11

Frequency-Domain Analysis and Design of Control Systems

11–1 INTRODUCTION

This chapter deals with the frequency-response approach to the analysis and design of control systems. By the term *frequency response*, we mean the steady-state response of a system to a sinusoidal input.

An advantage of the frequency-response approach is that frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often, the transfer functions of complicated components can be determined experimentally by frequency-response tests.

Frequency-response analysis and design of linear control systems is based on the Nyquist stability criterion, which enables us to investigate both the absolute and relative stabilities of linear closed-loop systems from a knowledge of their open-loop frequency-response characteristics.

In this chapter, we first present Bode diagrams (logarithmic plots). We then discuss the Nyquist stability criterion, after which the concept of the phase margin and gain margin is introduced. Finally, the frequency-response approach to the design of control systems is treated. MATLAB approaches to obtain Bode diagrams and Nyquist plots are included in this chapter.

It is noted that although the frequency response of a control system presents a qualitative picture of the transient response, the correlation between the frequency

and transient responses is indirect, except in the case of second-order systems. In designing a closed-loop system, we adjust the frequency-response characteristic of the open-loop transfer function by using several design criteria in order to obtain acceptable transient-response characteristics for the system.

Outline of the chapter. Section 11–1 has given introductory remarks. Section 11–2 presents Bode diagrams of transfer-function systems. In particular, first-order systems and second-order systems are examined in detail. The determination of static error constants from Bode diagrams is also discussed. Section 11–3 treats plotting Bode diagrams with MATLAB. Section 11–4 deals with Nyquist plots and the Nyquist stability criterion. The concept of phase margin and gain margin is introduced. Section 11–5 discusses plotting Nyquist diagrams with MATLAB. Finally, Section 11–6 presents the Bode diagram approach to the design of control systems. Specifically, we discuss the design of the lead compensator, lag compensator, and lag–lead compensator.

11–2 BODE DIAGRAM REPRESENTATION OF THE FREQUENCY RESPONSE

A useful way to represent frequency-response characteristics of dynamic systems is the *Bode diagram*. (Bode diagrams are also called *logarithmic plots of frequency response*.) In this section we treat basic materials associated with Bode diagrams, using first- and second-order systems as examples. We then discuss the problem of identifying the transfer function of a system from the Bode diagram.

Bode diagrams. A sinusoidal transfer function may be represented by two separate plots, one giving the magnitude versus frequency of the function, the other the phase angle (in degrees) versus frequency. A Bode diagram consists of two graphs: a curve of the logarithm of the magnitude of a sinusoidal transfer function and a curve of the phase angle; both curves are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20 \log |G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semilog paper, using a logarithmic scale for frequency and a linear scale for either magnitude (but in decibels) or phase angle (in degrees). (The frequency range of interest determines the number of logarithmic cycles required on the abscissa.)

The main advantage of using a Bode diagram is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. The method, based on asymptotic approximation by straight-line asymptotes, is sufficient if only rough information on the frequency-response characteristics is needed. Should exact curves be desired, corrections can be made easily. The phase-angle curves are readily drawn if a template for the phase-angle curve of $1 + j\omega$ is available.

Note that the experimental determination of a transfer function can be made simple if frequency-response data are presented in the form of a Bode diagram.

The logarithmic representation is useful in that it shows both the low- and high-frequency characteristics of the transfer function in one diagram. Expanding the low-frequency range by means of a logarithmic scale for the frequency is highly advantageous, since characteristics at low frequencies are most important in practical systems. Although it is not possible to plot the curves right down to zero frequency (because $\log 0 = -\infty$), this does not create a serious problem.

Number–decibel conversion line. A number–decibel conversion line is shown in Figure 11–1. The decibel value of any number can be obtained from this line. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20. This relationship may be seen from the formula

$$20 \log (K \times 10^n) = 20 \log K + 20n$$

Note that, when expressed in decibels, the reciprocal of a number differs from its value only in sign; that is, for the number K,

$$20 \log K = -20 \log \frac{1}{K}$$

Bode diagram of gain K. A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value. The log-magnitude curve of a constant gain K is a horizontal straight line at the magnitude of $20 \log K$ decibels. The phase angle of the gain K is zero. Varying K in the transfer function

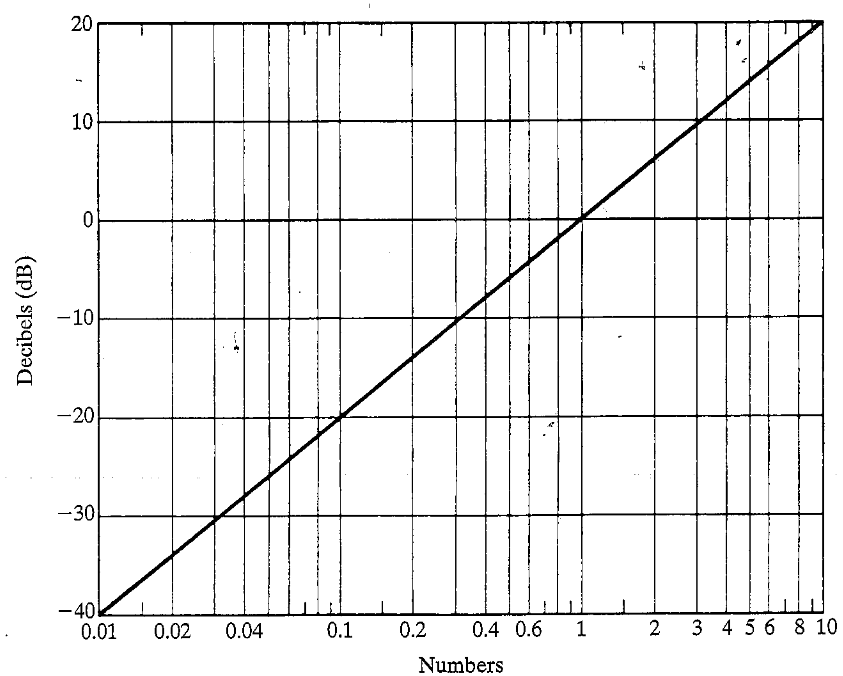


Figure 11–1 Number–decibel conversion line.

raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but does not affect the phase angle.

Bode diagrams of integral and derivative factors. The log magnitude of $1/(j\omega)$ in decibels is

$$\boxed{20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \text{ dB}}$$

The phase angle of $1/j\omega$ is constant and equal to $-90°$.

If the log magnitude $-20 \log \omega$ dB is plotted against ω on a logarithmic scale, the resulting curve is a straight line. Since

$$(-20 \log 10\omega) \text{ dB} = (-20 \log \omega - 20) \text{ dB}$$

the slope of the line is -20 dB/decade.

Similarly, the log magnitude of $j\omega$ in decibels is

$$20 \log |j\omega| = 20 \log \omega \text{ dB}$$

The phase angle of $j\omega$ is constant and equal to $90°$. The log-magnitude curve is a straight line with a slope of 20 dB/decade. Figures 11–2 and 11–3 show Bode diagrams of $1/j\omega$ and $j\omega$, respectively.

