APMA 4301: Problem Set 4

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1.

(a)

i.

The two elements, e_1 and e_2, will have the same Mass Matrix due to the symmetry of the problem. Additionally both diagonal terms will be he same as will the two off-diagonal terms. Thus we only need to evaluate two entries to find the Mass Matrices. The diagonal terms are:

$$\begin{aligned}\int_{e_i} \phi_i^2 \, dx &= \int_0^{\frac{1}{2}} (1-2x)^2 \, dx \\ &= \int_0^{\frac{1}{2}} (4x^2 - 4x + 1) \, dx = \frac{4x^3}{3} - 2x^2 + x \Big|_0^{\frac{1}{2}} = \frac{1}{6}\end{aligned} \tag{1}$$

The off-diagonal terms are:

$$\begin{aligned}\int_{e_i} \phi_i \phi_{i+1} \, dx &= \int_0^{\frac{1}{2}} (1-2x)2x \, dx \\ &= \int_0^{\frac{1}{2}} (-4x^2 + 2x) \, dx = -\frac{4x^3}{3} + x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{12}\end{aligned} \tag{2}$$

The overall mass matrix for each element is thus:

$$\boxed{M_{e_i} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}} \tag{3}$$

Let $\vec{f}^{(1)}$ and $\vec{f}^{(2)}$ be the element forcing vectors for e_1 and e_2. $\vec{b}^{(1)}$ can be calculated as:

$$\begin{aligned}f_1^{(1)} &= \int_{e_1} f\phi_1 \, dx = \int_0^{\frac{1}{2}} x(1-x)(1-2x)/2 \, dx \\ &= \int_0^{\frac{1}{2}} \left(x^3 - \frac{3x^2}{2} + \frac{x}{2}\right) dx = \frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\Big|_0^{\frac{1}{2}} = \frac{1}{64}\end{aligned} \tag{4}$$

$$
\begin{aligned}
f_2^{(1)} &= \int_{e_1} f\phi_2\,dx = \int_0^{\frac{1}{2}} x(1-x)x\,dx \\
&= \int_0^{\frac{1}{2}} \left(-x^3 + \frac{x}{2}\right)\,dx = -\frac{x^4}{4} - \frac{x^3}{3}\bigg|_0^{\frac{1}{2}} = \frac{5}{192}
\end{aligned} \tag{5}
$$

$\vec{f}^{(2)}$ will simply be $\vec{f}^{(1)}$ but with the values in reverse order due to the symmetry of the problem. So our element forcing vectors are:

$$
\boxed{\vec{f}^{(1)} = \begin{bmatrix} \frac{1}{64} \\ \frac{5}{192} \end{bmatrix}, \quad \vec{f}^{(2)} = \begin{bmatrix} \frac{5}{192} \\ \frac{1}{64} \end{bmatrix}} \tag{6}
$$

ii.

The Mass Matrix for the entire system is:

$$
\boxed{M = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{6} \end{bmatrix}} \tag{7}
$$

and the element forcing vector for the system is:

$$
\boxed{\vec{f} = \begin{bmatrix} \frac{1}{64} \\ \frac{5}{96} \\ \frac{1}{64} \end{bmatrix}} \tag{8}
$$

iii.

$$
M\vec{w} = \vec{f} \implies \vec{w} = M^{-1}\vec{f} \tag{9}
$$

$$
M^{-1} = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 7 \end{bmatrix} \tag{10}
$$

$$
\vec{w} = M^{-1}\vec{f} = \boxed{\begin{bmatrix} \frac{1}{48} \\ \frac{7}{48} \\ \frac{1}{48} \end{bmatrix}} \tag{11}
$$

iv.

$$
||e||_{L2} = \left[\int_0^1 (\tilde{f} - f)^2 dx\right]^{1/2} \tag{12}
$$

Since the integrand is symmetric we only have to find the integral from 0 to $\frac{1}{2}$ and double it. In this interval, we have $\tilde{f} = \frac{1}{48}(14x + 1 - 2x) = \frac{1}{48}(12x + 1)$.

