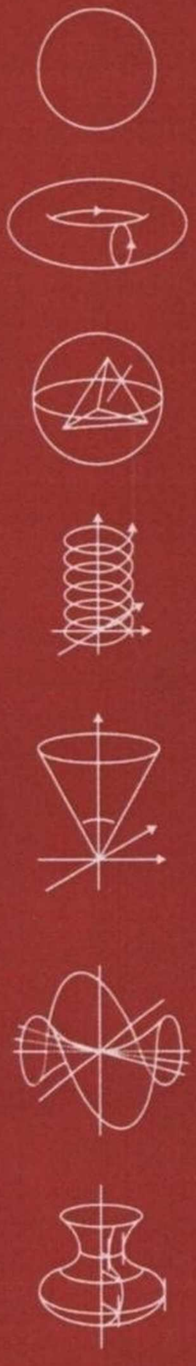


A First Course in Geometric Topology and Differential Geometry

Ethan D. Bloch

Birkhäuser

Dedicated to my parents,

in appreciation for all they have taught me

Ethan D. Bloch

A First Course in Geometric Topology and Differential Geometry

Birkhäuser
Boston • Basel • Berlin

Ethan D. Bloch
Department of Natural Sciences and Mathematics
Bard College
Annandale, New York 12504
USA

Library of Congress Cataloging In-Publication Data

Bloch, Ethan, 1956-
A first course in geometric topology and differential geometry / Ethan Bloch.
p. cm.
Includes bibliographical references (p. -) and index.
ISBN 0-8176-3840-7 (h : alk. paper). -- ISBN 3-7643-3840-7 (H : alk. paper)
1. Topology. 2. Geometry, Differential. I. Title.
QA611.ZB55 1996 95-15470
516.3'63--dc20 CIP

Printed on acid-free paper

Birkhäuser

ISBN 0-8176-3840-7
ISBN 3-7643-3840-7

Typeset from author's disk in $\mathcal{AMS}$-TEX by TEXniques, Inc., Boston, MA
Illustrations by Carl Twarog, Greenville, NC
Printed and bound by Maple-Vail, York, PA
Printed in the U.S.A.
9 8 7 6 5 4 3 2 1

Contents

Introduction

This text is an introduction to geometric topology and differential geometry via the study of surfaces, and more generally serves to introduce the student to the relation of the modern axiomatic approach in mathematics to geometric intuition. The idea of combining geometry and topology in a text is, of course, not new; the present text attempts to make such a combination of subjects accessible to the junior/senior level mathematics major at a university or college in the United States. Though some of the deep connections between the topology and geometry of manifolds can only be dealt with using more advanced techniques than those presented here, we do reach the classical Gauss–Bonnet Theorem — a model theorem for the relation of topology and geometry — at the end of the book.

The notion of a surface is the unifying thread of the text. Our treatment of point set topology is brief and restricted to subsets of Euclidean spaces; the discussion of topological surfaces is geometric rather than algebraic; the treatment of differential geometry is classical, treating surfaces in $\mathbb{R}^3$. The goal of the book is to reach a number of intuitively appealing definitions and theorems concerning surfaces in the topological, polyhedral and smooth cases. Some of the goodies aimed at are the classification of compact surfaces, the Gauss–Bonnet Theorem (polyhedral and smooth) and the geodesic nature of length-minimizing curves on surfaces. Only those definitions and methods needed for these ends are developed. In order to keep the discussion at a concrete level, we avoid treating a number of technicalities such as abstract topological spaces, abstract simplicial complexes and tensors. As a result, at times some proofs seem a bit more circuitous than might be standard, though we feel that the gain in avoiding unnecessary technicalities is worthwhile.

There are a variety of ways in which this book could be used for a semester course. For students with no exposure to topology, the first three chapters, together with a sampling from Chapters IV and V, could be used as a one-semester introduction to point set and geometric topology, with a taste of smooth surfaces thrown in. Alternately, Chapters IV–VIII could be used as a quite leisurely first course on differential geometry (skipping the few instances where the previous chapters are used, and adding an intuitive discussion of the Euler characteristic for the Gauss–Bonnet Theorem). Students who have had a semester of point set topology (or a real analysis course in which either $\mathbb{R}^n$ or metric spaces are discussed), could cover a fair bit of Chapters II–VIII in one semester, though some material would probably have to be dropped. It is also hoped that the book could be used for individual study.

This book developed out of lecture notes for a course at Bard College first given in the spring of 1991. I would like to thank Bard students Melissa Cahoon, Jeff Bolden, Robert Cutler, David Steinberg, Anne Willig, Farasat Bokhari, Diego Socolinsky and Jason Foulkes for helpful comments on various drafts of the original lecture notes. Thanks are also due to Matthew Deady, Peter Dolan, Mark Halsey, David Nightingale and Leslie Morris, as well as to the Mathematics Institute at Bar-Ilan University in Israel and the Mathematics Department at the University of Pennsylvania, who hosted me when various parts of this book were written. It is, of course, impossible to acknowledge every single topology and differential geometry text from which I have learned about the subject, and to credit the source of every definition, lemma and theorem (especially since many of them are quite standard); I have acknowledged in the text particular sources for lengthy or non-standard proofs. See the section entitled Further Study for books that have particularly influenced this text. For generally guiding my initial development as a mathematician I would like to thank my professors at Reed College and Cornell University, and in particular my advisor, Professor David Henderson of Cornell. Finally, I would like to thank Ann Kostant, mathematics editor at Birkhäuser, for her many good ideas, and the helpful staff at Birkhäuser for turning my manuscript into a finished book.

To the Student

Surfaces

Surfaces can be approached from two viewpoints, topological and geometric, and we cover both these approaches. There are three different categories of surfaces (and, more generally, "manifolds," a generalization of surfaces to all dimensions) that we discuss: topological, simplicial and smooth. In contrast to higher dimensional analogs of surfaces, in dimension two (the dimension of concern in this book), all three types of surfaces turn out to have the same topological properties. Hence, for our topological study we will concentrate on topological and simplicial surfaces. This study, called geometric topology, is covered in Chapters II–III.

Geometrically, on the other hand, the three types of surfaces behave quite differently from each other. Indeed, topological surfaces can sit very wildly in Euclidean space, and do not have sufficient structure to allow for manageable geometric analysis. Simplicial surfaces can be studied geometrically, as, for example, in Section 3.9. The most interesting, deep and broadly applicable study of the geometry of surfaces involves smooth surfaces. Our study of smooth surfaces will thus be fundamentally different in both aim and flavor than our study of topological and simplicial surfaces, focusing on geometry rather than topology, and on local rather than global results. The methodology for smooth surfaces involves calculus, rather than point-set topology. This study, called differential geometry, is studied in Chapters IV–VIII. Although apparently distinct, geometric topology and differential geometry come together in the amazing Gauss–Bonnet Theorem, the final result in the book.

Prerequisites

This text should be accessible to mathematics majors at the junior or senior level in a university or college in the United States. The minimal prerequisites are a standard calculus sequence (including multivariable calculus and an acquaintance with differential equations), linear algebra (including inner products) and familiarity with proofs and the basics of sets and functions. Abstract algebra and real analysis are not required. There are two proofs (Theorem 1.5.2 and Proposition 1.6.7) where the Least Upper Bound Property of the real numbers is used; the reader who has not seen this property (for example, in a real analysis course) can skip these proofs. If the reader has had a course in point-set topology, or a course in real analysis where the setting is either $\mathbb{R}^n$ or metric spaces, then much of Chapter I could probably be skipped over. In a few instances we make use of affine linear maps, a topic not always covered in a standard linear algebra course; all the results we need concerning such maps are summarized in the Appendix.

Rigor vs. Intuition

The study of surfaces from topological, polyhedral and smooth points of view is ideally suited for displaying the interaction between rigor and geometric intuition applied to objects that have inherent appeal. In addition to informal discussion, every effort has been made to present a completely rigorous treatment of the subject, including a careful statement of all the assumptions that are used without proof (such as the triangulability of compact surfaces). The result is that the material in this book is presented as dictated by the need for rigor, in contrast to many texts which start out more easily and gradually become more difficult. Thus we have the odd circumstance of Section 2.2, for example, being much more abstract than some of the computational aspects of Chapter V. The reader might choose to skip some of the longer proofs in the earlier chapters upon first reading.

At the end of the book is a guide to further study, to which the reader is referred both for collateral readings (some of which have a more informal, intuitive approach, whereas others are quite rigorous), and for references for more advanced study of topology and differential geometry.

Exercises

Doing the exercises is a crucial part of learning the material in this text. A good portion of the exercises are results that are needed in the text; such exercises have been marked with an asterisk (*). Exercises range from routine computations (particularly in the chapters on differential geometry) to rather tricky proofs. No attempt has been made to rate the difficulty of the problems, since doing so is highly subjective. There are hints for some of the exercises in the back of the book.

A First Course in Geometric Topology and Differential Geometry

CHAPTER IV

Curves in $\mathbb{R}^3$

4.1 Introduction

Though our main topic of concern is surfaces, prior to studying smooth surfaces we take a small detour through the study of smooth curves in $\mathbb{R}^3$ to develop some important tools. Our treatment of curves will be brief; more about curves, including such results such as the pretty Milnor–Fary Theorem, can be found in [M-P] or [DO1].

For the rest of the book we will be in the realm of differentiable functions. Section 4.2 reviews some basic facts concerning such functions, including the Inverse Function Theorem and some existence and uniqueness theorems for the solutions of ordinary differential equations, which play a foundational role for smooth surfaces.

4.2. Smooth Functions

We start with some assumptions about differentiable functions.

Definition. Let $U \subset \mathbb{R}^n$ be a set, and let $F: U \to \mathbb{R}^m$ be a map. We say F is **smooth** if

(1) the set U is open in $\mathbb{R}^n$;

(2) all partial derivatives of F of all orders exist and are continuous.

We can write F using coordinate functions as

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix},$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $F_1, \ldots, F_m: U \to \mathbb{R}$ are smooth functions. The

Jacobian matrix of F is the matrix of partial derivatives

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}.$$

The openness of the set U in the above definition will often be unstated, but will be assumed nonetheless. The following definition is the smooth analog of the notion of homeomorphism.

Definition. Let $U, V \subset \mathbb{R}^n$ be open sets. A function $f: U \to V$ is a **diffeomorphism** if it is bijective, and if both f and f^{-1} are smooth.

If $f: U \to V$ is a diffeomorphism then the Jacobian matrix Df is non-singular at each point in U (see Exercise 4.2.2).

We now turn to the Inverse Function Theorem and differential equations; the reader should feel free to skip this material until it is needed is subsequent sections. The Inverse Function Theorem addresses the question of whether a smooth function $f: U \to \mathbb{R}^n$ has a smooth inverse (that is, whether it is a diffeomorphism). The one-dimensional case is simple. Let $f: J \to \mathbb{R}$ be a smooth function for some open interval J. If $f'(x_0) \neq 0$ for point $x_0 \in J$, then the function is either strictly increasing or strictly decreasing near x_0, and it follows that near x_0 the function has an inverse. Of course, having $f'(x_0) \neq 0$ does not imply that the whole function f has an inverse, but only that the function restricted to some (possibly very small) open neighborhood of x_0 has an inverse. Since the graph of an inverse function is simply the reflection in the line $y = x$ of the original graph, we see that if $f'(x_0) \neq 0$ then the inverse function of f restricted to a neighborhood of $f(x_0)$ will also be smooth. The Inverse Function Theorem is the higher-dimensional analog of what we have just discussed. The condition $f'(x_0) \neq 0$ is replaced by the condition that the Jacobian matrix has non-zero determinant at the given point.

Theorem 4.2.1 (Inverse Function Theorem). *Let $U \subset \mathbb{R}^n$ be an open set and let $F: U \to \mathbb{R}^n$ be a smooth map. If $p \in U$ is a point such that* $\det DF(p) \neq 0$, *then there is an open set $W \subset U$ containing p such that $F(W)$ is open in $\mathbb{R}^n$ and F is a diffeomorphism from W onto $F(W)$.*

See [SK1, p. 34] and [BO, p. 42] for proofs, as well as other information concerning the Inverse Function Theorem. We will also need the following result, the proof of which is lengthy and might be skipped. This theorem is a special case of a more general result known as the Rank Theorem (see [BO, p. 47]); another special case of the Rank Theorem is given in Exercise 4.2.1.

Theorem 4.2.2. *Let $U \subset \mathbb{R}^2$ be an open set and let $f: U \to \mathbb{R}^3$ be a smooth map. If $p \in U$ is a point such that the matrix $Df(p)$ has rank 2, then there are open subsets $W \subset U$ and $V \subset \mathbb{R}^3$ containing p and $f(p)$ respectively and a smooth map $G: V \to \mathbb{R}^3$ such that $G(V)$ is open in $\mathbb{R}^3$, that G is a diffeomorphism from V onto $G(V)$, that $f(W) \subset V$ and that*

$$G \circ f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

for all $\binom{x}{y} \in W$.

Proof of Theorem 4.2.2. Let $\overline{U} \subset \mathbb{R}^2$ be defined by $\overline{U} = \{x - p \mid x \in U\}$. We define a function $\overline{f}: \overline{U} \to \mathbb{R}^3$ by $\overline{f}(v) = f(v + p) - f(p)$ for all $v \in \overline{U}$. Observe that $\overline{U}$ is open in $\mathbb{R}^2$, that $\overline{f}$ is smooth, that $D\overline{f}(v) = Df(v + p)$, that $O_2 \in \overline{U}$, that $\overline{f}(O_2) = O_3$ and that $D\overline{f}(O_2)$ has rank 2. If the function $\overline{f}$ is given in coordinates by

$$\overline{f}(\bar{u}) = \begin{pmatrix} \overline{f}_1(\bar{u}) \\ \overline{f}_2(\bar{u}) \\ \overline{f}_3(\bar{u}) \end{pmatrix},$$

where $\bar{u} = \binom{u_1}{u_2}$, then the Jacobian matrix of $\overline{f}$ is

$$D\overline{f} = \begin{pmatrix} \frac{\partial \overline{f}_1}{\partial u_1} & \frac{\partial \overline{f}_1}{\partial u_2} \\ \frac{\partial \overline{f}_2}{\partial u_1} & \frac{\partial \overline{f}_2}{\partial u_2} \\ \frac{\partial \overline{f}_3}{\partial u_1} & \frac{\partial \overline{f}_3}{\partial u_2} \end{pmatrix}.$$

Since the rank of the matrix $D\overline{f}(O_2)$ is 2, it follows from standard results in linear algebra that $D\overline{f}(O_2)$ has a 2×2 submatrix with non-zero determinant. By relabeling the coordinates of $\mathbb{R}^3$ if necessary, we may assume without loss of generality that the top two rows of $D\overline{f}(O_2)$ have non-zero determinant, that is

$$\det \begin{pmatrix} \frac{\partial \overline{f}_1}{\partial u_1}|_{\bar{u}=O_2} & \frac{\partial \overline{f}_1}{\partial u_2}|_{\bar{u}=O_2} \\ \frac{\partial \overline{f}_2}{\partial u_1}|_{\bar{u}=O_2} & \frac{\partial \overline{f}_2}{\partial u_2}|_{\bar{u}=O_2} \end{pmatrix} \neq 0. \tag{4.2.1}$$

We now define a function $H: \overline{U} \times \mathbb{R} \to \mathbb{R}^3$ by

$$H(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}) = \overline{f}(\bar{u}) + \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} = \begin{pmatrix} \overline{f}_1(\bar{u}) \\ \overline{f}_2(\bar{u}) \\ \overline{f}_3(\bar{u}) + u_3 \end{pmatrix},$$

where $\bar{u}$ is as above. The domain of H is an open subset of $\mathbb{R}^3$, and since the coordinate functions of $\overline{f}$ are smooth, so is the map H. Further, note that $H(\mathrm{O}_3) = \mathrm{O}_3$. The Jacobian matrix of H is

$$DH = \begin{pmatrix} \frac{\partial \overline{f}_1}{\partial u_1} & \frac{\partial \overline{f}_1}{\partial u_2} & 0 \\ \frac{\partial \overline{f}_2}{\partial u_1} & \frac{\partial \overline{f}_2}{\partial u_2} & 0 \\ \frac{\partial \overline{f}_3}{\partial u_1} & \frac{\partial \overline{f}_3}{\partial u_2} & 1 \end{pmatrix}.$$

It follows from Equation 4.2.1 that $\det DH(\mathrm{O}_3) \neq 0$. Applying the Inverse Function Theorem to H at the point O_3, we conclude that there is an open set $T \subset \overline{U} \times \mathbb{R}$ containing O_3 such that $H(T)$ is open in $\mathbb{R}^3$ and H is a diffeomorphism from T onto $H(T)$. Observe that $\mathrm{O}_3 \in H(T)$.

We now define the sets V and W and the map G as follows. Let $V = \{x + f(p) \mid x \in H(T)\}$. Note that V is open in $\mathbb{R}^3$ and that $f(p) \in V$. Next, define

$$W = f^{-1}(V) \cap \{x + p \mid x \in H(T) \cap \mathbb{R}^2\}.$$

Observe that W is open in $\mathbb{R}^2$, that $p \in W$ and that $f(W) \subset V$. We now define $G: V \to \mathbb{R}^3$ by

$$G(v) = (H|T)^{-1}(v - f(p)) + \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix}$$

for all $v \in V$, where $p = \binom{p_1}{p_2}$. Since $H|T$ is a diffeomorphism so is $(H|T)^{-1}$, and it follows that $G(V)$ is an open subset of $\mathbb{R}^3$ and that G is a diffeomorphism from V onto $G(V)$.

From the definitions of $\overline{f}$ and H it follows that $f(v) = \overline{f}(v-p)+f(p)$ for all $v \in U$, and that $\overline{f}(\binom{u_1}{u_2}) = H(\begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix})$ for all $\binom{u_1}{u_2} \in \overline{U}$. For each $\binom{x}{y} \in W$, we now compute

$$
\begin{aligned}
G \circ f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= H^{-1}\left(f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - f(p)\right) + \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix} \\
&= H^{-1}\left(\overline{f}\left(\begin{pmatrix} x \\ y \end{pmatrix} - p\right) + f(p) - f(p)\right) + \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix} \\
&= H^{-1}\left(\overline{f}\left(\begin{pmatrix} x - p_1 \\ y - p_2 \end{pmatrix}\right)\right) + \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix} \\
&= H^{-1}\left(H\left(\begin{pmatrix} x - p_1 \\ y - p_2 \\ 0 \end{pmatrix}\right)\right) + \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \quad \square
\end{aligned}
$$

The following result can be deduced from the above theorem.

Corollary 4.2.3. *Let $U \subset \mathbb{R}^2$ be an open set and let $f: U \to \mathbb{R}^3$ be a smooth map. If $p \in U$ is a point such that the matrix $Df(p)$ has rank 2, then there is an open set $W \subset U$ containing p such that $f|W$ is injective, and $Df(q)$ has rank 2 for all $q \in W$.*

Proof. Exercise 4.2.5. □

The other foundational material we need is the following three existence and uniqueness theorems for the solutions of ordinary differential equations. The first of these results is the standard such existence and uniqueness theorem; the second is a stronger version, which shows how solutions of ordinary differential equations depend upon the initial conditions; the third is a theorem concerning the special case of linear differential equations, where we have a slightly better result than for arbitrary differential equations. See [LA2, Chapter XVIII] for proofs of all three theorems, or [HZ] for the first two.