Bode diagram of first-order system. Consider the sinusoidal transfer function

$$\frac{X(j\omega)}{P(j\omega)} = G(j\omega) = \frac{1}{Tj\omega + 1}$$

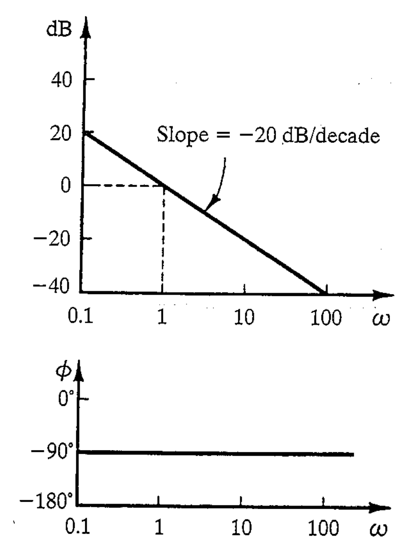


Figure 11–2 Bode diagram of $G(j\omega) = 1/(j\omega)$.

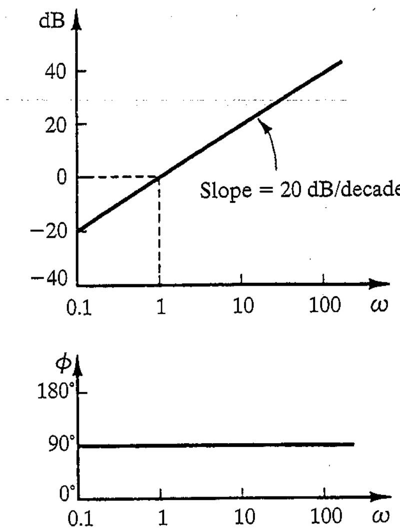


Figure 11–3 Bode diagram of $G(j\omega) = j\omega$.

The log magnitude of this first-order sinusoidal transfer function, in decibels, is

$$20 \log \left| \frac{1}{Tj\omega + 1} \right| = -20 \log \sqrt{\omega^2 T^2 + 1} \text{ dB}$$

$1+iT\omega$

For low frequencies such that $\omega \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{\omega^2 T^2 + 1} \doteqdot -20 \log 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line. For high frequencies such that $\omega \gg 1/T$,

$$-20 \log \sqrt{\omega^2 T^2 + 1} \doteqdot -20 \log \omega T \text{ dB}$$

This is an approximate expression for the high-frequency range. At $\omega = 1/T$, the log magnitude equals 0 dB; at $\omega = 10/T$, the log magnitude is -20 dB. Hence, the value of $-20 \log \omega T$ dB decreases by 20 dB for every decade of ω. For $\omega \gg 1/T$, the log-magnitude curve is therefore a straight line with a slope of -20 dB/decade (or -6 dB/octave).

The preceding analysis shows that the logarithmic representation of the frequency-response curve of the factor $1/(Tj\omega + 1)$ can be approximated by two straight-line asymptotes: a straight line at 0 dB for the frequency range $0 < \omega < 1/T$ and a straight line with slope -20 dB/decade (or -6 dB/octave) for the frequency range $1/T < \omega < \infty$. The exact log-magnitude curve, the asymptotes, and the exact phase-angle curve are shown in Figure 11–4.

The frequency at which the two asymptotes meet is called the *corner* frequency or *break* frequency. For the factor $1/(Tj\omega + 1)$, the frequency $\omega = 1/T$ is the corner frequency, since, at $\omega = 1/T$, the two asymptotes have the same value. (The low-frequency asymptotic expression at $\omega = 1/T$ is $20 \log 1$ dB $= 0$ dB, and the

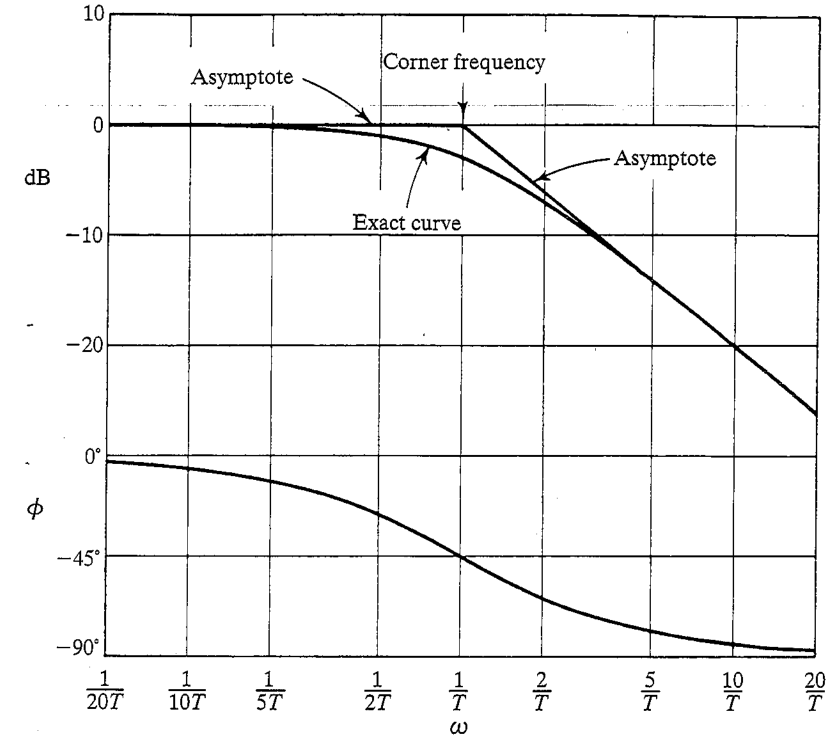


Figure 11–4 Log-magnitude curve together with the asymptotes and phase-angle curve of $1/(j\omega T + 1)$.

high-frequency asymptotic expression at $\omega = 1/T$ is also $20 \log 1$ dB $= 0$ dB.) The corner frequency divides the frequency-response curve into two regions: a curve for the low-frequency region and a curve for the high-frequency region. The corner frequency is very important in sketching logarithmic frequency-response curves.

The exact phase angle ϕ of the factor $1/(Tj\omega + 1)$ is

$$\phi = -\tan^{-1} \omega T$$

At zero frequency, the phase angle is 0°. At the corner frequency, the phase angle is

$$\phi = -\tan^{-1} \frac{T}{T} = -\tan^{-1} 1 = -45°$$

At infinity, the phase angle becomes $-90°$. Because it is given by an inverse-tangent function, the phase angle is skew symmetric about the inflection point at $\phi = -45°$.

The error in the magnitude curve caused by the use of asymptotes can be calculated. The maximum error occurs at the corner frequency and is approximately equal to -3 dB, since

$$-20 \log \sqrt{1 + 1} + 20 \log 1 = -10 \log 2 = -3.03 \text{ dB}$$

The error at the frequency one octave below the corner frequency, that is, at $\omega = 1/(2T)$, is

$$-20 \log \sqrt{\frac{1}{4} + 1} + 20 \log 1 = -20 \log \frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$

The error at the frequency one octave above the corner frequency, that is, at $\omega = 2/T$, is

$$-20 \log \sqrt{2^2 + 1} + 20 \log 2 = -20 \log \frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$

Thus, the error one octave below or above the corner frequency is approximately equal to -1 dB. Similarly, the error one decade below or above the corner frequency is approximately -0.04 dB. The error, in decibels, involved in using the asymptotic expression for the frequency response curve of $1/(Tj\omega + 1)$ is shown in Figure 11–5. The error is symmetric with respect to the corner frequency.

