The transcendence degree of the reals over certain set-theoretical subfields

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Abstract

It is a well-known result that, after adding one Cohen real, the transcendence degree of the reals over the ground-model reals is continuum. We extend this result for a set X of finitely many Cohen reals, by showing that, in the forcing extension, the transcendence degree of the reals over a combination of the reals in the extension given by each proper subset of X is also maximal. This answers a question of Kanovei and Schindler [2].

This article investigates a folklore result about the forcing extension by one Cohen real: the transcendence degree of the reals over the set of reals in the ground model is of cardinality $\mathfrak{c}$ (in the extension). This is obtained by noticing that one Cohen real can be split in a perfect set of Cohen reals which are mutually generic and, therefore, algebraically independent over the ground model reals.

It turns out it is possible to generalize this statement in two directions: taking into account other type of reals added (observed independently by Ben de Bondt and Elliot Glazer, see Corollary 2.5) and adding more than one Cohen real (see Theorem 3.13), which is the main result of this paper:

Main Theorem (Theorem 3.13)
Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then, in $V[X]$ the transcendence degree of $\mathbb{R}$ over F is continuum.

In [2], the authors asked whether a certain set-theoretical subfield of reals is a proper subfield in the Cohen model. Theorem 3.13 shows that it is a proper

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subfield and, moreover, the reals have transcendence degree continuum over this subfield.

Actually, one of the main obstacles to prove Theorem 3.13 is to show that the field F is a proper subfield of $\mathbb{R}$. In the case of $n = 1$, i.e., in the folklore result, this is for free: the Cohen real is a *new real*, namely it does not belong to the ground model reals, witnessing that the corresponding field F is a proper subfield.

In the case of $n = 2$, and $X = \{x, y\}$, it is tempting to take $x \oplus y$ as a witness for F being a proper subfield of $\mathbb{R}$. Nevertheless, after a closer examination, one can see that $x \oplus y$ actually belongs to the field F, since $x \oplus y = (x \oplus \vec{0}) + (\vec{0} \oplus y)$. The proof of Theorem 3.13 shows that an appropriate witness is the *composition* of the reals in X as functions from ω to ω.

This paper is organized as follows. In Section 1, we will go through the basic prerequisites of field theory and real analysis needed for the proofs of the auxiliary results that will lead to our main theorem. Section 2 contains the proof of the first direction of generalization of the folklore result, when one is allowed to add any real, not necessarily Cohen. Finally, Section 3 contains the proof of the main theorem, generalizing the folklore result in a second direction: allowing more than one Cohen real. We conclude by including a result about the relationship between the transcendence bases of $\mathbb{R}^{V[x,y]}$, $\mathbb{R}^{V[x]}$ and $\mathbb{R}^{V[y]}$ over $\mathbb{R}^{V}$.

1 Prerequisites

1.1 Field theory

Definition 1.2. A field F is an **extension** of a field K if K is a subfield of F, namely, $K \subseteq F$ and the operations on K are the ones on F restricted to K.

If F is a field and $X \subseteq F$, then the **subfield** (resp. **subring**) **generated by** X is the intersection of all subfields (resp. subrings) of F that contain X. If F is an extension of K and $X \subseteq F$, then the subfield (resp. subring) generated by $K \cup X$ is denoted by $K(X)$ (resp. $K[X]$).

Notation: Given a field K and $n \in \omega$, we denote by $K[x_0, \dots, x_{n-1}]$ the ring of polynomials in n variables over K.

First, let us recall the construction of a field extension for a given set X of generators.

Theorem 1.3 ([1, Theorem 1.3 Chapter V])
If F is an extension field of a field K and $X \subseteq F$, then the subfield $K(X)$ consists

of all elements of the form

$$\frac{f(u_0, \ldots, u_{n-1})}{g(u_0, \ldots, u_{n-1})} = f(u_0, \ldots, u_{n-1})g(u_0, \ldots, u_{n-1})^{-1},$$

where $n \in \omega$, $f, g \in K[x_0, \ldots, x_{n-1}]$, $u_0, \ldots, u_{n-1} \in X$ *and* $g(u_0, \ldots, u_{n-1}) \neq 0$.

Definition 1.4. Let F be an extension field of K. An element u of F is said to be **algebraic** over K iff u is a root of some nonzero polynomial $f \in K[x]$. If u is not a root of any nonzero $f \in K[x]$, u is **transcendental** over K.

Definition 1.5. A field F is **algebraically closed** if every non-constant polynomial $f \in F[x]$ has a root in F. If F is an extension field of K, F is algebraically closed, and every element of F is algebraic over K, we say that F is an **algebraic closure** of K.

Remark 1.6. If K is a subfield of $\mathbb{R}$, we will often talk about *the* algebraic closure of K and it will mean the algebraic closure of K that is a subfield of $\mathbb{C}$. Moreover, we will be interested in the algebraic closure of K relative to $\mathbb{R}$, which is the algebraic closure of K ($\subseteq \mathbb{C}$) intersected with $\mathbb{R}$. We will denote the algebraic closure of K relative to $\mathbb{R}$ simply by $\overline{K}$.

Definition 1.7. Let F be an extension field of K and S a subset of F. S is **algebraically dependent** over K if for some $n < \omega$ there is a nonzero polynomial $f \in K[x_0, \ldots, x_{n-1}]$ such that $f(s_0, \ldots, s_{n-1}) = 0$ for some distinct $s_0, \ldots, s_{n-1} \in S$. S is **algebraically independent** over K if S is not algebraically dependent over K.

Definition 1.8. Let F be an extension field of K. A **transcendence base** of F over K is a subset S of F which is algebraically independent over K and is a maximal (with respect to $\subseteq$) set with this property.

Let F be an extension field of K. Let S be a transcendence base of F over K. Then every transcendence base of F over K has the same cardinality as S [1, Theorems 1.8-1.9 Chapter VI], which allows a notion of dimension of a field extension, called the *transcendence degree*.

Definition 1.9. Let F be an extension field of K. The **transcendence degree** of F over K is the cardinal given by $|S|$, where S is any transcendence base of F over K.

Remark 1.10. Let F be an extension field of K and S a subset of F of cardinality strictly less than the transcendence degree of F over K. Then, it follows from [1, Theorem 1.5 Chapter VI] that there is u in F such that u is a transcendental over $K(S)$. In other words, if C is an algebraic closure of $K(S)$, then $F \backslash C \neq \emptyset$.

1.11 Real analysis

Definition 1.12. Let $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that f is **uniformly continuous** on A if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in A\ (\|x - y\| < \delta \implies |f(x) - f(y)| < \varepsilon)\,,$$

where $\|\cdot\|$ denotes the norm of a vector in $\mathbb{R}^n$, and $|\cdot|$ denotes the absolute value of a real number.

Recall that a continuous function on a compact set is uniformly continuous [3, Theorem 4.19], and $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded (see for example [3, Theorem 2.41]). In particular, a continuous function on the closure of a ball in $\mathbb{R}^n$ is uniformly continuous. This will be used later.

An essential element for the proof of our main theorem is proving a version of the Implicit Function Theorem (Lemma 3.1 and Lemma 3.5). So we start by stating the usual one.

