MATH 239 — Lecture 21

Bartosz Antczak *Instructor: Luke Postle* *February 27, 2017*

Midterm Info

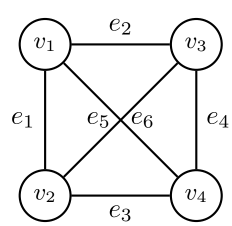
The midterm will cover only graph theory — **up to Matching and Covers**. This means that there's *no Konig's Theorem, Bipartite Matching Algorithm, or Hall's theorem.*
About half the exam is algorithmic (i.e., find an Eulerian circuit, find a particular colouring). The other half is proofs, some are easy (i.e., using Euler's formula) and others are more challenging.

21.1 Intro to Enumerative Combinatorics

One big topic in Combinatorics is graph theory (which was the first part of this course). Another big topic is counting, or *enumerative combinatorics*. It involves counting discrete objects (e.g., a graph, binary strings, compositions).

Example 21.1.1. *Examples for counting graphs:*

- How many graphs are there on n vertices?
- How many trees are there on n vertices?
- How many matchings (or perfect matchings) of K_n are there?
  For instance, in K_4, there's ***3*** *perfect matchings on labelled vertices:*



The matchings are $M_1 = \{e_1, e_4\}, M_2 = \{e_2, e_3\}, M_3 = \{e_5, e_6\}$.

21.1.1 How to Count

The basic operations for numbers are:

- Addition (+) [Plus its inverse, subtraction]
- Multiplication (×) [Plus its inverse, division]
- Equality (=)

We can apply these operations on objects in sets

Numbers	**Sets**
+	*Disjoint Union*
×	*Cartesian Product*
=	*Bijection*

Disjoint Unison

B is the **disjoint union** of A_1 and A_2 if $B = A_1 \cup A_2$ and A_1, A_2 are disjoint (i.e., $A_1 \cap A_2 = \emptyset$. Denoted as

$$B = A_1 \sqcup A_2$$

We can also extend this definition to many sets, B is the disjoint union of $A_1, A_2, \cdots, A_k$ if $B = A_1 \cup A_2 \cup \cdots A_k$ and $\forall i \neq j, A_i \cap A_j = \emptyset$. A proposition arises from this:

$$\textit{If } B = A_1 \sqcup A_2 \sqcup \cdots A_k, \textit{ then } |B| = |A_1| + |A_2| + \cdots |A_k| = \sum_{i=1}^{k} |A_i|$$

Remark: if $B = A_1 \sqcup A_2 \sqcup \cdots A_k$, then we say that $(A_1, A_2, \cdots, A_k)$ is a *partition* of B.

Cartesian Product

B is the **Cartesian product** of A_1 and A_2 is B is the set of ordered pairs whose 1st elements are in A_1 and the 2nd element in A_2. This is denoted as

$$B = A_1 \times A_2 = \{(a_1, a_2) \,:\, a_1 \in A_1,\, a_2 \in A_2\}$$

A common example of this is the plane of all real numbers, $\mathbb{R}^2 = \{(x, y) \,:\, x, y \in \mathbb{R}\}$. From this, a proposition arises:

$$\textit{If } B = A_1 \times A_2 \times \cdots \times A_k, \textit{ then } |B| = |A_1| \cdots |A_2| \cdots |A_k| = \prod_{i=1}^{k} |A_i|$$

If the same set is used in the product, this is called a **Cartesian power**. For example, if

$$B = A \times A \times \cdots_{k \text{ times}} \times A = A^k$$

A common example is R^n.

Bijection

A **bijection** from a set A to a set B is a 1-1 (i.e., one-to-one) mapping (or function) from A to B. Let's consider a proposition:

$$\textit{If there exists a bijection from } A \textit{ to } B\textit{, then } |A| = |B|$$

21.1.2 Binary Strings

A **binary string** is a sequence of zeros and ones. Its length is the number of digits. Some examples include: $001, 1010, 00101, 1101, \cdots$.

How many binary strings are there of length n?

The answer is 2^n. This works even when $n = 0$. This returns a value of 1, and this is correct. When $n = 0$, we have only the empty string ε.
There exists a formal proof showing that there are 2^n possible binary strings of length n. In short, it involves a bijection of $\{0, 1\}^n$. More on this next lecture.