a Cohen real.

Theorem 3.12
Let x and y be mutually generic Cohen reals over V. In $V[x,y]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Then, in $V[x,y]$ the transcendence degree of $\mathbb{R}$ over F is continuum.

Proof: Work in $V[x,y]$. Let $S = \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Notice that $\overline{S}^{\exp} \supseteq \overline{F}$, where $\overline{F}$ denotes the real-algebraic closure of F. By Lemma 3.11, there is a real z that does not depend V-continuously on S. Applying Lemma 3.9, we deduce that $z \notin \overline{S}^{\exp}$. Recall that $\overline{S}^{\exp}$ is a really closed subfield of $\mathbb{R}^{V[x,y]}$. Using Proposition 3.4, we get that the transcendence degree of $\mathbb{R}^{V[x,y]}$ over $\overline{S}^{\exp}$ is continuum. Therefore the transcendence degree of $\mathbb{R}^{V[x,y]}$ over $\overline{F}$ is also continuum. □

Now we will prove the general version of Theorem 3.12 for more Cohen reals. For this we need also a version of Lemma 3.11.

Lemma 3.13. *Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the set $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then there is $\mathsf{z} \in \mathbb{R}$ that does not depend V-continuously on S.*

Proof: We will define z from X in a similar way to the one in Lemma 3.11. Let $c\colon {}^{\omega}\omega \to [0,1]$ the same function defined in the proof of Lemma 3.11.

Think of X as a finite subset of the Baire space, $X = \{x_0, \dots, x_{k-1}\}$ and $k < \omega$. Consider $x_{k-1} \circ \cdots \circ x_0 \in {}^{\omega}\omega$ (composition of functions) and let $\mathsf{z} = c(x_{k-1} \circ \cdots \circ x_0)$.

Suppose on the contrary that z depends V-continuously on S. Then there are $n \in \omega$, $U \subseteq \mathbb{R}^n$ open, $F\colon U \to \mathbb{R}$ continuous function in V and $\mathsf{r} \subseteq S$ such that $F(\mathsf{r}) = \mathsf{z}$. Restrict F to a closed ball C in V such that (r, s) belongs to (the version in $V[g]$ of) C. Let $\delta\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be a witness for the uniform continuity of F restricted to C (Theorem 2.43).

To simplify notation we will write X_i^* for $X \backslash \{x_i\}$. Notice that $S = \bigcup_{i<k} \mathbb{R}^{V[X_i^*]}$. We can then partition $r \subseteq S$ in $\{r_i\}_{i<k}$ so that $r = \bigcup_{i<k} r_i$ and $r_i \in \mathbb{R}^{V[X_i^*]}$. Assume without loss of generality that the coordinates of $r \in \mathbb{R}^n$ are ordered with respect to this partition, namely, $r = (r_0, \dots, r_k)$, where $r_i \in \mathbb{R}^{\mathrm{lh}(r_i)}$ for $i < k$ and $\sum_{i<k} \mathrm{lh}(r_i) = n$.

Given a name $\sigma \in V^{\mathbf{C}^{\mathbf{k}}}$, we define σ_i^* in $V^{\mathbf{C}^{\mathbf{k}}}$ for $i < k$ as follows:

$$\sigma_i^* = \{(\pi_i^*, p) \mid p \in \mathbf{C}^{\mathbf{k}} \text{ and } \exists s \in \mathbf{C}\, (\pi, (p_0, \dots, p_{i-1}, s, p_{i+1}, \dots, p_{k-1})) \in \sigma\}.$$

Let τ be a name for z and for each $i < k$ let $\dot{\mathsf{r}}_i$ be a nice name for r_i that only depends on the coordinates $k \backslash \{i\}$, namely, $\dot{\mathsf{r}}_i = \sigma_i^*$ for some σ name for r_i. Let us write $\dot{\mathsf{r}}$ for the name $(\dot{\mathsf{r}}_0, \dots, \dot{\mathsf{r}}_{k-1})$.

Let q be a condition in $\mathbf{C}^k$ such that

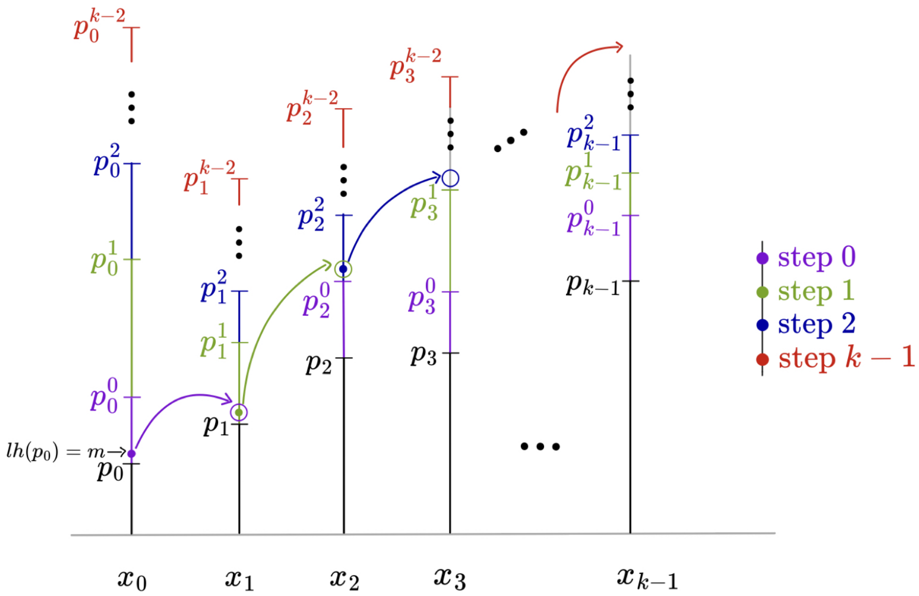

Figure 3.2: Steps 0 to $(k-1)$ of the construction. The first arrow represents that in Step 0 we take $p_0^0(m) = \operatorname{dom}(p_1)$, where $m = \operatorname{lh}(p_0)$, and similarly for the other arrows.

$$q \Vdash_V^{\mathbf{C}^k} \dot{r} \in \check{C} \text{ and } \check{F}(\dot{r}) = \tau.$$

We will find a condition $\tilde{q}$ below q that forces a contradictory statement, reaching a contradiction.

First let us assume $\operatorname{lh}(q_i) = m$ for all $i < k$. Extend q to $p = (p_0, \ldots, p_{k-1})$ so that $p_{k-1} \circ \cdots \circ p_0 \restriction m$ is defined, setting $p_0 = q_0$ and recursively $\operatorname{lh}(p_{i+1}) = \max(\operatorname{im} p_i) + 1$ for $i < k-1$. Thus, it is forced by p that $\tau \in \check{I}$, where I is some interval given by the code c. More precisely, $I = I_t$ where t is $(p_{k-1} \circ \cdots \circ p_0(0), \ldots, p_{k-1} \circ \cdots \circ p_0(m-1))$. Let $\varepsilon = \frac{\operatorname{lh}(I)}{6}$ and let $\delta = \delta(\varepsilon)$, which is computed in V.

Make $p \backslash \{p_0\}$ also decide $\dot{r}_0 \in \check{I}_0$, where I_0 is a ball of radius less than $\frac{\delta}{2k}$.

Notice that $p_{k-1} \circ \cdots \circ p_0(m)$ is not defined yet because we did not extend p_0.

We will extend sequentially p to conditions $p^0, \ldots, p^{k-1}$ in $\mathbf{C}^{k-1}$ as Figure 3.2 shows.

We denote coordinates with subindices and the different steps with superindices.

Step 0. Define $p^0 \in \mathbf{C}^k$ extending p such that

- $p_0^0(m) = \mathrm{dom}(p_1)$,
- $p_1^0 = p_1$, and
- $p^0 \setminus \{p_1^0\}$ decides $\dot{\mathsf{r}}_1 \in \check{I}_1$, where I_1 is a ball in V of radius less than $\frac{\delta}{2k}$.

Step 1. Define $p^1 \in \mathbf{C}^k$ extending p_0 such that

- $p_1^1 \circ p_0^0(m) = \mathrm{dom}(p_2^0)$,
- $p_2^1 = p_2^0$, and
- $p^1 \setminus \{p_2^1\}$ decides $\dot{\mathsf{r}}_2 \in \check{I}_2$, where I_2 is a ball in V of radius less than $\frac{\delta}{2k}$.

In general, for $i = 1, \ldots, k-2$:

Step i. Extend $p^{i-1} \in \mathbf{C}^k$ to p^i so that

- $p_i^i \circ \cdots \circ p_0^0(m) = \mathrm{dom}(p_{i+1}^{i-1})$,
- $p_{i+1}^i = p_{i+1}^{i-1}$, and
- $p^i \setminus \{p_{i+1}^i\}$ decides $\dot{\mathsf{r}}_{i+1} \in \check{I}_{i+1}$, where I_{i+1} is a ball in V of radius less than $\frac{\delta}{2k}$.

In V, take $r \in \prod_{i<k} I_i$, and let $z = F(r)$. Notice that:

$$\begin{aligned}
p^{k-2} &\Vdash_{V}^{\mathbf{C}^k} \dot{\mathsf{r}}, \check{r} \in \prod_{i<k} \check{I}_i, \\
p^{k-2} &\Vdash_{V}^{\mathbf{C}^k} \|\dot{\mathsf{r}} - \check{r}\| < \delta, \\
p^{k-2} &\Vdash_{V}^{\mathbf{C}^k} \left|\check{F}(\dot{\mathsf{r}}) - \check{F}(\check{r})\right| < \varepsilon, \\
p^{k-2} &\Vdash_{V}^{\mathbf{C}^k} \tau \in \check{I}'.
\end{aligned} \tag{3.2}$$

Here, $I' := (z - \varepsilon, z + \varepsilon)$. Notice that I' is an interval with end points in V and $\mathrm{lh}(I') = 2\varepsilon = \frac{\mathrm{lh}(I)}{3}$.

Step k-1. Extend $p^{k-2} \in \mathbf{C}^k$ to p^{k-1} such that $p_{k-1}^{k-1} \circ \cdots \circ p_0^0(m)$ is a number l that makes z avoid I', namely, there is $l \in \{0, 1, 2\}$ such that $I_{t^\frown l} \cap I' = \emptyset$. This is because $\mathrm{lh}\, I' = \mathrm{lh}\, I_{t^\frown 0} = \mathrm{lh}(I_{t^\frown 1}) = \mathrm{lh}(I_{t^\frown 2}) = \frac{\mathrm{lh}(I)}{3}$.

Define $\tilde{q} = p^{k-1}$. Then we get

$$\tilde{q} \Vdash_{V}^{\mathbf{C}^k} \tau \notin \check{I}',$$

which contradicts Equation 3.2.

Therefore, z does not depend V-continuously on $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. $\square$

Theorem 3.14
Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then, in $V[X]$ the transcendence degree of $\mathbb{R}$ with respect to F is continuum.

Proof: Work in $V[X]$. Let $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Notice that $\overline{S}^{\exp} \supseteq \overline{F}$, where $\overline{F}$ denotes the real-algebraic closure of F. By Lemma 3.13, there is a real z that does not depend V-continuously on S. Applying Lemma 3.9, we deduce that $z \notin \overline{S}^{\exp}$. Recall that $\overline{S}^{\exp}$ is a really closed subfield of $\mathbb{R}^{V[X]}$. Using Proposition 3.4, we get that the transcendence degree of $\mathbb{R}^{V[X]}$ over $\overline{S}^{\exp}$ is continuum. Therefore the transcendence degree of $\mathbb{R}^{V[X]}$ over $\overline{F}$ is also continuum. $\square$

Question 4. *Is Theorem 3.14 true for other forcings that add reals?*

We finish this chapter with another algebraic lemma, that will also be used in Part II, specifically in Lemma 7.7 (Amalgamation for PUC).

Lemma 3.15. *Let x, y be Cohen-mutually generic filters over V, where V is a model of* ZFC*. Let $B \subseteq \mathbb{R}^{V[x]}$ be an algebraically independent set over $\mathbb{R}^V$, then B is also algebraically independent over $\mathbb{R}^{V[y]}$.*

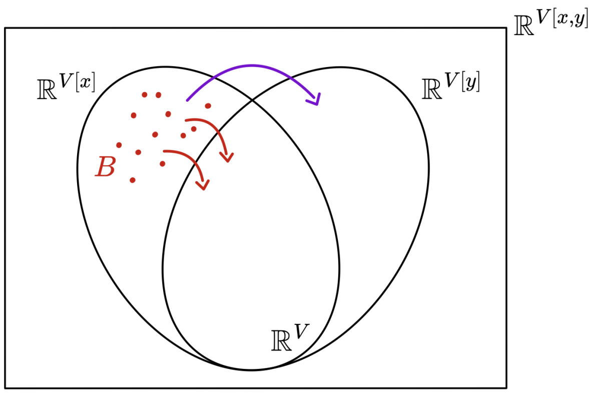

Figure 3.3: If $B \subseteq \mathbb{R}^{V[x]}$ is algebraically independent over $\mathbb{R}^V$, then B is also algebraically independent over $\mathbb{R}^{V[y]}$.

Proof: Suppose not, i.e. B is algebraically dependent over $\mathbb{R}^{V[y]}$ and without loss of generality assume B is finite. Let n be the cardinality of B. This means there are some finite multi index set $\mathcal{J} \subseteq {}^n\omega$ and some non-zero coefficients $\{d_J\}_{J\in\mathcal{J}} \subseteq \mathbb{R}^{V[y]}$ such that

$$\sum_{J\in\mathcal{J}} d_J B^J = 0, \tag{3.3}$$

where $J = (j_0, \ldots, j_{n-1})$, $B = \{b_0, \ldots, b_{n-1}\}$ and $B^J : b_0^{j_0} \cdots b_{n-1}^{j_{n-1}}$.

On the other hand, we can think of $\mathbb{R}^{V[y]}[B]$ (the minimal ring containing $\mathbb{R}^{V[y]}$ and B) as a vector space over the field $\mathbb{R}^{V[y]}$. In this context, it is clear that $\{B^J\}_{J\in{}^n\omega}$ spans $\mathbb{R}^{V[y]}(B)$. Therefore, there is $\mathcal{S} \subseteq \{B^J\}_{J\in{}^n\omega}$ such that $\mathcal{S}$ is a basis of $\mathbb{R}^{V[y]}[B]$ as $\mathbb{R}^{V[y]}$-vector space. Let $\mathcal{I}$ be the corresponding index set, i.e, $\mathcal{I} \subseteq {}^n\omega$ and $\mathcal{S} = \{B^J\}_{J\in\mathcal{I}}$.

Note that $\mathcal{J} \nsubseteq \mathcal{I}$. $\mathcal{J}$ cannot be a subset of $\mathcal{I}$, since the equation 3.3 shows a linear dependence of $\{B^J\}_{J\in\mathcal{J}}$ over $\mathbb{R}^{V[y]}$, and $\mathcal{S} = \{B^J\}_{J\in\mathcal{I}}$ is a basis, in particular, it is linearly independent over $\mathbb{R}^{V[y]}$.

Then, there must be an $I \in \mathcal{J}\backslash\mathcal{I}$, or equivalently, there is an $I \in \mathcal{J}$ such that $B^I \notin \mathcal{S}$. Now, since $B^I \in \mathbb{R}^{V[y]}[B]$, there are unique coefficients $\{c_i\}_{i=0}^{m-1}$ in $\mathbb{R}^{V[y]}$ and vectors $\{S_i\}_{i=0}^{m-1}$ in $\mathcal{S}$ such that

$$\sum_{i=0}^{m-1} c_i S_i = B^I. \tag{3.4}$$

Since we know

$$V[x,y] \models \text{“} \sum_{i=0}^{m-1} c_i S_i = B^I, \text{ with } \{c_i\}_{i=0}^{m-1} \subseteq \mathbb{R}^{V[y]}\text{”} \tag{3.5}$$

there is some condition $p \in y \subseteq \mathbf{C}$ such that

$$p \Vdash^{V[x]}_{\mathbf{C}} \text{“} \sum_{i=0}^{m-1} \tau_i \check{S}_i = \check{B}^I, \text{ with } \{\tau_i\}_{i=0}^{m-1} \subseteq \mathbb{R}^{V[\dot{g}]}, \text{”} \tag{3.6}$$

where $\dot{g}$ is the usual name for the generic filter, and τ_i is a name for c_i, i.e., $\tau_i^y = c_i$ for $i = 0, \ldots, m-1$. Note also that $S_i \in \mathcal{S} = \{B^J\}_{J\in\mathcal{I}} \subseteq V[x]$, which justifies the “check” on S_i, for $i = 0, \ldots, m-1$.

Now, let split y according to p into two mutually Cohen generics y_1, y_2 over $V[x]$ such that $p \in y_1, y_2$ (see Definition 2.21). Then, in $V[x, y_1, y_2] = V[x, y]$, we have

$$B^I = \sum_{i=0}^{m-1} \tau_i^1 S_i = \sum_{i=0}^{m-1} \tau_i^2 S_i,$$

where τ_i^1 and τ_i^2 are the interpretations of the name τ_i by y_1 and y_2 respectively, for each $i < m$. In particular, $\tau_i^1 \in V[y_1]$ and $\tau_i^2 \in V[y_2]$. But on the other hand, by uniqueness of the coefficients $\{c_i\}_{i=o}^{m-1}$, and taking into account that $V[x, y_1], V[x, y_2] \subseteq V[x, y]$, we have that $\tau_i^1 = \tau_i^2 = \tau_i^y = c_i$ for $i = 0, \ldots, m-1$. In particular, $c_i \in \mathbb{R}^{V[y_1]} \cap \mathbb{R}^{V[y_2]} = \mathbb{R}^V$. In other words, $B^I = \sum_{i=0}^{m-1} c_i S_i$, where $c_i \in \mathbb{R}^V$. But this means $0 = B^I - \sum_{i=0}^{m-1} c_i S_i$, where $c_i \neq 0$ and the right hand side is not trivial (we chose I such that $I \notin \mathcal{J}$, i.e., $S_i \neq B^I$ for all $i = 0, \ldots, m-1$). This contradicts B being algebraically independent over $\mathbb{R}^V$. $\square$

Remark.
Note that the same proof shows that if $B \subseteq \mathbb{R}^{V[x]}$ is *linearly* independent over $\mathbb{R}^V$, then B is also a linearly independent set over $\mathbb{R}^{V[y]}$.

Part II

Paradoxical sets of reals

Chapter 4

General setup

In part II there will be several results of the form "there is a model of ZF + there exist P + no C", where P is some notion of paradoxical set and C is a certain choice principle. When C is the principle of existence of a well-order of the reals, the proofs will have the same structure, which we develop in this chapter.

Each model will be an inner model of $V[g][h]$ where g is a $\mathbf{Q}$-generic filter over V, and h is a $\mathbf{P}$-generic filter over $V[g]$. Usually $\mathbf{P}$ will be a forcing notion approximating the paradoxical set considered, $\mathcal{P} = \cup h$ will be the paradoxical set added by $\mathbf{P}$, and $\mathbf{Q}$ will be an adequate forcing that adds reals, for example, the forcing adding $\aleph_1$-many Cohen reals using finite support.

Theorem 4.6 is based on is [12, Lemma 5.1]. We want to use the structure of that proof, but write it for a more general set up.

Definition 4.1. A forcing notion $\mathbf{P}$ is **real absolute** if it is absolute, each condition is a subset of the reals, and the order $\leq_{\mathbf{P}}$ is subset of the order given by $\supseteq$. Namely, there are formulas ψ and ψ' absolute between inner models, such that

$$\begin{aligned} p \in \mathbf{P} &\iff p \subseteq \mathbb{R} \text{ and } \psi(p) \\ \forall p_1, p_2 \in \mathbf{P}, p_1 \leq_{\mathbf{P}} p_2 &\iff p_1 \supseteq p_2 \text{ and } \psi'(p_1, p_2) \end{aligned}$$

Remark.
If a forcing $\mathbf{P}$ in M is real absolute and N is an inner model of M, then

$$\mathbf{P}^N = \mathbf{P}^M \cap N.$$

Definition 4.2. Let $\mathbf{Q}$ be a forcing notion in V, and let $\mathbf{P}$ be a forcing notion in $V[g]$ where g is a $\mathbf{Q}$-generic filter over V. Then we say that $\mathbf{P}$ and $\mathbf{Q}$ are **real alternating** if the following conditions hold (in $V[g]$):

1. for all $p \in \mathbf{P}$, $p \subseteq \mathbb{R}^{V[g]}$ and there is $r \in \mathbb{R}^{V[g]}$ such that $p \in V[r]$ and r can be computed from finitely many elements of p; and

2. for all $r \in \mathbb{R}^{V[g]}$, for all $p \in \mathbf{P}$, there is $\bar{p} \in \mathbf{P}$ such that $\bar{p} \leq p$ and $r \in V[\bar{p}]$.

