Advanced Set Theory

Winter Semester 21/22
Universität Münster

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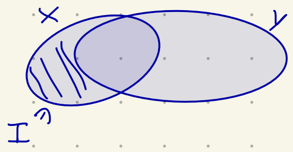
First topic: $\mathbb{P}_{max}$

Definition: An ideal I on κ:
- $I \subseteq \mathcal{P}(\kappa)$ s.t. $I \neq \emptyset$.
- $X, Y \in I \Rightarrow X \cup Y \in I$
- $X \in I$, $Y \subseteq X \Rightarrow Y \in I$.

Notation: I^+ = the positive sets $= \mathcal{P}(\kappa)/I$.

- This gives rise to a forcing:

$\mathbb{P}_I = I^+$ with $X \leq Y$ iff $X \setminus Y \in I$ (i.e. $X \subseteq Y$ mod I)



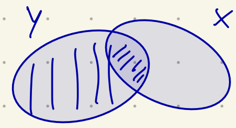
Lemma 1
Let g be $\mathbb{P}_I$-generic over V. Then g is a V-ultrafilter on κ.

Proof

Claim 1: $X \in \mathcal{P}(\kappa) \cap V$. Then $X \in g$ or $\kappa \setminus X \in g$.

Proof Claim 1:
Fix X. $D := \{Z \in I^+ : Z \subseteq X \text{ or } Z \subseteq \kappa \setminus X\}$.

- D is dense: Let $Y \in I^+$. Look at $Y \cap X$, $Y \setminus X$.
If $Y \cap X \in I$, $Y \setminus X \in I$, then $Y = (Y \cap X) \cup (Y \setminus X) \in Y$ ↯.
So at least one of $Y \cap X$, $Y \setminus X$ in in I^+.

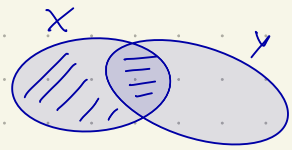


└ Then if $Z \in D \cap g$, $Z \subseteq X$ or $Z \subseteq \kappa \setminus X$. So $X \in g$ or $\kappa \setminus X \in g$.

Claim 2: $X \in g$, $Y \supseteq X$, $Y \subseteq \kappa$, then $Y \in g$. ✓

Claim 3: $X, Y \in g \Rightarrow X \cap Y \in g$.

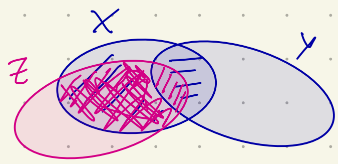
Proof Claim 3:
Fix X, Y. $D := \{Z \in I^+ : Z \subseteq X \cap Y \text{ or } Z \subseteq X \setminus Y\}$



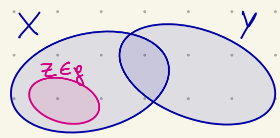
- D is dense below X: Let $Z \leq X$. Look at
$Z \cap (X \cap Y)$, $Z \cap (X \setminus Y)$.

$$\underset{I^+}{\overset{\cap}{Z}} = \underbrace{Z \cap (X \cap Y)} \cup \underbrace{Z \cap (X \setminus Y)} \cup \underbrace{Z \setminus X}_{\in I}$$

one of them has to be in I^+



- Let $Z \in D \cap g$. If $Z \subseteq X \setminus Y$, then $\emptyset = \underset{g}{\underset{\cap}{Z}} \cap \underset{g}{\underset{\cap}{Y}}$ but
$\emptyset \in I$, so $\emptyset \in I^+$.
└



Definition: I an ideal of κ. Say that I is $<\lambda$-complete iff for all $\{X_\xi : \xi < \theta\} \subseteq I$, $\theta < \lambda$, then $\bigcup_{\xi<\theta} X_\xi \in I$.

Every ideal is trivially $<\omega$-complete.

Lemma 2

Suppose I on κ is $<\kappa$-complete. Let g be $\mathbb{P}_I$-generic over V. Then g is V-$<\kappa$-complete, i.e., if $\{X_\xi : \xi < \theta\} \subseteq g$, $\theta < \kappa$, then $\bigcap_{\xi<\theta} X_\xi \in g$.

Proof

Fix $\{X_\xi : \xi<\theta\} \subseteq g$. Let $D = \{ Z \in I^+ : Z \subseteq \bigcap_{\xi<\theta} X_\xi$ or

$$(\exists \xi < \theta \; Z \subseteq X_\xi \setminus \bigcup_{\xi' \neq \xi} X_{\xi'}) \text{ or } Z \subseteq \kappa \setminus \bigcup_{\xi<\theta} X_\xi \}$$

D is dense and this gives Lemma 2.

Definition: I ideal on κ. I is normal iff for all $X \in I^+$ and for all regressive functions $f: X \to I$ (i.e., $f(\xi) < \xi$ for $\xi \in X$), then there is some $\xi < \kappa$ s.t. $f^{-1}{}''\{\xi\} \in I^+$.

Lemma 3

I an ideal on κ. TFAE:

a) I is normal

b) I is closed under diagonal unions, i.e., if $\{X_\xi : \xi < \kappa\} \subseteq I$, then $\nabla_{\xi<\kappa} X_\xi = \{\xi < \kappa : \xi \in \bigcup_{\xi'<\xi} X_{\xi'}\} \in I$.

Proof

($\Rightarrow$) Let $\{X_\xi : \xi < \kappa\} \subseteq I$, but $Y := \nabla_{\xi<\kappa} X_\xi \in I^+$. Let $f: Y \to \kappa$ s.t. $f(\xi) < \xi$ and $\xi \in X_{f(\xi)}$ for $\xi \in Y$.

By normality, we have ξ s.t. $f^{-1}{}''\{\xi\} \in I^+$. But $f^{-1}{}''\{\xi\} \subseteq X_{f(\xi)} \in I$ $\lightning$.

($\Leftarrow$) Fix $X \in I^+$, $f: X \to \kappa$ regressive. Write $X_\xi = f^{-1}{}''\{\xi\}$. If $X_\xi \in I$ for all $\xi < \kappa$, then $\nabla_{\xi<\kappa} X_\xi \in I$.

Pick $\xi \in X \setminus \nabla_{\xi'<\kappa} X_{\xi'}$. $\xi \notin X_{\xi'}$ for all $\xi' < \xi$. So $f(\xi) \geq \xi$. $\lightning$

Lemma 4

I an ideal on κ, g $\mathbb{P}_I$-generic over V. If I is normal, then g is V-normal (i.e., if $(X_\xi : \xi < \kappa) \in V$, $\{X_\xi : \xi < \kappa\} \subseteq g$, then $\Delta_{\xi<\kappa} X_\xi \in g$).

$$\Delta_{\xi<\kappa} X_\xi = \{\xi < \kappa : \xi \in \bigcap_{\eta<\xi} X_\eta\}$$

Let I be an ideal on κ, let g be $\mathbb{P}_I$-generic over V. We may define (in $V[g]$) the generic ultrapower:

$$j = j_g : V \to \underbrace{\mathrm{ult}(V, g)}$$

the elements are $[f] = \{h \in {}^{\kappa}V \cap V : \{\xi : h(\xi) = f(\xi)\} \in g\}$, $f \in {}^{\kappa}V \cap V$; and $[f] \tilde{\in} [h]$ iff $\{\xi : f(\xi) \in h(\xi)\} \in g$.

$j(x) = [c_x]$, c_x being the constant function with value x.

Łoś Theorem

$\mathrm{ult}(V, g) \models \varphi([f_1], \dots, [f_k]) \iff \{\xi < \kappa : V \models \varphi(f_1(\xi), \dots, f_k(\xi))\} \in g$.

Corollary

j is an elementary embedding.

Lemma 5

Let I be an ideal on κ, let g be $\mathbb{P}_I$-generic over V. If I is $<\kappa$-complete, then κ is the critical point, i.e., the least ordinal moved (i.e., κ is the least ξ s.t. $(\xi, \in \restriction \xi) \not\cong (j(\xi), \tilde{\in} \restriction j(\xi))$).

Lemma 6

If I is normal, then $([id], \tilde{\in} \restriction [id]) \cong (\kappa, \in \restriction \kappa)$

Examples of ideals which we will consider

- $NS_\kappa = \{X \subseteq \kappa : X \text{ is nonstationary}\}$, κ a regular uncountable cardinal.

  As we shall do $\mathbb{P}_{max}$, we will mostly consider ideals on ω_1.

Definition: Let θ be sufficiently big. Let I be an ideal on ω_1. A countable $X \prec H_\theta$ is called self-generic (w.r.t. I) iff $I \in X$ and:

Let $g = \{Y \in \mathcal{P}(\omega_1) \cap X : X \cap \omega_1 \in Y\}$. For all dense $D \subseteq \mathbb{P}_I$, $D \in X$, $D \cap g \neq \emptyset$.

Exercise:
For a countable $X \prec H_\theta$, $I \in X$, to be self-generic w.r.t. I is equivalent to:

Let $\pi : N \cong X \prec H_\theta$, N transitive, let $\alpha = X \cap \omega_1 = \pi^{-1}(\omega_1)$, let $\bar{g} = \{Y \in \mathcal{P}(\alpha) \cap N : \alpha \in \pi(Y)\}$. Then $\bar{g}$ is $\pi^{-1}(\mathbb{P}_I)$-generic over N. (Take $g = \pi''\bar{g}$)

21.10.21

Definition: • I an ideal on κ. Say that I is precipitous iff for all g $\mathbb{P}_I$-generic over V, $\mathrm{ult}(V,g)$ is well-founded.

• I an ideal on κ. Say that I is saturated iff $\mathbb{P}_I$ has the κ^+-chain condition. (i.e., there is no collection $(S_i : i<\kappa^+)$, $S_i \in I^+$ for all i, but $S_i \cap S_j \in I$ for all $i \neq j$.

Lemma 1

The following are equivalent:

(1) NS_{ω_1} is precipitous

(2) $\{X \prec H_\theta : X$ is countable and self-generic w.r.t. $NS_{\omega_1}\}$ is <u>projective stationary</u> for all suff. large θ.

$\mathcal{S} \subseteq [H_\theta]^\omega$ is projective stationary iff for all stationary $S \subseteq \omega_1$, $\{X \in \mathcal{S} : X \cap \omega_1 \in S\}$ is stationary.

(Trivially, every projective stationary is stationary).

Lemma 2

TFAE

(1) NS_{ω_1} is saturated

(2) $\{X \prec H_\theta : X$ is countable + self-generic$\}$ contains a club for all suff. large θ.

Corollary

NS_{ω_1} saturated $\Rightarrow$ NS_{ω_1} precipitous

Proof (of Lemma 1)

(2) $\Rightarrow$ (1) Assume (2), and assume (1) is false. Then there is some stationary $S \subseteq \omega_1$ and some θ s.t.

$S \Vdash$ "$\mathrm{ult}(H_\theta^{\check{}}, \dot{g})$ is ill-founded".

Pick $X \prec H_{(2^\theta)^+}$ countable, self-generic, $H_\theta, S \in X$ and such that $\alpha = X \cap \omega_1 \in S$. Let $\pi : M \cong X \prec H_{(2^\theta)^+}$, M transitive. $\alpha = \omega_1^M$.

By self-genericity: Let $g = \{X \in \mathcal{P}(\alpha) \cap M : \alpha \in \pi(X)\}$, g is $\pi^{-1}(\mathbb{P}_{NS_{\omega_1}})$-generic over M.

By $\alpha \in S \in X$, $S \cap \alpha = \pi^{-1}(S) \in g$.

By elementarity, $S \cap \alpha \Vdash_M$ "$\mathrm{ult}(\pi^{-1}(H_\theta); \dot{g})$ is ill-founded". So $\mathrm{ult}(\pi^{-1}(H_\theta); g)$ is ill-founded. Will have:

$$\begin{tikzcd}
M \arrow[r] & H_{(2^\theta)^+} \\
\pi^{-1}(H_\theta) \arrow[r, mapsto] \arrow[dr] & H_\theta \\
 & \mathrm{ult}(\pi^{-1}(H_\theta); g) \arrow[u, "k"']
\end{tikzcd}$$

We may define $k : \text{ult}(\pi^{-1}(H_\theta); g) \longrightarrow H_\theta$ by $[f] \longmapsto \pi(f)(\alpha)$. This is well-defined and elementary, as

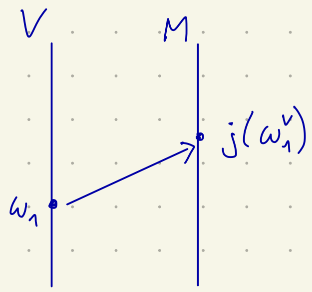
$$\text{ult}(\pi^{-1}(H_\theta), g) \models \varphi([f_1], \dots, [f_\ell]) \overset{\text{Łos}}{\iff}$$
$$\{\xi < \alpha : \pi^{-1}(H_\theta) \models \varphi(f_1(\xi), \dots, f_\ell(\xi))\} \in g$$
$$\iff H_\theta \models \varphi(\pi(f_1)(\alpha), \dots, \pi(f_\ell)(\alpha))$$

Hence $\text{ult}(\pi^{-1}(H_\theta); g)$ is well-founded. Contradiction!

(1) $\Rightarrow$ (2): We assume NS_{ω_1} is precipitous. Let $S \subseteq \omega_1$ be stationary. Let g be $\mathbb{P}_{NS_{\omega_1}}$-generic, $S \in g$.

Here, $\underset{\substack{\uparrow \\ \text{associated} \\ \text{embedding}}}{j} : V \longrightarrow \underset{\substack{\shortparallel \\ \text{transitive collapse} \\ \text{of ult}(V, g)}}{M} \subseteq V[g]$

Let θ be big enough, and consider $(H_\theta; \underset{\substack{\uparrow \\ \text{countably many} \\ \text{relations}}}{\vec{R}})$.



$$\underbrace{\text{Hull}^{j((H_\theta, \vec{R}))}(\omega_1^V)}_{\substack{\text{is countable} \\ \text{in } V[g]}} \prec j'' H_\theta \prec j((H_\theta; \vec{R}))$$

As $\text{Hull}^{j((H_\theta, \vec{R}))}(\omega_1)^V \in M$, we have in M some countable $X \prec j((H_\theta; \vec{R}))$ s.t. $\underset{\substack{\cap \\ j(S)}}{\omega_1^V} = X \cap \underset{\substack{\shortparallel \\ \omega_1^{j(H_\theta)}}}{\omega_1^M}$ and X is self-generic.

(namely, $X = \text{Hull}^{j((H_\theta, \vec{R}))}(\omega_1^V)$).

By elementarity, then $V \models \exists \alpha \in S \; \exists X \prec (H_\theta, \vec{R})$ countable + self-gen. + $\alpha = X \cap \omega_1$".

Proof of Lemma 2

(1) $\Rightarrow$ (2):

<u>Fact</u>: TFAE

(a) NS_{ω_1} is saturated

(b) If $\mathcal{A} \subseteq \mathbb{P}_{NS_{\omega_1}}$ is a maximal antichain, then we may write $\mathcal{A} = \{S_i : i < \omega_1\}$ and there is some club $C \subseteq \omega_1$ s.t. $C \subseteq \underset{i<\omega_1}{\nabla} S_i$ = the diagonal union of the S_i

(b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b): Let $\mathcal{A}$ be maximal antichain. Let $\mathcal{A} = \{S_i : i < \omega_1\}$. Suppose $\nabla_{i<\omega_1} S_i$ does not contain a club. So $\omega_1 \setminus \nabla_{i<\omega_1} S_i$ is stationary. But for each $j < \omega_1$, $(S_j \cap \omega_1 \setminus \nabla_{i<\omega_1} S_i) \subseteq j+1$. So $\mathcal{A}$ was not a maximal antichain. $\lrcorner$

Now, let $X \prec H_\theta$, θ large enough, and X countable. Let $\alpha = X \cap \omega_1$.

Let $g = \{Y \in \mathcal{P}(\omega_1) \cap X : \alpha \in Y\}$. To verify that X is self-generic, it suffices to prove that $g \cap \mathcal{A} \neq \emptyset$ for all $\mathcal{A} \subseteq \mathbb{P}_{NS_{\omega_1}}$ which are maximal antichains, $\mathcal{A} \in X$.

Fix such an $\mathcal{A}$. $X \models$ "We may write $\mathcal{A} = (S_i : i < \omega_1)$ and $\exists$ club $C \subseteq \omega_1$, $C \subseteq \nabla_{i<\omega_1} S_i$" Have $\alpha = \omega_1 \cap X \in C$. Hence have some $i < \alpha$ s.t. $\alpha \in S_i$. So $S_i \in X$ and $\alpha \in S_i$, so $S_i \in g \cap \mathcal{A}$.

We're left with having to prove (2) $\Rightarrow$ (1) of Lemma 2. In order to do that:

Definition: Let M be a transitive model of ZFC, and let $M \models$ "I is an ideal on ω_1". A pair $((M_i, \pi_{ij} : i \leq j \leq \delta), (g_i : i < \delta))$ is called putative generic iteration of M iff:

- $M_0 = M$
- all M_i, $i \leq \delta$, are transitive
- g_i is $\mathbb{P}^{M_i}_{\pi_{0i}(I)}$-generic over M_i
- $M_{i+1} \cong \mathrm{ult}(M_i; g_i)$ and $\pi_{i\,i+1}$ is the associated ultrapower map.
- the system commutes and direct limits are taken at limit stages.

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Lemma 2
TFAE
(1) NS_{ω_1} is saturated
(2) $\{X \prec H_\theta : X$ is countable + self-generic$\}$ contains a club for all suff. large θ.

Last time: (1) $\Rightarrow$ (2).

Proof of (2) $\Rightarrow$ (1):

Assume (2), and let $\mathcal{A} \subseteq \mathbb{P}_{NS_{\omega_1}}$ be a maximal antichain.
Let $X \prec H_\theta$, θ big enough, $\mathcal{A} \in X$. Suffices to get some $(S_\xi : \xi < \omega_1)$ and some club $C \subseteq \omega$ s.t. $(S_\xi : \xi < \omega_1) \subseteq \mathcal{A}$, also $C \subseteq \nabla_{\xi<\omega_1} S_\xi$. (From this, it follows that $\{S_\xi : \xi < \omega_1\} = \mathcal{A}$:
Let $S \subseteq \omega_1$ be stationary. $S \cap C$ is stationary. $\eta \in S \cap C \mapsto$ the least $\xi < \eta$ s.t. $\eta \in S_\xi$ is a regressive function, so by Fodor there is some $\xi < \omega_1$ and some stationary $\bar{S} \subseteq S \cap C$ s.t. $\bar{S} \subseteq S_\xi$. So S, S_ξ are compatible.

