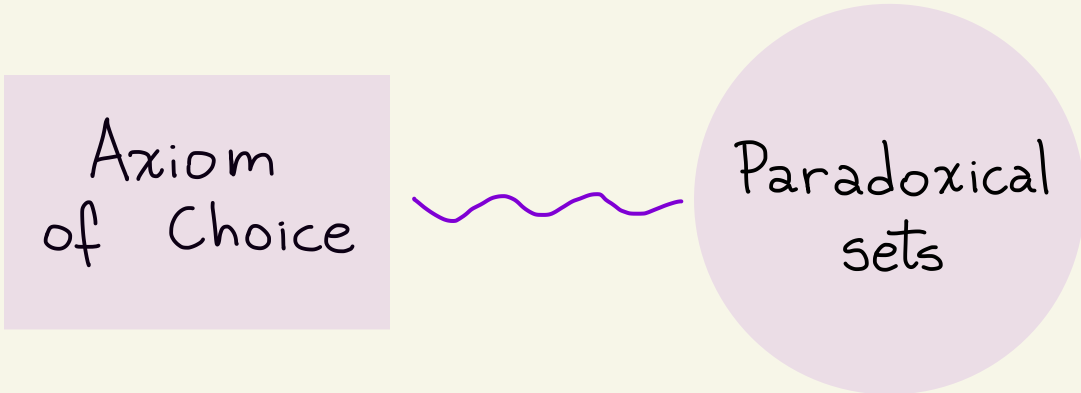
WWU MÜNSTER

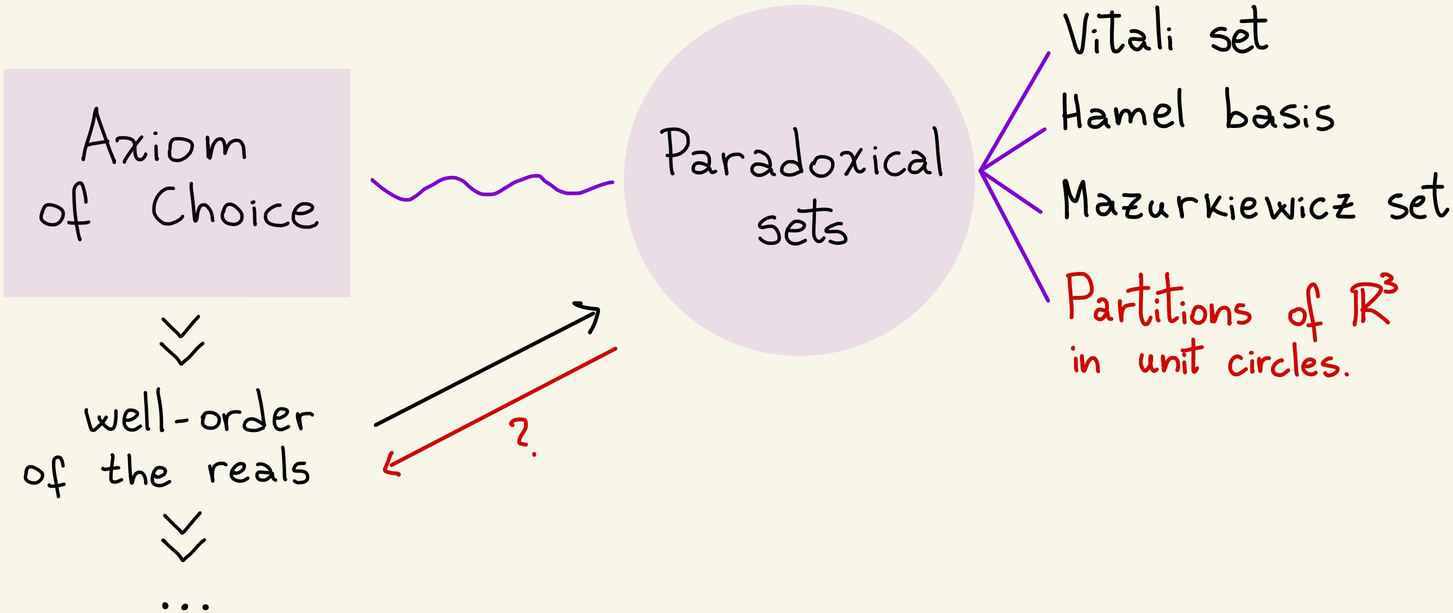
Paradoxical sets and the Axiom of Choice

Azul Lihuen Fatalini

Joint work with Prof. Ralf Schindler


Axiom of Choice
Paradoxical sets

My research


Axiom of Choice
well-order of the reals
...
Paradoxical sets
?
Vitali set
Hamel basis
Mazurkiewicz set
Partitions of $\mathbb{R}^3$ in unit circles.

