2-as lab. darbas

9-as variantas

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Problema 1. Ar duotieji trys vektoriai yra tiesiškai nepriklausomi?

$$\vec{v}_1 = \begin{pmatrix} -3-i \\ -5-i \\ -4-2i \\ 2-3i \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 3-3i \\ -5-5i \\ 3-4i \\ 4+i \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} -3-3i \\ -3+4i \\ -4-5i \\ 3+3i \end{pmatrix}.$$

Sprendimas. Vektoriai $v_1, v_2, \ldots, v_k$ yra tiesiškai nepriklausomi jei:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Taigi, turime lygčių sistemą:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \iff \begin{cases} (-3-i)c_1 + (3-3i)c_2 + (-3-3i)c_3 = 0, \\ (-5-i)c_1 + (-5-5i)c_2 + (-3+4i)c_3 = 0, \\ (-4-2i)c_1 + (3-4i)c_2 + (-4-5i)c_3 = 0, \\ (2-3i)c_1 + (4+i)c_2 + (3+3i)c_3 = 0. \end{cases}$$

Lygčių sistemą paverčiam į Gauso eliminacijos matricą:

$$\left[\begin{array}{ccc|c} -3-i & 3-3i & -3-3i & 0 \\ -5-i & -5-5i & -3+4i & 0 \\ -4-2i & 3-4i & -4-5i & 0 \\ 2-3i & 4+i & 3+3i & 0 \end{array}\right]$$

Pirmiausia panaikinam c_1 koeficientus 2-oje, 3-ioje ir 4-oje eilutėse, iš i-tosios $i \in \{2, 3, 4\}$ eilutės atėmę 1-ą eilutę padaugintą iš m_i, kur:

$$m_2 = \frac{-5-i}{-3-i} = \frac{8-i}{5}, \qquad m_3 = \frac{-4-2i}{-3-i} = \frac{7+i}{5}, \qquad m_4 = \frac{2-3i}{-3-i} = \frac{-3+11i}{10}.$$

Gauname:

$$\sim \left[\begin{array}{ccc|c} -3-i & 3-3i & -3-3i & 0 \\ 0 & -\frac{46}{5} + \frac{2}{5}i & \frac{12}{5} + \frac{41}{5}i & 0 \\ 0 & -\frac{9}{5} - \frac{2}{5}i & -\frac{2}{5} - \frac{1}{5}i & 0 \\ 0 & \frac{8}{5} - \frac{16}{5}i & -\frac{6}{5} + \frac{27}{5}i & 0 \end{array}\right].$$

Toliau taip pat panaikinam c_2 koeficientus 3-ioje ir 4-oje eilutėse:

$$n_3 = \frac{-\frac{9}{5} - \frac{2}{5}i}{-\frac{46}{5} + \frac{2}{5}i} = \frac{41 + 11i}{212}, \qquad n_4 = \frac{\frac{8}{5} - \frac{16}{5}i}{-\frac{46}{5} + \frac{2}{5}i} = \frac{-10 + 18i}{53}.$$

Gaunam:

$$\sim \left[\begin{array}{ccc|c} -3-i & 3-3i & -3-3i & 0 \\ 0 & -\frac{46}{5}+\frac{2}{5}i & \frac{12}{5}+\frac{41}{5}i & 0 \\ 0 & 0 & -\frac{93}{212}-\frac{405}{212}i & 0 \\ 0 & 0 & \frac{108}{53}+\frac{325}{53}i & 0 \end{array}\right].$$

Matosi, kad vienintelis sistemos sprendinys yra $(c_1, c_2, c_3) = (0, 0, 0)$.
$\therefore$ vektoriai v_1, v_2, v_3 yra tiesiškai nepriklausomi.

$\square$

Problema 2. Raskite $B^2 + 3A^\dagger C^{-2} + (A^{-1})^\dagger$, jeigu:

$$A = \begin{pmatrix} 4+i & 3-6i \\ 4+i & -9+6i \end{pmatrix}, \quad B = \begin{pmatrix} -2+i & -6-4i \\ -4-2i & 3+i \end{pmatrix}, \quad C = \begin{pmatrix} -1+4i & -1-6i \\ -8+4i & 5-4i \end{pmatrix}.$$

Sprendimas.