So now:

$$||e||_{L2} = \left(2\int_0^{\frac{1}{2}} \left[\frac{1}{48}(12x+1) - x(1-x)/2 \right]^2 dx \right)^{1/2} \approx 0.0093 \tag{13}$$

(b)

Since the problem is quadratic, we will be able to exactly represent it with a quadratic P2. We can find the Galerkin projection then by matching the function at 3 points. We see $f(0) = f(1) = 0$ and $f(\frac{1}{2}) = \frac{1}{8}$ so our projection should be $\tilde{f} = 0N_1 + \frac{1}{8}N_2 + 0N_3$ as N_i is 1 on the i-th point and 0 elsewhere. We see that this gives us $\tilde{f} = f$ exactly! This also implies that our L2-error is 0 as the integrand is always 0.

(c)

First, we can easily see that the analytic solution to this problem is:

$$u = \frac{-x^2 + 3x}{2} \tag{14}$$

Now we can solve using the Galerkin projection for the P1 element case.

$$\int vr = \int v \left(\frac{d^2u}{dx^2} - f \right) dx = 0 \tag{15}$$

$$\int v \frac{d^2u}{dx^2} dx = \int vf dx \tag{16}$$

$$\int \frac{du}{dx}\frac{dv}{dx} dx + v\frac{du}{dx}|_{\partial\Omega} = \int vf dx \tag{17}$$

Now substitute $u = \sum_j w_j \phi_j$, $v = \phi_i$ and switch the sum and integral:

$$\sum_j w_j \int \frac{d\phi_j}{dx}\frac{d\phi_i}{dx} dx = \int \phi_i f dx \tag{18}$$

$$a\vec{w} = \vec{f} \tag{19}$$

We can find a by looking at the contribution of each element. Once again, this will be symmetric so we can just look at the first element:

$$a_{11} = \int_0^{\frac{1}{2}} -2(-2)dx = 2 \tag{20}$$

$$a_{21} = \int_0^{\frac{1}{2}} -2(2d)x = -2 \tag{21}$$

So the overall matrix is:

$$a = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \tag{22}$$

Our $\vec{f}$ can be simply solved for geometrically as f is 1 so they are simply the areas of the basis functions:

$$\vec{f} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \tag{23}$$

Due to our boundary conditions, we already know 2 values of $\vec{w}$:

$$\vec{w} = \begin{bmatrix} 0 \\ w_2 \\ 1 \end{bmatrix} \tag{24}$$

Therefore the only constraint is from the middle row of a:

$$0 + 4w_2 - 2 = \frac{1}{2} \implies w_2 = \frac{5}{8} \tag{25}$$

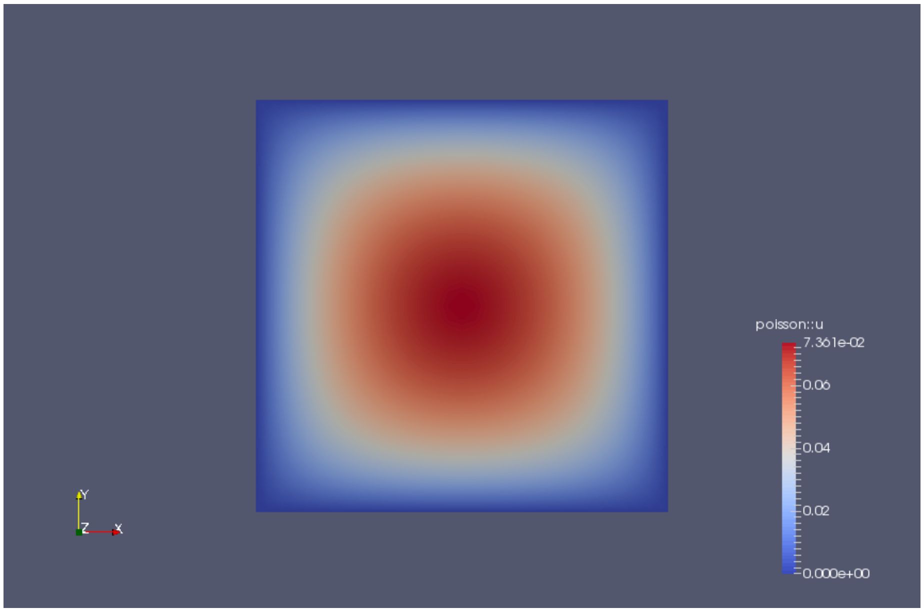
So now we can calculate the error:

$$\begin{aligned} ||e||_{L2} = \Big(&\int_0^{\frac{1}{2}} \left[\frac{5x}{4} - \frac{-x^2 + 3x}{2} \right]^2 dx \\ &+ \int_{\frac{1}{2}}^1 \left[2x - 1 + \frac{5x}{4} - \frac{-x^2 + 3x}{2} \right]^2 dx \Big)^{1/2} \approx 0.091 \end{aligned} \tag{26}$$

For the P2 case, we are once again trying to represent a quadratic function with a quadratic basis so our projection will be exact and we will have 0 error.