Theorem 1.13 (Implicit Function Theorem [3, Theorem 9.28])
Let $f\colon A \to \mathbb{R}$ be a function on an open set $A \subseteq \mathbb{R}^n \times \mathbb{R}$ such that $f(\mathsf{a}, \mathsf{z}) = 0$ for some point $(\mathsf{a}, \mathsf{z}) \in A$ and $\frac{\partial f}{\partial z}(\mathsf{a}, \mathsf{z}) \neq 0$. Then there are open sets $U \subseteq \mathbb{R}^{n+1}$ and $W \subseteq \mathbb{R}^n$ with $(\mathsf{a}, \mathsf{z}) \in U$ and $\mathsf{a} \in W$ having the following property:

To every $a \in W$ corresponds a unique z such that $(a, z) \in U$ and $f(a, z) = 0$. If this z is defined to be $g(a)$ then $g\colon W \to \mathbb{R}$ is continuously differentiable, $g(\mathsf{a}) = \mathsf{z}$ and $f(g(a), a) = 0$ for all $a \in W$.

Notation: We use normal letters a, z, r, s as variables and $\mathsf{a}, \mathsf{z}, \mathsf{r}, \mathsf{s}$ as fixed points. This will be also the case later in the text.

Remark 1.14. Let $f, A, (\mathsf{a}, \mathsf{z})$ as in Theorem 1.13. Suppose there is another point $(\mathsf{a}_0, \mathsf{z}_0) \in U$, $\mathsf{a}_0 \in W$ such that $f(\mathsf{a}_0, \mathsf{z}_0) = 0$ and $\frac{\partial f}{\partial z}(\mathsf{a}_0, \mathsf{z}_0) \neq 0$. Using the Implicit Function Theorem for the point $(\mathsf{a}_0, \mathsf{z}_0)$, we get two open sets $U_0 \subseteq \mathbb{R}^{n+1}$ and $W_0 \subseteq \mathbb{R}^n$ with $(\mathsf{a}_0, \mathsf{z}_0) \in U_0$ and $\mathsf{a}_0 \in W_0$, and a continuously differentiable function $g_0\colon W_0 \to \mathbb{R}$ such that $g_0(\mathsf{a}_0) = \mathsf{z}_0$ and $f(g_0(a), a) = 0$ for all $a \in W$. By uniqueness, g and g_0 coincide in $W \cap W_0$, which is an open set. Then $\tilde{g} = g \cup g_0\colon W \cup W_0 \to \mathbb{R}$ is continuously differentiable and $f(\tilde{g}(a), a) = 0$ for all $a \in W \cup W_0$.

Definition 1.15. Let $f\colon U \to \mathbb{C}$ be a function defined in an open set U of the complex plane $\mathbb{C}$. We say that f is **analytic** in U iff it is representable by power series, namely, for every $a \in U$ there are $r \in \mathbb{R}_{>0}$ and coefficients $\{c_n\}_{n<\omega}$ such that the following series converges

$$\sum_{n=0}^{\infty} c_n (z - a)^n,$$

and it is equal to $f(z)$ for all z in the disk $D(a, r) = \{z \colon |z - a| < r\}$.

Notice that, if f is analytic, then it has derivatives of all orders, and

$$c_n = \frac{f^{(n)}(a)}{n!}$$

for every $n < \omega$.

Remark 1.16. Sums, products and compositions of analytic functions are analytic. The exponential function and polynomials are analytic in the whole plane, $\frac{1}{z}$ is analytic in $\mathbb{C}\backslash\{0\}$, each branch of $\log z$ is analytic in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

We will also use the following property of analytic functions.

Theorem 1.17 ([4, Theorem 10.18])
Let U be an open connected set, and let $f \colon U \to \mathbb{C}$ be an analytic function. If f vanishes in a set that has a limit point in U, then $f \equiv 0$ in U.

2 The transcendence degree of really closed fields

In this section we will approach some algebraic questions related to different subfields of $\mathbb{R}$, which are actually $\mathbb{R}$ in some inner model. First, we state a folklore result which is the starting point of this paper. Then we state a generalization in one of the directions mentioned: taking out the condition that the real added is a Cohen real. We present the proof given by Ben de Bondt in private conversations. For the main theorem, only Proposition 2.4 will be used.

Lemma 2.1 (Folklore)**.** *Let M be a model of ZFC and g be a $\mathbb{C}$-generic filter over M. In $M[g]$, the transcendence degree of $\mathbb{R}$ over the algebraic closure relative to $\mathbb{R}$ of $\mathbb{R} \cap M$ is maximal (i.e. the cardinality of $\mathbb{R}$).*

In particular, if y is a Cohen real over $V[x]$, then in $V[x, y]$ the transcendence degree of $\mathbb{R}$ over $\overline{\mathbb{R} \cap V[x]}$ is $\mathfrak{c}$.

It is natural to ask for which forcings that add reals Lemma 2.1 holds. By private (independent) conversations with Ben de Bondt and Elliot Glazer, we know that Lemma 2.1 is true for any type of reals added to V. This result will follow from Proposition 2.4, but before that we need some definitions. We will give the proofs of Proposition 2.4 and Corollary 2.5 by Ben De Bondt.

Definition 2.2. Let $P \colon \mathbb{R}_{>0} \to \mathbb{R}$. We say P is a **generalized polynomial with coefficients** $\{a_i, r_i\}_{i<k} \subseteq \mathbb{R}$ for some $k < \omega$ iff $r_i \neq r_j$ for $i \neq j$, $a_i \neq 0$ for $i < k$, and

$$P(x) = \sum_{i<k} a_i x^{r_i}.$$

Definition 2.3. Let $R \subseteq \mathbb{R}$ be a subfield of the reals. We say R is **really closed** if it is closed under roots of generalized polynomials with coefficients in R. Namely, if P is a generalized polynomial with coefficients $\{a_i, r_i\}_{i<k} \subseteq R$ and $x \in \mathbb{R}_{>0}$ is such that $P(x) = 0$, then $x \in R$.

If P is a generalized polynomial then the set of roots $S = \{x \mid P(x) = 0\}$ is finite. Clearly S is definable over $(\mathbb{R}, +, \cdot, <, \exp)$ with parameters $\{a_i, r_i\}_{i<k}$. By o-minimality [5], S is a finite union of singletons and intervals. Since P is an analytic function over $\mathbb{R}_{>0}$, it cannot vanish in an interval, otherwise it would be constantly 0 (see Theorem 1.17). Therefore S is a finite union of singletons, i.e., a finite set.

Proposition 2.4. *If $R \subsetneq \mathbb{R}$ is a really closed subfield of $\mathbb{R}$, the transcendence degree of $\mathbb{R}$ over R is continuum.*

Proof: If $|R| < \mathfrak{c}$, the transcendence degree of $\mathbb{R}$ over R is $\mathfrak{c}$ by cardinality. Namely, if B is algebraically independent over R, and

$$S = \{x \mid \exists P \text{ with coefficients in } R \text{ such that } P(x, b) = 0 \text{ for some } b \in B^{<\omega}\},$$

then $|S| = \max\{|B|, |R|\}$. If B is a transcendence base, then $S = \mathbb{R}$ and therefore we have $|B| = \mathfrak{c}$.

