AMATH 250 — Lecture 2

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2.1 Classification Schemes for DEs

1. *Number of variables*

   (a) Ordinary DEs

   (b) Partial DEs

   (c) Systems of DEs

   (we'll only look at (a) and (c) in this course)

2. *Order of an ordinary DE for some function $y(x)$*

   - The order of an ordinary DE $y(x)$ is the order of the highest derivative of $y(x)$. In general, an $n-$th order DE has the form:

     $$F(x, y, y', \ldots, y^{(n)}) = 0$$

     or also

     $$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

3. *Linearity*

   - If F is a linear function of $y, y', \ldots, y^{(n)}$, then

     $$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = h(x) \tag{2.1}$$

     where $a_i(x)$ and $h(x)$ are given functions of x.

4. *Homogeneity of a linear DE*

   - For (2.1), if $h(x) = 0$ for all x, then the equation is a **homogeneous DE**; otherwise, the DE is non-homogeneous.

Example 2.1.1. *Classify the given DEs for $y(x)$*

a) $y' = x\sqrt{y}$
   Solution: 1st order and non-linear

b) $y'' + 2y' - 3y = 0$
   Solution: 2nd order, linear, homogeneous

2.1.1 Satisfying a DE

What does it mean when we say that a function $y(x)$ satisfies a DE? It simply means that it satisfies the equation itself.

Example 2.1.2. *Show that the given functions satisfy the DEs in the previous example*

a) $y = \frac{x^4}{16}$

Solution: $\frac{1}{4}x^3 = x\frac{x^2}{4} = \frac{1}{4}x^3 \quad \square$

b) $y_1 = e^x$ and $y_2 = e^{-3x}$

Solution:

1) it's trivial to see that $y_1 = e^x$ satisfies the DE
2) $9e^{-3x} - 6e^{-3x} - 3e^{-3x} = 0 \quad \square$

For (b), notice that any function of the form

$$y(x) = c_1e^x + c_2e^{-3x} \qquad c_1, c_2 \in \mathbb{R} \tag{2.2}$$

is also a solution of the DE.

Proof: Plugging in $y(x) = c_1e^x + c_2e^{-3x}$ into our DE yields:

$$(c_1e^x + c_2e^{-3x})'' + 2(c_1e^x + c_2e^{-3x})' - 3(c_1e^x + c_2e^{-3x}) = 0 \tag{2.3}$$

$$(c_1y_1 + c_2y_2)'' + 2(c_1y_1 + c_2y_2)' - 3(c_1y_1 + c_2y_2) = 0 \tag{2.4}$$

$$c_1(y'' + 2y' - 3y) + c_2(y'' + 2y' - 3y) = 0 \tag{2.5}$$

$$0c_1 + 0c_2 = 0 \quad \square \tag{2.6}$$

We refer to (2.2) as the **general solution** of the DE. The general solution for (a) is $\left(\frac{x^2}{4} + c\right)^2$ for $c \in \mathbb{R}$.

2.2 Mathematical Aspects of 1st-order DEs

The general form of a first order DE is

$$\frac{dy}{dx} = f(x, y)$$

where y represents an *unknown function* $y = y(x)$ (not a variable), and x is an *independent variable.*

2.2.1 Separable DEs

The general form of a first order separable DE is:

$$\frac{dy}{dx} = A(x)B(y)$$

where $A(x)$ and $B(y)$ are arbitrary functions. The general approach to solving these equations is shown:

$$\begin{aligned}
\frac{1}{B(y)} \cdot \frac{dy}{dx} &= A(x) && \text{(Divide by } B(y) \neq 0\text{)} \\
\int \frac{1}{B(y)} \cdot \frac{dy}{dx}\, dx &= \int A(x)\, dx \\
\int \frac{1}{B(y)}\, dy &= \int A(x)\, dx
\end{aligned}$$

Example 2.2.1. *Solve* $\frac{dy}{dx} = x\sqrt{y}$

$$\begin{aligned}
\frac{dy}{\sqrt{y}} &= x \cdot dx \\
\int \frac{1}{\sqrt{y}}\, dy &= \int x\, dx \\
2\sqrt{y} + c_1 &= \frac{1}{2}x^2 + c_2 \\
y &= \left(\frac{1}{4}x^2 + c\right) \quad \square
\end{aligned}$$

Example 2.2.2. *Solve* $\frac{dy}{dx} = -\frac{x}{y}$

$$\begin{aligned}
\frac{dy}{y} &= -x \cdot dx \\
\int \frac{1}{y}\, dy &= -\int x\, dx \\
x^2 + y^2 &= c \quad \square
\end{aligned}$$

This is an implicit solution.