Since the asymptotes are easy to draw and are sufficiently close to the exact curve, the use of such approximations in drawing Bode diagrams is convenient in establishing the general nature of the frequency-response characteristics quickly and with a minimum amount of calculation. Any straight-line asymptotes must have slopes of $\pm 20n$ dB/decade ($n = 0, 1, 2, \ldots$); that is, their slopes must be 0 dB/decade, ± 20 dB/decade, ± 40 dB/decade, and so on. If accurate frequency-response curves are desired, corrections may easily be made by referring to the curve given in Figure 11–5. In practice, an accurate frequency-response curve can be drawn by introducing a correction of 3 dB at the corner frequency and a correction of 1 dB at points one octave below and above the corner frequency and then connecting these points by a smooth curve.

Note that varying the time constant T shifts the corner frequency to the left or to the right, but the shapes of the log-magnitude and the phase-angle curves remain the same.

The transfer function $1/(Tj\omega + 1)$ has the characteristics of a low-pass filter. For frequencies above $\omega = 1/T$, the log magnitude falls off rapidly toward $-\infty$,

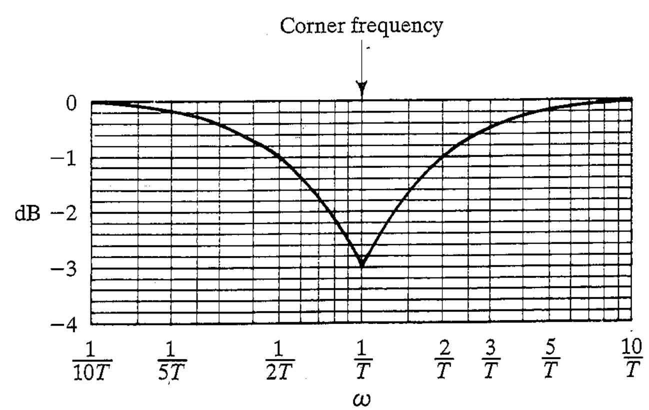


Figure 11–5 Log-magnitude error in the asymptotic expression of the frequency-response curve of $1/(j\omega T + 1)$.

essentially because of the presence of the time constant. In the low-pass filter, the output can follow a sinusoidal input faithfully at low frequencies, but as the input frequency is increased, the output cannot follow the input because a certain amount of time is required for the system to build up in magnitude. Therefore, at high frequencies, the amplitude of the output approaches zero and the phase angle of the output approaches $-90°$. Therefore, if the input function contains many harmonics, then the low-frequency components are reproduced faithfully at the output, while the high-frequency components are attenuated in amplitude and shifted in phase. Thus, a first-order element yields exact, or almost exact, duplication only for constant or slowly varying phenomena.

The shapes of phase-angle curves are the same for any factor of the form $(Tj\omega + 1)^{\pm 1}$. Hence, it is convenient to have a template for the phase-angle curve on cardboard, to be used repeatedly for constructing phase-angle curves for any function of the form $(Tj\omega + 1)^{\pm 1}$. If no such template is available, we have to locate several points on the curve. The phase angles of $(Tj\omega + 1)^{\pm 1}$ are

$$\pm 45° \quad \text{at} \quad \omega = \frac{1}{T}$$

$$\pm 26.6° \quad \text{at} \quad \omega = \frac{1}{2T}$$

$$\pm 5.7° \quad \text{at} \quad \omega = \frac{1}{10T}$$

$$\pm 63.4° \quad \text{at} \quad \omega = \frac{2}{T}$$

$$\pm 84.3° \quad \text{at} \quad \omega = \frac{10}{T}$$

Bode diagram of second-order system. Next, we shall consider a second-order system in the standard form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The sinusoidal transfer function $G(j\omega)$ is

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

or

$$G(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1} \tag{11–1}$$

If $\zeta > 1$, $G(j\omega)$ can be expressed as a product of two first-order terms with real poles. If $0 < \zeta < 1$, $G(j\omega)$ is a product of two complex-conjugate terms. Asymptotic approximations to the frequency-response curves are not accurate for this $G(j\omega)$

with low values of ζ, because the magnitude and phase of $G(j\omega)$ depend on both the corner frequency and the damping ratio ζ.

Noting that

$$|G(j\omega)| = \left|\frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}\right|$$

or

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{11–2}$$

we may obtain the asymptotic frequency-response curve as follows: Since

$$20 \log \left|\frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}\right| = -20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

for low frequencies such that $\omega \ll \omega_n$, the log magnitude becomes

$$-20 \log 1 = 0 \text{ dB}$$

The low-frequency asymptote is thus a horizontal line at 0 dB. For high frequencies $\omega \gg \omega_n$, the log magnitude becomes

$$-20 \log \frac{\omega^2}{\omega_n^2} = -40 \log \frac{\omega}{\omega_n} \text{ dB}$$

The equation for the high-frequency asymptote is a straight line having the slope −40 dB/decade, since

$$-40 \log \frac{10\,\omega}{\omega_n} = -40 - 40 \log \frac{\omega}{\omega_n}$$

The high-frequency asymptote intersects the low-frequency one at $\omega = \omega_n$ since at this frequency

$$-40 \log \frac{\omega_n}{\omega_n} = -40 \log 1 = 0 \text{ dB}$$

This frequency, ω_n, is the corner frequency for the quadratic function considered.

The two asymptotes just derived are independent of the value of ζ. Near the frequency $\omega = \omega_n$, a resonant peak occurs, as may be expected from Equation (11–1). The damping ratio ζ determines the magnitude of this resonant peak. Errors obviously exist in the approximation by straight-line asymptotes. The magnitude of the error is large for small values of ζ. Figure 11–6 shows the exact log-magnitude curves, together with the straight-line asymptotes and the exact phase-angle curves for the quadratic function given by Equation (11–1) with several values of ζ. If corrections

