AMATH 250 — Tutorial 1

Bartosz Antczak *May 8, 2017*

1.1 Logistic Model for Population Growth

Define $p(t) > 0$ as the population of a particular species with respect to time t.
Define b as the birth rate. We assume $b = b_0$ is constant
Define d as the death rate. We assume d is **not** a constant:

$$d(t) = d_0 + d_1 p(t)$$

where $d_0, d_1 > 0$ are both constants.
Define our model as

$$\begin{aligned} \frac{dp}{dt} &= (b - d)p && (1.1) \\ &= (b_0 - d_0)p - d_1 p^2 && (1.2) \end{aligned}$$

Define $r = (b_0 - d_0) > 0$ as the growth rate.
Lastly, define k as the carrying capacity of the system:

$$k = \frac{b_0 - d_0}{d_1}$$

From this, we have the following DE:

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{k}\right)$$

1.1.1 Qualitative analysis

We first observe two things:

1. $\frac{dp}{dt} > 0 \implies 0 < p < k$
2. $\frac{dp}{dt} < 0 \implies p > k$

We also see that the equilibrium solution occurs when $\frac{dp}{dt} = 0 \implies p = k$.
Now let's solve our DE.

1.1.2 Solving our DE

It's a separable DE.

$$\begin{aligned} \int \frac{dp}{dt} \cdot \frac{1}{p\left(1 - \frac{p}{k}\right)}\, dt &= \int r\, dt && (1.3) \\ \int \frac{k}{p(k-p)}\, dp &= \int r\, dt && (1.4) \end{aligned}$$

Aside

We integrate the left-hand side by partial fractions:

$$\begin{aligned}\frac{k}{p(k-p)} &= \frac{A}{p} + \frac{B}{k-p} = \frac{Ak - Ap + Bp}{p(k-p)} \\ &= \frac{1}{p} + \frac{1}{k-p}\end{aligned}$$

Back to integrating:

$$\int \frac{1}{p}\,dp + \int \frac{1}{k-p}\,dp = rt + c \tag{1.5}$$

$$\ln|p| - \ln|k-p| = rt + c \tag{1.6}$$

$$\left|\frac{p}{p-k}\right| = e^{rt+c} \tag{1.7}$$

$$\frac{p}{p-k} = c_2 e^{rt} \qquad \text{(Let } c_2 = \pm e^c\text{)} \tag{1.8}$$

$$p = (p-k)c_2 e^{rt} \tag{1.9}$$

$$p = -\frac{kc_2 e^{rt}}{1 - c^2 e^{rt}} \tag{1.10}$$

$$p = \frac{kc_2 e^{rt}}{c^2 e^{rt} - 1} \tag{1.11}$$