Geometric Inversion

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These notes are basically a translation of [3], adding some flavour from the other references. They were made for the training of the German EGMO team in 2021.

1 The idea: reflection over a circle

Rotations, translations and reflections are transformations of the plane. These transformatios move a segment to a different location, while preserving properties such as shape (it will be still a segment) and distance (the lenght of the segment will be the same). See Figure 1.

In the other hand, the inversion is a transformation that changes the shape of the figures, and the distance between them. To invert, we need a circle. In fact, some people call the geometric inversion a reflection over a circle[1]. See Figure 2.

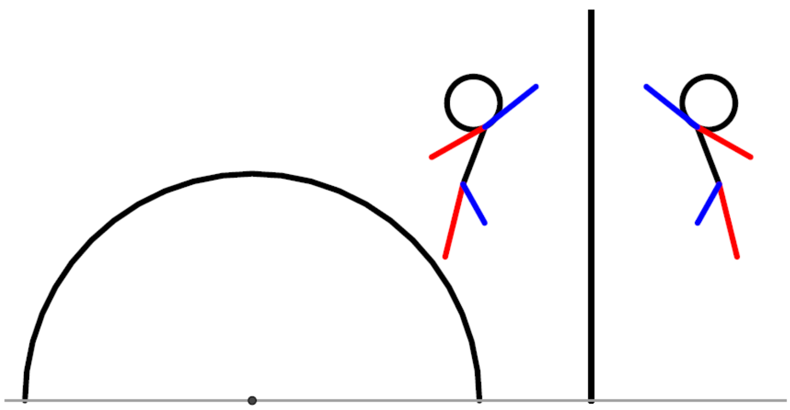

Figure 1: Reflection of a little person across a line.

I like to think inversion as if I had a mirror ball [2]. Everything from the outside goes inside

[1] See, for example, [4].

[2] Of course, inversion does not work exactly as a mirror ball. To begin with, we have a circle and not a ball. It is just a metaphor that I use to have something to start with.

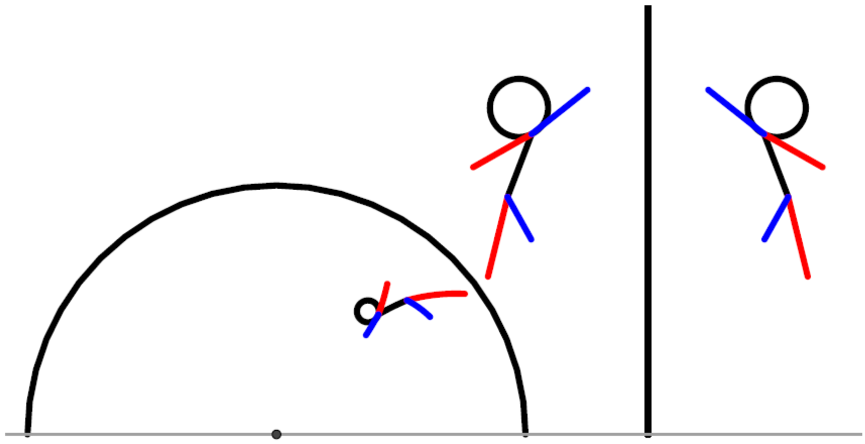

Figure 2: Reflection of a little person over a line and over a circle.

the ball, and of course, the images of the things need to bend to fit inside it. The shape of things change a lot, but some properties are preserved.

2 The mathematical definition

Let Γ be a circumference of radius R and center O. Let P be a point in the plane. The inversion ι over Γ sends the point P to the point P' in the half-line $\overrightarrow{OP}$ such that $OP \cdot OP' = R^2$. See Figure 3.