Set Theory

Sets

Consistency

Cardinality

ZFC

$\emptyset, \cap, \cup, \subseteq$

Axiom of Choice

Paradoxes

Intersectionality

Models

Venn diagrams

Continuum hypothesis

Infinity

Induction

Zorn's lemma

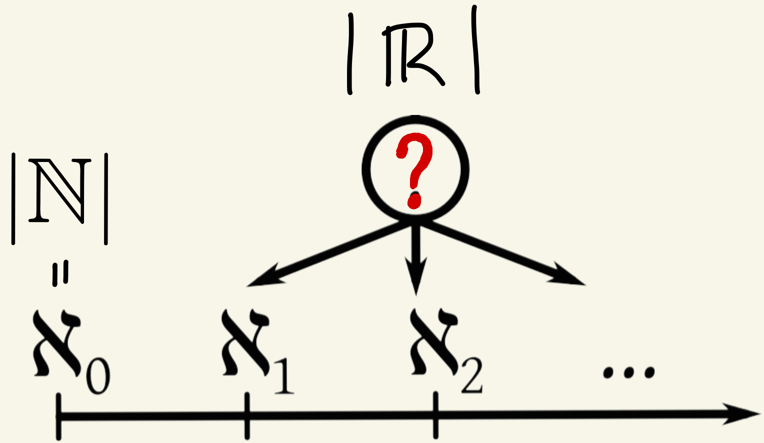
Forcing

$\aleph_0$

Set Theory

- $\mathfrak{c} = |\mathbb{R}|$

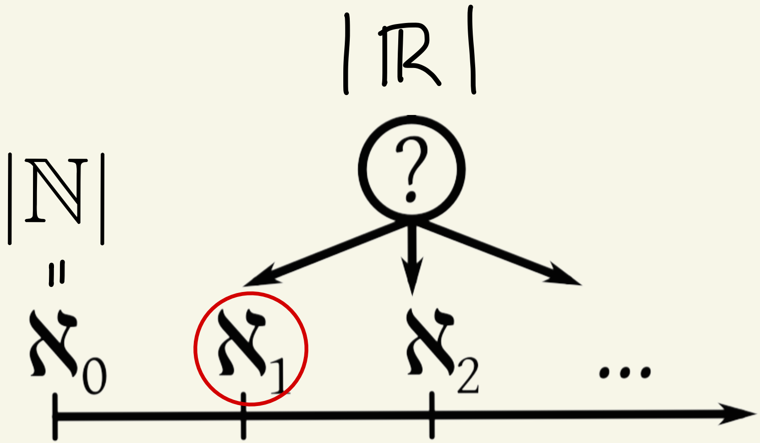
  “continuum”