Theorem 4.2.4 (Existence and uniqueness of solutions of ordinary differential equations). *Let $U \subset \mathbb{R}^n$ be an open set, let $F: U \to \mathbb{R}^n$ be a smooth map and let $t_0 \in \mathbb{R}$ and $v_0 \in U$ be points. Then there is a number $\epsilon > 0$ and a smooth map $c: (t_0 - \epsilon, t_0 + \epsilon) \to U$ such that*

$$c'(t) = F(c(t)) \quad \textit{and} \quad c(t_0) = v_0 \tag{4.2.2}$$

for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$*; if* $\tilde{c}\colon (t_0 - \delta, t_0 + \delta) \to U$ *is any other map that satisfies Equation 4.2.2 for some* $\delta > 0$*, then* $\tilde{c}(t) = c(t)$ *for all* t *in the intersection of the domains of the two maps.*

Theorem 4.2.5. *Let* $U \subset \mathbb{R}^n$ *be an open subset, let* $F\colon U \to \mathbb{R}^n$ *be a smooth map and let* $t_0 \in \mathbb{R}$ *and* $v_0 \in U$ *be points. Then there is a number* $\epsilon > 0$*, an open subset* $V \subset \mathbb{R}^n$ *containing* v_0 *and a smooth map* $C\colon (t_0 - \epsilon, t_0 + \epsilon) \times V \to U$ *such that*

$$C'(t, v) = F(C(t, v)) \quad \textit{and} \quad C(t_0, v) = v$$

for all $(t, v) \in (t_0 - \epsilon, t_0 + \epsilon) \times V$.

Let $M_{nn}(\mathbb{R})$ denote the set of real $n \times n$ matrices.

Theorem 4.2.6. *Let* (a, b) *be an open interval, let* $A\colon (a, b) \to M_{nn}(\mathbb{R})$ *be a smooth map and let* $t_0 \in (a, b)$ *and* $v_0 \in \mathbb{R}^n$ *be points. Then there is a unique smooth function* $c\colon (a, b) \to \mathbb{R}^n$ *such that*

$$c'(t) = A(t)c(t) \quad \textit{and} \quad c(t_0) = v_0$$

for all $t \in (a, b)$.

Exercises

4.2.1*. Let $c\colon (a, b) \to \mathbb{R}^3$ be a smooth map. Suppose that $p \in (a, b)$ is a point such that the matrix $c'(p) \neq 0$. Show that there is a number $\epsilon > 0$, an open subset $V \subset \mathbb{R}^3$ containing $c(p)$ and a smooth map $G\colon V \to \mathbb{R}^3$ such that $G(V)$ is open in $\mathbb{R}^3$, that G is a diffeomorphism from V onto $G(V)$, that $c((p - \epsilon, p + \epsilon)) \subset V$ and that

$$G \circ c(t) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

for all $t \in (p - \epsilon, p + \epsilon)$.

4.2.2*. Let $U, V \subset \mathbb{R}^n$ be open sets, and suppose $f\colon U \to V$ is a diffeomorphism. Show that $Df(p)$ is a non-singular matrix for all $p \in U$.

4.2.3*. Let $U, V \subset \mathbb{R}^n$ be open sets, and suppose $f\colon U \to V$ is a smooth bijective map. Show that if $Df(p)$ is a non-singular matrix for all $p \in U$, then f is a diffeomorphism.

4.2.4*. Let $c\colon (a, b) \to \mathbb{R}^2$ be a smooth function. Suppose that the tangent vectors to c are never the zero vector and are never parallel to the y-axis. Show that the image of c is the graph of a function of the form $y = f(x)$ for some smooth function $f\colon (p, q) \to \mathbf{R}$.

4.2.5*. Prove Corollary 4.2.3. State and prove the analog of this corollary for smooth functions $c\colon (a, b) \to \mathbb{R}^3$.

4.2.6. Give an example of a function $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has non-zero Jacobian matrix at all points, and yet is not injective in every neighborhood of any point.

4.3 Curves in $\mathbb{R}^3$

The concept of a curve in $\mathbb{R}^3$ is intuitively quite simple; imagine a twisted piece of string, as in Figure 4.3.1. A smooth curve is, pictorially, one that bends nicely and has no kinks or corners. To deal with smooth curves rigorously, however, we need to think of a curve slightly differently; rather than thinking of a curve as an object that simply sits in $\mathbb{R}^3$, we should view it as the path of a moving object. Every point on the curve corresponds to the location of the moving object at a particular time. We could imagine traversing the same path at a variety of different speeds, not to mention changing direction; we will deal with this issue shortly. Finally, rather than thinking about the points on the curve as simply points in $\mathbb{R}^3$, it is technically more useful to think of points on a curve as the endpoints of vectors starting at the origin. Putting these observations together we arrive at the following definition.

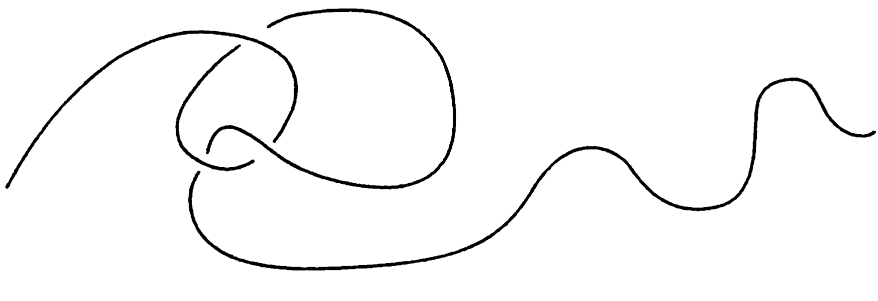


Figure 4.3.1

Definition. A **smooth curve** (or simply **curve**) in $\mathbb{R}^3$ is a smooth function $c\colon (a, b) \to \mathbb{R}^3$, where (a, b) is an interval (possibly infinite) in $\mathbb{R}$. For each $t \in (a, b)$ the **velocity vector** of the curve at t is the vector $c'(t)$, and the **speed** at t is the real number $\|c'(t)\|$. A curve is **unit speed** if $\|c'(t)\| = 1$ for all $t \in (a, b)$.

Example 4.3.1. Consider the curve $c\colon (0, 1) \to \mathbb{R}^3$ given by

$$c(t) = \begin{pmatrix} \cos t \\ \sin t \\ t^2 \end{pmatrix}.$$

Then

$$c'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 2t \end{pmatrix} \quad \text{and} \quad \|c'(t)\| = \sqrt{1 + 4t^2},$$

so that c is not unit speed. $\Diamond$

The above definition is actually not quite enough to insure that the image of the curve will look geometrically "smooth." Imagine a bug flying around in $\mathbb{R}^3$, and assume that the flight is smooth (in the sense of infinite differentiability). While maintaining smooth motion the bug could slow down till it stops altogether, turn 90° in some direction, and then take off again, gradually accelerating from its initial speed of zero. The path taken by the bug after executing this maneuver has a corner in it, even though its flight could be described as a smooth curve as we have defined it. The following definition eliminates the problem, and describes the class of curves with which we will be working.

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a smooth curve. The curve c is **regular** if $c'(t) \neq 0$ for all $t \in (a, b)$, that is, if $\|c'(t)\| \neq 0$ for all $t \in (a, b)$.

Example 4.3.2. The curve in Example 4.3.1 is regular, since $\|c'(t)\|$ is never zero. $\Diamond$

The following definition definition is the formal relation of "different ways of traversing a string."

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ and $\tilde{c}\colon (d, e) \to \mathbb{R}^3$ be smooth curves. We say that $\tilde{c}$ is a **reparametrization** of c if there is a diffeomorphism $h\colon (d, e) \to (a, b)$ such that $\tilde{c} = c \circ h$.

Observe that a curve and any reparametrization of it have the same image set in $\mathbb{R}^3$.

Example 4.3.3. Let $c\colon (1, 5) \to \mathbb{R}^3$ and $\tilde{c}\colon (0, 2) \to \mathbb{R}^3$ be defined by

$$c(t) = \begin{pmatrix} t^2+3 \\ t-7 \\ \sin t \end{pmatrix}, \quad \text{and} \quad \tilde{c}(t) = \begin{pmatrix} 4t^2+4t+4 \\ 2t-6 \\ \sin(2t+1) \end{pmatrix}.$$

Then $\tilde{c} = c \circ h$, where $h\colon (0, 2) \to (1, 5)$ is given by $h(t) = 2t + 1$. It is straightforward to verify that h is smooth, bijective, and has a smooth inverse, so that h is a diffeomorphism. $\lozenge$

The following lemma shows that any regular curve can be reparametrized in a particularly simple way. The proof of the lemma might at first appear to be pulled out of thin air, though there is actually an intuitive idea behind it, namely that a curve will be unit speed if the parameter corresponds to arc-length along the curve.

Proposition 4.3.4. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve.*

(i) *There is a reparametrization of c that is a unit speed curve.*
(ii) *Let $c \circ h_1$ and $c \circ h_2$ be unit speed reparametrizations of c, for appropriate functions $h_1\colon (d_1, e_1) \to (a, b)$ and $h_2\colon (d_2, e_2) \to (a, b)$. Then the function $h_2^{-1} \circ h_1\colon (d_1, e_1) \to (d_2, e_2)$ has the form $h_2^{-1} \circ h_1(s) = \pm s + k$ for some constant k.*

Proof. (i) Pick some point $t_0 \in (a, b)$. Define a function $q\colon (a, b) \to \mathbb{R}$ by

$$q(t) = \int_{t_0}^{t} \|c'(u)\|\, du.$$

By the Fundamental Theorem of Calculus the function q is smooth and $q'(t) = \|c'(t)\| > 0$, the inequality following from the regularity of c. Hence q is a strictly increasing function, and q is therefore a bijective map from (a, b) onto its image. The image of q will be the interval (d, e), where

$$d = \int_{t_0}^{a} \|c'(u)\|\, du\,, \qquad e = \int_{t_0}^{b} \|c'(u)\|\, du.$$

Let $h\colon (d, e) \to (a, b)$ be the inverse function of q. Since the derivative of q is never zero it follows from a standard theorem in Calculus that h is also smooth, and $h'(s) = 1/q'(s)$ for all $s \in (d, e)$. Let $\tilde{c}\colon (d, e) \to \mathbb{R}^3$ be defined by $\tilde{c} = c \circ h$. By definition $\tilde{c}$ is a reparametrization of c. Further, for each

$s \in (d, e)$ we have

$$\tilde{c}'(s) = c'(h(s))\, h'(s) = c'(h(s))\, \frac{1}{q'(h(s))} = c'(h(s))\, \frac{1}{\|c'(h(s))\|}.$$

Hence $\|\tilde{c}'(s)\| = 1$ for all $s \in (d, e)$.

(ii) For each $i = 1, 2$ we have

$$1 = \|(c \circ h_i)'(s)\| = \|c'(h_i(s))\|\, |h_i'(s)|$$

for $s \in (d_i, e_i)$. Hence

$$h_1'(s) = \pm\frac{1}{\|c'(h_1(s))\|} = \pm\frac{1}{\|c'(h_2(h_2^{-1} \circ h_1(s)))\|} = \pm h_2'(h_2^{-1} \circ h_1(s))$$

for each $s \in (d_1, e_2)$, and thus

$$(h_2^{-1} \circ h_1)'(s) = (h_2^{-1})'(h_1(s))\, h_1'(s) = \frac{h_1'(s)}{h_2'(h_2^{-1} \circ h_1(s))} = \pm 1.$$

Since $h_2^{-1} \circ h_1$ is smooth, then it is either constantly 1 or constantly -1. The desired result now follows. □

Though in theory the proof of part (i) of the above lemma gives a procedure for finding unit speed reparametrizations, in practice doing so is not always possible since it involves computing integrals and inverses of functions.

Example 4.3.5. The unit right circular helix is the curve $c\colon (-\infty, \infty) \to \mathbb{R}^3$ given by

$$c(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

See Figure 4.3.2. It is not hard to see that $\|c'(t)\| = \sqrt{2}$ for all t. Choosing $t_0 = 0$, we have

$$q(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}t,$$

and hence

$$h(t) = \frac{t}{\sqrt{2}}.$$

Thus our unit speed reparametrization is

$$\tilde{c}(t) = (c \circ h)(t) = \begin{pmatrix} \cos \frac{t}{\sqrt{2}} \\ \sin \frac{t}{\sqrt{2}} \\ \frac{t}{\sqrt{2}} \end{pmatrix}. \quad \diamond$$

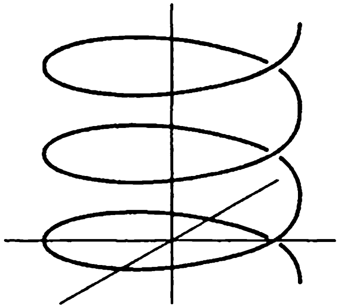


Figure 4.3.2

Suppose two smooth curves $c\colon (a, b) \to \mathbb{R}^3$ and $\tilde{c}\colon (d, e) \to \mathbb{R}^3$ have the same image. Can we realize $\tilde{c}$ as a reparametrization of c? Although the situation could be tricky if the curves were not injective, we do have the following lemma, which will suffice for our purposes.

Lemma 4.3.6. *Let $c\colon (a, b) \to \mathbb{R}^3$ and $\tilde{c}\colon (d, e) \to \mathbb{R}^3$ be injective regular curves with the same image. Then $\tilde{c}$ is a reparametrization of c.*

Proof. Since c is injective it must be a bijection onto its image, and hence there is a function $c^{-1}\colon c((a, b)) \to (a, b)$. Define the function $h\colon (d, e) \to (a, b)$ by letting $h = c^{-1} \circ \tilde{c}$. The function h is a bijection, and by Exercise 4.3.11 it is smooth. Doing this whole procedure in the other direction also shows that h^{-1} is smooth. Evidently $\tilde{c} = c \circ h$, and thus $\tilde{c}$ is a reparametrization of c. □

We now calculate the length of the image of a curve, which for convenience we will refer to as "length of a curve." It ought to be the case that the length depends only upon the image of the curve, and not upon the particular parametrization used. However, it is easier to make use of parametrizations in our definition of the length of a curve, and then to show that the quantity defined in fact does not depend upon the parametrization used. The idea is to approximate the image of the curve with a finite number of small straight line segments, add up the lengths of the segments to get an approximate length of the curve, and take the limit of these sums as smaller and smaller segments are used. In the limit the sum becomes an integral, and the term $\|c'(t)\|\,dt$ in the definition below comes from the lengths of the line segments. Such argumentation does not "prove" that our formula for the length of a curve equals our intuitive notion of what is meant by the length of such curves; it really only pushes back where the leap of faith is made.

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a smooth curve. The **length** of c is defined to be the number Length(c) given by

$$\text{Length}(c) = \int_a^b \|c'(t)\|\, dt, \tag{4.3.1}$$

provided the integral exists.

Example 4.3.7. Let $c\colon (1, 2) \to \mathbb{R}^3$ be given by

$$c(t) = \begin{pmatrix} \frac{t^2}{2} \\ 1 \\ \frac{t^3}{3} \end{pmatrix}.$$

It can be computed that $\|c'(t)\| = t\sqrt{1+t^2}$ (observe that $t > 0$). The length of c is thus

$$\text{Length}(c) = \int_1^2 t\sqrt{1+t^2}\, dt = \frac{5^{3/2} - 2^{3/2}}{3}. \quad \lozenge$$

The following lemma says that our definition of the length of curves behaves as we hoped it would with respect to parametrizations.

Lemma 4.3.8. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a smooth curve. If $\tilde{c}\colon (d, e) \to \mathbb{R}^3$ is a reparametrization of c, then Length$(\tilde{c})$ = Length(c).*

Proof. Exercise 4.3.6. $\square$

Exercises

4.3.1. Which of the following curves are regular?

(i) $c\colon (-\infty, \infty) \to \mathbb{R}^3$ given by $c(t) = \begin{pmatrix} 1 \\ t^3 \\ t^4 \end{pmatrix}$;

(ii) $d\colon (0, \infty) \to \mathbb{R}^3$ given by $d(t) = \begin{pmatrix} t\ln t - t \\ 5 \\ 2t\ln t - 2t \end{pmatrix}$.

4.3.2. The curve $c\colon (-\infty, \infty) \to \mathbb{R}^3$ defined by

$$c(t) = \begin{pmatrix} Be^{kt}\cos t \\ Be^{kt}\sin t \\ 0 \end{pmatrix}$$

is called the **logarithmic spiral**; this curve appears to appear in nature, describing, for example, the shape of a nautilus shell. Show that this curve has the property that the angle between the vector $c(t)$ and the vector $c'(t)$ is a constant. (This property in fact characterizes the logarithmic spiral.)

4.3.3*. Let $c: (a, b) \to \mathbb{R}^3$ be a smooth curve. Show that there is a diffeomorphism $h: (d, e) \to (a, b)$ for some interval (d, e) in $\mathbb{R}$ such that $\tilde{c} = c \circ h$ is unit speed and $h'(t) > 0$ for all $t \in (d, e)$.

4.3.4. Find unit speed reparametrizations of the following curves.

(i) $c: (0, \infty) \to \mathbb{R}^3$ given by $c(t) = \frac{1}{2}\begin{pmatrix} t \\ 1/t \\ \sqrt{2}\ln t \end{pmatrix}$.

(ii) The logarithmic spiral in Exercise 4.3.2.

4.3.5. The logarithmic spiral can be broken into segments from $t = 2n\pi$ to $t = 2(n+1)\pi$ for each $n \in \mathbb{Z}$. Find the length of such a segment. What is the ratio of the length of one such segment to the length of the previous segment? Intuitively, why would a nautilus shell would have this property?

4.3.6*. Prove Lemma 4.3.8.

4.3.7. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be points in $\mathbb{R}^3$. Choose a parametrization of the line segment from x to y and calculate the length of this curve. (There are many such parametrizations, so chose one you think will be most convenient to work with.)

4.3.8. Show that the circumference of a circle of radius r is $2\pi r$.

4.3.9. Let $y = f(x)$ be a function $f: (a, b) \to \mathbb{R}$. The graph of this function can be parametrized by the curve $c: (a, b) \to \mathbb{R}^3$ given by

$$c(t) = \begin{pmatrix} t \\ f(t) \\ 0 \end{pmatrix}.$$

Find a formula for the length of this curve. How does it compare to the standard formula for arc-length found in most Calculus texts?

4.3.10*. Let $c: (a, b) \to \mathbb{R}^3$ be a regular curve. Suppose that $c|[p, q]$ is injective for some closed interval $[p, q] \subset (a, b)$. Show that there exists a number $\epsilon > 0$ such that $c|(p-\epsilon, q+\epsilon)$ is a homeomorphism from $(p-\epsilon, q+\epsilon)$ to $c((p-\epsilon, q+\epsilon))$.

4.3.11*. Prove that $h = c^{-1} \circ \tilde{c}$ in the proof of Lemma 4.3.6 is smooth.

