MATH 408: Computational Methods for Differential Equations

Unbounded Grids and the Semi-Discrete Fourier Transform

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Outline

1. Unbounded Grids and the Semi-Discrete Fourier Transform
2. Spectral Differentiation

Our goal in the next few lessons will be to understand and compute differentiation matrices for spectral methods.

As with our earlier FD approach, we will get the entries of the differentiation matrices by differentiating an interpolant of samples of the solution u of our BVPs.

One of the main tools to achieve this for the globally defined spectral methods is the Fourier transform, in a few different variants:

- continuous FT (only as a starting point)
- semi-discrete FT
- discrete FT
- fast FT (as an implementation of DFT)

The Fourier Transform and its Inverse

The Fourier transform $\hat{u}$ of a function u that is square-integrable on $\mathbb{R}$ is defined as:

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} u(x)\mathrm{d}x, \quad \xi \in \mathbb{R}. \tag{1}$$

The FT allows us to decompose u into a set of waves $\hat{u}$ with wavenumbers (or spatial frequency) ξ.
The FT $\hat{u}$ "lives" on Fourier space (or frequency space).

The inverse Fourier transform lets us reconstruct the signal u from its Fourier transform $\hat{u}$:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} \hat{u}(\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}. \tag{2}$$

The inverse FT u "lives" on physical space (we think "space", but this could also be "time").

Remark

Other popular definitions of the Fourier transform are (see Appendix E.3.1 of LeVeque's book [1])

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx, \qquad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{u}(\xi) d\xi,$$

or

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi x} u(x) dx, \qquad u(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi x} \hat{u}(\xi) d\xi.$$

Remark

Also, note that the notation $\hat{u}$ refers to the Fourier transform of u and has nothing to do with the notation $\widehat{U} = [u(x_1), u(x_2), \ldots, u(x_m)]^T$ used earlier to denote a vector of values of the exact solution of an ODE.

Example

Compute the Fourier transform of the square pulse function

$$u(x) = \begin{cases} 1, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

The definition of the FT, the definition of u, and Euler's formula yield

$$\begin{aligned} \hat{u}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\xi x} dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\cos(\xi x) - i \sin(\xi x)\right] dx \\ &= 2 \int_0^{\frac{1}{2}} \cos(\xi x) dx \\ &= 2 \frac{\sin(\xi x)}{\xi} \bigg|_0^{\frac{1}{2}} = \frac{\sin\left(\frac{\xi}{2}\right)}{\frac{\xi}{2}} \quad \text{(sinc function).} \end{aligned}$$

The semi-discrete Fourier Transform and its Inverse

Let's restrict our attention to the discrete (unbounded) physical space $h\mathbb{Z}$, i.e., we now consider **infinitely many** uniformly spaced grid points $x_j = jh$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$ (or $j \in \mathbb{Z}$).

The continuous function u is now replaced by the (infinite) vector

$$\boldsymbol{u} = [\ldots, u_{-1}, u_0, u_1, \ldots]^T$$

of discrete values $u_j = u(x_j)$, $j \in \mathbb{Z}$.

We can think of the value u_j as a sample of the signal u at x_j.

Now we need the semi-discrete Fourier transform of $\boldsymbol{u}$ given by the (continuous) function

$$\hat{u}(\xi) = h \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x_j} u_j, \quad \xi \in [-\frac{\pi}{h}, \frac{\pi}{h}], \tag{3}$$

and the inverse semi-discrete Fourier transform, given by the (discrete) infinite vector $\boldsymbol{u}$ whose components are of the form

$$u_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \mathrm{e}^{\mathrm{i}\xi x_j} \hat{u}(\xi) \mathrm{d}\xi, \quad j \in \mathbb{Z}. \tag{4}$$

Remark

Note that the notion of a semi-discrete Fourier transform is just a different name for a Fourier series based on the complex exponentials $\mathrm{e}^{-\mathrm{i}\xi x_j}$ with Fourier coefficients u_j (see Appendix).

