CS 240 Midterm Review

Bartosz Antczak

Disclaimer: these questions were taken from past midterms and assignments but may not reflect what's going to be on this current midterm.

Problem Set 1 — Runtime Notation

1.0.1 O-notation

Show that $f(n) = 5n^3 + n^2 + 10n + 7 \in O(n^3)$.

Solution:

We must provide n_0, c such that $\forall n \geq n_0$, $f(n) \leq cn^3$. For all $n \geq 7$

$$\begin{aligned}
n^3 &\geq 7 \\
5n^3 &\geq 5n^3 \\
n^3 &\geq n^2 \\
10n^3 &\geq 10n \\
17n^3 &\geq f(n) && \text{(add all of the inequalities)}
\end{aligned}$$

Choose $c = 17$, $n_0 = 7$. Thus, $f(n) \in O(n^3)$

1.0.2 Big-Theta Notation

Show that $f(n) = \frac{n^3}{n+10} \in \Theta(n^2)$

Solution:

Provide an upper **and** lower bound:

$$\frac{n^3}{n+10} \leq \frac{n^3}{n} = n^2 \quad \forall n \geq 1$$
$$\frac{n^3}{n+10} \geq \frac{n^3}{n+n} = \frac{n^2}{2} \quad \forall n \geq 10$$

Choose $c_1 = \dfrac{1}{2}$, $c_2 = 1$, $n_0 = 10$. Thus, $f(n) \in \Theta(n^2)$.

1.0.3 Recurrence

What is the runtime of $T(n) = 2T(n-1) + 1$ with $T(1) = 1$?

Solution:

Expand this relation:

$$
\begin{aligned}
2(2T(n-2)+1)+1 &= 2^2T(n-2)+3 \\
2(2(2T(n-3)+1)+1)+1 &= 2^3T(n-3)+5 \\
2(2(2(2T(n-4)+1)+1)+1)+1 &= 2^4T(n-4)+7
\end{aligned}
$$

We observe a pattern. The relation is modelled as

$$2^{n-1} + \sum_{i=0}^{n-1} 2^i + 1$$

Observe that $\sum_{i=0}^{n-1} 2^i \in O(2^n)$ (as shown in our course notes). Thus, this recurrence relation has a runtime of $O(2^n)$.

Problem Set 2 — Loop Analysis

1.0.4 Loop 1

What is the runtime of this algorithm?

```
for (i = 1 to n)
  j = n^3
  while (j >= 1)
    j = j/3
```

Solution:

Observe that the while loop runs in

$$\log_3(n^3) = 3\log_3(n)$$

And the outer for loop runs in

$$\sum_{i=1}^{n} 3\log_3(n) = 3n\log_3(n) \in O(n\log n)$$

1.0.5 Loop 2

```
s = 1
for (i - 1 to n^2)
  for (j = 1 to n^3)
    for (k = 1 to j)
      s = 2 * s
```

Solution:

Focusing on the first two inner-most loops, the runtime is structured as

$$\sum_{j=1}^{n^3}\left(\sum_{k=1}^{j}1\right)=\sum_{j=1}^{n^3}j=\frac{n^3(n^3+1)}{2}$$

Analysing the outer loop, it has a runtime of

$$\sum_{i=1}^{n^2}\frac{n^3(n^3+1)}{2}\in O(n^8)$$

Therefore, the algorithm has a runtime of $O(n^8)$.

Problem Set 3 — Heaps

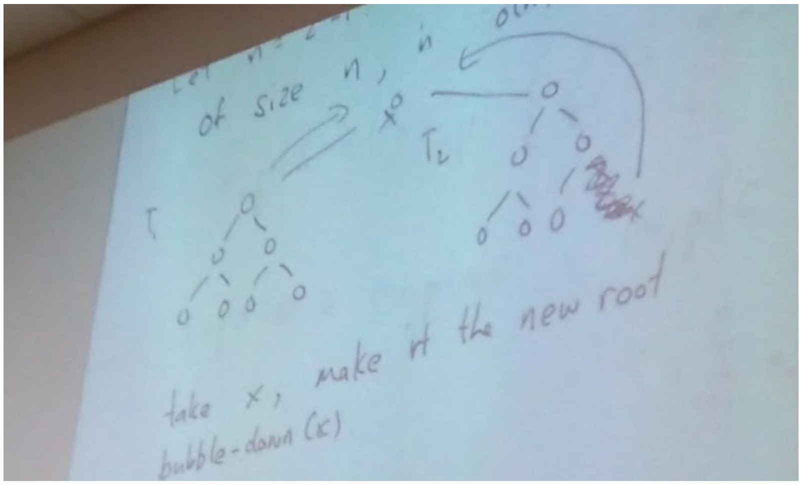
Let $n=2^k-1$. Write an algorithm that merges two heaps, T_1, T_2, both of size n, in $o(n)$.

Solution:

Observe that both of the heaps are full (since their size is 2^k-1, and by the structural property of heaps, every level must be filled). Our approach is as follows:

1. Take x, the smallest node in in T_2, and make it a new root, where its left and right children are T_1 and T_2 respectively.
2. Bubble down on x.

The visual representation of the new heap created is shown:



This process has a runtime of $\log(n)\in o(n)$, as required.

Problem Set 4 — Tries

Suppose we have a set of strings (hex numbers) $X = \{x_1, \cdots, x_n\}$, where $x_i < t$. If we store x in a compressed trie,

a) What is the space needed?

Solution:

Similar to how every node in a trie containing binary numbers will have **2** children, every node in this hexadecimal trie will have at most **16** children. For n leaves in the trie, the total number of internal nodes is $n - 1$. So, the internal nodes require $16(n - 1)$ amount of space, and the leaves require dn amount of space (where d is a constant). In total, the amount of memory required is

$$16(n - 1) + dn \in O(n)$$

Thus, we need $O(n)$ amount of space.

b) What is the height?

Solution:

The height is $O(\log_{16}(t))$

Problem Set 5 - Randomized Algorithms

Given the randomized algorithm, which finds the minimum value of an array A of size n:

```
find-min(A)
  i = random(n) // O(1)
  if (min(A, i)) // O(n)
    return A[i] // O(1)
  else
    return find-min(A) // O(?)
```

Find the best-case, worst-case, and expected-case.

Solution:

- **Best-case:** the first randomly generated number is the correct index, so we never recurse, which results in $O(n)$
- **Worst-case:** we never generate the correct index number, so the algorithm does not terminate
- **Expected Case:** let $T(n)$ be the expected runtime. By definition (and also from our course notes),

$$T(n) = \sum_{n} \Pr(n) \cdot n$$

Solving the equation, we get

$$
\begin{aligned}
T(n) &= \frac{1}{n}(cn) + \frac{n-1}{n}\left(T(n) + cn\right) \\
nT(n) &= nc + (n-1)T(n) + nc(n-1) && \text{(Doing some algebra)} \\
T(n) &= 2(n-1)nc \in O(n^2)
\end{aligned}
$$

Therefore, the expected runtime is $O(n^2)$.

Best of luck on the midterm! :)