4.4 Tangent, Normal and Binormal Vectors

The tangent vector to a curve is the vector that best approximates the curve at the point of tangency. See Figure 4.4.1. Given a smooth curve $c\colon (a, b) \to \mathbb{R}^3$, the tangent vector at point $t \in (a, b)$ turns out to be nothing other than the velocity vector $c'(t)$ defined previously. However, whereas we would like to think of a tangent vector as "starting" at the point of tangency on the curve, in our present situation the tangent vector is translated so that it starts at the origin. The use of the following definition will become apparent shortly.

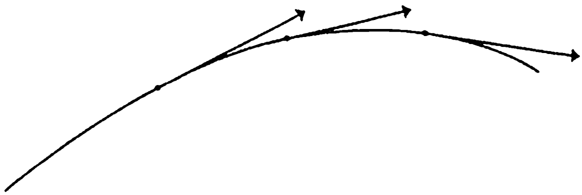


Figure 4.4.1

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a smooth curve. For each $t \in (a, b)$ such that $\|c'(t)\| \neq 0$ the **unit tangent vector** to the curve at t is the vector

$$T(t) = \frac{c'(t)}{\|c'(t)\|}.$$

If a curve is regular then the unit tangent vector is defined at all points. Also, if a curve is unit speed then the unit tangent vector is just the velocity vector.

Example 4.4.1. Let $c\colon (-\infty, \infty) \to \mathbb{R}^3$ be given by

$$c(t) = \begin{pmatrix} 1 \\ t \\ t^2/2 \end{pmatrix}.$$

Then

$$c'(t) = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix}, \quad \|c'(t)\| = \sqrt{1 + t^2} \quad \text{and} \quad T(t) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} \end{pmatrix}. \quad \diamond$$

Consider a regular curve $c\colon (a, b) \to \mathbb{R}^3$. Although the image of the curve need not lie in a single plane, at any point $c(t)$ on the curve there is a plane that is the closest thing to a plane containing the curve. See Figure 4.4.2. The unit tangent vector to the curve will be contained in this plane; we need to find another unit vector contained in the plane and linearly independent from the unit tangent vector. To find this other unit vector, we start by noting that the unit tangent vector function $T\colon (a, b) \to \mathbb{R}^3$ is also smooth. Observing that $\|T(t)\| = 1$ for all t, we have

$$\langle T(t), T(t)\rangle = 1,$$

where $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{R}^3$. Taking the derivative of both sides, and using the standard properties of derivatives and inner products (see Lemma 5.6.1), we deduce that

$$2\langle T'(t), T(t)\rangle = 0.$$

Thus $T'(t)$ is orthogonal to $T(t)$ for all t. If $T'(t) = 0$ then this whole business does not do us much good, so we will generally assume that $T'(t) \neq 0$. (This last assumption rules out the usual parametrization of a straight line, for example.) We can now define a new vector that is always orthogonal to $T(t)$.

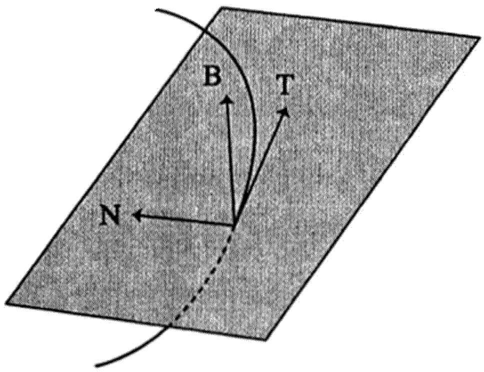


Figure 4.4.2

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve. For each $t \in (a, b)$ such that $\|T'(t)\| \neq 0$, the **unit normal vector** to the curve at t is the vector

$$N(t) = \frac{T'(t)}{\|T'(t)\|}.$$

Whenever the vectors $T(t)$ and $N(t)$ are both defined, we consider the plane that they span to be the plane that best fits the curve (referred to as the "osculating plane"), just as the tangent line is the line that best fits the curve.

Example 4.4.2. We continue Example 4.4.1, computing

$$T'(t) = \begin{pmatrix} 0 \\ \frac{-t}{(1+t^2)^{3/2}} \\ \frac{1}{(1+t^2)^{3/2}} \end{pmatrix}, \quad \|T'(t)\| = \frac{1}{1+t^2} \quad \text{and} \quad N(t) = \begin{pmatrix} 0 \\ \frac{-t}{\sqrt{1+t^2}} \\ \frac{1}{\sqrt{1+t^2}} \end{pmatrix}.$$

◊

It is often inconvenient to verify whether $\|T'(t)\| \neq 0$, since $T(t)$ is often a fraction with a complicated denominator. The following lemma makes life a bit easier.

Lemma 4.4.3. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve. For each $t \in (a, b)$ the following are equivalent:*

(1) $\|T'(t)\| \neq 0$;
(2) *The vectors $c'(t)$ and $c''(t)$ are linearly independent;*
(3) $c'(t) \times c''(t) \neq 0$.

Proof. Exercise 4.4.3. □

For convenience we adopt the following terminology.

Definition. A regular curve $c\colon (a, b) \to \mathbb{R}^3$ is **strongly regular** if any of the three equivalent conditions in Lemma 4.4.3 holds for all $t \in (a, b)$.

Example 4.4.4. For the curve in Example 4.4.1 we compute

$$c''(t) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \text{and} \quad c'(t) \times c''(t) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},$$

and thus the curve is strongly regular. ◊

For every t such that $\|T'(t)\| \neq 0$, we have now defined two orthogonal unit vectors $T(t)$ and $N(t)$. Given that our curve is in $\mathbb{R}^3$, and that three orthonormal vectors in $\mathbb{R}^3$ form a basis, we complete the picture by defining for each appropriate t a third unit vector orthogonal to both $T(t)$ and $N(t)$.

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve. For each $t \in (a, b)$ such that $\|T'(t)\| \neq 0$, the **unit binormal vector** to the curve at t is the vector

$$B(t) = T(t) \times N(t).$$

A few observations about the above definition. First, except for the sign, there is really no choice in the definition of $B(t)$ if we want the set of vectors $\{T(t), N(t), B(t)\}$ to form an orthonormal set. Second, the definition of $B(t)$ makes crucial use of the fact that our curve is in $\mathbb{R}^3$, since the cross product is only defined in three dimensions (in higher dimensions, by contrast, there are many possible choices for a unit vector orthogonal to any two given vectors). The vectors $\{T(t), N(t), B(t)\}$ are defined for all t in the domain of a strongly regular curve. These three vectors are often called the **Frenet frame** of the curve.

Example 4.4.5. Continuing Example 4.4.1, we compute

$$B(t) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{-t}{\sqrt{1+t^2}} \\ \frac{1}{\sqrt{1+t^2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The significance of the fact that $B(t)$ turns out to be a constant in this example will be clarified by Exercise 4.4.4. $\Diamond$

It would be nice to have a simpler way to compute the Frenet frame of a curve, since the often complicated denominator in the expression for $T(t)$ can make finding the necessary derivatives quite messy. An alternate method will be given in Lemma 4.5.7; although the statement of the relevant parts of this lemma could be given now, some additional concepts and results are needed prior to the proof of the lemma.

Exercises

4.4.1. For each of the following curves, determine whether the curve is strongly regular, and, if so, find T, N and B.

(i) The circle in the x-y plane of radius 2 centered at the origin which we parametrize by the curve $g\colon (-\infty, \infty) \to \mathbb{R}^3$ given by $g(t) = \begin{pmatrix} 2\cos(t/2) \\ 2\sin(t/2) \\ 0 \end{pmatrix}$;

(ii) $c\colon (-\infty, \infty) \to \mathbb{R}^3$ given by $c(t) = \begin{pmatrix} 1 \\ t \\ 3t \end{pmatrix}$;

(iii) $d\colon (0, \infty) \to \mathbb{R}^3$ given by $d(t) = \begin{pmatrix} \ln t \\ t \\ 0 \end{pmatrix}$.

4.4.2*. Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve lying entirely in a plane. Show that whenever $T(t)$ and $N(t)$ are both defined they are parallel to the plane containing the curve.

4.4.3*. Prove Lemma 4.4.3.

4.4.4*. Let $c\colon (a, b) \to \mathbb{R}^3$ be a strongly regular curve whose image lies entirely in a plane. Show that $B(t)$ is a constant.

4.5 Curvature and Torsion

If we look at the image of a curve in $\mathbb{R}^3$, as in Figure 4.5.1, we see that there are points on the curve at which the curve is bending more (point A) and others at which the curve is bending less (point B). We wish to quantify this bending. As in Section 3.9, we begin with a discussion of the expected properties of curvature before stating our definition. Curvature ought to be an assignment of a number to each point of the curve to tell us how much the curve is bending at that point. Although curvature should only depend upon the image of the curve, and not upon any particular choice of parametrization, it will be much more convenient to assign the curvature to each value t in the domain of the curve $c\colon (a, b) \to \mathbb{R}^3$. Thus curvature will be a function of the form $\kappa\colon (a, b) \to \mathbb{R}$. The function κ should be smooth, and it should have the property that whenever the image of the curve is a straight line in a neighborhood of a point $c(t)$, then $\kappa(t)$ should be zero.

Consider the velocity vector to a curve $c\colon (a, b) \to \mathbb{R}^3$. The faster the velocity vector changes direction as we move along the curve, the more the curve appears to be bending. Thus the measure of curvature ought to be something like the derivative of the velocity vector, or, better, the length of the derivative of the velocity vector (since curvature ought to be a scalar). The problem with this proposed definition is that it does depend upon the parametrization of the curve, since if we traverse a curve faster the derivative of the velocity vector will be larger. To overcome this problem, we first look at unit speed curves,

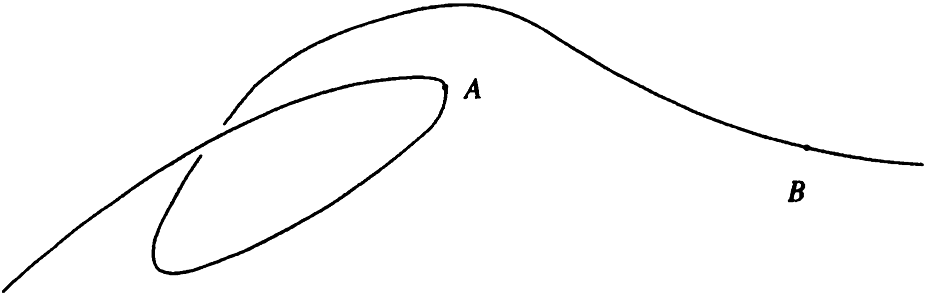


Figure 4.5.1

which gets rid of the problem of traversing a curve at differing speeds. For the following definition recall that for a unit speed curve $c: (a, b) \to \mathbb{R}^3$ the unit tangent vector $T(t)$ equals the velocity vector $c'(t)$.

Definition. Let $c: (a, b) \to \mathbb{R}^3$ be a unit speed curve. For each $t \in (a, b)$ the **curvature** of the curve at t is the number

$$\kappa(t) = \|T'(t)\| = \|c''(t)\|.$$

Observe that curvature is a smooth function $\kappa: (a, b) \to \mathbb{R}$, and that $\kappa(t) \geq 0$.

Example 4.5.1. (1) Any straight line in $\mathbb{R}^3$ can be parametrized by $c: (-\infty, \infty) \to \mathbb{R}^3$ of the form

$$c(t) = \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \\ a_3 t + b_3 \end{pmatrix};$$

the added condition that $a_1^2 + a_2^2 + a_3^2 = 1$ insures that this curve is unit speed. Clearly $c''(t)$ is the zero vector for all t, so $\kappa(t) = 0$ for all t. Hence we see that condition (2) for curvature suggested above is satisfied for this parametrization of a straight line.

(2) The circle of radius 2 in the x-y plane with center at the origin can be parametrized by the curve $d: (-\infty, \infty) \to \mathbb{R}^3$ given by

$$d(t) = \begin{pmatrix} 2\cos\frac{t}{2} \\ 2\sin\frac{t}{2} \\ 0 \end{pmatrix}.$$

It is seen that this curve is unit speed. We then compute

$$d'(t) = \begin{pmatrix} -\sin\frac{t}{2} \\ \cos\frac{t}{2} \\ 0 \end{pmatrix} \quad \text{and} \quad d''(t) = \frac{1}{2}\begin{pmatrix} -\cos\frac{t}{2} \\ -\sin\frac{t}{2} \\ 0 \end{pmatrix}.$$

It follows that $\kappa(t) = \frac{1}{2}$ for all t. The symmetry of the circle makes it reasonable that the curvature to be constant. ◊

The curvature function need not be constant, as seen in Exercise 4.5.1 (iii).

For a non-unit speed curve, we use reparametrization to reduce the problem to the previous definition.

Definition. Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve. Let $\tilde{c} = c \circ h$ be a unit speed reparametrization of c for some diffeomorphism $h\colon (d, e) \to (a, b)$, and let $\tilde{\kappa}$ be the curvature function for $\tilde{c}$. For each $t \in (a, b)$ the **curvature** of the curve c at t is the number $\kappa(t) = \tilde{\kappa}(h^{-1}(t))$.

The following lemma shows that the choice of unit speed reparametrization in the above definition does not affect the computation of curvature.

Lemma 4.5.2. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a regular curve, and let $c \circ h_1$ and $c \circ h_2$ be unit speed reparametrizations of c, where $h_1\colon (d_1, e_1) \to (a, b)$ and $h_2\colon (d_2, e_2) \to (a, b)$ are diffeomorphisms. If $\kappa_1(t)$ and $\kappa_2(t)$ are the curvature functions for $c \circ h_1$ and $c \circ h_2$ respectively, then*

$$\kappa_1(h_1^{-1}(t)) = \kappa_2(h_2^{-1}(t))$$

for all $t \in (a, b)$.

Proof. It follows from Proposition 4.3.4 (ii) that $h_2^{-1} \circ h_1(s) = \pm s + k$ for some constant k. Thus $h_1(s) = h_2(\pm s + k)$, so $c \circ h_1(s) = c \circ h_2(\pm s + k)$. Differentiating twice yields $(c \circ h_1)''(s) = (c \circ h_2)''(\pm s + k)$, so $\kappa_1(s) = \kappa_2(\pm s + k)$. If we let $s = h_1^{-1}(t)$ then $\kappa_1(h_1^{-1}(t)) = \kappa_2(\pm h_1^{-1}(t) + k)$. It is straightforward to verify that $h_2^{-1}(t) = \pm h_1^{-1}(t) + k$, and the result follows. □

Example 4.5.3. We compute the curvature for the curve in Example 4.3.5. Using the formula obtained for $\tilde{c}(t)$, we see

$$\tilde{c}'(t) = \begin{pmatrix} \frac{-\sin(t/\sqrt{2})}{\sqrt{2}} \\ \frac{\cos(t/\sqrt{2})}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \tilde{c}''(t) = \begin{pmatrix} \frac{-\cos(t/\sqrt{2})}{2} \\ \frac{-\sin(t/\sqrt{2})}{2} \\ 0 \end{pmatrix} \quad \text{and } \kappa(t) = \|\tilde{c}''(t)\| = \frac{1}{2}. \ ◊$$

Finally, we need to verify that curvature is a function of the points in the image of the curve in $\mathbb{R}^3$, rather than a function of the particular choice of parametrization. We will need to assume, however, that our parametrization is injective, since at a point where the curve intersects itself there is not necessarily a single value for curvature (this problem does not arise when curvature is a function of the parametrization). Thus, we need to show that if we have two injective parametrized curves with the same images, then they yield the same curvature at each point in the image. This fact can be seen to follow from Lemmas 4.3.6 and 4.5.2; details are left to the reader.

It would be nice to have a formula for curvature for arbitrary regular curves that does not involve reparametrization (which can be difficult to carry out in practice). Such a formula will be given in Lemma 4.5.7. For later use we note that, combining the definitions of $N(t)$ and $\kappa(t)$, we obtain

$$T'(t) = \kappa(t)N(t). \tag{4.5.1}$$

Though curvature tells us a great deal about curves, it does not tell us all we need to know. There are different curves with the same curvature functions, for example the curves in part (2) of Example 4.5.1 and Example 4.5.3. Observe that one of these curves is contained in a plane whereas the other is not. What we wish to measure is the extent to which a curve is twisting out of the plane spanned by $T(t)$ and $N(t)$ for each t in the domain of the curve. Just as the bending of the curve is measured by the change in $T(t)$, using the length of $T'(t)$, it seems plausible that the change in the length of $B'(t)$ will tell us something about how the curve is twisting out of the plane spanned by $T(t)$ and $N(t)$. The quantity $\|B'(t)\|$ almost works, but like curvature it would always be non-negative, and it turns out that in the present case we can do a bit better and get a signed quantity. What we need is the analog for $B'(t)$ of Equation 4.5.1.

Recall the proof of the fact that $T'(t)$ is perpendicular to $T(t)$. Since $B(t)$ is also a unit vector, we can similarly deduce that $B'(t)$ is perpendicular to $B(t)$. Since $\{T(t), N(t), B(t)\}$ form an orthonormal basis for $\mathbb{R}^3$ for all t at which all three vectors are defined, it follows that $B'(t)$ is a linear combination of $T(t)$ and $N(t)$. Next, taking the derivative of both sides of the equation $\langle B(t), T(t)\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle B'(t), T(t)\rangle + \langle B(t), T'(t)\rangle \\ &= \langle B'(t), T(t)\rangle + \langle B(t), \kappa(t)N(t)\rangle = \langle B'(t), T(t)\rangle, \end{aligned}$$

making use of Equation 4.5.1 and the fact that $\langle B(t), N(t)\rangle = 0$. It follows that

$B'(t)$ is a multiple of $N(t)$, which leads us to the following definition.

Definition. Let $c\colon (a,b) \to \mathbb{R}^3$ be a strongly regular unit speed curve. For each $t \in (a,b)$ the **torsion** of the curve at t is the unique real number $\tau(t)$ such that

$$B'(t) = -\tau(t)N(t). \tag{4.5.2}$$

The minus sign in the above equation is chosen for later convenience. Observe that $\tau(t) = -\langle B'(t), N(t)\rangle$, and thus torsion is a smooth function $\tau\colon (a,b) \to \mathbb{R}$. Finally, note that $|\tau(t)| = \|B'(t)\|$, which is analogous to the definition of $\kappa(t)$, though torsion can be negative.

Example 4.5.4. (1) We continue Example 4.5.1 part (2). It is not hard to see that

$$T(t) = \begin{pmatrix} -\sin\frac{t}{2} \\ \cos\frac{t}{2} \\ 0 \end{pmatrix}, \quad N(t) = \begin{pmatrix} -\cos\frac{t}{2} \\ -\sin\frac{t}{2} \\ 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence $B'(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for all t, and therefore $\tau(t) = 0$ for all t.

(2) We continue Example 4.5.3, where our calculations refer to the unit speed reparametrization $\tilde{c}$. It can be computed that

$$T(t) = \begin{pmatrix} \frac{-\sin(t/\sqrt{2})}{\sqrt{2}} \\ \frac{\cos(t/\sqrt{2})}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad N(t) = \begin{pmatrix} -\cos\frac{t}{\sqrt{2}} \\ -\sin\frac{t}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \sin\frac{t}{\sqrt{2}} \\ -\cos\frac{t}{\sqrt{2}} \\ 1 \end{pmatrix}.$$

Hence $B'(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(t/\sqrt{2}) \\ -\sin(t/\sqrt{2}) \\ 0 \end{pmatrix}$ for all t, and therefore $\tau(t) = \frac{1}{\sqrt{2}}$ for all t.

