AMATH 250 — Lecture 12

Bartosz Antczak *Instructor: Zoran Miskovic* *May 28, 2017*

12.1 Finishing up Buckingham Pi Theorem

Example 12.1.1. *Throwing a ball vertically in the air*

We list our parameters and their respective dimensions

	h	t	v_0	g	m
M	0	0	0	0	1
L	1	0	1	1	0
T	0	1	-1	-2	0

Here, we have $P = 5 - 3 = 2$ dimensionless parameters. What happens if we forget to list m? In that case, we'll have $N = 4$ and the rank of the new matrix would be $n = 2$ (one row of 0's). So in this case, we'd still have $p = 4 - 2 = 2$ parameters, but this is risky, so don't try it.
What happens if we forget t? It's totally fine to leave t out as well. Here's why: we use conservation of energy to solve for v_0 given h_{max}. Initially, we have $h(0) = 0$ and $v(0) = v_0$. At this point in time, all of our energy is in the form of kinetic energy: $\frac{1}{2}mv_0^2$.
At the max height, we have $v = 0$ and $h = h_{max}$. Here, all of our energy is converted to potential energy: mgh_{max}, and so our solution is

$$\frac{1}{2}mv_0^2 = mgh_{max} \implies h_{max} = \frac{v_0^2}{2g}$$

By Newton's law

$$\frac{d^2h}{dt^2} = -g = \frac{dv}{dt} \tag{12.1}$$

To find a DE relating v and h, assume $v = v(h(t))$. By Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt} = v \cdot \frac{dv}{dh} \tag{12.2}$$

Sub (12.2) into (12.1)

$$v \cdot \frac{dv}{dh} = -g \tag{12.3}$$

$$\int v\, dv = -\int g\, dh \tag{12.4}$$

$$\frac{1}{2}mv^2 = -mgh + d \qquad \text{(Multiply by } m\text{)} \tag{12.5}$$

$$\frac{1}{2}mv^2 + mgh = d \tag{12.6}$$

This equation is independent of time, ad it states that energy difference is constant (i.e., energy is conserved).

12.1.1 What does Buckingham Pi Theorem tell us?

	h	v_0	g	m
M	0	0	0	1
L	1	1	1	0
T	0	-1	-2	0

We have $N = 4$ and $r = 3$, and so we have one dimensionless product Π.

$$\Pi = h_{max}^a \cdot v_0^c \cdot g^d \cdot m^e \tag{12.7}$$

Solving DP $= 0$, we have

$$\begin{aligned} e &= 0 \\ a + c + d &= 0 \\ -c - 2d &= 0 \end{aligned}$$

Choosing d as an arbitrary constant, we have

$$h_{max}^d \cdot v_0^{-2d} \cdot g^d$$

Let $d = 1$

$$\Pi = \frac{h_{max} g}{v_0^2} = \text{const}$$

This solution gives us the conversation of energy

$$\frac{1}{2}v^2 = gh_{max} \implies \frac{1}{2} = \frac{gh_{max}}{v^2}$$

12.2 2nd order DEs (3.1.1, 3.1.3)

The general form of a 2nd order DE is

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

Initial value problems (IVPs) for this equation contain **two** initial conditions at some point x_0. Given y_0 and v_0,

$$\begin{aligned} y(x_0) &= y_0 \\ \frac{dy}{dx}(x_0) &= v_0 \end{aligned}$$

This means that at point (x_0, y_0), the slope is v_0.
We'll study linear 2nd order DEs. The general form is

$$\frac{d^2y}{dx^2} + P(x)\frac{dx}{dy} + Q(x)y = F(x)$$

where P, Q, F are given functions of x. If $F(x) = 0$, then our formula is homogeneous. If P and Q are constant on some interval I, then our formula has constant coefficients.
The Existence and Uniqueness Theorem for the solutions of the IVP states that if P, Q, F are all continuous on some interval I, then there exists a unique solution of the general DE.

12.2.1 Mechanical Oscillator

Consider a spring-loaded mass (just like grade 12 physics example). Let $y(t)$ be the displacement of m from its equilibrium position. Newton's law is

$$m\frac{d^2y}{dt^2} = F_{total}$$

We also define

$$\begin{aligned} F_d &= -\alpha\frac{dy}{dt} \\ F_r &= ky \qquad (k = \text{spring constant}) \\ F_{ext}(t) &= F(t) \end{aligned}$$

where $F(t)$ is constant. From this we define a 2nd order DE

$$\begin{aligned} m\frac{d^2y}{dx^2} &= -\alpha\frac{dy}{dt} - ky + F_{ext} \\ F_{ext} &= m\frac{d^2y}{dx^2} + \frac{dy}{dt} + ky \end{aligned}$$