Activity - Cubic Splines

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Problem

Given the data set below, write a system of equations to determine the coefficients of the natural cubic splines passing through the data points. Construct the natural spline model and graph the splines together with the data points.

x_i	2	4	7
y_i	2	8	12

Solution

$$
\begin{aligned}
S_i(x) &= y_i : \\
S_1(x) &= a_1 + b_1 x + c_1 x^2 + d_1 x^3, \quad x \in [2,4] \\
a_1 + 2b_1 + 4c_1 + 8d_1 &= 2 \\
S_2(x) &= a_2 + b_2 x + c_2 x^2 + d_2 x^3 \\
S_3(x) &= a_3 + b_3 x + c_3 x^2 + d_3 x^3 \\
S_4(x) &= a_4 + 7b_4 + 49c_4 + 343d_4 = 12 \\
S_1'(x) &= b_1 + 2c_1 x + 3d_1 x^2, \quad x \in [2,4] \\
2c_1 + 6d_1(4) &= 2c_2 + 6d_2(4) \\
b_2 - b_1 &= 8c_1 + 48d_1 - 8c_2 - 48d_2 = 0 \\
2c_1 + 24d_1 - 2c_2 - 24d_2 &= 0 \\
2c_1 + 12d_1 &= 0 \\
2c_4 + 12d_4 &= 0
\end{aligned}
$$

Linear Algebraic System of Equations

$$
\begin{aligned}
a_1 + 2b_1 + 4c_1 + 8d_1 &= 2 \\
a_1 + b_1 + c_1 + d_1 &= 2 \\
a_1 + 6b_1 + 36c_1 + 216d_1 &= 0 \\
a_1 + 7b_1 + 49c_1 + 343d_1 &= 2 \\
a_2 + 8b_2 + 64c_2 + 512d_2 &= 2 \\
b_2 + 8c_2 + 96d_2 - 8c_2 - 96d_2 &= 0 \\
2c_1 + 12d_1 - 2c_2 - 12d_2 &= 0 \\
2c_1 + 12d_1 &= 0 \\
2c_2 + 42d_2 &= 0
\end{aligned}
$$

Matrix

a_1	b_1	c_1	d_1	a_2	b_2	c_2	d_2	
1	2	4	8	0	0	0	0	2
1	1	1	1	0	0	0	0	2
1	6	36	216	0	0	0	0	0
1	7	49	343	0	0	0	0	2
0	0	0	0	1	8	64	512	2
0	0	0	0	0	1	0	0	0
0	0	2	12	0	0	−2	−12	0
0	0	2	12	0	0	0	0	0
0	0	0	0	0	0	2	42	0

$$
\begin{aligned}
a_1 &= -4 \\
b_1 &= 2.3333 \\
c_1 &= 0.5 \\
d_1 &= -0.0833 \\
a_2 &= -12.8888 \\
b_2 &= 0 \\
c_2 &= 1.6666 \\
d_2 &= -0.0555
\end{aligned}
$$

Interval Model

$$2 \leq x < 4 \quad S_1(x) = -4 + 2.333x + 0.5x^2 - 0.0833x^3$$
$$4 \leq x \leq 7 \quad S_2(x) = -12.8888 + 9x - 1.6666x^2 + 0.0555x^3$$

MATLAB Code