Now suppose we start with the mirror image of the unit right circular helix, obtained by reflecting the unit right circular helix in the y-z plane, resulting in the curve $f\colon (-\infty,\infty) \to \mathbb{R}^3$ given by

$$f(t) = \begin{pmatrix} -\cos t \\ \sin t \\ t \end{pmatrix}.$$

It can be found by a similar computation that the torsion for this curve is constantly $-\frac{1}{\sqrt{2}}$, which is the negative of the torsion for the original helix. It is to detect such differences of handedness that we made sure to allow torsion to be positive or negative. ◊

Although the torsion functions in the above example are constant, because we chose simple curves, torsion is not constant in general, as will be seen in some of the exercises. Just as for curvature, torsion is independent of the choice of parametrizations, and it can be computed for non-unit speed curves either by reparametrization or by the formula that will be given in Lemma 4.5.7.

Consider Equations 4.5.1 and 4.5.2. You will notice that we are missing a third equation, namely one giving the derivative of $N(t)$. The following theorem, which for completeness includes the two equations just mentioned, completes the picture, and really sums up much of what there is to say about curves in $\mathbb{R}^3$. For convenience we drop the argument t in the statement and proof of the following theorem.

Theorem 4.5.5 (Frenet–Serret Theorem). *Let $c\colon (a, b) \to \mathbb{R}^3$ be a strongly regular unit speed curve. Then*

$$\begin{aligned} T' &= && \kappa N && \\ N' &= -\kappa T && + && \tau B \\ B' &= && -\tau N. && \end{aligned}$$

Proof. Only the second equation remains to be proved. Just as we saw in Section 4.4 that $\langle T', T\rangle = 0$, the same argument shows that $\langle N', N\rangle = 0$, since N is a unit vector. Hence N' is a linear combination of T and B. If we write $N' = aT + bB$, take the inner product of this equation with each of T and B, and solve for a and b, we deduce that

$$N' = \langle N', T\rangle\, T + \langle N', B\rangle\, B.$$

Since $\langle N, T\rangle = 0$, we compute

$$0 = \langle N, T\rangle' = \langle N', T\rangle + \langle N, T'\rangle = \langle N', T\rangle + \langle N, \kappa N\rangle = \langle N', T\rangle + \kappa,$$

using Equation 4.5.1. Hence $\langle N', T\rangle = -\kappa$. Since $\langle N, B\rangle = 0$, we can similarly deduce that $\langle N', B\rangle = \tau$. □

The formulas in the above theorem are called the Frenet–Serret formulas. An easy way of remembering these formulas is to write them

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

though this expression is not strictly meaningful because the terms T, N and B are not numbers but column vectors.

A typical application of the Frenet–Serret Theorem is the following result.

Proposition 4.5.6. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a strongly regular unit speed curve. The following are equivalent:*

(1) *The image of c lies in a plane;*
(2) *$B(t)$ is a constant vector;*
(3) *$\tau(t) = 0$ for all $t \in (a, b)$.*

Proof. (1) $\Rightarrow$ (2). This follows from Exercise 4.4.4.
(2) $\Rightarrow$ (1). Let $p \in (a, b)$ be a point. We compute

$$\begin{aligned}\frac{d}{dt}\langle c(t) - c(p), B(t)\rangle &= \langle c'(t), B(t)\rangle + \langle c(t) - c(p), B'(t)\rangle \\ &= \langle T(t), B(t)\rangle = 0,\end{aligned}$$

using the fact that $B(t)$ is constant. It follows that $\langle c(t) - c(p), B(t)\rangle$ is constant for all $t \in (a, b)$. If we plug in $t = p$ we deduce that this constant must be zero. Hence $c(t) - c(p)$ is perpendicular to the constant vector $B(t)$, and therefore the image of c lies entirely in the plane containing the point $c(p)$ and perpendicular to the constant vector $B(t)$.
(2) $\Leftrightarrow$ (3). This follows immediately from the third of the Frenet–Serret equations. □

Finally, we give the promised formulas for computing the Frenet frame, curvature and torsion of a non-unit speed curve that avoids reparametrization. Once again we drop the argument t in the following lemma.

Lemma 4.5.7. *Let $c\colon (a, b) \to \mathbb{R}^3$ be a strongly regular curve. Then*

(i) $T = \dfrac{c'}{\|c'\|}$
(ii) $B = \dfrac{c' \times c''}{\|c' \times c''\|}$
(iii) $N = B \times T$
(iv) $\kappa = \dfrac{\|c' \times c''\|}{\|c'\|^3}$
(v) $\tau = \dfrac{\langle c' \times c'', c'''\rangle}{\|c' \times c''\|^2}$.

Proof. Part (i) is true by definition. We prove part (iv), leaving the other parts to the reader as Exercise 4.5.6. Let $\tilde{c} = c \circ h$ be a unit speed reparametrization of c, where h is an appropriate diffeomorphism. Let $g = h^{-1}$, so that $c = \tilde{c} \circ g$; let $\tilde{T}$, $\tilde{N}$ and $\tilde{B}$ denote the Frenet frame for $\tilde{c}$. Then

$$c'(t) = \tilde{c}'(g(t))\, g'(t) = \tilde{T}(g(t))\, g'(t),$$

and hence

$$\|c'(t)\| = \|\tilde{T}(g(t))\|\, |g'(t)| = |g'(t)|.$$

We now have

$$\begin{aligned} c''(t) &= \tilde{T}'(g(t))\,(g'(t))^2 + \tilde{T}(g(t))\, g''(t) \\ &= \tilde{\kappa}(g(t))\tilde{N}(g(t))\,(g'(t))^2 + \tilde{T}(g(t))\, g''(t)' \end{aligned}$$

and thus

$$\begin{aligned} c'(t) \times c''(t) &= g'(t)\, \tilde{T}(g(t)) \times \left\{\tilde{\kappa}(g(t))\tilde{N}(g(t))\,(g'(t))^2 + \tilde{T}(g(t))\, g''(t)\right\} \\ &= \tilde{\kappa}(g(t))\tilde{B}(g(t))\,(g'(t))^3. \end{aligned}$$

Therefore

$$\|c'(t) \times c''(t)\| = \tilde{\kappa}(g(t))|g'(t)|^3 = \tilde{\kappa}(g(t))\|c'(t)\|^3,$$

and hence

$$\tilde{\kappa}(g(t)) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}.$$

The desired result now follows, since by definition $\kappa(t) = \tilde{\kappa}(h^{-1}(t)) = \tilde{\kappa}(g(t))$.

□

Exercises

4.5.1. Compute the curvature and torsion for the following curves.

(i) A circle of radius R (without loss of generality in the x-y plane, centered at the origin).

(ii) $c\colon (0, \infty) \to \mathbb{R}^3$ given by $c(t) = \frac{1}{2}\begin{pmatrix} t \\ 1/t \\ \sqrt{2}\ln t \end{pmatrix}$.

(iii*) The logarithmic spiral in Exercise 4.3.2.

(iv) $d\colon (0, \infty) \to \mathbb{R}^3$ given by $d(t) = \frac{1}{2}\begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$.

4.5.2*. Let $c: (a, b) \to \mathbb{R}^3$ be a strongly regular curve. If A is a rotation matrix for $\mathbb{R}^3$ (that is, A is an orthogonal matrix with positive determinant), and if q is a vector in $\mathbb{R}^3$, then the curve $\widehat{c}: (a, b) \to \mathbb{R}^3$ given by

$$\widehat{c}(t) = Ac(t) + q$$

is the result of rotating and translating the image of c by A and q respectively. Show that the curvature and torsion of functions of $\widehat{c}$ are the same as for c.

4.5.3*. Let K and T be any real numbers such that $K > 0$. Show that there are numbers $a > 0$ and b such that the right circular helix

$$c(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}$$

has constant curvature K and constant torsion T.

4.5.4. Let $c: (a, b) \to \mathbb{R}^3$ be a unit speed curve such that $\kappa(t) = 0$ for all $t \in (a, b)$. Show that the image of c lies in a straight line.

4.5.5. Let $c: (a, b) \to \mathbb{R}^3$ be a strongly regular unit speed curve. Show that the image of c lies in a straight line iff there is a point $x_0 \in \mathbb{R}^3$ such that every tangent line to c goes through x_0.

4.5.6*. Prove Lemma 4.5.7 parts (ii), (iii) and (v).

4.5.7. Let $c: (a, b) \to \mathbb{R}^3$ be a strongly regular curve. Show that

$$\begin{array}{rclcccc} T' & = & & & \|c'\|\,\kappa\,N & & \\ N' & = & -\|c'\|\,\kappa\,T & & + & & \|c'\|\,\tau\,B \\ B' & = & & & -\|c'\|\,\tau\,N, & & \end{array}$$

where for convenience we drop the argument t.

4.6 Fundamental Theorem of Curves

Given a curve, we can clearly compute its curvature and torsion; can we go the other way? That is, if we are given curvature and torsion functions, is there a curve which has these values of curvature and torsion? We see from Exercise 4.5.5 that for any constant curvature and torsion functions there is at least one curve with the given curvature and torsion. The following theorem shows that in fact arbitrary curvature and torsion functions completely determine the curve up to translation and rotation of the image of the curve. Note the restriction in the theorem to positive curvature, to avoid things like straight lines, for which torsion is not defined.

Theorem 4.6.1 (Fundamental Theorem of Curves). *Let $\bar{\kappa}, \bar{\tau}: (a, b) \to \mathbb{R}$ be smooth functions with $\bar{\kappa}(t) > 0$ for all $t \in (a, b)$. Then there is a strongly regular unit speed curve $c: (a, b) \to \mathbb{R}^3$ whose curvature and torsion functions are $\bar{\kappa}$ and $\bar{\tau}$ respectively. If $c_1, c_2: (a, b) \to \mathbb{R}^3$ are two such curves, then c_2 can be obtained from c_1 by a rotation and translation of $\mathbb{R}^3$.*

Proof. We essentially follow [M-P] (though the idea of the proof is standard). Let $p \in (a, b)$ be a point. We will show that there exists a unique strongly regular unit speed curve $c: (a, b) \to \mathbb{R}^3$ whose curvature and torsion functions are $\bar{\kappa}$ and $\bar{\tau}$ respectively, and such that

$$c(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad T(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad N(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.6.1}$$

The precise statement of the theorem then follows straightforwardly using Exercise 4.5.2.

The idea of the proof is to solve the Frenet–Serret equations, which are differential equations, in order to find the purported tangent, normal and binormal vectors of the desired curve. Integrating the tangent vector will then give us the curve we are looking for. More precisely, consider the following system of linear differential equations with initial conditions, which are just the Frenet–Serret equations written out in coordinates:

$$\begin{aligned}
u_1'(t) &= \bar{\kappa}(t)\, u_4(t) \\
u_2'(t) &= \bar{\kappa}(t)\, u_5(t) \\
u_3'(t) &= \bar{\kappa}(t)\, u_6(t) \\
u_4'(t) &= -\bar{\kappa}(t)\, u_1(t) + \bar{\tau}(t)\, u_7(t) \\
u_5'(t) &= -\bar{\kappa}(t)\, u_2(t) + \bar{\tau}(t)\, u_8(t) \\
u_6'(t) &= -\bar{\kappa}(t)\, u_3(t) + \bar{\tau}(t)\, u_9(t) \\
u_7'(t) &= -\bar{\tau}(t)\, u_4(t) \\
u_8'(t) &= -\bar{\tau}(t)\, u_5(t) \\
u_9'(t) &= -\bar{\tau}(t)\, u_6(t),
\end{aligned} \tag{4.6.2}$$

$$\begin{aligned} u_1(p) &= 1, u_2(p) = 0, u_3(p) = 0, \\ u_4(p) &= 0, u_5(p) = 1, u_6(p) = 0, \\ u_7(p) &= 0, u_8(p) = 0, u_9(p) = 1. \end{aligned} \tag{4.6.3}$$

By Theorem 4.2.6 there are smooth functions $u_1, \ldots, u_9 \colon (a, b) \to \mathbb{R}$ satisfying Equations 4.6.2 and 4.6.3, and these functions are unique. For convenience, we define smooth vector-valued functions $X_1, X_2, X_3 \colon (a, b) \to \mathbb{R}^3$ by

$$X_1(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} u_4(t) \\ u_5(t) \\ u_6(t) \end{pmatrix}, \quad X_3(t) = \begin{pmatrix} u_7(t) \\ u_8(t) \\ u_9(t) \end{pmatrix},$$

where we think of X_1, X_2 and X_3 as the tangent, normal and binormal vectors respectively. Since the u_i satisfy Equations 4.6.2 and 4.6.3, we have

$$\begin{aligned} X_1'(t) &= \bar{\kappa}(t)\, X_2(t) \\ X_2'(t) &= -\bar{\kappa}(t)\, X_1(t) + \bar{\tau}(t)\, X_3(t) \\ X_3'(t) &= -\bar{\tau}(t)\, X_2(t), \end{aligned} \tag{4.6.4}$$

$$X_1(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.6.5}$$

We see in Exercise 4.6.2 that for all $t \in (a, b)$ the three vectors $\{X_1(t), X_2(t), X_3(t)\}$ form an orthonormal basis for $\mathbb{R}^3$.

We now define a curve $c \colon (a, b) \to \mathbb{R}^3$ by

$$c(t) = \int_p^t X_1(s)\, ds.$$

From the Fundamental Theorem of Calculus it follows that $c'(t) = X_1(t)$. Thus c is smooth (since X_1 is smooth) and is unit speed (since $X_1(t)$ is a unit vector for all t by the claim). Let T and κ denote the unit tangent vector and curvature of c respectively (for convenience we will drop the argument t throughout most of this proof). Evidently $T = X_1$. Using Equation 4.6.4 we further compute that

$$T' = X_1' = \bar{\kappa}\, X_2. \tag{4.6.6}$$

Since $\bar{\kappa} > 0$ and X_2 is never the zero vector (by the claim) we deduce that T' is never the zero vector. Hence c is a strongly regular curve, so the unit normal, unit binormal and torsion of c are all defined; we denote these quantities N, B and τ respectively. Using the Frenet–Serret Theorem (Theorem 4.5.5) and Equation 4.6.4, once again, we have

$$\kappa N = T' = X_1' = \bar{\kappa} X_2. \tag{4.6.7}$$

Taking the norm of both sides, and using the facts that $\|N\| = \|X_2\| = 1$, $\kappa \geq 0$ and $\bar{\kappa} > 0$, we deduce that $\kappa = \bar{\kappa}$. Cancelling by κ on both sides of Equation 4.6.7 yields $N = X_2$.

Since $\{T, N, B\}$ and $\{X_1, X_2, X_3\}$ are both orthonormal bases for $\mathbb{R}^3$, and since $T = X_1$ and $N = X_2$, it follows that $B(t) = \pm X_3(t)$; the continuity of $B(t)$ and $X_3(t)$ imply that the $\pm$ sign is independent of t. However, we observe that $B(p) = T(p) \times N(p)$ by definition and $X_3(p) = X_1(p) \times X_2(p)$ by Equation 4.6.5; hence $B(p) = X_3(p)$, and it follows that $B = X_3$ for all $t \in (a, b)$. Finally, using the Frenet-Serret Theorem and Equation 4.6.4 yet again, we have

$$-\tau N = B' = X_3' = -\bar{\tau} X_2. \tag{4.6.8}$$

Since $N = X_2$, and this vector is never the zero vector, we deduce that $\tau = \bar{\tau}$. We thus see that the curvature and torsion of the curve c are as desired. That c satisfies Equation 4.6.1 follows from the definition of c and Equation 4.6.5. Thus c has all the properties it is supposed to have. As for the uniqueness of c, we note that the functions $\{T, N, B\}$ are uniquely determined by the differential equation and initial conditions given in Equations 4.6.2 and 4.6.3. Thus c is uniquely determined since it is the unique solution to the differential equation and initial condition

$$c' = T, \quad c(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \square$$

Although in theory the above proof actually gives a procedure for finding the curve if the curvature and torsion are known, the bulk of the procedure involves solving some differential equations, which in practice can be quite difficult. A worked out example may be found in [M-P, §2-5].

Exercises

4.6.1. Show that curvature alone does not determine a curve up to rotation and translation.

4.6.2*. In this exercise we complete the missing piece of the proof of Theorem 4.6.1, namely to show that for all $t \in (a, b)$ the three vectors $\{X_1(t), X_2(t), X_3(t)\}$ form an orthonormal basis for $\mathbb{R}^3$. This proof has a few steps.

Step 1: For each pair of numbers $i, j \in \{1, 2, 3\}$, define a function $p_{ij}\colon (a, b) \to \mathbb{R}$ by

$$p_{ij}(t) = \langle X_i(t), X_j(t) \rangle.$$

Show that

$$\begin{aligned}
p'_{11} &= \bar{\kappa}\, p_{21} + \bar{\kappa}\, p_{12} \\
p'_{12} &= \bar{\kappa}\, p_{22} - \bar{\kappa}\, p_{11} + \bar{\tau}\, p_{13} \\
p'_{13} &= \bar{\kappa}\, p_{23} - \bar{\tau}\, p_{12} \\
p'_{21} &= -\bar{\kappa}\, p_{11} + \bar{\tau}\, p_{31} + \bar{\kappa}\, p_{22} \\
p'_{22} &= -\bar{\kappa}\, p_{12} + \bar{\tau}\, p_{32} - \bar{\kappa}\, p_{21} + \bar{\tau}\, p_{23} \\
p'_{23} &= -\bar{\kappa}\, p_{13} + \bar{\tau}\, p_{33} - \bar{\tau}\, p_{22} \\
p'_{31} &= -\bar{\tau}\, p_{21} + \bar{\kappa}\, p_{32} \\
p'_{32} &= -\bar{\tau}\, p_{22} - \bar{\kappa}\, p_{31} + \bar{\tau}\, p_{33} \\
p'_{33} &= -\bar{\tau}\, p_{23} - \bar{\tau}\, p_{32},
\end{aligned} \tag{4.6.9}$$

and

$$p_{ij}(p) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.6.10}$$

Step 2: For each pair of numbers $i, j \in \{1, 2, 3\}$ define a function $\delta_{ij}\colon (a, b) \to \mathbb{R}$ by

$$\delta_{ij}(t) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Show that these functions satisfy Equations 4.6.9 and 4.6.10.

Step 3: Deduce the desired result.

4.7 Planar Curves

In our discussion of surfaces we will encounter certain curves with images in planes in $\mathbb{R}^3$. Without loss of generality we will assume throughout this section that the curves under consideration have their images in $\mathbb{R}^2$, except when stated otherwise. Everything that we have said about curves in $\mathbb{R}^3$ certainly applies to curves whose images lie in planes. Planar curves all have zero torsion by Proposition 4.5.6, so we loose torsion as a useful concept. On the other hand, we can take advantage of planarity to strengthen the concept of curvature. By definition the curvature function of curves in $\mathbb{R}^3$ is always non-negative; there is no meaningful geometric way to define positive versus negative curvature for a curve in $\mathbb{R}^3$, since there is no way to say that it is bending in a particular direction. We cannot say that the curve is bending "away" from itself as opposed to bending "toward" itself, since such a description depends entirely upon how we look at the curve. For curves in $\mathbb{R}^2$, however, there is an inherent way to describe bending, namely as either clockwise or counterclockwise.

