AMATH 250 — Tutorial 4

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Problem 1

An object is thrown up in the air and it falls back down. We define all of the usual parameters with the addition of t_r which represents the time it takes the ball to come back to the ground after it was thrown.

(a) What can we say about t_r in terms of m, g, v_0 using $B\Pi T$?

Since we have $N = 4$, $r = 3$, there is one dimensionless variable. Let's define

$$\Pi = \frac{g}{v_0} t_r$$

By $B\Pi T$, $c = \Pi = \frac{g t_r}{v_0}$, which means that

$$t_r = \frac{c v_0}{g} \qquad \square$$

(b) Solve the DE and find t_r

By Newton's 2nd law, $F = ma \implies a = -g$.

$$\begin{aligned} \frac{dv}{dt} &= -g \\ v &= -gt + c_1 \end{aligned}$$

Using $v(0) = v_0$, we have $c_1 = v_0$

$$\begin{aligned} \frac{dh}{dt} &= -gt + v_0 \\ h &= -\frac{1}{2}gt^2 + v_0 t + c_2 \end{aligned}$$

Applying $h(0) = 0$, we have $c_2 = 0$, and so

$$h(t) = -\frac{1}{2}gt^2 + v_0 t$$

Setting $h(t) = 0$, we solve for t_r, which results in

$$t_r = \frac{2v_0}{g} \qquad \square$$

(c) Consider a drag force now

We define $f_{drag} = \alpha v$. Now we have the following variables

$$
\begin{aligned}
[t_r] &= T \\
[m] &= M \\
[v_0] &= LT^{-1} \\
[g] &= LT^{-2} \\
[\alpha] &= MT^{-1}
\end{aligned}
$$

We have $N - r = 2$ dimensionless variables. Define then as

$$\Pi_1 = \frac{gt_r}{v_0} \qquad \Pi_2 = \frac{\alpha v_0}{mg}$$

Note: writing Π_2 in terms of t_r wouldn't work because by $B\Pi T$, we'd then have

$$\Pi_1 = f\left(\Pi_2\right)$$

would would have t_r in terms of t_r.
So using our defined dimensionless variables, we have

$$
\begin{aligned}
\Pi_1 &= f(\Pi_2) \\
\frac{gt_r}{v_0} &= f\left(\frac{\alpha v_0}{mg}\right) \\
t_r &= \frac{v_0}{g} f\left(\frac{\alpha v_0}{mg}\right)
\end{aligned}
$$

(d) Solve the new DE

By Newton's 2nd law

$$
\begin{aligned}
F &= ma \\
ma &= -mg - \alpha v \\
a &= -g - \frac{\alpha v}{m} \\
\frac{dv}{dt} &= -g - \frac{\alpha v}{m} \\
\frac{dv}{dt} + \left(\frac{\alpha}{m}\right) v &= -g
\end{aligned}
$$

This DE is in standard form. We solve $v_h(t) = Ce^{-\frac{\alpha}{m}t}$ and $v_p(t) = A \implies v_p(t) = -\frac{gm}{\alpha}$. So our solution to our DE after ICs is

$$v(t) = \left(v_0 + \frac{gm}{\alpha}\right) e^{-\frac{\alpha}{m}t} - \frac{gm}{\alpha}$$

Now we integrate to solve for $h(t)$:

$$h(t) = -\frac{m}{\alpha}\left(v_0 + \frac{gm}{\alpha}\right) e^{-\frac{\alpha}{m}t} - \frac{gm}{\alpha}t + C$$

where $C = \frac{mv_0}{\alpha} + \frac{gm^2}{\alpha^2}$ Solve for $h(t_r) = 0$. To do this, we can apply an approx Taylor expansion to simplify the exponent

$$e^{-\frac{\alpha}{m}t} \approx 1 - \frac{\alpha}{m}t + \frac{\alpha^2}{2m^2}t^2$$

And so we have

$$t_r = \left(\frac{v_0}{g}\right)\left[\frac{2}{\frac{\alpha v_0}{mg} + 1}\right] = \frac{v_0}{g} f\left(\frac{\alpha v_0}{mg}\right) \qquad \square$$