Remark.
Notice that $V[r]$ is a forcing extension of V and a ground of $V[g]$ by Theorem 2.16, since $r \subseteq \omega \subseteq V$ and $r \in V[g]$. Let p and r be as in item 1 of Definition 4.2. Since r can be computed from finitely many elements of p then r will be an element of any model containing p. Therefore $V[r] \subseteq V[p]$, and since $p \in V[r]$, then $V[p] = V[r]$.

Definition 4.3. Let V be a model of ZF, and let $V[g_0]$, $V[g_1]$, and $V[g_2]$ be three forcing extensions of V, not necessarily obtained by the same forcing notion. We say that $(V[g_0], V[g_1], V[g_2])$ is a **real bifurcation** if

1. $\mathbb{R}^{V[g_o]} \subsetneq \mathbb{R}^{V[g_1]}$, $\mathbb{R}^{V[g_0]} \subsetneq \mathbb{R}^{V[g_2]}$, and

2. $V[g_0] = V[g_1] \cap V[g_2]$.

Definition 4.4. Let $\mathbf{Q}$ be a forcing notion in V, let g be a $\mathbf{Q}$-generic filter over V and $\mathbf{P}$ a real absolute forcing notion such that $\mathbf{P}$ and $\mathbf{Q}$ are real alternating. We say $\mathbf{P}$ is **$\mathbf{Q}$-balanced over** V (in $V[g]$) iff the following statement holds in $V[g]$:

For densely many $p \in \mathbf{P}^{V[g]}$, there exist g_1, g_2 (in $V[g]$) both $\mathbf{Q}$-generic over $V[p]$ such that

1. $V[\tilde{g}_i] = V[p, g_i]$ is a $\mathbf{Q}$-ground for $i = 1, 2$;

2. $(V[p], V[\tilde{g}_1], V[\tilde{g}_2])$ is a real bifurcation; and

3. for all $p_1 \in \mathbf{P}^{V[\tilde{g}_1]}$, $p_2 \in \mathbf{P}^{V[\tilde{g}_2]}$ extending p, p_1 and p_2 are compatible.

In this case we say that p is a **$\mathbf{Q}$-balanced condition**. See the representation of this situation in Figure 4.1.

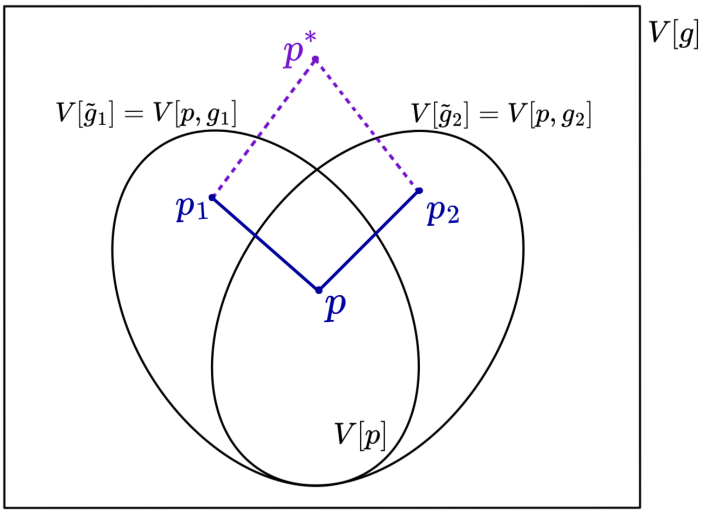

Figure 4.1: The condition $p \in \mathbf{P}$ is $\mathbf{Q}$-balanced. The compatibility of p_1 and p_2 is witnessed by p^*.

Definition 4.4 is based on the definition of *balanced* of the book Geometric Set Theory [42, Definition 5.2.1 and Proposition 5.2.2] (hence the name). There are differences, the biggest one that we only require this amalgamation property (item 3) for only one pair g_1 and g_2 instead of *for all*, but the spirit is the same.

Lemma 4.5. *In the context of Definition 4.4, let $p \in \mathbf{P}$ be $\mathbf{Q}$-balanced. Suppose $\mathbf{Q} \times \mathbf{Q} \cong \mathbf{Q}$. Then $V[p]$ is a $\mathbf{Q}$-ground of $V[g]$.*

Moreover, if $\mathbf{Q}$ is homogeneous, for any g' $\mathbf{Q}$-generic filter over $V[p]$, there are densely many conditions $\bar{p}$ in $\mathbf{P}$ such that $\bar{p}$ is $\mathbf{Q}$-balanced (in $V[g']$) and $V[\bar{p}]$ is a $\mathbf{Q}$-ground of $V[g']$.

Proof: By definition of balanced, there is a $\mathbf{Q}$-generic filter g_1 over $V[p]$. In other words, $V[p]$ is a $\mathbf{Q}$-ground of $V[g_1]$. Also, $V[p, g_1]$ is a $\mathbf{Q}$-ground of $V[g]$. Therefore, $V[p]$ is a $\mathbf{Q} \times \mathbf{Q}$-ground of $V[g]$. Since $\mathbf{Q} \times \mathbf{Q} \cong \mathbf{Q}$, we obtain that $V[p]$ is a $\mathbf{Q}$-ground of $V[g]$.

For the second part notice that:

$$V[g] \models \forall p \in \mathbf{P}\, \exists \bar{p} \leq p \text{ such that } \bar{p} \text{ is } \mathbf{Q}\text{-balanced and } V[\bar{p}] \text{ is a } \mathbf{Q} - \text{ground.}$$

Fix some balanced condition p in $V[g]$. In particular, p is a $\mathbf{Q}$-ground of $V[g]$. Then there is a condition $q \in \mathbf{Q}$ such that

$$q \Vdash^{\mathbf{Q}}_{V[p]} \forall p' \in \mathbf{P}\, \exists \bar{p} \leq p' \text{ such that } \bar{p} \text{ is } \check{\mathbf{Q}}\text{-balanced and } V[\bar{p}] \text{ is a } \check{\mathbf{Q}} - \text{ground.}$$

By homogeneity of $\mathbf{Q}$,

$$\mathbb{1} \Vdash_{V[p]}^{\mathbf{Q}} \forall p' \in \mathbf{P}\, \exists \bar{p} \leq p' \text{ such that } \bar{p} \text{ is } \check{\mathbf{Q}}\text{-balanced and } V[\bar{p}] \text{ is a } \check{\mathbf{Q}} - \text{ground.}$$

Hence, for any g' $\mathbf{Q}$-generic filter over $V[p]$, in $V[g']$ there are densely many conditions $\bar{p} \in \mathbf{P}$ such that $\bar{p}$ is $\mathbf{Q}$-balanced and $V[\bar{p}]$ is a $\mathbf{Q}$-ground of $V[g']$. $\square$

Now we are ready to state the main theorem of this section.

Theorem 4.6
Let V be a model of ZFC*. Let $\mathbf{Q}$ be a forcing notion over V, and g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be a forcing notion over $V[g]$, h be a $\mathbf{P}$-generic filter over $V[g]$, and $\mathcal{P} = \cup h$. Suppose the following conditions hold:*

1. *$\mathbf{Q}$ is homogeneous and $\mathbf{Q} \times \mathbf{Q} \cong \mathbf{Q}$.*
2. *$\mathbf{P}$ is real absolute and σ-closed.*
3. *$\mathbf{P}$ and $\mathbf{Q}$ are real-alternating,*
4. *$\mathbf{P}$ is a $\mathbf{Q}$-balanced forcing over V.*

Then

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{DC} + \neg \mathsf{WO}(\mathbb{R}).$$

Remark.
In Theorem 4.6, notice that the definition of balanced already includes real alternation and $\mathbf{P}$ being real absolute. We include it in the hypotheses of this theorem to facilitate the reading of the proof.

Proof: First, let us show that $L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{DC}$. Let $R \subseteq A \times A$ a relation on a nonempty set A such that for every $x \in A$ there is $y \in A$ with xRy. Since $R, A \in L(\mathbb{R}, \mathcal{P})$, there are some (real and ordinal) parameters that are used together with $\mathcal{P}$ to define R and A by some formulas. Let us call this set of finitely many reals and ordinals by P. We want to show that there is a sequence $\{x_n\}_{n<\omega}$ such that $x_n R x_{n+1}$ for all $n < \omega$.

Work in $V[g,h]$. Fix $x_0 \in A \in L(\mathbb{R}, \mathcal{P})^{V[g,h]}$. Then x_0 is definable from $\mathcal{P}$, some (finite) reals, and some ordinals as parameters. Consider x_0 as a set definable from γ_0, z_0 and $\mathcal{P}$, where γ_0 is the least ordinal (that encodes finite ordinals) such that (the decoding of) γ_0 appears as parameters for the least formula ϕ that defines x_0. This formula would have some real parameters to define x_0. Given ϕ and γ_0, there are some real parameters that work to define x_0. Choose[1] such real parameters and encode them by one real z_0. Notice that x_0 is now definable from $z_0 \in \mathbb{R}$ and $\mathcal{P}$. For each $n > 0$, consider the set $\{y \in A \mid x_n R y\}$. Take x_{n+1} an element of this set that can be defined from the least formula and the least ordinals. Pick some reals that make this formula and these ordinals work for a definition of x_{n+1}, encode them in one real z_{n+1}. Similarly, x_{n+1} is definable from z_{n+1}, $\mathcal{P}$ and the set of parameters P (to define A and R). Take $z = \bigoplus_{n<\omega} z_n$. Then z is a real number that encodes the sequence $\{z_n\}_{n<\omega}$. Since $z \in \mathbb{R} \ \in L(\mathbb{R}, \mathcal{P})^{V[g,h]}$, inside $L(\mathbb{R}, \mathcal{P})^{V[g,h]}$ we can decode the sequence $\{z_n\}_{n<\omega}$, which together with $\mathcal{P}$ and P helps as define the sequence $\{x_n\}_{n<\omega}$. Then $\{x_n\}_{n<\omega} \in L(\mathbb{R}, \mathcal{P})^{V[g,h]}$ (it is definable from $\mathcal{P}$ and $P \cup \{z\}$), and $x_n R x_{n+1}$ for all $n < \omega$, as we wanted.

Secondly, we want to show that $L(\mathbb{R}, \mathcal{P})^{V[g,h]}$ does not have a well-ordering or the reals. Suppose the contrary, i.e. that there is some $\mathbf{P}$-generic filter h over $V[g]$, and a formula $\phi(\cdot, \cdot, \vec{x}, \vec{\alpha}, \mathcal{P})$ with $\vec{x}$ a finite sequence in $\mathbb{R}^{V[g,h]}$ and $\vec{\alpha}$ a finite sequence of ordinals such that

$$V[g,h] \models \phi(\cdot, \cdot, \vec{x}, \vec{\alpha}, \mathcal{P}) \text{ defines a well-ordering of } 2^\omega. \tag{4.1}$$

Then there is a condition $p_0 \in h$ that forces such statement, namely,

$$p_0 \Vdash^{\mathbf{P}}_{V[g]} \phi(\cdot, \cdot, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}) \text{ defines a well-ordering of } 2^\omega, \tag{4.2}$$

where $\dot{\mathcal{P}}$ is the name given by $\dot{\mathcal{P}} = \{\langle p, \check{x}\rangle \mid x \in p \text{ and } p \in \mathbf{P}\}$. Notice that we can write $\check{\vec{x}}$ in 4.2 because $\mathbf{P}$ does not add reals, since it is σ-closed.

On the other hand, since $\mathbf{P}$ is real absolute, there is a formula ψ such that $p \in P \leftrightarrow \psi(p)$ and ψ is absolute between inner models. From now onwards, every time we write $\mathbf{P}$, we are actually interpreting the formula ψ in the corresponding model $V[\cdot]$, which is nothing more than $\mathbf{P}^{V[g]} \cap V[\cdot]$ by absoluteness of ψ. Also, when we write $\dot{\mathcal{P}}$, we mean the formula defining $\dot{\mathcal{P}} = \{\langle p, \check{x}\rangle \mid x \in p \text{ and } \psi(p)\}$.

Now, since $\mathbf{P}$ and $\mathbf{Q}$ are real-alternating, there is $r \in \mathbb{R}$ such that $p_0 \in V[r]$. Let us write $\vec{x}$ as $(x_0, \ldots, x_{m-1})$, where $m < \omega$. Take $s = r \oplus \bigoplus_{i \in m} x_i$, s is a real in $V[g]$. Again by the property of being real alternating, there is $\bar{p} \le p_0$ such that $s \in V[\bar{p}]$. Because $\mathbf{P}$ is $\mathbf{Q}$-balanced, there is $p \le \bar{p}$ which is a balanced condition. By real absoluteness, $V[p] \supseteq V[\bar{p}]$. Notice that then $s, r, \vec{x}, p_0 \in V[p]$. From 4.2, we can write

$$p \Vdash^{\mathbf{P}}_{V[g]} \phi(\cdot, \cdot, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}) \text{ defines a well-ordering of } 2^\omega, \tag{4.3}$$

[1] Notice that $V[g,h]$ satisfies ZFC.

By Lemma 4.5, $V[p]$ is a $\mathbf{Q}$-ground of $V[g]$. There is g' a $\mathbf{Q}$-generic over $V[p]$ such that $V[g] = V[p][g']$, and observe that Equation 4.3 is a statement in $V[g] = V[p][g']$. By definability of forcing, we can write this statement as a formula with parameters $\mathbf{P}, p, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}$:

$$V[p, g'] \models \Phi\left(p, \mathbf{P}, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right)$$

where Φ is the formula given by

$$\Phi(\cdot) \iff p \Vdash^{\mathbf{P}}_{V[g]} \phi(\cdot, \cdot, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}) \text{ defines a well-ordering of } 2^\omega.$$

The statement Φ has to be forced over $V[p]$ by some condition in $\mathbf{Q}$. Because $\mathbf{Q}$ is homogeneous and all the variables are definable or check names, we get that $\mathbb{1}_{\mathbf{Q}}$ already forced it:

$$\mathbb{1}_{\mathbf{Q}} \Vdash^{\mathbf{Q}}_{V[p]} \Phi\left(\check{p}, \mathbf{P}, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \dot{\mathcal{P}}\right).$$

Namely,

$$\mathbb{1}_{\mathbf{Q}} \Vdash^{\mathbf{Q}}_{V[p]} \check{p} \Vdash^{\mathbf{P}}_{V[p,\dot{g}]} \phi(\cdot, \cdot, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \dot{\mathcal{P}}) \text{ defines a well-ordering of } 2^\omega \tag{4.4}$$

Since p is $\mathbf{Q}$-balanced, there are g_1, g_2 $\mathbf{Q}$-generic filters over $V[p]$ such that the real bifurcation $(V[p], V[p, g_1], V[p, g_2])$ has the corresponding property of compatibility of conditions. To shorten the notation, we write $V[p, g_i] = V[\tilde{g}_i]$ for $i = 1, 2$. Notice that $\tilde{g}_i$ does not need to be $\mathbf{Q}$-generic. For $i = 1, 2$ we get

$$p \Vdash^{\mathbf{P}}_{V[\tilde{g}_i]} \phi(\cdot, \cdot, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}) \text{ defines a well-ordering of } 2^\omega \tag{4.5}$$

Take h_i $\mathbf{P}$-generic filter over $V[\tilde{g}_i]$ such that $p \in h_i$, for $i = 1, 2$. Let $\mathcal{P}_i = (\dot{\mathcal{P}})_{h_i}$. Then

$$V[\tilde{g}_i, h_i] \models \phi(\cdot, \cdot, \vec{x}, \vec{\alpha}, \mathcal{P}_i) \text{ defines a well-ordering of } 2^\omega.$$

Remember that $(V[p], V[\tilde{g}_1], V[\tilde{g}_2])$ is a real bifurcation and that $\mathbf{P}$ is σ-closed. Notice that, by homogeneity of $\mathbf{Q}$ and $V[p]$ being a $\mathbf{Q}$-ground of both $V[g]$ and $V[\tilde{g}_i]$, we get

$$(\mathbf{P} \text{ is } \sigma\text{-closed})^{V[g]} \implies (\mathbf{P} \text{ is } \sigma\text{-closed})^{V[\tilde{g}_i]}$$

for $i = 1, 2$. Therefore, we obtain

$$\mathbb{R}^{V[p]} \subsetneq \mathbb{R}^{V[\tilde{g}_i]} = \mathbb{R}^{V[\tilde{g}_i, h_i]},$$

and

$$\mathbb{R}^{V[p]} = \mathbb{R}^{V[\tilde{g}_1]} \cap \mathbb{R}^{V[\tilde{g}_2]}.$$

Since the set of reals is different in each model, the respective well orders have to differ at some point. Namely, there is some $\eta \in \mathrm{OR}$ for which the η^{th}-real given by ϕ is different in each model. We then have some digit $n \in \omega$ in which the respective η^{th} reals differ. Without loss of generality we can write:

$$V[\tilde{g}_i, h_i] \models \text{ the } n^{\mathrm{th}} \text{ digit of the } \eta^{\mathrm{th}} \text{ real given by } \phi \text{ is } i-1.$$

We can find then conditions $p_i \leq p$ in $h_i \subseteq P$ that force such a statement for $i = 1, 2$. Namely,

$$p_i \Vdash^{\mathbf{P}}_{V[\tilde{g}_i]} \text{ the } \check{n}^{\mathrm{th}} \text{ digit of the } \check{\eta}^{\mathrm{th}} \text{ real given by } \phi \text{ is } (i \check{-} 1). \tag{4.6}$$

We are exactly in the situation of the definition of $\mathbf{P}$ being $\mathbf{Q}$-balanced over V. We get then that p_1 and p_2 are compatible in $V[g]$.

To obtain a contradiction, we still have to work a bit more. We could be tempted to say that there is a contradiction already, looking at two compatible conditions that force incompatible statements. However, after a closer look, the conditions are forcing incompatible statements over different models. The rest of the proof consists in fixing this obstacle in order to get the desired contradiction.

By Lemma 4.5, for each $i = 1, 2$ there is $\bar{p}_i \in \mathbf{P}^{V[\tilde{g}_i]}$ such that $\bar{p}_i \leq p_i$ and $V[\bar{p}_i]$ is a $\mathbf{Q}$-ground of $V[\tilde{g}_i]$. Then,

$$\bar{p}_i \Vdash^{\mathbf{P}}_{V[\tilde{g}_i]} \text{ the } \check{n}^{\mathrm{th}} \text{ digit of the } \check{\eta}^{\mathrm{th}} \text{ real given by } \phi \text{ is } (i \check{-} 1). \tag{4.7}$$

By homogeneity of $\mathbf{Q}$ we have that, for $i = 1, 2$:

$$\mathbb{1} \Vdash^{\mathbf{Q}}_{V[\bar{p}_i]} \check{\bar{p}}_i \Vdash^{\mathbf{P}}_{V[\bar{p}_i, \dot{g}]} \text{ the } \check{\check{n}}^{\mathrm{th}} \text{ digit of the } \check{\check{\eta}}^{\mathrm{th}} \text{ real given by } \phi \text{ is } (i \check{\check{-}} 1). \tag{4.8}$$

Notice that $V[\bar{p}_i] \supseteq V[p_i] \supseteq V[p]$ and $\vec{x} \in V[p]$, so the variables of ϕ are check names.

Since $V[\tilde{g}_i]$ is a $\mathbf{Q}$-ground of $V[g]$ as well, and $\mathbf{Q} \times \mathbf{Q} \cong \mathbf{Q}$, $V[\bar{p}_i]$ is also a $\mathbf{Q}$-ground of $V[g]$. This gives us:

$$\bar{p}_i \Vdash^{\mathbf{P}}_{V[g]} \text{ the } \check{n}^{\mathrm{th}} \text{ digit of the } \check{\eta}^{\mathrm{th}} \text{ real is } (i \check{-} 1). \tag{4.9}$$

More explicitly,

$$\bar{p}_1 \Vdash_{V[g]}^{\mathbf{P}} \text{ the } \check{n}^{\text{th}} \text{ digit of the } \check{\eta}^{\text{th}} \text{ real is } \check{0}, \text{ and}$$
$$\bar{p}_2 \Vdash_{V[g]}^{\mathbf{P}} \text{ the } \check{n}^{\text{th}} \text{ digit of the } \check{\eta}^{\text{th}} \text{ real is } \check{1}.$$

Let p^* be a witness for the compatibility of $\bar{p}_1$ and $\bar{p}_2$ in $P \cap V[g]$. Then p^* forces contradictory statements. Therefore, there is no well-order of the reals in $L(\mathbb{R}, \mathcal{P})^{V[g,h]}$. $\square$

If we restrict our forcing $\mathbf{Q}$ and request more from $\mathbf{P}$, we can get a model with no non-principal ultrafilter on ω by the following theorem.