A curve, being parametrized, comes with a direction in which it is traversed; in Figure 4.7.1 (i) we see a curve with a given direction, and in Figure 4.7.1 (ii) is a curve with the same image, but parametrized in the other direction. The curve in Figure 4.7.1 (i) is bending in a counterclockwise direction from the point of view of a bug walking along the curve in the given direction; from the point of view of a bug walking along the curve in Figure 4.7.1 (ii), the bending is clockwise. The notion of clockwise vs. counterclockwise bending, which will give us positive or negative planar curvature, is thus seen to depend upon the given parametrization of the curve.

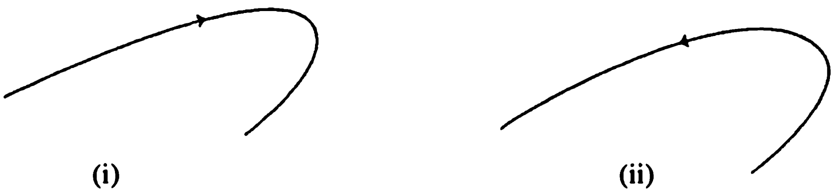


Figure 4.7.1

Technically, we proceed by defining variants of $T(t)$ and $N(t)$ for planar curves. For a planar curve $B(t)$ is constant, so we will not make use of it.

Definition. Let $c\colon (a, b) \to \mathbb{R}^2$ be a smooth curve. For each $t \in (a, b)$ such that $\|c'(t)\| \neq 0$, the **planar unit tangent vector** and **planar unit normal vector** to the curve at t, denoted $\tilde{T}(t)$ and $\tilde{N}(t)$ respectively, are defined by letting $\bar{T}(t) = T(t)$ and letting $\bar{N}(t)$ be the unit vector obtained by rotating $\bar{T}(t)$ counterclockwise by 90°. (See Figure 4.7.2.)

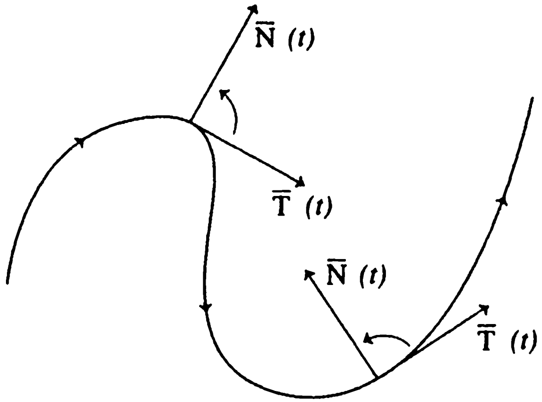


Figure 4.7.2

Note that $\bar{N}(t) = \pm N(t)$. Recall that rotating a vector in $\mathbb{R}^2$ counterclockwise by 90° is obtained by multiplying the vector (when written as a column vector with respect to the standard basis) by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and is thus a smooth operation.

Example 4.7.1. The unit circle in $\mathbb{R}^2$ centered at the origin can be parametrized in various ways; consider two such parametrizations, namely $c_a, c_b\colon (-\infty, \infty) \to \mathbb{R}^2$ given by

$$c_a(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \text{and} \quad c_b(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Both these parametrizations are unit speed, but c_a traverses the unit circle in the counterclockwise direction, whereas c_b traverses the unit circle in the clockwise direction. For c_a we compute

$$\bar{T}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \bar{N}(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix},$$

and for c_b we compute

$$\bar{T}(t) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} \quad \text{and} \quad \bar{N}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}. \quad \Diamond$$

It can be seen that $\bar{T}'(t)$ is perpendicular to $\bar{T}(t)$, using the same proof as for $T(t)$. Since the vectors $\{\bar{T}(t), \bar{N}(t)\}$ form an orthonormal basis for $\mathbb{R}^2$, it follows that $\bar{T}'(t)$ is a multiple of $\bar{N}(t)$. We are thus led to the following definition, which is analogous to the definition of torsion for curves in $\mathbb{R}^3$.

Definition. Let $c\colon (a, b) \to \mathbb{R}^2$ be a unit speed curve. The **planar curvature** of c at $t \in (a, b)$ is defined to be the unique real number $\bar{\kappa}(t)$ such that

$$\bar{T}'(t) = \bar{\kappa}(t)\bar{N}(t). \tag{4.7.1}$$

The above equation is entirely analogous to Equation 4.5.1, although in this case the equation is taken as the definition of planar curvature $\bar{\kappa}(t)$. Observe that $\bar{\kappa}(t)$ can be negative. However, since $\|\bar{N}(t)\| = 1$ for all t it follows that

$$|\bar{\kappa}(t)| = \|\bar{T}'(t)\| = \|T'(t)\| = \kappa(t).$$

Hence the only new information $\bar{\kappa}(t)$ brings is that it takes into account the direction of bending by being positive or negative.

Example 4.7.2. We continue Example 4.7.1. For c_a we have

$$\bar{T}'(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix},$$

and hence $\bar{\kappa}(t) = 1$ for all t. For c_b we have

$$\bar{T}'(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix},$$

and hence $\bar{\kappa}(t) = -1$ for all t. Observe that the difference in sign of the planar curvature of these two parametrizations of the unit circle corresponds to the difference in the orientation of the two parametrizations. $\Diamond$

As before, if we start with a non-unit speed curve we can calculate planar curvature by reparametrizing the curve so that it is unit speed and then calculating the planar curvature. See Exercises 4.7.3 and 4.7.4 for formulas for the planar curvature of a non-unit speed plane curve.

We will need to be able to measure the planar curvature of curves in arbitrary planes in $\mathbb{R}^n$, not just in $\mathbb{R}^2$. In an arbitrary plane there is no inherent notion of which direction of rotation is "clockwise" and which is "counterclockwise," since it depends upon how we look at the plane. So, for each plane we can arbitrarily choose a direction of rotation and call it clockwise. Such a choice is equivalent to choosing an ordered basis for the plane; if the plane does not contain the origin, consider a plane parallel to it that does contain the origin, and choose an ordered basis for this parallel plane. If the ordered basis is $\{x_1, x_2\}$, then we consider that counterclockwise rotation is given by a rotation of the plane taking x_1 to x_2 via the angle between the vectors that is less than π. If the plane is sitting in $\mathbb{R}^3$ (as will be the case later on) this choice is also equivalent to choosing a perpendicular direction to the plane, and using the right hand rule. No matter which approach we take, there are always two possible ways of making the choice. Such a choice is called an **orientation** of the plane.

Once we have made a choice of orientation for a given plane in $\mathbb{R}^n$, we can then compute planar curvature of the curve in the plane just as for curves in $\mathbb{R}^2$. If we were to choose the opposite orientation it is not hard to see that the planar curvature of the curve would change its sign. Thus planar curvature in arbitrary planes is well-defined only with respect to a chosen orientation of the plane.

Exercises

4.7.1. Find $\bar{T}(t)$ and $\bar{N}(t)$ and $\bar{\kappa}(t)$ for the following curves.

(i) $c: (-\infty, \infty) \to \mathbb{R}^2$ given by $c(t) = \binom{t}{t^2}$;

(ii) $d: (0, \infty) \to \mathbb{R}^3$ given by $d(t) = \binom{\ln t}{e^t}$.

4.7.2. Find the planar curvature of the logarithmic spiral in Exercise 4.3.2.

4.7.3*. Find a formula for planar curvature analogous to the formula for curvature given in Lemma 4.5.7 (iv). In particular, if a curve $c: (a, b) \to \mathbb{R}^2$ is given by $c(t) = \binom{c_1(t)}{c_2(t)}$, where $c_1, c_2: (a, b) \to \mathbb{R}$ are smooth functions, express the planar curvature in terms of c_1 and c_2.

4.7.4*. Find a formula for the planar curvature of the graph of a function of the form $y = f(x)$.

4.7.5. Let $c: (a, b) \to \mathbb{R}^2$ be a smooth curve such that the image of c is entirely contained in the closed ball in $\mathbb{R}^2$ of radius R centered at the origin. If $\|c(q)\| = R$ for some $q \in (a, b)$, show that $\kappa(q) \geq \frac{1}{R}$.

Endnotes

Notes for Section 4.2

(A) The openness of the domains of smooth functions is crucial if we are to use the standard definition of derivatives. It is possible to extend the definition of what it means to be differentiable to non-open subsets of Euclidean space, but we will avoid doing so to help clarify the nature of smoothness, and to point the way more clearly to smooth manifolds.

(B) Though it may seem like a stringent requirement that all smooth functions used are infinitely differentiable, that is, all partial derivatives of all orders exist and are continuous, such functions are actually quite plentiful. It would be possible to deal with functions that are only twice or thrice differentiable, but the gain in doing so is negligible, and is outweighed by the nuisance of having to pay closer attention in all statements of theorems and proofs to the exact level of differentiability.

(C) See [MU1, Chapter I] for a clarification of the relation between functions of various degrees of differentiability.

Notes for Section 4.3

(A) Some books use the terminology "parametrized by arc-length" to mean what we call "unit speed."

(B) See [JU] for a literary look at the smoothness of curves.

Notes for Section 4.4

In single variable Calculus, the curves used are the graphs of functions of the form $y = f(x)$. Such functions have the form $f\colon (a, b) \to \mathbb{R}$. Graphs in the x-y plane of such functions have one axis representing the independent variable (namely x) and one axis representing the dependent variable (namely y). By contrast, when we view the "graph" of a function of the form $c\colon (a, b) \to \mathbb{R}^3$ we are actually looking at the image of the function, since $\mathbb{R}^3$ only has room for the dependent variables (namely x, y and z). Hence our definition of the tangent vector looks slightly different than that seen in the Calculus of a single variable.

Further Study

The material in this book is merely an introduction to a number of branches of mathematics, and here we recommend some books for each topic. This list is certainly not exhaustive.

1. Collateral Reading

A number of books can be used as supplements to the present text. A classic expository work on geometry, which includes some nice illustrations in differential geometry, is [H-CV]. Of more recent vintage is the highly recommended [WE], which gives an intuitive treatment of the geometry and topology of surfaces and 3-manifolds (very much inspired by the work of Weeks' adviser, W. Thurston). The first few chapters are particularly germane to our topic, though for our purposes one need not pay attention to the emphasis on hyperbolic geometry. This is also the only book I know that includes tic-tac-toe on the torus and the Klein bottle.

To do [WE] correctly, read the classic [AB] first. This little volume, written by a Victorian schoolmaster, was as much a satire of Victorian society as a mathematics text, but it has served as an inspiration to many mathematicians and non-mathematicians alike in thinking about higher-dimensional space. A more recent sequel to [AB] is [BU], which contains some nice mathematical topics, but which does not have the satirical style of its predecessor. Another book on higher dimensions for a popular audience is [RU], although some of the more speculative parts of this book should be taken with many grains of salt. A very nice recent book on higher dimensions, of interest to mathematicians and non-mathematicians alike, is [BA4].

On a more standard note, a textbook that might make a nice complement to the present book is [NA]. This book, like the present text, focuses on geometric questions concerning subsets of Euclidean spaces, but it covers topics we do not, such as the fundamental group, simplicial homology, and differential topology. A nice little volume that not only discusses the topology of surfaces but also gives many historical references to the development of the subject is [F-F]. Two very recent books that discuss a number of topics concerning surfaces that we have skipped (such as group actions and covering spaces) are [ST] and [KY].

The recent book [FO] treats a variety of topics, including some differential geometry and geometric topology of surfaces, and supplements rigor with remarkable drawings. Another phenomenal source of drawings of surfaces in various stages of deformation, with accompanying explanations, is [FC]. The text [MCL] nicely puts the differential geometry of surfaces in the context of the development of non-Euclidean geometry, in much more detail than our brief discussion in Section 8.5. Another text that expands upon our treatment of smooth surfaces in $\mathbb{R}^3$ is [MG], which gives a very concrete introduction to Riemannian geometry via surfaces in $\mathbb{R}^n$, discussing general relativity in the process.

2. Point Set Topology (also known as General Topology)

For more advanced work it is necessary to discuss topological properties of sets that do not naturally sit in any Euclidean space. The most general setting for such study is the notion of a topological space; the study of the axiomatic properties of such spaces is called point set topology. Point set topology is both the foundational material for all branches of topology and a good place to practice proof techniques. An excellent text on point set topology is [MU2]; it would suffice to read Chapters 2–4, although the reader familiar with groups should definitely read Chapter 8 about the fundamental group. Another nice text is [AR], which may not quite match [MU2] for pure expository style, but which has the advantage of moving as quickly as possible to geometric topics. Another such text is [JA], although its lack of exercises is a serious drawback. A classic point set textbook often recommended by authors of an earlier generation is [KE]. This book covers important material and is considered to have good problems; on the other hand, it does not have a single figure.

3. Algebraic Topology

Algebraic topology is the study of topological problems using tools from abstract algebra such as groups and rings. The basic idea is to associate with each topological space various algebraic objects that reflect the properties of the original space. The first topics usually studied in algebraic topology are the fundamental group, covering spaces, homology groups and homotopy groups.

Even someone primarily interested in geometric questions will need these tools for advanced work. Algebraic topology has become a subject in its own right as well as a tool for other branches of mathematics. A classic text on the fundamental group and covering spaces (as well as surfaces) is [MS1]. Other introductory texts covering simplicial homology and the fundamental group (among other things) are [NA], [AR] and [CR]. Of the many advanced texts on algebraic topology, two of the more accessible are [MU3], and [MS2]. The text [MU3] is particularly recommended for its geometric approach, including a nice treatment of simplicial cohomology. A slightly older book that covers point set and algebraic topology from a geometric point of view is [H-Y]. The ultimate reference book for algebraic topology is [SP], though as a first exposure to the subject one should proceed at one's own risk.

4. Geometric Topology

Geometric topology focuses primarily on the study of manifolds, which are the higher-dimensional analog of surfaces. The restriction to manifolds, as opposed to general topological spaces, allows for a more geometric flavor (of the "rubber-sheet" variety) than in point set topology. Manifolds come in three varieties: topological; piecewise linear (abbreviated PL), which generalize what we have been calling simplicial surfaces; and differential (also known as smooth). In the two-dimensional case (that is, surfaces) these three categories essentially coincide, in that any surface of one type is homeomorphic to a surface of any of the other two types. In higher dimensions the three categories of manifolds behave quite differently from one another; for example, there are topological manifolds that are not homeomorphic to any differential manifold. The topological properties of differential manifolds are the subject of differential topology and will be discussed in Item 6 below. Geometric topology focusses on topological and PL manifolds. One needs to learn about general topological spaces and the fundamental group (at least) before attempting the books mentioned here on topological and PL manifolds.

A very nice text, and one upon which the current book draws fairly heavily, is [MO], dedicated to surfaces and 3-manifolds. The book contains proofs of the triangulability of 2-manifolds and 3-manifolds, the latter being quite difficult (and which was first proved by Moise). Also discussed are things such as wild spheres and wild arcs. Another book on geometric topology that maintains a low-dimensional, geometric point of view is [BI]. (Moise and Bing were classmates at the University of Texas, working under R.L. Moore.) Some standard texts on PL topology, all at the graduate level, are [ZE], [R-S], [HU], and [GL]. PL topology often appears quite formal at first encounter, especially [HU]. The text [ZE] is quite nice, and served as an inspiration for later texts on the subject, but is only available as unpublished lecture notes.

One very pretty geometric topic is knot theory, a branch of geometric topology but with a flavor all its own. Not only is this subject geometrically appealing, but there have recently been found some connections between it and such applied fields as quantum mechanics and DNA. Some books on the subject are [RO], [B-Z], and [KA].

Though closer to geometry and combinatorics than geometric topology, the study of polyhedra is both of inherent geometric interest and of use in applications. A nice discussion of the history of the study of polyhedra, which goes back to the ancient world, is given in [S-F, §4]. The combinatorial approach to polyhedra is taken in [GR1], [GR2] and [BD]. Applications to optimization can be found in [Y-K-K].

5. Differential Geometry

Differential geometry is an older subject than topology, having received a major impetus from the work of Gauss and Riemann. For historical comments see the appendix of [M-P]. Classical differential geometry is concerned with curves and surfaces in $\mathbb{R}^3$, as discussed in the present text. Three books taking the classical point of view, which contain material not covered here and upon which the current text has relied, are [M-P], [KL] and [DO1]. The last is particularly recommended; the text [M-P] is the most elementary, though not always elegant; there is much nice material in [KL], but the discussion is often rather terse.

Two main changes occur when moving beyond classical differential geometry: higher-dimensional manifolds are treated, and more advanced technologies (such as moving frames, differential forms, lie groups and vector bundles) are used. Although these more advanced techniques may be applied to surfaces, the advanced techniques are crucial in higher dimensions, where there are complications that do not arise in the case of surfaces in $\mathbb{R}^3$. Three undergraduate texts, slightly more advanced than the three mentioned above, are [ON], [S-T] and [TR]. The first of these two books treats moving frames, and the second discusses point set and algebraic topology as well as differential geometry, and includes the famous deRham Theorem. Some graduate level differential geometry texts are [DO2], [HI], [BO] and [K-N]. The ultimate introduction to differential geometry is the five volume opus [SK3 vols. I– V]. The coverage in this work is as follows: vol. I — the basics of smooth manifolds, differential forms, etc.; vol. II — an extremely thorough treatment of curvature and connections, in which the same topic is discussed via a sequence of approaches that roughly follows historical development, starting with the work of Gauss and Riemann; vol. III — classical surface theory (although one needs tools from the first two volumes); vol. IV — higher-dimensional manifolds; vol. V — advanced topics, including the generalized Gauss–Bonnet Theorem. The bibliography in vol. V is quite thorough. The five volumes [SK3 vols. I–V] are known for their exploratory and sometimes humorous style.

6. Differential Topology

This area is at the intersection of the various parts of the current text: the study of the topological properties of differential manifolds. Though certainly serving as foundational material for advanced differential geometry, differential topology has become a subject area distinct from either geometric topology or differential geometry and has seen major advances in the past 40 years. To study differential topology, advanced Calculus is definitely needed; see, for example, the classic [SK1] or the recent [MU4]. Some point set topology is also needed, and basic algebraic topology is necessary for the more advanced texts. An excellent place to start is the beautiful little book [MI3]. Two other introductory texts are [WA] and [B-J]. There is also some introductory material on differential topology in [NA]. Other books to look at, all at the graduate level, are [BO], [HR], [MU1], [WR] and [SK3 vol. I].

Finally, two books to which any student interested in the study of smooth manifolds should aspire are [MI2] and [M-S]. Both these books are influential graduate level texts, covering beautiful material and written in a style many mathematicians seek to emulate. Both books are based on notes taken during lectures by J. Milnor, one of the most important topologists of the last 40 years; one of the note-takers for [MI2] was M. Spivak, author of [SK3].

References

[AB] Abbott, E. A., *Flatland*, Dover, New York, 1952.

