AMATH 250 — Lecture 8

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Last Time

Section 1.3 in the notes.

8.1 Chapter 2: Dimensional Analysis

8.1.1 Characteristic scales and dimensionless variables

Example 8.1.1. *Continuously compounded interest at 5% / year*

Recall that

$$\frac{dV}{dt} = rV$$

with $V(t)$ = value of investment, $V(0) = 0$, and r is our interest rate of 5%/year. We have $[r] = T^{-1}$. We define the *characteristic time* as

$$t_c = \frac{1}{r} \implies [t_c] = T$$

What does this characteristic time represent? Recall the solution to our DE:

$$V(t) = V_0 e^{rt} \implies V(t_c) = V_0 e$$

This means that t_c is the time it takes investment to increase by a factor of e.
We now define *dimensionless time* as a new independent variable:

$$\tau = \frac{t}{t_c} = rt \implies [\tau] = 1$$

We want to change our DE:

$$V(t) = \widetilde{V}(\tau) = \widetilde{V}(\tau(t))$$

where $\tau(t) = rt$. We use chain rule:

$$\frac{dV}{dt} = \frac{d\widetilde{V}}{d\tau} \cdot \frac{d\tau}{dt} = \frac{d\widetilde{V}}{d\tau} r$$

Substitute into our DE:

$$\frac{d\widetilde{V}}{d\tau} r = r\widetilde{V}$$

$$\implies \frac{d\widetilde{V}}{d\tau} = \widetilde{V}$$

We often abuse notation: $V(t) = V(\tau)$.

Example 8.1.2. *Exponential decay of radioactive substance*

Carbon dating uses radioactive isotope ^{14}C with half-life $t_{\frac{1}{2}} = 5,730$ years. Let $m(t)$ be the amount (mass) of ^{14}C at time $t \geq 0$ with $m(0) = m_0$. By definition, $m\left(t_{\frac{1}{2}}\right) = \frac{1}{2}m_0$. Our DE for radioactive decay is

$$\frac{dm}{dt} = -km$$

where $k > 0$, a constant. We know that the solution to this DE is

$$m(t) = m_0 e^{-kt}$$

Let's express k in terms of $t_{\frac{1}{2}}$:

$$m\left(t_{\frac{1}{2}}\right) = m_0 e^{-kt_{\frac{1}{2}}} = \frac{1}{2}m_0 \tag{8.1}$$

$$e^{-kt_{\frac{1}{2}}} = \frac{1}{2} \tag{8.2}$$

$$-kt_{\frac{1}{2}} = \ln 2 \tag{8.3}$$

$$k = \frac{\ln 2}{t_{\frac{1}{2}}} \tag{8.4}$$

So we see that

$$\frac{dm}{dt} = -\frac{\ln 2}{t_{\frac{1}{2}}}m$$

Let's write it in dimensionless form

$$t_c = \frac{1}{k}$$

$$\tau = \frac{t}{t_c} = kt = \ln 2 \frac{t}{t_{\frac{1}{2}}}$$

If we let $m(t) = m(\tau), \tau = kt$, then we use the chain rule to solve

$$\frac{dm}{dt} = \frac{dm}{d\tau}\frac{d\tau}{dt} = k\frac{dm}{d\tau} \tag{8.5}$$

$$\implies \frac{dm}{d\tau} = -m \tag{8.6}$$

Example 8.1.3. *Motion of a baseball thrown vertically up from surface of Earth with initial velocity* v_0

We see that $h(0) = 0$ and $h'(0) = v_0$. By Newton's 2nd law, we have

$$m\frac{d^2h}{dt^2} = -mg \tag{8.7}$$

$$\frac{d^2h}{dt^2} = -g \tag{8.8}$$

Now we simply integrate

$$\frac{dh}{dt} = \int \frac{d^2h}{dt^2}\, dt = -gt + c_1$$

$$h(t) = g\frac{t^2}{2} + c_1 t + c_2$$

Using our ICs, we see that $c_2 = 0$ and $c_1 = v_0$.