2.

(a)

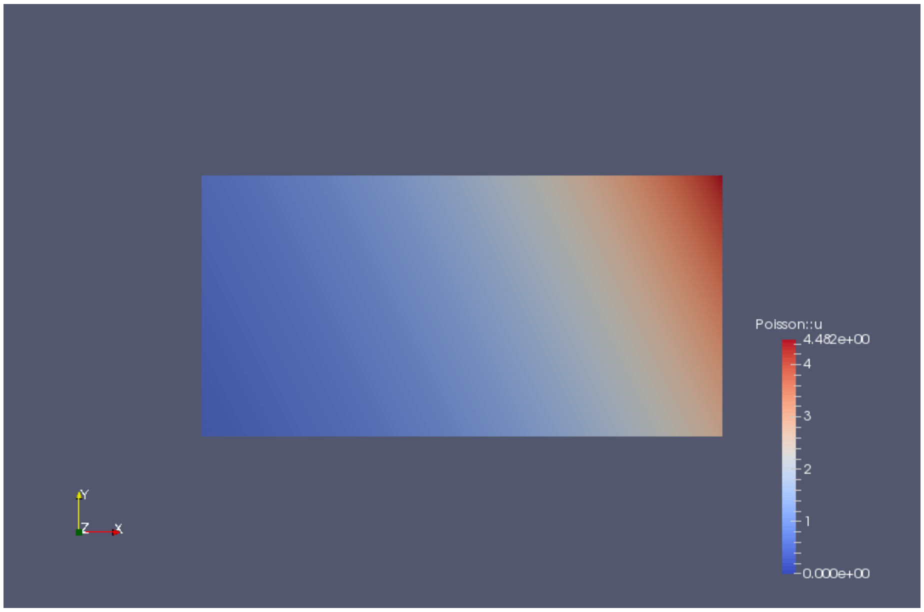


This image shows the solution to the problem generated by `2a/poisson.tfml`.

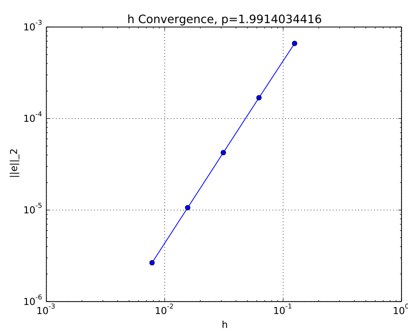
(b)

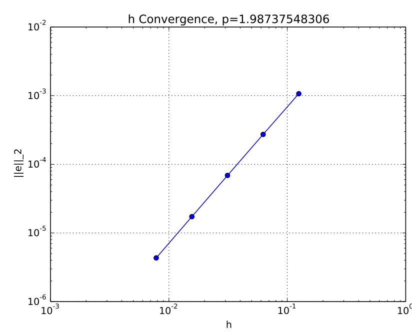
i.

To change this program I simply modified the mesh geometry to be a rectangle with the given dimensions in the `tfml` file. The spud path in the `shml` file also needed to be modified to a rectangular mesh.



This image shows the solution to the problem on the rectangle generated by `2bi/poisson.tfml`.