Have:
$$M \overset{\pi}{\cong} \overset{\overset{\mathcal{A}}{\cap}}{X} \prec H_\theta, \quad M \text{ transitive.}$$

Let $g = \{Y \in \mathcal{P}(\alpha) \cap M : \alpha \in \pi(Y)\}$, where $\alpha = X \cap \omega_1 = \omega_1^M$.
By hypothesis, g is $(NS_\alpha)^M$-generic over M.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\sigma"'] & H_\theta \\ & \mathrm{ult}(M,g) \arrow[u, "k"'] \end{tikzcd}$$

Let $\sigma : M \to \mathrm{ult}(M,g)$ be the associated ultrapower map. May define k;
$k : \mathrm{ult}(M,g) \to H_\theta$ like last time:

$$\underset{{}^\alpha M \cap M}{\underset{\cap}{[f]}} \overset{k}{\longmapsto} \pi(f)(\alpha)$$

This is well-defined and elementary:

$$\mathrm{ult}(M,g) \models \varphi([f], \dots) \overset{\text{Łoś}}{\Longleftrightarrow} \{\xi < \alpha : M \models \varphi(f(\xi), \dots)\} \in g$$

$$\Leftrightarrow \alpha \in \pi(\{\xi < \alpha : M \models \varphi(f(\xi), \dots)\}$$
$$= \{\xi < \omega_1 : H_\theta \models \varphi(f(\xi), \dots)\}$$

$$\Leftrightarrow H_\theta \models \varphi(\pi(f)(\alpha), \dots).$$

We may now repeat the process $\omega_1 + 1$ times.

$$
\begin{tikzcd}
 & M_0 = M \arrow[r, "\pi=\pi_0"] \arrow[d, "\sigma_0=\sigma"'] & H_\theta \\
 & \mathrm{ult}(M, g),\ g_0 = g \arrow[ur, "\pi_1 = k"'] \arrow[d, dotted] & \\
 & M_\alpha \arrow[uur, bend right, "\pi_\alpha"'] \arrow[d, "\sigma_{\alpha,\alpha+1}"'] & \\
 & M_{\alpha+1} := \mathrm{ult}(M_\alpha, g_\alpha) \arrow[uuur, bend right, "\pi_{\alpha+1}"'] \arrow[d, dotted] & \\
\text{direct limit of the } M_\alpha,\ \alpha<\lambda. = & M_\lambda \arrow[d, dotted] & \\
 & M_{\omega_1} \arrow[uuuuur, bend right, "\pi_{\omega_1}"'] &
\end{tikzcd}
$$

$$\left(M_\alpha ,\ \sigma_{\alpha\beta} ,\ \alpha \leq \beta \leq \omega_1 \right),\ \left(g_\alpha : \alpha < \omega_1 \right)$$

$$g_\alpha = \{ Y \in \mathcal{P}(\omega_1^{M_\alpha}) \cap M_\alpha : \ \omega_1^{M_\alpha} \in \pi_\alpha(Y) \}$$

$$(k(\sigma(x)) = k([c_x]) = \pi(c_x)(\alpha) = \pi(x))$$

For all $\alpha < \omega_1$, there is some $S \in \pi_\alpha^{-1}(\mathcal{A})$ s.t. $\omega_1^{M_\alpha} \in \sigma_{\alpha\,\alpha+1}(S)$ $\left(\Leftrightarrow S \in g_\alpha \cap \pi_\alpha^{-1}(\mathcal{A}) \right)$

$$\Updownarrow$$

$$\omega_1^{M_\alpha} \in \pi_\alpha(S)$$

$$
\begin{tikzcd}
S \arrow[d, mapsto] \arrow[rrr, bend left, mapsto] & M_\alpha \arrow[r] \arrow[d] & H_\theta & \pi_\alpha(S) \\
\sigma_{\alpha\,\alpha+1}(S) \arrow[urr, bend right] & M_{\alpha+1} \arrow[ur] \arrow[u, phantom, shift left=6, "\cong"] & &
\end{tikzcd}
$$

In other words:

$\exists T$ (namely $\pi_\alpha(S)$) $T \in \mathrm{ran}(\pi_\alpha) \cap \mathcal{A} \ \wedge\ \omega_1^{M_\alpha} \in T$) $(*)$

now pick $(S_\xi : \xi < \omega_1)$ s.t. $\{ S_\xi : \xi < \omega \cdot \alpha \} = \mathrm{ran}(\pi_\alpha) \cap \mathcal{A}$ for all α.

Let $C = \{ \alpha : \omega\cdot\alpha = \alpha \ \wedge\ \omega_1^{M_\alpha} = \alpha \}$. C is a club.

If $\alpha \in C$, then by $(*)$, $\alpha = \omega_1^{M_\alpha} \in S_\xi$ for some $\xi < \omega\cdot\alpha = \alpha$. So $C \subseteq \nabla_{\xi<\omega_1} S_\xi$

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next goal:

Theorem (Woodin)

Suppose NS_{ω_1} is saturated and $\mathcal{P}(\omega_1)^\#$ exists, then $\underset{\sim}{\delta}{}^1_2 = \omega_2$.

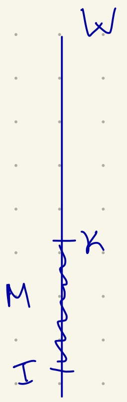
Definition: Let M be a countable transitive model of ZFC, and let $I \in M$ s.t. $M \models$ "I is an ideal on ω_1". We say that $(M; I)$ is iterable iff for all $\alpha < \omega_1$ and for all putative generic iterations $\left((M_\xi, \sigma_{\xi\eta} : \xi \leq \eta \leq \alpha), (g_\xi : \xi < \alpha) \right)$ of M by I and its images, the last model M_α is well-founded.

Comment: This implies that every putative generic iteration $\left((M_\xi, \sigma_{\xi\eta} : \xi \leq \eta \leq \omega_1), (g_\xi : \xi < \omega_1) \right)$ of M by I and its images is s.t. M_{ω_1} is well-founded.

We're interested in the existence of iterable countable structures. For many purposes, the following suffices:

Lemma

Suppose M is a countable transitive model of ZFC, $I \in M$ is s.t. $W \models$ "I is a precipitous ideal", where W is an inner model (class sized) of ZFC with $M = V_\kappa^{OR}$, $\kappa = M \cap OR$, κ inaccessible in W.
Then (M, I) is generically iterable.



Proof:

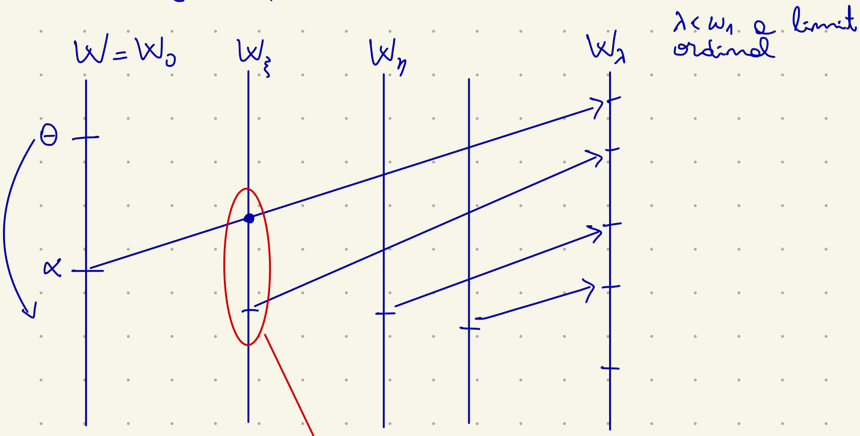
Let's first observe the following:

Let g be the $\mathbb{P}_I$-generic over W. Let $j : W \to \mathrm{ult}(W, g)$ be the associated ultrapower map. Then $j(V_\kappa^W) = \mathrm{ult}(V_\kappa^W, g)$.
$\quad$ '' M

Let I be an ideal on λ. Let $[f] \in j(V_\kappa^W) \in \mathrm{ult}(W, g)$. Then $f : \lambda \to W$, $f \in W$.
$\{\xi < \lambda : f(\xi) \in V_\kappa^W\} \in g$.
Let $\bar{f} : \lambda \to W$, $\bar{f}(\xi) = \begin{cases} f(\xi), & \text{if } f(\xi) \in V_\kappa^W \\ \emptyset, & \text{otherwise.} \end{cases}$,

then $\{\xi : \bar{f}(\xi) = f(\xi)\} \in g$, so $[\bar{f}] = [f]$. Also $\bar{f} \in V_\kappa^W$.

<u>How do you prove the lemma?</u>



Here is the contradiction.

- if something goes wrong it has to be a limit stage.
- Take λ the minimum?
- If such a picture exists, it exists in $W^{Col(\omega, \theta)}$ (by absoluteness)
  bad iteration
- Take the least α s.t. you get this.

Theorem (H. Woodin)

NS_{ω_1} is saturated + $\mathcal{P}(\omega_1)^{\#}$ exists $\Rightarrow$ $\underset{\sim}{\delta}_2^1 = \omega_2$

The existence of $\mathcal{P}(\omega_1)^{\#}$ follows from the existence of a measurable cardinal. So let's simply assume κ is a measurable cardinal.

What is $\underset{\sim}{\delta}_1^2$?

For our purposes, let us define:

Definition: $\underset{\sim}{\delta}_2^1 = \sup\{(\omega_1^V)^{+L[x]} : x \in \mathbb{R}\}$

In the presence of large cardinals (or just if every real has a sharp), $\underset{\sim}{\delta}_2^1 = u_2$, where:

for a real x, let C_x = the class of Silver indiscernibles for $L[x]$, and (a club class)
$u_1 = \omega_1^V = \min \bigcap C_x$
$u_2 = \min(\bigcap C_x \setminus (\omega_1+1))$.

$\underset{\sim}{\delta}_2^1 = \sup\{\alpha : \exists$ pre well-order R on $\mathbb{R}$ s.t. $R \in \underset{\sim}{\Delta}_2^1$ and $\|R\| = \alpha\}$

$= \sup\{\alpha : \exists f : \mathbb{R} \to \alpha$ onto s.t. $\{(x,y) : f(x) \leq f(y)\} \in \underset{\sim}{\Delta}_2^1\}$

Remark: $\underset{\sim}{\delta}_2^1 = \omega_2 \Rightarrow \neg CH$.

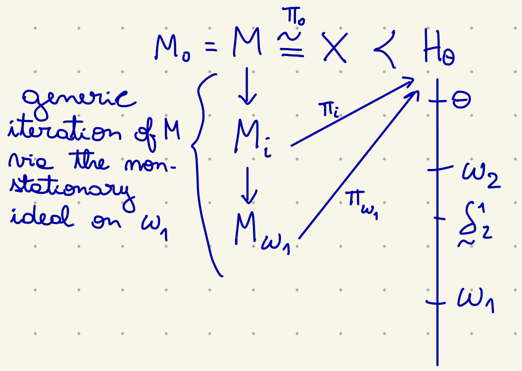
For every real x, $C_x \cap \omega_2$ is club in ω_2. So if CH holds true, $\bigcap_{x \in \mathbb{R}} (C_x \cap \omega_2)$ is club on ω_2. In particular, $u_2 < \omega_2$.

Related questions:

① Can $\sup\{\alpha : \exists f : \mathbb{R} \to \alpha$ onto s.t $\{(x,y) : f(x) \leq f(y)\} \in L(\mathbb{R})\}$ be $> \omega_3$?

② Does NS_{ω_1} saturated $\Rightarrow \neg CH$?

Proof (of Woodin's Theorem)

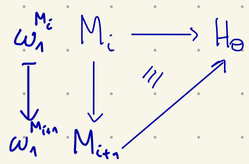


Assume $\underset{\sim}{\delta}_2^1 < \omega_2$.
Let $X \prec H_\Theta$ be countable. X is self-generic, etc.
$\mathrm{ran}(\pi_{\omega_1}) \supseteq X = \mathrm{ran}(\pi_0)$
$\cup$
$(\omega_1+1) \cup \{\underset{\sim}{\delta}_1^2\}$ ($\Rightarrow \underset{\sim}{\delta}_2^1 \subseteq \mathrm{ran}(\pi_{\omega_1})$)

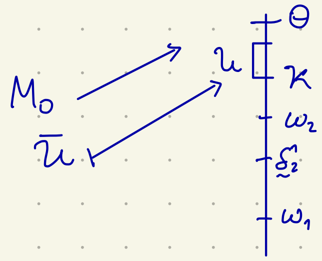
$\Rightarrow$ $\boxed{M_{\omega_1} \cap OR > \underset{\sim}{\delta}_2^1}$

In fact, $\omega_2^{M_{\omega_1}} > \underset{\sim}{\delta}_2^1$.

$\mathrm{crit}(\pi_i) = \omega_1^{M_i}$

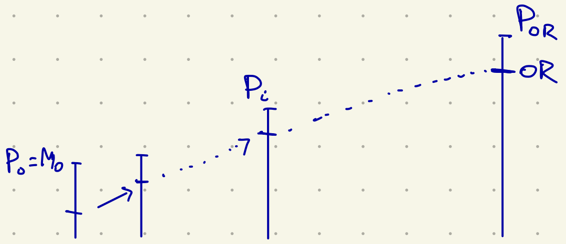


Let's assume $\theta > 2^{\kappa}$ (κ = the measurable cardinal).
w.l.o.g. have a measure $\mathcal{U}$ on κ in ran(π_0).



Let $\bar{\mathcal{U}} = \pi_0^{-1}(\mathcal{U}) \in M_0$, $M_0 \models$ "$\bar{\mathcal{U}}$ is a measure".
Due to the existence of $\pi_0 : M_0 \to H_\theta$, M_0 is iterable w.r.t. $\bar{\mathcal{U}}$ and its images.

Let $(P_i, \sigma_{ij} ; i \leq j \leq OR)$ be the iteration of $M_0 = P_0$ w.r.t. $\bar{\mathcal{U}}$ and its images.
Let $N = (V_{OR})^{P_{OR}}$.

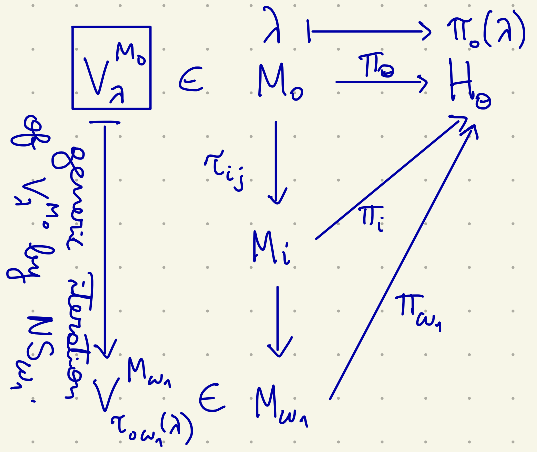


Let $\lambda < \pi_0^{-1}(\kappa)$ be inaccessible in M_0. Then $V_\lambda^{M_0} = V_\lambda^{P_i} = V_\lambda^{N}$ for all i. Because $H_\theta \models$ "NS_{ω_1} is precipitous", $N \models$ "NS_{ω_1} is precipitous".

By the statement from last time,

① $V_\lambda^{M_0}$ is generically iterable w.r.t. $(NS_{\omega_1})^{M_0}$

② $\tau_{0\omega_1}(\lambda) > \underset{\sim}{\delta}^1_2$.
$\|$
(the ω_1^{th} iterate of $V_\lambda^{M_0}$ according to the gen. iteration we define) $\cap$ OR

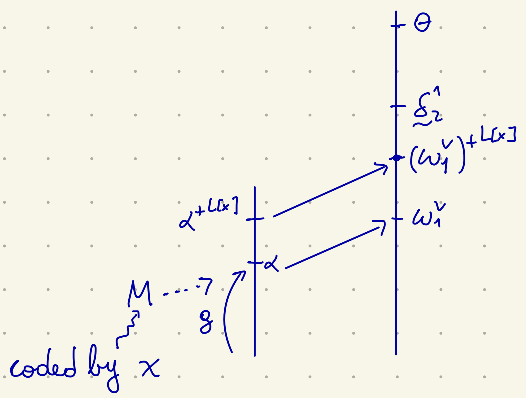


To summarize: We have a (countable) generically iterable model M together with a generic iteration $(M_i, \tau_{ij} : i \leq j \leq \omega_1)$ of $M = M_0$ s.t. $M_{\omega_1} \cap OR > \underset{\sim}{\delta}^1_2$.

Let's derive a contradiction from just that:

Let $x \in \mathbb{R}$ code M. Let $H \overset{\pi}{\cong} Y \prec H_\theta$, (trans. clle.)
$(M_i, \tau_{ij} : i \leq j \leq \omega_1) \in Y$, $x, x^{\#} \in Y$.
$\alpha = \omega_1^H = Y \cap \omega_1$.
$\pi^{-1}((\omega_1^V)^{+L[x]}) = \alpha^{+(L[x])^H} = \alpha^{+L[x]}$
π^{-1} (gen. iteration of length ω_1) $= (M_i, \pi_{ij} : i \leq j \leq \alpha)$

By elementarity, $M_\alpha \cap OR > \alpha^{+L[x]}$.



Let $g \in V$ be $Col(\omega, \alpha)$ - generic over $L[x]$.
$x \oplus g$ is a real.

$\alpha^{+L[x]} = \omega_1^{L[x,g]}$. Let $\beta < \alpha^{+L[x]}$.

By Shoenfield, $L[x,g]$ has a generic iteration $(P_i, \sigma_{ij} : i \leq j \leq \alpha)$ of $P_0 = M$ s.t. $P_\alpha \cap OR > \beta$.

This makes every $\Pi^1_1(x \oplus g)$ set a $\underset{\sim}{\Sigma}^1_1(x \oplus g)$ set inside $L[x,g]$:

Let's work inside $L[x,g]$. Let A be $\Pi^1_1(x \oplus g)$. Have a tree T on $\omega \times \omega$ (rec. on $\omega \times \omega$) s.t. for all reals z,

$z \in A$ iff T_z is well-founded.
iff $\exists f : T_z \longrightarrow \|T_z\| (< \omega_1)$ order preserving.
iff $\exists$ generic iteration $(P_i, \sigma_{ij} : i \leq j \leq \underline{\underline{\alpha}})$ of M,
$\exists f : T_z \longrightarrow P_\alpha \cap OR$ order preserving. $\lightning$

Definition: $\mathbb{P}_{max}$ consists of countable structures p of the form $p = (p, \in, I, a)$ where:
a) $p \models ZFC + MA_{\omega_1}$; $I, a \in p$; $a \leq \omega_1^p$
b) $p \models \omega_1 = \omega_1^{L[a]}$
b') $p \models$ "I is a normal precipitous ideal on ω_1"
c) p is generically iterable.

We'll define the order on $\mathbb{P}_{max}$ some other time

Today: How do you construct $\mathbb{P}_{max}$ conditions from large cardinals?

Theorem 1
If $\varkappa$ is a measurable cardinal and if g is $Col(\omega, <\varkappa)$ - generic over V, then $V[g] \models$ "there is a normal precipitous ideal on ω_1"

Theorem 2
If δ is a Woodin cardinal then there is some $\mathbb{P}$ s.t. if g is $\mathbb{P}$-generic over V then $\omega_1^{V[g]} = \omega_1^V$, in fact $NS_{\omega_1}^{V[g]} \cap V = NS_{\omega_1}^V$, and $V[g] \models$ "NS_{ω_1} is saturated".

Proof of Theorem 1
In V, let $\mathcal{U}$ be a normal $<\varkappa$-complete nontrivial ultrafilter on $\varkappa$.
In $V[g]$, we define an ideal on $\omega_1^{V[g]} = \varkappa$ as follows:

- For $X \subseteq \omega_1^{V[g]}$, let $X \in I \Leftrightarrow X \cap Y = \emptyset$ for some $Y \in \mathcal{U}$,

i.e. $X \in I^+$ iff $\forall Y \in \mathcal{U}\ X \cap Y \neq \emptyset$. /

I can be checked to be a uniform normal ideal on $\varkappa$ inside $V[g]$.
We claim that $V[g] \models$ "I is precipitous".

1st step: Let $j: V \to ult(V, \mathcal{U}) =: M$ be the ultrapower embedding, where M is transitive.
Let h be $Col(\omega, [\varkappa, j(\varkappa)))$-generic over $V[g] \Rightarrow h$ is also $Col(\omega, [\varkappa, j(\varkappa)))$-generic over $M[g]$.
$Col(\omega, X)$ = finite support product of $Col(\omega, \alpha)$, $\alpha \in X$.

- Inside $V[\underbrace{g, h}_{\text{may be construed as a } V\text{-generic filter for } Col(\omega, <j(\varkappa))}]$:

$\hat{j}: V[g] \to M[g, h]$
$\tau^g \mapsto j(\tau)^{g*h}$ where $\tau \in V^{Col(\omega, <\varkappa)}$.

> $p \in Col(\omega, <\varkappa)$ iff
> p is a function, dom $p = \varkappa$
> $p(\xi) \in Col(\omega, \xi)$ and
> $\{\xi < \varkappa : p(\xi) \neq \emptyset\}$ is finite

This is well-defined and elementary as
$V[g] \models \varphi(\tau^g, \dots) \Leftrightarrow \exists p \in g\ p \Vdash \varphi(\tau, \dots)$
$\Leftrightarrow M \models j(p) \Vdash \varphi(j(\tau), \dots)$ as $p \in g$ and $p \approx j(p)$, $j(p) \in g * h$
$\Rightarrow M[g, h] \models \varphi(j(\tau)^{g*h}, \dots)$

2^{nd} step: Define inside $V[g,h]$: $\mathcal{U}^* = \{X \in \mathcal{P}(\kappa) \cap V[g] : \kappa \in \hat{j}(X)\}$.
We aim to show that $\mathcal{U}^*$ is $\mathbb{P}_I$-generic over $V[g]$.
$\quad$ ↳ forcing consisting of elements of I^+

Claim: $\mathcal{U}^* \subseteq I^+$
Proof: If $X \in I$ then $\exists Y \in \mathcal{U}$ $X \cap Y = \emptyset$. But $\kappa \in j(Y) = \hat{j}(Y)$. As $\hat{j}(X) \cap \hat{j}(Y) = \emptyset$, $\kappa \notin \hat{j}(X)$.

