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# Partitions of R^3 in unit circles

*and the Axiom of Choice*

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Joint work with Prof. Ralf Schindler

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MM Mathematics Münster Cluster of Excellence

Set Theory in the UK

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# Partitions of R^3 in unit circles

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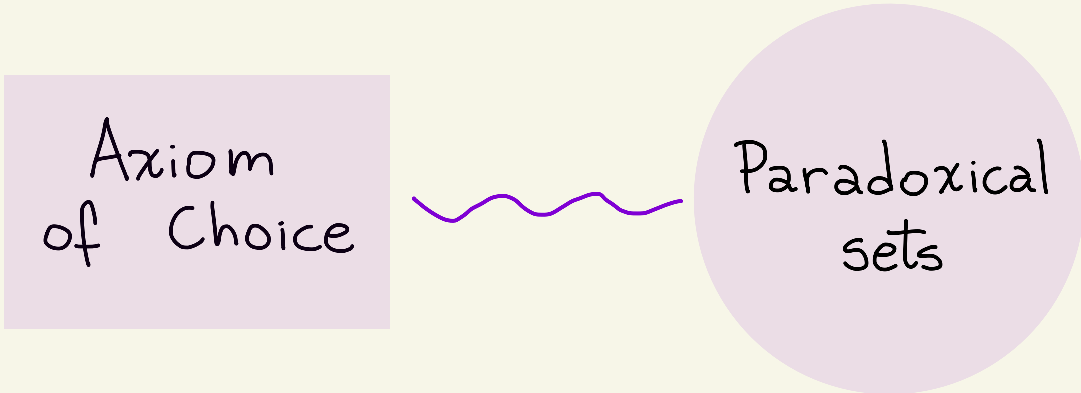
# Coming back

mathoverflow

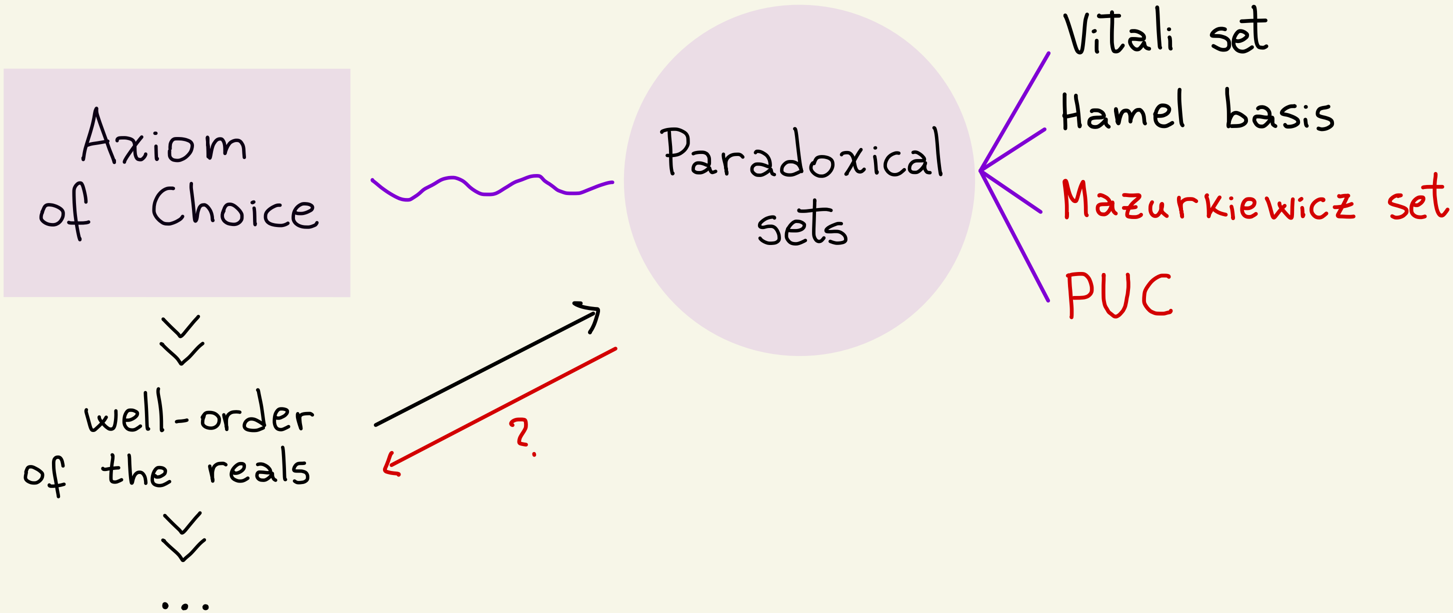
Axiom of choice and a set in the plane that intersects every line in two points