1) Apskaičiuojame B^2.

$$B^2 = BB, \text{ todėl:}$$

$$(B^2)_{11} = (-2+i)^2 + (-6-4i)(-4-2i) = (3-4i) + (16+28i) = 19 + 24i,$$

$$(B^2)_{12} = (-2+i)(-6-4i) + (-6-4i)(3+i) = (16+2i) + (-14-18i) = 2 - 16i,$$

$$(B^2)_{21} = (-4-2i)(-2+i) + (3+i)(-4-2i) = (10+0i) + (-10i) = -10i,$$

$$(B^2)_{22} = (-4-2i)(-6-4i) + (3+i)^2 = (16+28i) + (8+6i) = 24 + 34i.$$

Taigi

$$B^2 = \begin{pmatrix} 19+24i & 2-16i \\ -10i & 24+34i \end{pmatrix}.$$

2) Apskaičiuojame $A^\dagger$.

Transponuojam matricą ir pakeičiam visus elementus jungtiniais:

$$A^\dagger = \begin{pmatrix} 4-i & 4-i \\ 3+6i & -9-6i \end{pmatrix}.$$

3) Apskaičiuojame C^{-1} ir C^{-2}.

Tikrinam ar atvirkštinė gali egzistuoti:

$$\det(C) = (-1+4i)(5-4i) - (-1-6i)(-8+4i) = (11+24i) - (32+44i) = -21 - 20i \neq 0.$$

Panaudosim formulę 2×2 matricos atvirkštinei rasti:

$$C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} 5-4i & 1+6i \\ 8-4i & -1+4i \end{pmatrix}.$$

”Racionalizuojam” trupmenos apačią:

$$\frac{1}{-21-20i} = \frac{-21+20i}{(-21)^2+20^2} = \frac{-21+20i}{841}.$$

Todėl atvirkštinė yra:

$$C^{-1} = \begin{pmatrix} \frac{-25+184i}{841} & \frac{-141-106i}{841} \\ \frac{-88+244i}{841} & \frac{-59-104i}{841} \end{pmatrix}.$$

Jau labai nusibodo skaičiuot ranka, panaudojau (`https://matrix.reshish.com/matrix-multiplication/`).

$$C^{-2} = (C^{-1})^2 = \begin{pmatrix} \frac{5041-34276i}{707281} & \frac{20324-2376i}{707281} \\ \frac{-12128-27536i}{707281} & \frac{30937-12804i}{707281} \end{pmatrix},$$

4) Apskaičiuojame $(A^{-1})^\dagger$.

$$\det(A) = (4+i)(-9+6i) - (3-6i)(4+i) = (-42+15i) - (18-21i) = -60+36i \neq 0.$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} -9+6i & -(3-6i) \\ -(4+i) & 4+i \end{pmatrix} = \frac{1}{-60+36i} \begin{pmatrix} -9+6i & -3+6i \\ -4-i & 4+i \end{pmatrix}.$$

Kadangi

$$\frac{1}{-60+36i} = \frac{-60-36i}{60^2+36^2} = \frac{-5-3i}{408},$$

gaunam:

$$A^{-1} = \begin{pmatrix} \frac{21-i}{136} & \frac{11-7i}{136} \\ \frac{1+i}{24} & -\frac{1+i}{24} \end{pmatrix}.$$

Tada

$$(A^{-1})^\dagger = \begin{pmatrix} \frac{21+i}{136} & \frac{1-i}{24} \\ \frac{11+7i}{136} & -\frac{1-i}{24} \end{pmatrix}.$$

5) Apskaičiuojame $3A^\dagger C^{-2}$.

Vėl naudojam (`https://matrix.reshish.com/matrix-multiplication/`).

$$3A^\dagger C^{-2} = \begin{pmatrix} \frac{-270480-720483i}{707281} & \frac{569592-335943i}{707281} \\ \frac{494145+744030i}{707281} & \frac{-840087+133290i}{707281} \end{pmatrix}.$$

6) Sudedam viską į vieną.

