AMATH 250 — LECTURE 21

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21.1 Laplace Transformations (Ch. 4)

Recall our *swing problem* that we covered in rotational dynamics (for which I did not write notes):

$$y'' + 2\gamma y' + \omega_0^2 y = F(t)$$

We studied when $F(t) = f_0 \cos(\omega t)$.

Notation

We introduce $\mathcal{L}$ as an operator which takes a function $y(t)$ and produces another function $Y(s)$. This function simplifies hard DE problems.

Formal Definition

Given a real-(or complex-)valued function $y(t)$, defined on $t \in [0, \infty)$, the Laplace Transform $\mathcal{L}[y(t)]$ is defined as

$$Y(s) = \int_0^\infty e^{-st} y(t)\, dt$$

for all values of s for which the improper integral exists.
Even though $y(t)$ and s may be generally complex-valued, we'll assume for them to be **real** values, so $Y(s)$ is also real valued.

Example 21.1.1. *Find $\mathcal{L}[y(t)]$ for $y(t) = e^{\alpha t}$ with α being constant*

$$\begin{aligned}
Y(s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{\alpha t} dt \\
&= \lim_{T\to\infty} \int_0^T e^{(\alpha - s)t} dt \\
&= \lim_{T\to\infty} \begin{cases} \left.\frac{e^{(\alpha-s)t}}{\alpha - s}\right|_0^T & s \neq \alpha \\ T & s = \alpha \text{ (not defined)} \end{cases}
\end{aligned}$$

For $s \neq \alpha$:

$$\lim_{T\to\infty} \frac{1 - e^{-(s-\alpha)T}}{s - \alpha} = \begin{cases} \frac{1}{s-\alpha} & s > \alpha \\ DNE & s < \alpha \end{cases}$$

And so

$$\mathcal{L}[e^{\alpha t}] = Y(s) = \frac{1}{s - \alpha} \qquad \square$$

Linearity

If $\mathcal{L}[y_1(t)]$ and $\mathcal{L}[y_2(t)]$ exist, then

$$\mathcal{L}[c_1 y_1(t) + c_2 y_2(t)] = c_1 \mathcal{L}[y_1(t)] + c_2 \mathcal{L}[y_2(t)]$$

Conditions to Exist

$\mathcal{L}[f]$ exists if

- $f(t) \in O(e^{\alpha t})$ for some $\alpha \in \mathbb{R}$
- f is a piecewise continuous function