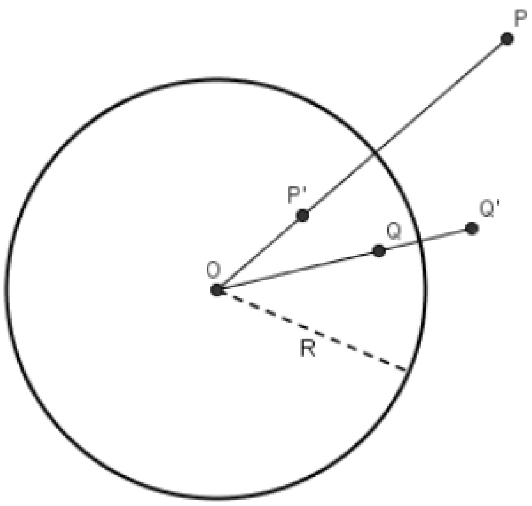

Figure 3: P' is in the half-line OP.

It is important to have in mind the following:

1. The inverse of point outside the circle is a point inside it.

2. The inverse of point inside the circle is a point outside it.

3. If the inverse of the point Q is Q', then the inverse of Q' is Q.

4. Each point in the plane has one (and only one) inverse, except for the center of the circle of inversion. This point has NO inverse.

5. The points lying on the circumference of inversion are fixed by the inversion, $\iota(P) = P$ for every P in Γ.

3 Geometric construction

Suppose we have the circle of inversion ω with center O and radius r and let P be a point outside the circle. Then the inversion of P, P' is a point obtained in the following way:

- Draw the tangent lines to the circle that pass through P.
- These lines intersect ω in X and Y.
- Let Q be the intersection between XY and OP.

It is clear from the construction that Q is in the half line $\overrightarrow{OP}$. Notice that the triangles $\triangle OXQ$ and $\triangle OPX$ are similar. Then we know that $\frac{OX}{OP} = \frac{OQ}{OX}$ and therefore $OP \cdot OQ = OX^2 = r^2$. In conclusion, Q must be P', the inverse of P through ω. See Figure 4.

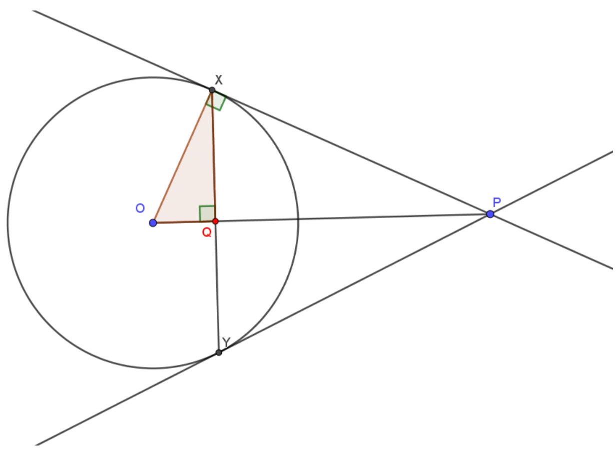

Figure 4: Q is the inversion of the point P over ω.

4 Some basic properties

We are going to prove two basic properties that follow from the definition.

Let P and Q two points outside a circumference centered in O with radio R. Let ι be the inversion over this circumference. Let P' and Q' the inverses of P and Q respectively.

Now we have $R^2 = OP \cdot OP' = OQ \cdot OQ'$. Because of the last equality, we obtain that the quadrilateral $PP'Q'Q$ is cyclic. Then we have $\angle OPQ = \angle OQ'P'$. See Figure 5.