Theorem 4.7
Suppose $\mathbf{Q}$ is the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be a forcing poset in $V[g]$ of the form

$$p \in \mathbf{P} \iff \exists x \in \mathbb{R}\; V[x] \models \text{“}p \subseteq \mathbb{R} \textit{ and } \psi(p)\text{''},$$

such that:

1. *$\mathbf{P}$ is σ-closed and real absolute.*
2. *For all $\beta < \omega_1$, there are densely many $p \in \mathbf{P}$ such that*
   - *(a) there is some γ, with $\beta \leq \gamma < \omega_1$, such that $V[g \restriction \gamma] \models p \subseteq \mathbb{R}$ and $\psi(p)$, and*
   - *(b) for all g_1, g_2 $\mathbf{Q}$-generic filters over V such that $(V[g \restriction \gamma], V[g_1], V[g_2])$ is a real bifurcation, and for every $p_1 \in \mathbf{P}^{V[g_1]}$, $p_2 \in \mathbf{P}^{V[g_2]}$ extending p, p_1 and p_2 are compatible.*

Let h be a $\mathbf{P}$-generic filter over $V[g]$ and $\mathcal{P} = \cup h$. Then

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{DC} + \neg\mathsf{UI}(\omega).$$

By how $\mathbf{P}$ is defined, “$p \in \mathbf{P}$” is already absolute between inner models, so $\mathbf{P}^M = \mathbf{P}^{V[g]} \cap M$ for any inner model $M \subseteq V[g]$. Notice that we ask $\mathbf{P}$ to be real absolute to ensure that the relation $\leq_P$ is also absolute between inner models.

Proof: **Step I:** Suppose the contrary. Then there is some formula $\phi(\cdot, \vec{x}, \vec{\alpha}, h)$, where $\vec{x} \subseteq \mathbb{R}^{V[g,h]}$ and $\vec{\alpha} \subseteq \mathrm{OR}$, such that

$$V[g,h] \models \phi\left(\cdot, \vec{x}, \vec{\alpha}, \mathcal{P}\right) \text{ defines a non principal ultrafilter on } \omega. \tag{4.10}$$

Since $\mathbf{P}$ is σ-closed, $\mathbb{R}^{V[g,h]} = \mathbb{R}^{V[g]}$. Applying Lemma 2.12, there is $\beta < \omega_1$ such that $\vec{x} \in \mathbb{R}^{V[g\restriction\beta]}$.

There is a condition p that forces the statement 4.10, namely,

$$p \Vdash^{\mathbf{P}}_{V[g]} \phi\left(\cdot, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right) \text{ defines a non principal ultrafilter on } \omega, \tag{4.11}$$

where by $\dot{\mathcal{P}}$ we mean the formula defining $\dot{\mathcal{P}} = \{\langle p, \check{x}\rangle \mid x \in p \text{ and } p \in \mathbf{P}\}$.

By hypothesis we can assume that p has the properties described in 2, in particular, $p \in V[g \restriction \gamma]$ for some $\gamma > \beta$. Now consider the γ^{th} Cohen real added by g, call it r. Notice that $r \colon \omega \to 2$, so

$$r^{-1}(0) \dot\cup r^{-1}(1) = \omega.$$

Therefore, in $V[g,h]$ one of these sets must be in the ultrafilter given by ϕ. Suppose it is the first one and let us name it $\bar{r}$. Pick $\bar{p} \leq p$ such that it forces that $\bar{r} \subseteq \omega$ is in the ultrafilter, namely,

$$\bar{p} \Vdash^{\mathbf{P}}_{V[g]} \phi\left(\check{\bar{r}}, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right). \tag{4.12}$$

By the choice of p, we know that (in $V[g]$) $p = \bar{p} \cap V[g \restriction \gamma]$.

By definability of forcing there is some formula that expresses the forcing relation in 4.12 in $V[g]$. Now think of g as

$$g = (g \restriction \gamma) \times g \restriction (\omega_1 \backslash \gamma).$$

Then there is a condition $q \in \mathbf{C}(\omega_1\backslash\gamma) = \mathbf{Q} \restriction (\omega_1\backslash\gamma)$ such that

$$q \Vdash^{\mathbf{C}(\omega_1\backslash\gamma)}_{V[g\restriction\gamma]} \text{“}\check{p} = \dot{\bar{p}} \cap V[g \restriction \gamma], \dot{\bar{p}} \leq_P \check{p} \text{ and } \dot{\bar{p}} \Vdash^{\mathbf{P}}_{V[g\restriction\gamma,\cdot]} \phi\left(\dot{\check{\bar{r}}}, \dot{\check{\vec{x}}}, \dot{\check{\vec{\alpha}}}, \dot{\mathcal{P}}\right).\text{”} \tag{4.13}$$

Notice that $\gamma > \beta$ implies that $\vec{x} \in V[g \restriction \beta] \subseteq V[g \restriction \gamma]$, and also $p \in V[g \restriction \gamma]$.

Let $\dot{\bar{p}}$ be any $\mathbf{C}(\omega_1\backslash\gamma)$-name for $\bar{p}$ and $\dot{\check{\bar{r}}}$ the following name for $\bar{r}$:

$$\dot{\check{\bar{r}}} = \left\{\left\langle \langle \check{n}, \mathbb{1}_{\mathbf{P}}\rangle^{\vee}, q\right\rangle \mid q \in \mathbf{C}(\omega_1\backslash\gamma), \langle n, 0\rangle \in q(\gamma)\right\}$$

Step II: We want to get a contradiction from 4.13. The idea is to extend $L[g \restriction \gamma]$ in two different (but still amalgamable) ways so that we get two sets which are in the corresponding ultrafilter, but whose intersection is finite. The rest of the proof consists in carefully constructing such extensions.

Let s be $q(\gamma) \in 2^{<\omega}$. Consider the forcing $\mathbf{C}_s$ over $V[g \restriction \gamma]$ given by

$$\begin{aligned}(a,b) \in \mathbf{C}_s \iff & a, b \in \mathbf{C} \text{ with } a, b \supseteq s,\\ & \mathrm{lh}(a) = \mathrm{lh}(b); \text{ and}\\ & \{n \geq \mathrm{lh}(s) \mid a(n) = b(n) = 0\} = \emptyset;\end{aligned}$$

and ordered by reverse inclusion in each coordinate.

Claim. *The following hold:*

i. $\mathbf{C}_s$ *is absolute.*

ii. $\mathbf{C}_s \cong \mathbf{C}$.

iii. If g_s *is a* $\mathbf{C}_s$*-generic filter over some transitive model* M*, consider* j_1 *and* j_2 *the projections of* g_s *to the first and second coordinate respectively. Then* j_1 *and* j_2 *are* $\mathbf{C}$*-generic over* M *and* $M[g_s]$ *is a* $\mathbf{C}$*-extension of* $M[j_1]$ *(or* $M[j_2]$*). So, for* $i = 1, 2$*, we have*

$$M \overset{\mathbf{C}}{\hookrightarrow} M[j_i] \overset{\mathbf{C}}{\hookrightarrow} M[g_s].$$

iv. $M[j_1] \cap M[j_2] = M$.

v. If r_1 *and* r_2 *are the Cohen reals added by* j_1 *and* j_2*, then* $\{n \mid r_0(n) = r_1(n) = 0\}$ *is finite.*

Proof:

i. It is clear.

ii. $\mathbf{C}_s$ is countable and without atoms, by Lemma 2.15 we get that $\mathbf{C}_s \cong \mathbf{C}$.

iii. For each $i = 1, 2$, $j_i \subseteq M$ and $j_i \in M[g_s]$, so by Theorem 2.16 we obtain

$$M \hookrightarrow M[j_i] \hookrightarrow M[g_s].$$

Moreover, consider $r_1 = \cup j_1$ and $r_2 = \cup j_2$. For each $i = 1, 2$, r_i is a real and $M[r_i] = M[j_i]$. We know also that $\mathbf{C}_s \cong \mathbf{C}$ so we can think of $M[g_s]$ as a Cohen extension of M. Applying 2.17, the corresponding forcings have to be forcing equivalent to $\mathbf{C}$.

iv. Suppose that $y \in M[j_1] \cap M[j_2]$. Then there are $\mathbf{C}$-names σ and τ such that $y = \sigma_{j_1} = \tau_{j_2}$. Let us assume by $\in$-induction that $x \subseteq M$. Since $y \in M[g_s]$, there is a condition $(a, b) \in \mathbf{C_s}$ such that

$$(a, b) \Vdash^{\mathbf{C_s}}_{M} \tilde{\sigma} = \tilde{\tau} \text{ and } \tilde{\sigma} \subseteq M,$$

where $\tilde{\sigma}$ and $\tilde{\tau}$ are the $\mathbf{C_s}$-names that resemble σ and τ respectively but each name has everything possible in the second, respectively first, coordinate. We will describe $\tilde{\sigma}$ and $\tilde{\tau}$ precisely as follows.

Given a name $\sigma \in M^{\mathbf{C}}$, we define σ^* and ${}^*\sigma$ in $M^{\mathbf{C_s}}$ recursively:

$$\sigma^* = \{(\pi^*, (a, b)) \mid (a, b) \in \mathbf{C_s} \text{ and } (\pi, a) \in \sigma\}$$
$${}^*\sigma = \{({}^*\pi, (a, b)) \mid (a, b) \in \mathbf{C_s} \text{ and } (\pi, b) \in \sigma\}$$

Set $\tilde{\sigma} = \sigma^*$ and $\tilde{\tau} =^* \tau$.

We claim that a decides "$\check{z} \in \sigma$" for every $z \in M$. If this holds, then $y = \sigma_{j_1} \in M$. Suppose it does not. Then there is some $z \in M$ for which a does not decide whether $\check{z}$ belongs to σ. Pick a_0, $a_1 \in \mathbf{C}$ extending a such that

$$a_0 \Vdash^{\mathbf{C}}_{M} \check{z} \notin \sigma, \text{ and}$$
$$a_1 \Vdash^{\mathbf{C}}_{M} \check{z} \in \sigma.$$

Let b' the extension of b of length $\max\{\mathrm{lh}(a_0), \mathrm{lh}(a_1)\}$ such that $b' = b^\frown\vec{1}$. Extend b' to b'' so that b' decides "$\check{z} \in \tau$". Let us assume without loss of generality that $b'' \Vdash^{\mathbf{C}}_{M} \check{z} \notin \tau$. Then let a' be the extension of a_1 of length $\mathrm{lh}(b'') \geq \mathrm{lh}(a_1)$ such that $a' = a_1{}^\frown\vec{1}$, as Figure 4.2 shows. If $b'' \Vdash^{\mathbf{C}}_{M} \check{z} \in \tau$, we would have taken $a' = a_0{}^\frown\vec{1}$. By construction, $(a', b'') \in \mathbf{C}_s$ and $(a', b'') \Vdash^{\mathbb{C}_s}_{M} \check{z} \in \tilde{\sigma} \wedge \check{z} \notin \tau$.

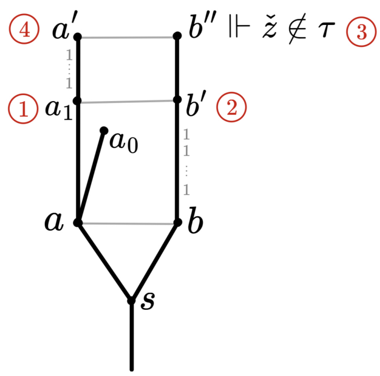

Figure 4.2: $(a', b'') \in \mathbf{C}_s$, but $a' \Vdash \check{z} \in \sigma$ and $b'' \Vdash \check{z} \notin \tau$.

This is a contradiction, since $(a', b'') \leq (a, b)$.

v. Let r_1, r_2 be the reals added by j_1 and j_2 respectively. Then $\{n \mid r_1(n) = r_2(n) = 0\} = \{n \mid s(n) = 0\}$, which is finite.

■

Step III: Let g_s be a $\mathbf{C}_s$-generic filter over $V[g \restriction \gamma]$ and let $\tilde{g}_1, \tilde{g}_2$ be $\mathbf{C}(\omega_1 \backslash (\gamma+1))$-mutually generic over $V[g \restriction \gamma, g_s]$ containing $q \restriction (\omega_1 \backslash (\gamma+1))$. For $i = 1,2$ we have that $\tilde{g}_i$ is also $\mathbf{C}(\omega_1 \backslash (\gamma+1))$-generic over $V[g \restriction \gamma, j_i]$. In other words, for $i = 1,2$, $j_i \times \tilde{g}_i$ is $\mathbf{C}(\omega_1 \backslash \gamma)$-generic over $V[g \restriction \gamma]$ and also $q \in j_i \times \tilde{g}_i$.

We have now our two different (but amalgamable) paths:

$$\begin{tikzcd}
& V[g \restriction \gamma][j_1][\tilde{g}_1] \arrow[dr, hook] & \\
V[g \restriction \gamma] \arrow[ur, hook] \arrow[dr, hook] & & V[g \restriction \gamma][g_s][\tilde{g}_1, \tilde{g}_2] \\
& V[g \restriction \gamma][j_2][\tilde{g}_2] \arrow[ur, hook] &
\end{tikzcd}$$

For $i = 1,2$ let g_i be $g \restriction \gamma \times j_i \times \tilde{g}_i$.

Now, recall 4.13:

$$q \Vdash^{\mathbf{C}(\omega_1 \backslash \gamma)}_{V[g \restriction \gamma]} \text{“}\check{p} = \dot{\bar{p}} \cap V[g \restriction \gamma], \dot{\bar{p}} \leq_P \check{p} \text{ and } \dot{\bar{p}} \Vdash^{\mathbf{P}}_{V[g \restriction \gamma, \cdot]} \phi\left(\dot{\check{\bar{r}}}, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \dot{\mathcal{P}}\right).\text{”}$$

Let $p_1 = (\dot{\bar{p}})_{g_1}$ and $p_2 = (\dot{\bar{p}})_{g_2}$. Let τ_i be $(\dot{\check{\bar{r}}})_{g_i}$ of $i = 1,2$. Looking at the definition $\dot{\check{\bar{r}}}$, and computing τ_i, we know that τ_1 and τ_2 are just check names of the preimage of 0 of the reals r_1 and r_2 respectively. Let us then write $\check{\bar{r}}_i = \tau_i$. Then we get that $p_1, p_2 \leq p$ and $p_1 = p \cap V[g \restriction \gamma]$ in $V[g]$ (or any other model that has p, p_1 and p_2 as elements, since these notions are absolute). Moreover,

$$p_1 \Vdash^{\mathbf{P}}_{V[g_1]} \phi\left(\check{\bar{r}}_1, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right), \text{ and} \tag{4.14}$$

$$p_2 \Vdash^{\mathbf{P}}_{V[g_2]} \phi\left(\check{\bar{r}}_2, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right). \tag{4.15}$$

Notice that $V[g_1] \cap V[g_2] = V[g \restriction \lambda]$ using the item iv of the claim and mutual genericity of $\tilde{g}_1$ and $\tilde{g}_2$. Then we are in the condition of the hypothesis 2b so we know p_1 and p_2 are compatible conditions.

Since $p_1 \in \mathbf{P}^{V[g_1]}$, there is $x \in \mathbb{R}^{V[g_1]}$ such that $p_1 \in V[x]$. Since g_1 is $\mathbf{Q}$-generic over V, there is $\delta < \omega_1$ such that $\delta > \gamma$, and $x \in V[g_1 \restriction \delta]$ (see Theorem 2.12). Then $p_1 \in V[g_1 \restriction \delta]$. From Equation 4.15 and the homogeneity of $\mathbf{Q}$, we get

$$\mathbb{1}_{\mathbf{Q}} \Vdash^{\mathbf{C}(\omega_1 \backslash \delta)}_{V[g_1 \restriction \delta]} \check{p}_1 \Vdash^{\mathbf{P}}_{V[g_1 \restriction \delta, \cdot]} \phi\left(\check{\check{\bar{r}}}_1, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \dot{\mathcal{P}}\right). \tag{4.16}$$

Now, consider $g_1 \upharpoonright \delta$ as one real. Notice that we can use Theorem 2.18 to say that $V[g_1 \upharpoonright \delta]$ is a $\mathbf{C}(\omega_1)$-ground of $V[g]$. Since $\mathbf{C}(\omega_1) \cong \mathbf{C}(\omega_1 \setminus \delta)$, we have

$$p_1 \Vdash^{\mathbf{P}}_{V[g]} \phi\left(\check{\bar{r}}_1, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right), \text{ and} \tag{4.17}$$

$$p_2 \Vdash^{\mathbf{P}}_{V[g]} \phi\left(\check{\bar{r}}_2, \check{\vec{x}}, \check{\vec{\alpha}}, \dot{\mathcal{P}}\right). \tag{4.18}$$

Remember that both p_1 and p_2 extend $p \in V[g \upharpoonright \gamma]$. Then we are in the same situation of hypothesis 2b, and in $V[g]$ the conditions p_1 and p_2 are compatible. Nevertheless p_1 and p_2 force incompatible statements, namely, $\bar{r}_1$ and $\bar{r}_2$ being elements of the ultrafilter given by ϕ. This leads to a contradiction. □

We are interested in applying Theorem 4.6 for specific paradoxical sets that are partitions of euclidean spaces in some way. For example, a Hamel basis of the reals is a *partition* of the reals in the following sense: each real is *covered* by one finite subset of the Hamel basis which spans it and this finite subset is unique. So we can think of the reals partitioned in pieces depending on which subset of the Hamel basis spans the real.

We wanted to state Theorem 4.6 in full generality for future applications, but for the purpose of this text we will use Corollary 4.11. To state it we will need some definitions.

Definition 4.8. Let V be a model of ZFC. Let $\mathbf{Q}$ be a forcing notion over V and g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be a forcing notion in $V[g]$. We say that **$\mathbf{P}$ adds a real partition** if it is of the form

$$p \in \mathbf{P} \iff \exists x \in \mathbb{R}\ V[x] \models \psi(p),$$

where there are $n, m < \omega$ and formulas ψ_1 and ψ_2 absolute between transitive models of set theory such that

$$\psi(p) : p \subseteq \mathbb{R}^n \wedge \left(\forall s \in [p]^{<\omega} \psi_1(s)\right) \wedge \left(\forall r \in \mathbb{R}^m \exists s \in [p]^{<\omega} \psi_2(r, s)\right),$$

and for any pair (x, p) as before, x can be computed from finitely many elements of p. We also request that $\leq_P$ is a subset of the reverse inclusion on $\mathbf{P}$, i.e.,

$$p_0 \leq_{\mathbf{P}} p_1 \iff p_0 \supseteq p_1 \wedge \phi(p_0, p_1)$$

where ϕ is absolute between transitive models.

Remark.

In principle it could be that there is no such pair (x, p), and then $\mathbf{P}$ would be the empty set. Of course, we do not want to consider such $\mathbf{P}$. In fact, we will assume that for any real x there is p such that $V[x] \models \psi(p)$. This holds in all the cases considered, in which ψ is the definition of a paradoxical set, and then $\mathsf{ZFC} \vdash \exists p \psi(p)$. We will assume this implicitly for the rest of the text.

Definition 4.9. Let $\mathbf{P}$ be a forcing notion that adds a real partition as in Definition 4.8. We say that $p \in V[g]$ is **partial condition** if there is $x \in \mathbb{R}^{V[g]}$ such that

$$V[x] \models p \subseteq \mathbb{R}^n, \forall s \in [p]^{<\omega} \psi_1(s).$$

We say that $\mathbf{P}$ satisfies **Extendability** if for any partial condition p in $V[x]$ there is a condition $\bar{p} \in \mathbf{P}$ witnessed by $\bar{x} \in \mathbb{R}$ such that $\bar{p} \supseteq p$ and $x \in V[\bar{x}]$. Moreover, if $p \in \mathbf{P}$ then we can pick $\bar{p}$ such that additionally $\bar{p} \leq_{\mathbf{P}} p$.

Remark.
If $\mathbf{P}$ is a forcing notion adding a Hamel basis on the reals, a condition is a Hamel basis in some $V[x]$, and a partial condition is a linearly independent set.

Notice that the reals associated with p and $\bar{p}$ in 4.9 may differ. They *will* differ for the cases of Mazurkiewicz sets and partitions in unit circles, but we can take $\bar{p}$ so that p and $\bar{p}$ share the same associated real x for the case of Hamel bases. This is because Hamel bases are exactly the maximal linearly independent sets. However, it is not true that every partial Mazurkiewicz set is extendable to a full Mazurkiewicz set (see discussion in the second paragraph of Section 6.1).

Definition 4.10. Let $\mathbf{P}$ be a forcing notion that adds a real partition as in Definition 4.8. We say that $\mathbf{P}$ satisfies **Amalgamation** (in $V[g]$) if for densely many $p \in \mathbf{P}$, for any g_1, g_2 mutually $\mathbf{Q}$-generic over $V[p]$ and for all $p_1 \in \mathbf{P} \cap V[p, g_1]$, $p_2 \in \mathbf{P} \cap V[p, g_2]$ such that $p_1 \leq_P p$ and $p_2 \leq_P p$, p_1 and p_2 are compatible.

Theorem 4.11 (Corollary of Theorem 4.6)
Let V be a model of ZFC. *Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, and let g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be a forcing notion over $V[g]$ that adds a real partition, let h be a $\mathbf{P}$-generic filter over $V[g]$, and let $\mathcal{P} = \cup h$.*

If $\mathbf{P}$ is σ-closed and satisfies Extendability and Amalgamation, then

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \psi(\mathcal{P}).$$

Proof: First, we need to prove that the hypotheses of Theorem 4.6 are satisfied.

