APMA 4301: Problem Set 4

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October 21, 2014

1.

(a)

i.

The two elements, e_1 and e_2, will have the same Mass Matrix due to the symmetry of the problem. Additionally both diagonal terms will be he same as will the two off-diagonal terms. Thus we only need to evaluate two entries to find the Mass Matrices. The diagonal terms are:

$$\int_{e_i} \phi_i^2 \, dx = \int_0^{\frac{1}{2}} (1-2x)^2 \, dx \tag{1}$$

$$= \int_0^{\frac{1}{2}} (4x^2 - 4x + 1) \, dx = \frac{4x^3}{3} - 2x^2 + x \Big|_0^{\frac{1}{2}} = \frac{1}{6} \tag{2}$$

The off-diagonal terms are:

$$\int_{e_i} \phi_i \phi_{i+1} \, dx = \int_0^{\frac{1}{2}} (1-2x)2x \, dx \tag{3}$$

$$= \int_0^{\frac{1}{2}} (-4x^2 + 2x) \, dx = -\frac{4x^3}{3} + x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{12} \tag{4}$$

The overall mass matrix for each element is thus:

$$\boxed{M_{e_i} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}} \tag{5}$$

Let $\vec{f}^{(1)}$ and $\vec{f}^{(2)}$ be the element forcing vectors for e_1 and e_2. $\vec{b}^{(1)}$ can be calculated as:

$$f_1^{(1)} = \int_{e_1} f\phi_1 \, dx = \int_0^{\frac{1}{2}} x(1-x)(1-2x)/2 \, dx \tag{6}$$

$$= \int_0^{\frac{1}{2}} \left(x^3 - \frac{3x^2}{2} + \frac{x}{2} \right) dx = \frac{x^4}{4} - \frac{x^3}{2} + x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{64} \tag{7}$$

$$f_2^{(1)} = \int_{e_1} f\phi_2 \, dx = \int_0^{\frac{1}{2}} x(1-x)x \, dx \tag{8}$$

$$= \int_0^{\frac{1}{2}} \left(-x^3 + \frac{x}{2}\right) \, dx = -\frac{x^4}{4} - \frac{x^3}{3}\Big|_0^{\frac{1}{2}} = \frac{5}{192} \tag{9}$$

$\vec{f}^{(2)}$ will simply be $\vec{f}^{(1)}$ but with the values in reverse order due to the symmetry of the problem. So our element forcing vectors are:

$$\boxed{\vec{f}^{(1)} = \begin{bmatrix} \frac{1}{64} \\ \frac{5}{192} \end{bmatrix}, \quad \vec{f}^{(2)} = \begin{bmatrix} \frac{5}{192} \\ \frac{1}{64} \end{bmatrix}} \tag{10}$$

ii.

The Mass Matrix for the entire system is:

$$\boxed{M = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{6} \end{bmatrix}} \tag{11}$$

and the element forcing vector for the system is:

$$\boxed{\vec{f} = \begin{bmatrix} \frac{1}{64} \\ \frac{5}{96} \\ \frac{1}{64} \end{bmatrix}} \tag{12}$$

iii.

$$M\vec{w} = \vec{f} \implies \vec{w} = M^{-1}\vec{f} \tag{13}$$

$$M^{-1} = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 7 \end{bmatrix} \tag{14}$$

$$\vec{w} = M^{-1}\vec{f} = \boxed{\begin{bmatrix} \frac{1}{48} \\ \frac{7}{48} \\ \frac{1}{48} \end{bmatrix}} \tag{15}$$

iv.

$$||e||_{L2} = \left[\int_0^1 (\tilde{f} - f)^2 dx\right]^{1/2} \tag{16}$$

Since the integrand is symmetric we only have to find the integral from 0 to $\frac{1}{2}$ and double it. In this interval, we have $\tilde{f} = \frac{1}{48}(14x + 1 - 2x) = \frac{1}{48}(12x + 1)$.

So now:

$$||e||_{L2} = \left(2\int_0^{\frac{1}{2}} \left[\frac{1}{48}(12x+1) - x(1-x)/2\right]^2 dx\right)^{1/2} \approx 0.0093 \qquad (17)$$