If $|R| = \mathfrak{c}$, pick $x \in \mathbb{R} \backslash R$ such that $x > 0$, and take $T \subseteq R$ linearly independent over $\mathbb{Q}$ of cardinality $\mathfrak{c}$. Let $B = \{x^t\}_{t \in T}$. We claim that B is algebraically independent over R. Suppose not, then there are $k \in \omega$, $\{t_i\}_{i<k} \subseteq T$ and P a non-zero polynomial over R with k variables such that

$$P\left(x^{t_0}, \ldots, x^{t_{k-1}}\right) = 0.$$

Specifically, there are $J \subseteq \omega^k$ and non-zero coefficients $\{a_j\}_{j \in J} \subseteq R$, such that

$$\sum_{j \in J} a_j x^{t_0 j_0} \cdots x^{t_{k-1} j_{k-1}} = 0,$$

where for each $j \in J \subseteq \omega^k$ we denote its coordinates by $(j_0, \ldots, j_{k-1})$. We can rewrite this by grouping the exponents, namely,

$$\sum_{i \in J} a_j x^{s_j} = 0,$$

where $s_j = \sum_{i<k} j_i t_i$. Notice that all of these $\{s_j\}_{j \in J}$ are different since $\{t_i\}_{i<k} \subseteq T$ and T is linearly independent over $\mathbb{Q}$. So x is the root of a generalized polynomial with coefficients in R. Since R is a really closed subfield of $\mathbb{R}$, we get that $x \in R$, which is a contradiction. Therefore, B is algebraically independent over R and the transcendence degree of $\mathbb{R}$ over R is $\mathfrak{c}$. □

Corollary 2.5. *If M and N are models of set theory such that $M \subseteq N$ and $\mathbb{R}^M \subsetneq \mathbb{R}^N$, then in N the transcendence degree of $\mathbb{R}^N$ over $\mathbb{R}^M$ is $(2^{\aleph_0})^N$. In particular, if N is a forcing extension of M via a forcing that adds reals, the same conclusion holds.*

Proof: In N, $\mathbb{R}^M$ is a subfield of $\mathbb{R}^N$. Recall that a generalized polynomial has always finitely many roots. If a generalized polynomial P with coefficients $\{a_i, r_i\}_{i<k}$ in $\mathbb{R}^M$ has n roots in N, this is expressed by the formula

$$\exists z_0, \dots, z_{n-1} \in \mathbb{R}_{>0}\ P(z_0) = \cdots = P(z_{n-1}) = 0,$$

which is $\Sigma^1_1(c)$ for $c = \bigoplus_{i<k}(a_i \oplus r_i)$, and therefore absolute between M and N. So the roots of P belong to M and $\mathbb{R}^M$ is a really closed subfield of $\mathbb{R}^N$. By applying Proposition 2.4 in N, we get that the transcendence degree of $\mathbb{R}^N$ over $\mathbb{R}^M$ is continuum. □

3 The Main Theorem

In this section we present the elements that are necessary for the proof of our main result (Theorem 3.13). We also include the proof for $|X| = 2$ (see Theorem 3.11), which has all the relevant ideas without the obstacle of notation. We introduce the concept of *V-continuously dependence* (Definition 3.4), which will be the key element to distance the fields F and $\mathbb{R}^{V[X]}$.

Notation: Let $S \subseteq \mathbb{R}$. We write $\overline{S}^{\text{field}}$ to denote the minimal subfield of $\mathbb{R}$ containing S. We write $\overline{S}^{\text{gp}}$ to denote the (real) closure under roots of generalized polynomials with coefficients in S. Finally, we write $\overline{S}^{\exp}$ for the set defined as follows:

$$\overline{S}^{\exp} := \bigcup_{n<\omega} S_n,$$

where $S_0 := S$, and $S_{n+1} := \overline{\overline{S_n}^{\text{gp}}}^{\text{field}}$.

Let S be a subset of $\mathbb{R}$. Then $\overline{S}^{\exp}$ is the smallest really closed subfield of $\mathbb{R}$ containing S. Clearly, $0, 1 \in \overline{S}^{\exp}$. If $a, b, c \in \overline{S}^{\exp}$ then there is $n \in \omega$ such that $a, b, c \in S_{n+1}$. Since S_{n+1} is a field, $ab^{-1} - c \in S_{n+1} \subseteq \overline{S}^{\exp}$. Thus, $\overline{S}^{\exp}$ is a field. Let P be a generalized polynomial with coefficients on $\overline{S}^{\exp}$ and let $z \in \mathbb{R}_{>0}$ be such that $P(z) = 0$. Let $\{a_i, r_i\}_{i\in k} \subseteq \overline{S}^{\exp}$ be the coefficients of P. There is n such that $\{a_i, r_i\}_{i\in k} \subseteq S_n$. Then $z \in \overline{S_n}^{\text{gp}} \subseteq S_{n+1} \subseteq \overline{S}^{\exp}$, and therefore $\overline{S}^{\exp}$ is closed under roots of generalized polynomials.

Lemma 3.1. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuously differentiable function in V. Let $V[g]$ be a forcing extension of V and $\bar{f}$ the version of f in $V[g]$. Suppose that in $V[g]$ there is $\mathsf{a} \in \mathbb{R}^n$ and $\mathsf{z} \in \mathbb{R}$ such that $\bar{f}(\mathsf{a}, \mathsf{z}) = 0$ and $\frac{\partial \bar{f}}{\partial z}|_{(\mathsf{a},\mathsf{z})} \neq 0$.*

Then in V there is an open set $U \subseteq \mathbb{R}^n$ and a continuously differentiable function $h : U \to \mathbb{R}$ such that its version $\bar{h}$ in $V[g]$ satisfies $\bar{h}(\mathsf{a}) = \mathsf{z}$ and $\bar{f}(a, \bar{h}(a)) = 0$ for all $a \in \bar{U}$, where $\bar{U}$ is the version of U in $V[g]$, and $\mathsf{a} \in \bar{U}$.

Lemma 3.1 is just an application of the Implicit Function Theorem. But for our purposes we need to check precisely that in this case the implicit function comes from the ground model V.

Remark 3.2. If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuously differentiable function in V, then its version $\bar{f}$ in $V[g]$ is also continuously differentiable, as will be proved below.

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function in V, we can represent it by a real $r \in \mathbb{R}^V$ coding $f \upharpoonright \mathbb{Q}^{n+1}$ (which is a sequence of countably many reals). Notice that "r codes a continuous function f_r" is equivalent to

$$\varphi(r)\colon \forall (q_m)_{m\in\omega} \subseteq \mathbb{Q}^n \text{ Cauchy } \exists y \in \mathbb{R} \forall \varepsilon \in \mathbb{Q}_{>0} \exists N \in \omega\, \forall n \geq N\, |f_r(q_n) - y| < \varepsilon.$$

This is a $\mathbf{\Pi}^1_2$ formula so absolute between V and $V[g]$ (by Shoenfield's absoluteness) and $\bar{f}$ is well defined and continuous.

Let $\phi(f, f')$ be the following formula stating that f' is the derivative of f.

$$\begin{aligned}\phi(f, f')\colon\ & \forall x_0 \in \mathbb{R}^n \forall \varepsilon \in \mathbb{Q}_{>0} \exists \delta \in \mathbb{Q}_{>0} \forall x \in \mathbb{R}^n \\ & \|x - x_0\| < \delta \to \left| \frac{f(x) - f(x_0) - \langle f'(x_0), x - x_0 \rangle}{\|x - x_0\|} \right| < \varepsilon\end{aligned}$$

which is equivalent to a $\mathbf{\Pi}^1_1$ formula.

Let f be as in Lemma 3.1. Let r be a real that codes this (continuous) function and let s be a real that codes the (continuous) function $g = Df : \mathbb{R}^n \to \mathbb{R}^n$.