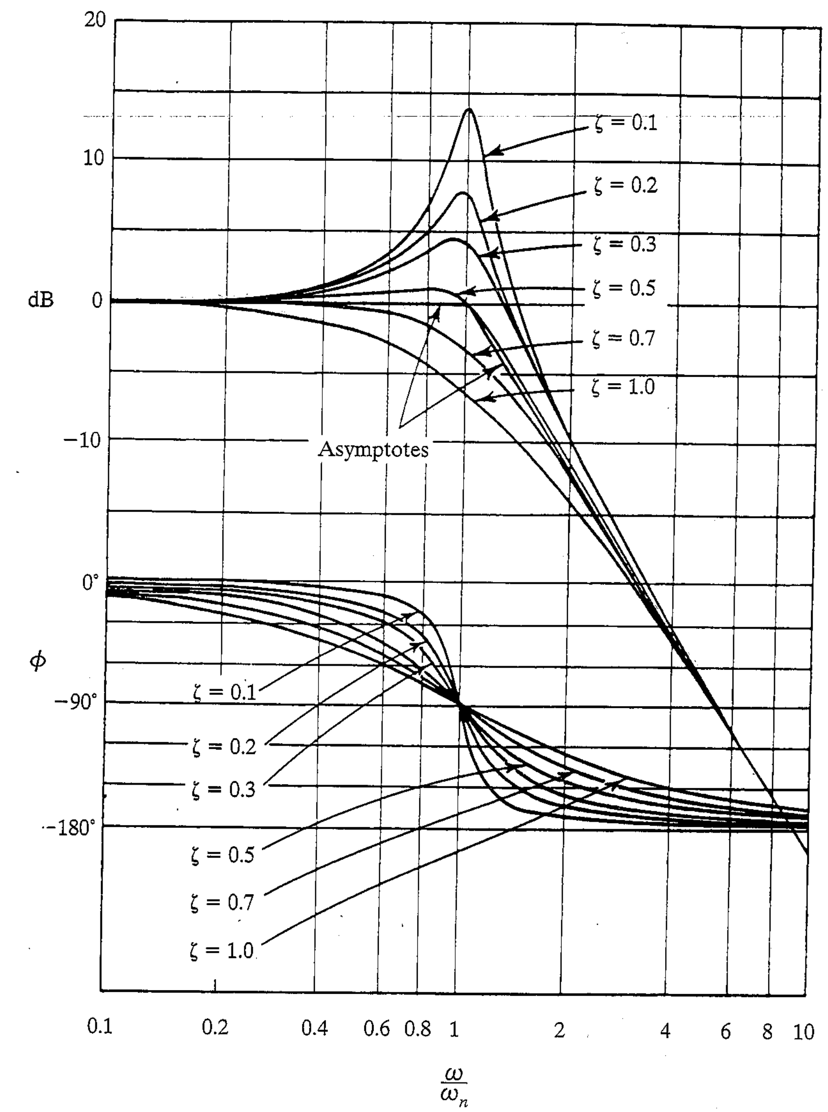


Figure 11–6 Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic sinusoidal transfer function given by Equation (11–1).

are desired in the asymptotic curves, the necessary amounts of correction at a sufficient number of frequency points may be obtained from Figure 11–6.

The phase angle of the quadratic function given by Equation (11–1) is

$$\phi = \Bigg/ \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1} = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{11–3}$$

The phase angle is a function of both ω and ζ. At $\omega = 0$, the phase angle equals 0°. At the corner frequency $\omega = \omega_n$, the phase angle is $-90°$, regardless of ζ, since

$$\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}\infty = -90°$$

At $\omega = \infty$, the phase angle becomes $-180°$. The phase-angle curve is skew symmetric about the inflection point, the point where $\phi = -90°$. There are no simple ways to sketch such phase curves; one needs to refer to the phase-angle curves shown in Figure 11–6.

To obtain the frequency-response curves of a given quadratic transfer function, we must first determine the value of the corner frequency ω_n and that of the damping ratio ζ. Then, by using the family of curves given in Figure 11–6, the frequency-response curves can be plotted.

Note that Figure 11–6 shows the effects of the input frequency ω and the damping ratio ζ on the amplitude and phase angle of the steady-state output. From the figure, we see that, as the damping ratio is increased, the amplitude ratio decreases. The maximum amplitude ratio for a given value of ζ occurs at a frequency that is less than the undamped natural frequency ω_n. Notice that the frequency ω_r at which the amplitude ratio is a maximum occurs at

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

This frequency is called the *resonant frequency*.

The value of ω_r can be obtained as follows: From Equation (11–2), since the numerator of $|G(j\omega)|$ is constant, a peak value of $|G(j\omega)|$ will occur when

$$g(\omega) = \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \qquad (11–4)$$

is a minimum. Since Equation (11–4) can be written as

$$g(\omega) = \left[\frac{\omega^2 - \omega_n^2(1 - 2\zeta^2)}{\omega_n^2}\right]^2 + 4\zeta^2(1 - \zeta^2)$$

the minimum value of $g(\omega)$ occurs at $\omega = \omega_n\sqrt{1 - 2\zeta^2}$. Thus the resonant frequency ω_r is

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \qquad 0 \le \zeta \le 0.707 \qquad (11–5)$$

As the damping ratio ζ approaches zero, the resonant frequency approaches ω_n. For $0 < \zeta \le 0.707$, the resonant frequency ω_r is less than the damped natural frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, which is exhibited in the transient response. From Equation (11–5), it can be seen that, for $\zeta > 0.707$, there is no resonant peak. The magnitude $|G(j\omega)|$ decreases monotonically with increasing frequency ω. (The magnitude is less than 0 dB for all values of $\omega > 0$; recall that, for $0.7 < \zeta \le 1$, the step response is oscillatory, but the oscillations are well damped and are hardly perceptible.)

The magnitude of the resonant peak M_r can be found by substituting Equation (11–5) into Equation (11–2). For $0 \le \zeta \le 0.707$,

$$M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{\;} \qquad (11–6)$$

As ζ approaches zero, M_r approaches infinity. This means that, if the undamped system is excited at its natural frequency, the magnitude of $G(j\omega)$ becomes infinite. For $\zeta > 0.707$,

$$M_r = 1 \qquad (11–7)$$

The relationship between M_r and ζ given by Equations (11–6) and (11–7) is shown in Figure 11–7.

The phase angle of $G(j\omega)$ at the frequency where the resonant peak occurs can be obtained by substituting Equation (11–5) into Equation (11–3). Thus, at the resonant frequency ω_r,

$$\underline{/G(j\omega_r)} = -\tan^{-1}\frac{\sqrt{1 - 2\zeta^2}}{\zeta} = -90° + \sin^{-1}\frac{\zeta}{\sqrt{1 - \zeta^2}}$$

Minimum-phase systems and nonminimum-phase systems. Transfer functions having neither poles nor zeros in the right-half s-plane are called *minimum-phase transfer functions*, whereas those having poles and/or zeros in the right-half s-plane are called *nonminimum-phase transfer functions*. Systems with minimum-phase transfer functions are called *minimum-phase systems*; those with nonminimum-phase transfer functions are called *nonminimum-phase systems*.

For systems with the same magnitude characteristic, the range in phase angle of the minimum-phase transfer function is minimum for all such systems, while the range in phase angle of any nonminimum-phase transfer function is greater than this minimum.

