AMATH 250 — LECTURE 10

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Last Time

Dimensionless forms of DEs. We looked at two examples

1. Salt concentration
$$\frac{dm}{dt} = \frac{f}{V}m + fc_{in}$$
We converted that DE into
$$\frac{d\mathcal{M}}{dt} + \mathcal{M} = 1$$
2. Skydiver
$$\frac{dv}{dt} = mg - \alpha g \implies \frac{dV}{dt} + V = 1$$

10.1 More on dimensionless forms of DEs

There is a third example that we'll cover on this topic: Newton's heating/cooling law. Define $T(t)$ as the temperature of the heating/cooling body. Define T_A as the ambient temperature. We have

$$\frac{dT}{dt} = -k(T - T_A) \tag{10.1}$$

We identify

- Characteristic time: $t_c = \frac{1}{k}$
- Characteristic temperature: $T_c = T_A$

Define

- Dimensionless time: $\tau = \frac{t}{t_c}$
- Dimensionless temperature: $\theta = \frac{T}{T_c}$

Sub our defined variables into (10.1) and we get

$$\frac{d\theta}{d\tau} + \theta = 1$$

10.2 Deducing physical relations using dimensionless analysis via Buckingham Pi Theorem [2.2.3] ($B\Pi T$)

Example 10.2.1. *Skydiver problem revisited*

For our defined DE for this problem, we have a relation between 4 physical quantities

$$F(v, m, g, \alpha) = 0 \tag{10.2}$$

$B\Pi T$ tells us that (10.2) must be equivalent to a relation of the form

$$f(\Pi_1, \Pi_2, \ldots) = c \qquad \exists c \in \mathrm{R} \tag{10.3}$$

where $\Pi_1, \Pi_2, \ldots$ are all possible independent dimensionless products of quantities v, m, g, α.
How do we find all possible products? First, we write the most general product of $\{v, m, g, \alpha\}$

$$\Pi = v^a m^b g^c \alpha^d \tag{10.4}$$

with unknown values a, b, c, d.
From there, we define the dimensions of Π as

$$[\Pi] = [v]^a [m]^b [g]^c [\alpha]^d \tag{10.5}$$

$$= L^{a+c} \cdot T^{-(a+2c+\alpha)} \cdot M^{b+d} \tag{10.6}$$

To make Π dimensionless, we need to cancel out the dimensions. This means we have a system of equations

$$\begin{aligned} a + C &= 0 \\ -a - 2c - d &= 0 \\ b + d &= 0 \end{aligned}$$

We see that there are actually an infinite number of solutions. We want to solve this system for a, c, d in terms of b, which is just some arbitrary real number. Solving, we have

$$\begin{aligned} a &= -b \\ c &= b \\ d &= -b \end{aligned}$$

Subbing in those values into (10.4) results

$$\Pi = v^{-b} m^b g^b \alpha^{-b} \tag{10.7}$$

$$= \left(\frac{mg}{v\alpha}\right)^b \tag{10.8}$$

Since b is arbitrary, let $b = 1$, so that $\Pi = \frac{mg}{v\alpha}$. by the $B\Pi T$, we must have some function f

$$f(\Pi) = c \implies f\left(\frac{mg}{v\alpha}\right) = c$$

If f is invertible, we can write $\frac{mg}{v\alpha} = f^{-1} = k$. From this, we conclude that the terminal velocity is $v = \frac{1}{k}\frac{mg}{\alpha}$.

10.3 Complete sets of dimensionless quantities

In general, if we have N physical quantities $Q_1, Q_2, \ldots, Q_N$ and the rank r of the system, then there will be $P = N - r$ independent dimensionless products $\Pi_1, \Pi_2, \ldots, \Pi_P$. Then the relation $F(Q_1, Q_2, \ldots, Q_N) = 0$

can be expressed as $f(\Pi_1, \Pi_2, \ldots, \Pi_P) = c$. Let's refer to our system in the previous example:

$$M = DP = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to write D by inspection:

	v	m	g	α
L	1	0	1	0
T	-1	0	-2	-1
M	0	1	0	1