[AL2] Alexander, J. W., *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 8–10.

[AL3] ———, *Remarks on a point set constructed by Antoine*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 10–12.

[AD] Allendoerfer, C. B., *The Euler number of a Riemann manifold*, Amer. J. Math. **62** (1940), 243–248.

[A-W] Allendoerfer, C. B., and Weil, A., *The Gauss–Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. **53** (1943), 101–129.

[AN1] ———, *Sur la possibilité d'étendre l'homéomorphie de deux figures à leurs voisinages*, C. R. Acad. Sci. Paris **171** (1920), 661–663.

[AN2] Antoine, L., *Sur l'homéomorphie de figures et de leurs voisinages*, J. Math. Pures Appl. **86** (1921), 221–324.

[AR] Armstrong, M. A., *Basic Topology*, Springer-Verlag, New York, 1983.

[BA1] Banchoff, T., *Critical points and curvature for embedded polyhedra*, J. Diff. Geom. **1** (1967), 245–256.

[BA2] ———, *Critical points and curvature for embedded polyhedral surfaces*, Amer. Math. Monthly **77** (1970), 475–485.

[BA3] ———, *Critical points and curvature for embedded polyhedra II*, Progress in Math. **32** (1983), 34–55.

[BA4] ———, *Beyond the Third Dimension*, Scientific American Library, New York, 1990.

[BP] Barr, S., *Experiments in Topology*, Crowell, New York, 1964.

[BT] Bartle, R. G., *The Elements of Real Analysis*, John Wiley & Sons, New York, 1964.

[BE] Berger, M., *Convexity*, Amer. Math. Monthly **97** (1990), 650–678.

[BI] Bing, R H, *The Geometric Topology of 3-Manifolds*, AMS Colloquium Publications, vol. 40, American Mathematical Society, Providence, RI, 1983.

[BL] Bloch, E. D., *A combinatorial Chern–Weil theorem for 2-plane bundles with even Euler characteristic*, Israel J. Math. **67** (1989), 193–216.

[BO] Boothby, W. M., *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, New York, 1975.

[BR] Brahana, H. R., *Systems of circuits on two-dimensional manifolds*, Ann. of Math. **23** (1922), 144–68.

[B-J] Bröcker, Th., and Jänich, K., *Introduction to Differential Topology*, Cambridge U. Press, Cambridge, 1982.

[BD] Brøndsted, A., *An Introduction to Convex Polytopes*, Springer-Verlag, New York, 1983.

[BN] Brown, M., *Locally flat imbeddings of topological manifolds*, Ann. of Math. **75** (1962), 331–341.

[B-Z] Burde, G., and Zieschang, H., *Knots*, De Gruyter, New York, 1985.

[BU] Burger, D., *Sphereland*, Perennial Library, Harper & Row, New York, 1965.

[BG] Burgess, C. E., *Classification of surfaces*, Amer. Math. Monthly **92** (1985), 349–454.

[CA1] Cairns, S. S., *Homeomorphisms between topological manifolds and analytic manifolds*, Ann. of Math. **41** (1940), 796–808.

[CA2] ———, *An elementary proof of the Jordan–Schoenflies theorem*, Proc. Amer. Math. Soc. **2** (1951), 860–867.

[CS] Cassels, J. W. S., *Economics for Mathematicians*, London Math. Soc. Lecture Note Series 22, Cambridge U. Press, Cambridge, 1981.

[CE] Cederberg, J. N., *A Course in Modern Geometries*, Springer-Verlag, New York, 1989.

[CH] Chern, S.-S., *What is geometry?*, Amer. Math. Monthly **97** (1990), 679–686.

[C-M-S] Cheeger, J., Muller, W., and Schrader, R., *On the curvature of piecewise flat spaces*, Commun. Math. Phys. **92** (1984), 405–454.

[CR] Croom, F. H., *Basic Concepts of Algebraic Topology*, Springer-Verlag, New York, 1978.

[DE] Debreu, G., *Theory of Value*, Yale U. Press, New Haven, CT, 1959.

[DI] Dierker, E., *Topological Methods in Walrasian Economics*, Lecture Notes in Economics and Mathematical Systems #92, Springer-Verlag, Berlin, 1974.

[DO1] Do Carmo, M., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.

[DO2] Do Carmo, M., *Riemannian Geometry*, Birkhäuser, Boston, 1992.

[D-F-N] Dubrovin, B. A., Fomenko, A. T., and Novikov, S. P., *Modern Geometry — Methods and Applications*, parts I–III, Springer-Verlag, New York, 1984.

[DU] Dugundji, J., *Topology*, Allyn & Bacon, Boston, 1966.

[EI] Eisenhardt, L. P., *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn, Boston, 1909.

[EU] Euclid, *The Thirteen Books of the Elements*, Dover, New York, 1956.

[FE] Federico, P. J., *Descartes on Polyhedra*, Springer-Verlag, New York, 1982.

[FN] Fenchel, W., *On total curvatures of Riemannian manifolds: I*, J. London Math. Soc. **15** (1940), 15–22.

[FL] Flanders, H., *Differential Forms*, Academic Press, New York, 1963.

[FO] Fomenko, A., *Visual Geometry and Topology*, Springer-Verlag, New York, 1994.

[FC] Francis, G. K., *A Topological Picture Book*, Springer-Verlag, New York, 1987.

[F-F] Fréchet, M., and Fan, K., *Initiation to Combinatorial Topology*, Prindle, Weber & Schmidt, Boston, 1967.

[F-L] Freedman, M. H., and Luo, F., *Selected Applications of Geometry to Low-Dimensional Topology*, American Mathematical Society, Providence, RI, 1989.

[F-Q] Freedman, M. H., and Quinn, F., *Topology of 4-manifolds*, Princeton U. Press, Princeton, 1990.

[FR] Friedberg, S., Insel, A., and Lawrence, E., *Linear Algebra*, 2nd ed., Prentice-Hall, Engelwood Cliffs, NJ, 1989.

[GA] Gauss, K. F., *General Investigations of Curved Surfaces*, Raven Press, New York, 1965.

[GL] Glaser, L. C., *Geometrical Combinatorial Topology*, vols. I–II, Van Nostrand Reinhold, New York, 1970.

[GE] Greenberg, M. J., *Euclidean and Non-Euclidean Geometry*, W. H. Freeman, San Francisco, 1974.

[GR1] Grünbaum, B., *Convex Polytopes*, Wiley, New York, 1967.

[GR2] ______, *Grassman angles of convex polytopes*, Acta. Math. **121** (1968), 293–302.

[HM] Hamilton, A. G., *Numbers, Sets and Axioms*, Cambridge U. Press, Cambridge, 1982.

[HE] Hempel, J., *3-manifolds*, Ann. of Math. Studies, vol. 86, Princeton U. Press, Princeton, 1976.

[HI] Hicks, N., *Notes on Differential Geometry*, Van Nostrand, Princeton, 1965.

[HIL] Hilbert, D., *Über Flächen von konstanten Gausscher Krümmung*, Trans. Amer. Math. Soc. **1** (1901), 87–99.

[H-CV] Hilbert, D., and Cohn-Vossen, S., *Geometry and the Imagination*, Chelsea, New York, 1956.

[HR] Hirsch, M. W., *Differential Topology*, Springer-Verlag, New York, 1976.

[H-Y] Hocking, J. G., and Young, G. S., *Topology*, Addison-Wesley, Reading, MA, 1961.

[HU] Hudson, J. F. P., *Piecewise Linear Topology*, Benjamin, Menlo Park, CA, 1969.

[HZ] Hurewicz, W., *Lectures on Ordinary Differential Equations*, M.I.T. Press, Cambridge, MA, 1966.

[H-W] Hurewicz, W., and Wallman, H., *Dimension Theory*, Princeton U. Press, Princeton, 1941.

[JA] Jänich, K., *Topology*, Springer-Verlag, New York, 1984.

[JU] Juster, N., *The Dot and the Line*, Random House, New York, 1963.

[KA] Kauffman, Louis H., *On Knots*, Ann. of Math. Studies, vol. 115, Princeton U. Press, Princeton, 1987.

[KE] Kelley, John L., *General Topology*, Van Nostrand, Princeton, 1955.

[KN] Kendig, K., *Elementary Algebraic Geometry*, Springer-Verlag, New York, 1977.

[KY] Kinsey, L. C., *Topology of Surfaces*, Springer-Verlag, New York, 1993.

[KI] Kirby, R., *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. **89** (1969), 575–582.

[K-S] Kirby, R., and Siebenmann, L., *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Ann. of Math. Studies, vol. 88, Princeton U. Press, Princeton, 1977.

[KL] Klingenberg, W., *A Course in Differential Geometry*, Springer-Verlag, New York, 1978.

[K-N] Kobayashi, S., and Nomizu, K., *Foundations of Differential Geometry*, I, II, Interscience, New York, 1963, 1969.

[LA1] Lang, S., *Linear Algebra*, Addison-Wesley, Reading, MA, 1966.

[LA2] Lang, S., *Analysis I*, Addison-Wesley, Reading, MA, 1968.

[LY] Lyusternik, L. A., *Convex Figures and Polyhedra*, Dover, New York, 1963.

[MA] Malitz, J., *Introduction to Mathematical Logic*, Springer-Verlag, New York, 1979.

[MK] Markov, A. A., *Unsolvability of the problem of homeomorphism*, Proc. Int. Cong. Math., 1958, pp. 300–306 (Russian).

[MT] Martin, G., *The Foundations of Geometry and the Non-Euclidean Plane*, Springer-Verlag, New York, 1975.

[MS1] Massey, W. S., *Algebraic Topology: An Introduction*, Springer-Verlag, New York, 1967.

[MS2] ———, *Singular Homology Theory*, Springer-Verlag, New York, 1980.

[MCL] McCleary, J., *Geometry from a Differentiable Viewpoint*, Cambridge U. Press, Cambridge, 1994.

[MC] McMullen, P., *Non-linear angle-sum relations for polyhedral cones and polytopes*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 247–261.

[M-P] Millman, R. S., and Parker, G. D., *Elements of Differential Topology*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

[MI1] Milnor, J. W., *Differential Manifolds which are Homotopy Spheres*, unpublished notes.

[MI2] ———, *Morse Theory*, Ann. of Math. Studies, vol. 51, Princeton U. Press, Princeton, 1963.

[MI3] ———, *Topology from the Differentiable Viewpoint*, U. of Virginia Press, Charlottesville, VA, 1965.

[MI4] ———, *Analytic proofs of the "Hairy Ball Theorem" and the Brouwer Fixed Point Theorem*, Amer. Math. Monthly **85** (1978), 521–524.

[M-S] Milnor, J.W., and Stasheff, J., *Characteristic Classes*, Ann. of Math. Studies, vol. 76, Princeton U. Press, Princeton, NJ, 1974.

[MO] Moise, E. E., *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, New York, 1977.

[MR] Moore, G. H., *Zermelo's Axiom of Choice*, Springer-Verlag, New York, 1982.

[MG] Morgan, F., *Riemannian Geometry: A Beginner's Guide*, Jones and Bartlett, Boston, 1993.

[MU1] Munkres, J. R., *Elementary Differential Topology*, Ann. of Math. Studies, vol. 54, Princeton U. Press, Princeton, 1966.

[MU2] ———, *Topology, A First Course*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[MU3] ———, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, CA, 1984.

[MU4] ———, *Analysis on Manifolds*, Addison-Wesley, Menlo Park, CA, 1991.

[NA] Naber, G., *Topological Methods in Euclidean Spaces*, Cambridge University Press, Cambridge, 1980.

[ON] O'Neill, B., *Elementary Differential Geometry*, Academic Press, New York, 1966.

[OS1] Osserman, R., *A Survey of Minimal Surfaces*, Van Nostrand Reinhold, New York, 1969.

[OS2] ———, *Curvature in the Eighties*, Amer. Math. Monthly **97** (1990), 731–756.

[PI] Pietsch, H., *Geodätische approximation einer topologischen triangulation*, Deutsche Math. 4 (1939), 583–589.

[RI] Richards, I., *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269.

[RO] Rolfsen, D., *Knots and Links*, Publish or Perish, Inc., Berkeley, CA, 1976.

[RT] Rotman, J. J., *Theory of Groups*, 2nd ed., Allyn & Bacon, Boston, 1973.

[R-S] Rourke, C., and Sanderson, B., *Introduction to Piecewise-Linear Topology*, Ergebnesse der Mathematik 69, Springer-Verlag, New York, 1972.

[RU] Rucker, R., *The Fourth Dimension*, Houghton Mifflin, Boston, 1984.

[RD] Rudin, M. E., *An unshellable triangulation of a tetrahedron*, Bull. Amer. Math. Soc. **64** (1958), 90–91.

[SA] Samelson, H., *Orientability of hypersurfaces in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **22** (1969), 301–302.

[S-F] Senechal, M., and Fleck, G., *Shaping Space*, Birkhäuser, Boston, 1988.

[S-T] Singer, I. M., and Thorpe, J. A., *Lecture Notes on Elementary Topology and Geometry*, Springer-Verlag, New York, 1967.

[SP] Spanier, E., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[SK1] Spivak, M., *Calculus on Manifolds*, Benjamin, New York, 1965.

[SK2] ______, *Calculus*, Benjamin, New York, 1967.

[SK3] ______, *A Comprehensive Introduction to Differential Geometry*, vols. I–V, Publish or Perish, Inc., Boston, 1975.

[ST] Stillwell, J., *Geometry of Surfaces*, Springer-Verlag, New York, 1992.

[SR] Struik, Dirk J., *Lectures on Classical Differential Geometry*, Addison-Wesley, Reading, MA, 1950.

[TH] Thomassen, C., *The Jordan–Schönflies theorem and the classification of surfaces*, Amer. Math. Monthly **99** (1992), 116–130.

[TR] Thorpe, J. A., *Elementary Topics in Differential Geometry*, Springer-Verlag, New York, 1979.

[TU] Trudeau, R. J., *The Non-Euclidean Revolution*, Birkhäuser, Boston, 1987.

[VA] Valentine, F. A., *Convex Sets*, McGraw-Hill, New York, 1964.

[WA] Wallace, A., *Differential Topology*, Benjamin/Cummings, Reading, MA, 1968.

[WR] Warner, F. W., *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, New York, 1983.

[WE] Weeks, J. R., *The Shape of Space*, Marcel Dekker, New York, 1985.

[WH1] Whitehead, J. H. C., *Convex regions in the geometry of paths*, Quart. J. Math. **3** (1932), 33–42, 226–227.

[WH2] ______, *On* C^1*-complexes*, Ann. of Math. **41** (1940), 809–824.

[WH3] ______, *Manifolds with transverse fields in Euclidean space*, Ann. of Math. **73** (1961), 154–212.

[Y-K-K] Yemelichev, V. A., Kovalev, M. M., and Kravtsov, M. K., *Polytopes, Graphs and Optimisation*, Cambridge U. Press, Cambridge, 1984.

[YU] Yu, Y.-L., *Combinatorial Gauss–Bonnet–Chern formula*, Topology **22** (1983), 153–163.

[ZE] Zeeman, E. C., *Seminar on Combinatorial Topology*, unpublished notes, I.H.E.S., Paris, 1963.

Hints for Selected Exercises

Section 1.2

1.2.2. Take a well-chosen nested family of open intervals in $\mathbb{R}$.

1.2.12. The goal is to show that $U = \mathbb{R}^n - A$ is open in $\mathbb{R}^n$. Let $p \in U$ be a point; show that $O_{D/2}(p, \mathbb{R}^n)$ contains at most one member of A. Now find a number $r > 0$ such that $O_r(p, \mathbb{R}^n) \subset U$.

1.2.17. Let $x = \operatorname{lub} S$, and suppose that $x \notin S$; obtain a contradiction by showing that $\mathbb{R} - S$.

Section 1.3

1.3.7. Look for a function $f\colon (0, 1) \to \mathbb{R}$ with slope that goes to infinity.

1.3.8. Divide $\mathbb{R}$ into two parts, one on which $f(x) \geq g(x)$, and one on which $g(x) \geq f(x)$.

1.3.9. Use Condition (3) of Proposition 1.3.3. Let $p \neq 0$ be a real number; first, find numbers $m, \delta_1 > 0$ such that $m < |xp|$ for all $|x - p| < \delta$; now find the desired δ.

1.3.10. Using Lemmas A.5 and 1.3.8, and writing F out in coordinates with respect to the standard basis of $\mathbb{R}^n$, it suffices to show that any function $f\colon \mathbb{R}^n \to \mathbb{R}$ of the form $f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = a_1x_1 + \cdots + a_nx_n + d$ (where $a_1, \ldots, a_n, d \in \mathbb{R}$ are any numbers) is continuous; the proof that f is continuous is similar to Example 1.3.4.

1.3.11. For each possible combination of continuous, open, and closed, an example is given, and sometimes a hint on how to prove that the desired properties are satisfied.

(1) Continuous, open and closed: The identity map $1_{\mathbf{R}}\colon \mathbb{R} \to \mathbb{R}$.

(2) Open, closed and not continuous: The map $f\colon [0, 2] \to [0, 1] \cup (2, 3]$ given by

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1]; \\ x + 1, & \text{if } x \in (1, 2]. \end{cases}$$

For non-continuity, consider the subset $U = (\frac{1}{2}, 1]$ of the codomain $[0, 1] \cup (2, 3]$. For openness, consider separately the open subset of $[0, 2]$ that contains the point 1 and those that do not. For closedness, use the bijectivity of f and Exercise 1.3.5.

(3) Continuous, closed and not open: Any constant map $f: \mathbb{R} \to \mathbb{R}$.

(4) Continuous, open and not closed: The projection map $\pi_1: \mathbb{R}^2 \to \mathbb{R}$, given by $\pi_1(\binom{x}{y}) = x$. For closedness, consider the subset $C \subset \mathbb{R}^2$ that is the sequence

$$C = \{\binom{1/2}{2}, \binom{1/3}{3}, \binom{1/4}{4}, \ldots\}.$$

Use Exercise 1.2.12.

(5) Open, not continuous and not closed: The map $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\binom{x}{y}) = \begin{cases} x + 1, & \text{if } y \geq 0; \\ x, & \text{if } y < 0. \end{cases}$$

This is similar to part (4).

(6) Closed, not continuous and not open: The map $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ -1, & \text{if } x < 0, \end{cases}$$

For non-continuity look at $f^{-1}((\frac{1}{2}, \frac{3}{2}))$.

(7) Continuous, not open and not closed: The inclusion map $i: [0, 1) \to \mathbb{R}$.

(8) Not continous, not open and not closed: The map $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2, & \text{if } x = 0; \\ x, & \text{if } x \neq 0. \end{cases}$$

For non-continuity look at $f^{-1}((1, 3))$. For non-openness look at $f((-1, 1))$. For non-closedness look at $f([0, 1])$.

Section 1.4

1.4.8. Use equivalence relations if you are familiar with the concept; otherwise, show directly that for any $x, y \in X$, if $[x]$ and $[y]$ have non-empty intersection then they are in fact equal sets.

