AMATH 250 — LECTURE 15

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Last time

General solution of DE with constant coefficient for $y(x)$

$$y'' + py' + qy = 0 \tag{15.1}$$

where p, q are constants. Our characteristic equation is

$$\lambda^2 + p\lambda + q = 0$$

Let the two solutions to that formula as λ_1 and λ_2. Let

$$y_1(x) = e^{\lambda_1 x}$$
$$y_2(x) = e^{\lambda_2 x}$$

From this, our solution to (15.1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

We have three cases for our constants

Case 1

$$p^2 > 4q \implies \lambda_1 \neq \lambda_2$$

Case 2

$$p^2 < 4q \implies \lambda_1 = a + ib, \quad \lambda_2 = a - ib$$

Here, our solutions are

$$y_1(x) = e^{ax}\cos(bx)$$
$$y_2(x) = e^{ax}\sin(bx)$$

Example 15.0.1. *Find general solution for $y'' + 4y' + 5y$*

Our characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

with solutions $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$. With here, we let $a = -2$ and $b = 1$ to yield a solution

$$y(x) = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x \qquad \square$$

Case 3

$$p^2 = 4q \implies \lambda_1 = \lambda_2 = \lambda = -\frac{p}{2}$$

Here we do something fancy. We define our trial function as

$$y(x) = U(x)e^{\lambda x}$$

With these, we have

$$\begin{aligned} y' &= U'e^{\lambda x} + U\lambda e^{\lambda x} \\ y'' &= U''e^{\lambda x} + 2U'\lambda e^{\lambda x} + U\lambda^2 e^{\lambda x} \end{aligned}$$

Plug into our general 2nd order DE (15.1)

$$\left[U'' + 2\lambda U' + U\lambda^2 + p(U' + U\lambda) + qU\right] e^{\lambda x} = 0 \tag{15.2}$$

$$\left[U(\lambda^2 + \lambda p + q) + U'(2\lambda + p) + U''\right] e^{\lambda x} = 0 \tag{15.3}$$

$$\implies U''(x) = 0 \tag{15.4}$$

From this, our trial function is

$$y(x) = (c_1 x + c_2)e^{\lambda x}$$

If one solution of (15.1) is $y_2(x) = e^{\lambda_2 x}$ and $\lambda_1 = \lambda_2$, then multiply $y_2(x)$ by a factor of x to get second linear independent solution.

Example 15.0.2. *Solve* $y'' - 2y' + y = 0$

The solution to our characteristic equation is $\lambda_2 = \lambda_2 = 1$ So $y_2(x) = e^x$, and using the result from last time $y_2(x) = xe^x$, so our given solution is

$$y(x) = c_1 e^x + c_2 x e^x \qquad \square$$

15.0.1 General solution of non-homogeneous DE with constant coefficients (3.2.4)

Our general formula is

$$y'' + py' + qy = f(x) \tag{15.5}$$

where $f(x)$ is generally given of the form:

- Polynomial in terms of x
- Exponent of x
- sin or cos function
- The product or sum of any combination of the previous functions

$y(x)$ is unknown. Recall the general solution is

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the associated homogeneous 2nd degree linear DE, and $y_p(x)$ is the particular solution of (15.5). We use the method of undetermined coefficients to find $y_p(x)$.

Example 15.0.3. *Solve the IVP:* $y'' - y' - 2y = 4x$ *with* $y(0) = 2$ *and* $y'(0) = 1$

There are 12 formal steps we need to follow on exams:

1. Define associated homogeneous DE
$$y'' - y' - 2y = 0$$

2. Define characteristic equation for
$$\lambda^2 - \lambda - 2 = 0$$

3. Solve
$$\lambda_1 = 2, \qquad \lambda_2 = -1$$

4. Find general solution of associated homogeneous DE
$$y_h(x) = c_1 e^{2x} + c_2 e^{-x}$$

5. Find $y_p(x)$. To do this, make a trial solution. We determine this by looking at our function $f(x)$. Assume $y_p(x) = A + Bx$.

6. Solve for A and B:
$$A = 1, \qquad B = -2$$

7. Some other simple step that I didn't bother writing down but wanted to maintain order number integrity

8. Build our particular solution
$$y_p(x) = 1 - 2x$$

9. General solution of DE is
$$y(x) = c_1 e^{2x} + c_2 e^{-x} + 1 - 2x$$

10. Impose the ICs
$$y(0) = c_1 + c_2 + 1 = 2$$
$$y'(0) = 2c_1 - c_2 - 2 = 1$$

11. Solve system:
$$c_1 = \frac{4}{3}, \qquad c_2 = -\frac{1}{3}$$

12. Plug back into general solution, and we're done
$$y(x) = \frac{4}{3}e^{2x} - \frac{1}{3}e^{-x} + 1 - 2x \qquad \square$$