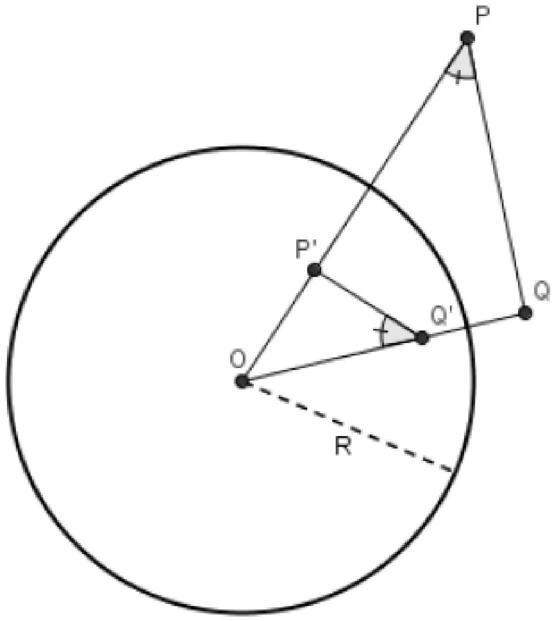

Figure 5: $\angle OPQ = \angle OQP'$.

Note that $\triangle OPQ \sim \triangle OQ'P'$, and then: $\frac{P'Q'}{PQ} = \frac{OP'}{OQ}$. We know that $OP' = \frac{R^2}{OP}$, using the lasts equalities we obtain $\frac{P'Q'}{PQ} = \frac{R^2}{OP \cdot OQ}$, or equivalently:

$$P'Q' = PQ \cdot \frac{R^2}{OP \cdot OQ}$$

In conclusion, for all points P y Q with inverses P' y Q':

1. $\angle OPQ = \angle OQ'P'$
2. $P'Q' = PQ \cdot \frac{R^2}{OP \cdot OQ}$

5 Inversion of figures

The most outstanding characteristic of inversion is how it can help us to solve problems that involve many circumferences. This is because an inversion transforms some circles into lines[3]. Let's look into it.

[3]And lines are usually easier to handle, right?

Line through O: Consider a circunferemce of center O and radius R. Let l be a line through O. If P is in l, the inverse of P will be in l. So the inversion[4] of l is l but all the points change place. See Figure 6.

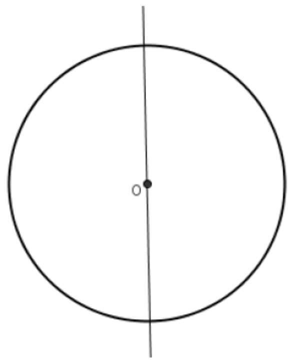

Figure 6: The inverse of a line through the center of inversion is the same line.

Line not passing through O**:** Let t be a line such that O is NOT on t. If we have n points $| \ P_1, P_2, \ldots, P_n$ in the line t, the their inverses $P_1', P_2', \ldots, P_n'$ are all lying in a circumference through O. Hence, a line t not passing through the center of the inversion O is transformed into a circumference that passes through O. See Figure 7.

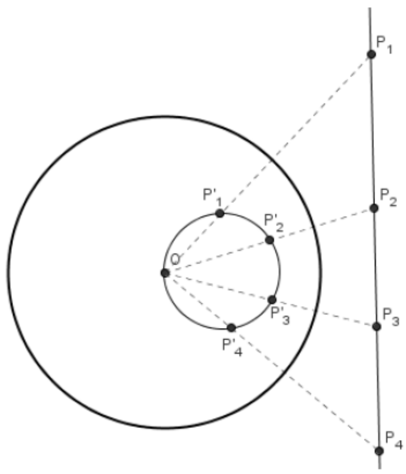

Figure 7: The inverse of a line not passing through the center of inversion is a circle passing through it.

Circle through O**:** Let Γ' a circle passing through O. Let $Q_1, \ldots, Q_n$ points in Γ'. Their inverses $Q_1', \ldots, Q_n'$ are collinear, they lay in a line that does NOT pass through O. See Figure 8.

[4]Actually, the inversion of $l \backslash \{O\}$ is $l \backslash \{O\}$, since O does NOT have an inverse. But it is ok if you say "The inverse of a line through O is the same line", everyone will understand what you are saying and there is usually no need no make the distinction.

