APMA4301: Problem Set 3

Brian Dawes

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Problem 1

(a)

i.

$$\begin{aligned}\nabla_5^2 u_{ij} = \frac{1}{h^2}\big[&(\sin[n\pi h(i-1)] + \sin[n\pi h(i+1)])\sin(m\pi hj) \\ &+ (\sin[m\pi h(j-1)] + \sin[m\pi h(j+1)])\sin(n\pi hi) \\ &- 4\sin(n\pi hi)\sin(m\pi hj)\big] \end{aligned} \tag{1}$$

But note that

$$\sin[n\pi h(i \pm 1)] = \sin(n\pi hi)\cos(n\pi h) \pm \cos(n\pi hi)\sin(n\pi h) \tag{2}$$

$$\implies \sin[n\pi h(i-1)] + \sin[n\pi h(i+1)] = 2\sin(n\pi hi)\cos(n\pi h) \tag{3}$$

This relationship still holds with $n \to m$ and $i \to j$. So now

$$\begin{aligned}\nabla_5^2 u_{ij} = \frac{1}{h^2}\big[&2\sin(n\pi hi)\cos(n\pi h)\sin(m\pi hj) \\ &+ 2\sin(m\pi hj)\cos(m\pi h)\sin(n\pi hi) \\ &- 4\sin(n\pi hi)\sin(m\pi hj)\big] \\ =& \frac{2}{h^2}[\cos(n\pi h) + \cos(m\pi h) - 2]u_{ij} \end{aligned} \tag{4}$$

$$\implies \boxed{\lambda_{mn} = \frac{2}{h^2}[\cos(n\pi h) + \cos(m\pi h) - 2]} \tag{5}$$

So the discretization of ϕ_{mn} is an eigenvector of A with eigenvalue λ_{mn} for $1 \le m \le \frac{1}{h}$ and $1 \le n \le \frac{1}{h}$ (since $m, n = 0$ corresponds to the zero function and $m, n > \frac{1}{h}$ will reduce to a lower value of m, n when discretized due to aliasing).

ii.

$$||A^{-1}||_2 = \max\left(\frac{1}{|\lambda_{mn|}}\right) = \max\left(\frac{h^2}{2|\cos(n\pi h) + \cos(m\pi h) - 2|}\right) \tag{6}$$

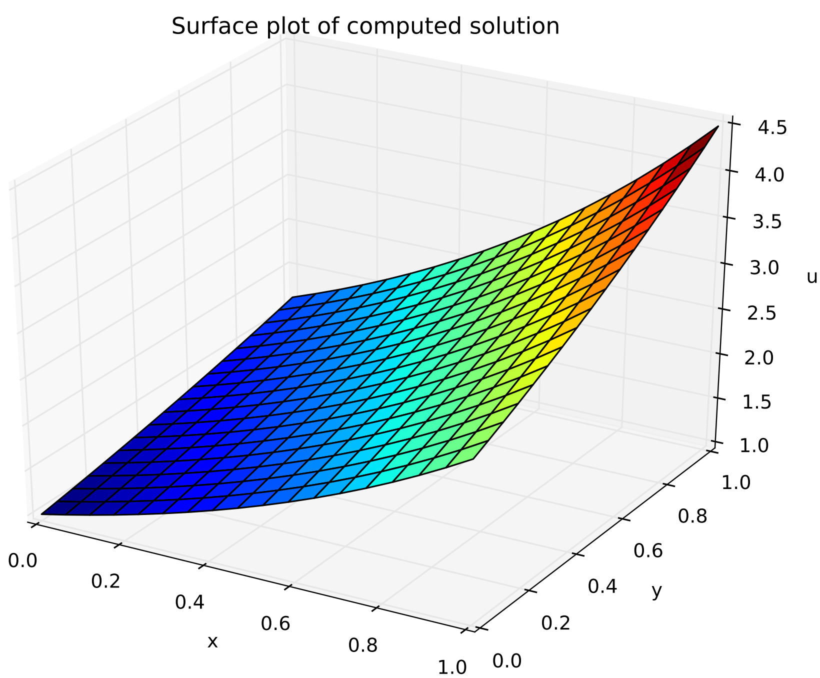
Since m, n range from 1 to $\frac{1}{h}$, the cos terms range from just below 1 to -1. The maximum occurs when the denominator is closest to one which is at $m = n = 1$.