Aliasing

Why are we allowed to work with a bounded Fourier space in the semi-discrete setting?

This can be explained by the phenomenon of aliasing.

Aliasing arises when a continuous function is sampled on a discrete set.

Consider the two (continuous) complex exponential functions

$$u(x) = \mathrm{e}^{\mathrm{i}\xi_1 x},$$
$$v(x) = \mathrm{e}^{\mathrm{i}\xi_2 x}.$$

We know that $u(x) \neq v(x)$ for all $x \in \mathbb{R}$ provided $\xi_1 \neq \xi_2$.

However, if we sample the two functions on the grid $h\mathbb{Z}$, then we get the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ with values

$$u_j = \mathrm{e}^{\mathrm{i}\xi_1 x_j},$$
$$v_j = \mathrm{e}^{\mathrm{i}\xi_2 x_j}.$$

If $\xi_2 = \xi_1 + 2k\frac{\pi}{h}$ for some integer k, then

$$\begin{aligned} v_j &= e^{i\xi_2 x_j} \\ &= e^{i(\xi_1 + 2k\frac{\pi}{h})x_j} \\ &= e^{i\xi_1 x_j} \underbrace{e^{i2k\frac{\pi}{h}jh}}_{=1} = u_j \end{aligned}$$

for any j and the two (different) continuous functions u and v appear identical in their discrete samples $\boldsymbol{u}$ and $\boldsymbol{v}$.

Thus, any complex exponential $e^{i\xi x}$ is matched on the grid $h\mathbb{Z}$ by infinitely many other complex exponentials (the aliases for the frequency ξ).

Therefore we can limit the representation of the Fourier variable (wavenumber) ξ to an interval of length $2\pi/h$. For reasons of symmetry we use $[-\pi/h, \pi/h]$.

Example

Let's consider two sine functions instead of complex exponentials, i.e.,

$$\begin{aligned} u(x) &= \sin(\pi x), \\ v(x) &= \sin(9\pi x). \end{aligned}$$

Thus $\xi_1 = \pi$ and $\xi_2 = 9\pi$. Also, let $h = \frac{1}{4}$, so that $x_j = jh = \frac{j}{4}$.
Then $\xi_2 = \xi_1 + 2k\frac{\pi}{h}$ corresponds to

$$\xi_2 = \xi_1 + 2k\frac{\pi}{1/4} = \xi_1 + 8k\pi.$$

Since $\xi_1 = \pi$ and $\xi_2 = 9\pi$ we let $k = 1$ and show

$$\begin{aligned} v_j &= \sin(9\pi x_j) = \sin(\pi + 8\pi)x_j \\ &= \sin(\pi x_j)\cos(8\pi x_j) + \cos(\pi x_j)\sin(8\pi x_j) \\ &= \sin(\pi x_j)\underbrace{\cos(8\pi \tfrac{j}{4})}_{=1} + \cos(\pi x_j)\underbrace{\sin(8\pi \tfrac{j}{4})}_{=0} = u_j. \end{aligned}$$

The MATLAB script `AliasDemo.m` illustrates this aliasing phenomenon.

Band-limited Functions

To get an interpolant of the (infinitely many) samples u_j we can now use a continuous extension of the inverse semi-discrete Fourier transform, i.e., we define the interpolant to be the function

$$p(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\xi x} \hat{u}(\xi) d\xi, \qquad x \in \mathbb{R}. \tag{5}$$

It is obvious (cf. (4)) from this construction that p interpolates the samples, i.e., $p(x_j) = u_j$, for any $j \in \mathbb{Z}$.
Moreover, the Fourier transform of the function p turns out to be

$$\hat{p}(\xi) = \begin{cases} \hat{u}(\xi), & \xi \in [-\pi/h, \pi/h], \\ 0, & \text{otherwise.} \end{cases}$$

This kind of function is known as a band-limited function, and p is called the band-limited interpolant of $\boldsymbol{u}$.