Then "g is the derivative of f and g is continuous" is equivalent to

$$\varphi(r) \wedge \varphi(s) \wedge \phi(f_r, f_s),$$

for some reals r and s. This is $\mathbf{\Pi}^1_2$ and therefore absolute between V and $V[g]$. All of this is to conclude that $\bar{f}$ is also continuously differentiable in $V[g]$.

Proof of Lemma 3.1: By Remark 3.2, $\bar{f}$ is also continuously differentiable in $V[g]$. We can apply then the Implicit Function Theorem in $V[g]$. There is then an open set $\bar{U} \subseteq \mathbb{R}^n$ and $\bar{h} : \bar{U} \to \mathbb{R}$ such that $\bar{h}(\mathsf{a}) = \mathsf{z}$ and $\bar{f}(a, h(a)) = 0$ for all $a \in \bar{U}$. Without loss of generality, let us assume $\bar{U}$ is an open set in V.

Notice that

$$\exists (a, z) \in \bar{U} \times \mathbb{R}\, \bar{f}(a, z) = 0 \wedge \frac{\partial \bar{f}}{\partial z}(a, z) \neq 0$$

is true in $V[g]$ and it is $\mathbf{\Sigma}^1_1$ (considering $\bar{f}$ and $\frac{\partial \bar{f}}{\partial z}$ as coded by a real) thus absolute. Therefore, in V there is $(\mathsf{a}', \mathsf{z}') \in \bar{U} \times \mathbb{R}$ such that $f(\mathsf{a}', \mathsf{z}') = 0$ and $\frac{\partial f}{\partial z}(\mathsf{a}', \mathsf{z}') \neq 0$.

By the Implicit Function Theorem (1.13) in V for the point $(\mathsf{a}', \mathsf{z}')$, there is an open set $U \subseteq \mathbb{R}^n$ and $h : U \to \mathbb{R}$ such that $\mathsf{a}' \in U$, $h(\mathsf{a}') = \mathsf{z}'$ and $f(a', h(a')) = 0$ for all $a' \in U$. By Remark 1.14, we can assume h is defined in all of $\bar{U}$. Then by uniqueness of the implicit function $\bar{h}$ is the version of h in $V[g]$ and we can assume that $\bar{U}$ is the version of U in $V[g]$. □

Remark 3.3. From now onwards, we will drop the notation $\bar{f}$ and we will just write f for a continuous function independently of the model in which the function is considered.

Definition 3.4. Working in $V[g]$ a forcing extension of V: Let $S \subseteq \mathbb{R}$ and $z \in \mathbb{R}$. We say that z **depends V-continuously on S** if there is $F \colon \mathbb{R}^k \to \mathbb{R}$ a continuous function in V with $k < \omega$ and $s \in S^k$ such that $F(s) = z$.

Notice that the interesting case is when $z \notin S$ and $z \notin \mathbb{R}^V$. Otherwise, $F = \mathrm{id}$ or $F \equiv z$ would trivially witness V-continuity. Observe that V-continuously dependence on some set S is a local property. So the function F that witnesses this does not need to be defined in the full space $\mathbb{R}^k$, but rather in an open set $U \subseteq \mathbb{R}^k$ such that $s \in U$.

Using Definition 3.4 we can restate Lemma 3.1 as follows:

Lemma 3.5. *Working in $V[g]$ a forcing extension of V: Let $f \colon \mathbb{R}^{n+1} \to \mathbb{R}$ a continuously differentiable function in V. Let $(\mathsf{a}, \mathsf{z}) \in \mathbb{R}^n \times \mathbb{R}$ such that $f(\mathsf{a}, \mathsf{z}) = 0$ and $\frac{\partial f}{\partial z}|_{(\mathsf{a},\mathsf{z})} \neq 0$. Then z depends V-continuously on a.*

Observe that if $z \in \overline{F}$ (the algebraic closure relative to $\mathbb{R}$ of a subfield F), by definition there is a polynomial P with coefficients in a field F such that $P(z) = 0$. Then we can assume P has also the property that $P'(z) \neq 0$. This is because a non-zero polynomial cannot vanish in all its derivatives, so we can exchange P for the highest derivative of P which vanishes at z, satisfying then the conditions of Lemma 3.5. As Proposition 3.6 shows, this will also be the case for the closure by roots of *generalized* polynomials.

Proposition 3.6. *Let $z \in \overline{F}^{\mathrm{gp}}$ (the closure of a field F by roots of generalized polynomials). Then there is a generalized polynomial P with coefficients in the field F such that $P(z) = 0$ and $P'(z) \neq 0$.*

Proof: By definition of $\overline{F}^{\mathrm{gp}}$ there is a generalized polynomial Q with coefficients in F such that $Q(z) = 0$. A generalized polynomial is in particular an analytic function on its domain (Definition 1.15), which in this case includes $\mathbb{R}_{>0}$. If it vanished in all its derivatives at z, then the power series of Q at z would have all coefficients equal to zero. Then, there would be an open set U containing z for which $Q \equiv 0$, then by Theorem 1.17 we get that $Q \equiv 0$ in its domain. This is a contradiction. Therefore Q has a non-zero derivative at z. Let P be the highest derivative of Q that vanishes at z. P is itself a generalized polynomial with coefficients in F, has z as a root, and its derivative at z is non-zero. □

Lemma 3.7. *Let $V[g]$ be a forcing extension of V. In $V[g]$, let $S \subseteq \mathbb{R}$ and $z \in \overline{S}^{\exp}$. Then z depends V-continuously on S.*

Proof: Recall that

$$\overline{S}^{\exp} := \bigcup_{n<\omega} S_n,$$

where $S_0 := S$, and $S_{n+1} := \overline{\overline{S_n}^{\mathrm{gp}}}^{\mathrm{field}}$.

Let $z \in \overline{S}^{\exp}$. Let us prove by induction on n that

$$z \in S_n \implies z \text{ depends } V\text{-continuously on } S.$$

For $n = 0$ and $z \in S_0 = S$, it is clear. Suppose the above statement is true for $n \in \omega$. Let $z \in S_{n+1} = \overline{\overline{S_n}^{\mathrm{gp}}}^{\mathrm{field}}$. Then there are polynomials P and Q with coefficients in $\mathbb{Q}$ and a finite set $t \subseteq \overline{S_n}^{\mathrm{gp}}$ such that $\frac{P(t)}{Q(t)} = z$ (See Theorem 1.3). Clearly the function $\frac{P}{Q}$ witnesses that z depends V-continuously on t. For each $u \in t$ there is $k < \omega$ and there is a generalized polynomial P_u with coefficients $\{a_u, r_u\} \subseteq S_n^k$ such that $P_u(u) = 0$. By Proposition 3.6, we can assume that $P'_u(u) \neq 0$. Consider the function $f\colon \mathbb{R}^{2k} \times \mathbb{R}_{>0} \to \mathbb{R}$ defined by

$$f(a, r, x) = \sum_{i=0}^{k-1} a_i x^{r_i}.$$

Then f is continuously differentiable, $f(a_u, r_u, u) = 0$ and $\frac{\partial f}{\partial x}(a_u, r_u, u) = P'_u(u) \neq 0$. Applying Lemma 3.5, u depends V-continuously on $a_u \cup r_u$.

