AMATH 250 — Lecture 13

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Last time

Finished $B\Pi T$ and started 2nd order DEs with the mechanical oscillation example.

13.1 World's Simplest 2nd order DE (3.1.2)

Here, we neglect friction $\alpha = 0$, and we also consider free oscillation $F(t) = F_{ext} = 0$. Our ICs are $y(0) = y_0$ and $\frac{dy}{dt}(0) = v_0$ (an initial bump). We see our "simplified" DE is

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \tag{13.1}$$

Define frequency as

$$\omega = \sqrt{\frac{k}{m}} \implies [\omega] = T^{-1}$$

Let's make our DE non-dimensional. Define

- Characteristic time: $\frac{1}{\omega}$
- Dimensionless time: $\tau = \frac{t}{t_c} = \omega t$

We now convert our DE to

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Let $y(t) = y(\tau(t))$. By Chain Rule

$$\frac{dy}{dt} = \frac{dy}{d\tau} \cdot \frac{d\tau}{dt} = \omega \frac{dy}{d\tau}$$

We need the 2nd derivative as well:

$$\frac{d^2y}{dt^2} = \omega \frac{d}{dt}\left(\frac{dy}{d\tau}\right) \tag{13.2}$$

$$= \omega \frac{d^2y}{d\tau^2}\frac{d\tau}{dt} \tag{13.3}$$

$$= \omega^2 \frac{d^2y}{d\tau^2} \tag{13.4}$$

Sub into our DE to get

$$\frac{d^2y}{d\tau^2} + y = 0$$

Recall that

$$\frac{d^2}{d\tau^2}(\cos\tau) = -\cos\tau$$
$$\frac{d^2}{d\tau^2}(\sin\tau) = -\sin\tau$$

By inspection $y_1(\tau) = \cos\tau$ and $y_1(\tau) = \sin\tau$ are 2 possible solutions to our simplest DE. Thanks to the fact that our DE is linear, we also have

$$y(\tau) = cy_1(\tau) + cy_2(\tau)$$

as the general solution $\forall\, c_1, c_2 \in$ R. The proof is trivial and avoided.
Let's solve our IVP with our IVs defined previously:

$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

- $y(0) = 0 \implies c_1 = y_0$
- $\frac{dy}{dt}(0) = v_0 \implies c_2 = \frac{v_0}{\omega}$

Our solution is then

$$y(t) = y_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

13.1.1 Solving 2nd order DEs (3.2)

Fundamental property of linear DEs

Given the 2nd order linear DE for $y(x)$

$$y'' + P(x)y' + Q(x)y = 0 \tag{13.5}$$

If $y_1(x)$ and $y_2(x)$ are solutions of the DE (13.5), then

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is a solution of (13.5) $\forall\, c_1, c_2 \in$ R.

Proof:
Let

$$\begin{aligned} &1. \quad c_1y_1'' + Pc_1y_1' + Qc_1y_1 = 0 \\ &2. \quad c_2y_2'' + Pc_2y_2' + Qc_2y_2 = 0 \end{aligned}$$

We see that we have

$$\begin{aligned} (c_1y_1'' + c_2y_2'') + P(c_1y_1' + c_2y_2') + Q(c_1y_1 + c_2y_2) &= 0 \\ y'' + Py' + Qy &= 0 \qquad \square \end{aligned}$$