Set Theory

- $\mathfrak{c} = |\mathbb{R}|$

"continuum" $\rightsquigarrow$ Continuum hypothesis



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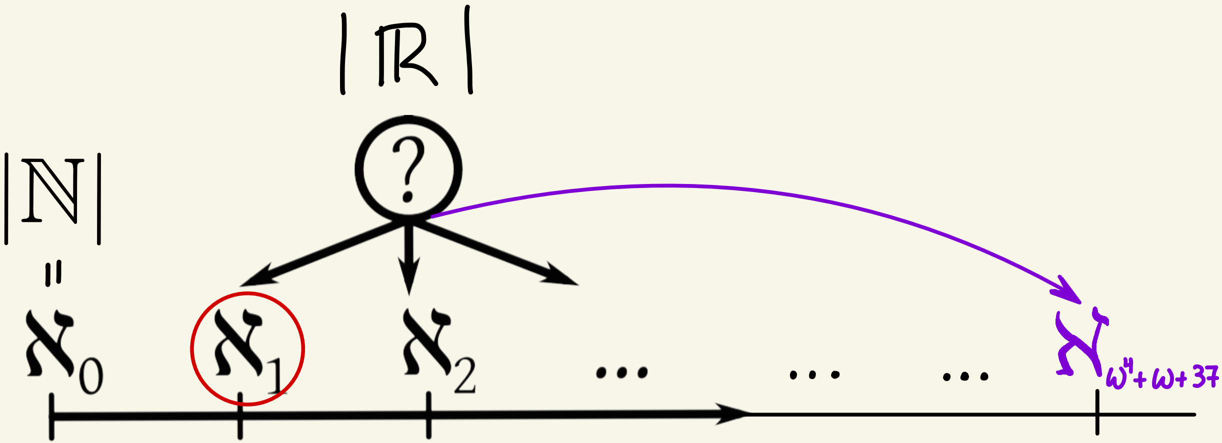
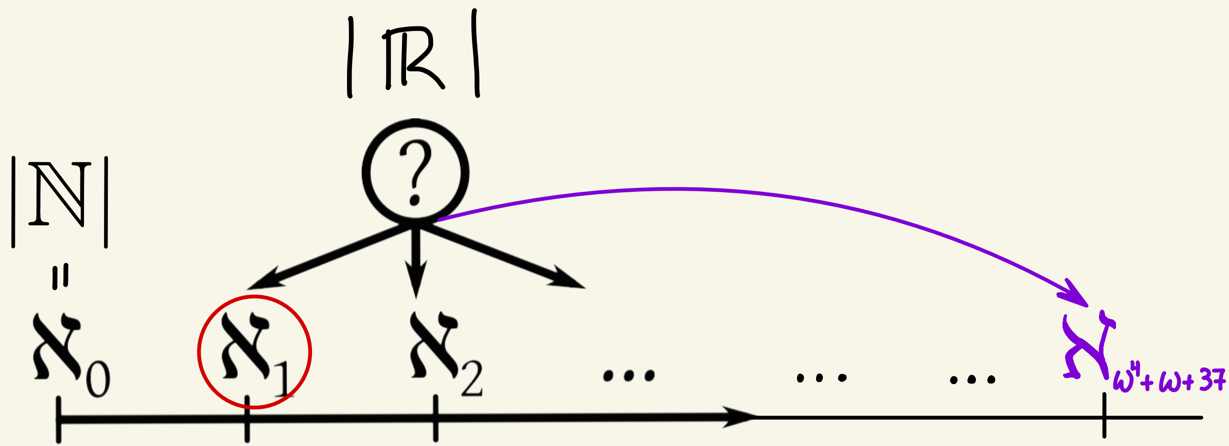


Set Theory

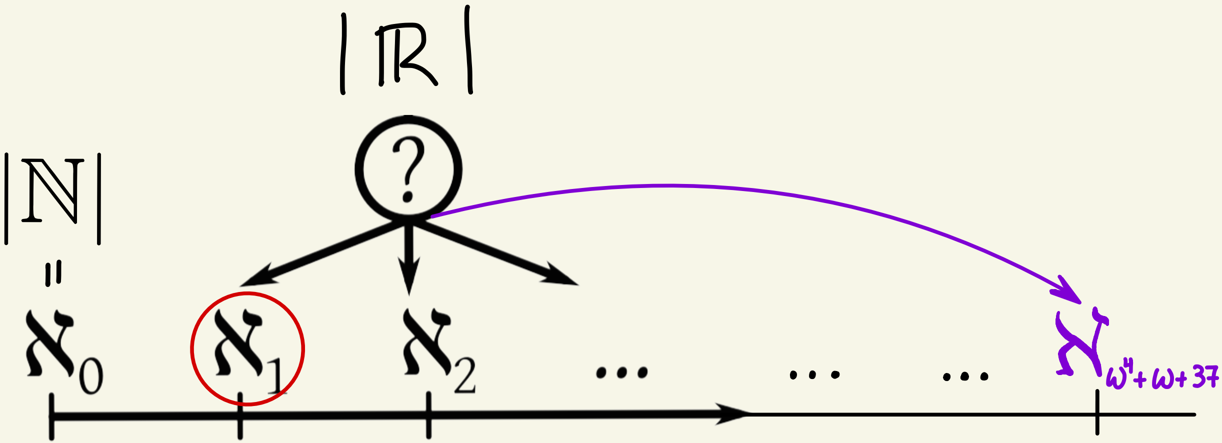
- Model of ZF(C) = "universe"



Set Theory

- Model of ZF(C) = "universe"