1. $\mathbf{Q}$ *is homogeneous and* $\mathbf{Q} \times \mathbf{Q} \cong \mathbf{Q}$. In this case, $\mathbf{Q} = \mathbf{C}(\omega_1)$ so it satisfies these properties (see Lemma 2.23).

2. $\mathbf{P}$ *is real absolute and* σ*-closed.* It is clear that it is real absolute, by noticing that $L[x]$ does not change through different models containing the same ordinals and x (see discussion after Definition 2.2), therefore its theory is absolute as well. $\mathbf{P}$ is σ-closed by hypothesis.

3. $\mathbf{P}$ *and* $\mathbf{Q}$ *are real-alternating.* The first condition of real-alternation is true because $\mathbf{P}$ adds a real partition. The second is due to $\mathbf{P}$ satisfying Extendability.

4. $\mathbf{P}$ *is a* $\mathbf{Q}$*-balanced forcing over* V. By Amalgamation, there are densely many $p \in \mathbf{P}$ that have the amalgamation property. For such a p, there is a real x such that $V[x] = V[p]$. Because of Lemma 2.12, there is some $\alpha < \omega_1$ such that $x \in V[g \upharpoonright \alpha]$.

   By Lemma 2.18, $V[x]$ is a $\mathbf{Q}$-ground of $V[g]$. Since $\mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}$, there are g_1, g_2 mutually $\mathbf{Q}$-generic over $V[x]$ such that $V[x, g_1, g_2] = V[g]$. Clearly, $(V[p], V[p, g_1], V[p, g_2])$ is a real bifurcation. Now, take any $p_1 \in \mathbf{P} \cap V[p, g_1]$ and $p_2 \in \mathbf{P} \cap V[p, g_2]$. By Amalgamation, the conditions p_1 and p_2 are compatible, and p is a $\mathbf{Q}$-balanced condition.

Applying Theorem 4.6, we get that

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}).$$

It is left to prove that

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \psi(\mathcal{P}).$$

Work inside $V[g, h]$. First, since $h \subseteq \mathbf{P}$, we have that for all $p \in h$ there some $x \in \mathbb{R}$ such that

$$V[x] \models p \subseteq \mathbb{R}^n, \forall s \in [p]^{<\omega} \psi_1(s) \wedge \forall r \in \mathbb{R}^m \exists s \in [p]^{<\omega} \psi_2(r, s).$$

From this and the fact that being a real number is absolute, we have that $\mathcal{P} \subseteq \mathbb{R}^n$. Let $s \in \mathcal{P}^{<\omega}$, $s = \{s_0, \ldots, s_{l-1}\}$. Then there is a finite set of conditions $p_0, \ldots, p_{l-1}$ in h such that $s_i \in p_i$ for all $i \in l$. Since h is a filter, there is $p \in h$ such that $p \leq_{\mathbf{P}} p_i$ for all $i \in l$. In particular, $p_i \subseteq p$ for all $i \in l$ and $s \subseteq p$. Since $p \in \mathbf{P}$, there is some x such that

$$V[x] \models \forall \tilde{s} \in [p]^{<\omega} \psi_1(\tilde{s})$$

We have then that $\psi_1(s)$ holds in $V[x]$. Because ψ_1 is absolute between inner models, we have that $\psi_1(s)$ holds in $V[g,h]$ as well as in $L(\mathbb{R},\mathcal{P})^{V[g,h]}$.

Secondly, notice first that $\mathbb{R}\cap L(\mathbb{R},h)^{V[g,h]} = \mathbb{R}\cap V[g,h] = \mathbb{R}\cap V[g]$ since $\mathbf{P}$ is σ-closed. Fix $r\in\mathbb{R}^m$. We claim that the set

$$D = \{p\in\mathbf{P}\mid r\in V[p]\}$$

is dense. Fix $p\in\mathbf{P}$. There is some real x that witnesses p is a condition. By absoluteness, p is a partial condition in $V[x\oplus r]$. By Extendability, there is $\bar{p}\supseteq p$ such that $x\oplus r\in V[\bar{p}]$, which implies $r\in V[\bar{p}]$. Since p is a condition, we can assume $\bar{p}\leq_P p$.

Since h is a generic filter, $D\cap h\neq\emptyset$. Let $p\in D\cap h$. By definition of $\mathbf{P}$ again, we have that

$$V[p]\models\forall\tilde{r}\in\mathbb{R}^m\exists s\in[p]^{<\omega}\psi_2(\tilde{r},s).$$

Since $r\in V[p]$ by definition of D,

$$V[p]\models\exists s\in[p]^{<\omega}\psi_2(r,s).$$

By absoluteness, there is $s\in[p]^{<\omega}\subseteq[\mathcal{P}]^{<\omega}$ such that $\psi_2(r,s)$ holds in $V[g,h]$ and also in $L(\mathbb{R},\mathcal{P})^{V[g,h]}$. Putting everything together, we have that

$$L(\mathbb{R},\mathcal{P})^{V[g,h]}\models\psi(\mathcal{P}),$$

as we wanted to show. □

Lemma 4.12. *Let $\mathbf{Q}$ and $\mathbf{P}$ be as in Corollary 4.11. If $\leq_{\mathbf{P}}=\supseteq\restriction(\mathbf{P}\times\mathbf{P})$, then $\mathbf{P}$ is σ-closed in $V[g]$.*

Proof: Let $\{p_n\}_{n<\omega}$ be a sequence of decreasing conditions. Let $\{x_n\}_{n<\omega}$ be a sequence of reals such that $V[x_n]\models\psi(p_n)$ for all $n<\omega$. We can do this since $V[g]\models\mathsf{AC}$. Take $x=\bigoplus_{n<\omega}x_n$ and $p=\bigcup_{n<\omega}p_n$.

We claim $p\in V[x]$ is a partial condition. Clearly, $p\subseteq\mathbb{R}^n$. Fix $s\in[p]^{<\omega}$. We know that s is finite, $\{p_n\}_{n<\omega}$ is decreasing sequence, and $\leq_{\mathbf{P}}$ is a subset of the reverse inclusion in $\mathbf{P}$. Therefore, there is $n<\omega$ such that $s\subseteq p_n$. Then $V[x_n]\models\psi_1(s)$ and by absoluteness we get that $V[x]\models\psi_1(s)$.

By Extendability, there is a condition $\bar{p}\in\mathbf{P}$ such that $\bar{p}\supseteq p$. Therefore $\bar{p}\leq_P p_n$ for all $n<\omega$. □

Remark.

In the last part of the proof of Theorem 4.11 where we showed that the model satisfies $\psi(\mathcal{P})$, we only needed to use that $\mathbf{P}$ is σ-closed and a bit of Extendability, for which condition 2a in Theorem 4.7 is enough. In other words, if in Theorem 4.7 we replace"real absolute" for the stronger condition of "$\mathbf{P}$ adds a real partition", then we obtain that

$$L(\mathbb{R}, \mathcal{P}^{V[g,h]}) \models \mathsf{DC} + \neg\mathsf{UI}(\omega) + \psi(\mathcal{P}).$$

Chapter 5

Hamel bases

In Section 5.1 we give an overview of some relevant results related to Hamel bases. In Section 5.2, we will show that we can apply the methods of Chapter 4 to this paradoxical set, getting a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{UI}(\omega)$ which has a Hamel basis.

Definition 5.1. Let $H \subseteq \mathbb{R}$. We say that H is a Hamel basis if it is a basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$, namely, a maximal linearly independent set over $\mathbb{Q}$.

It is clear that $\mathsf{ZF} + \mathsf{WO}(\mathbb{R})$ implies there is a Hamel basis: one can construct a Hamel basis by extending recursively a linearly independent set until it is maximal, always adding the first real which is not in the span of the linearly independent set taken so far.

5.1 Literature review

In ZF, the existence of a Hamel basis implies the existence of a Vitali set, which is the standard example of a non-measurable subset of $\mathbb{R}$. However, a Hamel basis itself can be measurable, and every measurable Hamel basis has null measure [56, Theorem I]. Yet there is no Hamel basis which is Borel [56, Theorem 2]. Actually, it is known that there is no analytic Hamel basis [31, Theorem 9]. The next best possible is being coanalytic, which is consistent and follows from $V = L$ [46, Theorem 9.26].

It is well-known that any uncountable analytic set has a perfect subset. There are Hamel bases with no perfect subset (any Burstin basis), but there is also a Hamel basis with a perfect subset [31, Example 1]. Vidnyánszky asks if it is consistent that there is a Hamel basis that is both Π^1_1 and contains a perfect subset [58, Problem 5.8].

The existence of a Hamel basis can be proven from $\mathsf{ZF} + \mathsf{WO}(\mathbb{R})$, and it is of course a corollary of the existence of bases for every vector space, which is equivalent to AC (see Figure 2.1). Having Question 2 in mind, a natural question is whether one can recover $\mathsf{WO}(\mathbb{R})$ from the existence of a Hamel basis. This was asked by Pincus and Prikry [48]. It turns out one cannot recover even Countable Choice, since in the Cohen-Halpern-Lévy model (Definition 2.27) there is a Hamel basis [9, Theorem 2.1]. If one is interested to have DC in the model, it is also possible: it consistent that there is a model of $\mathsf{ZF}+\mathsf{DC}+\neg\mathsf{WO}(\mathbb{R})$ with a Hamel basis [54, Theorem 1.1]. We will strengthen this result using the results on Chapter 4 by showing that in the same model there is no non-principal ultrafilter on ω (see Theorem 5.2). Moreover, in a following article [12, Theorem 5.4] the authors show that there is a model M of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$ in which there is a Hamel basis (moreover, a Burstin basis, i.e. a Hamel basis which has nonempty intersection with every perfect set) and several other paradoxical sets (Luzin, Sierpiński, and of course Vitali). Using the same methods from Chapter 4 we can also recover the existence of a Hamel basis in the model M. Furthermore, using an inaccessible cardinal Larson and Zapletal produced a model of $\mathsf{ZF}+\mathsf{DC} + \neg\mathsf{UI}(\omega)$ in which there is a Hamel basis [42, Corollary 12.2.10]. We actually can remove the requirement of the inaccessible cardinal and produce such a model, as Theorem 5.2 shows.

Considering the two approaches (choiceless set theory and the analytical hierarchy), one could ask how low in the hierarchy can a Hamel basis be while still not having a well-ordering on the reals. There is a version of the Cohen-Halpern-Lévy model (Definition 2.27) with Jensen reals instead of Cohen reals that satisfies $\mathsf{ZF}+\neg\mathsf{AC}_\omega(\mathbb{R})$ in which there is a Δ^1_3 Hamel basis [33, Theorem 0.1]. This was later improved by Schilhan, who constructs a model with a Δ^1_2 Hamel basis [52, Theorem 1.4].

5.2 Forcing a Hamel basis

In this section, we will show that we can apply Theorems 4.6 and 4.7 to the case of Hamel bases. We start by recovering the result that there is a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$ in which there is a Hamel basis [54, Theorem 1.1], but actually showing that in that model there is no non-principal ultrafilter on ω. Moreover, we can recover the result that there is a Hamel basis in the model of $\mathsf{ZF}+\mathsf{DC}+\neg\mathsf{WO}(\mathbb{R})$ presented in [12, Theorem 5.1], which uses Sacks reals instead of Cohen reals for the forcing $\mathbf{Q}$.

Theorem 5.2 (Corollary of Theorem 4.7)
Let $\mathbf{Q}$ *be the finite support product of* ω_1*-many copies of Cohen forcing, and let* g *be a* $\mathbf{Q}$*-generic filter over* V*. Let* $\mathbf{P_H}$ *be the forcing poset in* $V[g]$ *given by*

$$p \in \mathbf{P_H} \iff \exists x \in \mathbb{R} : \ V[x] \models p \textit{ is a Hamel basis},$$

ordered by reverse inclusion. Let h *be a* $\mathbf{P_H}$*-generic filter over* $V[g]$*, and let* $\mathcal{P} = \cup h$*. Then*

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{UI}(\omega) + \mathcal{P} \textit{ is a Hamel basis.}$$

Proof: We want to apply Theorem 4.7, so let us verify its hypotheses.

1. $\mathbf{P}$ *is* σ*-closed and real absolute.*
   It is clear that $\mathbf{P}$ is real absolute, by noticing that $L[x]$ does not change through different models containing the same ordinals and x therefore, the theory of $L[x]$ is absolute as well. It is easy to see that $\mathbf{P} = \mathbf{P_H}$ *adds a real partition* and satisfies *Extendability* because any partial condition (linearly independent set in some $V[x]$) can be extended to a full Hamel basis in $V[x]$. We can then use Lemma 4.12 and obtain that $\mathbf{P}$ is σ-closed.

2. *For all* $\beta < \omega_1$*, there are densely many* $p \in \mathbf{P}$ *such that*

   (a) *there is some* γ*, with* $\beta \leq \gamma < \omega_1$*, such that* $V[g \restriction \gamma] \models p \subseteq \mathbb{R}$ *and* $\psi(p)$*.*
   Fix $\tilde{p} \in \mathbf{P}$ and $r \in \mathbb{R}^{V[g]}$ such that $\tilde{p}$ is a Hamel basis in $V[r]$. By 2.12, there is $\gamma < \omega_1$ such that $r \in V[g \restriction \gamma]$. We can take here $\gamma > \beta$. Then,

   $$V[g \restriction \gamma] \models \tilde{p} \text{ is linearly independent over } \mathbb{Q}.$$

   In $V[g \restriction \gamma]$, which is a model of ZFC, extend $\tilde{p}$ to a p which is a Hamel basis. Then $p \in \mathbf{P}$ and $p \leq_{\mathbf{P}} \tilde{p}$.
   Moreover, we claim that *for all* $\bar{p} \leq_{\mathbb{P}} p$*,* $p = \bar{p} \cap V[g \restriction \gamma]$ *for* γ *as in item (a).*
   Since $\bar{p} \in \mathbf{P}$, in particular it is linearly independent. Then $\bar{p} \cap V[g \restriction \gamma] = \bar{p} \cap \mathbb{R}^{V[g \restriction \gamma]}$ is a set of reals containing p and linearly independent. But p is a Hamel basis in $V[g \restriction \gamma]$, therefore is a maximal linearly independent set, and hence $\bar{p} \cap \mathbb{R}^{V[g \restriction \gamma]} = p$.

   (b) *for all* g_1, g_2 $\mathbf{Q}$*-generic filters over* V *such that* $(V[g \restriction \gamma], V[g_1], V[g_2])$ *is a real bifurcation, and for every* $p_1 \in \mathbf{P}^{V[g_1]}$*,* $p_2 \in \mathbf{P}^{V[g_2]}$ *extending* p*,* p_1 *and* p_2 *are compatible (in* $V[g]$*):*

Fix g_1, g_2, p_1, p_2. We claim that $\bar{p} = p_1 \cup p_2$ is linearly independent. This is not a new argument (see [12, Claim 3]), but we reproduce it here for completeness. Suppose it is not linearly independent. Then there is $\vec{s} \in [\bar{p}]^{<\omega}$, $\vec{q} \in [\mathbb{Q}]^{<\omega}$ such that $\vec{s} \cdot \vec{q} = 0$. Separating terms accordingly, we can write

$$\vec{s_0} \cdot \vec{q_0} + \vec{s_1} \cdot \vec{q_1} + \vec{s_2} \cdot \vec{q_2} = 0,$$

where $\vec{s_0} \subseteq p$, $\vec{s_1} \subseteq p_1 \backslash p, \vec{s_2} \subseteq p_2 \backslash p$; $s = s_0 \cup s_1 \cup s_2$ and $q = q_0 \cup q_1 \cup q_2$. Now, notice that

$$\vec{s_1} \cdot \vec{q_1} = -\vec{s_0} \cdot \vec{q_0} - \vec{s_2} \cdot \vec{q_2} \in \mathbb{R}^{V[g_1]} \cap \mathbb{R}^{V[g_2]} = \mathbb{R}^{V[g \restriction \gamma]}.$$

Then, $-\vec{s_1} \cdot \vec{q_1}$ is a real number in $V[g \restriction \gamma]$. Since p is a Hamel basis there, there is $\vec{t_0} \in [p]^{<\omega} \vec{r_0} \in [\mathbb{Q}]^{<\omega}$ such that

$$\vec{t_0} \cdot \vec{r_0} = -\vec{s_1} \cdot \vec{q_1}.$$

Then we get

$$\vec{t_0} \cdot \vec{r_0} + \vec{s_1} \cdot \vec{q_1} = 0.$$

Since p_1 is linearly independent (and $t_0 \cap s_1 = \emptyset$), $\vec{r_0} = \vec{0}$ and $\vec{q_1} = \vec{0}$. Coming back to the first equation, we have

$$\vec{s_0} \cdot \vec{q_0} + \vec{s_2} \cdot \vec{q_2} = 0.$$

Since p_2 is linearly independent, $\vec{q_0} = \vec{0}$ and $\vec{q_2} = \vec{0}$. Therefore $\vec{q} = \vec{0}$, as we wanted.
Now, let r_1 and r_2 be such that p_1 and p_2 are Hamel bases in $V[r_1]$ and $V[r_2]$ respectively. In $V[r_1 \oplus r_2]$, we can extend $p_1 \cup p_2$ to a Hamel basis p^*. Then p^* witnesses the compatibility of p_1 and p_2 in $V[g]$.

Applying Theorem 4.7, we get that

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg \mathsf{UI}(\omega).$$

Finally, by the remark after Lemma 4.12 and noticing that $\mathbf{P}$ adds a real partition, we obtain that

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathcal{P} \text{ is a Hamel basis,}$$

as we wanted to show. $\square$

We can use Theorem 4.6 to recover a known result [12, Theorem 5.1] about Hamel basis after adding Sacks reals, here called Corollary 5.3. Notice that the hypotheses in Theorem 4.7 for the poset $\mathbf{P}$ are actually stronger than in Theorem 4.6, and the proof for Hamel bases in Theorem 5.2 does not use any relevant fact of Cohen reals. So the hypotheses 2, 3 and 4 of Theorem 4.6 are satisfied. For hypothesis 1, it is done in the paper mentioned [12, Claim 2].

Corollary 5.3 (of Theorem 4.6)**.** *Let $\mathbf{Q}$ be the countable support product of ω_1-many copies of Sacks forcing, and let g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be the forcing poset in $V[g]$ given by*

$$p \in \mathbf{P} \iff \exists x \in \mathbb{R} : \ V[x] \models p \textit{ is a Hamel basis of } \mathbb{R},$$

ordered by reverse inclusion. Let h be a $\mathbf{P}$-generic filter over $V[g]$, and let $\mathcal{P} = \cup h$. Then

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \mathcal{P} \textit{ is a Hamel basis.}$$

Chapter 6

Mazurkiewicz sets

In this chapter we will deal with another example of paradoxical set, this one with more of a geometrical flavor. It was defined for the first time by Mazurkiewicz in 1914 [45].

Definition 6.1. Let $M \subseteq \mathbb{R}^2$. We say that M is a **Mazurkiewicz set** (also called **two point set**) if for every line l in $\mathbb{R}^2$, $|M \cap l| = 2$.

Mazurkiewicz proved these particular sets existed, using the Axiom of Choice. We include the proof for completeness.

Notation: Let $p \subseteq \mathbb{R}^2$. By $\langle p \rangle$ we denote the set

$$\{l \text{ line } \mid \exists s_1, s_2 \in p \text{ and } l = l(s_1, s_2)\}.$$

We will frequently consider the set $\cup\langle p \rangle$. Notice that $\langle p \rangle$ is a set of lines and $\cup\langle p \rangle$ is instead a set of points in $\mathbb{R}^2$.

Theorem 6.2 (ZFC)
There is a Mazurkiewicz set.

Proof: Using a well ordering of $\mathbb{R}$ we can well order all the lines in $\mathbb{R}^2$. Let $\{l_\alpha\}_{\alpha<\mathfrak{c}}$ be such an enumeration. We will recursively define $p_\alpha \subseteq \mathbb{R}^2$ for $\alpha < \mathfrak{c}$. For $\alpha = 0$, take $p_0 = \emptyset$.

Now suppose p_β is defined for all $\beta < \alpha$. If α is a successor ordinal, namely $\alpha = \beta + 1$, take $r \subseteq l_\beta$ such that $|(p_\beta \cup r) \cap l_\beta| = 2$ and no element of r belongs to the lines *already covered by* p_β. Formally, we request that no element of r is in $\cup\langle p_\beta \rangle$. Define $p_\alpha = p_\beta \cup r$. If α is a limit ordinal, take $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

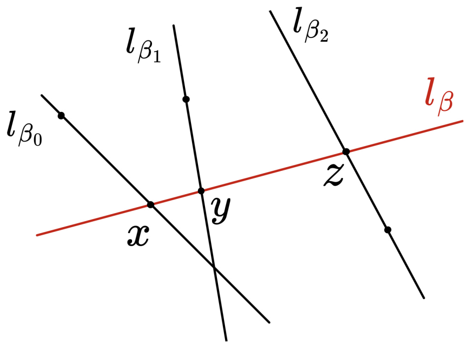

Figure 6.1: $z \in \cup\langle p_{\beta_2}\rangle$ since $z \in l(x, y)$.