Spectral Derivative

The spectral derivative vector $\boldsymbol{u}'$ of $\boldsymbol{u}$ can now be obtained by one of the following two procedures we are about to present.

Procedure #1:

1. Sample the function u at the (infinite set of) discrete points $x_j \in h\mathbb{Z}$ to obtain the data vector $\boldsymbol{u}$ with components u_j.
2. Compute the semi-discrete FT of the data via (3):

$$\hat{u}(\xi) = h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} u_j, \quad \xi \in [-\pi/h, \pi/h].$$

3. Find the band-limited interpolant p of the data u_j via (5):

$$p(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\xi x} \hat{u}(\xi) d\xi, \qquad x \in \mathbb{R}.$$

4. Differentiate p and evaluate at the grid points x_j.

From a computational point of view it is better to deal with this problem in the Fourier domain.

Note that the (continuous) FT of the (exact) derivative u' is given by

$$\widehat{u'}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} u'(x) \mathrm{d}x.$$

Applying integration by parts we get

$$\widehat{u'}(\xi) = \mathrm{e}^{-\mathrm{i}\xi x} u(x)\Big|_{-\infty}^{\infty} + \mathrm{i}\xi \underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} u(x) \mathrm{d}x}_{=\hat{u}(\xi)}.$$

If $u(x) \to 0$ for $x \to \pm\infty$ (which it has to for the FT of u to exist) then we see that

$$\widehat{u'}(\xi) = \mathrm{i}\xi \hat{u}(\xi). \tag{6}$$

Therefore, we obtain the spectral derivative $\boldsymbol{u}'$ by Procedure #2:

1. Sample the function u at the (infinite set of) discrete points $x_j \in h\mathbb{Z}$ to obtain the data vector $\boldsymbol{u}$ with components u_j.
2. Compute the semi-discrete FT of the data via (3):
$$\hat{u}(\xi) = h \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x_j} u_j, \quad \xi \in [-\pi/h, \pi/h].$$
3. Compute the FT of the derivative via (6):
$$\widehat{u'}(\xi) = i\xi \hat{u}(\xi).$$
4. Find the derivative vector via inverse semi-discrete FT (see (4)):
$$u_j' = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \mathrm{e}^{\mathrm{i}\xi x_j} \widehat{u'}(\xi) \mathrm{d}\xi, \quad j \in \mathbb{Z}.$$

How do we get the differentiation matrix D from these procedures?

Following Procedure #1, the semi-discrete FT of an arbitrary sample vector $\boldsymbol{u}$ is obtained by representing its components in terms of shifts of (discrete) delta functions, i.e.,

$$u_j = \sum_{k=-\infty}^{\infty} u_k \delta_{j-k}, \tag{7}$$

where the Kronecker delta function is defined by

$$\delta_j = \begin{cases} 1 & j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We use this approach since the semi-discrete FT of the delta function can be computed easily.

To find the semi-discrete FT of the delta function we recall (3), i.e.,

$$\hat{u}(\xi) = h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} u_j, \quad \xi \in [-\frac{\pi}{h}, \frac{\pi}{h}].$$

This implies

$$\begin{aligned} \hat{\delta}(\xi) &= h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} \delta_j \\ &= h e^{-i\xi x_0} = h \end{aligned}$$

for all $\xi \in [-\pi/h, \pi/h]$ since $x_0 = 0h$.