We know that z depends V-continuously on t and each element u of t depends V-continuously on $a_u \cup r_u$. Then z depends V-continuously on the finite set $w := \{a_u \cup r_u \mid u \in t\} \subseteq S_n$, witnessed by the composition of the functions that witness each step. By inductive hypothesis, each element of w depends V-continuously on S. Therefore z depends V-continuously on S. □

Definition 3.8. Let $f\colon C \subseteq \mathbb{R}^k \to \mathbb{R}$ be a continuous function defined on a compact set C. We say that $\delta\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ **witnesses the uniform continuity of** f if

$$\forall \varepsilon \in \mathbb{Q}_{>0} \forall x, x' \in C \; \|x - x'\| < \delta(\varepsilon) \to |f(x) - f(x')| < \varepsilon.$$

Notice this is a $\mathbf{\Pi}^1_1$ formula therefore absolute between transitive models containing its parameters. Recall that f can be coded by a real.

Now we are ready to prove the main theorem. For better readability, we first include the proof of it when $|X| = 2$ (see Lemma 3.9 and Theorem 3.11), which simplifies the notation but shows all the relevant ideas of the general proof.

Lemma 3.9. *Let x and y be mutually Cohen-generic reals over V. In $V[x, y]$, consider the set $S = \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Then there is $\mathsf{z} \in \mathbb{R}$ that does not depend V-continuously on S.*

Proof: We will define z from x and y. For this purpose we need a function $c\colon {}^{\omega}\omega \to [0, 1]$ that will help us translate properties of the Baire space to the real line in a nice way[1].

Let $a = (a_n)_{n<\omega}$ be an element of ${}^{\omega}\omega$. Then we define $c(a) \in [0, 1]$ as

$$c(a) = \sum_{n<\omega} \frac{a_n \bmod 3}{3^{n+1}},$$

where $m \bmod 3$ denotes the unique number in $\{0, 1, 2\}$ that has the same residue as $m \in \omega$ in the division by 3.

Let s be a sequence of natural numbers of length n. Then we know that there is a closed interval I_s of length $\frac{1}{3^n}$ such that for any $a \in {}^{\omega}\omega$: $c(a) \in \mathrm{int}(I_s)$ implies s is an initial segment of a; and if s is an initial segment of a, then $c(a) \in I_s$.

Think of x and y as elements of the Baire space. Consider $y \circ x \in {}^{\omega}\omega$ (composition of functions) and $\mathsf{z} = c(y \circ x)$. By definition of c, $\mathsf{z} \in \mathbb{R}^{V[x,y]}$.

Suppose on the contrary that z depends V-continuously on S. Then there are $k, l \in \omega$, U an open set in $\mathbb{R}^{k+l}$, $F\colon U \to \mathbb{R}$ a continuous function in V, and sets $\mathsf{r} \subseteq \mathbb{R}^{V[x]}$, $\mathsf{s} \subseteq \mathbb{R}^{V[y]}$ of sizes k and l such that $F(\mathsf{r}, \mathsf{s}) = \mathsf{z}$. Restrict F to a closed ball C in V such that (r, s) belongs to (the version in $V[g]$ of) C. Recall that C is compact, and therefore F is uniformly continuous in C. Let $\delta\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be a witness for the uniform continuity of F restricted to C.

[1] At a first glance, it seems that is not so relevant which is the c we choose. But, for example, a representation of reals as binary sequences does not work for the last part of the proof.

Given a name $\sigma \in V^{\mathbf{C}\times\mathbf{C}}$, we define σ^* and ${}^*\sigma$ in $V^{\mathbf{C}\times\mathbf{C}}$ recursively:

$$\sigma^* = \{(\pi^*, (p,q)) \mid (p,q) \in \mathbf{C} \times \mathbf{C} \text{ and } \exists q'(\pi, (p,q')) \in \sigma\}$$
$${}^*\sigma = \{({}^*\pi, (p,q)) \mid (p,q) \in \mathbf{C} \times \mathbf{C} \text{ and } \exists p'(\pi, (p',q)) \in \sigma\}$$

Let τ be a name for z and $\dot{\mathsf{r}}$ and $\dot{\mathsf{s}}$ nice names for r and s that only depend on the first and second coordinate respectively, namely, $\dot{\mathsf{r}} = \pi^*$ and $\dot{\mathsf{s}} = {}^*\sigma$ for some π, σ names for r and s respectively.

Let (p,q) be a condition in $\mathbf{C} \times \mathbf{C}$ such that

$$(p,q) \Vdash^{\mathbf{C}\times\mathbf{C}}_{V} (\dot{\mathsf{r}}, \dot{\mathsf{s}}) \in \check{C} \text{ and } \check{F}(\dot{\mathsf{r}}, \dot{\mathsf{s}}) = \tau.$$

We will find a condition $(\tilde{p}, \tilde{q})$ below (p,q) that forces the opposite statement, reaching a contradiction.

First let us assume $\mathrm{lh}(p) = \mathrm{lh}(q) = m$. We will extend (p,q) in several steps until getting to the desired $(\tilde{p}, \tilde{q})$, as Figure 1 shows.

1. Extend (p,q) to (p,q') so that it decides $(y \circ x) \restriction m$ by setting $\mathrm{lh}(q') = \max(\mathrm{im}\, p) + 1$. Notice that (p,q') does not decide $(y \circ x)(m)$.

   This implies that (p,q') forces that $\tau \in \check{I}$, where I is some interval given by the code c. More precisely, $I = I_t$ where t is the sequence given by $(q'_{p_0}, \ldots, q'_{p_{m-1}})$.

   Let $\varepsilon = \frac{\mathrm{lh}(I)}{6}$ and $\delta = \delta(\varepsilon)$ (this is computed in V).

2. Extend (p,q') to (p,q'') so that it decides $\dot{\mathsf{s}} \in \check{J}_\delta$ (some ball in $\mathbb{R}^l$ with a rational center and of radius less than $\delta/4$).

3. Extend (p,q'') to (p',q'') so that $p'(m) = \mathrm{dom}(q'')$ and $\mathrm{dom}(p') = m+1$ (see Figure 1). This implies $q''(p'(m))$ is not defined and therefore (p',q'') does not decide the value of $(y \circ x)(m)$.

4. Extend (p',q'') to (p'',q'') so that p' decides $\dot{\mathsf{r}} \in \check{I}_\delta$ (another ball in $\mathbb{R}^k$ of rational center and radius less than $\delta/4$).

   In V, take $(r,s) \in I_\delta \times J_\delta$. Notice that

   $$(p'',q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} (\dot{\mathsf{r}}, \dot{\mathsf{s}}), (\check{r}, \check{s}) \in \check{I}_\delta \times \check{J}_\delta \subseteq \check{C}.$$

   If (r, s) and (r,s) are elements of $I_\delta \times J_\delta$, then $\|r - \mathsf{r}\| < 2\delta/4 = \delta/2$ and $\|s - \mathsf{s}\| < \delta/2$. Then, $\|(r,s) - (\mathsf{r}, \mathsf{s})\| < \delta$. So we obtain

   $$(p'',q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \|(\dot{\mathsf{r}}, \dot{\mathsf{s}}) - (\check{r}, \check{s})\| < \delta.$$

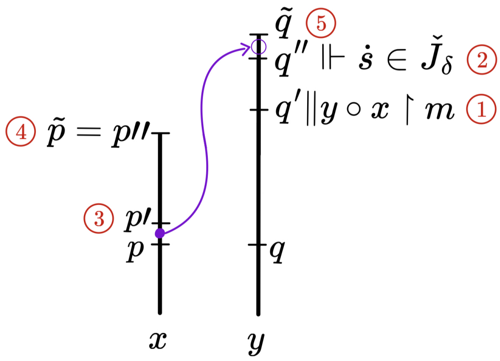

Figure 1: Steps 1–5. Here the arrow represents the fact that $p'(m) = \mathrm{dom}(q'')$, where $m = \mathrm{lh}(p)$.