*I think it's about time that you guys find the right general theorem for this (…). It seems like a source of a lot of papers, and it seems to me that there is some bigger theorem in the background that you're just scratching instances of each and every time.*

# Context



# Context



Axiom of Choice
well-order of the reals
...
Paradoxical sets
?
Vitali set
Hamel basis
Mazurkiewicz set
PUC

# Partitions in circles

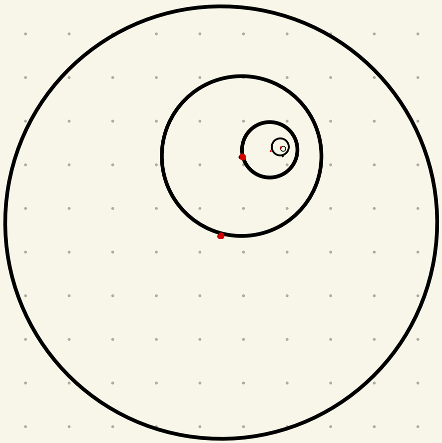
PUC := partition of $\mathbb{R}^3$ in unit circles

Question: (i) Why $\mathbb{R}^3$?

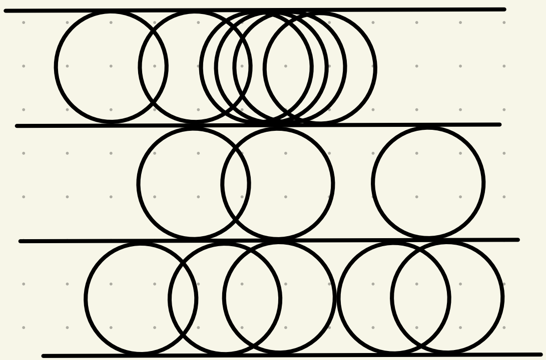
(ii) Why partitions? (1-covering)

(iii) Why circles? Why unit circles?

(i)



(ii)



# Partitions in circles

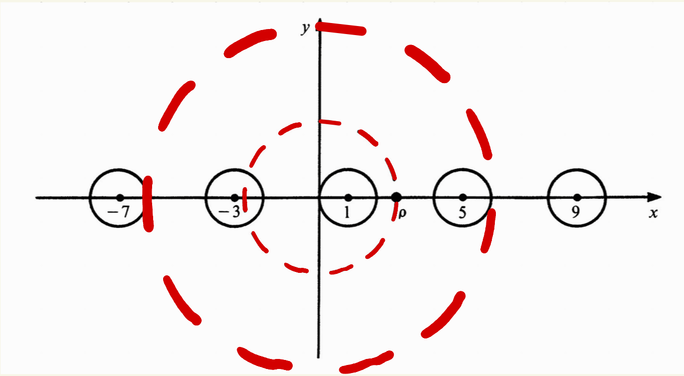
**Theorem (ZF) (Szulkin)**
$\mathbb{R}^3$ can be partitioned in circles.

