APMA 4301: Problem Set 5

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1.

(a) Butcher Tableaus

Forward Euler

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 0 & 1 \end{array} \tag{1}$$

Backward Euler

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 0 & 1 \\ \hline & 0 & 1 \end{array} \tag{2}$$

Mid-Point

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \tag{3}$$

Improved Euler (RK2)

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{4}$$

Trapezoidal

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 0 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{5}$$

Classical 4th order Runge-Kutta (RK4)

$$\begin{array}{c|cccc} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \tag{6}$$

TR-BDF2

This is a 2-step scheme first using the trapezoidal rule and then the BDF2 method.

$$u_{n+1/2} = u_n + k/4[f(u_n) + f(u_{n+1/2})] \tag{7}$$

$$3u_{n+1} - 4u_{n+1/2} + u_n = kf(u_{n+1}) \tag{8}$$

(b) Explicit Form

Forward Euler

$$u_1(k) = (1+z)u_0 \tag{9}$$

Backward Euler

$$\begin{aligned} u_1(k) &= u_0 + zu_1(k) \\ u_1(k) &= \frac{1}{1-z}u_0 \end{aligned} \tag{10}$$

Mid-Point

$$u_1(k) = (1 + z + z^2/2)u_0 \tag{11}$$

Improved Euler (RK2)

$$u_1(k) = (1 + z + z^2/2)u_0 \tag{12}$$

Trapezoidal

$$\begin{aligned} u_1(k) &= u_0 + z/2(u_0 + u_1(k)) \\ u_1(k) &= \frac{1+z/2}{1-z/2}u_0 \end{aligned} \tag{13}$$

Classical 4th order Runge-Kutta (RK4)

$$u_1(k) = (1 + z + z^2/2 + z^3/3! + z^4/4!)u_0 \tag{14}$$

TR-BDF2

$$u_{1/2} = \frac{1 + z/4}{1 - z/4} u_0 \tag{15}$$

$$3u_1 - 4\frac{1 + z/4}{1 - z/4} u_0 + u_0 = z u_1 \tag{16}$$

$$u_1 = \frac{1 + 5z/12}{1 - 7z/12 + z^2/12} u_0 \tag{17}$$

(c) Step Error

Forward Euler

$$e^z - R(z) = z^2/2 + O(z^3) \tag{18}$$

Backward Euler

$$\begin{aligned} e^z - R(z) &= (1 + z + z^2/2 + \dots) - (1 + z + z^2 + \dots) \\ &= -z^2/2 + O(z^3) \end{aligned} \tag{19}$$

Mid-Point

$$e^z - R(z) = z^3/6 + O(z^4) \tag{20}$$

Improved Euler (RK2)

$$e^z - R(z) = z^3/6 + O(z^4) \tag{21}$$

Trapezoidal

$$\begin{aligned} e^z - R(z) &= (1 + z + z^2/2 + z^3/6 \dots) - (1 + z + z^2/2 + z^3/4 \dots) \\ &= -z^3/12 + O(z^4) \end{aligned} \tag{22}$$

Classical 4th order Runge-Kutta (RK4)

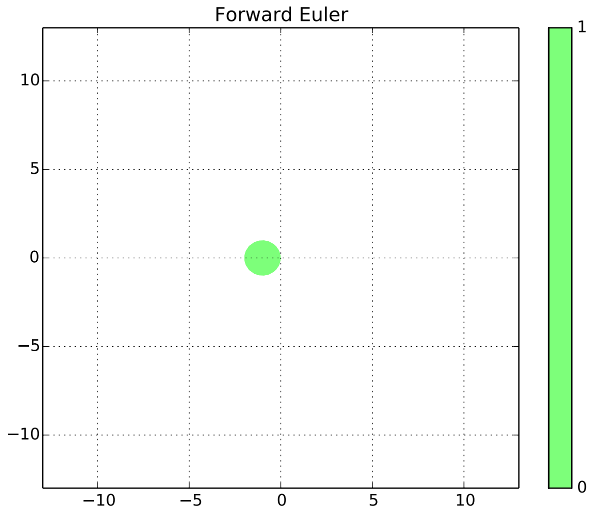
$$e^z - R(z) = z^5/5! + O(z^6) \tag{23}$$

TR-BDF2

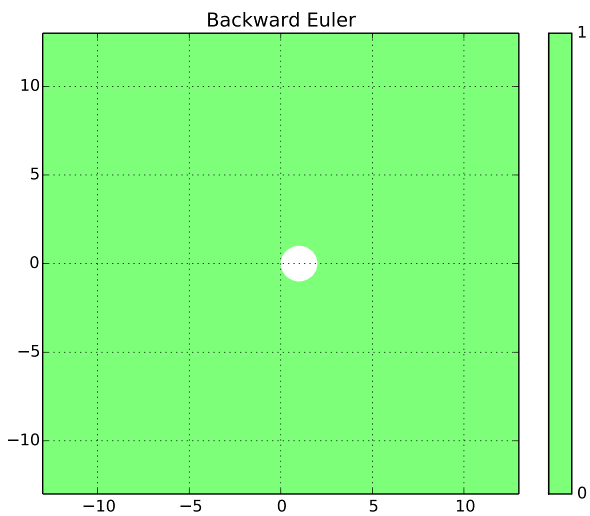
$$\begin{aligned} e^z - R(z) &= (1 + z + z^2/2 + z^3/6 \dots) - (1 + z + z^2/2 + 5z^3/24 \dots) \\ &= z^3/24 + O(z^4) \end{aligned} \tag{24}$$