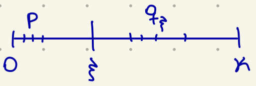
To do this, fix $\mathcal{A} \in V[g]$, a maximal antichain in $\mathbb{P}_I$. If $\mathcal{A} \cap \mathcal{U}^* = \emptyset$, then $\kappa \notin \bigcup \hat{j}\,``\mathcal{A}$.
Let $\dot{\mathcal{A}}$ be a V-name for $\mathcal{A}$, i.e. $\dot{\mathcal{A}}^g = \mathcal{A}$, and:

$$g \ni p \Vdash \text{"}\dot{\mathcal{A}} \text{ is a maximal antichain"}$$
$$g * h \ni (p,q) \Vdash \text{"for all } S \in \dot{\mathcal{A}} \ \check{\kappa} \notin \dot{\hat{j}}(S)\text{"}$$

For all names $\dot{S}$ for elements of $\dot{\mathcal{A}}$: $M \models (p,q) \Vdash^{Col(\omega,<j(\kappa))} \check{\kappa} \notin \dot{\hat{j}}(\dot{S})$,
$q \in Col(\omega, [\kappa, j(\kappa)))$, $q = [\underset{\hat{\kappa}}{\xi} \mapsto q_\xi]_{\mathcal{U}}$, $\xi \mapsto q_\xi$ being a function in V.
$\qquad \dot{\hat{j}}(\dot{S}) = j(\dot{S})^{g*h}$

$\Rightarrow$ for all $\dot{S}$ as above, $\{\xi : (p, q_\xi) \Vdash \check{\xi} \notin \dot{S}\} \in \mathcal{U}$.

Let $T = \{\xi < \kappa : q_\xi \in g\}$
$\qquad q_\xi \in Col(\omega, [\xi,\kappa)) \text{ "}\subseteq\text{" } Col(\omega, <\kappa)$



a) If $\xi \in T \cap Y(S)$, then $(p, q_\xi) \in g$, so $\xi \notin S$ (where $S = \dot{S}^g$).
So $\underbrace{T \cap S}_{\in I} \cap Y(S) = \emptyset$

b) Let $Z \in \mathcal{U}$. We want $\xi \in Z$, $q_\xi \in g$. (This proves $T \in I^+ \Rightarrow \mathcal{A}$ is not a maximal antichain ↯)
Let $D = \{r \in Col(\omega, <\kappa) : r \leq q_\xi \text{ some } \xi \in Z\}$. D is dense!
$\Rightarrow D \cap g \neq \emptyset \Rightarrow \exists r \in g \ \ r \leq q_\xi$ for some $\xi \in Z \Rightarrow q_\xi \in g$.
$\Rightarrow \mathcal{U}^*$ is $\mathbb{P}_I$-generic over $V[g]$.

Here:

$$\begin{tikzcd} & M[g,h] \\ S \in V[g] \arrow[ur, "\hat{j}"] \arrow[dr] \arrow[r, phantom, "///"] & \\ & Ult(V[g], \mathcal{U}^*) \arrow[uu] \end{tikzcd}$$

$\Rightarrow Ult(V[g], \mathcal{U}^*)$ is well-founded
$\Rightarrow \exists S \in \mathcal{U}^* \subseteq I^*$ $S \Vdash$ "$Ult(V[g], \dot{G})$ is well-founded"

3^{rd} step: Suppose it were not the case that $V[g] \models$ "I is precipitous".
Then $\exists S \in \mathbb{P}_I$ $S \Vdash$ "$Ult(V[g], \dot{G})$ is ill-founded".

Suppose there is some h $Col(\omega, [\kappa, j(\kappa)))$-generic over $V[g]$ s.t. if $\hat{j}: V[g] \to M[g,h]$ is the associated lift of $j: V \to M$, then $\kappa \in \hat{j}(S)$.

$\mathcal{U}^* = \{X \in \mathcal{P}(\kappa) \cap V[g] : \kappa \in \hat{j}(X)\}$. Then $S \in \mathcal{U}^*$, so $ult(V[g], \mathcal{U}^*)$ is ill-founded, but $ult(V[g], \mathcal{U}^*)$ is well-founded!

Let $\dot{S}^g = S$, $g \ni p \Vdash$ "$S \in I^+ \wedge \dot{S} \Vdash$ "$Ult(V[\dot{g}], \ddot{G})$ is ill-founded""

Suppose there were no $\underset{\mathrm{Col}(\omega,<j(\kappa))}{\underset{\cap}{q}} \leq p$ s.t. $q \Vdash \check{\kappa} \in \underbrace{j(\dot{S})}_{\in M^{\mathrm{Col}(\omega,<j(\kappa))}}$. Then $Y := \{\xi < \kappa : \neg \exists q \leq p \;\; q \Vdash \check{\xi} \in \dot{S}\} \in \mathcal{U}$.

Then $p \Vdash \check{Y} \cap \dot{S} \neq \emptyset$, so have $q \leq p$ and some ξ s.t. $q \Vdash \check{\xi} \in \check{Y} \cap \dot{S} \Rightarrow \xi \in Y$, but q witnesses $\xi \notin Y$. ↯

$\Rightarrow V[g] \models$ "I is precipitous". $\square$

Theorem

If δ is a Woodin cardinal then there is some $\mathbb{P} \subseteq V_\delta$ s.t. if g is $\mathbb{P}$-generic over V then $NS_{\omega_1}^{V[g]} \cap V = NS_{\omega_1}^V$ and $V[g] \models$ "NS_{ω_1} is saturated"

Definition: Let $\mathcal{A} \subseteq NS_{\omega_1}^+$ be a maximal antichain.
$\mathbb{P}_{\mathcal{A}} = S(\mathcal{A})$ = the sealing forcing for $\mathcal{A}$.

$(p,c) \in \mathbb{P}_{\mathcal{A}}$ iff $c \subseteq \omega_1$ is countable + closed
$p : \max(c) \longrightarrow \mathcal{A}$, and
$\forall \xi \in c \;\; \xi \in \bigcup \operatorname{ran}(p \upharpoonright \xi)$
(i.e. $\xi \in p(\eta)$ for some $\eta < \xi$)

Order on $\mathbb{P}_{\mathcal{A}}$: $(p,c) \leq (q,d)$ iff $c \cap (\max(d)+1) = d$ and $p \upharpoonright \max(d) = q$.

Lemma

Let $\mathcal{A} \subseteq NS_{\omega_1}^+$ be a maximal antichain. Then $\mathbb{P}_{\mathcal{A}}$ is ω-distributive and preserves stationary subsets of ω_1.

Proof:
Let $p \in \mathbb{P}_{\mathcal{A}}$, $\vec{D} = (D_n : n \in \omega)$ sequence of open dense sets in $\mathbb{P}_{\mathcal{A}}$. Let $T \subseteq \omega_1$ be stationary, and $p \Vdash$ "$\dot{C} \subseteq \check{\omega}_1$ is club".

We aim to construct $q \leq p$, $q \in \bigcap_{n<\omega} D_n$, $q \Vdash \check{T} \cap \dot{C} \neq \emptyset$.

Let $S \in \mathcal{A}$ be s.t. $S \cap T$ is stationary. Let $X \prec H_\theta$, θ large enough, X countable, $\mathcal{A}, \mathbb{P}_{\mathcal{A}}, p, \vec{D}, S, \dot{C} \in X$, with $\alpha = X \cap \omega_1 \in S \cap T$.

Let $(D_n' : n < \omega) \in V$ enumerate all the open dense sets which exist in X. We define a sequence $(p_n : n < \omega)$ as follows:

- $p_0 = p$
- given $p_n \in X$, pick $p_{n+1} \in X \cap D_n'$ s.t. $p_{n+1} \leq p_n$.

Write $p_n = (r_n, c_n)$ for $n < \omega$. Let $r = \bigcup r_n$, $\bar{c} = \bigcup_{n<\omega} c_n$. $\Rightarrow$ $\sup(\bar{c}) = \alpha$, $\operatorname{dom}(r) = \alpha$.

Consider $q = (r, \bar{c} \cup \{\alpha\})$. By density, $\operatorname{ran}(r) = \mathcal{A} \cap X$. Have $\alpha \in S \cap T$ so $\alpha \in \underset{r(\gamma),\text{ some } \gamma<\alpha}{\underset{\shortparallel}{S}} \in \mathcal{A} \cap X$.
$\Rightarrow q \in \mathbb{P}_{\mathcal{A}}$.
Have $q \in D_n$ for all $n < \omega$. Again by density, we have that $q \Vdash \check{\alpha} \in \dot{C}$.
Have $\alpha \in T$. So $q \Vdash \dot{C} \cap \check{S} \neq \emptyset$. $\square$

Question: Under which hypothesis is $\mathbb{P}_{\mathcal{A}}$ semiproper?

Definition: Let $\mathcal{A} \subseteq NS_{\omega_1}^+$ be a maximal antichain. We say that $\mathcal{A}$ is semiproper iff for all large enough θ and all $X \prec H_\theta$ with X countable, $\mathcal{A} \in X$, there is some $Y \prec H_\theta$, $Y \supseteq X$, $Y \cap \omega_1 = X \cap \omega_1$ (future notation: $Y \sqsupset X$), and there is some $S \in \mathcal{A} \cap Y$ s.t. $Y \cap \omega_1 = X \cap \omega_1 \in S$.

Definition: A forcing $\mathbb{P}$ is called semiproper iff for all large enough θ, for all $p \in \mathbb{P}$, for all countable $X \prec H_\theta$ $p, \mathbb{P} \in X$ there is some $q \leq p$ s.t.: if g is $\mathbb{P}$-generic over V, $q \in g$ then $\underbrace{X[g]}_{\{\tau^g : \tau \in V^{\mathbb{P}} \cap X\}} \cap \omega_1 = X \cap \omega_1$.

Lemma

Let $\mathcal{A} \subseteq NS^+_{\omega_1}$ be a maximal antichain. TFAE:

(1) $\mathcal{A}$ is semiproper

(2) $\mathbb{P}_{\mathcal{A}}$ is semiproper.

Proof:

(1) $\Rightarrow$ (2): Let $X \prec H_\theta$ like in the definition of semiproper (for forcing). Let $Y \sqsupseteq X$ be as in the definition of semiproper (for $\mathcal{A}$), i.e. $X \cap \omega_1 \in S \in \mathcal{A} \cap Y$, some S, Y countable.

Now we contruct q exactly as in the previous proof but w.r.t. Y. In particular, if $\mathbb{1}_{\mathbb{P}_{\mathcal{A}}} \Vdash \tau \in \check{\omega}_1$ then $D_\tau := \{r : \exists \beta < \omega_1\ r \Vdash \tau = \check{\beta}\}$ is open dense, $D_\tau \in Y$.

Then by the construction, there will be some $p \geq r \geq q$ s.t. $r \in Y \cap D_\tau$ so $Y \models r \in D_\tau$ so $r \Vdash \tau = \check{\beta}$, some $\beta < Y \cap \omega_1$. Hence $q \Vdash \tau \in (Y \cap \omega_1)^\vee$.

(2) $\Rightarrow$ (1): Exercise. □

Definition: δ is called a Woodin cardinal iff for all $\mathcal{A} \subseteq V_\delta$ there is some $\kappa < \delta$ s.t. κ is $<\delta, \mathcal{A}$-strong, i.e. for every $\alpha < \delta$ there is some $j : V \to M$ s.t.
- M is transitive
- j is elementary
- $\operatorname{crit} j = \kappa$
- $V_\alpha \subseteq M$, and
- $j(\mathcal{A}) \cap V_\alpha = \mathcal{A} \cap V_\alpha$
- (may also require ${}^{\kappa}M \subseteq M$).

Definition: δ is called Woodin with $\diamondsuit$ iff $\exists\ (A_\alpha : \alpha < \delta)$ s.t.
- $A_\alpha \subseteq V_\alpha$ for all $\alpha < \delta$
- for every $\mathcal{A} \subseteq V_\delta$ $\{\kappa < \delta : A_\kappa = \mathcal{A} \cap V_\kappa \wedge \kappa \text{ is } <\delta, \mathcal{A}\text{-strong}\}$ is stationary in δ.

Lemma

Let δ be Woodin. Then δ is Woodin with $\diamondsuit$ in $V^{Col(\delta,\delta)}$

Proof: Exercise.

In $L[E]$-models, every Woodin is Woodin with $\diamondsuit$.

We will use the fact that $\underbrace{\text{RCS}}_{\text{revised countable support}}$ iterations of semiproper forcings are semiproper.

Proof (of the theorem):

Fix δ a Woodin with $\diamond$, $(A_\alpha : \alpha < \delta)$.

We define a RCS iteration $(\mathbb{P}_\alpha : \alpha < \delta)$, $(\dot{\mathbb{Q}}_\alpha : \alpha < \delta)$. (In particular, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$).

For $\alpha < \delta$,

$$\dot{\mathbb{Q}}_\alpha \begin{cases} \dot{\mathbb{Q}} & \text{if } A_\alpha \in V^{\mathbb{P}_\alpha} \text{ is s.t. } \Vdash_{\mathbb{P}_\alpha} \text{"}A_\alpha \text{ is a maximal antichain in } NS^+_{\omega_1}\text{"} \\ & \text{and } \Vdash_{\mathbb{P}_\alpha} \text{"}\dot{\mathbb{Q}} \text{ is the associated sealing forcing and } \dot{\mathbb{Q}} \text{ is semiproper.} \\ Col(\omega_1, 2^{\aleph_2}) & \text{otherwise.} \end{cases}$$

$\to$ from the point of view of $V^{\mathbb{P}_\alpha}$

11.11.21

$\downarrow$

Virtual lecture

Definition: $p \in \mathbb{P}_{max}$ iff $p = (p; \in, I, a)$ where
a) p is countable + transitive
b) $p \models ZFC \wedge MA_{\omega_1}$ (Martin's Axiom)
c) $I \in p$, and $p \models$ "I is an uniform normal ideal on ω_1"
d) $a \in p$, $a \subseteq \omega_1^p$, and $\omega_1^p = \omega_1^{L[a]^p}$; and
e) p is generically iterable.

$\mathbb{P}_{max}$ is a Π^1_2 set (in the codes).

For $p, q \in \mathbb{P}_{max}$, $q < p$ iff
a) $p \in q$ and $q \models p$ is countable; and if $\alpha = \omega_1^q$, then
b) there is some generic iteration $((p_i, \pi_{ij} : i \leq j \leq \alpha), (g_i : i < \alpha)) \in q$ of p s.t. if $p_\alpha = (p_\alpha; \in, I', a')$ and $q = (q; \in, I'', a'')$, then $a'' = a'$ and $I' = I'' \cap p_\alpha$.

Let's first make the following observation:

Let $q < p$, and let's use the notation from the definition.

Claim: Let $A \in p_\alpha$ be s.t. $p_\alpha \models$ "A is a maximal antichain in $\mathbb{P}_{I'} = (I')^+$". Then in q, "there is some enumeration $f : \omega_1 \to A$ (onto) s.t. there is some club $C \subseteq \alpha$ with $C \subseteq \nabla_{\xi < \alpha} f(\xi)$.

Proof:
Let $\alpha_\xi = \omega_1^{p_\xi}$, $\xi < \alpha$; $\bar{C} = \{\alpha_\xi : \xi < \alpha\}$ is club in α. Let $A \in p_\alpha$ be an antichain as above.
Let's assume w.l.o.g. that $A \in ran(\pi_{0\alpha})$. Say $\bar{A} = \pi_{0\alpha}^{-1}(A)$

For every $0 \leq \xi < \alpha$, there is some $S \in \pi_{0\xi}(\bar{A})$ with $\alpha_\xi \in \pi_{\xi\xi+1}(S)$, hence $\alpha_\xi \in \pi_{\xi\alpha}(S) \in A$. So $\alpha_\xi \in \bigcup ran(\pi_{\xi\alpha}) \cap A$.

Let $C = \bar{C}'$ = the limit points of $\bar{C}$ s.t. $\xi = \alpha_\xi$.

Inside q, we may find some $f : \alpha \to A$ s.t. for all $\xi \in C$, $f''\xi = A \cap ran(\pi_{\xi\alpha})$.
Then if $\xi \in C$, $\xi = \alpha_\xi \in \bigcup f''\xi$.

$\dashv$ Claim

Now if $A \in p_\alpha$, $p_\alpha \models$ "A is a maximal antichain in $\mathbb{P}_{I'} = (I')^+$", let $f : \alpha \to A$, $C \subseteq \alpha$ club as just constructed.

We claim that A is still a maximal antichain in $\mathbb{P}_{I''}$ from the point of view of q:
Work in q: let $S \in q$, $S \in (I'')^+$. $S \cap C \in (I'')^+$. Define $h : S \cap C \to \omega_1 = \alpha$, $\xi \mapsto$ the least $\eta < \xi$ s.t. $\xi \in f(\eta)$.

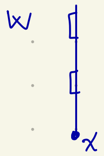
Have $T \in (I'')^+$, $h \upharpoonright T$ constant, say with value η_0. Then T is a positive subset of $S \cap f(\eta_0)$.

Lemma

If $\mathbb{R}$ is closed under $x \mapsto x^\dagger$, then for every $x \in \mathbb{R}$ there is $p \in \mathbb{P}_{max}$ with $x \in p$.

Proof sketch

Fix $x \in \mathbb{R}$. Think of $x^\dagger$ as an inner model W of ZFC, s.t.



$x \in W$, $W \models$ "there is a measurable cardinal κ below an inaccessible cardinal λ", and $\lambda < \omega_1^V$.

Let $g \in V$ be $Col(\omega, <\kappa) * \mathbb{P}$ - generic over W, where $\mathbb{P}$ is "the" standard forcing inside $W^{Col(\omega,<\kappa)}$ forcing MA_{ω_1}.

Exercise: In $W[g]$, there is a normal uniform precipitous ideal on $\omega_1 = \kappa$.

Then $V_\lambda^{W[g]}$ is generically iterable via I and its images. $\dashv$

Lemma ($\mathbb{R}$ is closed under $x \mapsto x^\dagger$)

$\mathbb{P}_{max}$ is σ-closed.

Proof:

Let $(p_n : n<\omega)$, $p_{n+1} < p_n$ $\forall n$.

$$\begin{array}{ccccccccccc}
p_0 & \xrightarrow{j} & p_0^1 & \dashrightarrow & p_0^2 & \dashrightarrow & p_0^3 & \dashrightarrow & p_0^4 & \cdots \dashrightarrow & p_0^\omega \\
 & & \cap & & \cap & & \cap & & \cap & & \cap \\
 & & p_1 & \xrightarrow{i} & p_1^2 & \dashrightarrow & p_1^3 & \dashrightarrow & p_1^4 & \cdots \dashrightarrow & p_1^\omega \\
 & & & & \cap & & \cap & & \cap & & \cap \\
 & & & & p_2 & \longrightarrow & p_2^3 & \dashrightarrow & p_2^4 & \cdots \dashrightarrow & p_2^\omega \\
 & & & & & & \cap & & \cap & & \cap \\
 & & & & & & p_3 & \longrightarrow & p_3^4 & \cdots \dashrightarrow & p_3^\omega \\
 & & & & & & & & \cap & & \cap \\
 & & & & & & & & p_4 & \cdots \dashrightarrow & \vdots
\end{array}$$

$$\bigcup_{n<\omega} p_n^\omega = p^\omega$$

$i(j) : p_0 \to i(p_0^1) =: p_0^2$

$$\begin{array}{ccccc}
p_0 & \xrightarrow{j} & p_0^1 & \dashrightarrow & p_0^2 \\
 & & \cap & & \cap \\
 & & p_1 & \xrightarrow{i} & p_1^2
\end{array}$$

Notation: If $p \in \mathbb{P}_{max}$, $p = (p; \in, I, a)$, write $I^p = I$, $a^p = a$.

Have $a^{p_0^\omega} = a^{p_1^\omega} = \dots$ Also, $I^{p_1^\omega} \cap p_0^\omega = I^{p_0^\omega}, \dots$ Let $a = a^{p_0^\omega}$, $I = \bigcup_{n<\omega} I^{p_n^\omega}$.

We may thus think of p^ω as a "$\mathbb{P}_{max}$ - condition", $p^\omega = (p^\omega; \in, I, a)$.

Claim: p^ω is generically iterable in the obvious sense.

Proof: Let g be $\mathbb{P}_I$-generic over P^ω.