$$||A^{-1}||_2 = \left(\frac{-h^2}{2[\cos(\pi h) + \cos(\pi h) - 2]}\right) \tag{7}$$

$$\begin{aligned} C = \lim_{h\to 0} ||A^{-1}||_2 &= \lim_{h\to 0}\left(\frac{-h^2}{2[\cos(\pi h) + \cos(\pi h) - 2]}\right) \\ &= \lim_{h\to 0}\left(\frac{h}{\pi\sin(\pi h) + \pi\sin(\pi h)}\right) \\ &= \lim_{h\to 0}\left(\frac{1}{\pi^2\cos(\pi h) + \pi^2\cos(\pi h)}\right) \\ &= \boxed{\frac{1}{2\pi^2}} \end{aligned} \tag{8}$$

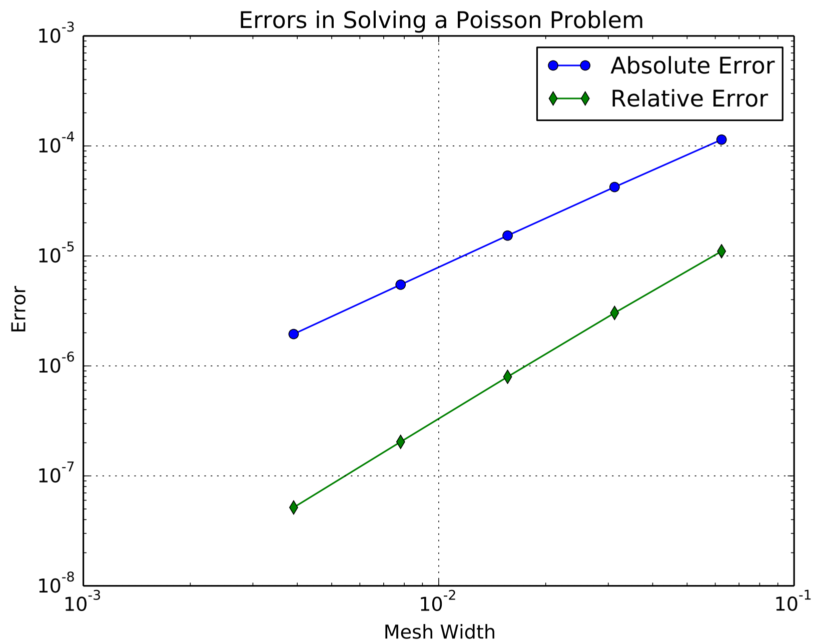
(b)

i.



This image was generated by poisson2d_mms.py using $h = 1/16$.

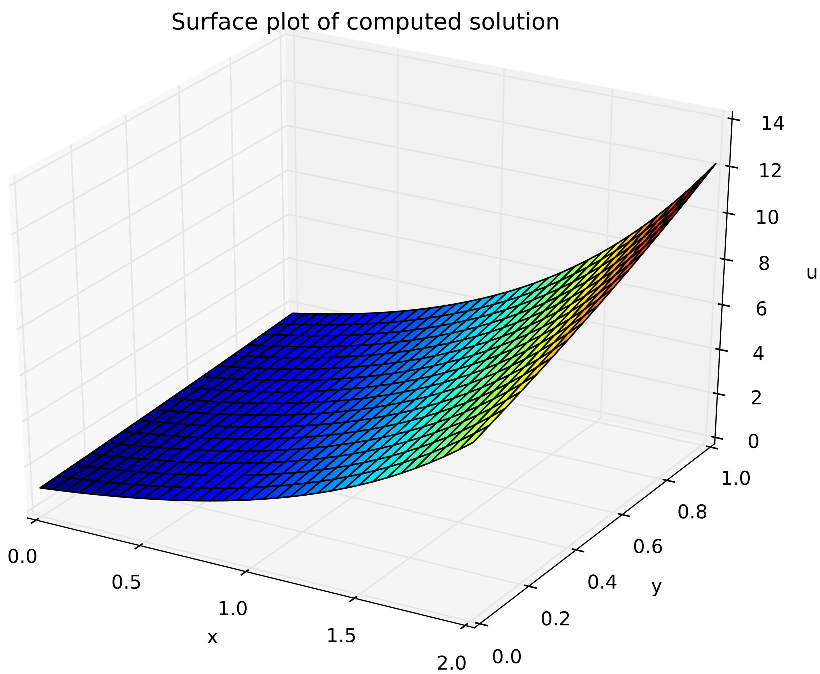
ii.