Section 1.5

1.5.3. It suffices to prove that the product of two connected subsets of Euclidean space is connected; the result for products of more than two connected sets would then follow by induction on the number of factors in the product; let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be connected; choose a point $(a, b) \in A \times B$; for each $x \in A$ let $T_x = (A \times \{b\}) \cup (\{x\} \times B)$; show that each T_x is connected (use Exercise 1.5.2); observe that $A \times B = \bigcup_{x \in A} T_x$, and deduce that $A \times B$ is connected.

1.5.4. The tricky part is showing that components are closed subsets; let C be a component and let $x \in A - C$ be a point; conclude that $C \cup \{x\}$ is not connected, and deduce that there is some number $\epsilon > 0$ such that $O_\epsilon(x, A) \subset A - C$.

1.5.6. Use the Intermediate Value Theorem, though you need to decide what to apply it to.

1.5.8. For the first part, let $r > 0$ be a number such that $O_r(x, \mathbb{R}^2) \subset U$; if $y, z \in U$ are any two points, then by hypothesis there is a path in U from y to z; if the path does not contain x there is nothing to prove; if the path does contain x, then show how to modify the path inside $O_r(x, \mathbb{R}^2)$ so that it misses x.

1.5.11. Let $x, y \in U$ be points for which there is no path from x to y in $U - \{a\}$; by hypothesis there must be a path from x to y in U, and hence this path must contain a; use openness find points x' and y' with the same properties as x and y but contained in V; now suppose that $V - \{a\}$ is path connected, and obtain a contradiction.

1.5.13. Suppose to the contrary that there is a component of A that intersects both B and $A - B$.

Section 1.6

1.6.4. This is a tricky problem; one possibility is to let $A = \mathbb{R} \subset \mathbb{R}^2$, to define an injective continuous map $f: A \to \mathbb{R}^n$ with the desired properties, and to let $B = f(A)$.

1.6.7. Cover the set $[a, b] \times \{0\}$ with open squares (rather than open disks) such that the function f is positive on each square.

1.6.8. To find a maximal element, consider the collection of all sets of the form $(-\infty, a)$ for $a \in A$.

1.6.10. Use the Extreme Value Theorem to find a point $x \in (a, b)$ at which f has a maximal or minimal value and $f(x) \neq f(a)$; use the Intermediate Value Theorem to prove the desired result.

1.6.11. Consider the function $d: A \times B \to \mathbb{R}$ defined by $d(a, b) = \|a - b\|$.

1.6.12. Pick any finite subcover of A by compactness; let U_{i_1} be any element of the finite subcover that contains p; if U_{i_1} contains q we are finished, so assume otherwise; use connectivity to show that there is some set in the finite subcover that intersects U_{i_1}, and call this set U_{i_2}; keep going until one of these sets contains q.

Section 2.2

2.2.5. For part (i), use the Schönflies Theorem to show that $\mathbb{R}^2 - C$ has precisely two components; show that at least one of these components must be unbounded, since $\mathbb{R}^2$ is unbounded and C is compact; show that not both of the components are unbounded, again using the compactness of C; for part (ii) use the hint for Exercise 2.2.6.

2.2.6. First reduce to the case where $B_1 = D^2$; then, by definition, there is some homeomorphism $g: D^2 \to B$; observe that $g(S^1) = \partial B$; using the compactness of D^2 show that $[\mathbb{R}^2 - \partial B] - g(\text{int } D^2)$ is open in $\mathbb{R}^2 - \partial B$; use Invariance of Domain to show that $g(\text{int } D^2)$ is open in $\mathbb{R}^2 - \partial B$; use the connectivity of $\text{int } D^2$ and Exercise 1.5.13 to deduce that $g(\text{int } D^2)$ is precisely one of the two components of $\mathbb{R}^2 - \partial B$; use the compactness of int D^2 to show that $g(\text{int } D^2)$ is the bounded component of $\mathbb{R}^2 - \partial B$; use a similar (though simpler) argument to show that $h(\text{int } D^2)$ is also the bounded component of $\mathbb{R}^2 - \partial B$; deduce the desired result.

2.2.8. The set B cannot be just the origin, so pick some point $z \in B$ other than the origin; show that the intersection of B with the line containing O_2 and z is a compact set; use Exercise 1.5.8 to find a point x in this intersection with maximal distance from the origin; use Invariance of Domain to show that $x \in \partial B$; show that x is as desired; to show that there are at least two such points x, use Invariance of Domain to show that B is not contained in a single line.

2.2.9. Reduce to the situation where $B_1 = D^2$; use Exercise 2.2.8 to find two points $x, y \in \partial B_2$ satisfying the conclusion of that exercise; break up $D^2 - \text{int } B_2$

into two disks by forming two 1-spheres using the points x and y and then using Corollary 2.2.5; use these disks to construct a homeomorphism from A onto $D^2 - \text{int}\, B_2$.

2.2.10. First suppose that $n > m$, and derive a contradiction to Invariance of Domain by thinking of $\mathbb{R}^m$ as a subset of $\mathbb{R}^n$ as usual; next assume $n < m$, and that A is open in $\mathbb{R}^m$; let $x \in A$ be a point, so that there exists some number $\epsilon > 0$ such that $O_\epsilon(x, \mathbb{R}^m) \subset A$; hence $\mathbb{R}^n$ contains a subset homeomorphic to $O_\epsilon(x, \mathbb{R}^m)$; think of $\mathbb{R}^n$ as a subset of $\mathbb{R}^m$ as usual; derive a contradiction using Invariance of Domain.

2.2.12. First show that for each point in ∂J there is a point arbitrarily close to it in int J; then show that if a point is in int B then there is a minimal positive distance from it to all points in ∂B.

2.2.13. Let $A \subset S^1$ be a proper subset homeomorphic to S^1; show that S^1 with a point removed is homeomorphic to $\mathbb{R}$, and hence a homeomorphic copy of A sits in $\mathbb{R}$; using the compactness and connectivity of A show that A must be a closed interval; show that a closed interval cannot be homeomorphic to S^1, yielding a contradiction.

Section 2.3

2.3.3. Let $V \subset Q$ be an open set containing p that is homeomorphic to int $D2$, and let $h\colon \text{int}\, D2 \to V$ be a homeomorphism; consider the set $h^{-1}(O_\epsilon(p, Q))$, and find appropriate disks there.

2.3.4. Let $p \in U$ be a point, and let $V \subset Q$ be an open subset containing Q that is homeomorphic to int $D2$; consider $U \cap V$, and use Invariance of Domain.

2.3.9. Let $H\colon Q_1 - \text{int}\, B_1 \to Q_2 - \text{int}\, B_2$ be a homeomorphism; show that $H(\partial B_1) = \partial B_2$; extend H over B_1 so that $H(B_1) = B_2$.

Section 2.4

2.4.2. The "obvious" thing you might try, namely cutting the rectangle in Figure 2.4.9 (i) in two with a horizontal line half-way up, does not work; a more judicious cut is needed.

Section 2.5

2.5.3. In the case of planes, let $p \in Q$ be a point, and let Π be a plane in $\mathbb{R}^n$ which contains an open neighborhood of p in Q; consider the set $T = \Pi \cap Q$; use the definition of relative closedness to show that T is closed in Q; use the hypothesis on Q to show that T is open in Q, the crucial observation being that a non-empty open subset of a plane in $\mathbb{R}^n$ cannot simultaneously be an open subset of a different plane in $\mathbb{R}^n$; conclude that $T = Q$.

Section 2.6

2.6.1. Use Lemma 2.4.5.

Appendix A2.2

A2.2.1. It suffices to show that the surfaces Q and Q_r in the proof of parts (i) and (ii) of the proposition are homeomorphic under the hypothesis that at least one of Q_1 and Q_2 is disk-reversible; assume without loss of generality that Q_2 is disk-reversible; the goal is to apply Exercise 1.4.9 to the maps f' and $r' \circ f'$; let $H\colon Q_2 \to Q_2$ be a homeomorphism such that $H(T_2') = T_2'$ and $H|\partial T_2'$ is an orientation reversing homeomorphism, which exists by Exercise 2.5.3; consider the map

$$d = r' \circ f' \circ (f')^{-1} \circ (H|\partial T_2')^{-1}\colon \partial T_2' \to \partial T_2',$$

and proceed as in the proof of parts (i) and (ii) of the proposition.

A2.2.3. Transfer everything to D^2 using Corollary 2.2.6 and Exercise 2.2.6, and use the method of Lemma A2.2.2.

A2.2.5. Find a homeomorphism $g\colon S^1 \to S^1$ such that $f_2 = f_1 \circ g$; then use Exercise A2.2.4.

A2.2.6. Pick any pair of antipodal points $a, b \in S^1$; define the function $f\colon \overrightarrow{ab} \to \mathbb{R}$ by letting $f(z)$ be the length of the arc from $f(-z)$ to $-f(z)$, where the length is positive if the arc is counterclockwise and negative if the arc is clockwise (note that by injectivity $f(-z) \neq f(z)$, so that there is never ambiguity in this definition); if $f(a)$ and $f(b)$ are antipodal there is nothing to prove, so assume otherwise; show that one of $f(a)$ and $f(b)$ is positive and the other is negative; since $\overrightarrow{ab}$ is an arc, the Intermediate Value Theorem (Theorem 1.5.4) can be applied to f; deduce the result.

A2.2.7. There are many ways to construct the map F, the simplest being as follows: The idea is to map each horizontal slice of $[-1, 1] \times [0, 1]$ homeomorphically to itself, squeezing the homeomorphism f to a point as one moves up $[0, 1]$, fixing the endpoints and increasing amounts of $[-1, 1]$; graphically this map F is suggested in Figure H.1; find an explicit formula.

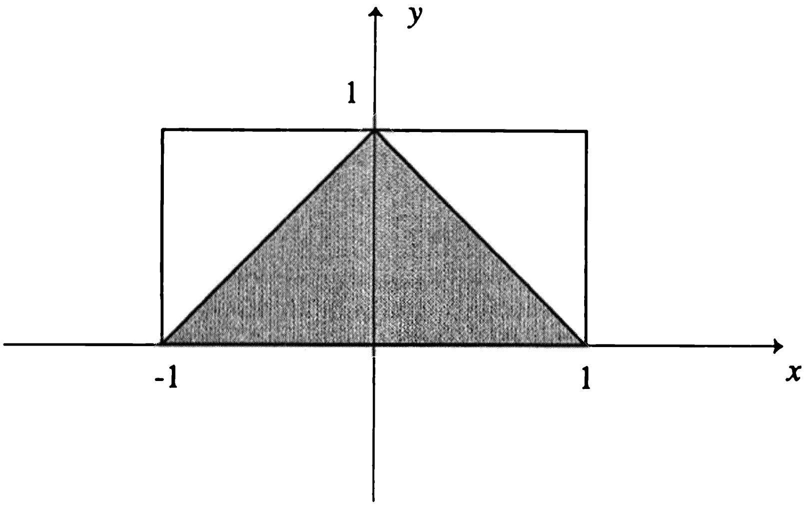


Figure H.1

A2.2.10. Use Proposition A2.2.6.

A2.2.11. Use Proposition A2.2.6.

Section 3.2

3.2.5. Linear independence.

3.2.6. Suppose $\eta = \langle a_0, \ldots, a_k \rangle$; translate the whole situation by a_0, and consider the issue of k-dimensional subspaces of $\mathbb{R}^n$ containing k given vectors.

3.2.8. Use Lemmas A.6 and A.7 and Exercise 1.3.10.

Section 3.3

3.3.6. The tricky part is (3) $\Rightarrow$ (2); suppose (2) holds but (3) is false, so that there exist simplices σ and τ of K for which there does not exist a connecting chain of simplices as in condition (3); let $C \subset |K|$ be the union of all simplices of K that can be connected to σ by a chain of simplices, and let D be the union of the rest of the simplices of K; show that $C \cup D = |K|$, that $C \cap D = \emptyset$ and that both C and D are non-empty and closed in $|K|$.

3.3.9. Although the idea is intuitively very straightforward — one simply chops up the polygonal disk into triangles — the proof is a bit trickier (though not longer) than might be expected; see [HO] for a proof and discussion of some false proofs that have been published.

Section 3.4

3.4.3. No need to repeat the proof of Theorem 3.4.1; use Exercise 3.3.10, although you need to check what happens with the links after subdivision.

Section 3.5

3.5.2. Use the method of Example 3.5.3 (2).

3.5.4. First show that no two of $S^2, T^2, T^2 \# T^2, T^2 \# T^2 \# T^2, \ldots$ are homeomorphic by using the Euler characteristic; then show that no two of $P^2, P^2 \# P^2, P^2 \# P^2 \# P^2, \ldots$ are homeomorphic; use Proposition 2.6.6 to show that no surface from one of these lists is homeomorphic to a surface in the other list.

Section 3.7

3.7.2. Break up $f_2(K)$ into parts.

3.7.5. Equation 3.7.1.

3.7.6. Let K be a simplicial surface with $f_0(K) = V_\zeta$; create new simplicial surfaces with $f_0(K)$ any integer greater than V_ζ by subdividing K, adding one 0-simplex at a time. Observe that if ζ and $f_0(K)$ are known, then $f_1(K)$ and $f_2(K)$ are determined.

Section 3.8

3.8.1. (i). Let m be the dimension of K, let τ be an m-simplex of K and let $x \in \operatorname{Int} \tau$ is a point; show that x can be chosen to be in $\operatorname{int}|K|$; show that $m = 2$, as in the proof of Theorem 3.4.1.

(ii) & (iii). Use the fact that $\mathbb{R}^2 \not\approx \mathbb{H}^2$ (Exercise 2.2.11), and that small enough open balls in $|K|$ about any two points in the interior of the same simplex of K are homeomorphic to deduce that the interior of each simplex of K is entirely contained either in $\operatorname{int}|K|$ or $\partial|K|$, as in the proof of Theorem 3.4.1; show that every 1-simplex of K, the interior of which is contained in $\operatorname{int}|K|$, is a face of two 2-simplices, and the underlying space of the link of every 0-simplex of K contained in $\operatorname{int}|K|$ is a 1-sphere.

Let η be a 1-simplex of K such that $\operatorname{Int} \eta \subset \partial|K|$; using an argument similar to that used in the proof of Theorem 3.4.1, show that η is the face of at least one 2-simplex of K; suppose η is contained in 2-simplices $\sigma_1, \ldots, \sigma_p$, where $p \geq 2$, let $x \in \operatorname{Int} \eta$ be a point and let $U \subset |K|$ be an open set containing x that is homeomorphic to $\mathbb{H}^2$; deduce that any point in $U \cap \partial|K|$ has an open neighborhood in $|K|$ that is homeomorphic to $\mathbb{H}^2$; by an argument to Exercise 2.3.3, choose U small enough so that it is entirely contained in $\operatorname{Int} \eta \cup \operatorname{Int} \sigma_1 \cup \cdots \cup \operatorname{Int} \sigma_p$; show that $U \cap \partial|K| \subset \operatorname{Int} \eta$; use Exercise 1.5.12 to show that $U \cap \operatorname{int}|K|$ is entirely contained in one of the $\operatorname{Int} \sigma_i$; deduce that U is entirely contained in σ_i, and derive a contradiction using Exercise 1.2.18 (ii) and an argument similar to that found in the proof of Theorem 3.4.1; deduce that η is contained in precisely one 2-simplex.

If w is a 0-simplex of K contained in $\partial|K|$, show that $|\operatorname{link}(w, K)|$ is an arc, as in the proof of Theorem 3.4.1; deduce that parts (ii) and (iii) of the theorem will both follow, except for the fact that $\operatorname{Bd} K$ is a subcomplex; deduce this remaining claim from Exercise 2.2.12.

3.8.2. Use induction on p.

3.8.5. Combine Corollary 2.2.5 (ii), Corollary A2.2.5 and Exercise 3.8.4.

Section 4.2

4.2.1. The proof is very similar to the proof of Theorem 4.2.2.

4.2.2. The proof is similar to the proofs of Proposition 5.2.5 (ii) and Lemma 7.2.2 (iii), using Exercise 4.2.1 instead of Theorem 4.2.2.

4.2.3. Use the chain rule.

4.2.5. One first needs to show that the image of c intersects each vertical line in $\mathbb{R}^2$ at most once; let c_1 and c_2 denote the x and y coordinate functions of c; show that c_1 is bijective and that its image is an open interval; then use Exercise 4.2.3 to conclude that c_1 is a diffeomorphism; then consider the function $c_2 \circ (c_1)^{-1}$.

Section 4.3

4.3.3. First show that if $h(d, e)(a, b)$ is any diffeomorphism (independent of c) then $h(t) \neq 0$ for all $t \in (d, e)$; hence either $h'(t) > 0$ for all $t \in (d, e)$ or $h'(t) < 0$ for all $t \in (d, e)$; now apply Proposition 4.3.4 (i) to c; if the derivative of h has the wrong sign, modify h.

4.3.10. First find $\epsilon_p > 0$ such that $c|(p - \epsilon_p, q]$ is injective as follows; use Exercise 4.2.5 to find a number $\epsilon_1 > 0$ such that $c|(p - \epsilon_1, p + \epsilon_1)$ is injective; by compactness find the minimal distance $D > 0$ from $c([p + \epsilon_1, q])$ to $c(p)$; find a number $\delta > 0$ be such that $c((p - \delta, p + \delta)) \subset O_{D/2}(c(p), \mathbb{R}^3)$; show that $\epsilon_p = \min\{\epsilon_1, \delta\}$ has the desired property; now use a similar argument to find a number $\epsilon_q > 0$ such that $c|[p - \epsilon_p/2, q + \epsilon_q)$ is injective; let $\epsilon = \frac{1}{2}\min\{\epsilon_p/2, \epsilon_q/2\}$; now use Proposition 1.6.14 (iii) and the analog for arc of Exercise 2.2.4 applied to $[p - \epsilon, q + \epsilon]$.

Section 4.4

4.4.2. First consider the case where the curve lies in the x-y plane, and then reduce the general case to the first case using rotations and translations of $\mathbb{R}^3$.

4.4.4. Use Exercise 4.4.2.

Section 4.5

4.5.2. Rotation matrices preserve inner product and cross product.

4.5.4. Observe that $c(t) = \int_p^t T(s)\, ds + c(p)$ for any fixed $p \in (a, b)$.

4.5.5. For the "if" part, for each $t \in (a, b)$ we have $c(t) - x_0 = \lambda(t)\, T(t)$ for some function λ: $(a, b) \to \mathbb{R}$; what can you say about $\kappa(t)$ in this case?

Section 4.6

4.6.1. Use a circle and a right circular helix or appropriate radii.

4.6.2. For step (1), use Equations 4.6.4 and 4.6.5; for step (3) use Theorem 4.2.6.

Section 4.7

4.7.4. Use Exercise 4.7.3.

4.7.5. Define $f\colon (a, b) \to \mathbb{R}$ to be $f(t) = \langle c(t), c(t)\rangle$; what can you say about $f'(q)$ and $f''(q)$? Use this information to show that $\bar{N}(q) = \pm c(q)/R$, and that $|\kappa(q)| \geq \frac{1}{R}$.

Section 5.2

5.2.6. Use the chain rule for partial derivatives.

5.2.8. Use Proposition 5.2.5 (ii).

Section 5.3

5.3.5. The c curve is in the x-z plane; it might help to sketch the surface.

5.3.6. Use the x-y plane as the surface, though the equation $z = 0$ for this surface does not do what we want.

Section 5.4

5.4.6. If $c\colon (-\epsilon, \epsilon) \to M$ is a curve in M, use the chain rule to show that $DF(c(t))$ is perpendicular to $c'(t)$ for all $t \in (-\epsilon, \epsilon)$.