<u>Key thing:</u> $g \cap P^\omega_n$ is $\mathbb{P}_{I^{P^\omega_n}}$-generic over P^ω_n.

$$\begin{tikzcd}
P_0 \arrow[r] & P_0^1 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_0^2 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_0^3 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_0^4 \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_0^\omega \arrow[d, phantom, "\cap"] & \mathrm{ult}(P_1^\omega; g\cap P_1^\omega) \arrow[dr] & \\
& P_1 \arrow[r] & P_1^2 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_1^3 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_1^4 \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_1^\omega \arrow[ur] \arrow[rr, dotted, "i \restriction P_1^\omega"] \arrow[d, phantom, "\cap"] & & i(P_1^\omega) \\
& & P_2 \arrow[r] & P_2^3 \arrow[r, dashed] \arrow[d, phantom, "\cap"] & P_2^4 \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_2^\omega \arrow[r, "i"] \arrow[d, phantom, "\cap"] & \mathrm{ult}(P_2^\omega; g\cap P_2^\omega) \arrow[r, dotted] & i'(P_2^\omega) \\
& & & P_3 \arrow[r] & P_3^4 \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_3^\omega \arrow[rr, "i'"] \arrow[d, phantom, "\cap"] & & \mathrm{ult}(P_3^\omega, g\cap P_3^\omega) \\
& & & & P_4 & \cdots \arrow[r, dotted] & \vdots & &
\end{tikzcd}$$

$$\bigcup_{n<\omega} P_n^\omega = P^\omega$$

$$g \cap P_1^\omega = \{X \in I^{P_1^\omega} : \omega_1^{P^\omega} \in i(X)\}$$

$$\begin{tikzcd}
P_0^\omega \arrow[r] \arrow[d, phantom, "\cap"] & \mathrm{ult}(P_0^\omega; g\cap P_0^\omega) \arrow[r, dotted] & P_0^{\omega+1} \arrow[r, dotted] \arrow[d, phantom, "\cap"] & P_0^{\omega+2} \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_0^{\omega+\omega} \\
P_1^\omega \arrow[rr] \arrow[d, phantom, "\cap"] & & \mathrm{ult}(P_1^\omega; g\cap P_1^\omega) \arrow[r, dotted] & P_1^{\omega+2} \arrow[d, phantom, "\cap"] & \cdots \arrow[r, dotted] & P_1^{\omega+\omega} \\
P_2^\omega \arrow[rrr] & & & \mathrm{ult}(P_2^\omega, g\cap P_2^\omega) & \cdots \arrow[r, dotted] & P_2^{\omega+\omega} \\
& & & & & \vdots
\end{tikzcd}$$

$$P^{\omega+\omega} = \bigcup_{n<\omega} P_n^{\omega+\omega}$$

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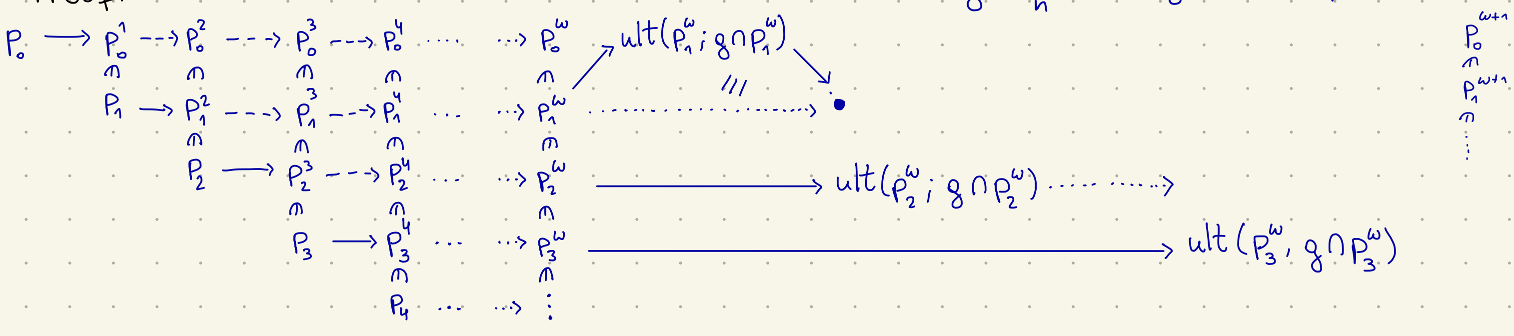
Lemma

$\mathbb{P}_{max}$ is σ-closed.

Proof:

g $\bigcup_n I^{P_n^\omega}$-generic / P^ω.



$$\bigcup_{n<\omega} P_n^\omega = P^\omega$$

Now we showed that if $x\in\mathbb{R}$, then there is a countable model $V_\lambda^{W[g]}$, where W is an inner model with a measure κ, $\lambda>\kappa$ inaccessible, $\lambda<\omega_1^V$.

$$\boxed{V_\lambda^{W[g]} \hookleftarrow Col(\omega,<\kappa)} \sim (V_\lambda^{W[g]}, \in, I, \text{—})$$

Take $x\in\mathbb{R}$ to code $(P_n : n\in\omega)$ (or $(P_n^\omega : n<\omega)$).

We aim to define an "iteration" of $(P_n^\omega : n<\omega)$ to get a final "iterate" $(P_n^* : n<\omega)$ s.t. for every $n<\omega$, $I^{P_n^*} = I\cap P_n^*$.

Simpler challenge: Let $p,q\in\mathbb{P}_{max}$, $p\in q$. There is then some generic iteration $((p_i,\pi_{ij} : i\le j\le\alpha),(g_i : i<\alpha))\in q$ $(\alpha=\omega_1^q)$ of $p_0=p$ s.t. $I^{p_\alpha} = I^q\cap p_\alpha$.

How do you prove this?

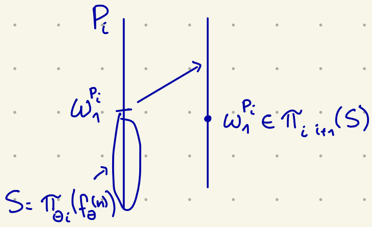
- $I^{p_\alpha}\subseteq I^q$: always true, no matter how we design the iteration.
- How do we arrange $(I^{p_\alpha})^+\cap p_\alpha \subseteq (I^q)^+$?

Inside q, let $\alpha = \underset{\text{disjoint union}}{\dot\bigcup}\{S_\xi : \xi<\alpha\}$, $S_\xi\in(I^q)^+$ for all ξ.

Suppose we already constructed $((p_i,\pi_{ij} : i\le j\le\theta),(g_i : i<\theta))$. Work inside q:

Pick $f_\theta:\omega\to(I^{p_\theta})^+\cap p_\theta$ bijective. Associate to each $f_\theta(n)$ a set S_ξ (e.g. $\xi=\omega\cdot\theta+n$).

Make sure later that if $i\in S_\xi$, then $\pi_{\theta i}(f_\theta(n))\in g_i$
$(\iff \omega_1^{p_i}\in\pi_{\theta\, i+1}(f_\theta(n))\subseteq\pi_{\theta\alpha}(f_\theta(n)))$.



$C=\{i<\alpha : \omega_1^{p_i}=i\}$ is a club, so $\underbrace{(S_\xi\cap C)}_{\in(I^q)^+}\setminus\theta\subseteq\pi_{\theta\alpha}(f_\theta(n))$.

⌞

Now do this with $(P_n^\omega : n<\omega)$, $(V_\lambda^{W[g]};\in,I)$ instead of (p,q).

Lemma

For $p \in \mathbb{P}_{max}$ and for every $a \subseteq \omega_1$ there is at most one generic iteration of p with last model p^* s.t. $a = a^{p^*}$

Proof

It's enough to prove:

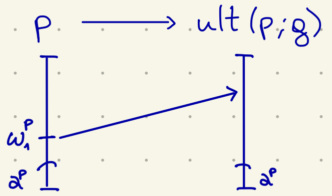
Let $p \in \mathbb{P}_{max}$, g $\mathbb{P}_{I^p}$-generic over p, and let $q = ult(p;g)$. Then g can be read off from p and a^q.

Recall $p = (p; \in, I^p, a^p)$ and $p \models$ "$\omega_1^{L[a^p]} = \omega_1$".

Inside $L[a^p]$, you may define a canonical sequence $(a_\xi : \xi < \omega_1^p = \omega_1^{L[a^p]^p})$ of pairwise almost disjoint subsets of ω_1^p.

Let $i : p \to ult(p;g) = q$ the ultrapower map.

Aside: For all $\mathbb{P}_{max}$-conditions p, a^p is bounded in ω_1^p.



From a^q, we may now read off $i((a_\xi : \xi < \omega_1^p)) = (a'_\xi : \xi < \omega_1^q \; (= \sup(a^q)))$, $a'_\xi = a_\xi$ for $\xi < \omega_1^p$.

Let $S \in \mathcal{P}(\omega_1^p) \cap p$. As $p \models MA_{\omega_1}$, there is some $x_S \in \mathcal{P}(\omega) \cap p$ s.t. $\forall \xi < \omega_1^p$ $(\xi \in S \iff \overline{\overline{x_S \cap a_\xi}} = \aleph_0)$.

Then $\omega_1^p \in i(S)$ iff $\overline{\overline{x_S \cap a'_{\omega_1^p}}} = \aleph_0$. But now $g = \{S \in (I^p)^+ \cap p : \omega_1^p \in i(S)\}$ can be read off from p, a^q. (or from p and any $b \supseteq a^q$, $b \cap \sup(a^q) = a^q$ for that matter). $\dashv$

Let g be $\mathbb{P}_{max}$-generic over V. For $\alpha < \omega_1$, $\{p \in \mathbb{P}_{max} : \alpha \leq \omega_1^p\}$ is dense. Hence, $A = \bigcup\{a^p : p \in g\}$ is unbounded in ω_1, and $A \cap \omega_1^p = a^p$ for all $p \in g$.

Notation: $A_g = \bigcup\{a^p : p \in g\}$.

<u>Next observation:</u>

For $p \in g$ and $\alpha < \omega_1$, pick $q \in g$, $\omega_1^q \geq \alpha$, and $p > q$. Hence (inside q) there is an iteration $((p_i, \pi_{ij} : i \leq j \leq \omega_1^q), (g_i : i < \omega_1^q))$ of $p = p_0$ s.t. $a^{p_{\omega_1^q}} = a^q = A_g \cap \omega_1^q$, and there is a <u>unique</u> such iteration.

There is then an (unique) iteration $((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ of p s.t. $a^{p_{\omega_1}} = A_g$, and $p_i \in g$ for all $i < \omega_1$.

We call this the iteration of p as being given by g.

Lemma

Let $p \in g$, g $\mathbb{P}_{max}$-generic over V and let $((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ be the iteration of p as being given by g.
Then $I^{p_{\omega_1}} = NS_{\omega_1}^{V[g]} \cap p_{\omega_1}$.

Proof:

- $I^{p_{\omega_1}} \subseteq NS_{\omega_1}^{V[g]}$ ✓
- Let $S \in (I^{p_{\omega_1}})^+ \cap p_{\omega_1}$, $S = \pi_{i\omega_1}(\bar{S})$, some $i < \omega_1$. Let $\tau \in V^{\mathbb{P}_{max}}$ be a name for a club.

Claim: $\{q \in \mathbb{P}_{max} : q < p_i \wedge \exists \alpha \; q \Vdash \check{\alpha} \in \dot{\pi}_{0\omega_1}(\check{\bar{S}}) \cap \tau\}$ is dense below p_i.

name for the iteration of p_i according to $\dot{g}$.

Recall: $p \in \mathbb{P}_{max}$ iff $p = (p; \in, I, a)$ where

a) p is countable + transitive

b) $p \models ZFC \wedge MA_{\omega_1}$ (Martin's Axiom)

c) $I \in p$, and $p \models$ "I is an uniform normal ideal on ω_1"

d) $a \in p$, $a \subseteq \omega_1^p$, and $\omega_1^p = \omega_1^{L[a]^p}$; and

e) p is generically iterable.

For $p, q \in \mathbb{P}_{max}$, $q < p$ iff

a) $p \in q$ and $q \models p$ is countable; and if $\alpha = \omega_1^q$, then

b) there is some generic iteration $((p_i, \pi_{ij} : i \leq j \leq \alpha), (g_i : i < \alpha)) \in q$ of p s.t. if $p_\alpha = (p_\alpha; \in, I', a')$ and $q = (q; \in, I'', a'')$, then $a'' = a'$ and $I' = I'' \cap p_\alpha$.

We showed:

- if $\mathbb{R}$ is closed under $x \mapsto x^\dagger$, then $\forall x \in \mathbb{R}\ \exists p \in \mathbb{P}_{max}\ x \in p$.
- $\mathbb{P}_{max}$ is σ-closed and that generic iterations of $p \in \mathbb{P}_{max}$ are determined by how a^p gets shifted.

Lemma

Let $p \in g$, g $\mathbb{P}_{max}$-generic over V and let $((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ be the iteration of p as being given by g.

Then $I^{p_{\omega_1}} = NS_{\omega_1}^{V[g]} \cap p_{\omega_1}$.

Proof (cont.):

"$\subseteq$": was easy

"$\supseteq$": We fixed $S \in (I^{p_{\omega_1}})^+ \cap p_{\omega_1}$. We want to see S is stationary in $V[g]$.

$S = \pi_{i\omega_1}(\bar{S})$, some $i < \omega_1$, $\bar{S} \in p_i \in g$. Let $\tau \in V^{\mathbb{P}_{max}}$ be a name for a club.

Claim: $\{q \in \mathbb{P}_{max} : q < p_i \wedge \exists \alpha\ q \Vdash \check{\alpha} \in \dot{\pi}_{0\omega_1}(\check{\bar{S}}) \cap \tau\}$ is dense below p_i.

↑ name for the (unique) gen. iteration map of p_i witnessing $p_i > q$.

If that's true, then $D \cap g \neq \emptyset$, so $V[g] \models \alpha \in S \cap \tau^g$, some α.

Proof (D is dense):

Let $q < p_i$. Let $\pi : p_i \longrightarrow p^* \in q$ witness $q < p_i$.

- $q_0 = q$, $\alpha_0 = 0$.
- given q_n, α_n, let $q_{n+1} < q_n$ and $\alpha_{n+1} > q_n \cap OR$ be s.t. $q_{n+1} \Vdash \check{\alpha}_{n+1} \in \tau$.

$$
\begin{array}{llllll}
\bar{S} \in p_i & & \xrightarrow{\qquad\qquad \sigma \qquad\qquad} & & & p_i^\omega \\
\pi(\bar{S}) \in q_0 & \longrightarrow q_0^1 & \longrightarrow q_0^2 & \longrightarrow & \cdots & q_0^\omega \\
 & & & & & \cap \\
 & q_1 & \longrightarrow q_1^2 & \longrightarrow & \cdots & q_1^\omega \\
 & & & & & \cap \\
 & & q_2 & \longrightarrow q_2^3 & & \vdots \\
 & & & & \ddots & q^\omega
\end{array}
$$

Let h be $\mathbb{P}_{(I^{q_*^\omega})^+ \cap q_*^\omega}$ s.t. $\sigma(\bar{S}) \in h$ where $\sigma : p_i \to p_i^\omega$.

Now continue and produce $r \in \mathbb{P}_{max}$, $r <$ all the q_n, $n < \omega$. Then, setting $\alpha = \omega_1^{p_i^\omega} = \omega_1^{q_n^\omega}$, all n, $\alpha = \sup_n \alpha_n$, then $r \Vdash \check{\alpha} \in \tau \cap \dot{\pi}_{0\omega_1^r}(\bar{S})$.

We proved:
Let g be $\mathbb{P}_{max}$-generic over V, then

$$g = \left\{ p \in \mathbb{P}_{max} : \begin{array}{l} \exists \text{ gen. iteration } ((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1)) \text{ of } p \text{ s.t.} \\ a^{p_{\omega_1}} = A_g \ \wedge \ I^{p_{\omega_1}} = NS_\omega^{V[g]} \cap p_{\omega_1} \end{array} \right\}.$$

Definition: Let $A \subseteq \mathbb{R}$. Let $p \in \mathbb{P}_{max}$. Then p is called A-iterable iff
a) $A \cap p \in p$, and
b) if $((p_i, \pi_{ij} : i \leq j \leq \alpha), (g_i : i < \alpha))$ is a generic iteration of p, $\alpha \leq \omega_1$, then $\pi_{0\alpha}(A \cap p) = A \cap p_\alpha$.

<u>Remark:</u> If A is Π_1^1 or Σ_1^1, then every $p \in \mathbb{P}_{max}$ is A-iterable.

We will prove:

Theorem
If $V = L(\mathbb{R}) \models AD$ (axiom of determinacy), then, for every A, $\{p \in \mathbb{P}_{max} : p \text{ is } A\text{-iterable and } (HC^p; \in, A \cap p) \prec (HC, \in, A)\}$ is dense.

<u>Remark:</u> In ZFC, there is trivially some $A \subseteq \mathbb{R}$ s.t. $\{p : p \text{ is } A\text{-iterable}\}$ is not dense.
Idea: Write $\omega_1 = S \cup \omega_1 \setminus S$, S stationary and costationary. Let S be coded by $A \subseteq \mathbb{R}$. Produce (say for NS_{ω_1} saturated + $\mathcal{P}(\omega_1)^\#$ ex.) $p \in \mathbb{P}_{max}$ together with a generic iteration $((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ of p s.t. $S \in \operatorname{ran}(\pi_{0\omega_1})$ and $I^{p_{\omega_1}} = NS_{\omega_1}^V \cap p_{\omega_1}$.
Say $\omega_1^{p_0} \in S$. Write $\bar{S} = \pi_{0\omega_1}^{-1}(S)$.

There is then a gen. iteration $((p_i', \pi_{ij}' ; i \leq j \leq \omega_1), (g_i' : i < \omega_1))$ of $p = p_0 = p_0'$ s.t. $\omega_1^{p_0} \setminus \bar{S} \in g_0'$. Then $\omega_1^{p_0} \notin \pi_{0\omega_1}'(\bar{S})$.

We will prove the theorem later. Suppose now its conclusion is true.

Definition: Let g be $\mathbb{P}_{max}$-generic over V.

$$\mathcal{P}(\omega_1)_g = \left\{ X \subseteq \omega_1 : \begin{array}{l} \exists p \in g \text{ s.t. if } ((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1)) \text{ is the generic iteration} \\ \text{of } p_0 = p \text{ as being given by } g, \text{ then } X \in p_{\omega_1} \end{array} \right\}$$

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Recall:

$\mathcal{P}(\omega_1)_g = \{X \subseteq \omega_1 : \exists p \in g$ s.t. if $((p_i, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ is the generic iteration of $p_0 = p$ as being given by g, then $X \in p_{\omega_1}\}$,

where g is $\mathbb{P}_{max}$-generic.

Lemma (under some hypothesis)
Let g be $\mathbb{P}_{max}$-generic over V, then $\mathcal{P}(\omega_1)^{V[g]} = \mathcal{P}(\omega_1)_g$.

Theorem
If $V = L(\mathbb{R}) \models AD$ (axiom of determinacy), then, for every A, $\{p \in \mathbb{P}_{max} : p$ is A-iterable and $(HC^p; \in, A \cap p) \prec (HC; \in, A)\}$ is dense.
(the underlying universe)

- $p \in \mathbb{P}_{max}$, $A \subseteq \mathbb{R}$. p is A-iterable iff:
  - $A \cap p \in p$, and
  - if $\pi : p \to p^*$ comes from a generic iteration of p (of length $\leq \omega_1$), then $\pi(A \cap p) = A \cap p^*$.

Proof (of the lemma under the hypothesis):

- $\mathcal{P}(\omega_1)_g \subseteq \mathcal{P}(\omega_1)^{V[g]}$: trivial

- $\supseteq$: Let $\tau \in V^{\mathbb{P}_{max}}$, say $\Vdash_{\mathbb{P}_{max}} \tau : \check{\omega}_1 \to 2$. Let $\bar{p} \in \mathbb{P}_{max}$ be given. Let $p < \bar{p}$ be as the above theorem, where
$$A = \{(r, \xi, h) : r \in \mathbb{P}_{max}, \xi < \omega_1, h \in \{0,1\} \text{ and } r \Vdash \tau(\check{\xi}) = \check{h}\},$$
construed as a set of reals. Write $\alpha = \omega_1^p$.

Inside p, construct a sequence $(q_\xi : \xi < \alpha)$ of $\mathbb{P}_{max}$ conditions as follows:
- $q_0 = \bar{p}$.
- Given a limit ordinal $\xi < \alpha$ and q_i, $i < \xi$, pick $q_\xi < q_i$ all $i < \xi$.
- Now let q_ξ be given. In V, for all $r \in \mathbb{P}_{max}$ there is some $s \in \mathbb{P}_{max}$, $s < r$, s.t. $s \parallel \tau(\check{\xi})$, i.e. $\exists h$ $s \Vdash \tau(\check{\xi}) = \check{h}$; so $(s, \xi, h) \in A$.

This statement is true in $(HC; \in, A)$, hence in $(p; \in, A \cap p)$. Pick $q_{\xi+1} \in p$ s.t. $(q_{\xi+1}, \xi, h) \in A$, some h, and $q_{\xi+1} < q_\xi$.

Inside of p, define $K = \{r \in \mathbb{P}_{max} : \exists \xi < \alpha \ r \geq q_\xi\}$. This is a filter.