We have to check the construction is possible, namely that such r exists. First, we will show that $|p_\beta \cap l_\beta| \leq 2$ for all $\beta < \mathfrak{c}$. Suppose β is the first ordinal such that $|p_\beta \cap l_\beta| \geq 3$. Let $\beta_0 + 1, \beta_1 + 1, \beta_2 + 1$ be the three steps in the construction in which the points x, y and z were added, respectively, with $\beta_0 < \beta_1 < \beta_2 < \beta$. The situation is shown in Figure 6.1. In step $\beta_2 + 1$, we requested $z \notin \cup\langle p_{\beta_2}\rangle$, but this is clearly a contradiction, since z belongs to the line $l(x, y)$ passing through x and y. Thus, $|p_\beta \cap l_\beta| \leq 2$.

Moreover, the lines in $\langle p_\beta\rangle$ are at most $|[p_\beta]^2| \leq |\beta| < \mathfrak{c}$. Each of these lines intersects l_β in at most one point, and $|l_\beta| = \mathfrak{c}$. Therefore $|l_\beta \setminus \bigcup\langle p_\beta\rangle| = \mathfrak{c}$. So, if $|p_\beta \cap l_\beta| < 2$, we can choose r as needed.

Take $M = \bigcup_{\alpha<\mathfrak{c}} p_\alpha$. We claim M is a Mazurkiewicz set. By construction, $|M \cap l| \geq 2$ for every line l. Suppose $|M \cap l_\beta| > 2$ for some $\beta < \mathfrak{c}$. We know that $|p_{\beta+1} \cap l_\beta| = 2$, call these points x and y. A third point $z \in M \cap l_\beta$ should have been added later, at step $\alpha+1$ with $\alpha \geq \beta$. But by construction we know that $z \notin \cup\langle p_\alpha\rangle \subseteq \cup\langle p_{\beta+1}\rangle$. Since $x, y \in p_{\beta+1}$, the line $l(x, y)$ passing through x and y is in $\langle p_{\beta+1}\rangle$, and then $z \notin l(x, y) = l_\beta$, which is a contradiction. Therefore M is a Mazurkiewicz set. $\square$

6.1 Literature review

It is clear from the proof given for Theorem 6.2 that ZFC proves that there is a three-point set, namely a subset of the plane that intersects every line in exactly 3 points. Similarly, one can construct n-point subsets of $\mathbb{R}^2$ for $n < \omega$. Even more, if for any line l of the plane we assign a cardinal α_l such that $2 \leq \alpha_l \leq \aleph_0$, then there exists a set of points in the plane that intersects every line l in precisely α_l points [2]. One year later this result was improved for $2 \leq \alpha_l \leq 2^{\aleph_0}$ [55]. Moreover, one can require the set to intersect each line in a set of certain order type or measure, instead of cardinality [21]; or intersect every line in a topological copy of a given zero-dimensional set (for example a Cantor set) [13, Corollary 5.2]; or intersect circles instead of lines, or both, to obtain a set of a given countable cardinality [2] (and also [36, Theorem 2]). The existence of Mazurkiewicz sets generalizes to vector spaces over infinite fields [38]. There is even a Mazurkiewicz set which is a Hamel bases of $\mathbb{R}^2$ [35, Lemma 12.4]. Mauldin raised the question of which are the conditions for which we can have a Borel set that meets every line l in exactly $2 \leq \alpha_l < \omega$ points. The function $l \mapsto \alpha_l$ needs to be Borel [43, Theorem 12], but what else can we say? There is a simple example of an $\aleph_0$-point set which is F_σ: the union of all the circles centered in the origin and with integer radii. Nevertheless, the case of α_l being a fix natural number n for any $n \geq 2$ is still open.

No n-point set can be contained in an $(n+1)$-point set, otherwise, their difference should be a 1-point set, which does not exist. However, for every k and n such that $2 \leq n \leq k-2 < \omega$ we have that each n-point set is contained in some k-point set [10, Theorem 5.2]. The reverse question naturally arises: is it possible to always find an n-point set inside a k-point set for $k \geq n+2$? Dijkstra gives a negative answer [20] for $n = 4$ and $k = 2$. Immediately after, a complete negative answer to this question was given [10, Theorem 5.5].

Different from the case of Hamel bases where any linearly independent subset of $\mathbb{R}$ can be extended to a Hamel basis, deciding whether a partial Mazurkiewicz set (a subset of $\mathbb{R}^2$ without three points on a line) can be extended to a (full) Mazurkiewicz set is very hard. The proof of Theorem 6.2 shows that any partial two-point set of cardinality strictly less than continuum can be extended to a full two-point set. However, there are *small* partial Mazurkiewicz sets of cardinality $\mathbf{c}$ that cannot be extended to a full Mazurkiewicz set. The simplest example of this is a circle. A circle has cardinality $\mathfrak{c}$, is a partial two-point set, but it is easy to see that it cannot be extended to a (full) two-point set. More on this topic is studied by Dijkstra, Kunen and van Mill [18, 19].

Going back to the question of whether a Mazurkiewicz set can be Borel, Larman claimed that a two-point set cannot be F_σ [41, Theorem 2]. Unfortunately, there was a mistake in the proof that was pointed out and fixed a few years later by Baston and Bostock [6, Theorem 3]. It is also true that a three-point set cannot be F_σ [10, Theorem 4.5]. Both proofs actually show that a two-point set and a three-point set can not contain an arc and derive a contradiction, an strategy that Larman introduced in his paper. However, for $n \geq 4$, n-point sets may contain arcs [10, Corollary 5.3], so a new strategy was needed. Nevertheless, Bouhjar, Dijkstra and Mauldin managed to overcome this obstacle, and proved that no n-point set is F_σ [11]. there is a generalization of this result to n-point sets in $\mathbb{R}^m$ for $m \geq 2$ [22]. Similar to the case of Hamel bases, there are measurable and non-measurable Mazurkiewicz sets, and every measurable Mazurkiewicz set has measure zero [23, II.10.21]. A Mazurkiewicz set must have topological dimension zero [39, Theorem 2], which answers a question of Mauldin [44, 1069 Problem 2.3]. The question whether 3-point sets have this property seems open, while n-point sets for $n \geq 4$ may be one-dimensional (they could contain a circle!).

It is also known that if an n-point set is analytic then it is Borel [46, Section 7]. As in the case of Hamel bases, if $V = L$ then there is a coanalytic Mazurkiewicz set [46, Theorem 7.21], and the same proof shows that the same holds for n-point sets.

Notice that the existence of a Mazurkiewicz set can be proven from $\mathsf{ZF} + \mathsf{WO}(\mathbb{R})$. Attending to the main question leading this text (Question 2), it is also true that one cannot recover a $\mathsf{WO}(\mathbb{R})$ from the existence of a Mazurkiewicz set, as it was the case for Hamel bases. The first to show this was Miller [47, Theorem 5]. As in the case of Hamel bases, one cannot even recover countable choice, since the Cohen-Halpern-Lévy model H contains a Mazurkiewicz set [8, Corollary 0.3]. The strategy for proving that H has a Hamel basis [9] and a partition of unit circles (Theorem 7.9) is proving that the object satisfies *Strong Amalgamation* (Definition 7.10). In the case of Mazurkiewicz sets, this strategy does not seem to work. Nevertheless, Beriashvili and Schindler gave a criteria for a model to have a Mazurkiewicz set by exploiting a geometrical construction of Chad, Knight and Suabedissen [14, Lemma 4.1], which was also used in the construction of the model by Miller. Furthermore, there is a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$ with a Mazurkiewicz set [7]. We can recover this result by using the methods in Chapter 4, and that is the main goal of Section 6.2.

6.2 Forcing a Mazurkiewicz set

Unlike the example of Hamel bases, partial two-point sets may not be extendable to a complete two-point set. Instead, any linearly independent set is extendable to a Hamel basis (under the Axiom of Choice). This makes that the conditions of *Extendability* and *Amalgamation* that Theorem 4.11 requires are harder to get.

In this section, we will show that there is a model of $\mathsf{ZF} + \mathsf{DC}$ with no well order of the reals in which there exists a Mazurkiewicz set. This was mentioned in [12] and the details are written in some unpublished notes by

Beriashvili and Schindler [7]. We recover this result in Theorem 6.6 using Theorem 4.11 for $\mathbf{P} = \mathbf{P_M}$ as in Definition 6.3. We need to prove that the hypotheses of Theorem 4.11 are fulfilled in this case. This will be taken care of by Lemmas 6.4 and 6.5.

Definition 6.3. Let V be a model of ZFC. Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Let us define $\mathbf{P_M}$ as the forcing poset in $V[g]$ given by

$$p \in \mathbf{P_M} \iff \exists x \in \mathbb{R}\ V[x] \models p \text{ is a Mazurkiewicz set,}$$

ordered by reverse inclusion.

Observe that $\mathbf{P_M}$ adds a real partition. For this, notice that

$$p \text{ is a Mazurkiewicz set} \iff p \subseteq \mathbb{R}^2 \wedge \Big(\forall s \in [p]^3 \psi_1(s)\Big) \wedge \Big(\forall r \in \mathbb{R}^3 \exists s \in [p]^2 \psi_2(r,s)\Big),$$

where $\psi_1(s)$ iff "the elements of s are not collinear" and $\psi_2(r,s)$ iff "the elements of s belong to the line given by r".

Clearly, ψ_1 and ψ_2 are Δ_0. Also, for any pair (x,p) as before, since p is a Mazurkiewicz set in $V[x]$ and x is a real, there is $s \in [p]^2$ such that s is contained in the line $l_x = \{(x,y) \mid y \in \mathbb{R}\}$. Then x can be computed from s, by taking the first coordinate of any of its elements.

Notice as well that the partial conditions relative to $\mathbf{P_M}$ are the subsets of $\mathbb{R}^2 \cap V[x]$ (for $x \in \mathbb{R}^{V[g]}$) such that no three points are collinear.

Remark.
To show that $\mathbf{P_M}$ adds a real partition we implicitly assumed that we have fixed a representation of the lines in $\mathbb{R}^2$ by points in $\mathbb{R}^3$. For example, let

$$S = \{(a,b,c) \in \mathbb{R}^3 \mid c = 1 \vee (c = 0 \wedge a = 1)\}.$$

Then for any $(a,b,c) \in S$ we can define a line l in $\mathbb{R}^2$ by

$$l = \{(x,y) \in \mathbb{R}^2 \mid ax + b = cy\}.$$

Conversely, for any line l there is a *unique* set of parameters $(a,b,c) \in S$ that determine l in this way. Formally, we define $\psi_2(r,s)$ so that also holds true in any case that r does not belong to the image of such representation.

NOTATION: For any line l, we will confuse it (the geometrical object) with its representation as an element of the set S described above. We will consider the **parameters** or **coordinates** of l as the set $\operatorname{coor}(l) = \{a, b, c\}$, where a, b, c are such that $(a, b, c) \in S$ and $l = \{(x, y) \in \mathbb{R}^2 \mid ax+b = cy\}$. Moreover, for any model M we will write "$l \in M$" as a short form of "$\operatorname{coor}(l) \in M$".

Similarly, if $r \in \mathbb{R}^n$ with $r = (r_0, \ldots, r_{n-1})$, we write $\operatorname{coor}(r)$ to denote the set $\{r_0, \ldots, r_{n-1}\}$. Furthermore, if $R \subseteq \mathbb{R}^n$ or R is a set of lines, we denote the set $\bigcup\{\operatorname{coor}(r) \mid r \in R\}$ by $\operatorname{coor}(R)$.

Lemma 6.4. *Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Then $\mathbf{P_M}$ in $V[g]$ satisfies Extendability.*

Proof: Looking at Definition 4.9, we need to prove that for any partial condition p in $V[x]$ ($x \in \mathbb{R}^{V[g]}$), there is a condition $\bar{p}$ and a real $\bar{x}$ such that $\bar{p} \supseteq p$, and $x \in V[\bar{x}]$. Recall that $\leq_{\mathbf{P_M}} = \supseteq \restriction \mathbf{P_M}$. So, if p is a condition, then $\bar{p} \leq_{\mathbf{P_M}} p$.

Fix $x \in \mathbb{R}^V[g]$, and let p be a partial condition in $V[x]$. Let $\gamma < \omega_1$ be such that $x \in V[g \restriction \gamma]$, let y be $\bigcup(g \restriction \{\gamma\})$, and define $\bar{x}$ as $x \oplus y$. We will use a variation of the proof of Theorem 6.2 so that we construct a Mazurkiewicz set $\bar{p}$ inside $V[\bar{x}]$ extending p.

Work inside $V[\bar{x}]$. By absoluteness, p is a partial condition in $V[\bar{x}]$, i.e. no three points in p are collinear. Notice that $\langle p\rangle \subseteq \{l \text{ line } \mid l \in V[x]\}$.

Let $\{l_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of all the lines excepting the ones in $\langle p\rangle$. We will recursively define $p_\alpha \subseteq \mathbb{R}^2$ for $\alpha < \mathfrak{c}$. For $\alpha = 0$, take $p_0 = p$. Now suppose p_β is defined for all $\beta < \alpha$. If α is a successor ordinal, namely $\alpha = \beta + 1$, we will take $r \subseteq l_\beta$ such that $|(p_\beta \cup r) \cap l_\beta| = 2$, and such that for each element of r, there is a coordinate of it that is not in the real algebraic closure of F_β, where

$$F_\beta = \text{the minimal field containing } (\mathbb{R} \cap V[x]) \cup \operatorname{coor}(p_\beta) \cup \operatorname{coor}(l_\beta)$$

This implies that no element of r is in $l_\beta \cup \langle p_\beta\rangle$ since any intersection point $l \cap l_\beta$ with $l \in \langle p_\beta\rangle$ would have both coordinates in $\overline{F_\beta}$. Define $p_\alpha = p_\beta \cup r$. If α is a limit ordinal, take $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

We have to check that the construction is possible, namely that such r exists. First, we will show that $|p_\beta \cap l_\beta| \leq 2$ for all $\beta < \mathfrak{c}$. The argument is the same as in the proof of Theorem 6.2. Suppose β is the first ordinal such that $|p_\beta \cap l_\beta| \geq 3$. Let x, y, z be three points in $p_\beta \cap l_\beta$, named alphabetically by the order of being added to the construction. If $x \in p$ we say x was added in the step 0. Since p is a partial condition, $z \notin p$ and z should have been added at some step $\delta + 1$ which is of course different from 0, which means

$z \in l_\delta$. By construction, $z \notin \cup\langle p_\delta\rangle$. This is a contradiction since $l(x,y) \in \langle p_\delta\rangle$ as Figure 6.1 (replacing β_2 for δ) shows. Thus, $|p_\beta \cap l_\beta| \le 2$ for all $\beta < \mathfrak{c}$.

The rest of the proof consists of showing that $\mathbb{R}\backslash\overline{F_\beta}$ has at least two points so that we can choose r. Notice that $p_\beta = p\dot{\cup}\tilde{p}_\beta$, where $|\tilde{p}_\beta| \le |\beta| < \mathfrak{c}$. Since $p \subseteq V[x]$, we can write

$$F_\beta = \text{the minimal field containing } (\mathbb{R} \cap V[x]) \cup \mathrm{coor}(\tilde{p}_\beta) \cup \mathrm{coor}(\{l_\delta\}_{\delta \le \beta}).$$

Thus, $F_\beta = \mathbb{R}^{V[x]}(S)$ (Definition 2.33), where S is a set of cardinality strictly less than $\mathfrak{c}$. Applying Lemma 3.1 and recalling that y was a Cohen real over $V[x]$, we know that the transcendence degree of $\mathbb{R} = \mathbb{R}^{V[\bar{x}]}$ over $\mathbb{R} \cap V[x]$ is $\mathfrak{c}$. Therefore $\mathbb{R}\backslash\overline{F_\beta}$ is actually of cardinality $\mathfrak{c}$, and there are enough possibilities to choose r from.

Take $\bar{p} = \bigcup_{\alpha<\mathfrak{c}} p_\alpha$. Then $\bar{p}$ is a Mazurkiewicz set in $V[\bar{x}]$ and it contains p. □

In the proofs of Theorem 6.2 and Theorem 6.4 we requested that the elements of r are not in $\cup\langle p_\beta\rangle$. In the first case, we argued that $l_\beta\backslash\langle p_\beta\rangle \neq \emptyset$ by cardinality. This does not work in the second proof, since $p_\beta \supseteq p$ and p can be of cardinality $\mathfrak{c}$. This happens, for example, in the case that p is a condition in $\mathbf{P_M}$.

Lemma 6.5. *Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be $\mathbf{Q}$-generic over V. Then $\mathbf{P_M}$ satisfies Amalgamation in $V[g]$.*

Proof: We need to prove that for densely many $p \in \mathbf{P_M}$, for any g_1, g_2 mutually $\mathbf{Q}$-generic over $V[p]$ and for all $p \in \mathbf{P_M} \cap V[p, g_1]$, $p_2 \in \mathbf{P_M} \cap V[p, g_2]$ extending p, p_1 and p_2 are compatible.

First, notice that $D = \{p \in \mathbf{P_M} \mid \exists\alpha < \omega_1\, V[g \restriction \alpha] \models p \text{ is a Mazurkiewicz set}\}$ is dense. For any condition $p \in \mathbf{P_M}$, there is a real x such that $p \in V[x]$. By Lemma 2.12, there is $\gamma < \omega_1$ such that $x \in V[g \restriction \gamma]$. Take $\bar{x} = \oplus_{\beta\le\gamma} \cup g \restriction \{\beta\}$, and repeat the proof of Extendability for this $\bar{x}$. Then there is $\bar{p}$ Mazurkiewicz set in $V[\bar{x}] = V[g \restriction \alpha]$ where $\alpha = \gamma + 1$. So $\bar{p} \le_{\mathbf{P_M}} p$ and $\bar{p} \in D$.

Now, fix $p \in D$ and let x be such that $V[x] \models p$ is a Mazurkiewicz set. Let g_1, g_2 be mutually $\mathbf{Q}$-generic filters over $V[p] = V[x]$, and fix $p_1 \in \mathbf{P_M} \cap V[x, g_1]$ and $p_2 \in \mathbf{P_M} \cap V[x, g_2]$ such that $p \subseteq p_1$, $p \subseteq p_2$. Let $y \in \mathbb{R} \cap V[x, g_1]$ and $z \in \mathbb{R} \cap V[x, g_2]$ be such that

$$V[x,y] \models p_1 \text{ is a Mazurkiewicz set, and}$$
$$V[x,z] \models p_2 \text{ is a Mazurkiewicz set.}$$

By Theorem 2.16, we can choose y and z such that they are Cohen generic over $V[x]$. Since g_1 and g_2 are mutually $\mathbf{Q}$-generic, y and z are mutually Cohen generic over $V[x]$.

We will show that $p_1 \cup p_2$ is a partial condition in $V[x, y, z]$. This is enough since, by Lemma 6.4, we can find a condition $\bar{p}$ that extends $p_1 \cup p_2$, and therefore witnesses the compatibility between p_1 and p_2.

Work in $V[x, y, z]$. Suppose $p_1 \cup p_2$ contains three different points on a line l. Since p_1 and p_2 are partial conditions, each of these sets does not contain three collinear points. Without loss of generality, we can assume $|l \cap p_2| = 2$ and $|l \cap p_1| \geq 1$. Notice that $|l \cap p_2| = 2$ implies that $l \in V[x, z]$. Since $p_1 \in V[x, y]$, we know that $|l \cap V[x, y]| \geq 1$. We divide in two cases, depending on whether $|l \cap V[x, y]| \geq 2$ or $|l \cap V[x, y]| = 1$.

Case 1. If $|l \cap V[x, y]| \geq 2$, then $l \in V[x, y]$, therefore $l \in V[x, y] \cap V[x, z] = V[x]$. Since p is a Mazurkiewicz set in $V[x]$, $|p \cap l| = 2$. Since $p_2 \supseteq p$ is a Mazurkiewicz set in $V[x, y]$, $p_2 \cap l = p \cap l$ and similarly for p_1. Therefore $(p_1 \cup p_2) \cap l = p \cap l$, contradicting the choice of l.

Case 2. If $|l \cap V[x, y]| = 1$, let r be the only element in $l \cap V[x, y]$. Let $s_1, s_2 \in l \cap p_2$. Then,

$$V[x, y, z] \models r \text{ is the only element of } V[x, y] \cap l(s_1, s_2).$$

Recall that y is generic over $V[x, z]$. There is a condition $t \in y \subseteq \mathbf{C}$ such that

$$t \Vdash^{\mathbf{C}}_{V[x,z]} \dot{r} \text{ is the only element of } V[x, \dot{g}] \cap l(\check{s}_1, \check{s}_2). \tag{6.1}$$

Split y in two mutually generic Cohen reals y_1, y_2 according to t as in Definition 2.21. From Equation 6.1, we get that

$$\begin{aligned} V[x, z, y_1] &\models r_1 \text{ is the only element of } V[x, y_1] \cap l(s_1, s_2), \text{ and} \\ V[x, z, y_2] &\models r_2 \text{ is the only element of } V[x, y_2] \cap l(s_1, s_2); \end{aligned}$$

where $r_1 = \dot{r}_{y_1}$ and $r_2 = \dot{r}_{y_2}$.

