APMA 4301: Problem Set 2

Brian Dawes

September 21, 2014

Problem 1

a)

Using the Taylor expansion, we can write:

$$u(x) = u(\bar{x}) + (x-\bar{x})u'(\bar{x}) + \frac{(x-\bar{x})^2}{2}u''(\bar{x}) + \frac{(x-\bar{x})^3}{6}u^{(3)}(\bar{x}) \tag{1}$$

Plugging in $x = 0, h, 2h$, we get:

$$u_0 \approx u(\bar{x}) - \bar{x}u'(\bar{x}) + \frac{\bar{x}^2}{2}u''(\bar{x}) - \frac{\bar{x}^3}{6}u^{(3)}(\bar{x}) \tag{2}$$

$$u_1 \approx u(\bar{x}) + (h-\bar{x})u'(\bar{x}) + \frac{(h-\bar{x})^2}{2}u''(\bar{x}) + \frac{(h-\bar{x})^3}{6}u^{(3)}(\bar{x}) \tag{3}$$

$$u_2 \approx u(\bar{x}) + (2h-\bar{x})u'(\bar{x}) + \frac{(2h-\bar{x})^2}{2}u''(\bar{x}) + \frac{(2h-\bar{x})^3}{6}u^{(3)}(\bar{x}) \tag{4}$$

where $u_0 = u(0)$, $u_1 = u(h)$, and $u_2 = u(2h)$.

Now we want to find stencil weights s_0, s_1, s_2 and s'_0, s'_1, s'_2 such that:

$$s_0u_0 + s_1u_1 + s_2u_2 = u'(\bar{x}) \tag{5}$$

$$s'_0u_0 + s'_1u_1 + s'_2u_2 = u''(\bar{x}) \tag{6}$$

This can be represented in matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ -\bar{x} & h-\bar{x} & 2h-\bar{x} \\ \bar{x}^2/2 & (h-\bar{x})^2/2 & (2h-\bar{x})^2/2 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

which can be solved from the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -\bar{x} & h-\bar{x} & 2h-\bar{x} & 1 & 0 \\ \bar{x}^2/2 & (h-\bar{x})^2/2 & (2h-\bar{x})^2/2 & 0 & 1 \end{bmatrix} \tag{8}$$

Case 1: $\bar{x} = 0$

Our matrix becomes:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & h & 2h & 1 & 0 \\ 0 & h^2/2 & 2h^2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & h & 2h & 1 & 0 \\ 0 & 0 & h^2 & -h/2 & 1 \end{bmatrix}$$

$$\rightarrow s_2 = -1/2h \text{ and } s_2' = 1/h^2$$
$$\rightarrow s_1 = 2/h \text{ and } s_1' = -2/h^2$$
$$\rightarrow s_0 = -3/2h \text{ and } s_0' = 1/h^2$$

$$\boxed{(s_0, s_1, s_2) = \frac{1}{2h}(-3, 4, -1) \text{ and } (s_0', s_1', s_2') = \frac{1}{h^2}(1, -2, 1)} \tag{9}$$

Case 2: $\bar{x} = h/2$

Our matrix becomes:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -h/2 & h/2 & 3h/2 & 1 & 0 \\ h^2/8 & h^2/8 & 9h^2/8 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & h & 2h & 1 & 0 \\ 0 & 0 & h^2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow s_2 = 0 \text{ and } s_2' = 1/h^2$$
$$\rightarrow s_1 = 1/h \text{ and } s_1' = -2/h^2$$
$$\rightarrow s_0 = -1/h \text{ and } s_0' = 1/h^2$$

$$\boxed{(s_0, s_1, s_2) = \frac{1}{h}(-1, 1, 0) \text{ and } (s_0', s_1', s_2') = \frac{1}{h^2}(1, -2, 1)} \tag{10}$$

Case 3: $\bar{x} = h$

Our matrix becomes:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -h & 0 & h & 1 & 0 \\ h^2/2 & 0 & h^2/2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -h & 0 & h & 1 & 0 \\ 0 & 0 & h^2 & h/2 & 1 \end{bmatrix}$$

$$\rightarrow s_2 = 1/2h \text{ and } s_2' = 1/h^2$$
$$\rightarrow s_0 = -1/2h \text{ and } s_0' = 1/h^2$$
$$\rightarrow s_1 = 0 \text{ and } s_1' = -2/h^2$$

$$\boxed{(s_0, s_1, s_2) = \frac{1}{2h}(-1, 0, 1) \text{ and } (s_0', s_1', s_2') = \frac{1}{h^2}(1, -2, 1)} \tag{11}$$

b)

Since we already accounted for the first and second derivatives in the calculation of our stencil weights, our leading error can only come from the third derivative or higher terms in Eqns. 2, 3, and 4.