5.4.7. For the case $k = 0$ compute the partial derivatives of $\langle x, n\rangle$ with respect to s and t; for the case $k \neq 0$ compute $\|x - (-\frac{1}{k}w\|$.

Section 5.5

5.5.9. For step (2), show that

$$\frac{\partial g_{jk}}{\partial u_i} = \langle x_{ij}, x_k \rangle + \langle x_{ik}, x_j \rangle$$
$$\frac{\partial g_{ik}}{\partial u_j} = \langle x_{ij}, x_k \rangle + \langle x_{jk}, x_i \rangle$$

by permuting the subscripts i, j and k in the equation found in Step (1) and using the equality of mixed partial derivatives. Now solve these two equations together with the equation in Step (1).

Section 5.6

5.6.4. In both cases, choose a curve $c\colon (-\epsilon, \epsilon) \to M$ such that $c(0) = p$ and $c'(0) = v$; then use the chain rule on $f \circ c$.

Section 5.7

5.7.1. For part (i), at each point $\bar{q} \in U$ one can find numbers $Z^1(\bar{q})$ and $Z^2(\bar{q})$ such that $Z \circ x(\bar{q}) = Z^1(\bar{q})x_1(\bar{q}) + Z^2(\bar{q})x_2(\bar{q})$ by using linear algebra; to show that the resulting functions Z^i are smooth, use Cramer's rule.

5.7.2. Assume without loss of generality that $\bar{p} = 0_2$; if $\{e_1, e_2\}$ are the standard basis vectors for $\mathbb{R}^2$, use the curve $c\colon (-\epsilon, \epsilon) \to M$ given by $c(t) = x(te_i)$ to compute $\widetilde{\nabla}_{x_i(\bar{p})} Z$.

5.7.8. Choose a coordinate patch, and do everything in coordinates; start off by stating and proving an analog of Exercise 5.2.6 for functions of two variables.

Section 5.9

5.9.2. One scheme is to show (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2); for (2) $\Rightarrow$ (1) show that for each point $p \in M$ there is an open subset $V \subset M$ containing p such that $f(V)$ is open in N and $f|V\colon V \to f(V)$ is a diffeomorphism; use Equation 5.9.1 to show that df_p is non-singular, and then use Proposition

5.9.3; for (4) $\Rightarrow$ (5) let $x: U \to M$ be a coordinate patch as in part (4); use Lemma 5.8.1 to show that letting $A = U$ works; for (5) $\Rightarrow$ (2) let $v \in T_pM$ be a vector; suppose that $c: (-\epsilon, \epsilon) \to M$ be a smooth curve such that $c(0) = p$ and $c'(0) = v$; for each $t \in (-\epsilon/2, \epsilon/2)$ show that

$$\int_{-\epsilon/2}^{t} \|c'(s)\| \, ds = \int_{-\epsilon/2}^{t} \|(f \circ c)'(s)\| \, ds;$$

deduce that $\|c'(t)\| = \|(f \circ c)'(t)\|$ for all $t \in (-\epsilon/2, \epsilon/2)$; conclude that $\|df_p(v)\| = \|v\|$; now use the fact that a linear map that preserves lengths of vectors also preserves inner products (see [LA1, chapter VIII §5]).

5.9.5. First show that Ψ is bijective; use the Inverse Function Theorem and Exercise 1.4.4 to show that Ψ is actually a homeomorphism.

5.9.9. Let $y: U \to M$ be a coordinate patch such that $p \in y(U)$; let $\bar{v} = (dy_p)^{-1}(v)$ and similarly for $\bar{w}$; let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map that sends $\binom{1}{0}$ to $\bar{v}$ and $\binom{0}{1}$ to $\bar{w}$; consider the map $x = y \circ F|F^{-1}(U)$; use Exercises 5.9.7 and 5.9.8.

Section 6.1

6.1.1. For part (2), start with any coordinate patch whose image contains p, and construct the monge patch from it.

Section 6.2

6.2.1. By symmetry it suffices to describe the Weingarten map at any one point in the cylinder; examine the effect of the Weingarten map on a well-chosen basis for the tangent plane at the point you choose.

7. Use the normal vector field $n = \frac{Z}{\|Z\|}$ to compute the Weingarten map.

Section 6.3

6.3.1. By symmetry, both kinds of curvature are constant.

6.3.2. One need not completely know the Weingarten map to compute Gaussian curvature in this case. What can be said about the normal vectors along each ruling?

6.3.5. For part (1) use Exercises 5.4.6 and 6.2.7.

Section 6.4

6.4.5. Use the Weingarten Equations.

6.4.6. First show that the L_{ij} are all zero; then show that n is constant, and apply Exercise 5.4.7.

6.4.7. To show that the Weingarten map is zero at all points use $K = H = 0$ to compute the principle curvatures at all points; the proof of Exercise 6.4.6 can now be used to show that if $x: U \to M$ is a coordinate patch with connected domain, then $x(U)$ is contained in a plane; now use Exercise 2.5.3.

6.4.8. If the statements of each of (i)–(iii) can be proved for the image of each coordinate patch $x: U \to M$ for which U is connected, then the result for all of M can be pieced together using Exercise 2.5.3; using Exercise 6.3.4 (ii) there must be a function $d: U \to \mathbb{R}$ such that $n_i(\bar{p}) = d(\bar{p})x_i(\bar{p})$ for all $\bar{p} \in U$ and $i = 1, 2$; show that the function d is smooth; show that $d_1(\bar{p})x_2(\bar{p}) = d_2(\bar{p})x_1(\bar{p})$ for all $\bar{p}$, where d_1 and d_2 denote the partial derivatives of d; deduce that d_1 and d_2 are constantly zero, and it follows that d is constant; now show that x satisfies the hypotheses, and hence the conclusion, of Exercise 5.4.7.

6.4.9. First find a general criterion for a vector being an eigenvector for a 2×2 matrix.

Section 6.5

6.5.1. For step (1), use Equation 6.5.6 and Exercise 5.7.4. For step (2), use step (1) together with Equation 6.4.1 and some manipulating of the expression we are trying to derive.

Section 7.2

7.2.4. Use Exercises 5.5.6 and 7.2.3.

7.2.6. For step (3), there must be numbers $p, q \in (x, y]$ such that $h^{-1} \circ c(p) = 0$ and $h^{-1} \circ c(q) = 1$; show that there must be some number u between p and q such that $c(u) = c(x)$, a contradiction to injectivity.

7.2.7. For part (iii) use Exercise 7.2.5.

Section 8.2

8.2.2. Write $\bar{p} = \binom{p_1}{p_2}$ and $\bar{q} = \binom{q_1}{q_2}$, and consider the matrix Df; use Exercises 4.2.2 and 4.2.3.

8.2.5. Some parts of the corollary follow from the compactness of M and the statement of Proposition 8.2.3, whereas other parts require looking at the proof of Proposition 8.2.3.

8.2.6. Use the Lebesgue Covering Lemma (Theorem 1.6.9).

Section 8.3

8.3.1. Use Exercise 6.5.2.

8.3.2. Define $\sqrt{G(\binom{0}{\theta})}, \ldots, \frac{\partial^3 \sqrt{G}}{\partial R^3}(\binom{0}{\theta})$ using the limiting values given in Lemma 8.3.3, and use the Taylor polynomial with remainder.

8.3.4. For the second part of the exercise suppose that both cases are false; then there is some closed interval $I \subset (a, b)$, where I may be a single point, such that I is a maximal set upon which the curve $(D_p)^{-1} \circ c$ has horizontal tangent vectors at all points in I; see Figure H.2; consider the images under D_p of lines of the form $\theta = k$ and the curve $(D_p)^{-1} \circ c$, and obtain a contradiction to Theorem 7.2.6 by looking at one of the endpoints of I.

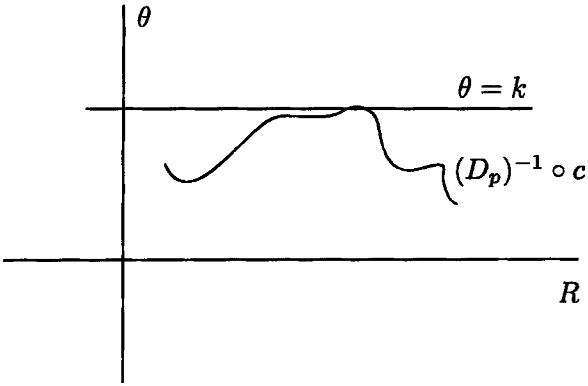


Figure H.2

8.3.5. The strategy is similar to the analogous part of the proof of Proposition 8.3.6; use Equation 8.3.6 to extend the function $\sqrt{G(\binom{R}{\theta})}$ smoothly over $(-\delta_p, \delta_p) \times (-\pi, 3\pi)$; take the derivative and restrict to an appropriate compact set.

Appendix A8.1

A8.1.1. If $\bar{q} = \binom{q_1}{q_2} \in O_{\epsilon_1}(\bar{p}, U)$ and $\bar{v} = \binom{v_1}{v_2} \in O_{\epsilon\epsilon_2/2}(\mathrm{O}_2, \mathbb{R}^2)$ are any two points, then show that

$$D\Phi(p_0) = \begin{pmatrix} \frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial q_2} & \frac{\partial q_1}{\partial v_1} & \frac{\partial q_1}{\partial v_1} \\ \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial q_2} & \frac{\partial q_2}{\partial v_1} & \frac{\partial q_2}{\partial v_1} \\ \frac{\partial c_1(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial q_1} & \frac{\partial c_1(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial q_2} & \frac{\partial c_1(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial v_1} & \frac{\partial c_1(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial v_1} \\ \frac{\partial c_2(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial q_1} & \frac{\partial c_2(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial q_2} & \frac{\partial c_2(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial v_1} & \frac{\partial c_2(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial v_1} \end{pmatrix},$$

where everything is evaluated at $p_0 = \begin{pmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix}$. The first two rows of this matrix are straightforward to compute. We discuss two of the other entries in the matrix; the remaining ones are similar to these two.

To compute $\frac{\partial c_1(\epsilon/2, \bar{q}, 2\bar{v}/\epsilon)}{\partial q_1}|_{p_0}$, hold all the variables other than q_1 constant (setting $q_2 = p_2$ and $v_1 = v_2 = 0$) and then taking the derivative; observe that $c_1(\frac{\epsilon}{2}, \binom{q_1}{p_2}, \mathrm{O}_2) = q_1$, since any geodesic of the form $c_{\bar{q},\mathrm{O}_2}(t)$ is the constant map $c_{\bar{q},\mathrm{O}_2}(t) = x(q)$ for all t by the uniqueness of geodesics.

To compute $\frac{\partial c_1(\epsilon/2, \bar{q}, 2\bar{v}/\epsilon)}{\partial v_1}|_{p_0}$, fix $q_1 = p_1$ and $q_2 = p_2$ and $v_2 = 0$, varying v_1; for notational ease let $t = v_1$, and take the derivative with respect to t at $t = 0$. Show that

$$\frac{\partial c_1(\frac{\epsilon}{2}, \bar{q}, \frac{2\bar{v}}{\epsilon})}{\partial v_1}|_{p_0} = \frac{d}{dt} c_1(\frac{\epsilon}{2}, p, t\frac{2}{\epsilon}\begin{pmatrix} 1 \\ 0 \end{pmatrix})|_{t=0}.$$

Use Exercise 7.2.5 to show that $c_{\bar{q},\bar{v}}(ta) = c_{\bar{q},t\bar{v}}(a)$ for any number a and all sufficiently small values of t, and use Exercise 5.9.7 to show that $c'_{\bar{p},\bar{v}}(0) = dx_{\bar{p}}(\bar{v}) = v_1 x_1(\bar{p}) + v_2 x_2(\bar{p})$, and hence $\frac{d c_i(s, \bar{p}, \bar{v})}{ds}|_{s=0} = v_i$ for $i = 1, 2$. Put these observations together to obtain the desired value.

A8.1.2. Suppose the result is false; it must then be the case that A intersects $GS_d(p, M)$; use the ideas of the proof of the length minimization part of Theorem A8.1.2 to show that, in fact, the length of A must be greater than d, a contradiction.

A8.1.3. First, show that $c([s_0 - \eta, s_0 + \eta])$ must have minimal length of all regular arcs in M with its endpoints; assume the contrary, and deduce that $c([a, b])$ could not be the regular arc of minimal length with endpoints p and q. Use the uniqueness in Theorem A8.1.2 (i) to deduce that $c([s_0 - \eta, s_0 + \eta])$ is a geodesic arc.

Appendix A8.2

A8.2.1. Use the Schönflies Theorem.

Index

$\mathrm{star}(\sigma, K)$	star of σ in K, 3.3
$\mathrm{link}(\sigma, K)$	link of σ in K, 3.3
$\lvert K \rvert$	underlying space, 3.3
$\mathcal{P}(\mathcal{V})$	partition induced by an admissible partition of vertices, 3.3
S^0	unit circle in $\mathbb{R}$, 3.4
$f_i(K)$	number of i-faces of K, 3.5
$\chi(K)$	Euler characteristic, 3.5
$\angle(v, \sigma)$	angle at v in σ, 3.7
$d(v)$	simplicial curvature at v, 3.7
$\mathrm{Bd}\, K$	simplicial boundary of asimplicial disk, 3.8
DF	Jacobian matrix of F, 4.2
$\mathrm{Length}(c)$	length of a curve, 4.3
$T(t)$	unit tangent vector, 4.4
$N(t)$	unit normal vector, 4.4
$B(t)$	unit binormal vector, 4.4
$\kappa(t)$	curvature of curves, 4.5
$\tau(t)$	torsion of curves, 4.5
$\bar{T}(t)$	planar unit tangent vector, 4.7
$\bar{N}(t)$	planar unit normal vector, 4.7
$\bar{\kappa}(t)$	planar curvature for curves, 4.7
x_1, x_2	partial derivatives of x, 5.2
$c(x, y)$	change of coordinate function, 5.2
$\bar{c}(t)$	pull-back of c, 5.2
$c_1(t), c_2(t)$	coordinate functions of c, 5.2
T_pM	tangent plane at p, 5.4
n	normal vector to a coordinate patch, 5.4
I_p	first fundamental form at p, 5.5
I	first fundamental form, 5.5
g_{ij}	metric coefficients, 5.5
(g_{ij})	matrix of metric coefficients, 5.5
$\nabla_v f$	directional derivative of f in the direction v, 5.6
$\widetilde{\nabla}_v Z$	directional derivative of Z in the direction v, 5.6
$\nabla_v Z$	covariant derivative of Z with respect to v, 5.6
$\frac{DZ}{dt}$	covariant derivative along a curve, 5.6
$\overline{\nabla}_v X$	covariant derivative, 5.7
Γ_{ij}^k	Christoffel symbols, 5.7

$\mathrm{Area}(S)$	area of a region in a surface, 5.8
df_p	differential of f at p, 5.9
$Tx(U)$	tangent space over $x(U)$, 5.9
$\hat{n}$	Gauss map, 6.1
$\mathrm{Area}_o(\hat{n}(T))$	oriented area of the image of the Gauss map, 6.1
L	Weingarten map, 6.2
II_p	second fundamental form at p, 6.2
II	second fundamental form, 6.2
(L_{ij})	matrix for Weingarten map, 6.2
(l_{ij})	matrix for second fundamental form, 6.2
n_i	partial derivative of n, 6.2
$K(p)$	Gaussian curvature at p, 6.3
$H(p)$	mean curvature at p, 6.3
k_1, k_2	principle curvatures, 6.3
Ω_v	oriented plane generated by v, 6.3
v_Ω	unit vector generated by Ω, 6.3
E, F, G	metric coefficients, 6.4
A, B, C	matrix entries for the second fundamental form, 6.4
ρ_v	bound for the domain of a geodesic, 7.2
$\Lambda(t)$	length of variation of a curve, 7.3
E_p	domain of exponential map, 8.2
$\exp_p$	exponential map, 8.2
δ_p	radius of ball in the domain for the exponential map, 8.2
δ_M	radius of ball in the domain for the exponential map, 8.2
ϵ_M	radius of ball in the image of the exponential map, 8.2
Υ_p	orthogonal map, 8.2
Exp_p	exponential coordinate patch, 8.2
$GS_r(p, M)$	geodesic circle, 8.2
$GO_r(p, M)$	open geodesic ball, 8.2
$\overline{GO_r}(p, M)$	closed geodesic ball, 8.2
rect	polar to rectangular map, 8.3
D_p	geodesic coordinate patch, 8.3
$(D_p)_i$	partial derivative of the geodesic coordinate patch, 8.3
$\overline{E}, \overline{F}, \overline{G}$	metric coefficients of Exp_p, 8.3
$\gamma_\theta(s)$	geodesic ray, 8.3
$\alpha_R(t)$	geodesic circle, 8.3
L_r	length of geodesic circle of radius r, 8.3
γ_p	radius of convex geodesic ball, A8.2

Index

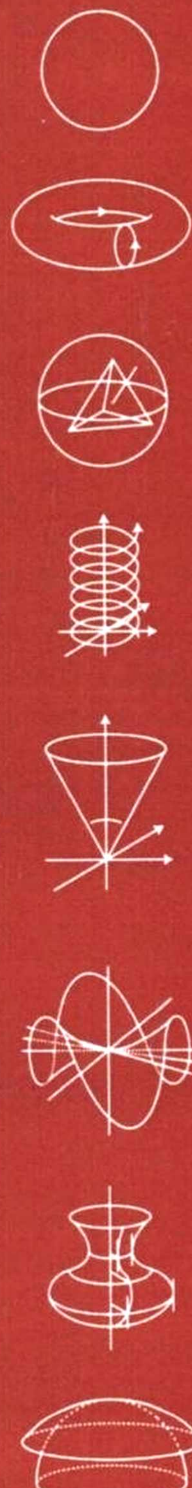
Ethan D. Bloch

A First Course in Geometric Topology and Differential Geometry

The uniqueness of this text in combining geometric topology and differential geometry lies in its unifying thread: the notion of a surface. With numerous illustrations, exercises and examples, the student comes to understand the relationship of the modern abstract approach to geometric intuition. The text is kept at a concrete level, avoiding unnecessary abstractions, yet never sacrificing mathematical rigor. The book includes topics not usually found in a single book at this level.

A number of intuitively appealing definitions and theorems concerning surfaces in the topological, polyhedral and smooth cases are presented from the geometric view. Point set topology is restricted to subsets of Euclidean spaces. The treatment of differential geometry is classical, dealing with surfaces in $\mathbf{R}^3$. Included are the classification of compact surfaces, the Gauss-Bonnet Theorem and the geodesic nature of length minimizing curves on surfaces.

The material here should be accessible to math majors at the junior/senior level in an American university or college, the minimal prerequisites being standard Calculus sequence (including multivariable Calculus and an acquaintance with differential equations), linear algebra (including inner products), and familiarity with proofs and the basics of sets and functions.



ISBN 0-8176-3840-7
90000
9 780817 638405
ISBN 0-8176-3840-7