

APMA 4301: Problem Set 5

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1.

(a)

Forward Euler

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 1 & 0 & \\ \hline & 0 & 1 & \end{array} \quad (1)$$

Backward Euler

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 0 & 1 & \\ \hline & 0 & 1 & \end{array} \quad (2)$$

Mid-Point

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{1}{2} & \frac{1}{2} & 1 & \\ \hline & 0 & 1 & \end{array} \quad (3)$$

Improved Euler (RK2)

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 1 & 0 & \\ \hline & \frac{1}{2} & \frac{1}{2} & \end{array} \quad (4)$$

Trapezoidal

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 0 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} & \end{array} \quad (5)$$

Classical 4th order Runge-Kutta (RK4)

$$\begin{array}{c|cccc}
 0 & 0 & & & \\
 \frac{1}{2} & \frac{1}{2} & 0 & & \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & \\
 1 & 0 & 0 & 1 & 0 \\
 \hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
 \end{array} \tag{6}$$

TR-BDF2

(b)

Forward Euler

$$u_1(k) = (1 + z)u_0 \tag{7}$$

Backward Euler

$$\begin{aligned}
 u_1(k) &= u_0 + zu_1(k) \\
 u_1(k) &= \frac{1}{1 - z}u_0
 \end{aligned} \tag{8}$$

Mid-Point

$$\begin{aligned}
 u_1(k) &= u_0 + z/2u_0(k) + zu_1(k) \\
 u_1(k) &= \frac{1 + z/2}{1 - z}u_0
 \end{aligned} \tag{9}$$

Improved Euler (RK2)

$$u_1(k) = (1 + z + z^2/2)u_0 \tag{10}$$

Trapezoidal

$$\begin{aligned}
 u_1(k) &= u_0 + z/2(u_0 + u_1(k)) \\
 u_1(k) &= \frac{1 + z/2}{1 - z/2}u_0
 \end{aligned} \tag{11}$$

Classical 4th order Runge-Kutta (RK4)

$$\begin{aligned}
 u_1(k) &= (1/6 + 1/3 + z/6 + 1/3 + z/6 + z^2/12 + 1/6 + z/6 + z^2/12 + z^3/12)u_0 \\
 &= (1 + z/2 + z^2/6 + z^3/12)u_0
 \end{aligned} \tag{12}$$

TR-BDF2

(c)

Forward Euler

$$e^z - R(z) = z^2/2 + O(z^3) \quad (13)$$

Backward Euler

$$\begin{aligned} e^z - R(z) &= (1 + z + z^2/2 + \dots) - (1 + z + z^2 + \dots) \\ &= -z^2/2 + O(z^3) \end{aligned} \quad (14)$$

Mid-Point

Improved Euler (RK2)

$$e^z - R(z) = z^3/6 + O(z^4) \quad (15)$$

Trapezoidal

$$\begin{aligned} e^z - R(z) &= (1 + z + z^2/2 + z^3/6 \dots) - (1 + z + z^2/2 + z^3/4 \dots) \\ &= -z^3/12 + O(z^4) \end{aligned} \quad (16)$$

Classical 4th order Runge-Kutta (RK4)

$$e^z - R(z) = z^5/5! + O(z^6) \quad (17)$$

TR-BDF2

(d)

Forward Euler

Backward Euler

Mid-Point

Improved Euler (RK2)

Trapezoidal

Classical 4th order Runge-Kutta (RK4)

TR-BDF2

(e)

Forward Euler

Backward Euler

Mid-Point

Improved Euler (RK2)

Trapezoidal

Classical 4th order Runge-Kutta (RK4)

TR-BDF2

2.

(a)

(b)

i.

ii.

iii.

iv.

v.