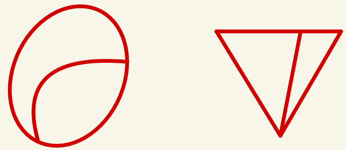
  (C) ↓ Axiom of Choice

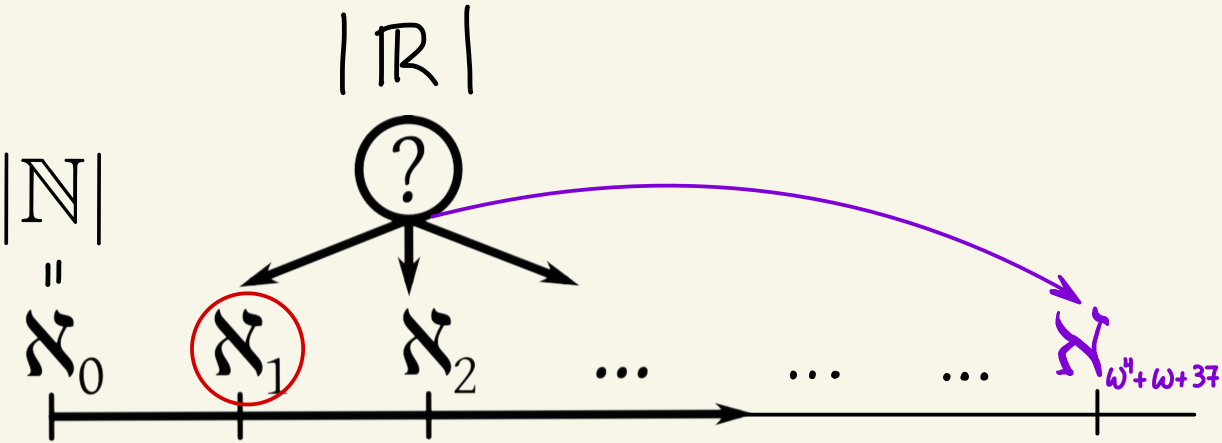


Set Theory

- Model of ZF(C) = "universe" → There are several

  C ↓ Axiom of Choice





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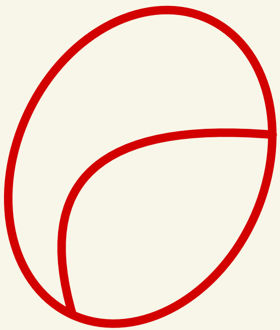
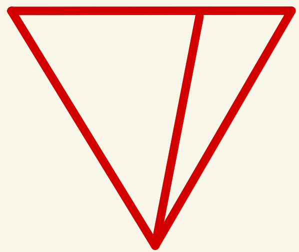




Choice principles

Axiom of Choice

$\Longleftrightarrow$

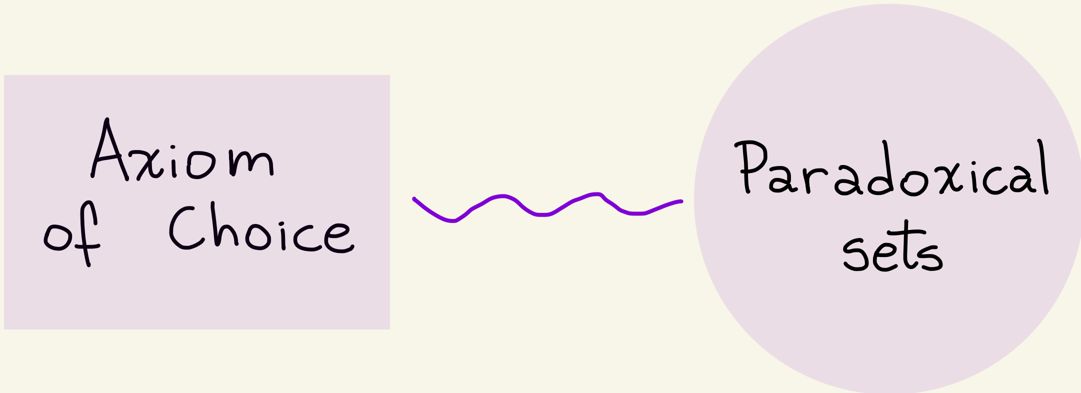
Zorn's lemma

$\Longleftrightarrow$

Well-ordering principle

$\Downarrow$

"$\mathbb{R}$ can be well-ordered"



Axiom of Choice
Paradoxical sets

Paradoxical sets

- $P \subseteq \mathbb{R}^n \quad n = 1, 2, 3.$

P satisfies some counterintuitive property.

The existence of such a P involves Axiom of Choice

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- $P \subseteq \mathbb{R}^n \quad n = 1, 2, 3.$

P satisfies some counterintuitive property.
The existence of such a P involves Axiom of Choice

$\Downarrow$

a well-order of the reals

$\downarrow$

allows induction "on the reals"