Consider as an example the two systems whose sinusoidal transfer functions are, respectively,

$$G_1(j\omega) = \frac{1 + j\omega T}{1 + j\omega T_1} \qquad \text{and} \qquad G_2(j\omega) = \frac{1 - j\omega T}{1 + j\omega T_1}, \qquad 0 < T < T_1$$

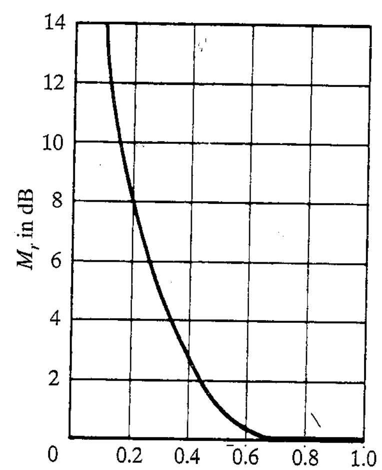


Figure 11–7 Curve of M_r versus ζ for the second-

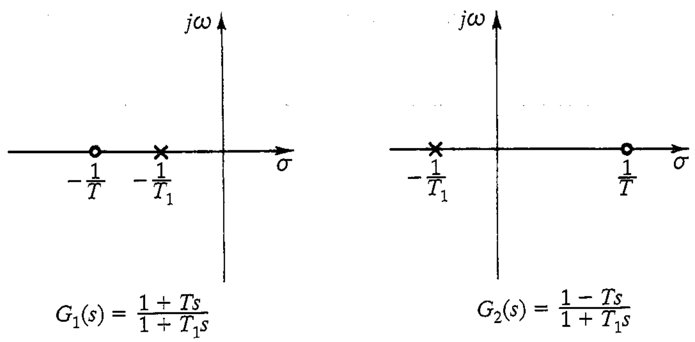


Figure 11–8 Pole–zero configurations of minimum-phase and nonminimum-phase systems (G_1: minimum phase, G_2: nonminimum phase).

The pole–zero configurations of these systems are shown in Figure 11–8. The two sinusoidal transfer functions have the same magnitude characteristics, but they have different phase-angle characteristics, as shown in Figure 11–9. The two systems differ from each other by the factor

$$G(j\omega) = \frac{1 - j\omega T}{1 + j\omega T}$$

The magnitude of the factor $(1 - j\omega T)/(1 + j\omega T)$ is always unity. But the phase angle equals $-2\tan^{-1}\omega T$ and varies from 0 to $-180°$ as ω is increased from zero to infinity.

For a minimum-phase system, the magnitude and phase-angle characteristics are directly related. That is, if the magnitude curve of a system is specified over the entire frequency range from zero to infinity, then the phase-angle curve is uniquely determined, and vice versa. This relationship, however, does not hold for a nonminimum-phase system.

Nonminimum-phase situations may arise (1) when a system includes a nonminimum-phase element or elements and (2) in the case where a minor loop is unstable.

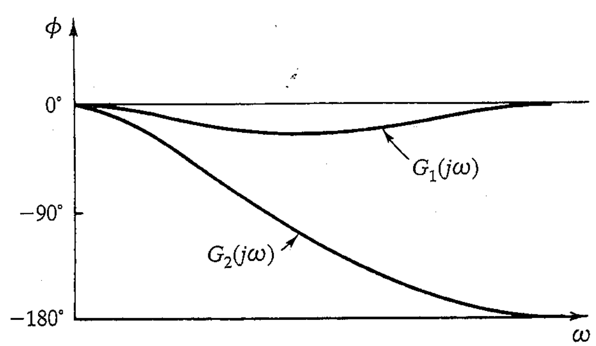


Figure 11–9 Phase-angle characteristics of minimum-phase and nonminimum-phase systems (G_1: minimum phase, G_2: nonminimum phase).

For a minimum-phase system, the phase angle at $\omega = \infty$ becomes $-90°(q - p)$, where p and q are, respectively, the degrees of the numerator and denominator polynomials of the transfer function. For a nonminimum-phase system, the phase angle at $\omega = \infty$ differs from $-90°(q - p)$. In either system, the slope of the log-magnitude curve at $\omega = \infty$ is equal to $-20(q - p)$ dB/decade. It is therefore possible to detect whether the system is minimum phase by examining both the slope of the high-frequency asymptote of the log-magnitude curve and the phase angle at $\omega = \infty$. If the slope of the log-magnitude curve as ω approaches infinity is $-20(q - p)$ dB/decade and the phase angle at $\omega = \infty$ is equal to $-90°(q - p)$, the system is minimum phase.

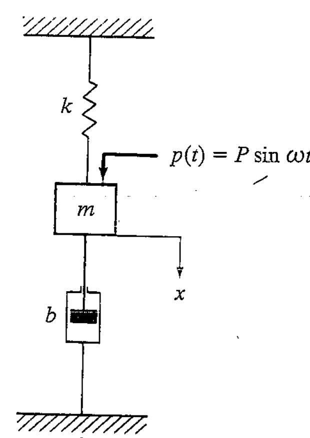
Nonminimum-phase systems are slow in response because of their faulty behavior at the start of the response. In most practical control systems, excessive phase lag should be carefully avoided. In designing a system, if a fast response is of primary importance, nonminimum-phase components should not be used.

Example 11–1

Consider the mechanical system shown in Figure 11–10. An experimental Bode diagram for this system is shown in Figure 11–11. The ordinate of the magnitude curve is the amplitude ratio of the output to the input, measured in decibels—that is, $|X(j\omega)/P(j\omega)|$ in dB. The units of $|X(j\omega)/P(j\omega)|$ are m/N. The phase angle is $\underline{/X(j\omega)/P(j\omega)}$ in degrees. The input is a sinusoidal force of the form

$$p(t) = P \sin \omega t$$

where P is the amplitude of the sinusoidal input force and the input frequency is varied from 0.01 to 100 rad/s. The displacement x is measured from the equilibrium position before the sinusoidal force is applied. Note that the amplitude ratio $|X(j\omega)/P(j\omega)|$ does not depend on the absolute value of P. (This is because, if the input amplitude is doubled, the output amplitude is also doubled. Therefore, we can choose any convenient amplitude P.) Determine the numerical values of m, b, and k from the Bode diagram.



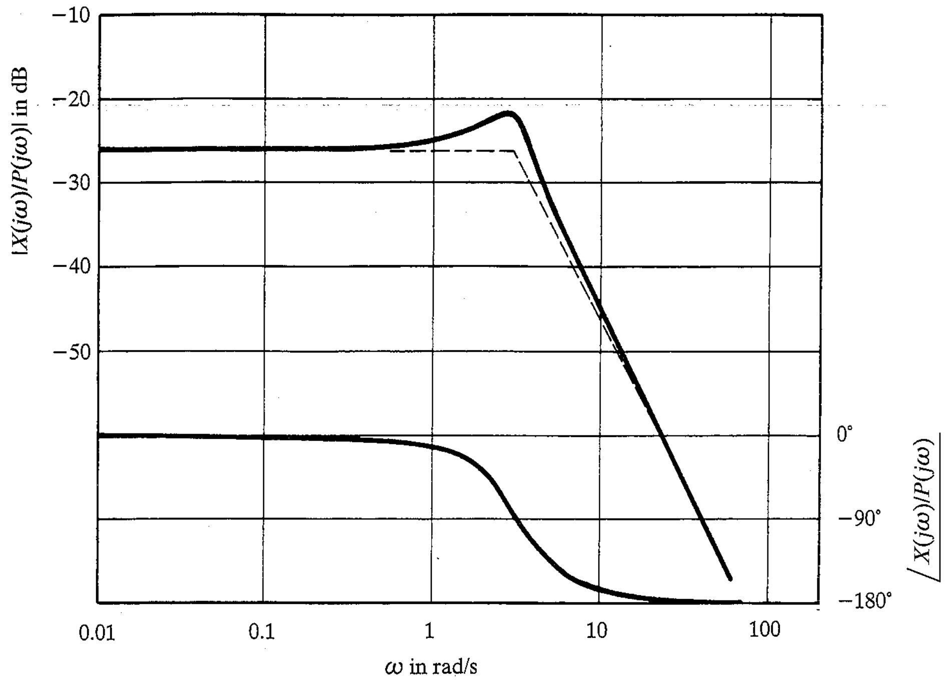


Figure 11–11 Experimental Bode diagram for the system shown in Figure 11–10.

