lab2

September 25, 2016

1 Lorenz attractor

Differential equation

$$\frac{d\overline{R}}{dt} = \begin{bmatrix} \sigma \cdot (y - x) \\ -y + x \cdot (r - z) \\ -b \cdot z + x \cdot y \end{bmatrix}, \qquad \overline{R} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

1.1 Fixpoints

Assume that

$$\begin{cases} \sigma > 0 \\ b > 0 \\ r > 0 \end{cases}$$

Equation to solve is

$$\frac{d\overline{R}}{dt} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which means

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma \cdot (y - x) \\ -y + x \cdot (r - z) \\ -b \cdot z + x \cdot y \end{bmatrix} \implies \begin{cases} \sigma \cdot (y - x) & = 0 \\ -y + x \cdot (r - z) & = 0 \\ -b \cdot z + x \cdot y & = 0 \end{cases}$$

It's obvious that $x = y = p$, where p is some new variable

$$\begin{cases} x = y & = p \\ p \cdot (r - z - 1) & = 0 \\ -b \cdot z + p^2 & = 0 \end{cases}$$

1.1.1 Solutions

When $p = 0$

$$\begin{cases} x & = 0 \\ y & = 0 \\ z & = 0 \end{cases}$$

Otherwise $r-z-1=0$

$$\begin{cases} z &= r-1 \\ p^2 &= b\cdot z \end{cases} \Longrightarrow \begin{cases} z &= r-1 \\ x &= \pm\sqrt{b\cdot z} \\ y &= \pm\sqrt{b\cdot z} \end{cases}$$

Finally

$$\begin{cases} z &= r-1 \\ x &= \pm\sqrt{b\cdot(r-1)} \\ y &= \pm\sqrt{b\cdot(r-1)} \end{cases}$$

1.2 Stability in center

1.2.1 Linearization

We cannot solve the equation because of $x\cdot y$ and $x\cdot z$ so it needs to be linearized

Just kick those products

$$\frac{d\overline{R}}{dt} = \begin{bmatrix} \sigma\cdot(y-x) \\ -y+x\cdot(r-z) \\ -b\cdot z+x\cdot y \end{bmatrix}$$

In matrix form

$$\frac{d\overline{R}}{dt} = \begin{bmatrix} -\sigma, & \sigma, & 0 \\ r, & -1, & 0 \\ 0, & 0, & -b \end{bmatrix} \cdot \overline{R}$$

It's obvious that $d\overline{C} = d\overline{R}$ so

$$\frac{d\overline{R}}{dt} = \begin{bmatrix} -\sigma, & \sigma, & 0 \\ r, & -1, & 0 \\ 0, & 0, & -b \end{bmatrix} \cdot \overline{R}$$

Solution is

$$\overline{R} = \overline{R}_0 \cdot \exp \begin{bmatrix} -\sigma, & \sigma, & 0 \\ r, & -1, & 0 \\ 0, & 0, & -b \end{bmatrix}$$

1.2.2 Solution

$$0 = \begin{vmatrix} -\sigma-\lambda, & \sigma, & 0 \\ r, & -1-\lambda, & 0 \\ 0, & 0, & -b-\lambda \end{vmatrix} = (-b-\lambda)\cdot[(-\sigma-\lambda)\cdot(-1-\lambda)-\sigma\cdot r] = -(b+\lambda)\cdot\big(\sigma+\sigma\cdot\lambda+\lambda+\lambda^2-\sigma\cdot r\big)$$

Obvious that one sulution is

$$\lambda_1 = -b$$

It's easy to solve

$$\lambda^2+\lambda\cdot(\sigma+1)+\sigma\cdot(1-r)=0$$

$$D = \sigma^2+2\cdot\sigma+1-4\cdot\sigma+4\cdot\sigma\cdot r = (\sigma-1)^2+4\cdot\sigma\cdot r$$

$$\lambda_{2,3} = \frac{-\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\cdot\sigma\cdot r}}{2}$$

Conition of a fixpoint

$$\pm\sqrt{(\sigma-1)^2 + 4\cdot\sigma\cdot r} < \sigma + 1$$

$$r < \frac{(\sigma-1)^2}{4\cdot\sigma}$$

Square both parts

$$(\sigma-1)^2 + 4\cdot\sigma\cdot r < (\sigma+1)^2$$

$$4\cdot\sigma\cdot r < 4\cdot\sigma$$

$$r < 1$$

1.3 Stability in second point

Fixpoints are

$$\overline{R}_0 = \begin{bmatrix} \pm\sqrt{b\cdot(r-1)} \\ \pm\sqrt{b\cdot(r-1)} \\ r+1 \end{bmatrix}$$