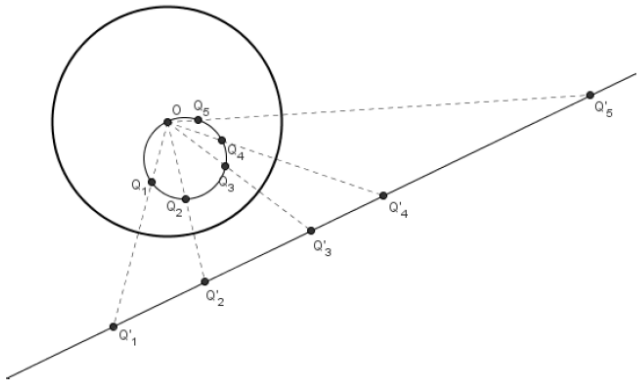

Figure 8: The inverse of a circle passing through the center of inversion is a line not passing through it.

Circle not passing through O: Let Γ_1 be a circle NOT passing through O. Let $A_1, \ldots, A_n$ points in Γ_1. Their inverses $A_1', A_2', \ldots, A_n'$ are all lying in a circumference[5] NOT passing through O. See Figure 9.

To sum up:

- A line through O maps to itself.
- A circle through O maps to a line not containing O and vice-versa.
- A circle not passing through O maps to a circle not passing through O (not necessarily the same).

The second property is the most interesting, since if we have a problem with several circles passing through a point, it is likely that an inversion over a circumference centered in this point will help us. In many problems, the important task is to find a suitable center of the inversion: you might say "Let's consider the inversion of center O and arbitrary radius R".

Remark: Intersection and tangency is preserved by inversion.

6 Comments

- Problems that invert to themselves are usually really easy.
- Usually, an inverted problem will not be easy. However, we often have good reason to believe that the inverted problem is simpler than the original.

[5]It is worth noting that the centers of these circles are also collinear. (However, keep in mind that the centers of the circle do not map to each other!).

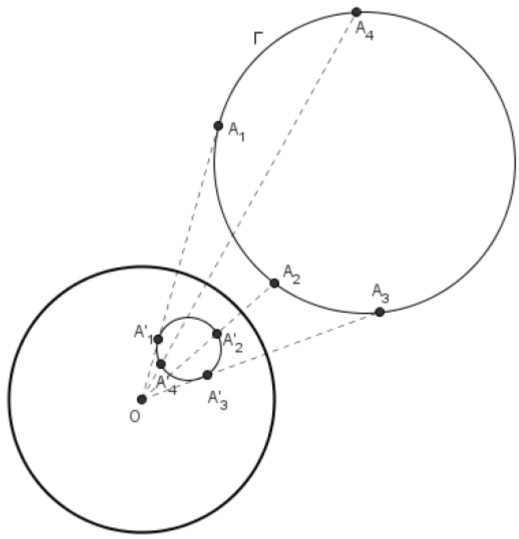

Figure 9: The inverse of a circle not passing through the center of inversion is (another) circle not passing through it.

- In a constest, we do not have to go through the full detail in explaining how to arrive at the inverted image.
- One of the most persuasive reasons to invert: Inversion lets us turn circles into lines.
- Sometimes you have to work a little bit in the original problem and then invert it.

7 Problems

Here there is a personal selection of problems that *can* be solved using inversion. I tried to add the source whenever I knew it. All the problems not from EGMO appear in the websites of the references. I tried to order them by difficulty, but of course, that is a subjective and complex task.

To start

1. Let $ABCD$ be a quadrilateral. If $ABCD$ is cyclic then $AB \cdot CD + BC \cdot AD = AC \cdot BD$[6].

2. Let I be the incenter of a triangle ABC and let D, E, F the points of tangency of the incircle at BC, AC and AB respectively. Prove that the circles passsing through AID, BIE and CIF intersect in another point different from I.

3. (Argentinian IMO Selection Test 2012 P5) In a acutangle triangle ABC, let M be the middle point of $\overline{AB}$ and let P in BC, Q in AC such that AP, BQ are heights of ABC. The circle through B, M, P is tangent to AC. Prove that the circle passing through A, M, Q is tangent to the line $\overleftrightarrow{BC}$.