From this graph, we see the absolute error scales with $h^{1.5}$, while the relative error scales with h^2. Both errors were computed with numpy.linalg.norm($\cdot$,2).

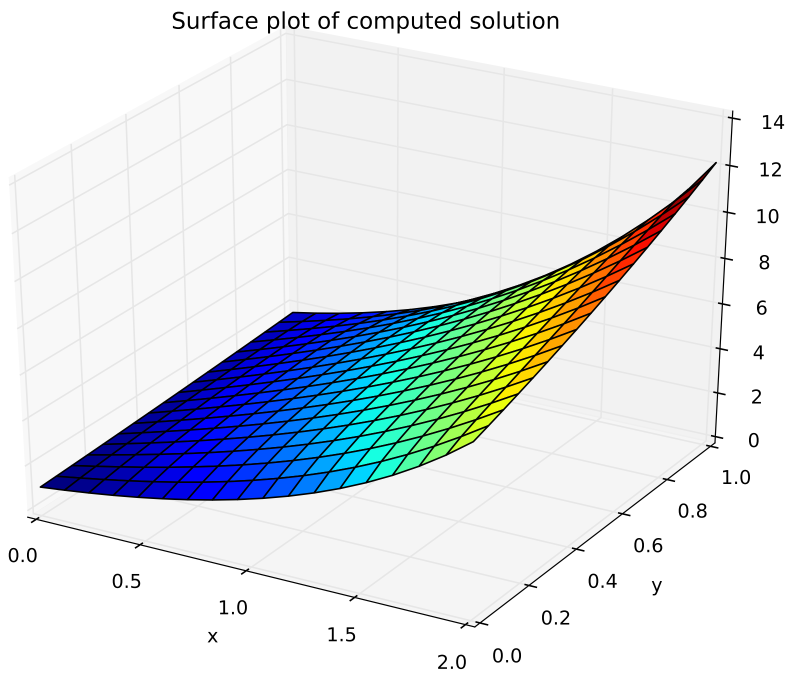
(c)

i.



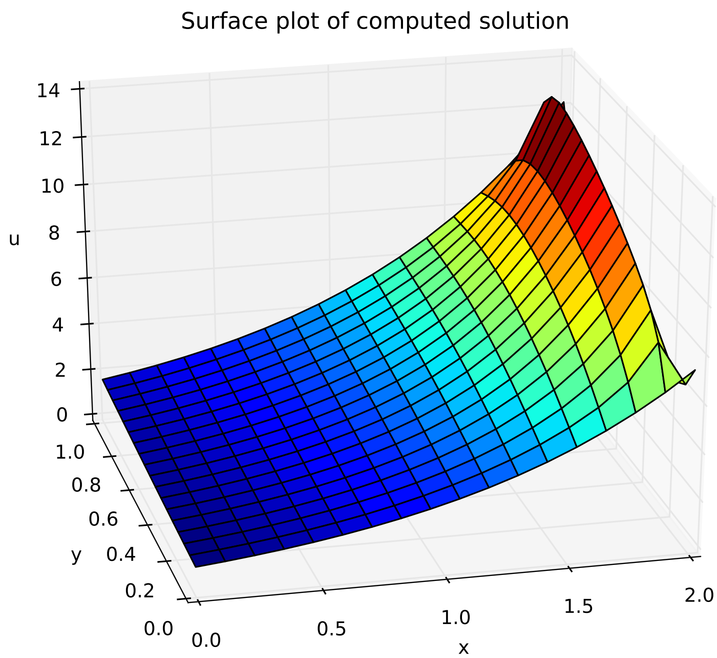
This image was generated by poisson2d_rectangle.py using $h = 1/16$.

ii.



This image was generated by poisson2d_nonuniformMesh.py using $m_x = m_y = 16$.

iii.



This image was generated by poisson2d_neumann.py using $m_x = m_y = 16$. Unfortunately, I was not able to get the neumann boundary condition to work properly. This can be seen in the lip on the $x = 2$ boundary.