

## ECE 650, Fall 2018

### A Polynomial-Time Reduction from VERTEX-COVER to CNF-SAT

A *vertex cover* of a graph  $G = (V, E)$  is a subset of vertices  $C \subseteq V$  such that each edge in  $E$  is incident to at least one vertex in  $C$ .

VERTEX-COVER is the following problem:

- Input: An undirected graph  $G = (V, E)$ , and an integer  $k \in [0, |V|]$ .
- Output: True, if  $G$  has a vertex cover of size  $k$ , false otherwise.

CNF-SAT is the following problem:

- Input: a propositional logic formula,  $F$ , in Conjunctive Normal Form (CNF).  
That is,  $F = c_1 \wedge c_2 \wedge \dots \wedge c_m$ , for some positive integer  $m$ . Each such  $c_i$  is called a “clause”. A clause  $c_i = l_{i,1} \vee \dots \vee l_{i,p}$ , for some positive integer  $p$ . Each such  $l_{i,j}$  is called a literal. A literal  $l_{i,j}$  is either an atom, or the negation of an atom.
- Output: True, if  $F$  is satisfiable, false otherwise.

We present a polynomial-time reduction from VERTEX-COVER to CNF-SAT. A polynomial-time reduction is an algorithm that runs in time polynomial in its input. In our case, it takes as input  $G, k$  and produces a formula  $F$  with the property that  $G$  has a vertex cover of size  $k$  if and only if  $F$  is satisfiable.

The use of such a reduction is that given an instance of VERTEX-COVER that we want to solve,  $(G, k)$ , we use the reduction to transform it to  $F$ , and provide  $F$  as input to a SAT solver. The true/false answer from the SAT solver is the answer to the instance of VERTEX-COVER. Assuming the SAT solver works efficiently (for some characterization of “efficient”), we now have an efficient way of solving VERTEX-COVER. Furthermore, the satisfying assignment from the SAT solver can be used to re-construct the solution to VERTEX-COVER.

#### The reduction

Given a pair  $(G, k)$  where  $G = (V, E)$ , denote  $|V| = n$ . Assume that the vertices are named  $1, \dots, n$ . Construct  $F$  as follows.

- The reduction uses  $n \times k$  atomic propositions, denoted  $x_{i,j}$ , where  $i \in [1, n]$  and  $j \in [1, k]$ . A vertex cover of size  $k$  is a list of  $k$  vertices. An atomic proposition  $x_{i,j}$  is true if and only if the vertex  $i$  of  $V$  is the  $j$ th vertex in that list.
- The reduction consists of the following clauses
  - At least one vertex is the  $i$ th vertex in the vertex cover:  
*and for every  $i$ , only one variable is true*  
 $\forall i \in [1, n], \text{ a clause } (x_{1,i} \vee x_{2,i} \vee \dots \vee x_{n,i})$

- No one vertex can appear twice in a vertex cover.

$$\forall m \in [1, n], \forall p, q \in [1, k] \text{ with } p < q, \text{ a clause } (\neg x_{m,p} \vee \neg x_{m,q})$$

In other words, it is not the case that vertex  $m$  appears both in positions  $p$  and  $q$  of the vertex cover.

- No more than one vertex appears in the  $m$ th position of the vertex cover.

$$\forall m \in [1, k], \forall p, q \in [1, n] \text{ with } p < q, \text{ a clause } (\neg x_{p,m} \vee \neg x_{q,m})$$

- Every edge is incident to at least one vertex in the vertex cover.

$$\forall \langle i, j \rangle \in E, \text{ a clause } (x_{i,1} \vee x_{i,2} \vee \dots \vee x_{i,k} \vee x_{j,1} \vee x_{j,2} \vee \dots \vee x_{j,k})$$

The number of clauses in the reduction is  $k + n \binom{k}{2} + k \binom{n}{2} + |E|$ .