Medium

1. (IMOmath P3) Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\angle PAB$.

2. (Shortlist IMO 2003) Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and $\Gamma_2; \Gamma_2$ and Γ_3; Γ_3 and $\Gamma_4; \Gamma_4$ and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}$$

3. (USAMO 1993/2) Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections across $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are concyclic.

4. (EGMO 2016 P4) Two circles, ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at a point T_1 and internally tangent to ω_2 at a point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

[6]In fact, the other direction is also true. This is known as *Der Satz des Ptolemäus*

Difficult

1. (EGMO 2013 P5) Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to Ω at the point P. A line parallel to $\boldsymbol{AB}$ and intersecting the interior of triangle $\boldsymbol{A}BC$ is tangent to ω at Q Prove that $\angle ACP = \angle QCB$

2. (EGMO 2018 P5) Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$

7.1 Hints

To start

1. Invert through a circle of center A and radius r (arbitrary radius).

2. Invert through the incircle.

3. (Argentinian IMO Selection Test 2012 P5) Invert with center in M and radius $r = MA$.

Medium

1. (IMOmath P3) Invert through a circle of center C and radius r (arbitrary radius).

2. (Shortlist IMO 2003) Invert through a circle of center P and radius r (arbitrary radius).

3. (USAMO 1993/2) Invert through a circle of center E and radius r (arbitrary radius).

4. (EGMO 2016 P4) Invert through a circle of center X_1 and radius r (arbitrary radius). (Notice that if you invert with other center, there is no way to use the condition that ω_1 and ω_2 have the same radius.)

Difficult

1. (EGMO 2013 P5) Work on the original picture to see if there is a suitable radius r to invert the picture and obtain the same one (or super similar). The center of the inversion will be C.

2. (EGMO 2018 P5) You should prove first that M, U, V are collinear; where M is the middle point of the arc AB that doesn't contain C. Invert through a circle of center M and radius $r = MA$ (arbitrary radius).

7.2 Further hints

To start

1. Use the relation between the lenght of xy and $x'y'$ for every pair of points in the picture.

2. Remember the *geometrical cosntruction* of the inverse of a point.

3. (Argentinian IMO Selection Test 2012 P5) Notice that the picture is the same, but on the other side. Then? Then nothing. It's done.

Medium

1. (IMOmath P3) Use a lot of times that if you invert two figures F_1 and F_2, the intersections (and therefore tangency properties) will be preserved.

2. (Shortlist IMO 2003) Use the relation between the lenght of xy and $x'y'$ for some suitable pairs of points.

3. (USAMO 1993/2) Remember that A, B, C, D will be in the same lines with respect to E. What does it say about the angle with origin E?

4. (EGMO 2016 P4) Do angle chasing to prove what you need, i.e., that *that quadrilateral* is cyclic.

Difficult

1. (EGMO 2013 P5) Choose r so that $r^2 = CY \cdot AC = BC \cdot CX$. Then use the fact that tangency is preserved by inversion to see that the inverted picture will be almost the same. Everything is reflected now.

2. (EGMO 2018 P5) Notice that you obtained the same picture but P andd Q are exchanged somehow. Use that plus the property of inversion that relates some old angles with new ones to get what you want.

References

[1] Evan Chen , *Euclidean Geometry in Mathematical Olympiads (Chapter 8))*, American Mathematical Society, 2016. Available here: `https://www.maa.org/sites/default/files/pdf/ebooks/pdf/EGMO_chapter8.pdf`

[2] Dušan Djukić, *Inversion.* Available here: `https://www.imomath.com/index.php?options=323&lmm=0`

[3] Luis Ferroni, *Geometría 3* (in Spanish), 2012. Available here: `https://omaforos.com.ar/viewtopic.php?f=2&t=1346`

[4] Isaak Moiseevich Yaglom, *Geometric Transformations IV: Circular Transformations.* Vol. 44. MAA, 2009.