First, we need to determine the transfer function of the system. The system equation is

$$m\ddot{x} + b\dot{x} + kx = p(t) = P \sin \omega t$$

The Laplace transform of this last equation, with zero initial condition, gives

$$(ms^2 + bs + k)X(s) = P(s)$$

where $P(s) = \mathscr{L}[p(t)]$. The transfer function for the system is

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k}$$

This mechanical system possesses poles only in the left-half s-plane, so it is a *minimum-phase system*. For a minimum-phase system, the transfer function can be uniquely determined solely from the magnitude curve of the Bode diagram.

The sinusoidal transfer function is

$$\frac{X(j\omega)}{P(j\omega)} = \frac{1}{m(j\omega)^2 + bj\omega + k} \tag{11–8}$$

Now, from the Bode diagram, we find that

$$\frac{X(j0+)}{P(j0+)} = -26 \text{ dB}$$

Hence,

$$\frac{X(j0+)}{P(j0+)} = \frac{1}{k} = -26 \text{ dB} = 0.0501$$

or

$$k = 19.96 \text{ N/m}$$

Also from the Bode diagram, the corner frequency ω_n is seen to be 3.2 rad/s. Since the corner frequency of the system given by Equation (11–8) is $\sqrt{k/m}$, it follows that

$$\omega_n = \sqrt{\frac{k}{m}} = 3.2$$

Thus,

$$m = \frac{k}{(3.2)^2} = \frac{19.96}{10.24} = 1.949 \text{ kg}$$

Next, we need to estimate the value of the damping ratio ζ. Comparing the Bode diagram of Figure 11–11 with the Bode diagram of the standard second-order system shown in Figure 11–6, we find the damping ratio ζ to be approximately 0.32, or $\zeta = 0.32$. Then, noting that

$$\frac{b}{m} = 2\zeta\omega_n$$

we obtain

$$b = 2\zeta\omega_n m = 2 \times 0.32 \times 3.2 \times 1.949 = 3.992 \text{ N-s/m}$$

We have thus determined the values of m, b, and k to be as follows:

$$m = 1.949 \text{ kg}, \qquad b = 3.992 \text{ N-s/m}, \qquad k = 19.96 \text{ N/m}$$

Relationship between system type and log-magnitude curve. Consider the unity-feedback control system. The static position, velocity, and acceleration error constants describe the low-frequency behavior of type 0, type 1, and type 2 systems, respectively. For a given system, only one of the static error constants is finite and significant. (The larger the value of the finite static error constant, the higher the loop gain is as ω approaches zero.)

The type of the system determines the slope of the log-magnitude curve at low frequencies. Thus, information concerning the existence and magnitude of the steady-state error in the response of a control system to a given input can be determined by observing the low-frequency region of the log-magnitude curve of the system.

Determination of static position error constants. Consider the unity-feedback control system shown in Figure 11–12. Assume that the open-loop transfer function is given by

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}$$

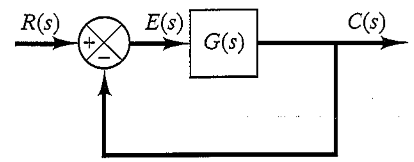


Figure 11–12 Unity-feedback control system.

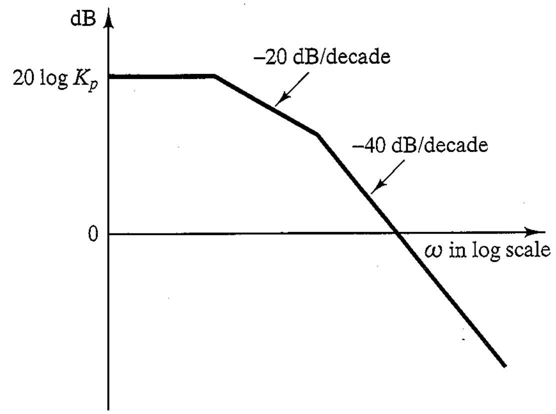


Figure 11–13 Log-magnitude curve of a type 0 system.

or

$$G(j\omega) = \frac{K(T_a j\omega + 1)(T_b j\omega + 1)\cdots(T_m j\omega + 1)}{(j\omega)^N (T_1 j\omega + 1)(T_2 j\omega + 1)\cdots(T_p j\omega + 1)}$$

Figure 11–13 is an example of the log-magnitude plot of a type 0 system. In such a system, the magnitude of $G(j\omega)$ equals K_p at low frequencies, or

$$\lim_{\omega \to 0} G(j\omega) = K_p$$

It follows that the low-frequency asymptote is a horizontal line at 20 log K_p dB.

Determination of static velocity error constants. Consider again the unity-feedback control system shown in Figure 11–12. Figure 11–14 is an example of the log-magnitude plot of a type 1 system. The intersection of the initial −20-dB/decade segment (or its extension) with the line $\omega = 1$ has the magnitude 20 log K_v. This may be seen as follows: For a type 1 system,

$$G(j\omega) = \frac{K_v}{j\omega} \qquad \text{for } \omega \ll 1$$

Thus,

$$20 \log \left|\frac{K_v}{j\omega}\right|_{\omega=1} = 20 \log K_v$$

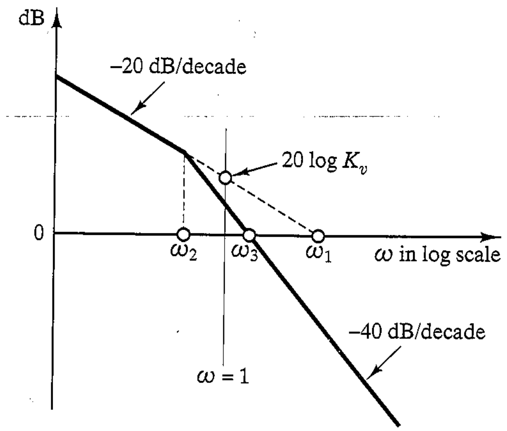


Figure 11–14 Log-magnitude curve of a type 1 system.

