

APMA4301: Problem Set 3

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Problem 1

(a)

i.

$$\begin{aligned}\nabla_5^2 u_{ij} = \frac{1}{h^2} & [(\sin[n\pi h(i-1)] + \sin[n\pi h(i+1)]) \sin(m\pi h j) \\ & + (\sin[m\pi h(j-1)] + \sin[m\pi h(j+1)]) \sin(n\pi h i) \\ & - 4 \sin(n\pi h i) \sin(m\pi h j)]\end{aligned}\quad (1)$$

But note that

$$\sin[n\pi h(i \pm 1)] = \sin(n\pi h i) \cos(n\pi h) \pm \cos(n\pi h i) \sin(n\pi h) \quad (2)$$

$$\implies \sin[n\pi h(i-1)] + \sin[n\pi h(i+1)] = 2 \sin(n\pi h i) \cos(n\pi h) \quad (3)$$

This relationship still holds with $n \rightarrow m$ and $i \rightarrow j$. So now

$$\begin{aligned}\nabla_5^2 u_{ij} = \frac{1}{h^2} & [2 \sin(n\pi h i) \cos(n\pi h) \sin(m\pi h j) \\ & + 2 \sin(m\pi h j) \cos(m\pi h) \sin(n\pi h i) \\ & - 4 \sin(n\pi h i) \sin(m\pi h j)] \\ & = \frac{2}{h^2} [\cos(n\pi h) + \cos(m\pi h) - 2] u_{ij}\end{aligned}\quad (4)$$

$$\implies \boxed{\lambda_{mn} = \frac{2}{h^2} [\cos(n\pi h) + \cos(m\pi h) - 2]} \quad (5)$$

So the discretization of ϕ_{mn} is an eigenvector of A with eigenvalue λ_{mn} for $1 \leq m \leq \frac{1}{h}$ and $1 \leq n \leq \frac{1}{h}$ (since $m, n = 0$ corresponds to the zero function and $m, n > \frac{1}{h}$ will reduce to a lower value of m, n when discretized due to aliasing).

ii.

$$\|A^{-1}\|_2 = \max \left(\frac{1}{|\lambda_{mn}|} \right) = \max \left(\frac{h^2}{2|\cos(n\pi h) + \cos(m\pi h) - 2|} \right) \quad (6)$$

Since m, n range from 1 to $\frac{1}{h}$, the cos terms range from just below 1 to -1. The maximum occurs when the denominator is closest to one which is at $m = n = 1$.

$$\|A^{-1}\|_2 = \left(\frac{-h^2}{2[\cos(\pi h) + \cos(\pi h) - 2]} \right) \quad (7)$$

$$\begin{aligned} \lim_{h \rightarrow 0} \|A^{-1}\|_2 &= \lim_{h \rightarrow 0} \left(\frac{-h^2}{2[\cos(\pi h) + \cos(\pi h) - 2]} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{h}{\pi \sin(\pi h) + \pi \sin(\pi h)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{\pi^2 \cos(\pi h) + \pi^2 \cos(\pi h)} \right) \\ &= \frac{1}{2\pi^2} \end{aligned} \quad (8)$$