Problem 1: Bicoid gradient scaling (40 points)

The generic reaction-diffusion equation for a system with one component and one spatial dimension is[1]:

$$\frac{\partial}{\partial t} c(x,t) = f\left[c(x,t)\right] + D\nabla^2 c(x,t)$$

When simulating such systems, we consider only a finite number of points on a one-dimensional grid.

a) Show, using the definition of the derivative, that the Laplacian term can be approximated on a grid with spacing h:

$$\nabla^2 c(x,t) \approx \frac{c(x-h,t) + c(x+h,t) - 2\,c(x,t)}{h^2}$$

We will model Bicoid reaction and diffusion in one dimension (representing the A-P axis of the embryo). Since Bicoid diffuses through the cytoplasm, it should not be able to "exit" the embryo at either end: the boundaries of our grid must therefore be reflective. Unfortunately, the MATLAB function which implements the discrete Laplacian, `del2()`, does not handle these boundary conditions.

b) Write a function to estimate the discrete Laplacian on a grid with reflective boundaries at either end. The Turing pattern example script in the Lecture 29 notes, which implements the discrete Laplacian on a torus, may be a helpful starting point.

c) Suppose that the Bicoid protein gradient is established over the course of two hours in a 500 μm embryo according to the reaction-diffusion equation:

$$\frac{\partial}{\partial t} c(x,t) = \alpha\,\delta_k(x) - \beta\,c(x,t) + D\frac{\partial^2}{\partial x^2} c(x,t)$$

where $\delta_k(x)$ is the Kronecker delta distribution, $\alpha = 10$ nM/s is the translation rate, $\beta = 1/1800$ s^{-1} is the degradation rate, and $D = 0.3$ $\mu m^2/s$ is the diffusion rate. Show that the final Bicoid gradient is approximately exponential by simulating the system using Euler's method and your subroutine from part (b). Assume that no Bicoid protein is present initially and use a step size[2] of h=5 μm.

d) Repeat for an embryo twice as long, and for a third embryo half as long as the original, maintaining the same step size and synthesis rate. Plot the Bicoid concentration profiles vs. fractional body length on the same axes. Does the concentration profile scale?

e) Implement a gradient scaling mechanism of your choice and demonstrate an improvement by creating a plot similar to part (c) for comparison.

[1] ∇^2 is the Laplace operator. In one dimension, it is simply the second spatial derivative, ∂_{xx}. On the Cartesian plane, it is the sum of the two spatial derivatives, $\partial_{xx} + \partial_{yy}$.

[2] If you do use a different grid spacing, ensure that Bicoid is still synthesized in a 5 μm region.

Problem 2: Growing snake (60 points)

Another Turing pattern mechanism proposed by Gierer and Meinhardt (1972) follows:

$$\frac{\partial A}{\partial t} = \frac{\alpha A^2}{B} - \beta A + \epsilon D \nabla^2 A \qquad\qquad \frac{\partial B}{\partial t} = \gamma A^2 - \delta B + D\nabla^2 B$$

where all constants are positive.

a) Show that the spatially-homogeneous solution for this system is:

$$(A^*, B^*) = \left(\frac{\alpha\delta}{\beta\gamma}, \frac{\alpha^2\delta}{\beta^2\gamma}\right)$$

b) Show that for Turing patterns to arise, the following two conditions must hold:

$$\delta > \beta \qquad \text{and} \qquad (\beta + \delta\epsilon)^2 > 8\beta\delta\epsilon$$

Consider the specific case where $\alpha = \beta = \gamma = 1$, $\delta = 1.1$, $\epsilon = 0.12$, and $D = 15$. You will use Euler's method to simulate this system on a two-dimensional grid representing the skin of a snake. The left and right boundaries of the grid (the "head" and "tail") should be reflective under diffusion, while the top and bottom should wrap around. You may wish to modify the example Turing pattern script on the course website for this purpose.

d) Simulate this system on a grid with dimensions 5 units x 10 units (a "baby snake"). Use a step size of one unit; choose the time step and duration appropriately to allow the system to reach its steady-state pattern. Include an image of your results. How many stripes does this snake have?

e) The snake grows up. Repeat the simulation on a grid with dimensions 10 units by 100 units. How many stripes does the snake have now? Include an image of the results.

f) The snake lives large. Repeat on a grid with dimensions 100 units by 100 units. Include an image of the results.

g) Determine which modes are unstable for organisms with the following lengths (i) $L = \sqrt{5^2 + 10^2}$, (ii) $L = \sqrt{10^2 + 100^2}$, and (iii) $L = \sqrt{100^2 + 100^2}$. How are these values reflected in the images you produced in (d-f)? Hint: see Iglesias section 3.3.2 and Figure 3.9 for a worked example.