The intersection of the initial −20-dB/decade segment (or its extension) with the 0-dB line has a frequency numerically equal to K_v. To see this, define the frequency at this intersection to be ω_1; then

$$\left|\frac{K_v}{j\omega_1}\right| = 1$$

or

$$K_v = \omega_1$$

As an example, consider the type 1 system with unity feedback whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + F)}$$

If we define the corner frequency to be ω_2 and the frequency at the intersection of the −40-dB/decade segment (or its extension) with the 0-dB line to be ω_3, then

$$\omega_2 = \frac{F}{J}, \qquad \omega_3^2 = \frac{K}{J}$$

Since

$$\omega_1 = K_v = \frac{K}{F}$$

it follows that

$$\omega_1 \omega_2 = \omega_3^2$$

or

$$\frac{\omega_1}{\omega_3} = \frac{\omega_3}{\omega_2}$$

On the Bode diagram,

$$\log \omega_1 - \log \omega_3 = \log \omega_3 - \log \omega_2$$

Thus, the ω_3 point is just midway between the ω_2 and ω_1 points. The damping ratio ζ of the system is then

$$\zeta = \frac{F}{2\sqrt{KJ}} = \frac{\omega_2}{2\omega_3}$$

Determination of static acceleration error constants. Consider once more the unity-feedback control system shown in Figure 11–12. Figure 11–15 is an example of the log-magnitude plot of a type 2 system. The intersection of the initial −40-dB/decade segment (or its extension) with the $\omega = 1$ line has the magnitude of $20 \log K_a$. Since, at low frequencies,

$$G(j\omega) = \frac{K_a}{(j\omega)^2} \qquad \text{for } \omega \ll 1$$

it follows that

$$20 \log \left| \frac{K_a}{(j\omega)^2} \right|_{\omega=1} = 20 \log K_a$$

The frequency ω_a at the intersection of the initial −40-dB/decade segment (or its extension) with the 0-dB line gives the square root of K_a numerically. This can be seen from the following:

$$20 \log \left| \frac{K_a}{(j\omega_a)^2} \right| = 20 \log 1 = 0$$

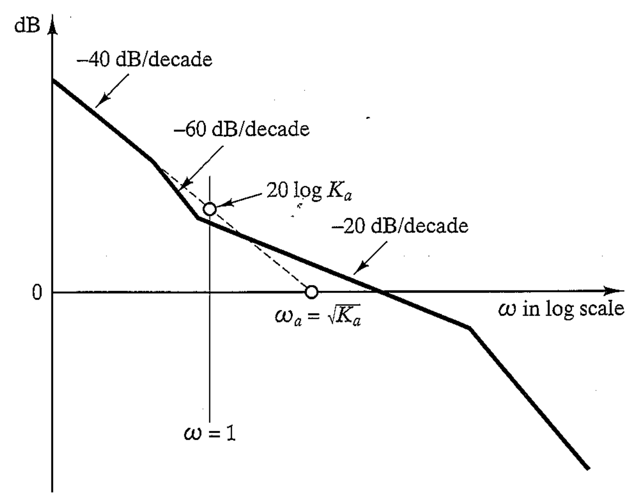


Figure 11–15 Log-magnitude curve of a type 2 system.

which yields

$$\omega_a = \sqrt{K_a}$$

Cutoff frequency and bandwidth. Referring to Figure 11–16, the frequency ω_b at which the magnitude of the closed-loop frequency response is 3 dB below its zero-frequency value is called the *cutoff frequency*. Thus,

$$\left| \frac{C(j\omega)}{R(j\omega)} \right| < \left| \frac{C(j0)}{R(j0)} \right| - 3 \text{ dB} \qquad \text{for } \omega > \omega_b$$

For systems in which $|C(j0)/R(j0)| = 0$ dB,

$$\left| \frac{C(j\omega)}{R(j\omega)} \right| < -3 \text{ dB} \qquad \text{for } \omega > \omega_b$$

The closed-loop system filters out the signal components whose frequencies are greater than the cutoff frequency and transmits those signal components with frequencies lower than the cutoff frequency.

The frequency range $0 \le \omega \le \omega_b$ in which the magnitude of the closed loop does not drop −3 dB is called the *bandwidth* of the system. The bandwidth indicates the frequency where the gain starts to fall off from its low-frequency value. Thus, the bandwidth indicates how well the system will track an input sinusoid. Note that, for a given ω_n, the rise time increases with increasing damping ratio ζ. On the other hand, the bandwidth decreases with increasing ζ. Therefore, the rise time and the bandwidth are inversely proportional to each other.

The specification of the bandwidth may be determined by the following factors:

1. The ability to reproduce the input signal. A large bandwidth corresponds to a small rise time, or a fast response. Roughly speaking, we can say that the bandwidth is proportional to the speed of the response.
2. The necessary filtering characteristics for high-frequency noise.

For the system to follow arbitrary inputs accurately, it is necessary that it have a large bandwidth. From the viewpoint of noise, however, the bandwidth should not be too large. Thus, there are conflicting requirements on the bandwidth, and a

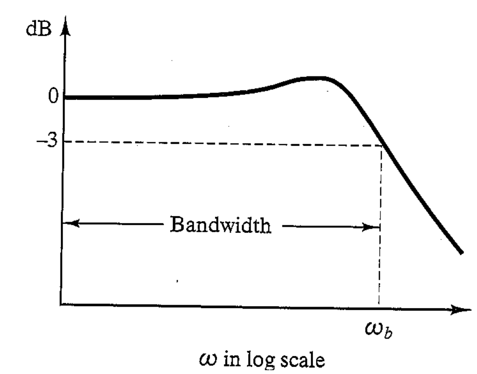


Figure 11–16 Logarithmic plot showing

compromise is usually necessary for good design. Note that a system with a large bandwidth requires high-performance components, so the cost of components usually increases with the bandwidth.

Cutoff rate. The cutoff rate is the slope of the log-magnitude curve near the cutoff frequency. The cutoff rate indicates the ability of a system to distinguish a signal from noise.

Note that a closed-loop frequency response curve with a steep cutoff characteristic may have a large resonant peak magnitude, which implies that the system has a relatively small stability margin.

Example 11–2

Consider the following two systems:

$$\text{System I: } \frac{C(s)}{R(s)} = \frac{1}{s+1}, \qquad \text{System II: } \frac{C(s)}{R(s)} = \frac{1}{3s+1}$$

Compare the bandwidths of these systems. Show that the system with the larger bandwidth has a faster speed of response and can follow the input much better than the system with a smaller bandwidth.

Figure 11–17(a) shows the closed-loop frequency-response curves for the two systems. (Asymptotic curves are represented by dashed lines.) We find that the bandwidth of system I is $0 \le \omega \le 1$ rad/s and that of system II is $0 \le \omega \le 0.33$ rad/s. Figures 11–17(b) and (c) show, respectively, the unit-step response and unit-ramp response curves for the two systems. Clearly, system I, whose bandwidth is three times wider than that of system II, has a faster speed of response and can follow the input much better.

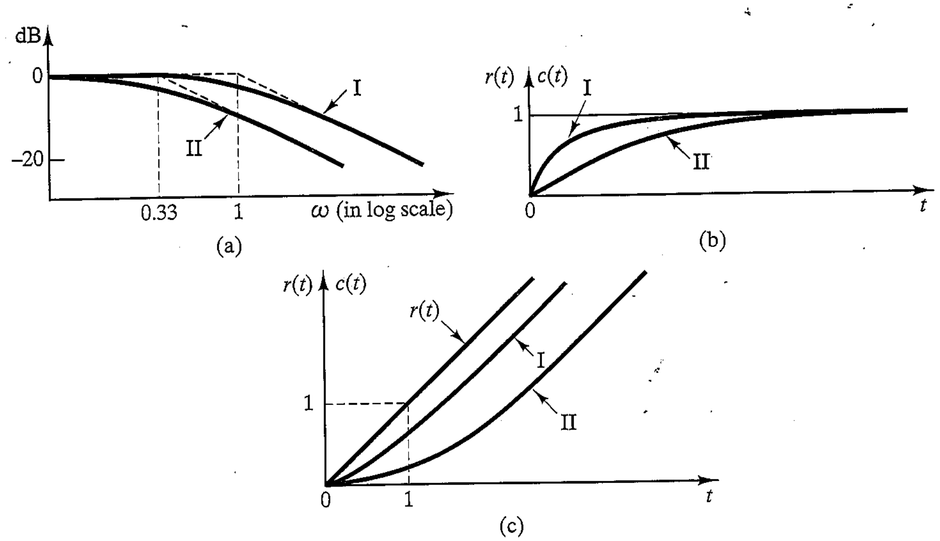


Figure 11–17 Comparison of the dynamic characteristics of the two systems considered in Example 11–2: (a) closed-loop frequency-response curves; (b) unit-step response curves; (c) unit-ramp response curves.