$$B^2 + 3A^\dagger C^{-2} = \begin{pmatrix} 19+24i & 2-16i \\ -10i & 24+34i \end{pmatrix} + \begin{pmatrix} \frac{-270480-720483i}{707281} & \frac{569592-335943i}{707281} \\ \frac{494145+744030i}{707281} & \frac{-840087+133290i}{707281} \end{pmatrix}$$

$$B^2 + 3A^\dagger C^{-2} = \begin{pmatrix} \frac{13167859+16254261i}{707281} & \frac{1984154-11652439i}{707281} \\ \frac{494145-6328780i}{707281} & \frac{16134657+24180844i}{707281} \end{pmatrix},$$

$$B^2 + 3A^\dagger C^{-2} + (A^{-1})^\dagger = \begin{pmatrix} \frac{13160859+16247801i}{707281} & \frac{1985014-10719058i}{707281} \\ \frac{494145+736949i}{707281} & \frac{1615057+24183744i}{707281} \end{pmatrix} + \begin{pmatrix} \frac{21+i}{136} & \frac{1-i}{24} \\ \frac{11+7i}{136} & -\frac{1-i}{24} \end{pmatrix}.$$

$$\boxed{B^2 + 3A^\dagger C^{-2} + (A^{-1})^\dagger = \begin{pmatrix} \frac{1805681725+2211286777i}{96190216} & \frac{48326977-280365817i}{16974744} \\ \frac{74983811-855763113i}{96190216} & \frac{386524487+581047537i}{16974744} \end{pmatrix}}$$

□

Problema 3. Užrašykite vektorių $\vec{v}_1$ bazėje $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

$$\vec{v}_1 = \begin{pmatrix} -2-i \\ 3+i \\ -2+4i \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3+4i \\ 3+2i \\ -5-i \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1+i \\ 3-4i \\ -4-3i \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -2-i \\ -2+i \\ 1-4i \end{pmatrix}.$$

Sprendimas. Ieškome skaliarų $c_2, c_3, c_4 \in \mathbb{C}$, kad

$$\vec{v}_1 = c_2\,\vec{v}_2 + c_3\,\vec{v}_3 + c_4\,\vec{v}_4.$$

Gauname lygčių sistemą:

$$\begin{cases} (3+4i)c_2 + (-1+i)c_3 + (-2-i)c_4 = -2-i, \\ (3+2i)c_2 + (3-4i)c_3 + (-2+i)c_4 = 3+i, \\ (-5-i)c_2 + (-4-3i)c_3 + (1-4i)c_4 = -2+4i. \end{cases}$$

Pasiverčiam į Gauso eliminacijos matricą:

$$\left[\begin{array}{ccc|c} 3+4i & -1+i & -2-i & -2-i \\ 3+2i & 3-4i & -2+i & 3+i \\ -5-i & -4-3i & 1-4i & -2+4i \end{array}\right].$$

Išsprendę sistemą su sympy gauname:

$$c_2 = -\frac{16721}{14893} - \frac{4997}{14893}i, \qquad c_3 = -\frac{1074}{14893} + \frac{6303}{14893}i, \qquad c_4 = -\frac{323}{281} - \frac{540}{281}i.$$

Taigi

$$\boxed{\vec{v}_1 = \left(-\frac{16721}{14893} - \frac{4997}{14893}i\right)\vec{v}_2 + \left(-\frac{1074}{14893} + \frac{6303}{14893}i\right)\vec{v}_3 + \left(-\frac{323}{281} - \frac{540}{281}i\right)\vec{v}_4}$$

$\square$

Problema 4. Iš bazės iš ankstesnio uždavinio gaukite ortonormuotą bazę naudodami Gramo-Šmidto procesą (`https://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process`) ir normuodami gautus vektorius. Detaliai aprašykite kiekvieną žingsnį.

$$\vec{v}_1 = \begin{pmatrix} 3+4i \\ 3+2i \\ -5-i \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1+i \\ 3-4i \\ -4-3i \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} -2-i \\ -2+i \\ 1-4i \end{pmatrix}.$$

> **Pastaba.** Pervadinau $\vec{v}_2$ į $\vec{v}_1$, $\vec{v}_3$ į $\vec{v}_2$, $\vec{v}_4$ į $\vec{v}_3$ (kiekvienam užd. skirtingas kontekstas).

Sprendimas.