Since $V[x, y_1], V[x, y_2] \subseteq V[x, y]$ and $r, r_1, r_2 \in l$, we obtain that $r = r_1 = r_2$. Then, $r \in V[x, y_1] \cap V[x, y_2]$, so $r \in V[x]$. Thus, $(p_1 \cup p_2) \cap l = p_2 \cap l$, contradicting the choice of l.

Finally, there is no such l, and therefore $p_1 \cup p_2$ is a partial condition. By Extendability (Lemma 6.4), there is $\bar{p} \in \mathbf{P_M}$ such that $\bar{p} \supseteq p_1 \cup p_2$. Since the order in $\mathbf{P_M}$ is the reverse inclusion, we get $\bar{p} \leq_{\mathbf{P_M}} p_1$ and $\bar{p} \leq_{\mathbf{P_M}} p_2$. Thus, p_1 and p_2 are compatible.

$\square$

Now we are ready to prove the main result of this chapter. The proof consists in putting all the results of this chapter together and applying Theorem 4.11.

Theorem 6.6 (Corollary of Theorem 4.11)
Let $\mathbf{Q}$ *be the finite support product of* ω_1*-many copies of Cohen forcing, let* g *be a* $\mathbf{Q}$*-generic filter over* V*. Let* $\mathbf{P}$ *be the forcing poset in* $V[g]$ *given by*

$$p \in \mathbf{P} \iff \exists x \in \mathbb{R}\; V[x] \models p \text{ is a Mazurkiewicz set},$$

ordered by reverse inclusion. Let h *be a* $\mathbf{P}$*-generic filter over* $V[g]$*, and let* $\mathcal{P} = \cup h$*. Then*

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \mathcal{P} \textit{ is a Mazurkiewicz set.}$$

Proof: We will use Theorem 4.11. Notice that $\mathbf{P} = \mathbf{P_M}$, and we have shown that this forcing adds a real partition. Since the order in $\mathbf{P}$ is the reverse inclusion, we know that $\mathbf{P}$ is σ-closed applying Lemma 4.12. Finally, $\mathbf{P}$ satisfies Extendability by Lemma 6.4, and Amalgamation by Lemma 6.5. We can then apply Theorem 4.11, and obtain the desired conclusion. □

Chapter 7

Partitions of $\mathbb{R}^3$ in unit circles

In this chapter we will consider another example of a paradoxical set. This time, a more recent and therefore less studied object: a partition of $\mathbb{R}^3$ in unit circles. In Section 7.1, we will give an overview of similar objects constructed with and without choice. In Sections 7.3 and 7.4 we will show models with this paradoxical set but with no well order of the reals. The first model will satisfy DC, and in the second model, AC_ω does not hold.

Definition 7.1. Let $\mathcal{C}$ denote the family of circles[1] of radius one in $\mathbb{R}^3$. We say that $\mathcal{P} \subseteq \mathcal{C}$ is a **partition of unit circles (PUC)** if $\mathcal{P}$ consists of disjoint circles that cover $\mathbb{R}^3$, namely, for all $C_1, C_2 \in \mathcal{P}$ we have that $C_1 \cap C_2 = \emptyset$, and $\cup\mathcal{P} = \mathbb{R}^3$.

Conway and Croft [17, Appendix] mentioned for the first time that this object exists using the Axiom of Choice. Actually the result they showed is more general and the existence of a partition of $\mathbb{R}^3$ is only a comment at the end of the appendix of the paper. Here we include the proof only for our case, which we took from Jonsson [32, Lemma 1.7].

Theorem 7.2 (ZFC)
There is a partition of $\mathbb{R}^3$ in unit circles.

Proof: Let $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of the points in $\mathbb{R}^3$. We will recursively define p_α for $\alpha < \mathfrak{c}$.

For $\alpha = 0$, set $p_0 = \emptyset$. Suppose that p_β is defined for all $\beta < \alpha$. If α is a successor ordinal of the form $\beta + 1$ and $x_\beta \in \cup p_\beta$, take $p_{\beta+1} = p_\beta$. If

[1] In this text, a circle is always of dimension 1, not to confuse with a *disk*.

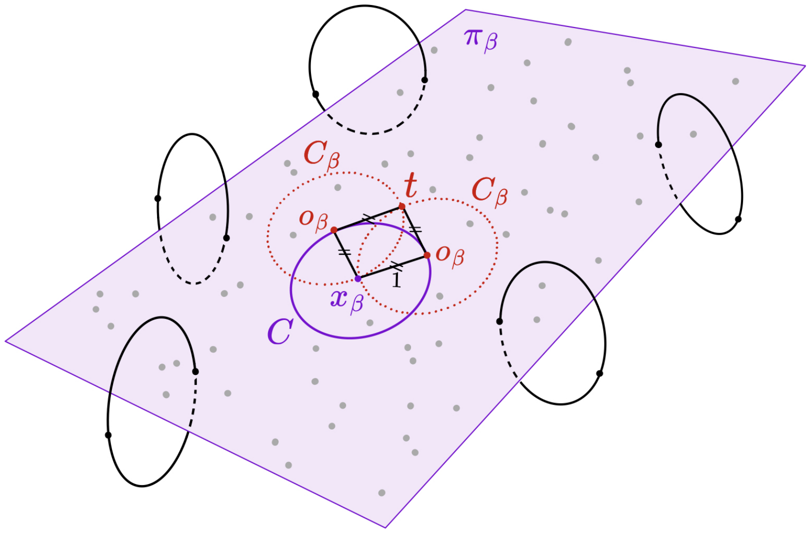

Figure 7.1: For each point t that we want to avoid, there are two options for o_β that we have to discard.

$x_\beta \notin \cup p_\beta$, we will choose a unit circle C_β such that $x_\beta \in C_\beta$ and $C_\beta \cap C = \emptyset$ for all $C \in p_\beta$. Supposing we can choose such a circle C_β, we define $p_{\beta+1} = p_\beta \cup \{C_\beta\}$. Finally, if α is a limit ordinal, define $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

As always, we need to check that the construction is legit, namely, we can choose such a circle C_β. Since we need C_β to have radius 1, we only need to choose a center o_β of the circle and a vector n_β normal to the plane in which C_β will be contained. If o_β and n_β are fixed, they determine exactly one unit circle.

First, choose n_β such that the plane π_β determined by n_β and the point x_β does not contain any of the circles $\{C_\delta\}_{\delta<\beta}$ in p_β. This is possible because there are less than $|\beta| < \mathfrak{c}$ such planes (at most one per circle) and $\mathfrak{c}$ possibilities to choose n_β.

Second, notice that $x_\beta \in C_\beta$ implies we need o_β to be at distance 1 from x_β. Since we fixed n_β, the possibilities for o_β are contained in the only unit circle C contained in π_β with center x_β. For each $\delta < \beta$, $C_\delta \cap \pi_\beta$ consists of at most two points, so there are at most $|\beta| < \mathfrak{c}$ points to avoid. For each of these points t, there are at most two options for o_β that we have to discard, because such o_β would give rise to a circle C_β that would contain t as Figure 7.1 shows. But we have $\mathfrak{c}$ choices for o_β so we can choose o_β so that $C_\beta \cap C_\delta = \emptyset$ for all $\delta < \beta$.

Take $\mathcal{P} = \bigcup_{\alpha<\mathfrak{c}} p_\alpha$. For any two circles C_β and C_α in $\mathcal{P}$ added in steps $\beta + 1$ and $\alpha + 1$ of the construction, if $\beta < \alpha$ then C_α was chosen so that

$C_\beta \cap C_\alpha = \emptyset$. Moreover, for any $r \in \mathbb{R}^3$ there is $\alpha < \mathfrak{c}$ such that $r = x_\alpha$. By construction, $r \in \cup p_{\alpha+1} \subseteq \cup \mathcal{P}$. Thus, $\mathcal{P}$ is a partition of $\mathbb{R}^3$ in unit circles. $\square$

7.1 Literature review

In 1964, Conway and Croft analyzed the problem of covering S^n or $\mathbb{R}^n$ with open/closed/half-closed arcs and segments respectively, of the same length [17]. They found many answers to these questions and provide several explicit such partitions (in ZF). However they could not find an explicit solution to the problem of partitioning S^n into closed arcs of the same length. They developed a more general theorem [17, Appendix] that could be applied to all of these problems for dimension $n \geq 3$, but it used the Axiom of Choice. A corollary of this theorem is the existence of a partition of $\mathbb{R}^3$ in unit circles.

There is no trace that somebody looked into that or similar objects (after all, the result only appears in the last sentence of the appendix in the paper of Conway and Croft), until Szulkin in 1983 showed in a surprisingly simple way that is possible to partition $\mathbb{R}^3$ in circles without the Axiom of Choice, but dropping the requirement of the circles to have the same radius [57]. Moreover, Jonsson [32] attributes also to Kharazishvili the result of the existence of a partition of $\mathbb{R}^3$ in unit circles, who seems to have proven it in 1985, but the present author could not recover this reference. Nevertheless, it is easy to check that $\mathbb{R}^2$ cannot be partitioned in circles (not even in Jordan curves). To the best of the author's knowledge, there are no more results regarding unit circles that did not use the Axiom of Choice until very recently: now we know that there is an open set of $\mathbb{R}^3$ for which there is an explicit partition of unit circles [1, Example 3.1].

In 1985, Bankston and Fox tried to expand Theorem 7.2 (but topologically) to bigger dimensions of the euclidean space as well as of the spheres that are used to tile it. By similar reasons to the case $n = 1$, S^n cannot partition $\mathbb{R}^{n+1}$ for any n, not even allowing the tiles to be *topological* copies of S^n [5, Theorem 2.3]. But Bankston and Fox proved (in ZF) that $\mathbb{R}^{n+2}$ (and any bigger dimension) can be partitioned in topological copies of S^n for all $n < \omega$ [4]. Additionally, the proof of Theorem 7.2 generalizes to prove (in ZFC) that $\mathbb{R}^{2n+1}$ can be partitioned in *isomorphic* copies of S^n [5, Theorem 2.5]. As far as we know, the question of whether $\mathbb{R}^m$ can be partitioned in isomorphic copies of S^n is open for $n + 2 \leq m < 2n + 1$ and $n \geq 2$, with or without using the Axiom of Choice, or even allowing different radii. As the simplest example, it is not known whether $\mathbb{R}^4$ can be partitioned in two dimensional spheres [5, Question 3.1.iv].

A natural question arises: Which are the possible pieces in which $\mathbb{R}^3$ can be partitioned? For example, $\mathbb{R}^3$ can be partitioned in: letters T [49], rhombi with edge length 1 [60, Theorem 3] (not known for filled squares), *unlinked* circles of the same radius [32, Theorem 2.3], unlinked circles with each positive real number appearing exactly once as a radius [32, Theorem 2.1], and even any family of cardinality $\mathfrak{c}$ of real analytic curves [32, Theorem 3.1]. However, it is not true that $\mathbb{R}^3$ can be partitioned in isometric copies of any fixed Jordan curve. A nice overview of (subsets of) $\mathbb{R}^n$ that can be partitioned in (subsets of) $\mathbb{R}^m$ is displayed by Jonsson and Wästlund [32, Section 4].

On the side of negative results, Cobb proved in 1995 that $\mathbb{R}^3$ cannot be continuously decomposed into circles [15]. Continuously, in this case, means that any sequence of points converging to some point x induces a convergence of the radii, planes, and centers of the circles associated with them in the partition, and they converge to the the magnitudes of the circle that passes through x.

In yet another direction, Kharazishvili explores the concept of k-homogeneous covering, i.e., each point of the euclidean space considered is covered by exactly k tiles. For $k = 1$, these are just partitions. While $\mathbb{R}^2$ cannot be partitioned by copies of S^1, there is a simple 2-homogeneous covering of $\mathbb{R}^2$ in circles of the same radius [36, Example 2]. We refer the reader to [34, 36] for more examples.

Different from the cases of Hamel bases and Mazurkiewicz sets, we do not much more about PUCs. The contribution of this chapter is to give similar results that were known for Hamel basis and Mazurkiewicz sets but for PUCs. Hamkins asked whether a partition of $\mathbb{R}^3$ in unit circles can be Borel [26]. Similarly to the Mazurkiewicz case, we will show that in case we find a partition in unit circles that is analytic, then it is Borel (Lemma 7.3). Using the strategy of Miller [46] for obtaining coanalytic Hamel bases and Mazurkiewicz sets under the assumption of $V = L$, which was later generalized by Vidnyánszky [58, Theorem 3.4], it can also be shown that if $V = L$, then there is a coanalytic PUC. In terms of our guiding question (Question 2), we will exhibit a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$ with a partition of $\mathbb{R}^3$ in unit circles (Theorem 7.8) by applying the methods of Chapter 4. Furthermore, we will show in Theorem 7.9 that the Cohen-Halpern-Lévy model has a PUC, so we cannot recover countable choice from the existence of this paradoxical set.

7.2 Properties of PUCs

Lemma 7.3. *If there is a partition of unit circles that is analytic, it is actually Borel.*

Proof: Suppose there is a PUC $\mathcal{P}$ which is $\boldsymbol{\Sigma}^1_1$. We show that its complement is also $\boldsymbol{\Sigma}^1_1$ therefore $\mathcal{P}$ is actually Borel. Notice that (unit) circles can be coded by one real.

$$C \notin \mathcal{P} \iff \exists C_1 \exists x_1 \exists C_2 \exists x_2 \text{ such that } C_1 \neq C_2, C_1, C_2 \in \mathcal{P},$$
$$x_1 \in C \cap C_1; x_2 \in C \cap C_2;$$

which is again $\boldsymbol{\Sigma}^1_1$. □

The same result holds for Mazurkiewicz sets, as mentioned in Section 6.1.

Remark.
Notice that the proof of Theorem 7.2 shows that any family of disjoint unit circles of cardinality less than $\mathfrak{c}$ can be extended to a partition of $\mathbb{R}^3$ in unit circles. This is not true for families of disjoint unit circles that have cardinality $\mathfrak{c}$ even if there are still $\mathfrak{c}$ points to be covered. For example, a similar proof of Theorem 7.2 shows that we can partition $\mathbb{R}^3 \setminus l$ in unit circles, where l is any line in $\mathbb{R}^3$. This is a family of disjoint unit circles, they cover exactly $\mathbb{R}^3 \setminus l$ so there are $|l| = \mathfrak{c}$ points not covered. Nevertheless, there is no circle that we can add to this family to cover all $\mathbb{R}^3$.

7.3 Forcing a PUC

We aim to apply Theorem 4.11 to partitions of unit circles. For this goal, we have to face the same challenge that Mazurkiewicz sets presented for the *Extendability* property, since it is not true that any family of disjoint circles (partial conditions) is extendable to a partition inside the same model. Furthermore, we have to work much more to show *Amalgamation*. This property does not hold if we only consider a forcing poset ordered by reverse inclusion, as it was the case for Mazurkiewicz sets and Hamel bases (see discussion after the proof of Lemma 7.7). Nevertheless, we will be able to show that there is a model of $\mathsf{ZF}+\mathsf{DC}$ with no well order of the reals in which there is a partition of $\mathbb{R}^3$ in unit circles in Theorem 7.8. We will start by setting some notation and defining the forcing that we will need to construct such a model.

Notation: As we did in Chapter 6, given $r \in \mathbb{R}^n$ we write $\operatorname{coor}(r)$ for the (unordered) set of coordinates. We will extend this notation for circles and planes.

We will think of a circle C as given by parameters (o, n), where $o \in \mathbb{R}^3$ is its center and $n \in \mathbb{R}^3$ is a normal vector of the unique plane that contains C. If we choose the normal vectors to be inside the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \text{ and } (z > 0 \vee (z = 0 \wedge y > 0))\} \cup \{(0, 0, 1)\},$$

then the assignment of a normal vector to any given plane is *unique*. Therefore any circle C has exactly one representation by parameters $(o, n) \in \mathbb{R}^3 \times S$.

If C is a circle with parameters $(o, n) \in \mathbb{R}^3 \times S$, then we write $\operatorname{coor}(C)$ for $\operatorname{coor}(o) \cup \operatorname{coor}(n)$. Given a model M, we will use "$C \in M$" as shorthand for "$\operatorname{coor}(C) \in M$".

Consider the set

$$\overline{S} = \{(a, b, c, d) \in \mathbb{R}^4 \mid a = 1 \vee (a = 0 \wedge (b = 1 \vee (b = 0 \wedge c = 1)))\}$$

Then every plane π can be represented *uniquely* by parameters $(a, b, c, d) \in \overline{S}$ such that

$$\pi = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz + d = 0\}.$$

In this case, we write $\operatorname{coor}(\pi) = \{a, b, c, d\}$. Similarly, "$\pi \in M$" is shorthand for "$\operatorname{coor}(\pi) \in M$".

If R is a set of circles, planes, or points, we write $\operatorname{coor}(R)$ to denote $\bigcup\{\operatorname{coor}(r) \mid r \in R\}$.

Definition 7.4. Let V be a model of ZFC. Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing. Let g be a $\mathbf{Q}$-generic filter over V. In $V[g]$, we define a partial order $\mathbf{P_C}$ as follows[2]:

- $p \in \mathbf{P_C}$ iff $\exists x \in \mathbb{R}$ such that $V[x] \models p$ is a PUC.
- $p \leq_{\mathbf{P_C}} q$ iff
  - i. $q \supseteq p$,
  - ii. there are reals x and y such that p is a PUC in $V[x]$, q is a PUC in $V[y]$, and $x \in V[y]$.
  - iii. q extends p in an *algebraically independent way*. Namely, in $V[y]$, for all $C \in q \setminus p$ with center o and contained in the plane π, we have that $\pi \notin V[x]$ and $o \notin \overline{\mathbb{R}^{V[x]}(\operatorname{coor}(\pi))}$.

[2]Here $\mathbf{C}$ stands for *circles*, as $\mathbf{P_{PUC}}$ would be too long and $\mathbf{P_P}$ would be too ugly.

Here, $\overline{F}$ denotes the algebraic closure of F relative to $\mathbb{R}$ and $\mathbb{R}^{V[x]}(\mathrm{coor}(\pi))$ is the minimal field containing $\mathbb{R}^{V[x]}$ and $\mathrm{coor}(\pi)$ as in Definition 2.33.

Notice that since "$\pi \in V[x]$" means "$\mathrm{coor}(\pi) \in V[x]$", and $\mathrm{coor}(\pi)$ is a finite set of reals, "$\pi \notin V[x]$" means there is at least one coordinate of π that is not in $V[x]$.

Observe that Condition ii in Definition 7.4 is well defined. Notice that we can *recover* the real x from the condition $p \in \mathbf{P}$, namely, if x and x' are reals such that $V[x] \models$ "p is a PUC" and $V[x'] \models$ "p is a PUC", then we have that $V[x] = V[x']$. This is due to the fact that the set

$$R = \{r \in \mathbb{R} \mid (0, 0, r) \in C \text{ where } C \text{ is a circle given by an element of } p\}$$

is absolute between models that contain p. In each model, p is a PUC and thus covers the respective z-axis. Therefore,

$$\mathbb{R}^{V[x]} = R^{V[x]} = R^{V[x']} = \mathbb{R}^{V[x']}.$$

Since x and x' are reals, we get $V[x] = V[x']$.

Moreover, $\leq_P$ is a partial order. Reflexivity and antisymmetry are clear. For transitivity, suppose you have conditions p, q, r witnessed by the reals x, y, z such that $p \leq_{\mathbf{P_C}} q$ and $q \leq_{\mathbf{P_C}} r$. By definition of $p \leq_{\mathbf{P_C}} q$,

$$V[y] \models \forall C \in q \backslash p \text{ given by } (o, \pi), \pi \notin V[x], \text{ and } o \notin \overline{\mathbb{R}^{V[x]}(\mathrm{coor}(\pi))}.$$

By absoluteness, this also is true in $V[z]$. Notice that $r\backslash p = r\backslash q \cup q\backslash p$. By definition of $q \leq_{\mathbf{P_C}} r$,

$$V[z] \models \forall C \in r \backslash p \text{ given by } (o, \pi), \pi \notin V[y], \text{ and } o \notin \overline{\mathbb{R}^{V[y]}(\mathrm{coor}(\pi))}.$$

Since $x \in V[y]$, we have $\mathbb{R}^{V[x]} \subseteq \mathbb{R}^{V[y]}$, and hence we get that $\pi \notin V[x]$ and $o \notin \overline{\mathbb{R}^{V[x]}(\mathrm{coor}(\pi))}$.

Remark.
If C is a unit circle in $V[x]$, its parameters (o, n) are elements of $\mathbb{R} \cap V[x]$. Let C' be the unit circle in $V[y]$ given by (o, n) where y is such that $\mathbb{R} \cap V[x] \subsetneq \mathbb{R} \cap V[y]$. If we look at C and C' as sets (and not as their definitions) we will get that $C \subsetneq C'$. In other words, the same parameters produce different sets in different models.