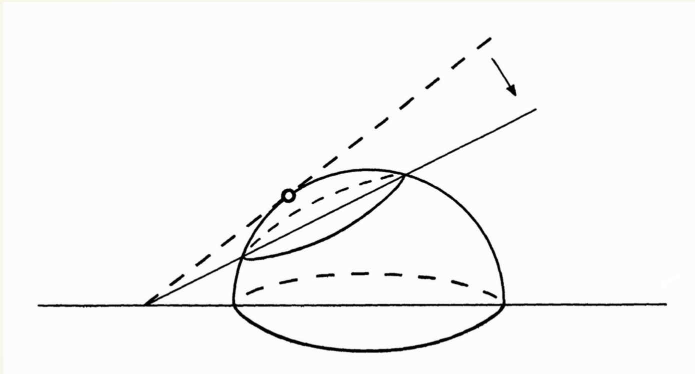
**Theorem (ZFC) (Conway-Croft / Kharazishvili)**
$\mathbb{R}^3$ can be partitioned in unit circles.

# Partitions in circles

**Theorem (ZF) (Szulkin)**

$\mathbb{R}^3$ can be partitioned in circles.



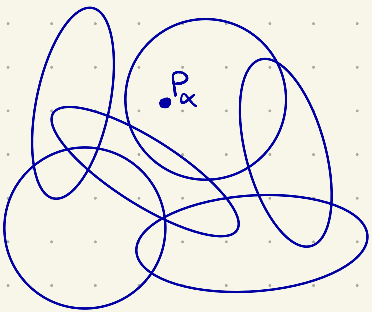



# Partitions in unit circles

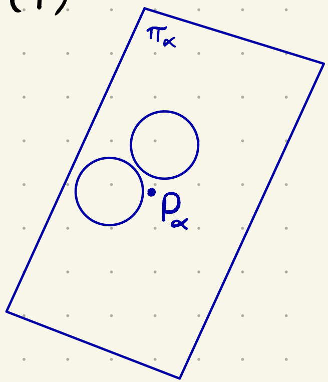
**Theorem** (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Let $\mathbb{R}^3 = \{p_\alpha\}_{\alpha < \mathfrak{c}}$.

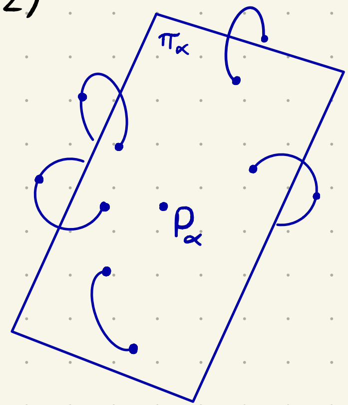
(0)




(1)




(2)



# Partitions in unit circles

**Observation:** The proof shows that any partial PUC of cardinality $< \mathfrak{c}$ can be extended to a (complete) PUC.

**Question:** Can we always extend a partial PUC to a (complete) PUC?

# Partitions in unit circles

**Observation:** The proof shows that any partial PUC of cardinality $< \mathfrak{c}$ can be extended to a (complete) PUC.

**Question:** Can we always extend a partial PUC to a (complete) PUC?

- Sometimes there is not enough "space" to extend a partial PUC.

# The result

**Theorem**

There is a model of ZF + no well-order of $\mathbb{R}$ + $\exists$ PUC

# The model(s)

1. Cohen–Halpern–Lévy model:

$$H := \mathrm{HOD}^{L[g]}_{A \cup \{A\}}$$

where $g$ is $\mathbb{C}(\omega)$-generic over $L$, and $A = \{c_n : n < \omega\}$ is the set of Cohen reals added by $g$.

2.

$$W = L(\mathbb{R}, b)^{L[\tilde{g}, h]}$$

where $\tilde{g}$ is $\mathbb{C}(\omega_1)$-generic over $L$,
$h$ is $\mathbb{P}$-generic over $L[\tilde{g}]$, and
$b = \bigcup h$ is the PUC added by $h$.