Set Theory

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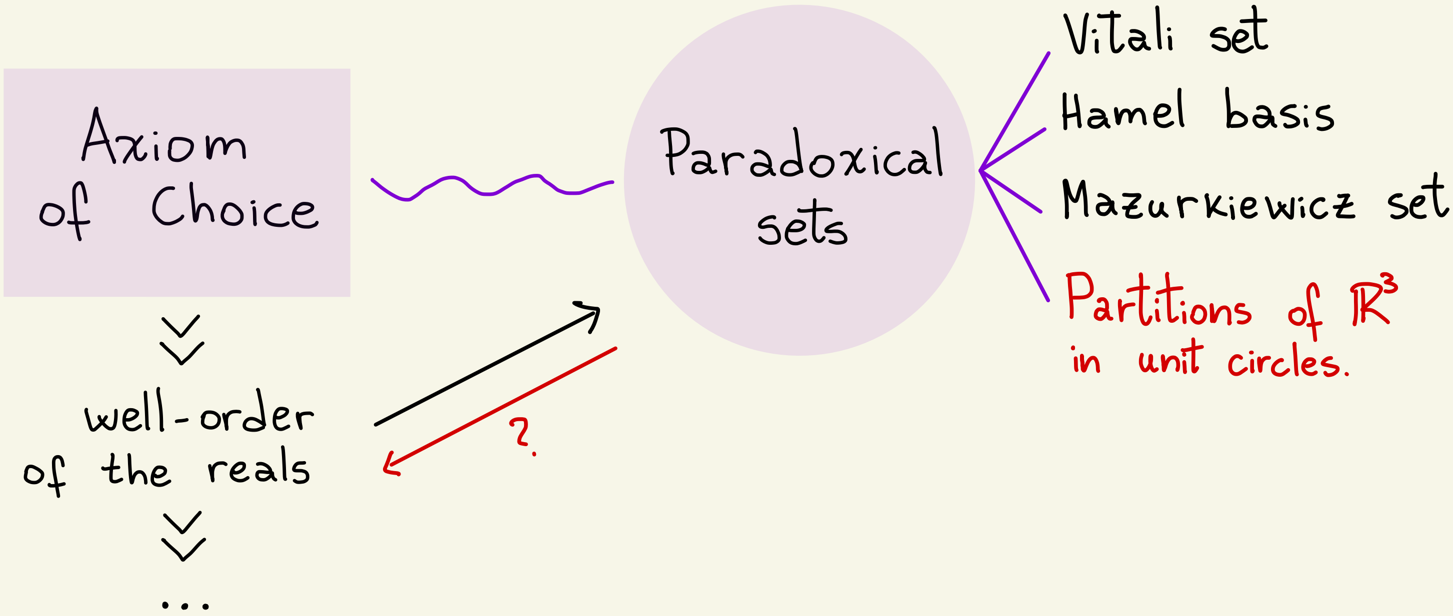

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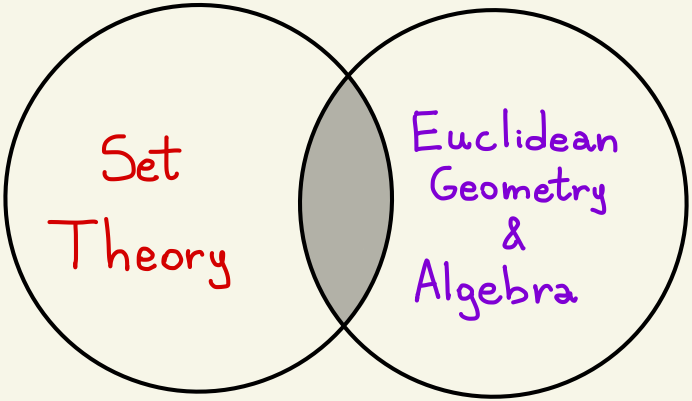
Partitions of $\mathbb{R}^3$ in unit circles.

Contribution

Negative answer $\rightarrow$ Model of $ZF + \neg C + \exists P$



Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Let $\mathbb{R}^3 = \{p_\alpha\}_{\alpha < \mathfrak{c}}$.

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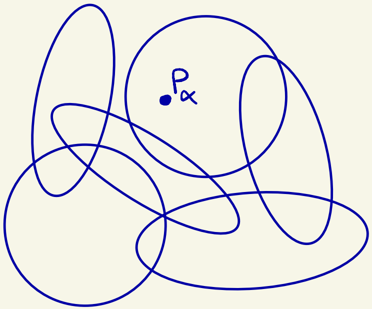

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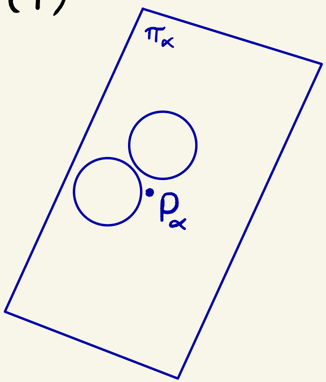

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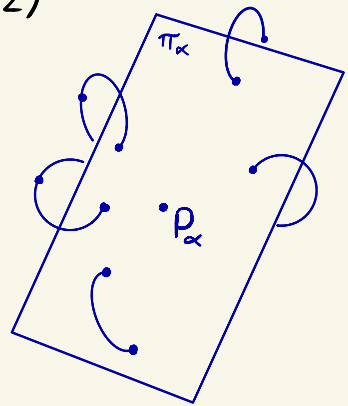


