AMATH 250 — LECTURE 9

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Last Time

Dimensional analysis, baseball thrown up in air example. We saw that

$$h = -\frac{1}{2}gt^2 + v_0 t \tag{9.1}$$

9.1 Continuing baseball example

Define characteristic time t_c with $[t_c] = T$

$$t_c = \frac{v_0}{g}$$

Also define characteristic length ℓ_c with $[\ell_c] = L$

$$\ell_c = \frac{v_0^2}{g}$$

Define dimensionless time τ with $[\tau] = 1$

$$\tau = \frac{t}{t_c}$$

Define dimensionless height H

$$H = \frac{h}{\ell_c}$$

Insert $h = H\ell_c$ and $t = \tau t_c$ into (9.1)

$$H\ell_c = -\frac{1}{2}gt^2\tau^2 + v_0 t_c \tau \tag{9.2}$$

$$H\ell_c = -\frac{1}{2}\ell_c\tau^2 + \ell_c\tau \tag{9.3}$$

$$H\ell_c = -\frac{1}{2}\ell_c\tau^2 + \ell_c\tau \tag{9.4}$$

$$H = -\frac{1}{2}\tau^2 + \tau \quad \square \tag{9.5}$$

9.2 Converting DEs to dimensionless form

Example 9.2.1. *Mixing tank with constant flow rates $f_{in} = f_{out} = f$, which also means that the volume V is also constant.*

The DE for mass of salt $m(t)$ is

$$\frac{dm}{dt} = -\frac{f}{V}m + fc_{in} \tag{9.6}$$

By inspection (will be made more formal in later lectures), we define:

- characteristic time: $t_c = \frac{V}{f}$

- dimensionless time: $\tau = \frac{t}{t_c}$

Using the Chain Rule, we let $m = m(\tau) = m(\tau(t))$,

$$\frac{dm}{dt} = \frac{dm}{d\tau} \cdot \frac{d\tau}{dt} = \frac{1}{t_c} \cdot \frac{dm}{dt} \tag{9.7}$$

Sub (9.7) into (9.6)

$$\frac{1}{t_c}\frac{dm}{d\tau} = -\frac{1}{t_c}m + fc_{in} \tag{9.8}$$

$$\frac{dm}{d\tau} + m = t_c f c_{in} \tag{9.9}$$

$$\frac{dm}{d\tau} + m = V c_{in} \tag{9.10}$$

If c_{in} is constant, then we can define characteristic mass $m_c = Vc_{in}$ with $[m_c] = M$. Now we define dimensionless mass

$$\mathcal{M}(\tau) = \frac{m(t)}{m_c} = \frac{m(\tau)}{m_c} \tag{9.11}$$

We now solve

$$\frac{d\mathcal{M}}{d\tau} = \frac{d\mathcal{M}}{dm} \cdot \frac{dm}{d\tau} \tag{9.12}$$

$$= \frac{1}{m_c} \cdot \frac{dm}{d\tau} \tag{9.13}$$

Sub (9.13) into (9.10)

$$m_c\frac{d\mathcal{M}}{d\tau} + m_c\mathcal{M} = m_c \tag{9.14}$$

$$\frac{d\mathcal{M}}{d\tau} + \mathcal{M} = 1 \tag{9.15}$$

The general solution is

$$\mathcal{M}(\tau) = 1 + Ce^{-\tau}$$

This DE defines all mixing tank problems, it's the simplest possible DE. We can convert the general DE to a particular solution by letting $\tau = \frac{t}{t_c}$ and $\mathcal{M} = \frac{m}{m_c}$

$$\begin{aligned} m(t) &= m_c\left[1 + Ce^{-\frac{t}{t_c}}\right] \\ &= Vc_{in} + De^{-\frac{f}{V}t} \quad \square \end{aligned}$$

Example 9.2.2. *Skydiver problem*

As shown in previous lectures, we have the following DE

$$m\frac{dv}{dt} = mg - \alpha v \tag{9.16}$$

We have $[\alpha] = \frac{M}{T}$. By inspection (will formalize later), we define

- characteristic velocity: $v_{term} = v_c = \frac{mg}{\alpha} = t_c g$

- characteristic time: $t_c = \frac{m}{\alpha}$

Convert (9.16) in "one shot". Define

- dimensionless time: $\tau = \frac{t}{t_c}$
- dimensionless velocity: $V = \frac{v}{v_c}$

Consider $v = v(\tau)$ and so $V = V(\tau)$. By chain rule

$$\frac{dv}{dt} = \frac{dv}{dV} \cdot \frac{dV}{d\tau} \cdot \frac{d\tau}{dt} \tag{9.17}$$

$$= v_c \cdot \frac{dV}{d\tau} \cdot \frac{1}{t_c} \tag{9.18}$$

$$= \frac{v_c}{t_c} \cdot \frac{dV}{d\tau} \tag{9.19}$$

Sub (9.19) into (9.16)

$$m \cdot \frac{v_c}{t_c} \cdot \frac{dV}{d\tau} = mg - \alpha v_c V \tag{9.20}$$

$$m \cdot \frac{mg}{\alpha} \cdot \frac{\alpha}{m} \cdot \frac{dV}{d\tau} = mg - mgV \tag{9.21}$$

$$\frac{dV}{d\tau} + V = 1 \tag{9.22}$$

The result is exactly the same as the mixing tank problem!