# Cohen-Halpern-Lévy model

1. Cohen-Halpern-Lévy model:

$$H := \mathrm{HOD}^{L[g]}_{A \cup \{A\}}$$

where $g$ is $\mathbb{C}(\omega)$-generic over $L$, and $A = \{c_n : n < \omega\}$ is the set of Cohen reals added by $g$.

## Facts about $H$

(i) There is no well-ordering of the reals.

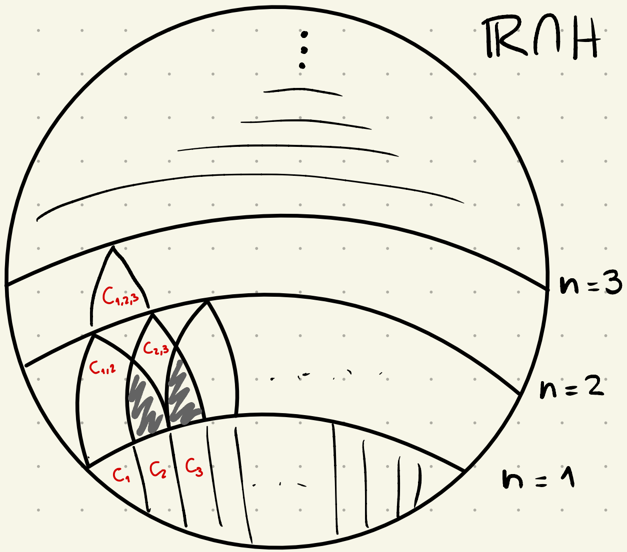
(ii) There is no countable subset of $A$.

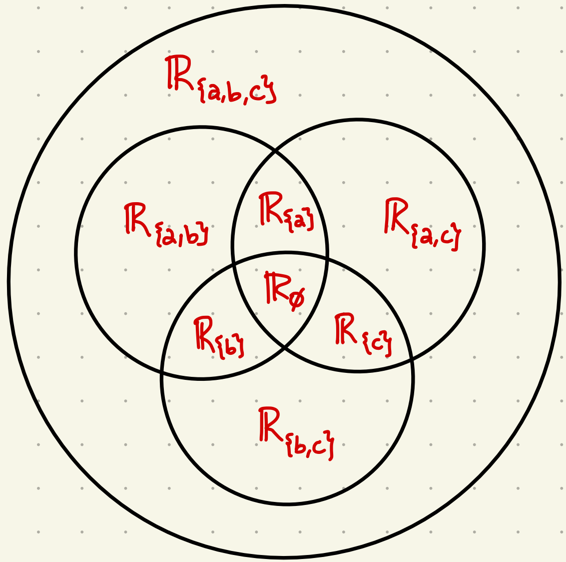
(iii) $$\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a])$$

# Construction of a PUC in H

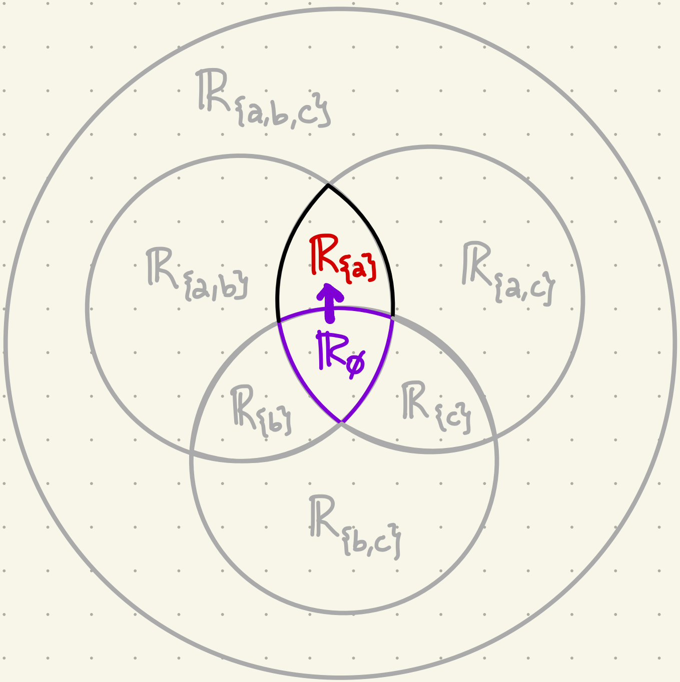
Q: How do we get a PUC in H?