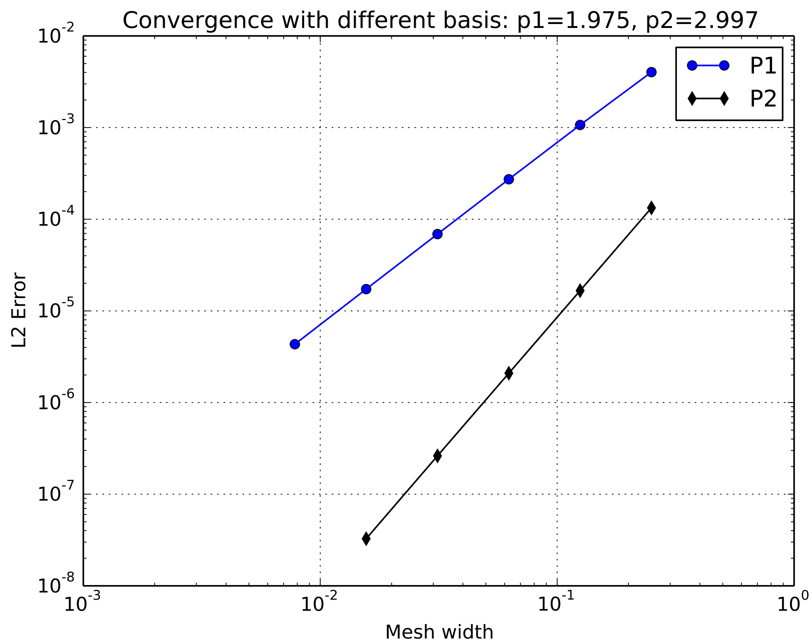




The left plot shows the convergence on the rectangle and was generated by `2bi/poisson_mms.shml`. The right plot shows the convergence on the unit square and was generated by the `poisson_mms.shml` file contained in the class src repository.

From these two images, we see that both problems converge approximately at the same rate with $p = 1.99$. However, for each h value, the error on the unit square is slightly larger. This is because the rectangle has twice the area of the unit square and thus for each h value it has twice as many degrees of freedom and thus has a smaller error.

ii.

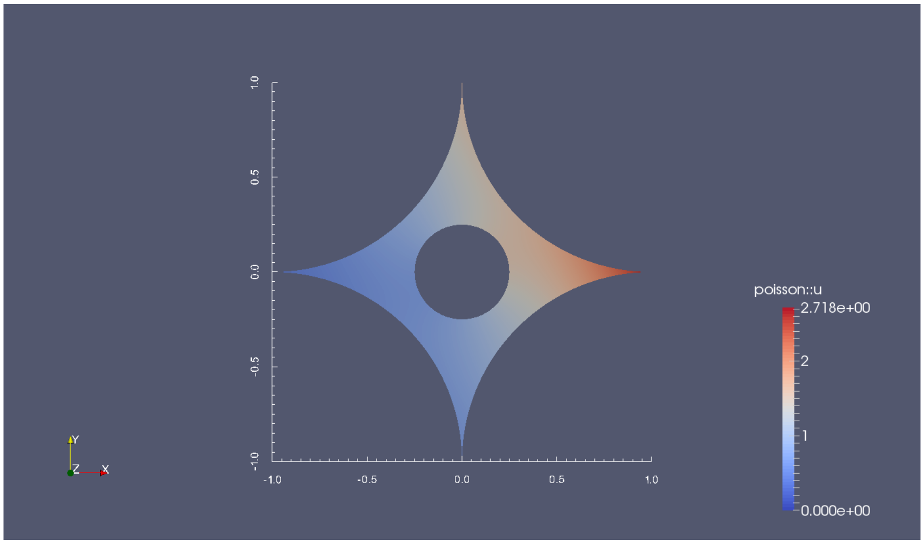


This image shows the convergence of the solution with P1 and P2 elements. We see that the P2 elements converge much more rapidly as a function of h.

For P1 elements, the error falls below $1e-6$ around $h = 4e-3$ which corresponds to 62500 elements. With P2 elements, we need only $h = 5e-2$ which corresponds to 400 elements. For P1 elements, there is roughly 1 degree of freedom per element so we have 62500. However for P2 elements, there are roughly 2 degrees of freedom per element so there are about 800 degrees of freedom. The $h = \frac{1}{128}$ P1 case has a wall-time of 4.96 while the $h = \frac{1}{16}$ P2 case has a wall-time of 6.91 and both have approximately the same error. This indicates that simply decreasing h is quicker than increasing p.

iii.

I did the second option for this problem. My domain was bounded by the points (1,0), (0,1), (-1,0), and (0,-1) connected by quarter circles and contained a circular hole at the origin with radius $\frac{1}{4}$. This mesh is described by the file `2biii/star.geo` and is run by `2biii/poisson.tfml`. The output is shown in the image below.


poisson::u
2.718e+00
2
1
0.000e+00
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0
Y
Z
X