Let $z = F(r, s) \in \mathbb{R}^V$. Notice that

$$\mathbb{1} \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \check{\delta} \text{ witnesses uniform continuity of } \check{F} \text{ on } \check{C}.$$

Then we get that

$$\begin{aligned} &(p'', q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} |\check{F}(\dot{\mathsf{r}}, \dot{\mathsf{s}}) - \check{F}(\check{r}, \check{s})| < \varepsilon, \text{ i.e.,} \\ &(p'', q'') \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} |\tau - \check{z}| < \varepsilon. \end{aligned} \tag{1}$$

5. Finally, extend (p'', q'') to $(\tilde{p}, \tilde{q})$ so that $\tilde{p} = p''$ and $\mathrm{lh}(\tilde{q}) = \mathrm{lh}(q'') + 1$ as follows. Let $I' = (z - \varepsilon, z + \varepsilon)$. We know that $\mathrm{lh}(I') = 2\varepsilon = \frac{\mathrm{lh}\, I}{3}$, so $I' \cap I \subsetneq I$. Notice that I and I' are intervals in V, in the sense of their endpoints being reals in V. Recall that I is an interval I_t for some finite sequence t and $I_t = I_{t^\frown 0} \cup I_{t^\frown 1} \cup I_{t^\frown 2}$ where each of the intervals $I_{t^\frown i}$ is of length $\frac{\mathrm{lh}(I)}{3}$. Since $\mathrm{lh}(I') = \frac{\mathrm{lh}\, I}{3}$ as well, there is a number $j \in \{0, 1, 2\}$ such that $I_{t^\frown j} \cap I' = \emptyset$. Let us assign to $\tilde{q}(\tilde{p}(m))$ the number j, so that z avoids being in I'.

   Then we get
   $$(\tilde{p}, \tilde{q}) \Vdash^{\mathbb{C}\times\mathbb{C}}_{V} \tau \notin \check{I}',$$
   which contradicts Equation 1.

Thus, z as we defined it does not depend V-continuously on $\mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. □

Remark 3.10. The composition of two mutually generic Cohen reals is a Cohen real.

Theorem 3.11
Let x and y be mutually generic Cohen reals over V. In $V[x,y]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Then, in $V[x,y]$ the transcendence degree of $\mathbb{R}$ over F is continuum.

Proof: Work in $V[x,y]$. Let $S = \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$. Notice that $\overline{S}^{\exp} \supseteq \overline{F}$, where $\overline{F}$ denotes the real-algebraic closure of F. By Lemma 3.9, there is a real z that does not depend V-continuously on S. Applying Lemma 3.7, we deduce that $z \notin \overline{S}^{\exp}$. Recall that $\overline{S}^{\exp}$ is a really closed subfield of $\mathbb{R}^{V[x,y]}$. Using Proposition 2.4, we get that the transcendence degree of $\mathbb{R}^{V[x,y]}$ over $\overline{S}^{\exp}$ is continuum. Therefore the transcendence degree of $\mathbb{R}^{V[x,y]}$ over $\overline{F}$ is also continuum. □

Now we will prove the version of Theorem 3.11 in which we allow to have more than two Cohen reals. For this purpose, we also need a more general version of Lemma 3.9, and that is the role of Lemma 3.12.

Lemma 3.12. *Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the set $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then there is $\mathsf{z} \in \mathbb{R}$ that does not depend V-continuously on S.*

Proof: We will define z from X in a similar way to the one in Lemma 3.9. Let $c \colon {}^{\omega}\omega \to [0,1]$ the same function defined in the proof of Lemma 3.9.

Think of X as a finite subset of the Baire space, $X = \{x_0, \ldots, x_{k-1}\}$ and $k < \omega$. Consider $x_{k-1} \circ \cdots \circ x_0 \in {}^{\omega}\omega$ (composition of functions) and let $\mathsf{z} = c(x_{k-1} \circ \cdots \circ x_0)$.

Suppose on the contrary that z depends V-continuously on S. Then there are $n \in \omega$, $U \subseteq \mathbb{R}^n$ open, $F \colon U \to \mathbb{R}$ continuous function in V and $\mathsf{r} \subseteq S$ such that $F(\mathsf{r}) = \mathsf{z}$. Restrict F to a closed ball C in V such that (r, s) belongs to (the version in $V[g]$ of) C. Since C is compact, F is uniformly continuous in C. Let $\delta \colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be a witness for the uniform continuity of F restricted to C.

To simplify notation we will write X_i^* for $X \backslash \{x_i\}$. Notice that $S = \bigcup_{i<k} \mathbb{R}^{V[X_i^*]}$. We can then partition $r \subseteq S$ in $\{r_i\}_{i<k}$ so that $r = \bigcup_{i<k} r_i$ and $r_i \in \mathbb{R}^{V[X_i^*]}$. Assume without loss of generality that the coordinates of $r \in \mathbb{R}^n$ are ordered with respect to this partition, namely, $r = (r_0, \ldots, r_k)$, where $r_i \in \mathbb{R}^{\mathrm{lh}(r_i)}$ for $i < k$ and $\sum_{i<k} \mathrm{lh}(r_i) = n$.

Given a name $\sigma \in V^{\mathbf{C}^{\mathbf{k}}}$, we define σ_i^* in $V^{\mathbf{C}^{\mathbf{k}}}$ for $i < k$ as follows:

$$\sigma_i^* = \{(\pi_i^*, p) \mid p \in \mathbf{C}^{\mathbf{k}} \text{ and } \exists s \in \mathbf{C}\, (\pi, (p_0, \ldots, p_{i-1}, s, p_{i+1}, \ldots, p_{k-1})) \in \sigma\}.$$

Let τ be a name for z and for each $i < k$ let $\dot{\mathsf{r}}_i$ be a nice name for r_i that only depends on the coordinates $k \backslash \{i\}$, namely, $\dot{\mathsf{r}}_i = \sigma_i^*$ for some σ name for r_i. Let us write $\dot{\mathsf{r}}$ for the name $(\dot{\mathsf{r}}_0, \ldots, \dot{\mathsf{r}}_{k-1})$.