We may assume the following:

- $\sup_{\xi < \alpha} \omega_1^{q_\xi} = \alpha$
- if $q_\xi \to q_\xi^\alpha$ is the generic iteration of q_ξ as being given by K, then $I^{q_\xi^\alpha} = I^p \cap q_\xi^\alpha$.

Let $a = \bigcup_{\xi < \alpha} a^{q_\xi} = a^{q_\xi^\alpha}$ for some/all $\xi < \alpha$. Write $p^* = (p; \in, I^p, a)$.
(underlying universe)

Then $K \in p^*$, K is a filter, every element of K is weaker than p^*, and for all $\xi < \alpha = \omega_1^{p^*}$ there is some $r \in K$ and some h s.t. $(r, \xi, h) \in A \cap p^*$.

By density, there is some such A-iterable p^* in g. Let $K \in p^*$ be as above.

Now let $((p_i^*, \pi_{ij} : i \leq j \leq \omega_1), (g_i : i < \omega_1))$ be the generic iteration of $p^* = p_0^*$ as being given by g.

If $r \in \pi_{0\omega_1}(K)$, then $r \in \pi_{0i}(K)$, some $i < \omega_1$, and then $r > p_i^* \in g$. So $r \in g$, i.e., $\pi_{0\omega_1}(K) \subseteq g$.

If $\xi < \omega_1$, then there are $r \in \pi_{0\omega_1}(K)$ and $h \in \{0,1\}$ s.t. $(r, \xi, h) \in \pi_{0\omega_1}(A \cap p^*) = A \cap p^*_{\omega_1} \subseteq A$, so $r \Vdash \tau(\check{\xi}) = \check{h}$.

Now $\tau^g = \{(\xi, h) : \exists r \in \pi_{0\omega_1}(K) \ (r, \xi, h) \in \pi_{0\omega_1}(A \cap p^*)\} \in p^*_{\omega_1}$.

Theorem $(V = L(\mathbb{R}) \models AD)$

$\forall x \in \mathbb{R} \ \forall A \subseteq \mathbb{R} \ \exists p \in \mathbb{P}_{max} \ \ x \in p$, p is A-iterable, and $(HC^p, \in, A \cap p) \prec_{\Sigma_1^1} (HC, \in, A)$.

If this is wrong, then $\exists A \subseteq \mathbb{R} \ \exists x \ \underbrace{\forall p \in \mathbb{P}_{max} \ (\ldots)}_{\text{a statement with parameters } \mathbb{R}, A \text{, and all quantifications over reals}}$.

So this is $\Sigma_1^{L(\mathbb{R})}(\{\mathbb{R}\})$, in fact $(\Sigma_1^2)^{L(\mathbb{R})}$.

↑ parameter allowed

Theorem 1 $(V = L(\mathbb{R}) \models AD)$

Every true Σ_1^2-statement is witnessed by some $A \subseteq \mathbb{R}$ which is a Δ_1 singleton, i.e., there are φ and ψ, Σ_1 formulae, s.t. $x \in A \Leftrightarrow \varphi(x, \mathbb{R}) \Leftrightarrow \neg\psi(x, \mathbb{R})$.

Theorem 2 $(V = L(\mathbb{R}) \models AD)$

If A is $\Sigma_1(\{\mathbb{R}\})$, then A is Suslin, i.e., there is some $\alpha \in OR$ and some tree T on $\omega \times \alpha$ s.t. $A = p[T] = \{x \in {}^\omega\omega : \exists (\beta_n : n < \omega) \ \forall k \ (x \upharpoonright k, (\beta_0, \ldots, \beta_k)) \in T\}$

Corollary

Every true Σ_1^2-statement is witnessed by an $A \subseteq \mathbb{R}$ which is Suslin and co-Suslin.

Theorem 3 $(V = L(\mathbb{R}) \models AD)$

Let T be a set of ordinals, and let $x \in \mathbb{R}$. There is a cone of reals y s.t. $L[y, T] \models$ "ω_1 is measurable in $HOD_{x,T}$".

So suppose there is some $A \subseteq \mathbb{R}$ and some $x \in \mathbb{R}$ s.t. no $p \in \mathbb{P}_{max}$ with $x \in p$ is A-iterable.

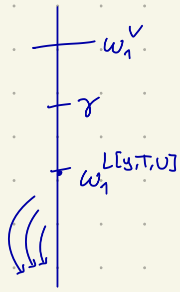
Then there is some such A which is Suslin and co-Suslin, say $A = p[T]$, $\mathbb{R} \setminus A = p[U]$.

Let $x \in \mathbb{R}$. There is some $y \in \mathbb{R}$ s.t.

$$x \in HOD_{x,T,U}^{L[y,T,U]} \models \text{"}\omega_1^{L[y,T,U]} \text{ is measurable"}.$$

Let g be $Col(\omega, <\omega_1^{L[y,T,U]}) * \mathbb{P}$ - generic over $HOD^{L[y,T,U]}_{x,T,U}$, $g \in V$, where $\mathbb{P}$ forces MA_{ω_1}.

Pick γ, $\omega_1^{L[y,T,U]} < \gamma < \omega_1^V$ s.t. $V_\gamma^{HOD^{L[y,T,U]}_{x,T,U}[g]} \models ZFC$.



Write $W = HOD^{L[y,T,U]}_{x,T,U}[g]$. Then for $x \in \mathbb{R} \cap W$, $x \in A$ iff $x \in p[T]$ iff $W \models$ "$x \in p[T]$". So $A \cap W \in W$.

Let $\pi : W \to W^*$ be the iteration map coming from a generic iteration of W of countable length.

Let $x \in W^*$. If $x \in A \cap W^*$, then $x \in p[T]$, so $x \in p[\pi(T)]$ (let $(\beta_n : n < \omega)$ be s.t. $(x \restriction k, (\beta_0, \dots, \beta_{k-1})) \in T$ $\forall k$; then $(x \restriction k, (\pi(\beta_0), \dots, \pi(\beta_{k-1}))) \in \pi(T)$ $\forall k$, so $x \in p[\pi(T)]$), then $W^* \models$ "$x \in p[\pi(T)]$".

By the same argument, if $x \in (\mathbb{R} \cap W^*) \setminus A$, then $W^* \models$ "$x \in p[\pi(U)]$".
But $W \models$ "$p[U] = \mathbb{R} \setminus p[T]$", so $W^* \models$ "$p[\pi(U)] = \mathbb{R} \setminus p[\pi(T)]$".

So if $x \in (\mathbb{R} \cap W^*) \setminus A$, then $W^* \models$ "$x \notin p[\pi(T)]$". But $\pi(A \cap W) = \pi(\{x \in W : W \models x \in p[T]\})$ $= \{x \in W^* : W^* \models x \in p[\pi(T)]\} = A \cap W^*$.

Theorem $(V = L(\mathbb{R}) \models AD)$
For all $x \in \mathbb{R}$, for all $A \subseteq \mathbb{R}$, there is some $p = (p; \in, I)$ generically iterable s.t. $x \in p$, p is A-iterable, and $(HC^p; \in, A \cap p) \prec_{\Sigma_1} (HC; \in, A)$.

Proof:
Let's prove the statement which we need for the application, i.e., for the proof of $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_g$.

We had $A = \{(r, \xi, h) : r \Vdash \tau(\check{\xi}) = \check{h}\}$. Let $B = \{(q, r, \xi, h) : r \leq q \wedge (r, \xi, h) \in A\}$.

Suppose there is no $A \subseteq \mathbb{R}$ as above s.t. for no $x \in \mathbb{R}$ there is $p = (p; \in, I)$ generically iterable, $x \in p$, p is A-iterable, and $p \models \forall \xi < \omega_1 \, \forall q \, \exists r \leq q \, \exists h \, (r, \xi, h) \in A$.

So $\exists A \subseteq \mathbb{R} \, \exists B \subseteq \mathbb{R} \, (\exists x \in \mathbb{R} \, \forall p$ either $p = (p; \in, I)$ is not A-iterable, or $x \notin p$, or $(q, r, \xi, h) \in B \leftrightarrow r \leq q \wedge (r, \xi, h) \in A$ and $p \not\models \forall q \, \forall \xi \, \exists r \, \exists h \, (q, r, \xi, h) \in B \cap p)$

The part in brackets has all quantifiers bounded to $\mathbb{R}$.

By theorems 1+2, have witnesses A, B which are Suslin, co-Suslin. Say $A = p[T]$, $\mathbb{R} \setminus A = p[U]$, $B = p[T']$, $\mathbb{R} \setminus B = p[U']$.

By Theorem 3, we have a real y s.t. $\varkappa = \omega_1^{L[y, T, U, T', U']}$ is measurable in $W = HOD_{x, T, U, T', U'}^{L[y, T, U, T', U']}$. Let k be $Col(\omega, <\varkappa)$-generic over W.

Look at $W[k]$. $x \in W[k]$, and by the argument from last time, $W[k]$ is A-iterable.
Let $q \in W[k]$ and $\xi < \varkappa = \omega_1^{W[k]}$. Write $(T')_{q, \xi} = \{(s, h, \vec{\alpha}) : (q \upharpoonright lh(s), s, \xi \upharpoonright lh(s), h, \vec{\alpha}) \in T'\}$, $(T')_{q, \xi} \in W[k]$. $(T')_{q, \xi}$ is ill-founded in V, so it's ill-founded in $W[k]$. So in $W[k]$ there are $r, h, \vec{\alpha}$ s.t. $(q, r, \xi, h, \vec{\alpha}) \in [T']$, i.e.,

$$(q, r, \xi, h) \in p[T'], \text{ i.e.,}$$
$$(q, r, \xi, h) \in \underbrace{B \cap W[k] \in W[k]}_{\text{from the argument last time}}$$

We reached a contradiction!

Theorem 3 $(V = L(\mathbb{R}) \models AD)$
Let T be a set of ordinals, and let $x \in \mathbb{R}$. There is a cone of reals y s.t.

$$HOD_{x, T}^{L[y, T]} \models \text{“}\omega_1^{L[y, T]} \text{ is measurable”}.$$

- $C \subseteq \mathbb{R}$ is a cone iff $\exists x \in \mathbb{R} \, (C = \{y \in \mathbb{R} : y \geq_T x\})$.

By Martin, under AD:
If $A \subseteq \mathbb{R}$ is Turing-invariant (i.e. $x \in A \wedge y \equiv_T x \rightarrow y \in A$), then there is a cone C s.t. $C \subseteq A$ or $C \cap A = \emptyset$.

• $C \subseteq \mathbb{R}$ is a T-cone iff $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x \in L[T,y] \to y \in C)$

Proof:

Now fix $T \subseteq OR$, $x \in \mathbb{R}$.

We first claim that there is a T-cone of y's s.t. in $L[T,y]$, every $OD_{T,x}$-set of reals is determined.

The set A of all y s.t. $L[T,y] \models$ "$OD_{T,x}$-determinacy holds" is T-invariant in the sense: if $y \in A$ and $L[T,y] = L[T,y']$, then $y' \in A$.

Therefore, by the proof of Martin's theorem, if the claim is false, then there is a T-cone of y s.t. $L[T,y] \models$ "$OD_{T,x}$-det. is false".

In $L[T,y]$, let A_y be the "least" non-determined $OD_{T,x}$-set. $A_{y'} = A_y$ if $L[T,y'] = L[T,y]$. Write G_{A_y} for the usual game associated to A_y.

Look at:

$$G: \begin{array}{c|cc} \mathrm{I} & z, a & \\ \hline \mathrm{II} & & y, b \end{array} \qquad \mathrm{I} \text{ wins iff } a \oplus b \in A_{z \oplus y}.$$

G is determined.

• Say I has a winning strategy, σ, in G. Pick $\sigma \in L[T,y]$, where y is from the above cone. We get a contradiction by showing $L[T,y] \models$ "A_y is determined".

Work in $L[T,y]$. Play a G_{A_y}. Say II plays b. Player I imagines a play of G in which II plays $y \oplus b, b$ and I replies by z, a according to σ.

$$G_{A_y} \begin{array}{c|cc} \mathrm{I} & a & \\ \hline \mathrm{II} & & b \end{array} \qquad\qquad G \begin{array}{c|cc} \mathrm{I} & \overset{\sigma * (y \oplus b, b)}{\widetilde{z, a}} & \\ \hline \mathrm{II} & & y \oplus b, b \end{array}$$

As σ wins, $a \oplus b \in A_{z \oplus (y \oplus b)}$. Have $b, \sigma, z \in L[T,y]$, so $z \oplus (y \oplus b) \in L[T,y]$. $y \in L[T, z \oplus (y \oplus b)]$ is trivial. So $A_{z \oplus (y \oplus b)} = A_y$. So the strategy described is in $L[y,T]$ a winning strategy in G_{A_y}. ↯.

• If II has a winning strategy in G then we get a ↯ in a similar fashion.

Now say $L[T,y] \models$ "$OD_{T,x}$-determinacy holds true". Work in $L[T,y]$.

If $z \in \mathbb{R}$, write $|z| = \sup\{ \|z'\| : z' \equiv_T z \wedge z' \in \overset{\text{codes for wellorders}}{\mathrm{WORD}} \}$.

Write $S = \{ |z| : z \in \mathbb{R} \}$. Let $\pi : \omega_1 \to S$ be order preserving. Define a measure μ by: If $A \subseteq \omega_1$, A $OD_{T,x}$. Say that $A \in \mu$ iff $\{ z \in \mathbb{R} : |z| \in \pi''A \}$ contains a cone.

$\mu \in OD_{T,x}$, so $\mu \in HOD_{T,x}$. μ is an ultrafilter in $HOD_{T,x}$. μ is σ-complete, so $HOD_{T,x} \models$ "μ is $< \omega_1^{L[y,T]}$-complete". So μ witnesses that $HOD_{T,x} \models \omega_1^{L[y,T]}$ is measurable. □

Theorem 1 (V = L(ℝ) ⊨ AD)

Every true Σ^2_1-statement is witnessed by some $A \subseteq \mathbb{R}$ which is a Δ_1 singleton, i.e., there are φ and ψ, Σ_1 formulae, s.t. $x \in A \Leftrightarrow \varphi(x, \mathbb{R}) \Leftrightarrow \neg\psi(x, \mathbb{R})$.

Theorem 2 (V = L(ℝ) ⊨ AD)

If A is $\Sigma_1(\{\mathbb{R}\})$, then A is Suslin, i.e., there is some $\alpha \in OR$ and some tree T on $\omega \times \alpha$ s.t. $A = p[T] = \{x \in {}^\omega\omega : \exists(\beta_n : n<\omega)\ \forall k\ (x \restriction k, (\beta_0, \dots, \beta_k)) \in T\}$

- Let's assume $V = L(\mathbb{R}) \models AD$ for now. Theorems 1 and 2 follow from the fact:

Theorem 0

The pointclass $\Sigma_1(\{\mathbb{R}\})$ has the scale property.

Theorem 0 gives that if $A \subseteq \mathbb{R}$ is $\Sigma_1(\{\mathbb{R}\})$, $A \neq \emptyset$, then there is some $\{x\} \subseteq A$ s.t. $\{x\}$ is $\Sigma_1(\{\mathbb{R}\})$

How does this consequence produce Theorem 1?

Suppose $\exists A \subseteq \mathbb{R}\ \underset{\Sigma_0}{\varphi}(A, \mathbb{R})$ holds true in $L(\mathbb{R})$. Let α be least such that $J_\alpha(\mathbb{R}) \models \exists A \subseteq \mathbb{R}\ \varphi(A, \mathbb{R})$. (Could also work with $\alpha = \delta^2_1$ = the least δ with $J_\delta(\mathbb{R}) \prec^{\mathbb{R}}_{\Sigma_1} L(\mathbb{R})$).

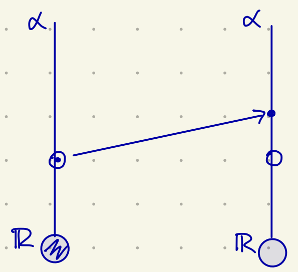
$\boxed{\text{The indices of the } J_\alpha(\mathbb{R}) \text{ hierarchy are always limit ordinals}}$

Let's define $h : \mathbb{R} \rightharpoonup J_\alpha(\mathbb{R})$ as follows.

$h(\underset{\text{Gödel number of a } \Sigma_1 \text{ formula}}{\ulcorner\psi\urcorner}, \underbrace{z, \vec{x}}_{\in \mathbb{R}}) = y$ iff $\exists \beta < \alpha\ \exists \vec{\gamma} < \beta$ [y is definable over $S_\beta(\mathbb{R})$ from $z, \vec{\gamma}$ as parameters $\wedge\ y \in S_\beta(\mathbb{R}) \wedge S_\beta(\mathbb{R}) \models \psi(y, \vec{x}) \wedge$ for all $\vec{\eta} <^* \vec{\gamma}$ and for all $y' \in S_\beta(\mathbb{R})$ definable over $S_\beta(\mathbb{R})$ for $z, \vec{\eta}$, $S_\beta(\mathbb{R}) \not\models \psi(y', \vec{x})$].

So h is partial and $\Sigma_1^{J_\alpha(\mathbb{R})}(\{\mathbb{R}\})$. Easy exercise: $\mathrm{ran}(h) \prec^{\mathbb{R}}_{\Sigma_1} J_\alpha(\mathbb{R})$.

So by condensation, $J_{\bar\alpha}(\mathbb{R}) \overset{\sigma}{\cong} \mathrm{ran}(h) \prec^{\mathbb{R}}_{\Sigma_1} J_\alpha(\mathbb{R})$, some $\bar\alpha \leq \alpha$. By the choice of α, $\bar\alpha = \alpha$.



σ is Σ_1-elementary. Then σ is the identity.

In other words, $h : \mathbb{R} \twoheadrightarrow J_\alpha(\mathbb{R})$ is surjective.

Now let $B \subseteq \mathbb{R}$ be defined by $x \in B$ iff $J_\alpha(\mathbb{R}) \models \varphi(\overset{\Sigma_1}{h}(x), \mathbb{R})$

(iff $J_\alpha(\mathbb{R}) \models \exists y\ (y = h(x) \wedge \varphi(y, \mathbb{R}))$)

(Our hypothesis was $J_\alpha(\mathbb{R}) \models \exists A \subseteq \mathbb{R}\ \underset{\Sigma_0}{\varphi}(A, \mathbb{R})$). So B is $\Sigma_1(\{\mathbb{R}\})$.

Now we can apply the "basis theorem", i.e., we get some $\{x_0\} \subseteq B$ s.t. $\{x_0\}$ is $\Sigma_1^{J_\alpha(\mathbb{R})}$, i.e., have $x = x_0$ iff $J_\alpha(\mathbb{R}) \models \overset{\Sigma_1}{\psi}(x)$

iff $J_\alpha(\mathbb{R}) \models \underbrace{\forall x'(\psi(x') \to x' = x)}_{\Pi_1}$

So $\{x_0\}$ is $\Delta_1^{J_\alpha(\mathbb{R})}$.

The existence of x_0 follows from the fact that $\Sigma_1^{J_\alpha(\mathbb{R})}$ has the scale property (if α was chosen as we did, and from the scale property of $\Sigma_1^{L(\mathbb{R})}$ if $\alpha = \underset{\sim}{\delta}_1^2$).

Let $A^* = h(x_0)$. Have $J_\alpha(\mathbb{R}) \models \psi(A^*, \mathbb{R})$. Also,

$x \in A^*$ iff $\exists \bar{x} \in \mathbb{R}\, (J_\alpha(\mathbb{R}) \models \psi(\bar{x}) \wedge \exists \bar{A} \subseteq \mathbb{R}\, (\bar{A} = h(\bar{x}) \wedge x \in \bar{A}))$ $\rightarrow \Sigma_1^{J_\alpha(\mathbb{R})}$

iff $\forall \bar{x} \in \mathbb{R}\ \forall \bar{A} \subseteq \mathbb{R}\ (J_\alpha(\mathbb{R}) \models \psi(\bar{x}) \wedge \bar{A} = h(\bar{x}) \rightarrow x \in \bar{A})$ $\rightarrow \Pi_1^{J_\alpha(\mathbb{R})}$

Definition: Let $A \subseteq \mathbb{R}$. A scale on A is a sequence $(f_n : n < \omega)$ s.t. $f_n : A \rightarrow OR$ (norms) for all $n < \omega$ and s.t. if $\{x_m : m < \omega\} \subseteq A$ is s.t. $\underbrace{x_m \nearrow x \mod \vec{f}}$, then $x \in A$ and $f_n(x) \leq$ the eventual value of $f_n(x_m)$ as $m \rightarrow \infty$.

meaning that $x_m \nearrow x$ and $\forall n\ f_n(x_m)$ is eventually constant.