Theorem

There is a model of

$$ZF + \exists \text{ partition of } \mathbb{R}^3 \text{ in unit circles}$$

$$+ \text{ no well order of the reals.}$$

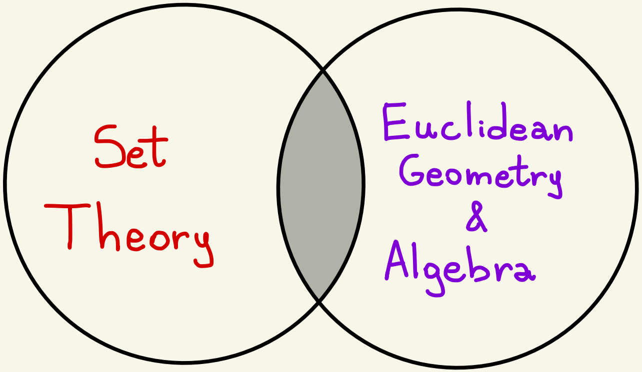
Set Theory

How does the proof look like?

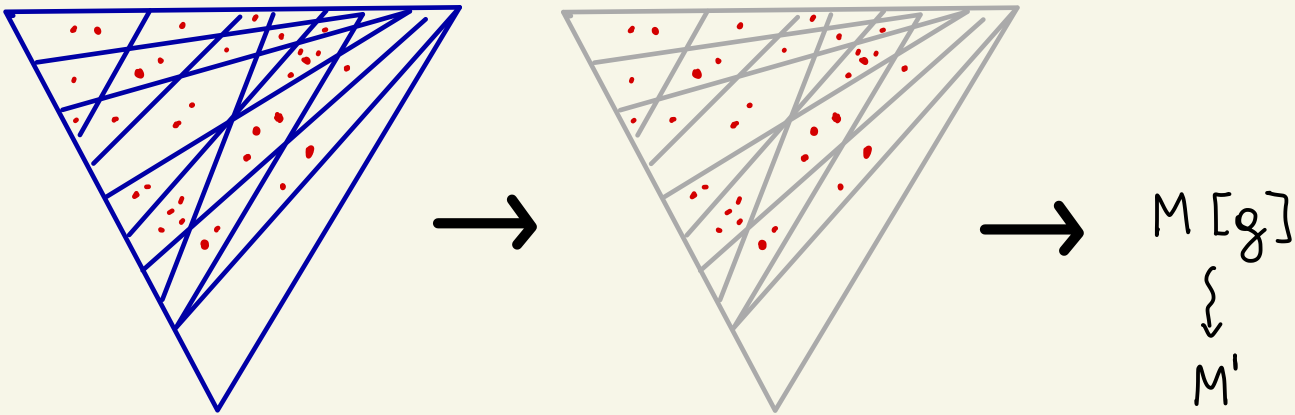
$(M, \mathbb{P}, g \subseteq \mathbb{P}) \longrightarrow \boxed{\text{Forcing}} \rightsquigarrow M[g]$

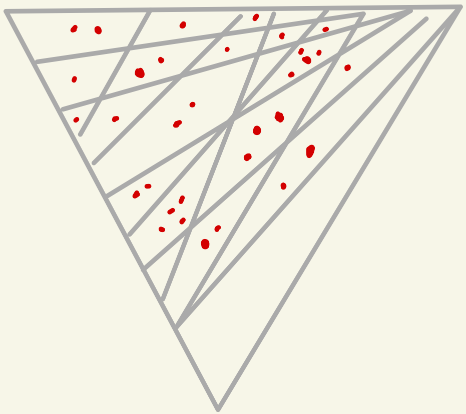
Strategy

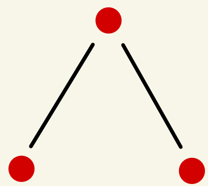
Negative answer → Model of ZF + $\neg$C + $\exists$P