We will alternate between considering the parameters of each circle and the circle itself (the geometrical object) whenever needed, hoping that the reader can perceive whenever this distinction is important.

Notice that we can construe $\mathbf{P_C}$ so that $\mathbf{P_C}$ adds a real partition (Definition 4.8). We can see the family $\mathcal{C}$ of circles of radii one in $\mathbb{R}^3$ as the set

$$\mathcal{C} = \{(o, n) \mid o \in \mathbb{R}^3 \text{ and } n \in S\} = \mathbb{R}^3 \times S.$$

Fixing this codification, each condition in $\mathbf{P_C}$ is a subset of $\mathbb{R}^6$.

Notice that

$$p \text{ is a PUC} \iff p \subseteq \mathbb{R}^6, \forall s \in [p]^2 \psi_1(s) \wedge \forall r \in \mathbb{R}^3 \exists s \in [p]^1 \psi_2(r, s),$$

where

$$\psi_1(s) \text{ iff "}s \subseteq \mathcal{C} \text{ and if } s = \{s_0, s_1\}, \text{ then the circles given by } s_0 \text{ and } s_1 \text{ do not intersect"},$$

and

$$\psi_2(r, s) \text{ iff "}s = \{s_0\}, s_0 \in \mathcal{C}, \text{ and } r \text{ is covered by the circle given by } s_0\text{"}.$$

First, the circles (given by) $s_0 = (o_0, n_0)$ and $s_1 = (o_1, n_1)$ intersect if and only if the following holds:

$$\exists x \in \mathbb{R}^3 \, \langle o_0 - x, n_0 \rangle = \langle o_1 - x, n_1 \rangle = 0 \text{ and } d(x, o_0) = d(x, o_1) = 1$$

where $\langle \cdot, \cdot \rangle$ here denotes the inner product and "$-$" is the subtraction of vectors in $\mathbb{R}^3$. This is a $\mathbf{\Sigma^1_1}$ property, with parameters $\operatorname{coor}(s_0) \cup \operatorname{coor}(s_1)$. By Mostowski's Absoluteness (Theorem 2.9), it is absolute between transitive models containing the parameters.

Second, ψ_2 is clearly $\mathbf{\Delta_0}$. Furthermore, if $p_1 \leq_{\mathbf{P_C}} p_2$ is given by $p_1 \supseteq p_2$ and $\phi(p_1, p_2)$ as in Definition 7.4, then the corresponding ϕ is absolute.

Finally, for every pair (x, p) such that $V[x] \models p$ is a PUC, there is a circle $C \in p$ which is the only circle in p intersecting the point $(0, 0, x)$ and we can compute x from (the parameters of) C. So $\mathbf{P_C}$ adds a real partition.

As we did in Chapter 6, we will construct our model using Theorem 4.11 and we will show that $\mathbf{P_C}$ satisfies the hypotheses of that theorem one by one, as shown in Lemmas 7.5, 7.6 and 7.7. Notice that the partial conditions of $\mathbf{P_C}$ (see Definition 4.9) are the subsets of $\mathcal{C} \cap V[x]$ for some $x \in \mathbb{R}^{V[g]}$ which consist of pairwise disjoint circles.

Lemma 7.5. *Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Then $\mathbf{P_C}$ in $V[g]$ satisfies Extendability.*

Proof: Let p be a family of unit circles in $V[x]$ that are pairwise disjoint, where $x \in \mathbb{R}^{V[g]}$. We need to show that we can extend p to $\bar{p} \in V[\bar{x}]$ such that $\bar{p} \supseteq p$ and $x \in V[\bar{x}]$.

Similar to the proof of Lemma 6.4, let $\gamma < \omega_1$ be such that $x \in V[g \upharpoonright \gamma]$ (see Lemma 2.12), let $y = \cup(g \upharpoonright \{\gamma\})$, and let $\bar{x} = x \oplus y$. Then $V[\bar{x}] = V[x, y]$, and y is $\mathbf{C}$-generic over $V[x]$. We will prove that there is a condition $\bar{p} \in V[\bar{x}]$ such that $\bar{p} \supseteq p$, by strengthening the construction of a PUC in ZFC shown in the proof of Theorem 7.2.

Work in $V[\bar{x}]$. Let $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of the points in $\mathbb{R}^3 \setminus \cup p$. Here $\cup p$ is the union of all the circles given by p as computed in $V[\bar{x}]$. We will recursively define p_α for $\alpha < \mathfrak{c}$. For $\alpha = 0$, set $p_0 = p$. Notice that p is still a family of disjoint unit circles in $V[\bar{x}]$ by the absoluteness of ψ_1. Suppose that p_β is defined for all $\beta < \alpha$. If α is a successor ordinal of the form $\beta + 1$ and $x_\beta \in \cup p_\beta$ (namely, x_β is covered by a circle in p_β), take $p_{\beta+1} = p_\beta$. If $x_\beta \notin \cup p_\beta$, we will pick a unit circle C_β such that $x_\beta \in C_\beta$ and $C_\beta \cap C = \emptyset$ for all $C \in p_\beta$. Assuming we can choose such a C_β, we define $p_{\beta+1} = p_\beta \cup \{C_\beta\}$. Finally, if α is a limit ordinal, define $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

We need to check that the construction is possible, namely, that we can choose such a circle C_β. Again, we only need to choose an origin o_β and a normal vector n_β.

Notice that if $C \in p \in V[x]$, then its parameters (o, n) would be in $V[x]$. First, we want to choose n_β different to all the normal vectors of circles in p_β. Since $|\mathbb{R} \setminus \mathbb{R}^{V[x]}| = \mathfrak{c}$, we have in principle continuum many options for n_β that are different from all the normal vectors of circles in p. Since $p_\beta = p \dot{\cup} \tilde{p}_\beta$ and $|\tilde{p}_\beta| \leq |\beta|$, we have to also avoid choosing at most $|\beta|$-many normal vectors (one per circle in $\tilde{p}_\beta$). Because $|\beta| < \mathfrak{c}$, we can choose n_β with the property needed. This means that the plane π_β (determined by n_β and x_β) in which C_β will be contained is different from all the planes containing circles in p_β. Therefore, $|\pi_\beta \cap C| \leq 2$ for every circle $C \in p_\beta$.

Secondly, notice that $x_\beta \in C_\beta$ implies that we need o_β to be at distance 1 from x_β. Since we fixed n_β, the possibilities for o_β are contained in the only unit circle C contained in π_β with center x_β. Let us choose $o_\beta \in \pi_\beta$ such that at least one coordinate of o_β is not in $\overline{F_\beta}$, where

$$F_\beta = \text{ the minimal field containing } (\mathbb{R} \cap V[x]) \cup \operatorname{coor}(\tilde{p}_\beta) \cup \operatorname{coor}(\pi_\beta, x_\beta),$$

and recall that $\operatorname{coor}(\tilde{p}_\beta) = \bigcup_{\delta<\beta} \operatorname{coor}(o_\delta, \pi_\delta)$.

We can choose such an o_β because of the Lemma 3.1, and because we still have one degree of freedom for a point in $\mathbb{R}^3$ after prescribing $o_\beta \in \pi_\beta$ and $d(o_\beta, x_\beta) = 1$. See Figure 7.1. We might not be able to choose all the coordinates of o_β to not be in $\overline{F_\beta}$, but we only need one of them to not be in $\overline{F_\beta}$.

Let C_β be the circle determined by (o_β, n_β). We need to check that it satisfies the requirements that we requested in the recursive definition. Clearly $x_\beta \in C_\beta$, since $o_\beta, x_\beta \in \pi_\beta$, $d(o_\beta, x_\beta) = 1$ and o_β is the center of C_β. Fix $\tilde{C} \in p_\beta$. We want to show $C_\beta \cap \tilde{C} = \emptyset$. Suppose there is some $t \in C_\beta \cap \tilde{C}$. If $\tilde{C} \in p$, its parameters belong to $\mathbb{R} \cap V[x]$. We can calculate o_β from $(o, \pi, x_\beta, \pi_\beta)$ "algebraically": t is one of the (at most two) intersection points of the only unit circle $\tilde{C}$ given by (o, π) and the plane π_β, and o_β is then one of the (at most two) points in π_β such that $d(o_\beta, x_\beta) = d(o_\beta, t) = 1$. See Figure 7.2. Moreover, in such a situation, there are polynomials P_i of degree 4 with coefficients in the minimal field containing $\mathrm{coor}(o, \pi, x_\beta, \pi_\beta)$ such that $P_i(o_\beta^{(i)}) = 0$ for $i = 1, 2, 3$. Here $o_\beta^{(i)}$ denotes the i^{th} coordinate of the point o_β. This implies that all the coordinates of o_β belong to $\overline{F_\beta}$, contradicting the choice of o_β.

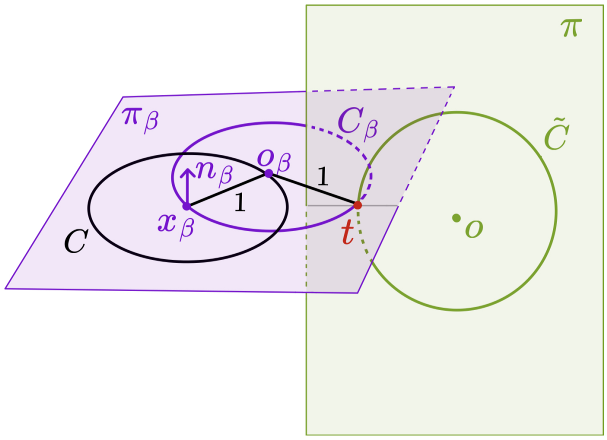

Figure 7.2: t is one of the (at most two) intersection points of the only unit circle $\tilde{C}$ given by (o, π) and the plane π_β, and o_β is then one of the (at most two) points in π_β such that $d(o_\beta, x_\beta) = d(o_\beta, t) = 1$.

The case in which $\tilde{C} \in \tilde{p}_\beta$ is analogous. $\tilde{C}$ must have been added in some step $\delta + 1$. We obtain a contradiction from $\tilde{P}_i(o_\beta^{(i)}) = 0$ for $i = 1, 2, 3$; where $\tilde{P}_i$ is some polynomial that has coefficients in the minimal field containing $\mathrm{coor}(o_\delta, \pi_\delta, p_\beta, \pi_\beta)$.

Take $\bar{p} = \bigcup_{\alpha < \mathfrak{c}} p_\alpha$. For any two circles $C, D \in \bar{p}$, we want to show that $C \cap D = \emptyset$. This is clear for C, D added in step 0, namely, $C, D \in p$. If they were added in different steps, for example, D strictly after C, there is $\alpha < \omega_1$ such that $D = C_\alpha$ and $C \in p_\alpha$. By construction, $C_\alpha \cap C = \emptyset$. Moreover, for any $r \in \mathbb{R}^3$ either $r \in \cup p$ or there is $\alpha < \mathfrak{c}$ such that $r = x_\alpha$. In the first case, $r \in \cup\bar{p}$ since $p = p_0 \subseteq \bar{p}$. In the second case, $r \in \cup p_{\alpha+1} \subseteq \cup\bar{p}$ by construction. Thus, $\bar{p}$ is a partition of $\mathbb{R}^3$ in unit circles that extends p.

Additionally, if p is a condition such that $V[x] \models p$ is a PUC, then by construction we have $\bar{p} \leq_{\mathbf{P_C}} p$. $\square$

As we discussed at the beginning of this chapter, it is not true that every partial PUC in a model M of ZFC can be extended to a (complete) PUC inside the model M. But Lemma 7.5 says that we can always do it when we make *more space* for it, namely, add more reals to the model.

Remark.
We will use and abuse the notation $p_0 \leq_{\mathbf{P}} p_1$ even when p_0 and p_1 are partial conditions. If the models where we are considering p_0 and p_1 are fixed, for example $p_0 \in V[y_0]$ and $p_1 \in V[y_1]$, we can reuse Definition 7.4. One may not be able to recover the real x from a partial condition, so $\leq_{\mathbf{P_M}}$ is not a relation between partial conditions. We could define it as a relation between pairs (y, p) where $V[y] \models$ "p is a family of disjoint unit circles". In this case, it will not be a partial order because antisymmetry fails, but the relation is transitive by the same argument that shows $\leq_{\mathbf{P_C}}$ as a relation on $\mathbf{P_C}$ is transitive.

Using this notation, the proof of Lemma 7.5 gives us that for any partial condition p in a model $V[x]$, there is a condition $\bar{p} \in \mathbf{P}$ and $\bar{x}$ such that $x \in V[\bar{x}]$ and $\bar{p} \leq_{\mathbf{P_C}} p$. We will need this for the proof of Lemma 7.6.

Lemma 7.6. *Let $\mathbf{Q}$=$\mathbf{C}(\omega_1)$. Let g be a $\mathbf{Q}$-generic filter over a model V of* ZFC*. Then $\mathbf{P_C}$ is σ-closed in $V[g]$.*

Proof: Work in $V[g]$. Let $\{p_n\}_{n<\omega}$ be a sequence of decreasing conditions. Let $\{x_n\}_{n<\omega}$ be a sequence of reals such that $V[x_n] \models$ "p_n is a PUC" for all $n < \omega$. We can do this since $V[g] \models$ AC. Take $x = \bigoplus_{n<\omega} x_n$ and $p = \bigcup_{n<\omega} p_n$. By the proof of Lemma 4.12, $V[x] \models p$ is a family of disjoint unit circles, so p is a partial condition.

Fix $n < \omega$. Then $p \leq_{\mathbf{P}} p_n$: for all $C \in p \backslash p_n$, there is $m > n$ such that $C \in p_m \backslash p_n$. Since $p_m \leq_{\mathbf{P}} p_n$, $\pi \notin V[x_n]$, and $o \notin \overline{\mathbb{R}^{V[x_n]}(\text{coor}(\pi))}$ in $V[x_m]$. This also holds in $V[x]$ by absoluteness.

Using Lemma 7.5, we find $\bar{x} \in \mathbb{R}$ and $\bar{p} \in \mathbf{P}$ such that $V[\bar{x}] \models \bar{p}$ is a PUC. By the Remark above we know that $\bar{p} \leq_{\mathbf{P_C}} p$. Since $p \leq_{\mathbf{P}} p_n$ for all $n < \omega$, by transitivity we get that $\bar{p} \leq_{\mathbf{P_C}} p_n$ for all $n < \omega$. $\square$

Lemma 7.6 implies that $\mathbf{P}$ does not add reals. It is very important to have this property for our purposes because, intuitively, we could have been trying to add a PUC h by partial versions of it, while at the end adding new reals which would not have been considered in the partial approximations. Therefore the forcing would not ensure that h covers all the points in $\mathbb{R}^3$ in the extension.

Finally, we are ready to prove the last lemma of this section.

Lemma 7.7. *Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Then $\mathbf{P_C}$ satisfies Amalgamation in $V[g]$.*

Proof: We need to prove that for densely many $p \in \mathbf{P_C}$, for any g_1, g_2 mutually $\mathbf{Q}$-generic over $V[p]$, and for all $p \in \mathbf{P_C} \cap V[p, g_1]$, $p_2 \in \mathbf{P_C} \cap V[p, g_2]$ such that $p_1, p_2 \leq_{\mathbf{P_C}} p$, we get that p_1 and p_2 are compatible.

Analogously to the Mazurkiewicz case (see beginning of the proof of Lemma 6.5), we can assume that there are $x, y, z \in \mathbb{R}^{V[g]}$ such that

$$V[x] \models p \text{ is a PUC,} \tag{7.1}$$
$$V[x, y] \models p_1 \text{ is a PUC, and} \tag{7.2}$$
$$V[x, z] \models p_2 \text{ is a PUC;} \tag{7.3}$$

and y and z are mutually generic Cohen reals over $V[x]$.

Work in $V[x, y, z]$. It is clear that $p_1 \cup p_2$ is a family of unit circles. We only need to prove that they are also disjoint to assert that $p_1 \cup p_2$ is a partial condition. We already showed that if two circles are disjoint in $V[x]$ then they are disjoint in $V[x, y, z]$. So let $C_1 \in p_1 \backslash p$, $C_2 \in p_2 \backslash p$, and suppose $C_1 \cap C_2 \neq \emptyset$. Let π_i, o_i be the plane and the origin, respectively, of C_i, for $i = 1, 2$. Notice that $\mathrm{coor}(\pi_1, o_1) \subseteq V[x, y]$ and $\mathrm{coor}(\pi_2, o_2) \subseteq V[x, z]$ We have two cases, given by the cardinality of $C_1 \cap C_2$. We aim to reach a contradiction.

Case 1. Assume $C_1 \cap C_2 = \{t, u\}$, with $t \neq u$. Observe that $\pi_1 \neq \pi_2$. Otherwise,

$$\mathrm{coor}(\pi_1) = \mathrm{coor}(\pi_2) \in V[x, y] \cap V[x, z] = V[x],$$

which contradicts the requirement $\pi_1 \notin V[x]$, given by $p_1 \leq_{\mathbf{P_C}} p$.

Since $\pi_1 \neq \pi_2$, we obtain that $C_1 \cap \pi_2 = C_2 \cap \pi_1 = \{t, u\}$. See Figure 7.3. Now, we can calculate o_2 algebraically using π_1, o_1, π_2: we get t and u from computing $C_1 \cap \pi_2$, and C_1 is given by π_1 and o_1. Now, o_2 is one of the two points in π_2 such that $d(o_2, t) = d(o_2, u) = 1$.

Moreover, in such a situation, there are polynomials P_i of degree 2 with coefficients in the minimal field containing $\mathrm{coor}(o_1, \pi_1, \pi_2)$ such that $P_i(o_2^{(i)}) = 0$,

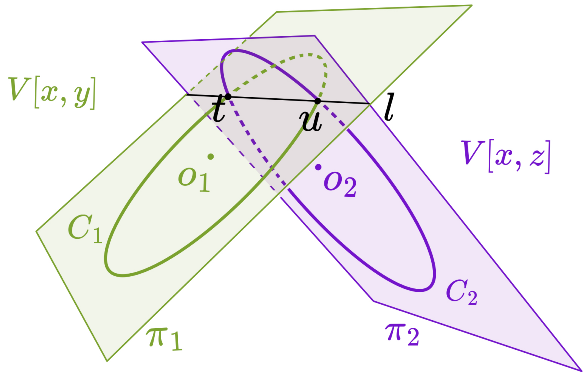

Figure 7.3: Two circles from different models intersecting in two points t and u.

for $i = 1, 2, 3$. Here $o_2^{(i)}$ denotes the i^{th} coordinate of the point o_2. Remember that $p_2 \leq_{\mathbf{P_C}} p$ so $o_2 \notin \overline{\mathbb{R}^{V[x]}(\mathrm{coor}(\pi_2))}$, namely, there is a coordinate of o_2 that does not belong to this field. Suppose without loss of generality that it is $o_2^{(1)}$. Take $B \subseteq \{o_2^{(1)}\} \cup \mathrm{coor}(\pi_2)$ maximal such that $B \subseteq \mathbb{R}^{V[x,z]}$ is algebraically independent over $\mathbb{R}^{V[x]}$ and contains $o_2^{(1)}$. Then, by Lemma 3.15, B is also algebraically independent over $\mathbb{R} \cap V[x,y]$. Recall that $\mathrm{coor}(o_1, \pi_1) \subseteq \mathbb{R} \cap V[x,y]$. This leads to a contradiction, since $P_1(o_2^{(1)}) = 0$.

Case 2. $C_1 \cap C_2 = \{t\}$.

Case 2a. Suppose that there is a circle $C \neq C_1$ with parameters in $V[x,y]$ such that $C \cap C_2 = \{u\}$ and $u \neq t$, as Figure 7.4 shows. Let o and π be the origin and plane of C, respectively. Then, similarly to Case 1, we can compute algebraically all the coordinates from o_2 using $\mathrm{coor}(o_1, \pi_1, o, \pi, \pi_2)$. The contradiction is analogous.

Case 2b. Suppose that there is a circle C with parameters in $V[x,y]$ such that $C \cap C_2 = \{t\}$. Then $t \in C \cap C_1$, so $t \in V[x,y]$. We know that t is the only point in $C_2 \cap V[x,y]$. If not, we could easily define a circle in $V[x,y]$ passing through a possible second point u, and this situation was discarded in Case 2a. So we know that

$$V[x,y,z] \models t \text{ is the only element of } V[x,y] \cap C(o_2, \pi_2),$$

where $C(o_2, \pi_2)$ describes the unique unit circle with origin o_2 contained in the plane π_2.