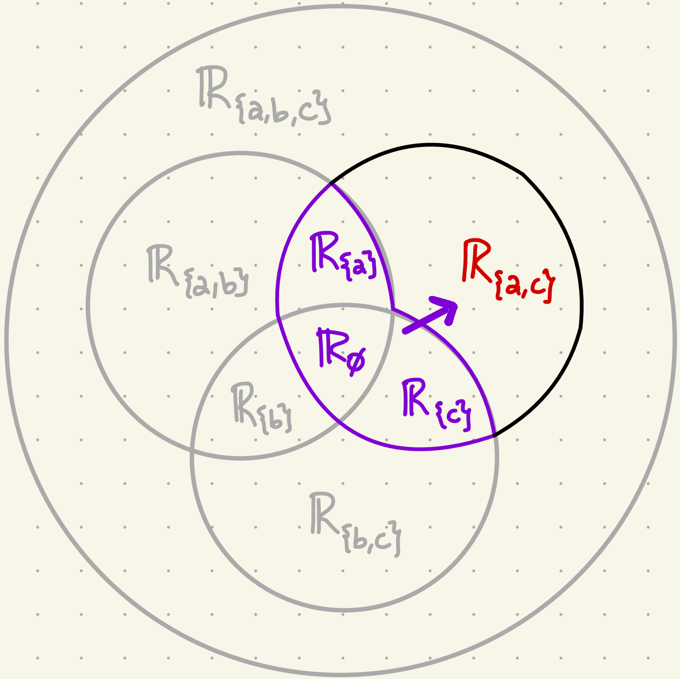
$$\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a]) = \bigcup_{n<\omega} \bigcup_{\substack{|a|=n \\ a \subseteq A}} \mathbb{R}_a$$



