MATH 239 — Tutorial 9

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Problem Set 1.7

d) Solve $[x^9]((1-4x)^5+(1-3x)^{-2})$.

Solution

We can simplify the expression to:

$$\begin{aligned}
[x^9]((1-4x)^5+(1-3x)^{-2}) &= [x^9](1-4x)^{-5}+[x^9](1-3x)^{-2} \\
&= 0+3^9[y^9](1-y)^{-2} \\
&= 3\binom{9+2-1}{2-1} \\
&= 3\binom{10}{1} = 10\cdot 3^9
\end{aligned}$$

f) Solve $[x^{n+1}]x^k(1-4x)^{-2k}$.

Solution

$$\begin{aligned}
[x^{n+1}]x^k(1-4x)^{-2k} &= [x^{n+1-k}](1-4x)^{2k} \\
&= 4^{n+1-k}[y^{n+1-k}](1-y)^{-2k} \\
&= 4^{n+1-k}\binom{n+1-k+2k-1}{2k-1} \\
&= 4^{n+1-k}\binom{n+k}{2k-1}
\end{aligned}$$

g) Solve $[x^n]x^k(1-x^2)^{-m}$.

Solution

$$\begin{aligned}
[x^n]x^k(1-x^2)^{-m} &= [x^{n-k}](1-x^2)^{-m} \\
&= [x^{\frac{n-k}{2}}](1-x)^{-m} \\
&= \binom{\frac{n-k}{2}+m-1}{m-1} \qquad \text{(if } n-k \text{ is even; it's 0 otherwise)}
\end{aligned}$$

9.1 Additional Problems

1) How many compositions of n are there with k parts? $(n \geq k \geq 1)$

Solution

Let $S = \mathbb{N}^k$. Then a composition $(c_1, \ldots, c_k)$ is just an element in S with $c_1 + \ldots + c_k = n$. Define $w(c_1, \ldots, c_k) = n$.

$$\begin{aligned}\Phi_S(x) = \Phi_{\mathbb{N}^k}(x) &= \Phi_{\mathbb{N}}(x)^k \\ &= (\sum_{i\geq 1} x^i)^k \\ &= (\frac{x}{1-x})^k \qquad \text{(Geometric Sequence)}\end{aligned}$$

So $[x^n](\frac{x}{1-x})^k$ is equal to:

$$\begin{aligned}&= [x^n]x^k(1-x)^{-k} \\ &= [x^{n-k}](1-x)^{-k} \\ &= \binom{n-k+k-1}{k-1} \qquad \text{(Negative binomial)} \\ &= \binom{n-1}{k-1}\end{aligned}$$

2) Let $k \in \mathbb{K}$ be fixed. How many compositions of n with k parts are there, where each part is congruent to $1 \mod 5$?

Solution

Let $S = \mathbb{N}^k_{\equiv 1}$, $\mathbb{N}^k_{\equiv 1} = \{1, 6, 11, 16, 21, \cdots\}$ (i.e., $5k+1$). Set $w(c_1, \ldots, c_k) = c_1 + \cdots + c_k$. We get:

$$\begin{aligned}\Phi_S(x) &= \Phi_{\mathbb{N}^k_{\equiv 1}}(x) \\ &= (\Phi_{\mathbb{N}_{\equiv 1}}(x))^k \\ &= \left(\sum_{i\geq 0} x^{5i+1}\right)^k \\ &= \left(x\sum_{i\geq 0} x^{5i}\right)^k \\ &=\end{aligned}$$