AMATH 250 — Lecture 3

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Last Time

1st order DEs and seperable DEs of the form

$$\frac{dy}{dx} = A(x)B(y)$$

We looked at two examples:

$$\frac{dy}{dx} = x\sqrt{y} \implies y = \left(\frac{x^2}{y} + c\right) \qquad c \in \mathbb{R} \tag{3.1}$$

$$\frac{dy}{dx} = -\frac{x}{y} \implies y = x^2 + y^2 = d \qquad d \in \mathbb{R} \tag{3.2}$$

3.1 Qualitative Sketch of a Solution (1.2.4)

When sketching DEs, we focus on just drawing certain graphs (since there may be infinitely many) at the "breakpoints".

Example 3.1.1. *Draw a qualitative sketch of equation 3.1*

Let's consider some cases:

1. $\frac{dy}{dx} = 0 \implies x = 0$ or $y = 0$

   Solution: draw the lines $x = 0, y = 0$

2. $c = 0$

   Solution: draw the graph $y = \frac{x^4}{16}$

3. $c > 0$ or $c < 0$

   Solution: pick some arbitrary value such as $c = 1$ or $c = -2$ and plot that graph

The qualitative sketch of equation 3.2 is trivial and therefore skipped.

3.2 Solving Linear DEs: integrating factor method (1.2.3)

This method follows a series of steps:

1. Set DE to standard form:

   $$\frac{dy}{dx} + k(x)y = f(x)$$

   where k and f are given functions.

2. Define the following integrating factor:

   $$I(x) = e^{\int k(x)\ dx}$$

Also notice that
$$\frac{dI}{dx} = e^{\int k(x)\, dx} \cdot \frac{d}{dx} \int k(x)\, dx = I(x)k(x)$$

3. Multiply the DE from step 1 by $I(x)$ and simplify:

$$I(x)\frac{dy}{dx} + I(x)k(x)y = I(x)f(x) \tag{3.3}$$
$$I(x)\frac{dy}{dx} + \frac{dI}{dx}y = I(x)f(x) \tag{3.4}$$
$$\frac{d}{dx}\left[I(x)y\right] = I(x)f(x) \tag{3.5}$$
$$I(x)y = \int I(x)f(x)\, dx \tag{3.6}$$

4. Solve (3.6) for $y(x)$.

Example 3.2.1. *Find the general solution of* $x\frac{dy}{dx} = x^2 + 2y$

$$\begin{aligned} x\frac{dy}{dx} &= x^2 + 2y \\ x\frac{dy}{dx} - \frac{2y}{x} &= x && \text{(Divide by } x \neq 0) \end{aligned}$$

Let $k(x) = -\frac{2}{x}$ and $f(x) = x$. We have:

$$\int k(x)\, dx = -2\int \frac{dx}{x} = -\ln x^2 = \ln\left(\frac{1}{x^2}\right)$$

This means that we set our integrating factor to

$$I(x) = e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$$

And so our solution is:

$$\begin{aligned} \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \frac{1}{x} \\ \frac{y}{x^2} &= \ln|x| \\ y &= (x^2 \ln|x|) + c \end{aligned}$$

Quick note

We did not add a constant when solving

$$\int k(x)\, dx = \ln\left(\frac{1}{x^2}\right)$$

This is okay, however, because this constant will get cancelled out later on in our calculations.