Case 1: $\bar{x} = 0$

For the first derivative, we get:

$$1/2h(0 + 2h^3/3 - 4h^3/3)u^{(3)}(0) = \boxed{-\frac{h^2}{3}u^{(3)}(0)} \tag{12}$$

and for the second derivative, we get:

$$1/h^2(0 - h^3/3 + 4h^3/3)u^{(3)}(0) = \boxed{hu^{(3)}(0)} \tag{13}$$

Case 2: $\bar{x} = h/2$

For the first derivative, we get:

$$1/h(h^3/48 + h^3/48 + 0)u^{(3)}(h/2) = \boxed{-\frac{h^2}{24}u^{(3)}(h/2)} \tag{14}$$

and for the second derivative, we get:

$$1/h^2(-h^3/48 - h^3/24 + 9h^3/16)u^{(3)}(h/2) = \boxed{\frac{h}{2}u^{(3)}(h/2)} \tag{15}$$

Case 3: $\bar{x} = h$

For the first derivative, we get:

$$1/2h(h^3/6 + 0 + h^3/6)u^{(3)}(h) = \boxed{\frac{h^2}{6}u^{(3)}(h)} \tag{16}$$

and for the second derivative, we get:

$$1/h^2(h^3/6 + 0 + h^3/6)u^{(3)}(h) = 0 \tag{17}$$

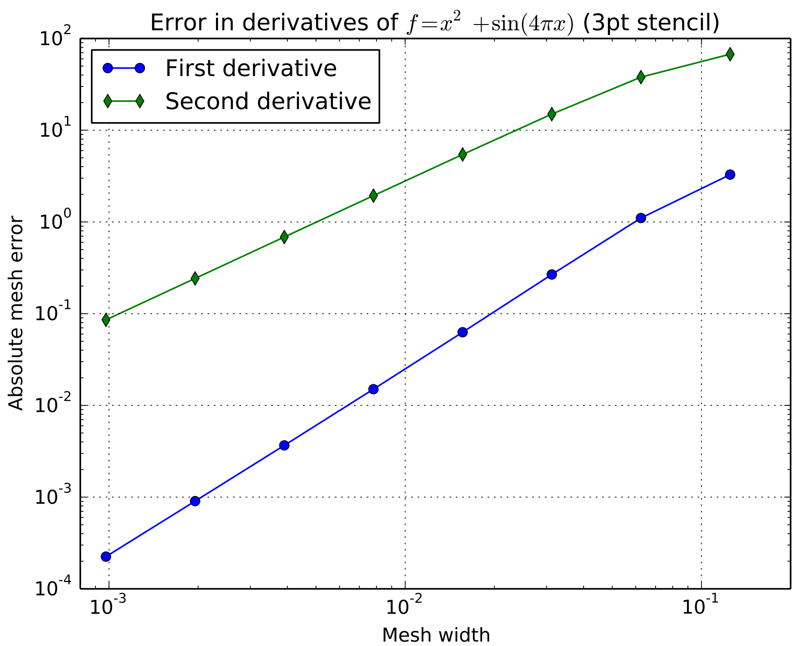
Since the third derivative cancels out, we must look at the fourth derivative to find the leading error term. The fourth derivative error term is:

$$(h^4 s0'/24 + 0 + h^4 s2'/24)u^{(4)}(h) = \boxed{\frac{h^2}{12}u^{(4)}(h)} \tag{18}$$

Problem 2

a)

i.

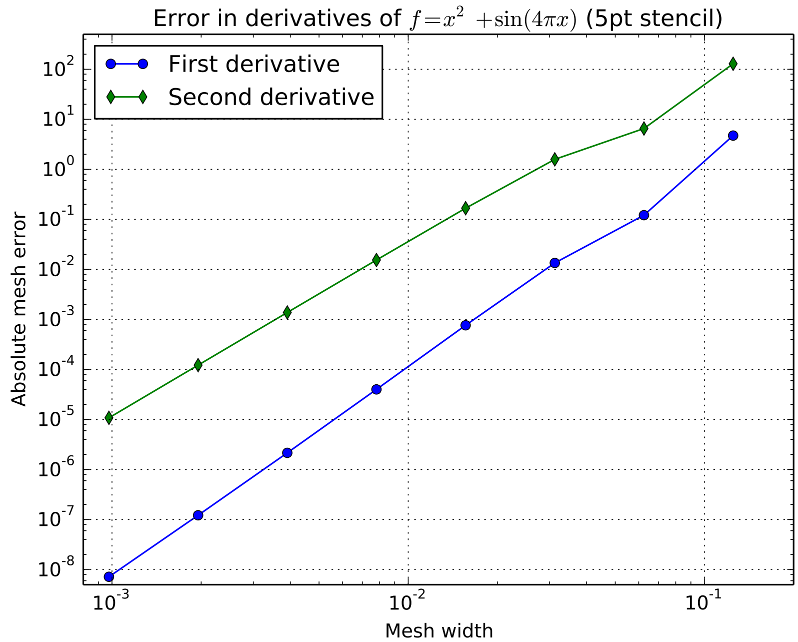


Error in derivatives of $f = x^2 + \sin(4\pi x)$ (3pt stencil)
First derivative
Second derivative
Absolute mesh error
Mesh width
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

ii.

iii.

b)



Problem 3

a)

b)

c)