AMATH 250 — Tutorial 3

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Problem 1

Make the following DE dimensionless. This DE models the velocity of a ball thrown upward with initial velocity v_{init} and R is radius of earth.

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2} \tag{3.1}$$

(a) Define characteristic variables

$$v_c = \sqrt{Rg}$$

$$r_c = R$$

(b) Define dimensionless variables

$$\mathcal{V} = \frac{v}{v_c} = \frac{v}{\sqrt{Rg}} \implies dV = \frac{dv}{\sqrt{Rg}}$$

$$\mathcal{R} = \frac{r}{r_c} = \frac{r}{R} \implies d\mathcal{R} = \frac{dr}{R}$$

(c)

$$\frac{dv}{dr} = \frac{dv}{d\mathcal{V}} \cdot \frac{d\mathcal{V}}{d\mathcal{R}} \cdot \frac{d\mathcal{R}}{dr} \tag{3.2}$$

$$= \sqrt{Rg} \cdot \frac{d\mathcal{V}}{d\mathcal{R}} \cdot \frac{1}{R} \tag{3.3}$$

$$= \sqrt{\frac{g}{R}} \cdot \frac{d\mathcal{V}}{d\mathcal{R}} \tag{3.4}$$

(d) Sub int original DE

$$\mathcal{V}\sqrt{Rg}\sqrt{\frac{g}{R}} \cdot \frac{d\mathcal{V}}{d\mathcal{R}} = -\frac{g}{\mathcal{R}^2} \tag{3.5}$$

$$\mathcal{V} \cdot \frac{d\mathcal{V}}{d\mathcal{R}} = -\frac{1}{\mathcal{R}^2} \tag{3.6}$$

Now we solve our dimensionless DE. We see that $v(R) = v_{init}$. Now $r = R\mathcal{R}$, and letting $r = R$ (at Earth surface level), we have $\mathcal{R} = 1$. So we have on IC

$$\mathcal{V}(1) = \frac{v_{init}}{\sqrt{Rg}}$$

So now we solve

$$\int \mathcal{V} d\mathcal{V} = -\int \mathcal{R}^2 d\mathcal{R} \tag{3.7}$$

$$\frac{1}{2}\mathcal{V}^2 = \mathcal{R}^{-1} + C \tag{3.8}$$

$$\mathcal{V} = \sqrt{\frac{2}{\mathcal{R}} + C} \tag{3.9}$$

Plugging in our DE, we have

$$\mathcal{V}(\mathcal{R}) = \sqrt{\frac{2}{R} + \frac{v_{init}^2}{Rg} - 2}$$

(e) Now re-dimensonalize the DE. Solve for $v(r)$. We do this by subbing in the values for $\mathcal{V}$ and $\mathcal{R}$.

$$\frac{v}{\sqrt{Rg}} = \sqrt{\frac{2R}{r} + \frac{v_{init}^2}{Rg} - 2} \tag{3.10}$$

$$v(r) = \sqrt{\frac{2R^2 g}{r} + v_{init}^2 - 2Rg} \qquad \square \tag{3.11}$$