Skaičiuojam $\vec{u}_1$

$$\vec{u}_1 = \vec{v}_1.$$

Normalizuojam $\vec{u}_1$, kad gauti $\vec{e}_1$:

$$\langle u_1, u_1 \rangle = 64.$$

$$\|\vec{u}_1\| = 8, \qquad \vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{8}\begin{pmatrix} 3+4i \\ 3+2i \\ -5-i \end{pmatrix}.$$

Skaičiuojam $\vec{u}_2$

$$\langle u_1, v_2 \rangle = 25.$$

Taigi projekcija:

$$\mathrm{proj}_{u_1}(v_2) = \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = \frac{25}{64} u_1.$$

Tada

$$\vec{u}_2 = \vec{v}_2 - \mathrm{proj}_{u_1}(v_2) = \vec{v}_2 - \frac{25}{64}\vec{v}_1.$$

Taigi:

$$\vec{u}_2 = \begin{pmatrix} -1+i \\ 3-4i \\ -4-3i \end{pmatrix} - \frac{25}{64}\begin{pmatrix} 3+4i \\ 3+2i \\ -5-i \end{pmatrix} = \frac{1}{64}\begin{pmatrix} -139-36i \\ 117-306i \\ -131-167i \end{pmatrix}.$$

Normalizuojam $\vec{u}_2$, kad gauti $\vec{e}_2$:

$$\langle u_2, u_2 \rangle = \frac{2703}{64}.$$

Vadinasi

$$\|\vec{u}_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{\frac{2703}{64}} = \frac{\sqrt{2703}}{8},$$

ir ortonormuotas vektorius

$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{\frac{1}{64}\begin{pmatrix} -139 - 36i \\ 117 - 306i \\ -131 - 167i \end{pmatrix}}{\frac{\sqrt{2703}}{8}} = \frac{1}{8\sqrt{2703}} \begin{pmatrix} -139 - 36i \\ 117 - 306i \\ -131 - 167i \end{pmatrix}.$$

Skaičiuojam $\vec{u}_3$

Pirmiausia randame projekciją į $\vec{u}_1$:

$$\langle u_1, v_3 \rangle = -15 + 33i$$

Taigi

$$\mathrm{proj}_{u_1}(v_3) = \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 = \frac{-15 + 33i}{64}\, u_1.$$

Tada randame projekciją į $\vec{u}_2$:

$$\langle u_2, v_3 \rangle = \frac{311 + 263i}{64}$$

Taigi

$$\mathrm{proj}_{u_2}(v_3) = \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 = \frac{311 + 263i}{2703}\, u_2.$$

Belieka įsistatyti ir suprastinti:

$$u_3 = v_3 - \mathrm{proj}_{u_1}(v_3) - \mathrm{proj}_{u_2}(v_3) = v_3 - \frac{-15 + 33i}{64} u_1 - \frac{311 + 263i}{2703} u_2$$

$$= \begin{pmatrix} -2 - i \\ -2 + i \\ 1 - 4i \end{pmatrix} - \frac{-15 + 33i}{64} \begin{pmatrix} 3 + 4i \\ 3 + 2i \\ -5 - i \end{pmatrix} - \frac{311 + 263i}{2703} \frac{1}{64} \begin{pmatrix} -139 - 36i \\ 117 - 306i \\ -131 - 167i \end{pmatrix}$$

Supaprastinus gauname

$$\vec{u}_3 = \frac{1}{51} \begin{pmatrix} 49 - 68i \\ -48 + 15i \\ -36 - 59i \end{pmatrix}.$$

Normalizuojam $\vec{u}_3$, kad gauti $\vec{e}_3$:

$$\langle u_3, u_3 \rangle = \frac{281}{51}.$$

Vadinasi

$$\|\vec{u}_3\| = \sqrt{\langle u_3, u_3 \rangle} = \sqrt{\frac{281}{51}}$$

ir ortonormuotas vektorius

$$\vec{e}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \frac{\frac{1}{51}\begin{pmatrix} 49-68i \\ -48+15i \\ -36-59i \end{pmatrix}}{\sqrt{\frac{281}{51}}} = \frac{\frac{1}{51}\begin{pmatrix} 49-68i \\ -48+15i \\ -36-59i \end{pmatrix}}{\frac{\sqrt{14331}}{51}} = \frac{1}{\sqrt{14331}}\begin{pmatrix} 49-68i \\ -48+15i \\ -36-59i \end{pmatrix}.$$

Rezultatas

Gavom ortonormuotą bazę iš pradinės bazės:

$$\boxed{\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \frac{1}{8}\begin{pmatrix} 3+4i \\ 3+2i \\ -5-i \end{pmatrix},\ \frac{1}{8\sqrt{2703}}\begin{pmatrix} -139-36i \\ 117-306i \\ -131-167i \end{pmatrix},\ \frac{1}{\sqrt{14331}}\begin{pmatrix} 49-68i \\ -48+15i \\ -36-59i \end{pmatrix} \right\}.}$$

□