Say that $\sqrt{b\cdot(r-1)}$ are multiplied by some constants $c \in \{-1, 1\}$ to ease further calculations
Also define $k = \sqrt{b\cdot(r+1)}$

$$\overline{R}_0 = \begin{bmatrix} -c\cdot k \\ -c\cdot k \\ r-1 \end{bmatrix}$$

Move the center of our coordinates system to these points

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \overline{C} = \overline{R} - \overline{R}_0 = \begin{bmatrix} x + c\cdot k \\ y + c\cdot k \\ z - r + 1 \end{bmatrix}$$

1.3.1 Jacobian

Here is Jacoby matrix of original equation

$$J = \begin{bmatrix} -\sigma, & \sigma, & 0 \\ r - z, & -1, & -x \\ y, & x, & -b \end{bmatrix}$$

Replace values by stable points

$$J' = \begin{bmatrix} -\sigma, & \sigma, & 0 \\ 1, & -1, & -c\cdot k \\ c\cdot k, & c\cdot k, & -b \end{bmatrix}$$

1.3.2 Solution

$$0 = \begin{vmatrix} -\sigma - \lambda, & \sigma, & 0 \\ 1, & -1-\lambda, & -c \cdot k \\ c \cdot k, & c \cdot k, & -b-\lambda \end{vmatrix}$$

We have a polynomial

$$P(\lambda) = \lambda^3 + \lambda^2 \cdot (1 + b + \sigma) + \lambda \cdot b \cdot (\sigma + r) + 2 \cdot b \cdot (r - 1) = 0$$

Since b and σ are positive and $r > 1$ out of critical value for $[0, 0, 0]$

$$\begin{cases} \frac{P(\lambda)}{d\lambda} > 0, & \lambda \geq 0 \\ P(\lambda) = 2 \cdot b \cdot (r-1) > 0 \end{cases} \implies P(\lambda) > 0, \qquad \lambda \in \mathbb{R}^+$$

This means that non-negative real λ is not a solution, thus $\forall r > 1 : \lambda_1 < 0$
Other two eigenvalues should occur as a complex conjugate pair
We need to explore the critical value where $Re(\lambda) = 0$

$$P(\pm i \cdot \Lambda) = \pm i \cdot \Lambda^3 - \Lambda^2 \cdot (1 + b + \sigma) \pm i \cdot \Lambda \cdot b \cdot (\sigma + r) + 2 \cdot b \cdot \sigma \cdot (r - 1) = 0$$

It doesn't matter whether Λ is positive or negative

$$P(i \cdot \Lambda) = \left[2 \cdot b \cdot \sigma \cdot (r-1) - \Lambda^2 \cdot (1 + b + \sigma)\right] + i \cdot \Lambda \cdot \left[b \cdot (\sigma + r) - \Lambda^2\right]$$

We have a system of equations

$$\begin{cases} \Lambda^2 = \frac{2 \cdot b \cdot (r-1)}{1+b+\sigma} \\ \Lambda^2 = b \cdot \sigma (\sigma + r) \end{cases}$$

Left parts are equal so right too

$$\frac{2 \cdot b \cdot \sigma \cdot (r-1)}{1 + b + \sigma} = b \cdot (\sigma + r)$$

$$2 \cdot b \cdot \sigma \cdot r - 2 \cdot b \cdot \sigma = b \cdot r \cdot (1 + b + \sigma) + b \cdot \sigma \cdot (1 + b + \sigma)$$

$$r \cdot (2 \cdot \sigma - 1 - b - \sigma) = \sigma \cdot (1 + b + \sigma + 2)$$

Finally

$$r_c = \frac{\sigma \cdot (\sigma + b + 3)}{\sigma - b - 1}$$

1.4 Phase volume

Need to calculate

$$\frac{1}{V} \cdot \frac{dV}{dt} =?, \qquad V = x \cdot y \cdot z$$

According to Liouville theorem

$$\frac{1}{V} \cdot \frac{dV}{dt} = \overline{\nabla} \cdot \overline{R} = -(\sigma + 1 + b)$$

Solution

$$V(t) = \exp\{V_0 - t \cdot (\sigma + 1 + b)\}$$

This means that phase volume shrinks exponentially and the system is dissipative

In [1]:
```python
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import pylab
pylab.rcParams['figure.figsize'] = (20.0, 20.0)
```