If A admits a scale, then $A = p[T]$, some T on $\omega \times \alpha$ (some α), i.e. A is Suslin.

- Why? Let $\alpha = \sup_n (\sup f_n)$ and $(s, \vec{\gamma}) \in T$ iff $s \in {}^{<\omega}\omega$, $\vec{\gamma} = (\gamma_0, \ldots, \gamma_{lh(s)-1}) \in {}^{lh(s)}\alpha$, $\exists x \supseteq s, x \in {}^\omega\omega$ $\forall n < lh(s)\ f_n(x) = \gamma_n$.

  T is called the tree from the scale.

For $\Sigma_1^{L(\mathbb{R})}$ (or $\Sigma_1^{J_\alpha(\mathbb{R})}$, α as in the proof before) to have the scale property you need scales on elements of $\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L(\mathbb{R})}$ (or $\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{J_\alpha(\mathbb{R})}$) with some definability condition on the prewellorderings associated with the "norms" from the scale.

- Modulo Moschovakis, how do Martin-Steel get the scale property on $\Sigma_1^{L(\mathbb{R})}$ (or $\Sigma_1^{J_\alpha(\mathbb{R})}$)?

Definition: Let $A \subseteq \mathbb{R}$. Say that A admits a closed game representation iff for all $x \in {}^\omega\omega$, have G_x:

I	x_0, β_0		x_2, β_1	$\ldots$
II		x_1		$x_3 \quad \ldots$

$x_i \in {}^\omega\omega$, $\beta_i \in OR$. G_x is closed and continuously associated with x in the following sense:

There is $\bar{A}$ s.t. I wins a play of G_x iff $\forall n < \omega\ \bar{A}(x \restriction n, (x_0 \restriction n, \ldots, x_n \restriction n), (\beta_0, \ldots, \beta_n))$, for all n.

Define $(x, n) \in A_k$ iff n is a position of length k in G_x from which on I has a winning strategy.

Want $A = \{x : (x, \emptyset) \in A_0\}$.

Using Moschovakis' arguments, if A admits a closed game representation, then A admits a scale.

Let A be $\Sigma_1^{J_\alpha(\mathbb{R})}$ (α as in the proof before). How do you get a closed game representation for A?

$\Sigma_1^{J_\alpha(\mathbb{R})}$ has the scale property:

- Proof of Theorem 2.1 in Steel "Scales in $L(\mathbb{R})$" produces a close game representation of $\Sigma_1^{J_\alpha(\mathbb{R})}$ sets.
- Construction in the proof of Theorem 2.2. in Moschovakis' "Scales on coinductive sets". This gives scales.
  This construction makes use of the proof of the 2^{nd} periodicity theorem 6C.3 in Moschovakis "Descriptive set theory"
  4E.3
- The rest is classical, e.g. Uniformization

Theorem

Let g be $\mathbb{P}_{max}$-generic on $L(\mathbb{R})$, where $L(\mathbb{R}) \models AD$. Then:

a) $L(\mathbb{R})[g] = L(\mathbb{R})[A_g] \models ZFC + 2^{\aleph_1} = \aleph_2$.

b) $L(\mathbb{R})[g]$ has the same cardinals as $L(\mathbb{R})$ outside of (ω_2, Θ).

c) $\underset{\sim}{\delta}{}_2^1 = \omega_2$ in $L(\mathbb{R})[g]$ $\quad (\Rightarrow 2^{\aleph_0} = \aleph_2)$

d) In $L(\mathbb{R})[g]$, NS_{ω_1} is saturated

Proof

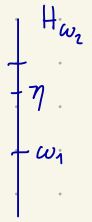
a) We first show Ψ_{AC} in $L(\mathbb{R})[g]$ where

Ψ_{AC}: for all $S, T \subseteq \omega_1$ stationary and costationary, there is some f_η, $\eta < \omega_2$, s.t. for club many $\xi < \omega_1$, $\xi \in S \leftrightarrow f_\eta(\xi) \in T$.

Here, $(f_\eta : \eta < \omega_2)$ is "the" sequence of canonical functions defined as follows: Let $\eta < \omega_2$. Let $f: \omega_1 \to \eta$ onto. Let $f_\eta(\xi) = otp(f''\xi)$ for $\xi < \omega_1$. So $f_\eta : \omega_1 \to \omega_1$. If $\bar{f}: \omega_1 \to \eta$ is also onto, then $\{\xi : f''\xi = \bar{f}''\xi\}$ contains a club. In particular, $f_\eta(\xi) = otp\ \bar{f}''\xi$ on a club.

We may also define f_η as follows. Fix $\eta < \omega_2$.

(Let $(X_i : i \leq \omega_1)$ be a continuous + increasing system of $X \prec H_{\omega_2}$ with $\eta \in X_0$, and X_i is countable for $i < \omega_1$.
Then $f_\eta(\xi) = otp(X_\xi \cap \eta)$. This defines the same function modulo a club.)



Let us prove that $\Psi_{AC} \Rightarrow 2^{\aleph_1} = \aleph_2$.

Write $\omega_1 = \dot{\bigcup}_{i<\omega_1} S_i$, every S_i stationary, $S_i \cap S_j = \emptyset$ for $i \neq j$. We can do that in $L(\mathbb{R})[A_g]$, as A_g yields $(\ell_\alpha : \alpha < \omega_1)$ where $\ell_\alpha : \omega \to \alpha$ cofinal for all α. Then Solovay's proof yields such a decomposition of ω_1.

Let T be stationary costationary. Let $X \subseteq \omega_1$. Write $S_X = \bigcup\{S_i : i \in X\}$. If $X \neq \emptyset, \omega_1$, then S_X is stationary costationary.

$\Psi_{AC} \Rightarrow$ have $\eta < \omega_2$ s.t. on a club $\xi \in S_X$ iff $f_\eta(\xi) \in T$. (Write η_X for the smallest such η)

- We claim that $X \mapsto \eta_X$ is injective. $(\Rightarrow 2^{\aleph_1} = \aleph_2)$.

Let $X \neq Y$, $\eta_X = \eta_Y = \eta$. We have clubs C, D s.t.

$$\forall \xi \in C \ (\xi \in S_X \Leftrightarrow f_\eta(\xi) \in T)$$
$$\forall \xi \in D \ (\xi \in S_Y \Leftrightarrow f_\eta(\xi) \in T)$$

Say $i \in X \setminus Y$. Let $\xi \in S_i \cap C \cap D$. Then $\underbrace{\xi \in S_i}_{\Downarrow} \Rightarrow f_\eta(\xi) \in T \Rightarrow \xi \in S_Y = \bigcup \{S_j : j \in Y\}$

$\xi \notin S_j$ for all $j \neq i$, in part., $\xi \notin S_Y$. $\lightning$

If you look at the proof, this shows:

$L(\mathbb{R})[g] = L(\mathbb{R})[A_g]$ has a well-order of $\mathcal{P}(\omega_1)$ (of order type ω_2), in particular, there is a well-order of $\mathbb{R}$.

But in $L(\mathbb{R})[A_g]$, everything is <u>ordinal</u> definable from <u>reals</u> and <u>A_g</u>; so $L(\mathbb{R})[g] \models ZFC$.

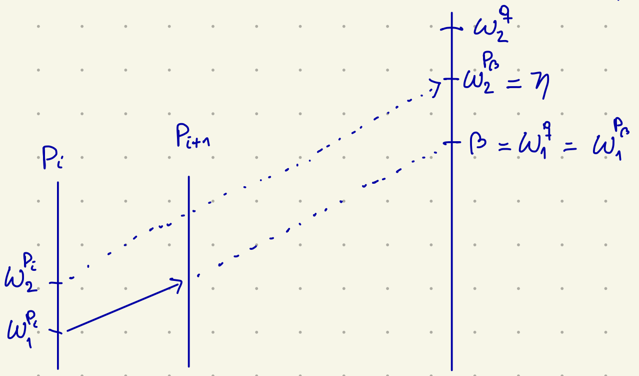
It remains to show ψ_{AC} in $L(\mathbb{R})[g]$.

Let us fix $S, T \in \mathcal{P}(\omega_1) \cap L(\mathbb{R})[g]$, stationary costationary. As $\mathcal{P}(\omega_1) \cap L(\mathbb{R})[g] = \mathcal{P}(\omega_1)_g$, there is some $p \in g$ together with some $p \to p^*$, an iteration as being given by g, s.t. $S, T \in p^*$. Will have, for $\alpha = \omega_1^p$, $S \cap \alpha$, $\alpha \setminus S$, $T \cap \alpha$, $\alpha \setminus T \in (I^+)^p$.

Let $q \in \mathbb{P}_{max}$, $p \in q$. Let $\beta = \omega_1^q$. Let's define an iteration $(p_i, \pi_{ij} : i \leq j \leq \beta) \in q$ of p s.t. $I^{p_\beta} = I^q \cap p_\beta$. We may actually choose the iteration in such a way that for all $i < \beta$, if $i = \omega_1^{p_i} \in \pi_{0i+1}(S \cap \alpha) \Rightarrow \omega_1^{p_{i+1}} \in \pi_{0i+2}(T \cap \alpha)$; and
if $i = \omega_1^{p_i} \in \pi_{0i+1}(\alpha \setminus S) \Rightarrow \omega_1^{p_{i+1}} \in \pi_{0i+2}(\alpha \setminus T)$.

Change a^q in such a way that $(p_i, \pi_{ij} : i \leq j \leq \beta)$ witnesses $p >$ <u>the new q.</u>
replace a^q by a^{p_β}

Assume that $I^p = (NS_{\omega_1})^p$, and I^p is saturated in p. Then $\omega_1^{p_{i+1}} = \omega_2^{p_i}$.



Let's look at f_η, as being defined in q. We may assume that f_η is defined via $(\mathrm{ran}(\pi_{i\beta}) : i < \beta)$, i.e. $f_\eta(i) = \mathrm{otp}(\mathrm{ran}(\pi_{i\beta}) \cap \eta) = \omega_2^{p_i}$.

Therefore, for $\boxed{i = \omega_1^{p_i}}$, $\left[\begin{array}{l} i \in \pi_{0\beta}(S \cap \alpha) \Leftrightarrow f_\eta(i) = \omega_2^{p_i} \in \pi_{0\beta}(T \cap \alpha) \\ i \in \pi_{0\beta}(\alpha \setminus S) \Leftrightarrow f_\eta(i) = \omega_2^{p_i} \in \pi_{0\beta}(\alpha \setminus T) \end{array}\right.$

Then there is such a q in g. $q \vDash \exists \eta < \omega_2 \ \exists \text{ club } C \ \forall i \in C \ (i \in (S \cap \beta) \Leftrightarrow f_\eta(i) \in (T \cap \beta))$

as being given by g

$q^* \vDash \exists \eta < \omega_2 \ \exists \text{ club } C \ \forall i \in C \ (i \in S \Leftrightarrow f_\eta(i) \in T)$

13.12.21

Theorem

Let g be $\mathbb{P}_{max}$-generic on $L(\mathbb{R})$, where $L(\mathbb{R}) \models AD$. Then:

a) $L(\mathbb{R})[g] = L(\mathbb{R})[A_g] \models ZFC + 2^{\aleph_1} = \aleph_2$.

b) $L(\mathbb{R})[g]$ has the same cardinals as $L(\mathbb{R})$ outside of $(\omega_2, \Theta^{L(\mathbb{R})})$, and every $L(\mathbb{R})$-cardinal $\kappa \in (\omega_2, \Theta^{L(\mathbb{R})})$ has cardinality $\aleph_2$ in $L(\mathbb{R})[g]$.

c) $\underset{\sim}{\delta}^1_2 = \omega_2$ in $L(\mathbb{R})[g]$ $(\Rightarrow 2^{\aleph_0} = \aleph_2)$

d) In $L(\mathbb{R})[g]$, NS_{ω_1} is saturated

In $L[A_g]$, we may pick $(l_\alpha : \alpha < \omega_1)$, $l_\alpha : \omega \to \alpha$ cofinal (for α a lim.)

Now work in $L(\mathbb{R})[g]$ and run the Solovay proof.

$$\boxed{\underset{\substack{\cap \\ g}}{p} = (p; \in, I, a) \xrightarrow[\text{given by } g]{} (p^*; \in, I^{p^*}, A_g)}$$

$p \models$ "$\omega_1^{L[a]} = \omega_1$" $\qquad$ $p^* \models$ "$\omega_1^{L[A_g]} = \omega_1$"

$\omega_1^{p^*} = \omega_1^{L(\mathbb{R})} = \omega_1^{L(\mathbb{R})[g]}$.

Proof of b):

- $\omega_1^{L(\mathbb{R})[g]} = \omega_1^{L(\mathbb{R})}$, as $\mathbb{P}_{max}$ is σ-closed.

- $\omega_2^{L(\mathbb{R})[g]} = \omega_2^{L(\mathbb{R})}$: Let $\alpha < \omega_2^{L(\mathbb{R})[g]}$. Let $X \subseteq \omega_1$ coding a well-order of order type α.

We know: $X \in p^*$, where $\underset{\substack{\cap \\ g}}{p} \longrightarrow p^*$.
generic iteration of p as being given by g of length ω_1

We showed: if $\underline{x \in \mathbb{R} \cap L(\mathbb{R})}$ codes p, then $\underline{\alpha < p^*} \cap OR < \omega_1^{+L[x]} < (\underset{\sim}{\delta}^1_2)^{L(\mathbb{R})} = \omega_2^{L(\mathbb{R})}$.
this follows from AD.

So $\omega_2^{L(\mathbb{R})} = \omega_2^{L(\mathbb{R})[g]}$

- Every $L(\mathbb{R})$-cardinal $\kappa \in (\omega_2, \Theta)$ has cardinality $\aleph_2$ in $L(\mathbb{R})[g]$:

By definition, $\Theta = \sup\{\alpha : \exists f : \mathbb{R} \to \alpha \text{ onto}\}$. Let $\alpha < \Theta^{L(\mathbb{R})}$. Let $f : \mathbb{R} \to \alpha$ onto, $f \in L(\mathbb{R})$. We showed $2^{\aleph_1} = \aleph_2$, so $2^{\aleph_0} \leq \aleph_2$, inside $L(\mathbb{R})[g]$. So in $L(\mathbb{R})[g]$, have $h : \omega_2 \to \mathbb{R}$ onto. Then $f \circ h : \omega_2 \to \alpha$ is surjective, so $\overline{\overline{\alpha}} \leq \aleph_2$ in $L(\mathbb{R})[g]$.

Now let us show that $\Theta^{L(\mathbb{R})}$ remains a cardinal in $L(\mathbb{R})[g]$. The same argument will actually show that all $L(\mathbb{R})$-cardinals $\geq \Theta^{L(\mathbb{R})}$ remain cardinals in $L(\mathbb{R})[g]$.

Before we continue, let's prove c).

c). In $L(\mathbb{R})$, $\underset{\sim}{\delta}^1_2 = \omega_2$. $\mathbb{P}_{max}$ doesn't add reals and it doesn't collapse ω_2, so $\underset{\sim}{\delta}^1_2 = \omega_2$ in $L(\mathbb{R})[g]$. So CH is false, so by $2^{\aleph_1} = \aleph_2$: $2^{\aleph_0} = \aleph_2$.

- To show $\Theta^{L(\mathbb{R})}$ remains a cardinal in $L(\mathbb{R})[g]$:

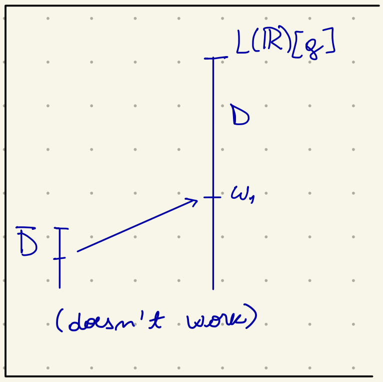
Let $f: \mathbb{R} \to \Theta^{L(\mathbb{R})}$, $f \in L(\mathbb{R})[g]$. We need to see that f is not surjective.
Let $f = \tau^g$, $\tau \in L(\mathbb{R})^{\mathbb{P}_{max}}$. Define $F: \mathbb{R} \times \mathbb{R} \to \Theta^{L(\mathbb{R})}$
$(p, x) \mapsto \alpha$ if p is (codes a) $\mathbb{P}_{max}$ condition and $p \Vdash \tau(\check{x}) = \alpha$.

$\mathrm{ran}(F) \supseteq \mathrm{ran}(f)$, and $F \in L(\mathbb{R})$, so that by the definition of Θ, there is some $\gamma < \Theta^{L(\mathbb{R})}$ with $\mathrm{ran}(f) \subseteq \mathrm{ran}(F) \subseteq \gamma$, so f is not cofinal, in particular not surjective.

Now let $\kappa > \Theta^{L(\mathbb{R})}$ be an $L(\mathbb{R})$-cardinal. Suppose κ is least s.t. κ is <u>not</u> an $L(\mathbb{R})[g]$-cardinal, so $\kappa = \lambda^{+L(\mathbb{R})}$, some λ. In $L(\mathbb{R})[g]$, pick $f: \lambda \to \kappa$ onto, $f \in L(\mathbb{R})[g]$.
$f = \tau^g$, $\tau \in L(\mathbb{R})^{\mathbb{P}_{max}}$. Define $F: \mathbb{R} \times \lambda \to \kappa$
$(p, \alpha) \mapsto \beta$ iff $p \in \mathbb{P}_{max}$ and $p \Vdash \tau(\check{\alpha}) = \check{\beta}$.
Again, $\mathrm{ran}(F) \supseteq \mathrm{ran}(f)$, $F \in L(\mathbb{R})$.

(To be continued next time)

D). Let $D \in L(\mathbb{R})[g]$ be dense in $NS^+_{\omega_1}$. Let $\tau^g = D$.



$p \in g$, $S \in (I^+)^p$. Let $i: p \to p^*$ be the iteration of length ω_1 as being given by g. Showed: $i(S)$ is stationary in $L(\mathbb{R})[g]$.
So have $T \in D$, $T \subseteq i(S)$.

By $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_g$, have some $q < p$, $q \in g$, s.t. if $j: q \to q^*$ is as being given by g, then $T \in q^* \cap D$, $T \subseteq i(S)$, $T = j(\bar{T})$, some $\bar{T}$.

Pick $x \in \mathbb{R} \cap q$ which codes $\bar{T}$ modulo $\dot{a}^q$ (using $q \models MA_{\omega_1}$). That also means x codes T modulo A_g. So we have $x \in \mathbb{R} \cap q$ s.t. if x codes T' via $A_g = \dot{A}^g$, then $T' \in D$, $T' \subseteq j(S)$, $q \Vdash$ "if $\check{x}$ codes T' via $\dot{A}$, then $T' \in \tau$."

$$B = \left\{ (p, s, x, q) : \begin{array}{l} p, q \in \mathbb{P}_{max},\ q < p,\ \text{let } i: p \to p^* \in q \text{ witness } q < p,\ s \in (I^+)^p,\ q \text{ has} \\ \text{some } x \in \mathbb{R},\ q \Vdash \text{"if } \check{x} \text{ codes } T' \text{ via } \dot{A}, \text{ then } T' \in \tau\text{"},\ \text{and} \\ q \models \text{"if } x \text{ codes } \bar{T}' \text{ via } \dot{a}, \text{ Then } i(S) \supseteq \bar{T}',\ \bar{T}' \in (I^+)^q\text{"} \end{array} \right\}$$

We showed: $\forall p, s$, $p \in \mathbb{P}_{max}$, $s \in (I^+)^+$ there are x, q s.t $(p, s, x, q) \in B$.

Let's now pick $r \in g$, r is B-iterable, $(HC^r; \in, B \cap r) \prec (HC; \in, B)$. Now build a filter $K \in r$ s.t. if $p \in K$, $s \in (I^p)^+$, have $x \in q \in K$ s.t. $(p, s, x, q) \in B$.

By changing $\dot{a}^r$ if necessary, may assume that $p > r$ for all $p \in K$.
Let $i: r \to r^*$ be the generic iteration of length ω_1 as being given by g. If $p \in i(K)$, $s \in (I^+)^p$, we have $x \in q \in i(K)$ s.t. $(p, s, x, q) \in i(B \cap r) = B \cap r^* \subseteq B$.

$i(K) \subseteq g$. So if $j: p \to p^*$ is as being given by g, $j(s) \supseteq$ the T coded by x, which in turn is an element of $\tau^g = D$.

We may build K in a way that there is a club $C \subseteq \omega_1^r$ in r s.t. if $\alpha \in C$, then $\alpha \in S$, where $S = i(\bar{S})$ for some $\bar{S} \in (I^+)^p$, $p \in K$, $\omega_1^p < \alpha$. ($i : p \to p^*$ as being given by K).
In fact, want to have $(p, \bar{S}, x, q) \in B$, $x \in q \in K$, s.t. if x codes T modulo $\ddot{a}^r$, then $\alpha \in T$.

