AMATH 250 — Lecture 4

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Last Time

We looked at sections 1.2.2 – 1.2.4. All those sections cover methods of solving 1st order DEs and sketching DEs.
We learned that the general solution of a 1st order DE has 1 arbitrary constant.

4.1 Existence-Uniqueness theorem

If $\frac{dy}{dx} = f(x, y)$ is of class C^1 (has continuous partial derivatives), then the solution curves of the DE do not intersect.

4.2 First order linear DEs with constant coefficient (1.2.5)

Example 4.2.1. *Suppose an amount of money $V_0 = 1,000$ is invested at time $t_0 = 0$ in a fund that pays interest at a constant rate of 5%/year. Assuming that the interest is compounded continuously in time, what is the value of investment $V(t)$ after $t = 10$ years?*

We want to derive a DE for $V(t)$. Let $\Delta V = V(t + \Delta t) - V(t)$. We have:

$$\frac{\Delta V}{V} \approx r\Delta t$$

with r = relative rate = 5%/year. If we rearrange and let $\Delta t \to 0$, we have:

$$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = rV \tag{4.1}$$

$$\frac{dV}{dt} = rV \tag{4.2}$$

$$\int \frac{dV}{V} = \int r\ dt \tag{4.3}$$

$$\ln |V| = rt + c \tag{4.4}$$

$$|V| = e^c e^{rt} \tag{4.5}$$

$$V = De^{rt} \qquad \text{(Let } D = \pm e^c\text{)} \tag{4.6}$$

What happens if I start adding/withdrawing money to/from my account at some rate $f(t)$? We have:

$$\frac{dV}{dt} + kV = f(t)$$

where $k = r$.

Aside

The general solution of a DE

$$\frac{dy}{dx} + k(x)y = f(x) \tag{4.7}$$

has the form $y(x) = y_p(x) + y_h(x)$, where $y_p(x)$ is a particular solution of (4.7), and $y_h(x)$ is a general solution of the associated homogeneous DE

$$\frac{dy}{dx} + k(x)y = 0 \tag{4.8}$$

Proposition

If $y_p(x)$ is a particular solution of (4.7) and $y(x)$ is any solution of that same DE, then

$$y_h(x) = y(x) - y_p(x)$$

is a solution of (4.8).

Proof: we have

$$\frac{dy}{dx} + k(x)y = f(x) \qquad \text{(i)}$$

$$\frac{dy_p}{dx} + k(x)y_p = f(x) \qquad \text{(ii)}$$

Solving (i) - (ii), we have:

$$\frac{d}{dx}(y - y_p) + k(x)[y - y_p] = 0 \implies y_h = y - y_p \quad \square$$

When $k(x) = k =$ constant, we have

$$\frac{dy_p}{dx} + ky_p = f(x)$$

The general solution is $y(x) = y_h(x) + y_p(x)$.
(refer to table on page 17 of textbook).