In [2]:
```python
T = 250
dt = 0.01
STEPS = int(T/dt)
```

In [3]:
```python
S = 10.
B = 8./3
INITIAL = (2., -1., 0.)
```

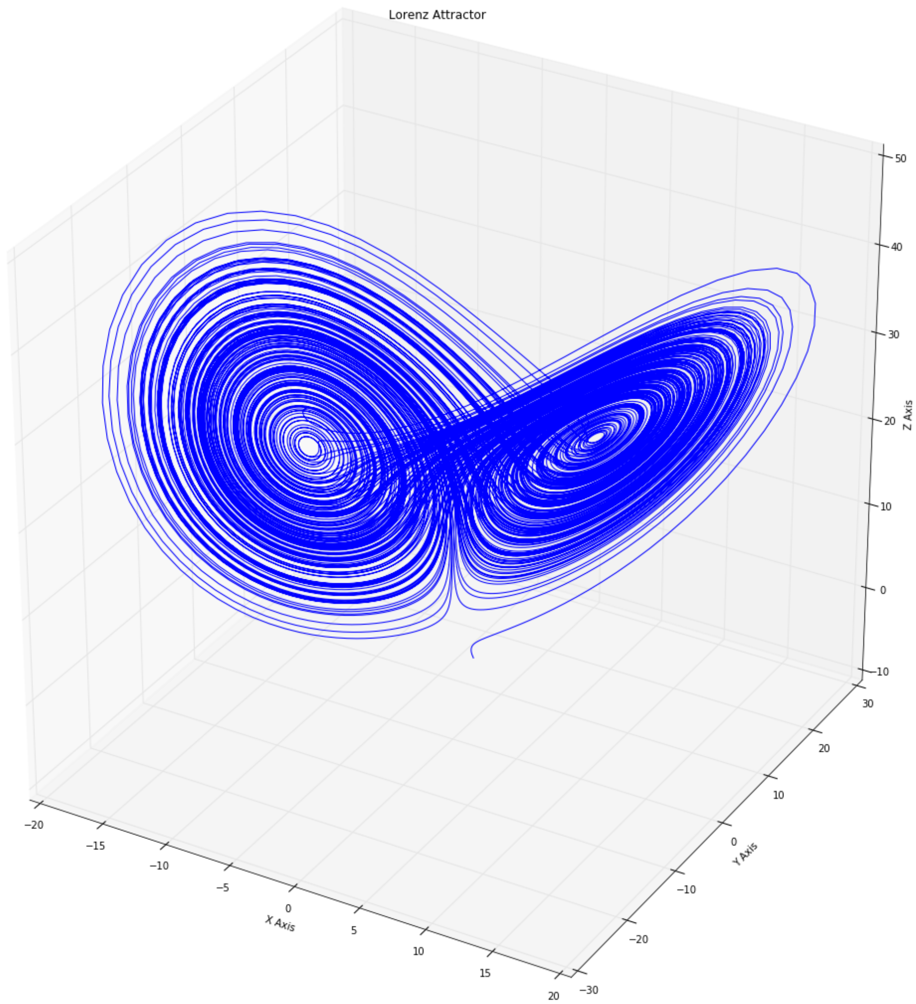
In [4]:
```python
R_C = S * (S + B + 3) / (S - B - 1)
R_L = R_C * .5
R_G = R_C * 2
```

In [5]:
```python
def lorenz(x, y, z, r=R_C, s=S, b=B):
    x_dot = s*(y - x)
    y_dot = r*x - y - x*z
    z_dot = x*y - b*z
    return x_dot, y_dot, z_dot
```

