Gram Schmidt process

Notes

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§1 Prerequisites

Definition 1 (Linearly independent vectors). A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if the only solution to the equation $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ is $c_1 = c_2 = \cdots = c_k = 0$.

Intuitively this means that no vector in the set can be expressed as a linear combination of the others.

Definition 2 (Vector basis). A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is a basis for a vector space if:

- It's linearly independent (each vector in the set is linearly independent of one another)
- It spans the vector space (every vector in the vector space can be expressed as a linear combination of the basis vectors)

Theorem 1

If you have n vectors in $\mathbb{F}^n$, place them as columns of an $n \times n$ matrix A. Then

$$\det(A) \neq 0 \iff \text{the vectors are linearly independent} \iff \text{they form a basis of } \mathbb{F}^n.$$

If the number of vectors is not n, or if the vectors are not in $\mathbb{F}^n$, then this determinant criterion does not apply (since there is no associated square matrix).

Definition 3 (Inner (dot) product for vectors over $\mathbb{R}$). The inner product of two vectors $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ is defined as: $\langle v, w\rangle = v_1w_1 + v_2w_2 + \cdots + v_nw_n$, or the matrix form: $\langle v, w\rangle = v^Tw$.

Definition 4 (Inner (dot) product for vectors over $\mathbb{C}$). The inner product of two vectors $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ is defined as: $\langle v, w\rangle = \bar{v_1}w_1 + \bar{v_2}w_2 + \cdots + \bar{v_n}w_n$, or the matrix form: $\langle v, w\rangle = v^\dagger w$.

Definition 5 (Orthogonal vectors)**.** A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is orthogonal if $\langle v_i, v_j\rangle = 0$ for all $i \neq j$.

Definition 6 (Orthonormal vectors)**.** A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is orthonormal if it's orthogonal and each vector has unit norm.

Definition 7 (Norm (length) of a vector)**.** The norm of a vector v is defined as $\|v\| = \sqrt{\langle v, v\rangle}$.

Definition 8 (Projection of a vector onto another)**.** The projection of a vector v onto a vector w is defined as: $\mathrm{proj}_w v = \frac{\langle w,v\rangle}{\langle w,w\rangle} w \qquad (w \neq 0)$.

§2 Gram Schmidt Process

Definition 9 (Gram Schmidt process)**.** Gram Schmidt process is a method for computing a set of orthogonal vectors from a set of linearly independent vectors.

Gram–Schmidt orthogonalization. Given k nonzero linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, the Gram–Schmidt process defines the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ as follows:

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|},\\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|},\\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|},\\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|},\\
&\;\;\vdots & &\;\;\vdots\\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}
$$

The sequence $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is the required system of orthogonal vectors, and the normalized vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ form an orthonormal set. The calculation of the sequence $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is known as Gram–Schmidt orthogonalization, and the calculation of the sequence $\mathbf{e}_1, \ldots, \mathbf{e}_k$ is known as Gram–Schmidt orthonormalization.