Let q be a condition in $\mathbf{C}^k$ such that

$$q \Vdash^{\mathbf{C}^k}_{V} \dot{\mathsf{r}} \in \check{C} \text{ and } \check{F}(\dot{\mathsf{r}}) = \tau.$$

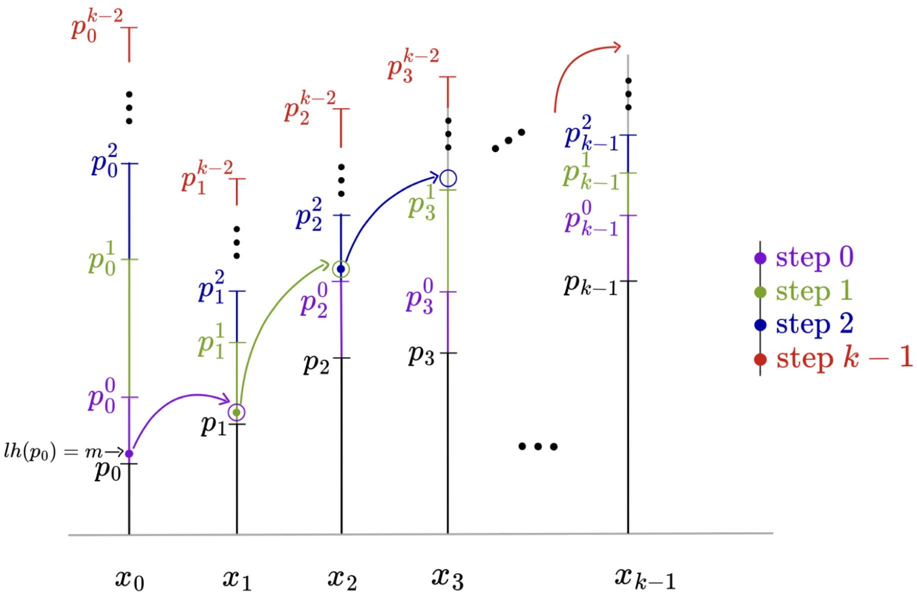

Figure 2: Steps 0 to $(k-1)$ of the construction. The first arrow represents that in Step 0 we take $p_0^0(m) = \mathrm{dom}(p_1)$, where $m = \mathrm{lh}(p_0)$, and similarly for the other arrows.

We will find a condition $\tilde{q}$ below q that forces a contradictory statement, reaching a contradiction.

First let us assume $\mathrm{lh}(q_i) = m$ for all $i < k$. Extend q to $p = (p_0, \ldots, p_{k-1})$ so that $p_{k-1} \circ \cdots \circ p_0 \restriction m$ is defined, setting $p_0 = q_0$ and recursively $\mathrm{lh}(p_{i+1}) = \max(\mathrm{im}\, p_i) + 1$ for $i < k-1$. Thus, it is forced by p that $\tau \in \check{I}$, where I is some interval given by the code c. More precisely, $I = I_t$ where t is $(p_{k-1} \circ \cdots \circ p_0(0), \ldots, p_{k-1} \circ \cdots \circ p_0(m-1))$. Let $\varepsilon = \frac{\mathrm{lh}(I)}{6}$ and let $\delta = \delta(\varepsilon)$, which is computed in V.

Make $p \backslash \{p_0\}$ also decide $\dot{r}_0 \in \check{I}_0$, where I_0 is a ball of radius less than $\frac{\delta}{2k}$.

Notice that $p_{k-1} \circ \cdots \circ p_0(m)$ is not defined yet because we did not extend p_0.

We will extend sequentially p to conditions $p^0, \ldots, p^{k-1}$ in $\mathbf{C}^{k-1}$ as Figure 2 shows.

We denote coordinates with subindices and the different steps with superindices.

Step 0. Define $p^0 \in \mathbf{C}^k$ extending p such that

- $p_0^0(m) = \mathrm{dom}(p_1)$,
- $p_1^0 = p_1$, and
- $p^0 \backslash \{p_1^0\}$ decides $\dot{r}_1 \in \check{I}_1$, where I_1 is a ball in V of radius less than $\frac{\delta}{2k}$.

Step 1. Define $p^1 \in \mathbf{C}^k$ extending p_0 such that

- $p^1_1 \circ p^0_0(m) = \mathrm{dom}(p^0_2)$,
- $p^1_2 = p^0_2$, and
- $p^1 \backslash \{p^1_2\}$ decides $\dot{\mathsf{r}}_2 \in \check{I}_2$, where I_2 is a ball in V of radius less than $\frac{\delta}{2k}$.

In general, for $i = 1, \ldots, k-2$:

Step i. Extend $p^{i-1} \in \mathbf{C}^k$ to p^i so that

- $p^i_i \circ \cdots \circ p^0_0(m) = \mathrm{dom}(p^{i-1}_{i+1})$,
- $p^i_{i+1} = p^{i-1}_{i+1}$, and
- $p^i \backslash \{p^i_{i+1}\}$ decides $\dot{\mathsf{r}}_{i+1} \in \check{I}_{i+1}$, where I_{i+1} is a ball in V of radius less than $\frac{\delta}{2k}$.

In V, take $r \in \prod_{i<k} I_i$, and let $z = F(r)$. Notice that:

$$
\begin{aligned}
&p^{k-2} \Vdash^{\mathbf{C}^k}_{V} \dot{\mathsf{r}}, \check{r} \in \prod_{i<k} \check{I}_i, \\
&p^{k-2} \Vdash^{\mathbf{C}^k}_{V} \|\dot{\mathsf{r}} - \check{r}\| < \delta, \\
&p^{k-2} \Vdash^{\mathbf{C}^k}_{V} \left|\check{F}(\dot{\mathsf{r}}) - \check{F}(\check{r})\right| < \varepsilon, \\
&p^{k-2} \Vdash^{\mathbf{C}^k}_{V} \tau \in \check{I}'.
\end{aligned} \tag{2}
$$

Here, $I' := (z - \varepsilon, z + \varepsilon)$. Notice that I' is an interval with end points in V and $\mathrm{lh}(I') = 2\varepsilon = \frac{\mathrm{lh}(I)}{3}$.

Step k-1. Extend $p^{k-2} \in \mathbf{C}^k$ to p^{k-1} such that $p^{k-1}_{k-1} \circ \cdots \circ p^0_0(m)$ is a number l that makes z avoid I', namely, there is $l \in \{0, 1, 2\}$ such that $I_{t^\frown l} \cap I' = \emptyset$. This is because $\mathrm{lh}\, I' = \mathrm{lh}\, I_{t^\frown 0} = \mathrm{lh}(I_{t^\frown 1}) = \mathrm{lh}(I_{t^\frown 2}) = \frac{\mathrm{lh}(I)}{3}$.

Define $\tilde{q} = p^{k-1}$. Then we get

$$\tilde{q} \Vdash^{\mathbf{C}^k}_{V} \tau \notin \check{I}',$$

which contradicts Equation 2.

Therefore, z does not depend V-continuously on $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. □

Theorem 3.13
Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then, in $V[X]$ the transcendence degree of $\mathbb{R}$ with respect to F is continuum.

Proof: Work in $V[X]$. Let $S = \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Notice that $\overline{S}^{\exp} \supseteq \overline{F}$, where $\overline{F}$ denotes the real-algebraic closure of F. By Lemma 3.12, there is a real z that does not depend V-continuously on S. Applying Lemma 3.7, we deduce that $z \notin \overline{S}^{\exp}$. Recall that $\overline{S}^{\exp}$ is a really closed subfield of $\mathbb{R}^{V[X]}$. Using Proposition 2.4, we get that the transcendence degree of $\mathbb{R}^{V[X]}$ over $\overline{S}^{\exp}$ is continuum. Therefore the transcendence degree of $\mathbb{R}^{V[X]}$ over $\overline{F}$ is also continuum. □

Question 3.14. *Is Theorem 3.13 true for other forcings that add reals?*

Theorem 3.11 shows that a base B of $\mathbb{R}^{V[x,y]}$ over F (the minimum subfield of $\mathbb{R}^{V[x,y]}$ such that $F \supseteq \mathbb{R}^{V[x]} \cup \mathbb{R}^{V[y]}$) has size continuum. If one would like to extend B to a base of $\mathbb{R}^{V[x,y]}$ over $\mathbb{R}^V$ instead, we need to produce a transcendence base C of F over $\mathbb{R}^V$. It turns out that there is a natural candidate for C: a transcendence base of $\mathbb{R}^{V[x]}$ over $\mathbb{R}^V$ union a transcendence base of $\mathbb{R}^{V[y]}$ over $\mathbb{R}^V$, as the following proposition shows.