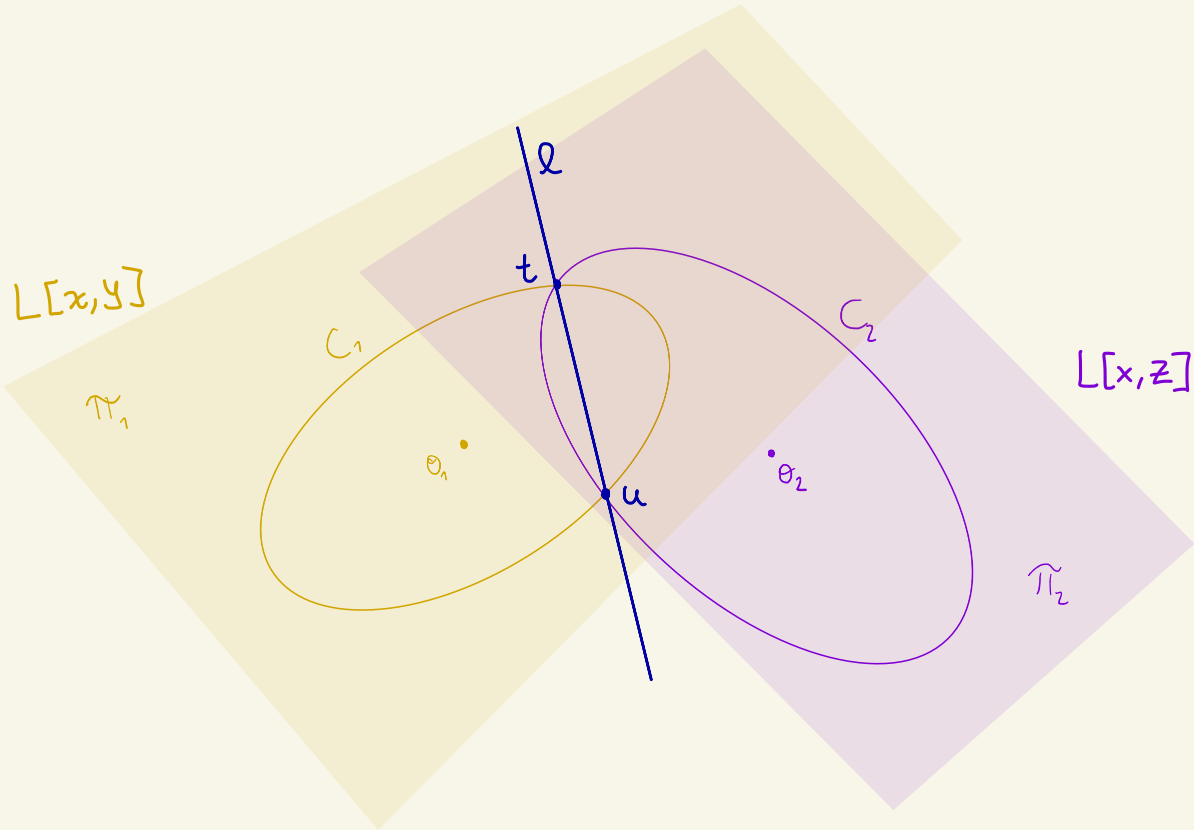


Strategy:
- Construct a model (of ZF + $\neg$C) which contains a nice structure of inner models satisfying AC.
- P will be a limit of the partial paradoxical sets.


M [g]
M'







$L[x,y]$
ℓ
t
C_1
C_2
$L[x,z]$
π_1
o_1
o_2
u
π_2

Contribution

	Hamel basis	Mazurkiewicz	Partition of $\mathbb{R}^3$ in unit circles
$DC + \neg WO(\mathbb{R})$	✔ [4]	✔ [4]	✔
$DC + \neg Uf(\omega)$	✔	✔	?
$\neg AC_\omega$	✔ [3]	✔ [2]	✔

References

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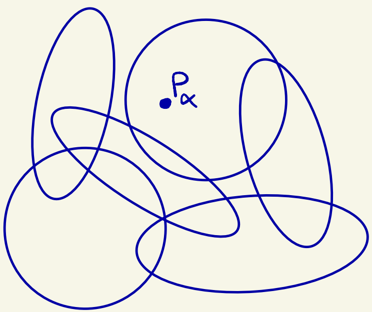

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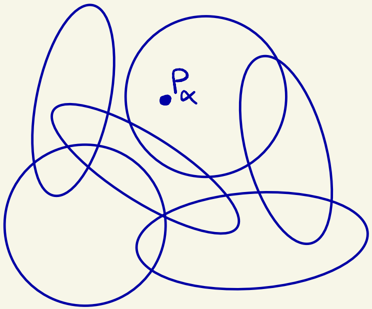

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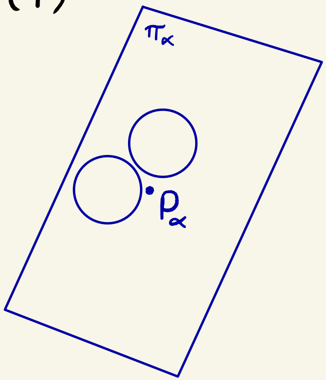