11–3 PLOTTING BODE DIAGRAMS WITH MATLAB

In MATLAB, the command 'bode' computes magnitudes and phase angles of the frequency response of continuous-time, linear, time-invariant systems.

When the command 'bode' (without left-hand arguments) is entered in the computer, MATLAB produces a Bode plot on the screen. When invoked with left-hand arguments, as in

```
[mag,phase,w] = bode(num,den,w)
```

'bode' returns the frequency response of the system in matrices mag, phase, and w. No plot is drawn on the screen. The matrices mag and phase contain the magnitudes and phase angles respectively, of the frequency response of the system, evaluated at user-specified frequency points. The phase angle is returned in degrees. The magnitude can be converted to decibels with the statement

```
magdB = 20* log10(mag)
```

To specify the frequency range, use the command logspace(d1,d2) or logspace(d1,d2,n). logspace(d1,d2) generates a vector of 50 points logarithmically equally spaced between decades 10^{d1} and 10^{d2}. That is, to generate 50 points between 0.1 rad/s and 100 rad/s, enter the command

```
w = logspace(−1,2)
```

logspace(d1,d2,n) generates n points logarithmically equally spaced between decades 10^{d1} and 10^{d2}. For example, to generate 100 points between 1 rad/s and 1000 rad/s, enter the following command:

```
w = logspace(0,3,100)
```

To incorporate these frequency points when plotting Bode diagrams, use the command bode(num,den,w) or bode(A,B,C,D,iu,w), each of which employs the user-specified frequency vector w.

Example 11–3

Plot a Bode diagram of the transfer function

$$G(s) = \frac{25}{s^2 + 4s + 25}$$

When the system is defined in the form

$$G(s) = \frac{\text{num}(s)}{\text{den}(s)}$$

use the command bode(num,den) to draw the Bode diagram. [When the numerator and denominator contain the polynomial coefficients in descending powers of s, bode(num,den) draws the Bode diagram.] MATLAB Program 11–1 plots the Bode diagram for this system. The resulting Bode diagram is shown in Figure 11–18.

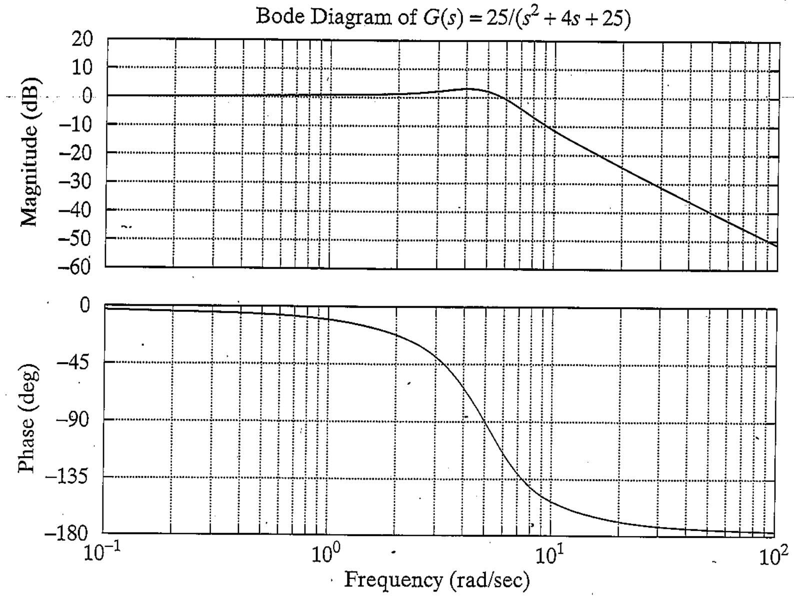


Figure 11–18 Bode diagram of $G(s) = \dfrac{25}{s^2 + 4s + 25}$.

MATLAB Program 11–1

```
>> num = [25];
>> den = [1   4   25];
>> bode(num,den)
>> grid
>> title('Bode Diagram of G(s) = 25/(s^2+4s+25)')
```

11–4 NYQUIST PLOTS AND THE NYQUIST STABILITY CRITERION

In this section, we first discuss Nyquist plots and then the Nyquist stability criterion. We then define the phase margin and gain margin, which are frequently used for determining the relative stability of a control system. Finally, we discuss conditionally stable systems.

Nyquist plots. The *Nyquist plot* of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ in polar coordinates as ω is varied from zero to infinity. Thus, the polar plot is the locus of vectors $|G(j\omega)|\underline{/G(j\omega)}$ as ω is varied from zero to infinity. Note that, in polar plots, a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis. The Nyquist plot is often called the *polar plot*. An example of such a

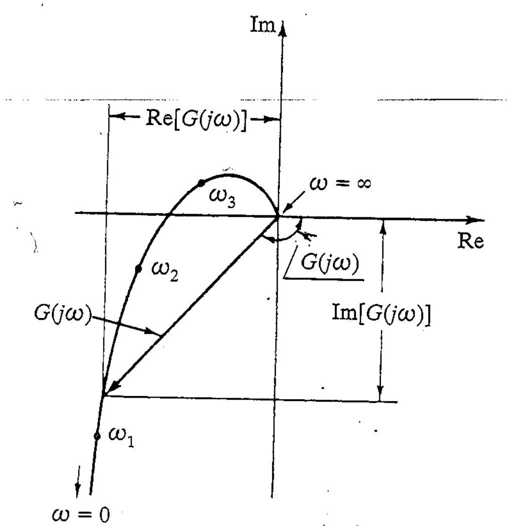


Figure 11–19 Nyquist plot.

plot is shown in Figure 11–19. Each point on the polar plot of $G(j\omega)$ represents the terminal point of a vector at a particular value of ω. The projections of $G(j\omega)$ on the real and imaginary axes are the real and imaginary components of the function.

An advantage in using a Nyquist plot is that it depicts the frequency-response characteristics of a system over the entire frequency range in a single plot. One disadvantage is that the plot does not clearly indicate the contribution of each individual factor of the open-loop transfer function.

Table 11–1 shows examples of Nyquist plots of simple transfer functions.

The general shapes of the low-frequency portions of the Nyquist plots of type 0, type 1, and type 2 minimum-phase systems are shown in Figure 11–20(a). It can be seen that, if the degree of the denominator polynomial of $G(j\omega)$ is greater than that of the numerator, then the $G(j\omega)$ loci converge clockwise to the origin. At $\omega = \infty$, the loci are tangent to one or the other axis, as shown in Figure 11–20(b).