$i : r \to r^*$ as being given by g

In the end, get $i(\overset{\overset{r}{\Downarrow}}{C})$, a club s.t. for all $\alpha \in i(C)$, have $\alpha \in T$, some $T \in D$, but also T has a preimage in some $q \in i(K)$ with $\omega_1^q < \alpha$.

16.12.21

Last time: Let g be $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$, then in $L(\mathbb{R})[g]$: ZFC, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, $\underset{\sim}{\delta}^1_2 = \omega_2$, NS_{ω_1} is saturated.

Also we claimed that the cardinals of $L(\mathbb{R})[g]$ are the cardinals of $L(\mathbb{R})$ outside of $(\omega_2, \Theta^{L(\mathbb{R})})$.

Remaining proof: If $\kappa > \Theta^{L(\mathbb{R})}$ is a cardinal of $L(\mathbb{R})$, then κ is a cardinal in $L(\mathbb{R})[g]$.

Why is this true?

Let $\kappa \geq \Theta^{L(\kappa)}$ be the least cardinal of $L(\mathbb{R})$ which is not a cardinal in $L(\mathbb{R})[g]$. So $\kappa = \lambda^{+L(\mathbb{R})}$. Let $f \in L(\mathbb{R})[g]$, $f : \lambda \to \lambda^{+L(\mathbb{R})}$ be onto, $f \in L(\mathbb{R})[g]$. $f = \tau^g$. Say $p_0 \Vdash \tau : \check{\lambda} \to \check{\lambda}^{+L(\mathbb{R})}$.

Define $h : \mathbb{R} \times \lambda \to \lambda^{+L(\mathbb{R})}$

$$(p, \xi) \mapsto \begin{cases} \text{the } \eta \text{ s.t. } p \Vdash \tau(\check{\xi}) = \check{\eta} \text{ where } p \leq p_0 \text{ (if it exists)} \\ 0 \text{ otherwise} \end{cases}$$

$h \in L(\mathbb{R})$ is surjective.

For all $\xi < \lambda$, $o_\xi = \mathrm{otp}(\{h(x,\xi) : x \in \mathbb{R}\}) < \Theta$.
omicron

Let $(h_\xi : \xi < \lambda)$, $h_\xi : o_\xi \to \lambda^{+L(\mathbb{R})}$ being the inverse of the transitive collapse of $\{h(x,\xi) : x \in \mathbb{R}\}$.

So let $\tilde{h} : \Theta \times \lambda \to \lambda^{+L(\mathbb{R})}$

$$(\zeta, \xi) \mapsto \begin{cases} h_\xi(\zeta) & \text{if } \downarrow \\ 0 & \text{otherwise} \end{cases}$$

This is a surjection. But there is in ZF a surjection $\lambda \to \lambda \times \lambda$ (so also $\lambda \to \Theta \times \lambda$). This gives a contradiction.

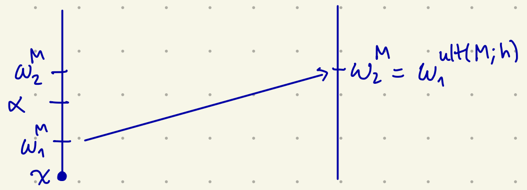
Lemma

Let g be a $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$, and let $A = A_g$. Let $X \in L(\mathbb{R})[g]$, $X \prec (H_\Theta; \in, NS_{\omega_1}, A)^{L(\mathbb{R})[g]}$, X countable; let $j : M \to X$ be the (inverse of the transitive) collapse of X. Then $\underline{M \in g}$ (assuming $H_\Theta \models ZFC$).

Proof:

We first claim that M is iterable. h is $(\mathbb{I}^M)$-generic$/M$.

$M \xrightarrow{i} ult(M; h)$. Have $i(\omega_1^M) = \omega_2^M$. In particular, $\omega_1^{ult} \in wfp(ult)$.
well-founded part



$\underset{\sim}{\delta}^1_2 = \omega_2 \Rightarrow$
for all $\alpha < \omega_2^M$ there is some $x^\# \in M$ s.t. the $(\omega_1^M)^{th}$ - iterate of $x^\#$ has height $\geq \alpha$.

So for every $\alpha < \omega_2^{ult}$ we have some $x^\# \in M$ s.t. the $(\omega_1^{ult})^{th}$-iterate of $x^\#$ has length $\geq \alpha$. This shows that $\omega_2^{ult} \in wfp(ult)$. $\underset{\displaystyle wfp(ult)}{\cap}$

From this it follows that ult is well-founded.

This argument shows that M is iterable.

We know that $g = \{p \in \mathbb{P}_{max} : \exists$ gen. iteration of length ω_1 of p giving rise to $j: p \to p^* = (p^*; \in, NS_{\omega_1}^{L(\mathbb{R})[g]}, A)\}$

$M \to (H_{\omega_2}; \in, NS_{\omega_1}, A)^{L(\mathbb{R})[g]}$.

It follows that $M \in g$.

Lemma

$\mathbb{P}_{max}$ is homogeneous in the following sense:

If $p, q \in \mathbb{P}_{max}$, there are $p^* < p$, $q^* < q$ and an automorphism $\pi : (\mathbb{P}_{max})_{\leq p^*} \to (\mathbb{P}_{max})_{\leq q^*}$

$(\mathbb{P}_{max})_{\leq p^*} = \{r \in \mathbb{P}_{max} : r \leq p^*\}$

Proof:

Let p, q be given. Let $r \in \mathbb{P}_{max}$, $p, q \in r$. Working inside r, we may construct iterations $p \to p' = (p'; \in, I^r \cap p', a^{p'})$

$q \to q' = (q'; \in, I^r \cap q', a^{q'})$

Let $p^* = (r; \in, I^r, a^{p'})$, $q^* = (r; \in, I^r, a^{q'})$. Then $p^* < p$, $q^* < q$. It's then clear how to get an automorphism as desired.

Corollary

If g is $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$, then $L(\mathbb{R})[g]$ does not have a well-order of $\mathbb{R}$ which is $OD_{\mathbb{R}}$.

The well-order of $\mathbb{R}$ we form (using Ψ_{AC}) is $\Sigma_1^{H_{\omega_2}}(\{A_g\})$.

- I'd like to turn to $\Pi_2^{H_{\omega_2}}$-maximality. $L(\mathbb{R})[g]$ satisfies a lot of interesting $\Pi_2^{H_{\omega_2}}$ statements.

Definition: $(*)$ is the statement:

$AD^{L(\mathbb{R})}$, and there is some g $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$ s.t. $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g]$.

We have:

$$\begin{tikzcd}
& & (*) \arrow[dll, Rightarrow] \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[drr, Rightarrow] & & \\
(A) & \mathrm{acg} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[drr, Rightarrow] & & \Psi_{AC} \arrow[d, Rightarrow] & (B) \\
x^\# \text{ exists for all } x \in \mathbb{R} & \delta_2^1 = \omega_2 \arrow[d, Rightarrow] & & \mathrm{CBP} & \\
& \neg \mathrm{CH} & & &
\end{tikzcd}$$

α is x-adm. iff $L_\alpha[x] \models KP$

acg: admissible club guessing: $\forall C \subseteq \omega_1$ club $\exists x \in \mathbb{R}$ $C \supseteq \{$countable x-admissibles$\}$.

CBP: club bounding principle: every function $f: \omega_1 \to \omega_1$ is bounded by a canonical function on a club.

(A): $\forall S \subseteq \omega_1$ stationary costationary there is some $x \in \mathbb{R}$ and some h $Col(\omega, <\omega_1^V)$-gen./$L[x]$ $L[x,S] = L[x,h]$.

(B): $\forall A \subseteq \omega_1\ \exists B \subseteq \omega_1\ (A \in L[B] \wedge \underbrace{B \text{ is amenably closed}}_{\forall D \subseteq \omega_1\ [(\forall \xi < \omega_1\ D \cap \xi \in L[B]) \to D \in L[B]]})$

Definition: Let φ be a statement. We call φ $M_1^\#$-consistent iff for all $x \in \mathbb{R}$ there is a (countable) transitive model M s.t.

- $x \in M$,
- $M \models ZFC$,
- M is closed under $x \mapsto M_1^\#(x)$, and
- $M \models \varphi$

$ZFC + \varphi \vdash_{M_1^\#} \psi$ iff $(\varphi \wedge \neg\psi)$ is not $M_1^\#$-consistent

If $ZFC + \varphi \vdash_{M_1^\#} \psi$, then $ZFC + \varphi \vdash_{\Omega} \psi$ (where $\vdash_\Omega$ refers to Woodin's Ω-logic).

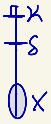
Theorem

Let g be $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$. Let φ be $\Pi_2^{H_{\omega_2}}$, and φ is $M_1^\#$-consistent. Then $L(\mathbb{R})[g] \models \varphi$.

17.01.22

Recall:

- φ is $M_1^\#$-consistent iff f.a. $x \in \mathbb{R}$ there is a (countable) transitive M s.t. $\underbrace{M \models ZFC}_{x \in M}$, M is closed under $\underset{(X \in \mathcal{P}(\omega_1^M) \cap M)}{X \mapsto M_1^\#(X)}$, and $M \models \varphi$
- $M_1^\#(X)$ = the least iterable model N of "there is a measurable cardinal above a Woodin cardinal" s.t. $X \in N$.



Theorem $(AD^{L(\mathbb{R})})$
Let g be $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$, and let φ be $\Pi_2^{H_{\omega_2}}$ and $M_1^\#$-consistent. Then $L(\mathbb{R})[g] \models \varphi$.

In particular, if V is closed under $X \mapsto M_1^\#(X)$, φ is $\Pi_2^{H_{\omega_2}}$ and φ holds in V, then $L(\mathbb{R})[g] \models \varphi$.

Proof:
Suppose $L(\mathbb{R})[g] \models \neg\varphi$. Say $\varphi \equiv \forall X\, \exists Y\, \bar{\varphi}(X,Y)$, where X, Y range over elements of H_{ω_2} and $\bar{\varphi}$ is bounded.

Let's pick $X \in H_{\omega_2}^{L(\mathbb{R})[g]} \cap L(\mathbb{R})[g]$ s.t. $L(\mathbb{R})[g] \models \forall Y\ \neg\bar{\varphi}(X,Y)$, say $X \subseteq \omega_1$, $X \in L(\mathbb{R})[g]$.
By "$\mathcal{P}(\omega_1)_g = \mathcal{P}(\omega_1)$", we have:

$$g \ni p \longrightarrow p^* \ni X$$

the generic iteration of p of length ω_1 as being given by g.

Let $X = \tau^g$, some τ. Have $q \leq p$, $q \in g$, $q \Vdash \dot{\pi}(X \cap \omega_1^q) = \tau \wedge \forall Y\, \neg\bar{\varphi}(\tau, Y)$.

name for the iteration map from the iteration of length ω_1 as being given by the generic

Let $r \leq q$. By the fact that φ is $M_1^\#$-consistent, there is some M, countable, transitive, closed under $X \mapsto M_1^\#(X)$, $r \in M$, $M \models \varphi$.

Working in M, produce a generic iteration of r of length ω_1^M, $r \underset{\text{iteration map}}{\overset{\in M}{\longrightarrow}} r^* = (r^*, \in, (NS_{\omega_1})^M \cap r^*, a^*)$

$$\text{coded by } Z \in \mathcal{P}(\omega_1^M) \cap M \left\{ \begin{array}{l} q \longrightarrow q' \longrightarrow q^* \ni i(X \cap \omega_1^q) \quad (q \xrightarrow{i} q^*) \\ \qquad\quad \cap \qquad\quad \cap \\ \qquad\quad r \longrightarrow r^* = (r^*, \in, (NS_{\omega_1})^M \cap r^*, a^*) \\ \qquad\qquad\quad \cap\, M \end{array} \right.$$

$M \models \exists Y\, \bar{\varphi}(i(X \cap \omega_1^q), Y)$. Pick such a Y, and consider $M_1^\#(Z, Y) \models \bar{\varphi}(i(X \cap \omega_1^q), Y)$.

Let $h \in V$ be generic over $M_1^\#(Z,Y)$ for the forcing to force NS_{ω_1} to be saturated.

Let $\theta <$ the top measurable cardinal of $M_1^\#(Z,Y)$ s.t. θ is inaccessible in $M_1^\#(Z,Y)$.

Then $s = \left(V_\theta^{M_1^\#(Z,Y)[h]}; \in, (NS_{\omega_1})^{M_1^\#(Z,Y)[h]}, a^{r^*}\right) \in \mathbb{P}_{max}$, $s < r$, $s \models \overline{\varphi}(i(X \cap \omega_1^q), Y)$. Contradiction!

Theorem (A proper class of Woodin cardinals)

Let $L^{M_1^\#}(\mathbb{R})$ be the least inner model of ZF which has all the reals and is closed under $X \mapsto M_1^\#(X)$. Let g be $\mathbb{P}_{max}$-generic over $L^{M_1^\#}(\mathbb{R})$. Then $L^{M_1^\#}(\mathbb{R})[g] \models ZFC + BMM^{++}$.

- By way of definition, BMM^{++} is the following statement:

Let φ be Σ_1 in $\mathcal{L}_{\dot\in, \dot{NS}_{\omega_1}}$ (= the 1st order language of set Theory equipped with a predicate for the nonstationary ideal), let $\mathbb{P}$ be a stationary set preserving forcing, and let $A \in H_{\omega_2} \cap V$. If $V^{\mathbb{P}} \models \varphi(A)$, then $V \models \varphi(A)$.

Proof:

Say $A \in \mathcal{P}(\omega_1) \cap \overbrace{L^{M_1^\#}(\mathbb{R})[g]}^{=W}$. Suppose $\mathbb{P} \in L^{M_1^\#}(\mathbb{R})[g]$ preserves stationary subsets of ω_1, and $W^{\mathbb{P}} \models \underset{\uparrow\, \Sigma_1 \text{ in } \mathcal{L}_{\dot\in, \dot{NS}_{\omega_1}}}{\varphi(A)}$.

Suppose $W \models \neg\varphi(A)$. Say $g \ni p \Vdash_{L^{M_1^\#}(\mathbb{R})} \neg\varphi(\underset{\text{name for the iteration as before}}{\dot\pi}(A \cap \omega_1^p))$.

Inside $W^{\mathbb{P} * Col(\omega,\theta)}$ (θ big enough) pick a generic h for the forcing in $M_1^\#(H_{\omega_2}^{W^{\mathbb{P}}})$ to make NS_{ω_1} saturated.

Inside $W^{\mathbb{P} * Col(\omega,\theta)}$, $s = \left(\underset{\text{some } \tau \text{ inaccessible in } M_1^\#(H_{\omega_2}^{W^{\mathbb{P}}}) \text{ above the Woodin}}{W_\tau^{M_1^\#(H_{\omega_2}^{W^{\mathbb{P}}})}}[h]; \in, NS_{\omega_1}^{M_1^\#(H_{\omega_2}^{W^{\mathbb{P}}})[h]}, A_g\right)$. $s \models \varphi(A)$.

Now look at:

"$\exists s < p \; s \models \varphi(\pi(A \cap \omega_1^p))$, where $\pi: p \to p^*$ witnesses $s < p$".

This is true in $W^{\mathbb{P} * Col(\omega,\theta)}$, and it is Σ_3^1 (code for p). So this is true in W. Contradiction!

20.01.22

Goal: Theorem: $MM^{++} \Rightarrow (*)$.

Definition: (Foreman-Magidor-Shelah) MM^{++} is the statement.
Let $\mathbb{P}$ be stationary set preserving, let $\mathcal{D} = \{D_i : i < \omega_1\}$ be a collection of dense sets, and let $\{\tau_i : i < \omega_1\}$ a collection of names of stationary subsets of ω_1 (i.e., $1\!\!1_{\mathbb{P}} \Vdash \tau_i \subseteq \check{\omega}_1$ is stationary). There is then a filter $g \subseteq \mathbb{P}$ s.t.
for all $i < \omega_1$ $\begin{cases} g \cap D_i \neq \emptyset \\ \tau_i^g = \{\xi : \exists p \in g \ p \Vdash \check{\xi} \in \tau_i\} \text{ is stationary.} \end{cases}$

Lemma (folcklore)
TFAE:
(1) MM^{++}
(2) Let $\Theta \geq \omega_1$ regular and let $(H_\Theta; \in, \vec{R})$ be a model. Let φ be Σ_1 in $\mathcal{L}_{\dot{\in}, \dot{NS}_{\omega_1}}$
at most $\aleph_1$ many
Let $\mathbb{P}$ be a stationary set preserving forcing, and assume that $V^{\mathbb{P}} \models \varphi((H_\Theta, \in, \vec{R}))$.
Then in V there is some $j : (H; \in, \vec{S}) \longrightarrow (H_\Theta; \in, \vec{S})$ $(\mathrm{Card}(H) = \aleph_1)$ s.t. $V \models \varphi((H; \in, \vec{S}))$

Proof: (Exercise)

Definition: Let Γ be a collection of universally Baire sets of reals. Γ-BMM^{++} is the following statement:
Let $\mathbb{P}$ be a stationary set preserving forcing, and let $A_1, \dots, A_k \in \Gamma$. Then,
$(H_{\omega_2}; \in, NS_{\omega_1}, A_1, \dots, A_k) \prec_{\Sigma_1} (H_{\omega_2}^{V^{\mathbb{P}}}; \in, NS_{\omega_1}^{V^{\mathbb{P}}}, A_1^*, \dots, A_k^*)$.

Here, a set $A \subseteq \mathbb{R}$ is called universally Baire iff there are trees T, U on $\omega \times OR$ s.t. $A = p[T]$ and for all Θ, $\Vdash^{Col(\omega,\Theta)} p[U] = \mathbb{R} \setminus p[T]$.

If $\mathbb{P}$ is a poset, then $A^* = p[T] \cap V^{\mathbb{P}}$ (this does not depend on the choice of trees).

Lemma:
If MM^{++} holds true, then Γ^∞-BMM^{++} is true, $\Gamma^\infty =$ all universally Baire sets of reals.

Proof: (Exercise)

Theorem (with Asperó) ($ZFC + NS_{\omega_1}$ is saturated $+$ V is closed under $M_\omega^\#$).
TFAE:
(1) $(*)$
(2) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$-$BMM^{++}$

Proof:
(1) $\Rightarrow$ (2) was basically done last time.
(2) $\Rightarrow$ (1):

Recall $(*)$:
$\exists$ $L(\mathbb{R})$-generic filter $g \subseteq \mathbb{P}_{max}$ s.t. $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g]$.

Fix $A \subseteq \omega_1$ s.t. $\omega_1^V = \omega_1^{L[A]}$. Let $g = \left\{ p \in \mathbb{P}_{max} : \exists \text{ gen. iteration } (p_i : i \le \omega_1), (g_i : i < \omega_1) \text{ of } p_0 = p \text{ s.t. } p_{\omega_1} = (p_{\omega_1}; \in, NS^V_{\omega_1} \cap p_{\omega_1}, A) \right\}$

- g is a filter:
  - $q > p \in g \Rightarrow q \in g$ ✓
  - Let $p, q \in g$. $p \longrightarrow p_{\omega_1} = (p_{\omega_1}; \in, NS_{\omega_1} \cap p_{\omega_1}, A) \in (H_{\omega_2}; \in, NS_{\omega_1}, A)$.

    $q \longrightarrow q_{\omega_1} = (q_{\omega_1}; \in, NS_{\omega_1} \cap q_{\omega_1}, A) \in$

Let $X \prec H_\theta$ countable. Let $M \overset{\sigma}{\cong} X$, M transitive.

$\underset{\text{the iterations}}{\ni}$

$\sigma(H; \in, I, a) = (H_{\omega_2}; \in, NS_{\omega_1}, A)$

- If g is $L(\mathbb{R})$-generic, then $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g]$:

  Let $Z \in \mathcal{P}(\omega_1)$. $(p; \in, I, a) = p \cong \underset{\ni Z}{X} \prec (H_{\omega_2}; \in, NS_{\omega_1}, A)$

- g is generic: Let $D \subseteq \mathbb{P}_{max}$ be dense, $D \in L(\mathbb{R})$. Let's identify D with a set of reals in $L(\mathbb{R})$, also denoted by D. D is universally Baire.