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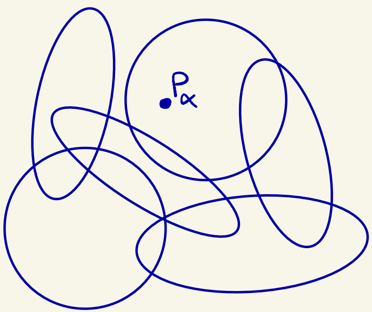

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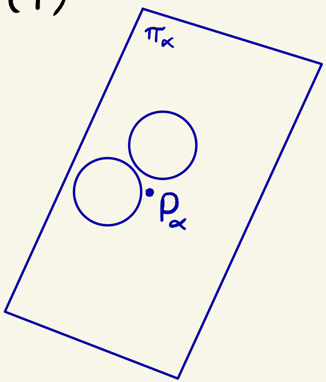

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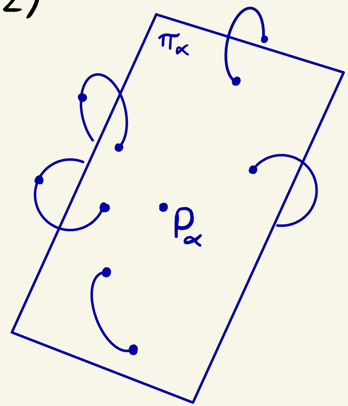
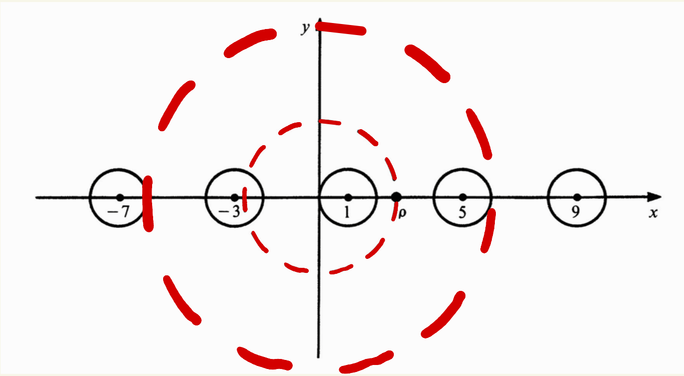
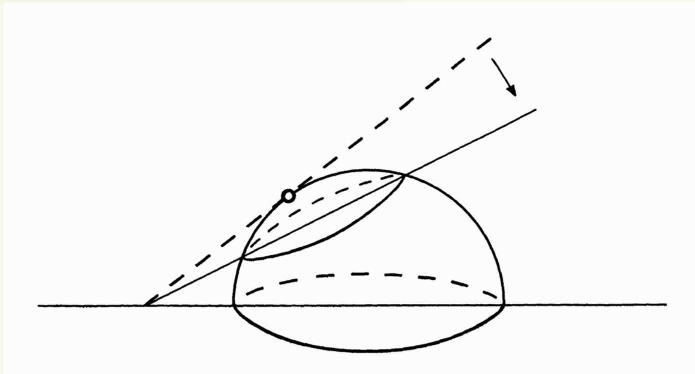
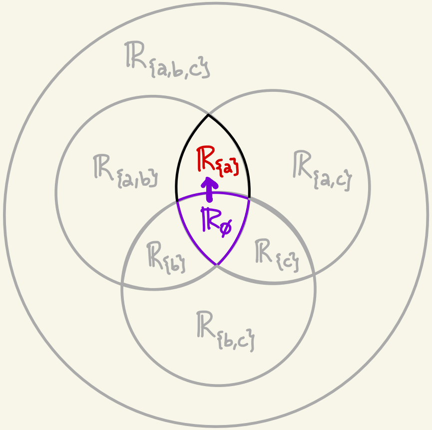


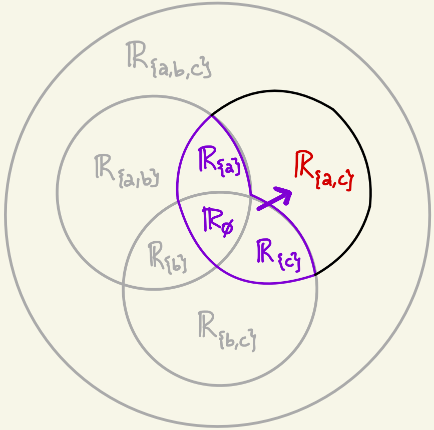
Theorem (ZF) (Szulkin)
$\mathbb{R}^3$ can be partitioned in circles.





Algebra





Theorem (F., Schindler; ~2024)

Let V be a model of ZFC and let S be a finite set of mutually generic Cohen reals over V.

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over $\overline{\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}}^{\,alg}$ is $\mathfrak{c}$.