These derivations follow van Kampen (3rd ed.) but summarize only points related to the description of Poisson processes.

Characteristic functions

The characteristic function for a random variable X on the reals is defined as:

$$G(k) = \left\langle e^{ikX} \right\rangle = \int_{-\infty}^{\infty} e^{ikx} P(x)\, dx$$

Its Taylor expansion is:

$$G(k) = \int_{-\infty}^{\infty} \left[\sum_{\ell=0}^{\infty} \frac{(ik)^\ell x^\ell}{\ell!} \right] P(x)\, dx = \sum_{\ell=0}^{\infty} \frac{(ik)^\ell}{\ell!} \int_{-\infty}^{\infty} x^\ell P(x)\, dx = \sum_{\ell=0}^{\infty} \frac{(ik)^\ell}{\ell!} \mu_\ell$$

where μ_ℓ is the ℓth moment of X. The characteristic function is therefore also the moment generating function.

For a random variable X defined only on the integers,

$$P(X = n) = \sum_{n\in\mathbb{Z}} p_n \delta(x - n) \implies G(k) = \sum_{n\in\mathbb{Z}} p_n e^{ikn}$$

Point processes

A *point process* is a "random set of points": for example, it might be the set of times at which a chemical reaction (or any other event) occurs. A sample is described by the ordered set of σ real numbers (times) $\tau_1, \tau_2, \ldots$. The first s events in this series are represented by $\tau_1, \ldots, \tau_s$. The normalized probability distribution is expressed in terms of functions $P_s(\tau_1, \ldots, \tau_s)$:

$$P_0 + \int_{-\infty}^{\infty} P_1(\tau_1)\, d\tau_1 + \int_{\tau_1}^{\infty} \left[\int_{-\infty}^{\infty} P_2(\tau_1, \tau_2)\, d\tau_1 \right] d\tau_2 + \ldots = 1$$

If we restrict the requirement that the τ_i are in chronological order, then noting that there are $s!$ permutations of the set $\tau_1, \ldots, \tau_s$ and requiring that P_s is symmetric in all its variables (i.e. one event is not more likely to occur at a certain time than any other), then we can rewrite the above normalization as:

$$P_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{\mathbb{R}^s} P_s(\tau_1, \ldots, \tau_s)\, d\tau_1 \ldots d\tau_s = 1 \tag{1}$$

Suppose we would like to know the average number of events occurring within an interval (t_a, t_b). We introduce the indicator function $f(t) = 1\, \forall\, t \in (t_a, t_b)$, and zero otherwise. Then of s events, the number that occur within this interval is $N = \sum_{k=1}^{s} f(\tau_k)$, and

$$\begin{aligned}
\langle N \rangle &= \left\langle \sum_{k=1}^{s} f(\tau_k) \right\rangle \\
&= \sum_{s=1}^{\infty} \int_{\mathbb{R}^s} \left(\sum_{k=1}^{s} f(\tau_k) \right) P_s(\tau_1, \ldots, \tau_s)\, d\tau_1 \cdots d\tau_s \\
&= \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} s f(\tau_1) \left[\int_{\mathbb{R}^{s-1}} P_s(\tau_1, \ldots, \tau_s)\, d\tau_2 \cdots d\tau_s \right] d\tau_1 \\
&= \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \int_{t_a}^{t_b} \left[\int_{\mathbb{R}^{s-1}} P_s(\tau_1, \ldots, \tau_s)\, d\tau_2 \cdots d\tau_s \right] d\tau_1
\end{aligned}$$

Poisson processes

For a process in which all of the τ_i are independent and identically distributed, the P_s can be factorized in the form

$$P_s(\tau_1, \ldots, \tau_s) = e^{-\nu} P(\tau_1) \cdots P(\tau_s), \qquad \text{where } \nu = \int_{-\infty}^{\infty} P(\tau)\, d\tau$$

is a normalization constant which follows from equation 1. The formula for the average number of events in an interval can now be simplified:

$$\begin{aligned} \langle N \rangle &= e^{-\nu} \int_{t_a}^{t_b} P(\tau)\, d\tau \left(\sum_{s=1}^{\infty} \frac{1}{(s-1)!} \left[\int_{-\infty}^{\infty} P(\tau)\, d\tau \right]^{s-1} \right) \\ &= \int_{t_a}^{t_b} P(\tau)\, d\tau \end{aligned} \tag{2}$$

It is no harder to show that for this type of process,

$$\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$$

To learn more about the probability distribution of N, we find its characteristic function:

$$\begin{aligned} \langle e^{ikN} \rangle &= \sum_{s=0}^{\infty} \frac{1}{s!} \int_{\mathbb{R}^s} e^{ikN} P_s(\tau_1, \ldots, \tau_s)\, d\tau_1 \ldots d\tau_s \\ &= e^{-\nu} \sum_{s=0}^{\infty} \frac{1}{s!} \left[\int_{-\infty}^{\infty} e^{ikf(\tau)} P(\tau)\, d\tau \right]^s \\ &= \exp\left[\int_{-\infty}^{\infty} e^{ikf(\tau)} P(\tau)\, d\tau - \int_{-\infty}^{\infty} P(\tau)\, d\tau \right] \\ &= \exp\left[\left(e^{ik} - 1\right) \int_{t_a}^{t_b} P(\tau)\, d\tau \right] \qquad \text{which by equation 2 is} \\ &= \exp\left[\left(e^{ik} - 1\right) \langle N \rangle \right] = e^{-\langle N \rangle} \sum_{k=0}^{\infty} \frac{\langle N \rangle^k e^{ikN}}{k!} \end{aligned}$$

But as we also know from equation 1 that $G(k) = \sum_{n \in \mathbb{Z}} p_n e^{ikn}$ since N can take on only integral values, then by comparing terms:

$$p(N = k) = \sum_{k=0}^{\infty} \frac{\langle N \rangle^k e^{-\langle N \rangle}}{k!}, \qquad \text{which is the Poisson pdf with } \lambda = \langle N \rangle$$

It is hopefully now clear why processes with events that are independent and identically distributed are called Poisson processes.

When $P(\tau) = \nu/(t_b - t_a)$ is constant on the interval, as ν, $t_b - t_a \to \infty$ with $P(\tau)$ fixed, the distribution of τ_i approaches a stationary distribution called *shot noise*.