# The problems that arise



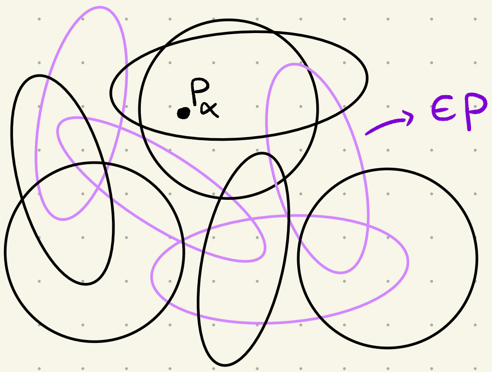

Extendability

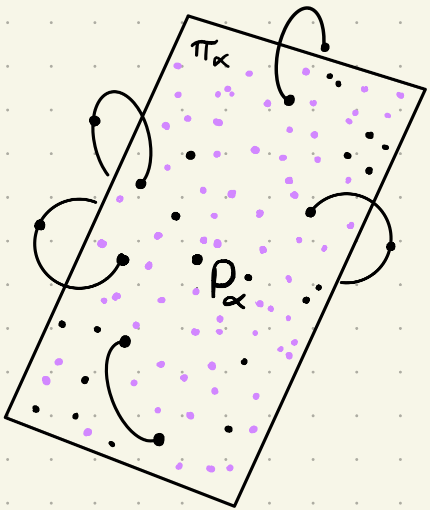
(Strong) Amalgamation

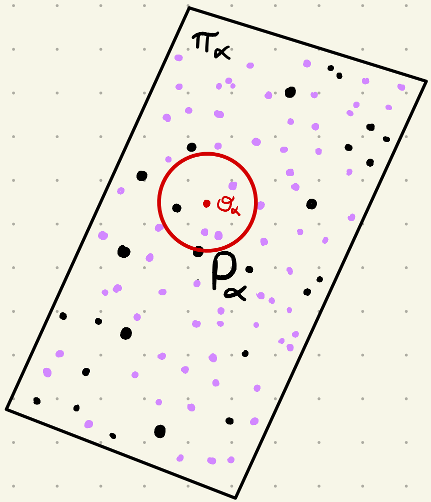
# Construction of a PUC in H

**Lemma 1 (Extendability)**
Let $V$ be a ZFC model and $p \in V$ such that $V \models$ "$p$ is a (partial) PUC". Let $c$ be a Cohen real over $V$. Then, there is $q \in V[c]$ s.t. $V[c] \models$ "$q \supseteq p \wedge q$ is a PUC"

$$\mathbb{R}^3 \setminus \bigcup p = \{P_\alpha\}_{\alpha < \mathfrak{c}}$$



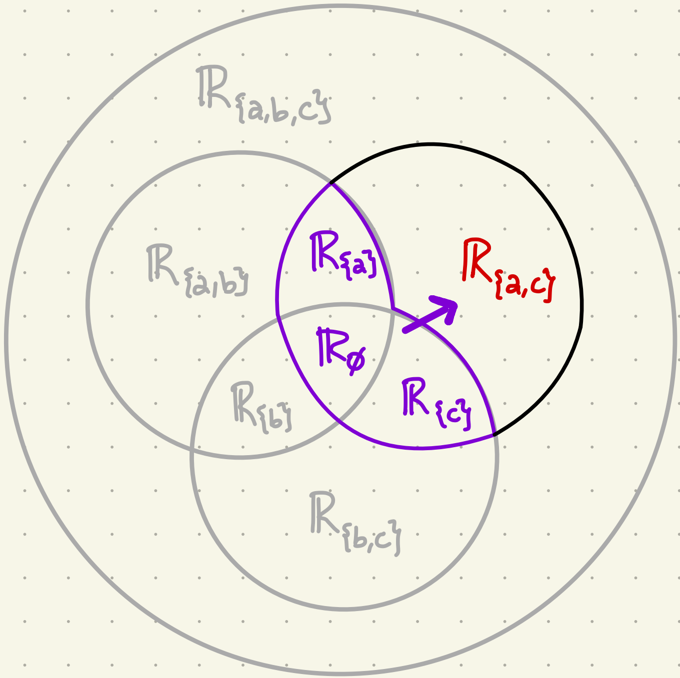



# Construction of a PUC in H

**Fact:** Let $V \models ZFC$ and $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

# Construction of a PUC in H




(Strong) Amalgamation ($n=2$)

# Construction of a PUC in H

## Lemma 2 (Strong Amalgamation)

Let $a, b, c$ mutually generic Cohen reals and let $p, q_1, q_2$ be such that

$$\begin{cases} L[a] \models p \text{ is a PUC} \\ L[a,b] \models q_1 \text{ is a PUC} \\ L[a,c] \models q_2 \text{ is a PUC} \end{cases}$$

and $q_1, q_2 \leqslant_{\mathbb{P}} p$.

Then $L[a,b,c] \models q_1 \cup q_2$ is a partial PUC and it can be extended to a PUC $q \leqslant_{\mathbb{P}} q_1 \cup q_2$

# Algebraic detour

**Fact:** Let $V \models ZFC$ and $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

**Lemma (Transcendence degree)**

Let $V$ be a model of ZFC and let $S$ be a finite set of mutually generic Cohen reals.

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over $\overline{\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}}^{alg}$ is $\mathfrak{c}$.

Q: What can we say if the reals are not Cohen reals?

# Coming back

Axiom of Choice

Paradoxical sets

Vitali set

Hamel basis

Mazurkiewicz set

PUC

$\Downarrow$

well-order of the reals

?

$\Downarrow$

...

Thank you for you attention!

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