Proposition 3.15. *Let x, y be Cohen-mutually generic filters over V, where V is a model of* ZFC*. Let $B \subseteq \mathbb{R}^{V[x]}$ be an algebraically independent set over $\mathbb{R}^V$, then B is also algebraically independent over $\mathbb{R}^{V[y]}$.*

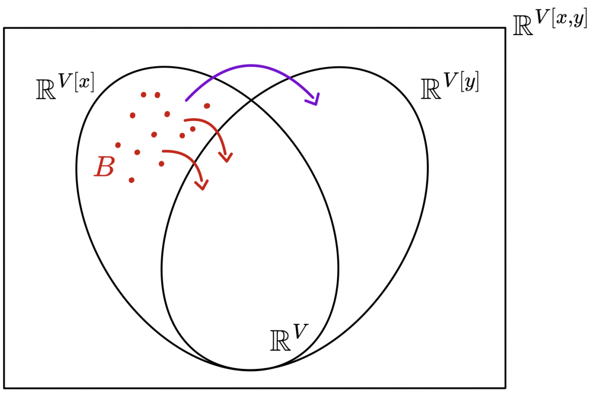

Figure 3: If $B \subseteq \mathbb{R}^{V[x]}$ is algebraically independent over $\mathbb{R}^V$, then B is also algebraically independent over $\mathbb{R}^{V[y]}$.

Proof: Suppose not, i.e. B is algebraically dependent over $\mathbb{R}^{V[y]}$ and without loss of generality assume B is finite. Let n be the cardinality of B. This means there are some finite multi index set $\mathcal{J} \subseteq {}^n\omega$ and some non-zero coefficients $\{d_J\}_{J \in \mathcal{J}} \subseteq \mathbb{R}^{V[y]}$ such that

$$\sum_{J\in\mathcal{J}} d_J B^J = 0, \tag{3}$$

where $J = (j_0, \ldots, j_{n-1})$, $B = \{b_0, \ldots, b_{n-1}\}$ and $B^J : b_0^{j_0} \cdots b_{n-1}^{j_{n-1}}$.

On the other hand, we can think of $\mathbb{R}^{V[y]}[B]$ (the minimal ring containing $\mathbb{R}^{V[y]}$ and B) as a vector space over the field $\mathbb{R}^{V[y]}$. In this context, it is clear that $\{B^J\}_{J\in{}^n\omega}$ spans $\mathbb{R}^{V[y]}(B)$. Therefore, there is $\mathcal{S} \subseteq \{B^J\}_{J\in{}^n\omega}$ such that $\mathcal{S}$ is a basis of $\mathbb{R}^{V[y]}[B]$ as $\mathbb{R}^{V[y]}$-vector space. Let $\mathcal{I}$ be the corresponding index set, i.e, $\mathcal{I} \subseteq {}^n\omega$ and $\mathcal{S} = \{B^J\}_{J\in\mathcal{I}}$.

Note that $\mathcal{J} \nsubseteq \mathcal{I}$. $\mathcal{J}$ cannot be a subset of $\mathcal{I}$, since the equation 3 shows a linear dependence of $\{B^J\}_{J\in\mathcal{J}}$ over $\mathbb{R}^{V[y]}$, and $\mathcal{S} = \{B^J\}_{J\in\mathcal{I}}$ is a basis, in particular, it is linearly independent over $\mathbb{R}^{V[y]}$.

Then, there must be an $I \in \mathcal{J}\backslash\mathcal{I}$, or equivalently, there is an $I \in \mathcal{J}$ such that $B^I \notin \mathcal{S}$. Now, since $B^I \in \mathbb{R}^{V[y]}[B]$, there are unique coefficients $\{c_i\}_{i=0}^{m-1}$ in $\mathbb{R}^{V[y]}$ and vectors $\{S_i\}_{i=0}^{m-1}$ in $\mathcal{S}$ such that

$$\sum_{i=0}^{m-1} c_i S_i = B^I. \tag{4}$$

Since we know

$$V[x,y] \models \text{“}\sum_{i=0}^{m-1} c_i S_i = B^I, \text{ with } \{c_i\}_{i=0}^{m-1} \subseteq \mathbb{R}^{V[y]}\text{”} \tag{5}$$

there is some condition $p \in y \subseteq \mathbf{C}$ such that

$$p \Vdash^{V[x]}_{\mathbf{C}} \text{“}\sum_{i=0}^{m-1} \tau_i \check{S}_i = \check{B}^I, \text{ with } \{\tau_i\}_{i=0}^{m-1} \subseteq \mathbb{R}^{V[\dot{g}]},\text{”} \tag{6}$$

where $\dot{g}$ is the usual name for the generic filter, and τ_i is a name for c_i, i.e., $\tau_i^y = c_i$ for $i = 0, \ldots, m-1$. Note also that $S_i \in \mathcal{S} = \{B^J\}_{J\in\mathcal{I}} \subseteq V[x]$, which justifies the “check” on S_i, for $i = 0, \ldots, m-1$.

Now, let split y into two mutually Cohen generics y_1, y_2 over $V[x]$ such that $p \in y_1, y_2$ as follows: considering y, y_1 and y_2 as functions from ω to ω, let y_1 and y_2 have p as initial segment and be such that $y\backslash p = (y_0\backslash p) \oplus (y_1\backslash p)$, where $\oplus$ is the operation of alternating digits between the reals.

Then, in $V[x, y_1, y_2] = V[x, y]$, we have

$$B^I = \sum_{i=0}^{m-1} \tau_i^1 S_i = \sum_{i=0}^{m-1} \tau_i^2 S_i,$$

where τ_i^1 and τ_i^2 are the interpretations of the name τ_i by y_1 and y_2 respectively, for each $i < m$. In particular, $\tau_i^1 \in V[y_1]$ and $\tau_i^2 \in V[y_2]$. But on the other hand, by uniqueness of the coefficients $\{c_i\}_{i=o}^{m-1}$, and taking into account that $V[x, y_1], V[x, y_2] \subseteq V[x, y]$, we have that $\tau_i^1 = \tau_i^2 = \tau_i^y = c_i$ for $i = 0, \ldots, m-1$. In particular, $c_i \in \mathbb{R}^{V[y_1]} \cap \mathbb{R}^{V[y_2]} = \mathbb{R}^V$. In other words, $B^I = \sum_{i=0}^{m-1} c_i S_i$, where $c_i \in \mathbb{R}^V$. But this means $0 = B^I - \sum_{i=0}^{m-1} c_i S_i$, where $c_i \neq 0$ and the right hand side is not trivial (we chose I such that $I \notin \mathcal{J}$, i.e., $S_i \neq B^I$ for all $i = 0, \ldots, m-1$). This contradicts B being algebraically independent over $\mathbb{R}^V$. □

Remark 3.16. Note that the same proof shows that if $B \subseteq \mathbb{R}^{V[x]}$ is *linearly* independent over $\mathbb{R}^V$, then B is also a linearly independent set over $\mathbb{R}^{V[y]}$.

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