(d) Stability Plots

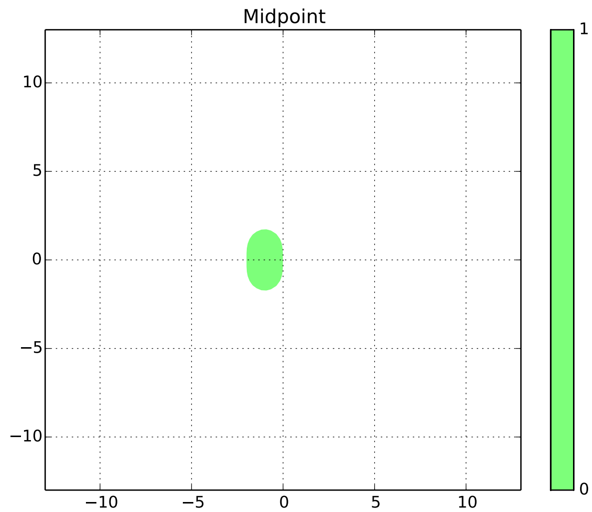
Forward Euler



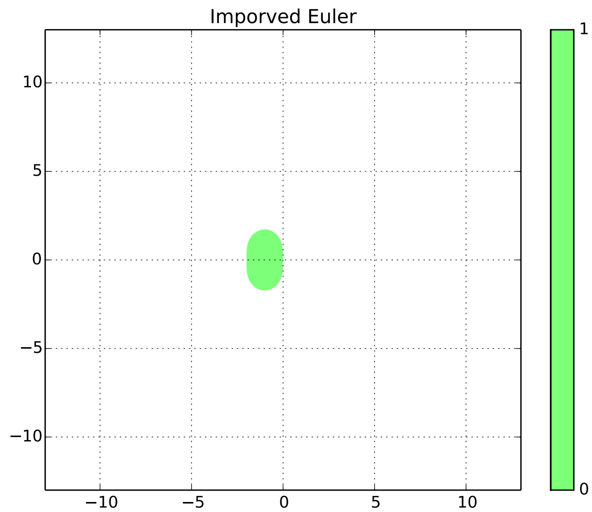
Backward Euler



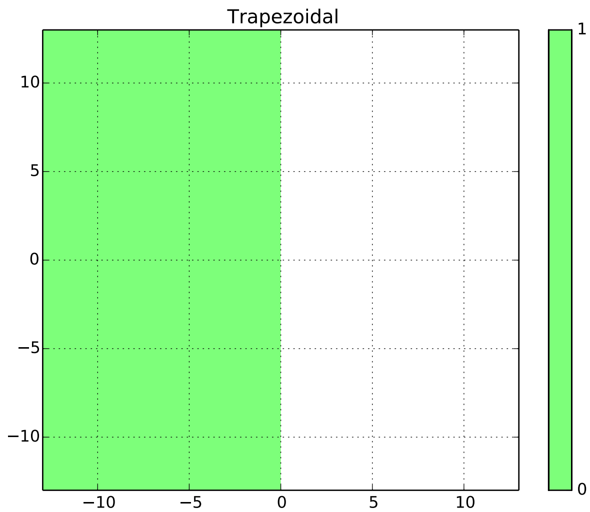
Mid-Point



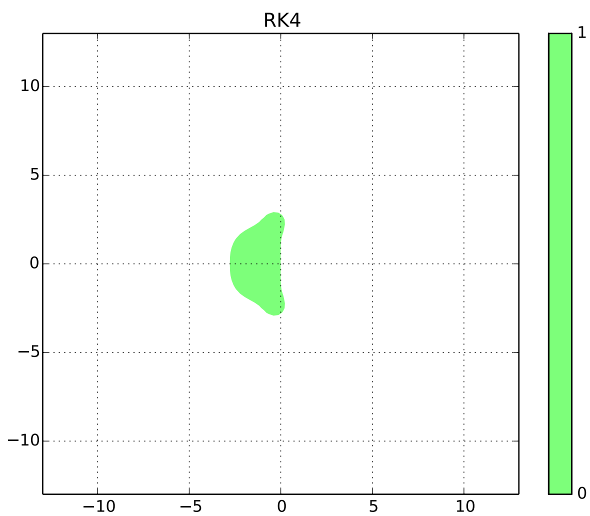
Improved Euler (RK2)