Then the band-limited interpolant of δ is of the form (see (5))

$$
\begin{aligned}
p(x) &= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\hat{\delta}(\xi)d\xi \\
&= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x} h\, d\xi \\
&= \frac{h}{\pi}\int_{0}^{\pi/h} \cos(\xi x)d\xi \\
&= \frac{h}{\pi}\left.\frac{\sin(\xi x)}{x}\right|_{0}^{\pi/h} \\
&= \frac{h}{\pi}\frac{\sin(\pi\frac{x}{h})}{x} = \frac{\sin(\frac{\pi x}{h})}{\frac{\pi x}{h}} = \operatorname{sinc}(\frac{\pi x}{h}).
\end{aligned}
$$

We now calculate the band-limited interpolant of an arbitrary data vector $\boldsymbol{u}$:

$$\begin{aligned} p(x) &\stackrel{(5)}{=} \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\hat{u}(\xi)d\xi \\ &\stackrel{(3)}{=} \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\left[h\sum_{j=-\infty}^{\infty} e^{-i\xi x_j}u_j\right]d\xi \\ &\stackrel{(7)}{=} \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\left[h\sum_{j=-\infty}^{\infty} e^{-i\xi x_j}\sum_{k=-\infty}^{\infty} u_k\delta_{j-k}\right]d\xi. \end{aligned}$$

Interchanging the summation results in

$$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\left[h\sum_{k=-\infty}^{\infty} u_k\sum_{j=-\infty}^{\infty} e^{-i\xi x_j}\delta_{j-k}\right]d\xi.$$

Now we use the definition of the delta function and the same calculation as for the band-limited interpolant of the delta function above to obtain the final form of the band-limited interpolant of an arbitrary data vector ***u***:

$$
\begin{aligned}
p(x) &= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x}\left[h\sum_{k=-\infty}^{\infty} u_k \sum_{j=-\infty}^{\infty} e^{-i\xi x_j}\delta_{j-k}\right] d\xi \\
&\overset{\text{Def.}\delta}{=} \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi x} h \sum_{k=-\infty}^{\infty} u_k e^{-i\xi x_k} d\xi \\
&\overset{\text{rearrange}}{=} \sum_{k=-\infty}^{\infty} u_k \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\xi(x-x_k)} h\, d\xi \\
&\overset{\text{bandlim}}{=} \sum_{k=-\infty}^{\infty} u_k \operatorname{sinc}\frac{(x-x_k)\pi}{h}.
\end{aligned}
$$

If u is a band-limited function then the result of this calculation is known as the Whittaker–Shannon Sampling Theorem and $p(x) = u(x)$, i.e., the reconstruction is exact for all $x \in \mathbb{R}$!

Example

Band-limited interpolation for the functions

$$u_1(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise}, \end{cases}$$

$$u_2(x) = \begin{cases} 1, & |x| \le 3 \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_3(x) = (1 - |x|/3)_+.$$

is illustrated in the MATLAB script `BandLimitedDemo.m`.

Remark

- *The accuracy of the reproduction is not very high.*
- *In particular, for $h \to 0$ we observe a Gibbs phenomenon caused by the low smoothness of the data functions.*

Calculating the Differentiation Matrix

To find the components of the derivative vector $\boldsymbol{u}'$ we need to differentiate the band-limited interpolant and evaluate at the grid points. By linearity this leads to

$$u_j' = p'(x_j) = \sum_{k=-\infty}^{\infty} u_k \frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{sinc}\frac{(x-x_k)\pi}{h}\right]_{x=x_j},$$

or in (infinite) matrix-vector form

$$\boldsymbol{u}' = \mathsf{D}\boldsymbol{u}$$

with the entries of D given by

$$\mathsf{D}_{jk} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{sinc}\frac{(x-x_k)\pi}{h}\right]_{x=x_j}, \quad j,k = -\infty,\ldots,\infty.$$

The entries in the $k = 0$ column of D are of the form (see next slide)

$$D_{j0} = \frac{d}{dx}\left[\operatorname{sinc}\frac{x\pi}{h}\right]_{x=x_j=jh} = \begin{cases} 0, & j = 0 \\ \frac{(-1)^j}{jh}, & \text{otherwise.} \end{cases}$$

