# AMATH 250 — LECTURE 5

*Bartosz Antczak* *Instructor: Zoran Miskovic* *May 11, 2017*

---

**Last Time**

We looked at sections 1.2.2 – 1.2.4. All those sections cover methods of solving 1st order DEs and sketching DEs.
We learned that the general solution of a 1st order DE has 1 arbitrary constant.

## 5.1 Existence-Uniqueness theorem

*If $\frac{dy}{dx} = f(x, y)$ is of class $C^1$ (has continuous partial derivatives), then the solution curves of the DE do not intersect.*

## 5.2 First order linear DEs with constant coefficient (1.2.5)

**Example 5.2.1.** *Suppose an amount of money $V_0 = 1,000$ is invested at time $t_0 = 0$ in a fund that pays interest at a constant rate of 5%/year. Assuming that the interest is compounded continuously in time, what is the value of investment $V(t)$ after $t = 10$ years?*

We want to derive a DE for $V(t)$. Let $\Delta V = V(t + \Delta t) - V(t)$. We have:

$$\frac{\Delta V}{V} \approx r\Delta t$$

with $r$ = relative rate = 5%/year. If we rearrange and let $\Delta t \to 0$, we have:

$$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = rV \tag{5.1}$$

$$\frac{dV}{dt} = rV \tag{5.2}$$

$$\int \frac{dV}{V} = \int r\ dt \tag{5.3}$$

$$\ln |V| = rt + c \tag{5.4}$$

$$|V| = e^c e^{rt} \tag{5.5}$$

$$V = De^{rt} \qquad \text{(Let } D = \pm e^c\text{)} \tag{5.6}$$

What happens if I start adding/withdrawing money to/from my account at some rate $f(t)$? We have:

$$\frac{dV}{dt} + kV = f(t)$$

where $k = r$.

**Aside**

The general solution of a DE

$$\frac{dy}{dx} + k(x)y = f(x) \tag{5.7}$$

has the form $y(x) = y_p(x) + y_h(x)$, where $y_p(x)$ is a particular solution of (4.7), and $y_h(x)$ is a general solution of the associated homogeneous DE

$$\frac{dy}{dx} + k(x)y = 0 \tag{5.8}$$

**Proposition**

If $y_p(x)$ is a particular solution of (4.7) and $y(x)$ is any solution of that same DE, then

$$y_h(x) = y(x) - y_p(x)$$

is a solution of (4.8).

*Proof:* we have

$$\begin{aligned} \frac{dy}{dx} + k(x)y &= f(x) && \text{(i)} \\ \frac{dy_p}{dx} + k(x)y_p &= f(x) && \text{(ii)} \end{aligned}$$

Solving (i) - (ii), we have:

$$\frac{d}{dx}(y - y_p) + k(x)[y - y_p] = 0 \implies y_h = y - y_p \quad \square$$

When $k(x) = k =$ constant, we have

$$\frac{dy_p}{dx} + ky_p = f(x)$$

The general solution is $y(x) = y_h(x) + y_p(x)$.
(refer to table on page 17 of textbook).

## 5.3 Applications of DEs (1.3) - Skydiver Example

Vertical motion of body of mass $m$ under the force of gravity with constant acceleration and the drag force of air with drag coefficient $\alpha$. Let $h_0$ be the initial height of the skydiver and $(t)$ be the position of the skydiver at time $t$ (unknown function). The net force acting on the skydiver is

$$F_{\text{net}} = F_g + F_d$$

where $F_g = mg$ and $F_d = -\alpha v$. We then have

$$F = mg - \alpha v \tag{5.9}$$

$$m\frac{dv}{dt} = mg - \alpha v \tag{5.10}$$

$$\frac{dv}{dt} + \frac{\alpha v}{m} = g \tag{5.11}$$

We'll use the method of undetermined coefficients to solve. $v_h = ce^{-\frac{\alpha}{m}t}$ and $v_p = A = \frac{gm}{\alpha}$. And therefore the general solution is

$$v(t) = ce^{-\frac{\alpha}{m}t} + \frac{gm}{\alpha}$$

## 5.4 Dimensions of Physical Quantities (1.1.2)

Very simple stuff.