# MATH 239 — TUTORIAL 8

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**Recall Binomial Theorem**

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

**Generating Series**

Suppose $S$ is a set of configurations and for each $\sigma \in S$, we have some weight $w(\sigma)$ (must be non-negative and an integer). For a given $k$, how many elements of $S$ have weight $k$?
We define the **generating series** of $S$ with respect to $w$ as:

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{k \geq 0} a_k x^k$$

Here, $a_k$ is the number of elements of $S$ with weight $k$.

**Example 8.0.1.** *Find $\Phi_S(x)$ if:*

a) $S = \{1, 2, \cdots\}$ and $w(i) = i$:

$$\begin{aligned}\Phi_S(x) &= 0x^0 + 1x + 1x^2 + \cdots \\ &= \sum_{i=0}^{\infty} x^i\end{aligned}$$

b) $S = \{1, 2, \cdots\}$ and $w(i) = i$ if $i$ is even, $w(i) = i - 1$ if $i$ is odd. Let's first construct a table:

| $i$ | $w(i)$ |
|---|---|
| 1 | 0 |
| 2 | 2 |
| 3 | 2 |
| 4 | 4 |
| 5 | 4 |
| 6 | 6 |
| 7 | 6 |

From here, our generating series is:

$$\begin{aligned}\Phi_s(x) &= 1x^0 + 0x^1 + 2x^2 + 0x^3 + 2x^4 + \cdots \\ &= 1 + \sum_{i=1}^{\infty} 2x^{(2i)}\end{aligned}$$

**Definition**

$[x^n]A(x)$ is notation that defines the coefficient of $x^n$ in $A(x)$.
Let $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $A(y) = a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots$. We say that $A(x) = A(y)$ if all of the coefficients are the same (which means that we don't care about the variables $x$ and $y$).

**Example 8.0.2.**

Prove that $(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$

**Solution**

Use induction.

- **Base Case:** let $k = 1$. It's clear that this statement holds
- **Hypothesis:** assume that this statement holds for some $m$. We'll try to show it works for $m+1$
- **Conclusion:** consider $m = m+1$. We have

$$(1-x)^{-(m+1)} = (1-x)^{-m}(1-x)^{-1}$$

By our hypothesis,

$$[x^i](1-x)^{-m} = \binom{i-m-1}{m-1}$$

Also, since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

We see that $[x^i](1-x)^{-1} = 1$.
So we have,

$$\begin{aligned}
[x^n](1-x)^{-(m+1)} &= \sum_{i=0}^{n} ([x^i](1-x)^{-m}[x^{n-i}](1-x)^{-1}) \\
&= \sum_{i=0}^{n} \binom{i+m-1}{m-1} \\
&= \binom{n+m}{m} \qquad \text{(Using the solution to problem 1.5.3 in course notes)}
\end{aligned}$$

From here, we see that every coefficient in $(1-x)^{-(m+1)}$ is equal to $\binom{n+m}{m}$, thus the entire polynomial is equal to:

$$(1-x)^{-(m+1)} = \sum_{n\geq 0} \binom{m+n}{m} x^n \qquad \square$$

*(Here, we have proven the negative binomial series)*

**Example 8.0.3.**

Solve $[x^8](1-x)^{-7}$.

**Solution**

Observe that $[x^8](1-x)^{-7} = [x^8]\sum_{n\geq 0}\binom{n+7-1}{7-1}x^n$. So the coefficient's value when $n=8$ is:

$$\binom{8+7-1}{7-1} = \binom{14}{6}$$

**Example 8.0.4.**

Solve $[x^{10}]x^6(1-2x)^{-5}$.

**Solution**

This is equivalent to solving for $[x^4](1-2x)^{-5}$. Now, let $y=2x$. We have:

$$\begin{aligned} &= \left[\left(\frac{y}{2}\right)^4\right](1-y)^{-5} \\ &= 2^4[y^4](1-y)^{-5} \end{aligned}$$

We now have our problem outlined in a way that we can solve it:

$$2^4\binom{4+5-1}{5-1}$$