MATH 239 — Lecture 32

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Recall from last lecture

$$\binom{1/2}{k} = \frac{(-1)^{k-1}(1\cdot 2\cdot 3\cdots(2k-3)(2k-2))}{2^k k! 2^{k-1}(1\cdot 2\cdots k-1)} = \frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\binom{2k-2}{k-1}$$

By Binomial theorem

$$(1+x)^{1/2} = 1 + \sum_{k\geq 1}\frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\binom{2k-2}{k-1}$$

We can use this to solve

$$(1-4x)^{1/2} = 2x\left(\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n\right) - 1$$

This content will **not** be on the final.

32.1 Recurrence Relations

32.1.1 Theorem 1

Let

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + \cdots \\ P(x) &= p_0 + p_1x + p_2x^2 + \cdots \\ Q(x) &= 1 - q_1x - q_2x^2 - \cdots \end{aligned}$$

be formal power series. Then

$$Q(x)A(x) = P(x) \iff \forall n \geq 0,\, a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + \cdots + q_na_0$$

Example 32.1.1. *Consider the following formal power series.*

$$\begin{aligned} A(x) &= \frac{1+x}{1-5x+6x^2} \\ P(x) &= 1 + x \implies p_0 = 1,\, p_1 = 1 \\ Q(x) &= 1 - 5x + 6x^2 \implies q_1 = 5,\, q_2 = 6 \end{aligned}$$

Using partial fractions, we have

$$A(x) = \frac{1+x}{1-5x+6x^2} = \frac{a}{1-2x} + \frac{b}{1-3x} \implies a = 3,\ b = 4$$

Finding the nth coefficient yields:

$$[x^n]A(x) = [x^n]\left(\frac{-3}{1-2x}\right) + [x^n]\left(\frac{4}{1-3x}\right) \implies a_n = (-3 \cdot 2^n + 4 \cdot 3^n)$$

For a_n, where do the 2 and 3 come from? They're the roots of

$$P = x^{\deg Q} Q(x^{-1})$$

We define P as the *characteristic polynomial of A*.

32.1.2 Theorem 2

Suppose $(a_n)_{n \geq 0}$ satisfies the recurrence relation $a_n + q_1 a_{n-1} + \cdots q_k a_{n-k} = 0,\ (n \geq k)$. If the characteristic polynomial of this has root r_1 for $i = 1, \ldots, j$, then the general solution is

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_j r_k^n$$

The previous theorem holds for **no repeated roots**.

32.1.3 Theorem 3

Let $A(x) = \sum_{n \geq 0} a_n x^n$ where a_n satisfies $a_n + q_1 a_{n-1} + \cdots + q_k q_{n-k} = 0\ (n \geq k)$. If $g(x) = 1 + q_1 x + q_2 x^2 + \cdots + q_k x^k$, then there exists a polynomial f of degree $< k$ such that

$$A(x) = \frac{f(x)}{g(x)}$$

Proof of Theorem 3

By the rule for finding coefficients of products of power series,

$$a_n + q_1 a_{n-1} + \cdots + q_k a_{n-k} = [x^n](1 + a_1 x + a_2 x^2 + \cdots + a_k x^k)A(x)$$

So $[x^n]g(x)A(x) = 0$ if $n \geq k$, and so

$$g(x)A(x) = f(x),\ \deg(f) < k$$

Example 32.1.2.

Recall the Fibonacci Numbers: $a_n = a_{n-1} + a_{n-2} \implies a_n - a_{n-1} - a_{n-2} = 0$
So $g(x) = 1 - x - x^2$. From this, our characteristic polynomial is $x^{\deg(g)} g(x^{-1}) = x^2 - x - 1$. From this, our roots to the characteristic polynomial are:

$$r_1, r_2 = \frac{1 \pm \sqrt{5}}{2}$$

That tells us that $a_n = c_1 r_1^n + c_2 r_2^n$. To find c_1 and c_2, use initial conditions (i.e., a_0 and a_1, both 1)

- For $n = 0$, $c_1 + c_2 = a_0 = 1$

- For $n = 1$, $c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$

From this,

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, \qquad c_2 = \frac{1-\sqrt{5}}{2\sqrt{5}}$$