Let's work in $V^{Col(\omega,\omega_2)}$ for a while:

Have $V \prec_{\Pi^1_2(D)} V^{Col(\omega,\omega_2)}$.

$$\begin{tikzcd}
 & D^* & \\
 & q \arrow[u, phantom, "\in"] \arrow[r, "j"', "\text{gen. iteration of length } \omega_3^V = \omega_1^{V^{Col(\omega,\omega_2)}}"] & q^* \\
P_0 = (H^V_{\omega_2}; \in, NS^V_{\omega_1}, A) \arrow[r, "\text{gen. iteration of length } \alpha = \omega_1^q"'] & p_\alpha \arrow[u, phantom, "\in"] \arrow[r, "i"] & p^* = (H^M_{\omega_2}; \in, NS^M_{\omega_1}, i(A)) \arrow[u, phantom, "\in"] \\
V \arrow[u, phantom, "\in"] \arrow[rr, "\tilde{\imath}"] & & M \arrow[u, phantom, "\in"]
\end{tikzcd}$$

$i(A) = j(A) = \tilde{\imath}(A)$

$q \in p[\tilde{\imath}(T)]$.

$M^{Col(\omega,\omega_2^M)} \models$ "there is some gen. iteration $(p_\kappa : \kappa \le \tilde{\imath}(\omega_1)), (g_\kappa : \kappa < \tilde{\imath}(\omega_1))$ with last model $p_{\tilde{\imath}(\omega_1)} = (H^V_{\omega_2}; \in, NS^M_{\omega_1}, \tilde{\imath}(A))$ and there is some generic iteration $(q_\kappa : \kappa \le \tilde{\imath}(\omega_1)), (g'_\kappa : \kappa < \tilde{\imath}(\omega_1))$ and some countable α, s.t. $(p_\kappa : \kappa \le \alpha), (g_\kappa : \kappa < \alpha) \in q_0$, $q_0 \in p[\tilde{\imath}(T)]$, and $q_{\tilde{\imath}(\omega_1)} = (q_{\tilde{\imath}(\omega_1)}; \in, I, \tilde{\imath}(A))$, some I."

$M^{Col(\omega,\omega_2^M)} \subseteq V^{Col(\omega,\omega_2^M)}$. This is true by Schoenfield.

Now we use $\tilde{\imath}$ to conclude:

$V^{Col(\omega,\omega_2)} \models$ "there is some gen. iteration $(p_\kappa : \kappa \le \omega_1)$ with last model $p_{\omega_1} = (H^V_{\omega_2}; \in, NS_{\omega_1}, A)$ and there is some generic iteration $(q_\kappa : \kappa \le \omega_1)$ and some countable α, s.t. $(p_\kappa : \kappa \le \alpha) \in q_0$, $q_0 \in p[T]$ and $q_{\omega_1} = (q_{\omega_1}; \in, I, A)$ some I."

In $V^{Col(\omega,\omega_2)}$:

$$\begin{tikzcd}
 & D^* = p[T] & \\
 & q_0 \arrow[u, phantom, "\in" description] \arrow[r] & q_{\omega_1} \\
P_0 \arrow[r] & P_\alpha \arrow[u, phantom, "\ni" description] \arrow[r] & P_{\omega_1} = (H_{\omega_2}; \in, NS_{\omega_1}, A) \arrow[u, phantom, "\ni" description] \\
 & & V \arrow[u, phantom, "\supset" description]
\end{tikzcd}$$

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In $V^{Col(\omega,\omega_2)}$:

$$\mathbb{P}_{max},\ D \text{ dense},\ D \in L(\mathbb{R})$$
$$D \supseteq D^* = p[T] \ni q_0$$

$$\begin{tikzcd}
 & q_0 \arrow[r] \arrow[d, phantom, sloped, "\in"] & q_{\omega_1} = (Q_{q_{\omega_1}}; \in, I, A) \arrow[d, phantom, sloped, "\in"] & \\
P_0 \arrow[r] & P_\alpha \arrow[r] & P_{\omega_1} = (H_{\omega_2}; \in, NS_{\omega_1}, A) & \alpha = \omega_1^{q_0}
\end{tikzcd}$$

$$P_{\omega_1} \subset V, \quad A \subseteq \omega_1$$

Suppose we can add objects as in the picture above by a forcing $\mathbb{P}$ which preserves stationary subsets of ω_1 s.t. $I = NS_{\omega_1}^{V^{\mathbb{P}}} \cap q_{\omega_1}$.

Then: $V^{\mathbb{P}} \models$ "$\exists q_0 \in D^*\ \exists$ gen. iteration $(q_i : i \leq \omega_1)$ of q_0 s.t. $q_{\omega_1} = (Q_{q_{\omega_1}}; \in, A, NS_{\omega_1}^{V^{\mathbb{P}}} \cap q_{\omega_1})$"

By $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$-$BMM^{++}$, we then have "$\exists q_0 \in D\ \exists$ gen. iteration $(q_i : i \leq \omega_1)$ s.t. $q_{\omega_1} = (Q_{q_{\omega_1}}; \in, NS_{\omega_1} \cap q_{\omega_1}, A)$" in V, i.e., $q_0 \in D \cap g_A \neq \emptyset$. So g_A is $L(\mathbb{R})$-generic.

So it remains to cook up a forcing $\mathbb{P}$ with the property as above.

First step: We may assume w.l.o.g. that $\diamondsuit_{\omega_3}$ holds true.

$\exists$ sequence $(\bar{A}_\delta : \delta < \omega_3)$ $\forall A \subseteq \omega_3$ $\{\delta < \omega_3 : \bar{A}_\delta = A \cap \delta\}$ is stationary.

This is the case as $\diamondsuit_{\omega_3}$ holds true in $V^{Col(\omega_3,\omega_3)}$.

2nd step: Fix our $\diamondsuit_{\omega_3}$-sequence $(A_\delta : \delta < \omega_3)$. Write $\varkappa = \omega_3$. Pick $e : \varkappa \to H_\varkappa$ bijective. Write $Q_\delta = e''\delta$, $A_\delta = e''\bar{A}_\delta \subseteq Q_\delta$.

Have q_0 (or rather, a real x coding q_0) $\in p[T] = D^*$.
I.e., $(x, \vec{\alpha}) \in [T]$ in $V^{Col(\omega,\omega_2)}$.

In V, we may cover the set of ordinals from the countable sequence $\vec{\alpha}$ by a set B of ordinals of size $\aleph_2$.
Let $T \in H_\theta$ (we may assume T is set sized).

Let $M \overset{\sigma}{\cong} X \prec H_\theta$. $Card(X) = \aleph_2$, M transitive. Let $\sigma(\bar{T}) = T$. Then $x \in p[\bar{T}]$,

$X \supseteq \{T\} \cup B$

where $\bar{T}$ may be construed as a tree on $\omega \times \omega_1$.

Let us write T for $\bar{T}$ from now on.

There is a club $C \subseteq \kappa$ s.t. for all $\delta \in C$:
(i) Q_δ is transitive,
(ii) $\{T, (H_{\omega_2}; \in, NS_{\omega_1}, A)\} \cup (\omega_2+1) \subseteq Q_\delta$.
(iii) $Q_\delta \cap OR = \delta$.
(iv) $(Q_\delta; \in) \prec (H_\kappa; \in)$

It's easy to verify that for all $A \subseteq H_\kappa$, $\{\delta < \omega_3 : (Q_\delta; \in, A_\delta) \prec (H_\kappa; \in, A)\}$ is stationary.
We fix $((Q_\delta; \in, A_\delta) : \delta \in C)$ throughout the rest of the proof.

We now define a sequence of forcings $\mathbb{P}_\lambda$, $\lambda \in C \cup \{\overset{\omega_3}{\kappa}\}$. Our final forcing will be $\mathbb{P} = \mathbb{P}_\kappa$. We will also have that $\mathbb{P}_\lambda \subseteq Q_\lambda$, $\mathbb{P}_\mu \subseteq \mathbb{P}_\lambda$ and $\leq_{\mathbb{P}_\mu} = \leq_{\mathbb{P}_\lambda} \cap \mathbb{P}_\mu$ for all $\mu \leq \lambda \in C \cup \{\kappa\}$.
Fix $\lambda \in C \cup \{\kappa\}$, and assume $\mathbb{P}_\mu$ has already been defined for $\mu < \lambda$.

We will be interested in objects of the following form:

$\varphi = ((M_i; \pi_{ij}, N_i, \sigma_{ij} : i \leq j < \omega_1), (\kappa_n, \alpha_n : n < \omega), ((\lambda_\delta, X_\delta) : \delta \in K))$, where:

- $M_0, N_0 \in \mathbb{P}_{max}$
- $x = (\kappa_n : n < \omega)$ is a real code for N_0, and $((\kappa_n, \alpha_n) : n \in \omega) \in [T]$
- $(M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0}) \in N_0$ is a gen. iteration of M_0 witnessing $N_0 < M_0$.
- $(N_i, \sigma_{ij} : i \leq j \leq \omega_1)$ is a gen. iteration of N_0 s.t. $N_{\omega_1} = (N_{\omega_1}; \in, I, A)$, some I.
- $(M_i, \pi_{ij} : i \leq j \leq \omega_1) = \sigma_{0\omega_1}((M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0}))$ and $M_{\omega_1} = (H_{\omega_2}; \in, NS_{\omega_1}, A)$,
- $K \subseteq \omega_1$,
- for all $\delta \in K$, $\lambda_\delta \in \lambda \cap C$, if $\gamma < \delta$ are in K, then $\lambda_\gamma < \lambda_\delta$ and $X_\gamma \cup \{\lambda_\gamma\} \subseteq X_\delta$.
- $X_\delta \prec (Q_{\lambda_\delta}; \in, A_{\lambda_\delta})$ and $\delta = \omega_1 \cap X_\delta$

The forcing $\mathbb{P}_\lambda$ comes with a 1st order language $\mathcal{L}_\lambda$. $\mathcal{L}$ will be the 1st order language with the following symbols:
$\underline{T}$ intended to denote T
$\underline{x}$ (for $x \in H_\kappa$) intended to denote x
$\dot{n}$ intended to denote elements of M_i, N_i $i < \omega_1$.
$\dot{M}_i / \dot{\pi}_{ij} / \vec{\dot{M}} / \dot{N}_i / \dot{\sigma}_{ij}$ " " $M_i / \pi_{ij} / (M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0}) / N_i / \sigma_{ij}$.
$\dot{a}$ " " " the distinguished a-predicate of M_i / N_i
$\dot{I}$ " " " " " I " " "
$\dot{X}_\delta$ " " " X_δ

Formulae of $\mathcal{L}$:

$\ulcorner \dot{N}_i \models \varphi(\underbrace{\underline{\xi_1}, \dots, \underline{\xi_\kappa}}_{\text{countable ordinals}}, \dot{n}_1, \dots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \dots, \dot{M}_{j_\ell}, \dot{\pi}_{q_1 r_1}, \dots, \vec{\dot{M}}) \urcorner$

$\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = \underline{x} \urcorner$, $x \in H_{\omega_2}$

$\ulcorner \dot{\sigma}_{ij}(\dot{n}) = \dot{m} \urcorner$; $\ulcorner (\underline{u}, \underline{\alpha}) \in \underline{T} \urcorner$

$\ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner$ $\delta < \omega_1$, $\mu < \kappa$; $\ulcorner \underline{x} \in \dot{X}_\delta \urcorner$ $\delta < \omega_1$, $x \in H_\kappa$.

$\mathcal{L}_\lambda$ = the fragment of $\mathcal{L}$ which doesn't mention objects outside of Q_λ.

An object $\mathcal{C} = ((M_i; \pi_{ij}, N_i, \sigma_{ij} : i \leq j < \omega_1), (\kappa_n, \alpha_n : n < \omega), ((\lambda_\delta, X_\delta) : \delta \in K))$ as before is precertified by a collection Σ of sentences from $\mathcal{L}_\lambda$ iff there are surjections $e_i : \omega \longrightarrow N_i$, $i < \omega_1$, s.t. a formula is in Σ if an only if it is true when interpreted in the obvious way.

e.g. $\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = \underset{\sim}{x} \urcorner \in \Sigma \iff e_i(n) \in M_i$ and $\pi_{i\omega_1}(e_i(n)) = x$.

$\ulcorner \sigma_{ij}(\dot{n}) = \dot{m} \urcorner \in \Sigma \iff \sigma_{ij}(e_i(n)) = e_j(m)$; etc.

We are concerned with the potential certificates:

$$\mathcal{C} = \big((M_i; \pi_{ij}, N_i, \sigma_{ij} : i \le j < \omega_1), (\kappa_n, \alpha_n : n < \omega), ((\overset{C\cap\lambda}{\overset{\in}{\lambda_\delta}}, \underset{\substack{(Q_{\lambda_\delta}; \in, A_{\lambda_\delta}) \\ X_\delta \cap \omega_1 = \delta}}{X_\delta}) : \delta \in K)\big),$$

$\mathcal{L}$ to describe such certificates
$\cup\!|$
$\mathcal{L}_\lambda$

- $\mathcal{C}$ is <u>precertified</u> by a collection Σ of $\mathcal{L}$-formulae iff Σ describes $\mathcal{C}$.

We are about to define $\mathbb{P}_\lambda$, $\lambda \in C \cup \{\varkappa\}$

- Say that a potential certificate $\mathcal{C}$ is certified by Σ iff $\mathcal{C}$ is precertified by Σ and, if $\delta \in K$, then $[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$ for all $E \subseteq \mathbb{P}_{\lambda_\delta}$ which is dense in $\mathbb{P}_{\lambda_\delta}$ and definable over $(Q_{\lambda_\delta}, \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$ from parameters in X_δ.

Definition: Let $\mathcal{C}$ be a potential certificate. Call $\mathcal{C}$ a semantic certificate iff $\exists$ collection Σ of formulae s.t. $\mathcal{C}$ is certified by Σ. Call Σ a syntactic certificate if it certifies some $\mathcal{C}$.
If $\Sigma \cup \{p\}$ is a set of formulae, say that p is certified by Σ iff there is an (unique) $\mathcal{C}$ s.t. $\mathcal{C}$ is certified by Σ and $p \in [\Sigma]^{<\omega}$

$p \in \mathbb{P}_\lambda$ iff $V^{Col(\omega,\lambda)} \models$ "there is some Σ s.t. p is certified by Σ"

$\mathbb{P} = \mathbb{P}_{\varkappa = \omega_3}$

Key fact: Forcing with $\mathbb{P}_\lambda$ adds a syntatic + semantic certificate.
basically, the union of the generic.

We are left to having to prove:

Let g be $\mathbb{P}$-generic over V, and let $((M_i, \pi_{ij}, N_i, \sigma_{ij} : i \le j \le \omega_1), \ldots)$ is the semantic certificated read off from $\cup g$ and if $N_{\omega_1} = (N_{\omega_1}; \in, I, A)$, then $I = NS_{\omega_1}^{V[g]} \cap N_{\omega_1}$.

- $\omega_4 \to$ preserved
- $\omega_3 \to$ cof ω_1 by Shelah (witnessed here by $K \ni \delta \mapsto \lambda_\delta$ as being given by $\cup g$)
- $\omega_2 \to$ cof ω $((M_0, -) \to (H_{\omega_2}; \in, -)$, $\pi_{0\omega_1}''(M_0 \cap OR)$ is cofinal in $\omega_2)$
- ω_1

Proof of this:

Let $\bar{p} \Vdash \dot{C}$ club in $\check{\omega}_1$

$\bar{p} \Vdash \dot{S} \in (\mathcal{P}(\check{\omega}_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}$

$\bar{p} \Vdash \dot{S}$ is represented by $[\dot{i}_0, \dot{n}_0]$ in the term model giving by $\dot{N}_{\omega_1}$

$$N_{i_0} \longrightarrow N_{\omega_1}$$
$$\bar{S} \longmapsto S$$
$$\overset{?}{[n_0]} \longmapsto [i_0, n_0]$$

$\ulcorner \dot{N}_{i_0} \models \dot{n}_0$ is a subset of ω_1, get $\dot{n}_0 \notin \dot{I}^{N_{i_0}} \urcorner \in \bar{p}$.

Let $p \leq \bar{p}$. We want $q \leq p$, $q \Vdash \check{\delta} \in \dot{S} \cap \dot{C}$, some δ.

$D_\xi = \{ q \leq p : \exists \eta \geq \xi \ (\eta < \omega_1 \wedge q \Vdash \check{\eta} \in \dot{C}) \}$, for $\xi < \omega_1$. Every D_ξ is open dense.

$E = \{ (q, \eta) : q \Vdash \check{\eta} \in \dot{C} \}$

$\tau = ((D_\xi : \xi < \omega_1), E) \subseteq H_\varkappa$.

By $\diamondsuit_{\omega_2}$, we may pick some $\lambda \in C$ s.t. $p \in \mathbb{P}_\lambda$. $(Q_\lambda, \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\varkappa; \in, \mathbb{P}, \tau)$.

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic$/V$, and pick $g' \ni p$ a filter $\subseteq \mathbb{P}_\lambda$ which meets all the dense sets definable (from parameters) over $(Q_\lambda, \ldots)$.

$U_{g'}$ is a syntetic certificate

$\wr$

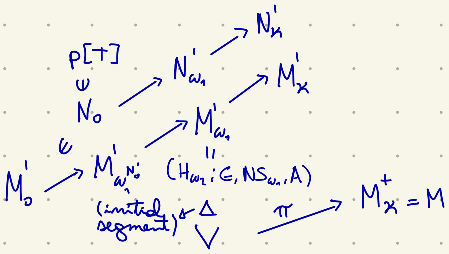
a semantic certificate

$((M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1), (\kappa'_n, \alpha'_n : n < \omega), (\lambda'_\delta, X'_\delta : \delta \in \kappa'))$

Let S be the set represented by $[i_0, n_0]$ in the term model giving N'_{ω_1}.

$S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I^{N_{\omega_1}}$

We may then expand the iteration $(N'_i, \sigma'_{ij} : i \leq j \leq \omega_1)$ to $(N'_i, \sigma'_{ij} : i \leq j \leq \varkappa) \in V[h]$ s.t. $\omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S)$.



We may now extend $\pi'' U_{g'}$ to a syntactic certificate which corresponds to the following semantical certificate:

$$((M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1), (\kappa'_n, \pi(\alpha'_n) : n < \omega), ((\pi(\lambda'_\delta), \pi'' X_\delta) : \delta \in \kappa')^\frown(\overset{\lambda_{\omega_1}}{\overset{\shortparallel}{\pi(\lambda)}}, \pi'' Q_\lambda))$$

$\hookrightarrow$ This certifies

$\pi(p) \cup \{ \ulcorner \omega_1 \longmapsto \pi(\lambda) \urcorner, \ulcorner \dot{\sigma}_{i_0 \omega_1+1}(\dot{n}_0) = \underset{\substack{\uparrow \\ \text{new}}}{\dot{m}_0} \urcorner, \ulcorner \underline{\omega_1} \in \dot{m}_0 \urcorner \}$

$\pi'' [U_g]^{<\omega}$

$\shortparallel$

$\pi'' g \cap \pi'' Q_\lambda \cap E \neq \emptyset$ for every $E \subseteq \pi(\mathbb{P}_\lambda)$ definable over $\pi''(Q_\lambda, \ldots)$ from param. in $\pi'' Q_\lambda$.

By pulling back, there will be a countable ordinal δ s.t. $q = p \cup \{ \ulcorner \delta \mapsto \lambda \urcorner, \ulcorner \sigma_{i_0 \delta+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \delta \in \dot{m}_0 \urcorner \} \in \mathbb{P}$.

Easy: $q \Vdash \check{\delta} \in \dot{S}$.

<u>Claim:</u> $q \Vdash \check{\delta} \in \dot{C}$.

Suppose $r \leq q$, $r \Vdash \sup(\dot{C} \cap \check{\delta}) = \check{\eta}$ (some $\eta < \delta$).

$$\left((M_i'', \pi_{ij}'', N_i'', \sigma_{ij}'' : i \leq j \leq \omega_1), (\kappa_n'', \alpha_n'' : n < \omega), (\lambda_\varepsilon'', X_\varepsilon'' : \varepsilon \in \underset{\delta}{\underset{\Psi}{K}}) \right)$$

$\lambda_\delta = \lambda$, $X_\delta \prec (Q_\lambda ; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa ; \in, \mathbb{P}, \tau)$, $X_\delta \cap \omega_1 = \delta$.

$$\tau \sim ((D_\xi : \xi < \omega_1), E)$$

$D_\eta \cap \mathbb{P}_\lambda$ is definable over $(Q_\lambda ; \ldots)$ from parameters in X_δ.

$\Rightarrow \exists s \in [\Sigma]^{<\omega} \cap D_\eta \cap \mathbb{P}_\lambda \cap X_\delta$.

↖ corresponding to the above semantic certificate

$s \Vdash \eta^* \in \dot{C}$, some $\eta^* \geq \eta$, $\eta^* \leq \delta$. But $s \| r$, as they are both certified by Σ. ⊣