The remaining columns are up or down shifts of this column and so the matrix is a Toeplitz matrix of the form (see slide #18 of `Notes408_15_HigherOrderMethods2ptBVPs`)

$$D = \frac{1}{h}\begin{bmatrix} & \vdots & \\ \ddots & \frac{1}{3} & \\ \ddots & -\frac{1}{2} & \\ \ddots & 1 & \\ & 0 & \\ & -1 & \ddots \\ & \frac{1}{2} & \ddots \\ & -\frac{1}{3} & \ddots \\ & \vdots & \end{bmatrix}.$$

The explicit formula for the derivative of the sinc function above is obtained using elementary calculations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{sinc}\frac{x\pi}{h}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin\frac{x\pi}{h}}{\frac{x\pi}{h}}\right] \\ &= \frac{1}{x}\cos\left(\frac{x\pi}{h}\right) - \frac{h}{x^2\pi}\sin\left(\frac{x\pi}{h}\right),\end{aligned}$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{sinc}\frac{x\pi}{h}\right]_{x=x_j=jh} &= \frac{1}{jh}\cos(j\pi) - \frac{1}{j^2h\pi}\sin(j\pi) \\ &= \frac{1}{jh}(-1)^j - \frac{1}{j^2h\pi}0.\end{aligned}$$

Working on an infinite grid is not very practical. So, next we consider periodic grids.

References I

- R. LeVeque.
  Finite Difference Methods for Ordinary and Partial Differential Equations.
  SIAM, Philadelphia, 2007.
- L. N. Trefethen.
  Spectral Methods in MATLAB.
  SIAM, Philadelphia, 2000.

Appendix: Complex Form of the Fourier Series

Fourier series are often expressed in terms of complex exponentials instead of sines and cosines.

The main ingredient for understanding this translation in notation is Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This, of course, implies

$$e^{-i\theta} = \cos\theta - i\sin\theta,$$

and so

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

We can therefore rewrite the Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

as

$$\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \right] \\ &= a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\left(a_n + \frac{b_n}{i} \right) e^{i\frac{n\pi x}{L}} + \left(a_n - \frac{b_n}{i} \right) e^{-i\frac{n\pi x}{L}} \right] \end{aligned}$$

We break this into two series and use $\frac{1}{\mathrm{i}} = -\mathrm{i}$ to arrive at

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - \mathrm{i}b_n)\,\mathrm{e}^{\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}}$$

Now we perform an index transformation, $n \to -n$, on the first series to get

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_{-n} - \mathrm{i}b_{-n})\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}}$$

Note that, using the symmetries of cosine and sine,

$$\begin{aligned} a_{-n} &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{(-n)\pi x}{L}\,\mathrm{d}x = a_n \\ b_{-n} &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{(-n)\pi x}{L}\,\mathrm{d}x = -b_n \end{aligned}$$

We can therefore rewrite

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_{-n} - \mathrm{i}b_{-n})\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}}$$

as

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_n + \mathrm{i}b_n)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,\mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}}$$

If we introduce new coefficients

$$c_0 = a_0 \qquad \text{and} \qquad c_n = \frac{a_n + \mathrm{i}b_n}{2}$$

then we get the exponential form of the Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{-\mathrm{i}\frac{n\pi x}{L}}$$

with Fourier coefficients

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x$$

and

$$
\begin{aligned}
c_n &= \frac{a_n + \mathrm{i} b_n}{2} \\
&= \frac{1}{2L}\left[\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x + \mathrm{i}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\frac{n\pi x}{L} + \mathrm{i}\sin\frac{n\pi x}{L}\right]\mathrm{d}x \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{e}^{\mathrm{i}\frac{n\pi x}{L}}\,\mathrm{d}x
\end{aligned}
$$

Note that this formula also gives the correct value for c_0.

Remark

The formula for the Fourier coefficients c_n is known as the inverse semi-discrete Fourier transform of f.