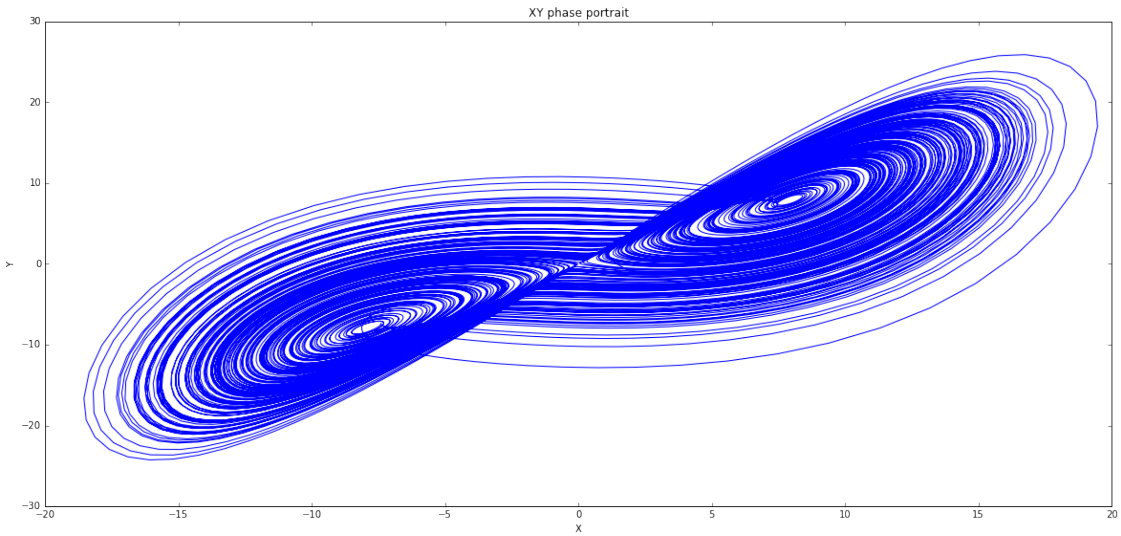
In [6]:
```python
# Need one more for the initial values
xs = np.empty((STEPS + 1,))
ys = np.empty((STEPS + 1,))
zs = np.empty((STEPS + 1,))

# Setting initial values
xs[0], ys[0], zs[0] = INITIAL
```

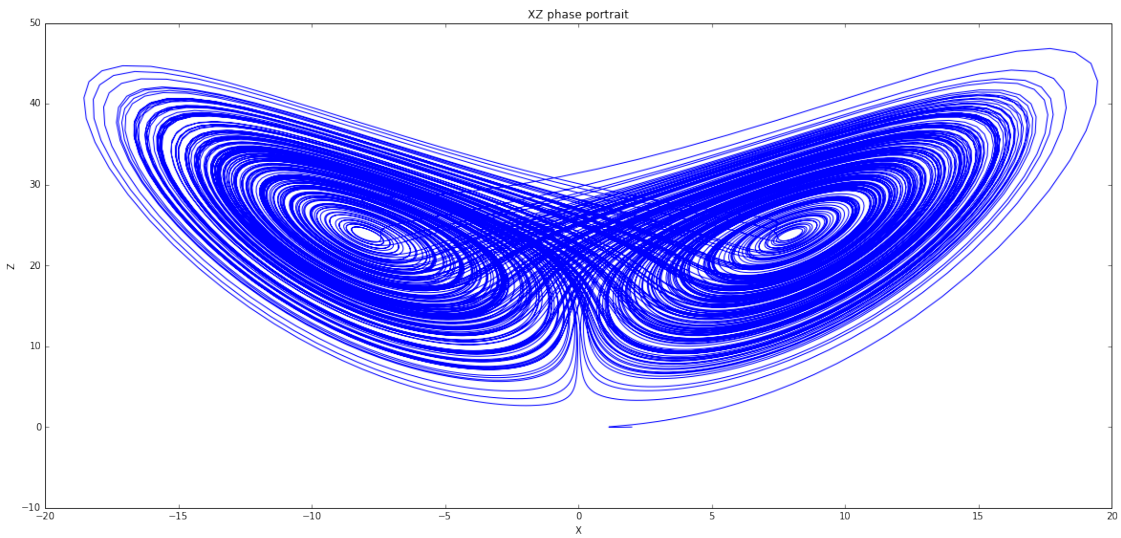
In [7]:
```python
# Stepping through "time".
for i in range(STEPS):
    # Derivatives of the X, Y, Z state
    x_dot, y_dot, z_dot = lorenz(xs[i], ys[i], zs[i], r=R_C)
    xs[i + 1] = xs[i] + (x_dot * dt)
    ys[i + 1] = ys[i] + (y_dot * dt)
    zs[i + 1] = zs[i] + (z_dot * dt)
```

In [8]:
```python
fig = plt.figure()
ax = fig.gca(projection='3d')

ax.plot(xs, ys, zs)
ax.set_xlabel("X Axis")
ax.set_ylabel("Y Axis")
ax.set_zlabel("Z Axis")
ax.set_title("Lorenz Attractor")

plt.show()
```

