# MATH 239 — Tutorial 2

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**Tutorial Plan**

Paths and cycles.

## Problem 1

Let $G$ be a graph with minimum degree $k$, $k \geq 2$. Prove that

a) $G$ contains a path of length $\geq k$

b) $G$ contains a cycle of length $\geq k+1$

*Note: a useful proof method in graph theory is to assume a "longest path" (similar to how induction is a useful proof method on natural numbers in algebra), so let's use it!*

**Solution**

Let $P = a_0, a_1, \cdots, a_\ell$ be a longest path in $G$. We see that every neighbour of $a_0$ is in the set $S = \{a_1, a_2, \cdots, a_\ell\}$, otherwise, if there is a neighbour $x \notin S$, then the path $x, a_0, a_1, \cdots, a_\ell$ would be longer than $P$.

Since $a_0$ has at least $k$ neighbours, $|S| \geq k$, thus $\ell \geq k$. This proves (a).

Because $|\{a_1, \cdots, a_{k-1}\}| = k-1$, $a_0$ has at least one neighbour $a_j \in S - \{a_1, \cdots, a_{k-1}\}$ (i.e., $j \geq k$). Take $C = a_0, a_1, \cdots, a_j, a_0, \cdots$. Since $j \geq k$, the length of $C$ must be greater than or equal to $k+1$. This proves (b).

Lenght of path = number of edges.

## Problem 2

Show that if there is a closed walk of odd length in the graph $G$, then $G$ contains a cycle of odd length.

**Solution**

Let $W$ be a closed odd walk in $G$. Let $W'$ be a closed subwalk of $W$ with odd length, and we choose $W'$ such that the length of $W'$ is as small as possible (remember that $W'$ is *closed*).

We claim that $W'$ is a cycle. We'll prove this by contradiction: suppose that $W'$ is not a cycle. Let $W' = u_0, u_1, \cdots, u_m = u_0$. Since $W'$ is not a cycle, there exist two indices $i$ and $j$ ($i < j \leq m-1$) such that $u_i = u_j$. Now consider $W_1 = u_i, u_{i+1}, \cdots, u_j$ and $W_2 = u_0, u_1, \cdots, u_i, u_{j+1}, cdots, u_m = u_0$ (here we skipped $u_j$). We see that

$$\text{length}(W_1) + \text{length}(W_2) = \text{length}(W')$$

But $W'$ has an odd length, which means that one of $W_1$ and $W_2$ is a closed subwalk of $W$ with odd length, contradicting the minimality of $W'$.
By the claim, $W'$ is the desired odd cycle.
To prove something is not bipartite, find an odd cycle.