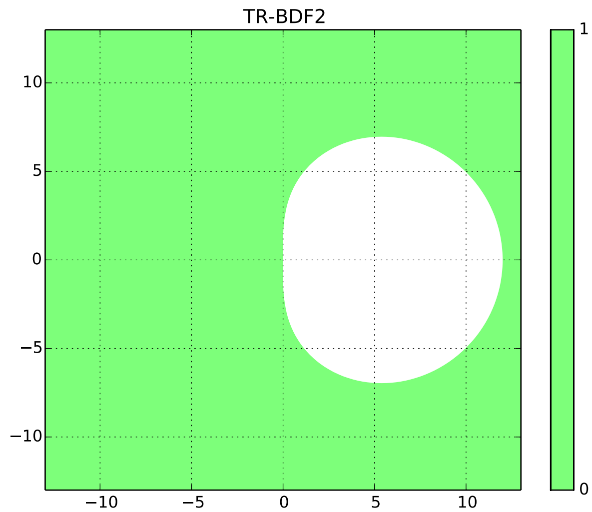
Trapezoidal



Classical 4th order Runge-Kutta (RK4)



TR-BDF2



(e) Stable Time Steps

Forward Euler

$$|1+\lambda k_{max}| = 1 \implies k_{max} = -\frac{2}{\lambda} \tag{25}$$

Backward Euler

$$|\frac{1}{1-\lambda k_{max}}| = 1 \implies k_{maz} = \infty \tag{26}$$

Mid-Point

$$|1+\lambda k_{max} + (\lambda k_{max})^2/2| = 1 \implies k_{max} = -\frac{2}{\lambda} \tag{27}$$

Improved Euler (RK2)

$$|1+\lambda k_{max} + (\lambda k_{max})^2/2| = 1 \implies k_{max} = -\frac{2}{\lambda} \tag{28}$$

Trapezoidal

$$|\frac{1+\lambda k_{max}/2}{1-\lambda k_{max}/2}| = 1 \implies k_{maz} = \infty \tag{29}$$

Classical 4th order Runge-Kutta (RK4)

$$|1+\lambda k_{max}+(\lambda k_{max})^2/2+(\lambda k_{max})^3/6+(\lambda k_{max})^4/24| = 1 \implies k_{max} \approx -\frac{-2.7853}{\lambda} \tag{30}$$

TR-BDF2

$$\frac{1+5\lambda k/12}{1-7\lambda k/12+(\lambda k)^2/12} = 1 \implies k_{max} = \infty \tag{31}$$

2.

$$\frac{\partial T}{\partial t} = \nabla^2 T; \quad T(\mathbf{x},0) = A\exp\left[\frac{-\mathbf{x}^T\mathbf{x}}{\sigma^2}\right] \tag{32}$$

(a)

$$T(\mathbf{x},t) = \frac{A}{1+4t/\sigma^2}\exp\left[\frac{\mathbf{x}^T\mathbf{x}}{\sigma^2+4t}\right] \tag{33}$$

This obviously satisfies the initial condition at $t = 0$.

$$\frac{\partial T}{\partial t} = \frac{A}{1+4t/\sigma^2}\left[\frac{4\mathbf{x}^T\mathbf{x}}{(\sigma^2+4t)^2} - \frac{4}{\sigma^2+4t}\right]\exp\left[\frac{\mathbf{x}^T\mathbf{x}}{\sigma^2+4t}\right] = \nabla^2 T \tag{34}$$

(b)

i.

$$\begin{aligned} F(u_i) &= \int_\Omega u_t[u_i - u_n + (1-\theta)\nabla^2 u_n + \theta\nabla u_i^2]\,d\mathbf{x} \\ &= \int_\Omega u_t(u_i-u_n) + k\nabla u_t\cdot[(1-\theta)\nabla u_n + \theta\nabla u_i]\,d\mathbf{x} \end{aligned} \tag{35}$$

Note that the surface terms are 0 due to the Neumann boundary condition.

In UFL, this can be written as:

```
F = (k*inner(grad(u_t),theta*grad(u_i)
    +(1-theta)*grad(u_n))+inner(u_t,(u_i-u_n)))*dx
```

ii.

Differentiating F, we get the Jacobian:

$$J(u_i) = u_t - k\theta\nabla^2 u_t \tag{36}$$

after using the identity $\frac{\partial}{\partial u}(f\cdot\nabla u) = -\nabla f$.

In UFL this can be done implicitly with:

```
J = derivative(F,us_i,us_a)
```

iii.

The file `diffusion.tfml` solves this problem with $\theta = 0$. The file `diffusion.shml` loops over the problem for $\theta = 0, 0.5, 1$. The L1 error functional is included in the diagnostics.

iv.

On a 100 point mesh, the highest frequency possible is $1/200\pi$. From part 1, the forward Euler scheme should be stable for $k = 2/200\pi \approx= 0.003$. However, the forward Euler scheme is unstable for this timestep and for every other timestep I tried.

v.

For $\theta = 1$, we have an L1 error of 8.7694862558e-05 at $t = 0.012$ and $k = 0.0002$. For $\theta = 0.5$, we have an L1 error of 3.3144604514e-05 at $t = 0.012$ and $k = 0.0002$. Since the step error of the backwards Euler method is second order, we can find the appropriate timestep by scaling the current timestep by the square of the ratio of the errors. Doing so, we get $k = 0.0000285$. Using this timestep, we get an error of 3.1004511669e-05, which is actually slightly better than the trapezoidal scheme.