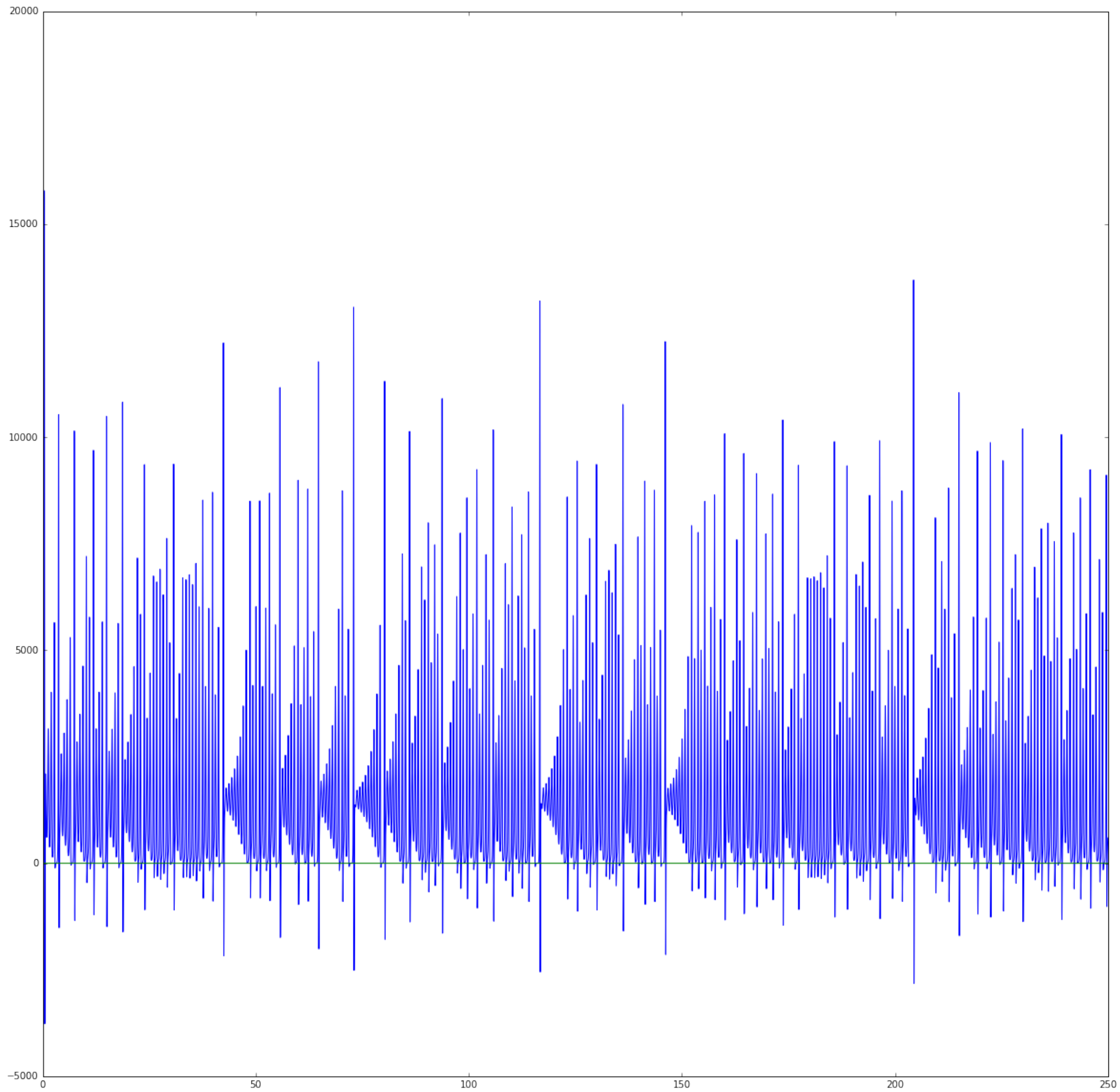


In [9]:
```
plt.figure(1)
ax = plt.subplot(211)
plt.plot(xs, ys)
plt.title('XY phase portrait')
ax.set_xlabel('X')
ax.set_ylabel('Y')

ax = plt.subplot(212)
plt.plot(xs, zs)
plt.title('XZ phase portrait')
ax.set_xlabel('X')
ax.set_ylabel('Z')
plt.show()
```






```
In [10]: INITIAL_VOLUME = INITIAL[0] * INITIAL[1] * INITIAL[2]
         TIME = np.empty((STEPS+1,))
         real_phase_volume = np.empty((STEPS+1,))
         time = 0
         for t in range(STEPS+1):
             time = t*dt
             TIME[t] = time
             real_phase_volume[t] = np.exp(INITIAL_VOLUME-t*(S + 1 + B))
         calculated_phase_volume = xs * ys * zs

         plt.plot(TIME, calculated_phase_volume)
         plt.plot(TIME, real_phase_volume)
         plt.show()
```


20000
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5000
0
−5000
0
50
100
150
200
250