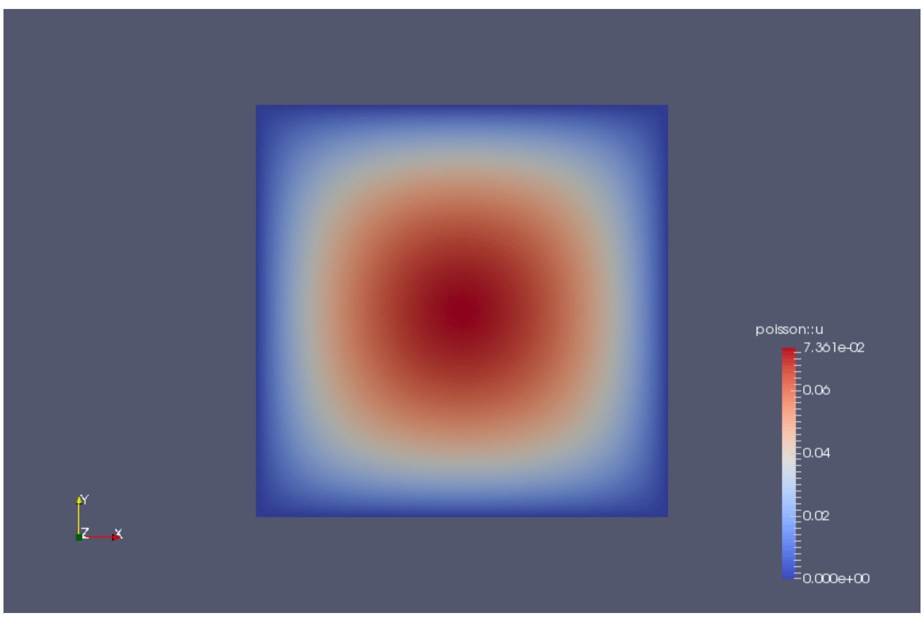
(b)

Since the problem is quadratic, we will be able to exactly represent it with a quadratic P2. We see $f(0) = f(1) = 0$ and $f(\frac{1}{2}) = \frac{1}{8}$ so our function should be $0N_1 + \frac{1}{8}N_2 + 0N_3$ as N_i is 1 on the i-th point and 0 elsewhere. We see that this gives us back f exactly. This also implies that our L2-error is 0 as the integrand is always 0.

(c)

2.

(a)



This image shows the solution to the problem generated by `2a/poisson.tfml`
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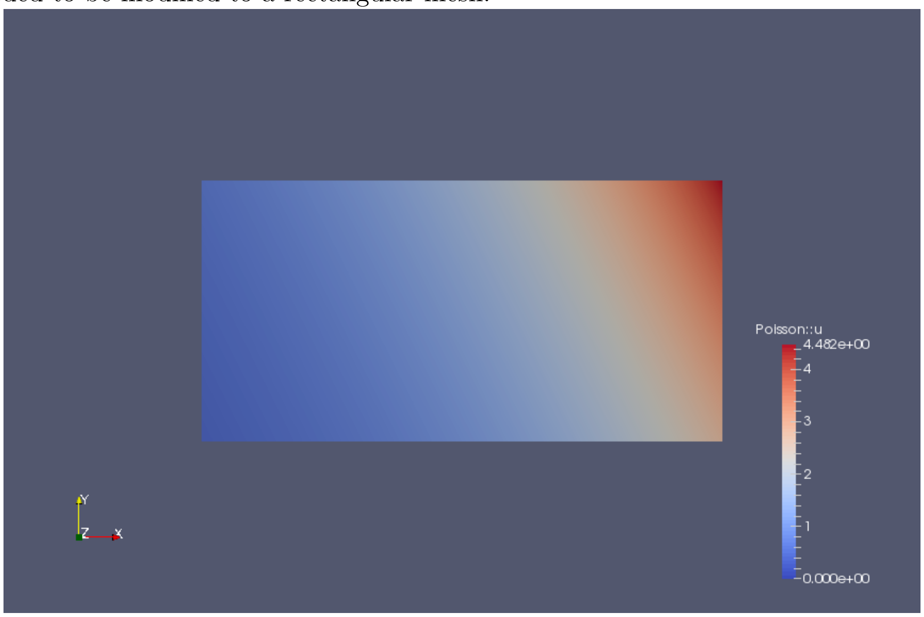
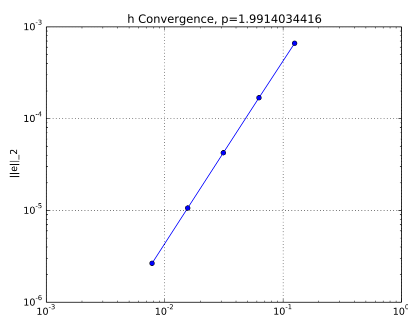
(b)

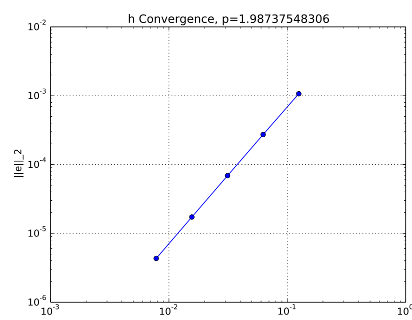
i.

To change this program I simply modified the mesh geometry to be a rectangle with the given dimensions in the `tfml` file. The spud path in the `shml` file also needed to be modified to a rectangular mesh.



This image shows the solution to the problem on the rectangle generated by `2bi/poisson.tfml`.