Recall that y is $\mathbf{C}$-generic over $V[x,z]$. There is then a condition $s \in y \subseteq \mathbf{C}$ and $\dot{t}$ a $\mathbf{C}$-name for t in $V[x,z]$ such that

$$s \Vdash^{\mathbf{C}}_{V[x,z]} \dot{t} \text{ is the only element of } V[x,\dot{g}] \cap C(\check{o}_2, \check{\pi}_2), \tag{7.4}$$

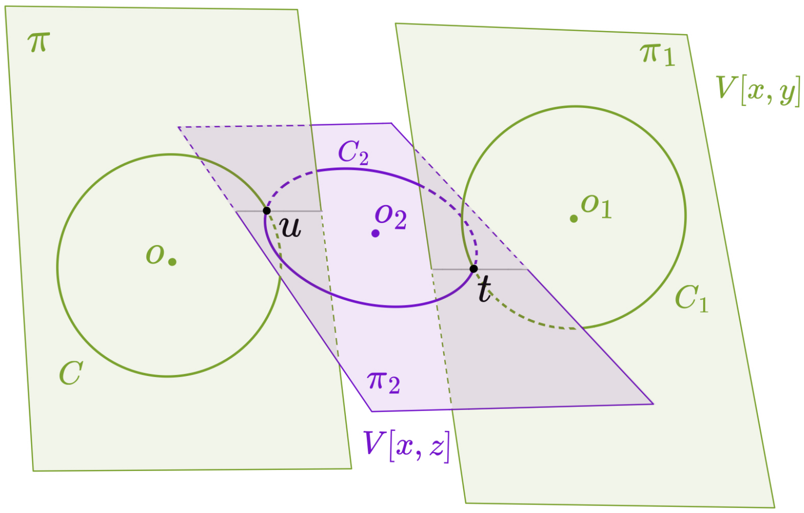

Figure 7.4: There are circles C and C_1 with parameters in $V[x, y]$ such that $C \cap C_2 = \{u\}$, $C_1 \cap C_2 = \{t\}$, and $u \neq t$.

where $\dot{g}$ is the usual name for the $\mathbf{C}$-generic real y. Split y in two mutually generic Cohen reals y_1, y_2 according to s as in Definition 2.21. From Equation 7.4, we get that

$$V[x, z, y_1] \models t_1 \text{ is the only element of } V[x, y_1] \cap C(o_2, \pi_2), \text{ and}$$
$$V[x, z, y_2] \models t_2 \text{ is the only element of } V[x, y_2] \cap C(o_2, \pi_2),$$

where $t_1 = \dot{t}_{y_1}$ and $t_2 = \dot{t}_{y_2}$.

Since $V[x, y_1], V[x, y_2] \subseteq V[x, y]$ and $t, t_1, t_2 \in C_2$, we obtain that $t = t_1 = t_2$. Then, $t \in V[x, y_1] \cap V[x, y_2]$, so $t \in V[x]$. Since $V[x] \models p$ is a PUC, t was covered by some circle C in p. Hence, $C \cap C_1 \neq \emptyset$, which contradicts $p_1 \leq_{\mathbf{P_C}} p$.

Case 2c. C_1 is the only circle (with parameters) in $V[x, y]$ such that $C_1 \cap C_2 \neq \emptyset$.

Similarly to Case 2b, we have that there is an $s \in y$ such that

$$s \Vdash^{\mathbf{C}}_{V[x,z]} \tau \text{ is the only circle from } V[x, \dot{g}] \text{ that intersects } C(\check{o}_2, \check{\pi}_2). \tag{7.5}$$

Split y again in two mutually generic Cohen reals y_1, y_2 containing s. From Equation 7.5, we get that

$$V[x, z, y_1] \models D_1 \text{ is the only circle from } V[x, y_1] \text{ that intersects } C(o_2, \pi_2), \text{ and}$$
$$V[x, z, y_2] \models D_2 \text{ is the only circle from } V[x, y_2] \text{ that intersects } C(o_2, \pi_2),$$

where $D_1 = \tau_{y_1}$ and $D_2 = \tau_{y_2}$.

Since $V[x, y_1], V[x, y_2] \subseteq V[x, y]$, and C_1, D_1, D_2 define circles that intersect C_2, we obtain that $C_1 = D_1 = D_2$. Then, $C_1 \in V[x, y_1] \cap V[x, y_2]$, so $C_1 \in V[x]$, i.e., $\mathrm{coor}(C_1) \in V[x]$. Since $C_1 \in p_1 \backslash p$, by definition of $p_1 \leq_{\mathbf{P_C}} p$ we get that $\pi \notin V[x]$ This is a contradiction.

Taking all the cases into account, we obtain that $p_1 \cup p_2$ is a family of disjoint unit circles, and therefore it is a partial condition with respect to $\mathbf{P_C}$. Moreover, considering $V[x, y, z]$ as the model containing $p_1 \cup p_2$ we claim that $p_1 \cup p_2 \leq_{\mathbf{P_C}} p_1, p_2$. If $C \in (p_1 \cup p_2)\backslash p_2$, namely, $C \in p_1 \backslash p$, we know that $o \notin \overline{\mathbb{R}^{V[x]}(\mathrm{coor}(\pi))}$. Take $B \subseteq \mathrm{coor}(\pi) \cup \mathrm{coor}(o)$ a maximal algebraically independent set over $\mathbb{R}^{V[x]}$ containing the coordinate of o that is not in $\overline{\mathbb{R}^{V[x]}(\mathrm{coor}(\pi))}$. Then B is also algebraically independent over $\mathbb{R}^{V[x,y]}$ by Lemma 3.15, and hence $o \notin \overline{\mathbb{R}^{V[x,y]}(\mathrm{coor}(\pi))}$.

Finally, by Lemma 7.5, we can obtain a condition $\bar{p} \in V[\bar{x}]$ such that $\bar{p} \leq_P p_1 \cup p_2$, and such that $V[x, y, z] \subseteq V[\bar{x}]$. By transitivity, $\bar{p} \leq_{\mathbf{P_C}} p_1, p_2$ as we wanted. □

It would be tempting to try to consider $\mathbf{P_C}$ with the order given just by reverse inclusion. However, using this forcing, Amalgamation does not not work. Consider $x, y, z \in \mathbb{R}$ and p, p_1, p_2 as in Equations 7.1–7.3 in the proof of Lemma 7.7. Let π_y and π_z the planes that consist of all the points in $\mathbb{R}^3$ with first, respectively second, coordinate y, respectively z. Assume $|z - y| < 1$, and that p_1 contains a circle C_1 and p_2 contains a circle C_2 described as follows:

C_1 is the only unit circle contained in π_y and origin $(y, y, 0)$, and

C_2 is the only unit circle contained in π_z and origin $(z, z, 0)$.

Figure 7.5 shows that the circles C_1 and C_2 will intersect in the points

$$\left(y, z, \sqrt{1 - (z - y)^2}\right), \left(y, z, -\sqrt{1 - (z - y)^2}\right).$$

This contradicts Amalgamation for $(\mathbf{P_C}, \supseteq)$.

Now we are ready to prove the main theorem of this section.

Theorem 7.8 (Corollary of Theorem 4.11)
Let $\mathbf{Q}$ be the finite support product of ω_1-many copies of Cohen forcing, let g be a $\mathbf{Q}$-generic filter over V. Let $\mathbf{P}$ be the forcing poset in $V[g]$ described in Definition 7.4. Let h be a $\mathbf{P}$-generic filter over $V[g]$, and let $\mathcal{P} = \cup h$. Then

$$L(\mathbb{R}, \mathcal{P})^{V[g,h]} \models \mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \mathcal{P} \textit{ is a partition of } \mathbb{R}^3 \textit{ in unit circles.}$$

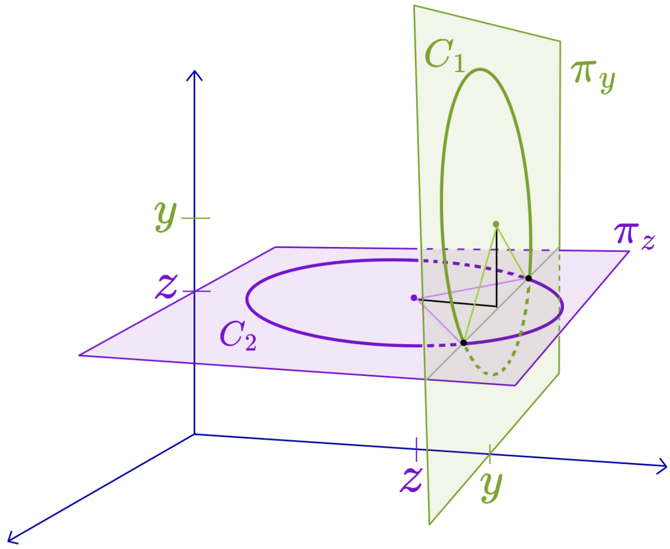

Figure 7.5: C_1 and C_2 intersect in the points $\left(y, z, \pm\sqrt{1-(z-y)^2}\right)$.

Proof: We will apply Theorem 4.11. Notice that $\mathbf{P} = \mathbf{P_C}$ and we have shown that this forcing adds a real partition. It is not a trivial forcing because for any $x \in \mathbb{R}^{V[g]}$, $V[x]$ is a model of AC, and thus has a PUC (see Theorem 7.2). Lemmas 7.5, 7.7 and 7.6 show, respectively, that $\mathbf{P}$ satisfies Extendability, Amalgamation, and is σ-closed. We can then apply Theorem 4.11 and obtain the desired conclusion. □

7.4 A PUC in the Cohen-Halpern-Lévy model

We follow the structure of the proof in [9], which shows that the Cohen-Halpern-Lévy model H there is a Hamel basis of $\mathbb{R}$.

For this, we will need an stronger version of the Lemma 7.7 and therefore a stronger version of the Lemma 3.1 to prove it, which is Theorem 3.14.

Theorem 7.9
Let $\mathbf{C}(\omega)$ denote the finite support of ω-many copies of $\mathbf{C}$, let g be a $\mathbf{C}(\omega)$-generic filter over L and A be the set of Cohen reals added by g. Let H be the Cohen-Halpern-Lévy model as described in Definition 2.27. Then

$$H = \mathsf{HOD}_A^{L[g]} \models \textit{There is a PUC} + \neg\mathsf{AC}_\omega.$$

Proof: Using Theorem 2.28, we deduce that AC_ω does not hold in H. So we only need to prove that there is a partition of unit circles in H.

Work inside H. We will construct a family $\{p_Y\}_{Y\in[A]^{<\omega}}$ so that each p_Y is a partition of unit circles in $L[Y]$, and for each $Y \in [A]^{<\omega}$ such that $Y \subseteq X \in [A]^{<\omega}$, then $p_X \leq p_Y$, where $\leq$ is defined as $\leq_{\mathbf{P_C}}$ in Definition 7.4. We will do so recursively on $n = |Y|$.

For $n = 0$: notice that L has a PUC by Theorem 7.2. Let $p_\emptyset \in L$ be the $<_\emptyset$-least PUC in L. We do so using the global well order $<_\emptyset$ (see Lemma 2.29). Suppose we already defined p_Y for all $Y \subseteq A$ with $|Y| \leq n$.

Let X be a subset of A of size $n+1$. Consider

$$p^* = \bigcup_{Y\subsetneq X} p_Y.$$

Let $Y, Y' \subsetneq X$ and $Y \neq Y'$. We claim that p_Y and $p_{Y'}$ are compatible, namely, its union is a family of disjoint unit circles. If either Y or Y' is a subset of the other, for example, $Y \subseteq Y'$, then by inductive hypothesis $p_{Y'} \leq p_Y$, and hence $p_{Y'} \cup p_Y = p_{Y'}$. If not, then consider $Z = Y \cap Y'$. Then $Z \subsetneq Y, Y'$. Recall that $\mathbf{C} \cong \mathbf{C}^k$ for any $k < \omega$ (Theorem 2.15). By the proof of Lemma 7.7, we get that p_Y and p'_Y are compatible. To check if p^* is a family of disjoint unit circles, we have to take two circles, and check whether they intersect. By the pairwise compatibility of $\{p_Y \mid Y \subsetneq X\}$ and recalling that intersection between two circles is absolute, we get that p^* is a family of disjoint unit circles.

We need to prove that in $L[X]$ there is a partition of unit circles p_X such that $p_X \leq p_Y$ for all $Y \subsetneq X$. In particular, we need $p^* \subseteq p_X$. We will proceed in a way similar to the proof of Lemma 7.5.

Work in $L[X]$. Let $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of the points in $\mathbb{R}^3 \setminus \cup p^*$. Here $\cup p^*$ is the union of all the circles given by p^* as computed in $L[X]$. We will recursively define p_α for $\alpha < \mathfrak{c}$. For $\alpha = 0$, set $p_0 = p^*$. Suppose that p_β is defined for all $\beta < \alpha$. If α is a successor ordinal of the form $\beta + 1$, and $x_\beta \in \cup p_\beta$ (namely, x_β is covered by a circle in p_β), take $p_{\beta+1} = p_\beta$. If $x_\beta \notin \cup p_\beta$, we will pick a unit circle C_β such that $x_\beta \in C_\beta$ and $C_\beta \cap C = \emptyset$ for all $C \in p_\beta$. Assuming we can choose such a C_β, we define $p_{\beta+1} = p_\beta \cup \{C_\beta\}$. Finally, if α is a limit ordinal, define $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$. We need to check that the construction is possible, namely, that we can choose such a circle C_β.

Since C_β will have radius 1, we only need to choose a center o_β of the circle and a vector n_β normal to the plane in which C_β will be contained in. We want to choose n_β to be different from all the normal vectors of circles in p_β. Let

$$\mathbb{R}^* = \bigcup_{Y\subsetneq X} (\mathbb{R} \cap L[Y]).$$

Notice that if $C \in p^*$, then its parameters (o, n) would be in $\mathbb{R}^*$. Since $|\mathbb{R}\backslash\mathbb{R}^*| = \mathfrak{c}$, we have continuum many options for n_β that are different from all the normal vectors of circles in p^*. Since $p_\beta = p^* \dot\cup \tilde{p}_\beta$ and $|\tilde{p}_\beta| \leq |\beta|$, we have to also avoid choosing at most $|\beta|$-many normal vectors (one per circle in $\tilde{p}_\beta$). Since $|\beta| < \mathfrak{c}$, we can choose n_β with the desired property. This implies that the plane π_β (determined by n_β and x_β) in which C_β will be contained is different from all the planes containing circles in p_β. Therefore, $|\pi_\beta \cap C| \leq 2$ for every circle $C \in p_\beta$.

Additionally, it is clear that we have to choose o_β in π_β so that the distance between x_β and o_β is equal to 1. The locus of such a point is then a circle C contained in π_β with center x_β and of radius 1.

Let us choose $o_\beta \in \pi_\beta$ such that $o_\beta \notin \overline{F_\beta}$ (i.e. at least one coordinate is not an element of $\overline{F_\beta}$), where

$$F_\beta = \text{ the minimal field containing } \mathbb{R}^* \cup \operatorname{coor}(\tilde{p}_\beta) \cup \operatorname{coor}(\pi_\beta, x_\beta).$$

Recall that $\operatorname{coor}(\tilde{p}_\beta) = \bigcup_{\delta<\beta} \operatorname{coor}(o_\delta, \pi_\delta)$, and therefore it has cardinality at most $|\beta| < \mathfrak{c}$. Applying Theorem 3.14 to this context, we know that $\mathbb{R} = \mathbb{R}^{L[X]}$ has transcendence degree $\mathfrak{c}$ over the minimal field containing $\mathbb{R}^*$. Also, $\operatorname{coor}(\tilde{p}_\beta) \cup \operatorname{coor}(\pi_\beta, x_\beta)$ has cardinality $|\beta| < \mathfrak{c}$. So we can conclude $|\mathbb{R}\backslash\overline{F_\beta}| = \mathfrak{c}$. Finally, we can choose o_β such that $o_\beta \notin \overline{F_\beta}$ because we still have one degree of freedom after prescribing $o_\beta \in \pi_\beta$ and $d(o_\beta, x_\beta) = 1$.

Let C_β be the circle determined by (o_β, n_β). We have to check that it satisfies the required properties for the recursive construction. Clearly $x_\beta \in C_\beta$. Now fix $\tilde{C} \in p_\beta$. We want to show $C_\beta \cap \tilde{C} = \emptyset$. Suppose there is some $t \in C_\beta \cap \tilde{C}$. If $\tilde{C} \in p^*$, its parameters (o, π) belong to $\mathbb{R}^*$. We can calculate o_β from $(o, \pi, x_\beta, \pi_\beta)$ "algebraically": $\tilde{C}$ can be computed from (o, π), t can be computed from $(\tilde{C}, \pi_\beta)$, and o_β can be computed from (t, x_β, π_β). See Figure 7.2. This means all the coordinates of o_β belong to $\overline{F_\beta}$, contradicting the choice of o_β.

The case in which $\tilde{C} \in \tilde{p}_\beta$ is analogous. $\tilde{C}$ must have been added in some step $\delta + 1 < \beta$. We can then calculate o_β from $(o_\delta, \pi_\delta, x_\beta, \pi_\beta)$ "algebraically" in the same fashion, from which we get the same contradiction.

Take $\tilde{p} = \bigcup_{\alpha<\mathfrak{c}} p_\alpha$. For any two circles $C, D \in \tilde{p}$, we want to show that $C \cap D = \emptyset$. This is clear for C, D added in step 0, namely, $C, D \in p^*$. If they were added in different steps, for example, D strictly after C, there is $\alpha < \omega_1$ such that $D = C_\alpha$ and $C \in p_\alpha$. By construction, $C_\alpha \cap C = \emptyset$, so we can conclude $\tilde{p}$ is a family of disjoint unit circles. Moreover, for any $r \in \mathbb{R}^3$, either $r \in \cup p^*$ or there is $\alpha < \mathfrak{c}$ such that $r = x_\alpha$. In the first case, $r \in \cup\tilde{p}$, since $p^* = p_0 \subseteq \tilde{p}$. In the second case, $r \in \cup p_{\alpha+1} \subseteq \cup\tilde{p}$ by construction. Thus, $\tilde{p}$ is a partition of $\mathbb{R}^3$ in unit circles that extends p^*.

Moreover, $\tilde{p} \leq p_Y$ for every $Y \supsetneq X$: Clearly, $\tilde{p} \supseteq p_Y$. Also, $Y \in L[X]$. Fix $C \in \tilde{p} \backslash p_Y$. By construction, $o \notin \overline{\mathbb{R}^*(\text{coor}(\pi))}$ and $\pi \notin \mathbb{R}^*$. Since $\mathbb{R}^* \supseteq \mathbb{R}^{L[Y]}$, then $o \notin \overline{\mathbb{R}^{L[Y]}(\text{coor}(\pi))}$ and $\pi \notin \mathbb{R}^{L[y]}$. Therefore $\tilde{p} \leq p_Y$ for every $Y \supsetneq X$ as we wanted.

We have just proved that in $L[X]$ there is a partition of unit circles that extends p^* and that is below p_Y (according to $\leq_{\mathbf{P_C}}$) for each $Y \subsetneq X$. In H, let p_X be the $<_X$-least such partition (see 2.29). Finally, define $p = \bigcup_{Y \in [A]^{<\omega}} p_Y$. We claim p is a partition of unit circles (in H). Clearly it is a family of unit circles. If $C_0 \in p_X$ and $C_1 \in p_Y$, then $C_0, C_1 \in p_{X \cup Y}$ which is a partition of unit circles in $L[X \cup Y]$; therefore, C_0 and C_1 are disjoint. Let $r \in \mathbb{R}^3$. Then by Theorem 2.28 there is some $Y \in [A]^{<\omega}$ such that $r \in \mathbb{R}^3 \cap L[Y]$. Therefore r is covered by some circle $C \in p_Y$. □

We will capture the main obstacle in the proof of Theorem 7.9 by the following definition.

Definition 7.10. Let g be a $\mathbf{C}(\omega)$-generic filter over L, and let A be the set of reals added by g. Assume $\mathbf{P} \in L[g]$ is a forcing that adds a real partition as in Definition 4.8.

Let X be a finite subset of A. We say that $\{p_Y\}_{Y \subsetneq X} \subseteq \mathbf{P}$ is a **compatible family of conditions** iff for every $Z \subseteq Y \subsetneq X$ we have that

$$L[Y] \models \psi(p_Y) \text{ and } p_Y \leq_P p_Z.$$

We say that $\mathbf{P}$ satisfies **Strong Amalgamation** if for all $X \in [A]^{<\omega}$ and for all family of compatible conditions $\{p_Y\}_{Y \subsetneq X}$ in $\mathbf{P}$, we have that

$$L[X] \models \bigcup_{Y \subsetneq X} p_Y \text{ is a partial condition (as in Def. 4.9)}$$

and moreover, there is $p_X \in L[X]$ such that

$$L[X] \models \psi(p_X) \text{ and } p_X \leq_{\mathbf{P}} p_Y \text{ for every } Y \subseteq X.$$

Remark.

The proof of Theorem 7.9 actually showed that $\mathbf{P_C}$ satisfies Strong Amalgamation in $L[g]$, where g is $\mathbf{C}(\omega)$-generic over L. The proof of the existence of a Hamel basis in H [9, Theorem 2.1] essentially shows that the partial order $\mathbf{P_H}$ defined in Theorem 5.2 satisfies Strong Amalgamation in $L[g]$ as well. This is a strategy that has been proven to work to find some paradoxical sets in H. However, it is hard to see whether the forcing $\mathbf{P_M}$ for

Mazurkiewicz sets satisfies Strong Amalgamation. It has been shown that the Cohen model H contains a Mazurkiewicz set [8, Corollary 0.3], but using other construction.

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