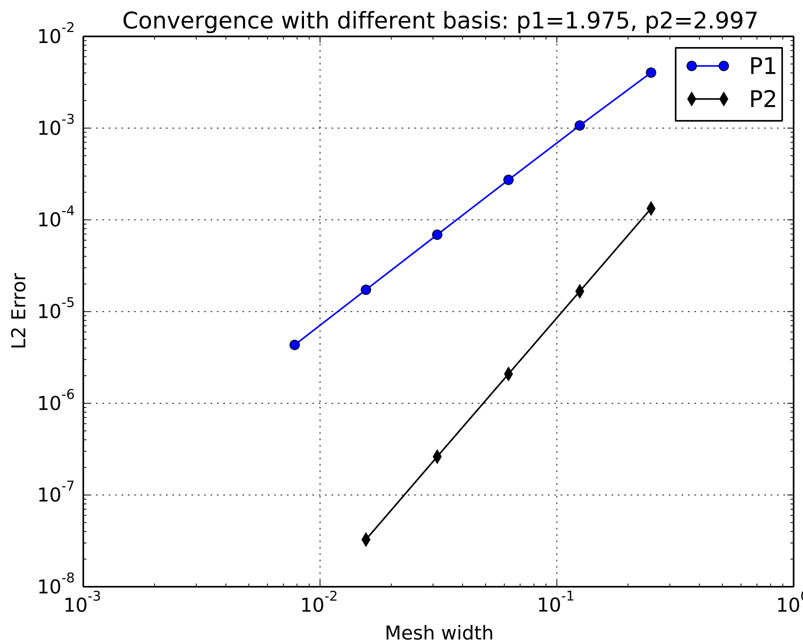




The left plot shows the convergence on the rectangle and was generated by `2bi/poisson_mms.shml`. The right plot shows the convergence on the unit square and was generated by the `poisson_mms.shml` file contained in the class src repository.

From these two images, we see that both problems converge approximately at the same rate with $p = 1.99$. However, for each h value, the error on the unit square is slightly larger. This is because the rectangle has twice the area of the unit square and thus for each h value it has twice as many degrees of freedom and thus has a smaller error.

ii.

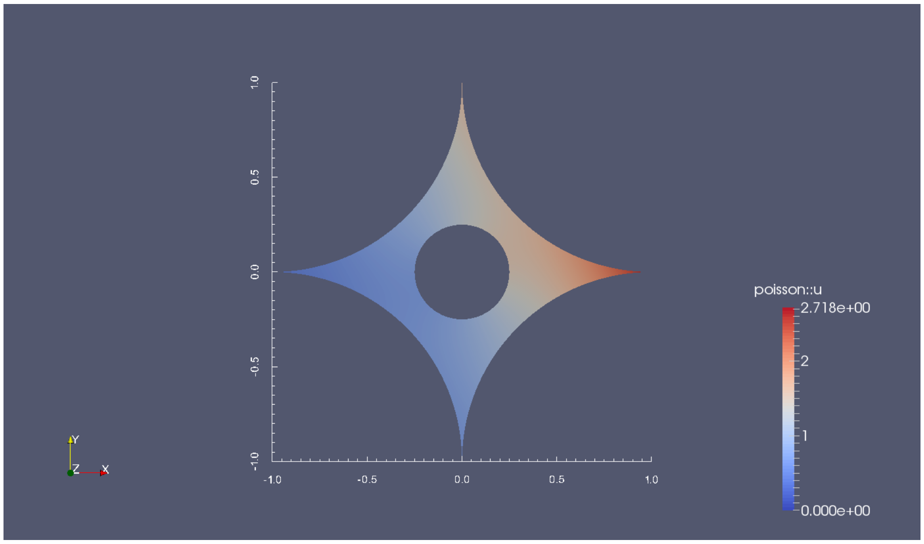


This image shows the convergence of the solution with P1 and P2 elements. We see that the P2 elements converge much more rapidly as a function of h.

For P1 elements, the error falls below $1e-6$ around $h = 4e-3$ which corresponds to 62500 elements. With P2 elements, we need only $h = 5e-2$ which corresponds to 400 elements. For P1 elements, there is roughly 1 degree of freedom per element so we have 62500. However for P2 elements, there are roughly 2 degrees of freedom per element so there are about 800 degrees of freedom. The $h = \frac{1}{128}$ P1 case has a wall-time of 4.96 while the $h = \frac{1}{16}$ P2 case has a wall-time of 6.91 and both have approximately the same error. This indicates that simply decreasing h is quicker than increasing p.

iii.

I did the second option for this problem. My domain was bounded by the points (1,0), (0,1), (-1,0), and (0,-1) connected by quarter circles and contained a circular hole at the origin with radius $\frac{1}{4}$. This mesh is described by the file `2biii/star.geo` and is run by `2biii/poisson.tfml`. The output is shown in the image below.


1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0
poisson::u
2.718e+00
2
1
0.000e+00
Y
Z
X