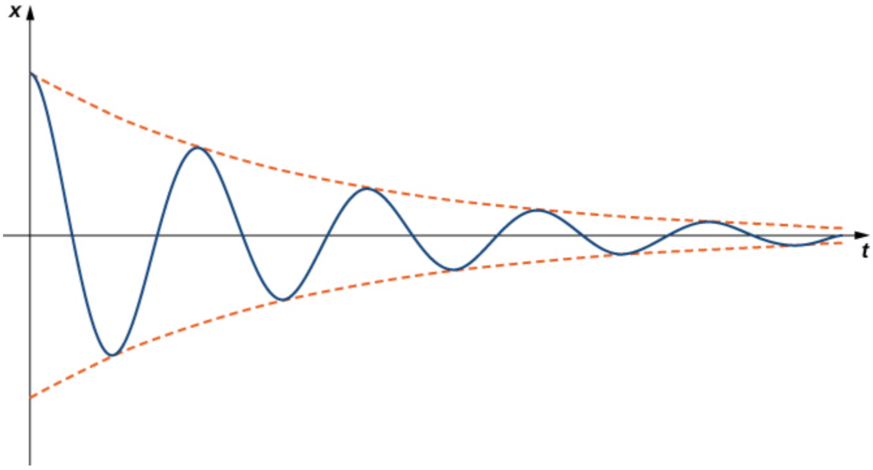

Figure 17.3.7: *Behavior of an underdamped spring-mass system.*

Note that for all damped systems, $\lim_{t\to\infty} x(t) = 0$. The system always approaches the equilibrium position over time.

EXAMPLE 17.3.5: UNDERDAMPED SPRING-MASS SYSTEM

A 16-lb weight stretches a spring 3.2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 9 in. below equilibrium.

Solution

We have $k = \frac{16}{3.2} = 5$ and $m = \frac{16}{32} = \frac{1}{2}$, so the differential equation is

$$\frac{1}{2}x'' + x' + 5x = 0, \text{ or } x'' + 2x' + 10x = 0. \tag{17.3.52}$$

This equation has the general solution

$$x(t) = e^{-t}(c_1\cos(3t) + c_2\sin(3t)). \tag{17.3.53}$$

Applying the initial conditions, $x(0) = \frac{3}{4}$ and $x'(0) = 0$, we get

$$x(t) = e^{-t}\left(\frac{3}{4}\cos(3t) + \frac{1}{4}\sin(3t)\right). \tag{17.3.54}$$

EXERCISE 17.3.5

A 1-kg mass stretches a spring 49 cm. The system is immersed in a medium that imparts a damping force equal to four times the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 24 cm above equilibrium.

Hint

First find the spring constant.

Answer

$x(t) = -0.24e^{-2t}\cos(4t) - 0.12e^{-2t}\sin(4t)$

EXAMPLE 17.3.6: CHAPTER OPENER: MODELING A MOTORCYCLE SUSPENSION SYSTEM

For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when they land could cost them the race.

Figure $\mathbf{17.3.8}$***:*** *(credit: modification of work by nSeika, Flickr)*

This suspension system can be modeled as a damped spring-mass system. We define our frame of reference with respect to the frame of the motorcycle. Assume the end of the shock absorber attached to the motorcycle frame is fixed. Then, the "mass" in our spring-mass system is the motorcycle wheel. We measure the position of the wheel with respect to the motorcycle frame. This may seem counterintuitive, since, in many cases, it is actually the motorcycle frame that moves, but this frame of reference preserves the development of the differential equation that was done earlier. As with earlier development, we define the downward direction to be positive.

When the motorcycle is lifted by its frame, the wheel hangs freely and the spring is uncompressed. This is the spring's natural position. When the motorcycle is placed on the ground and the rider mounts the motorcycle, the spring compresses and the system is in the equilibrium position (Figure 17.3.9).

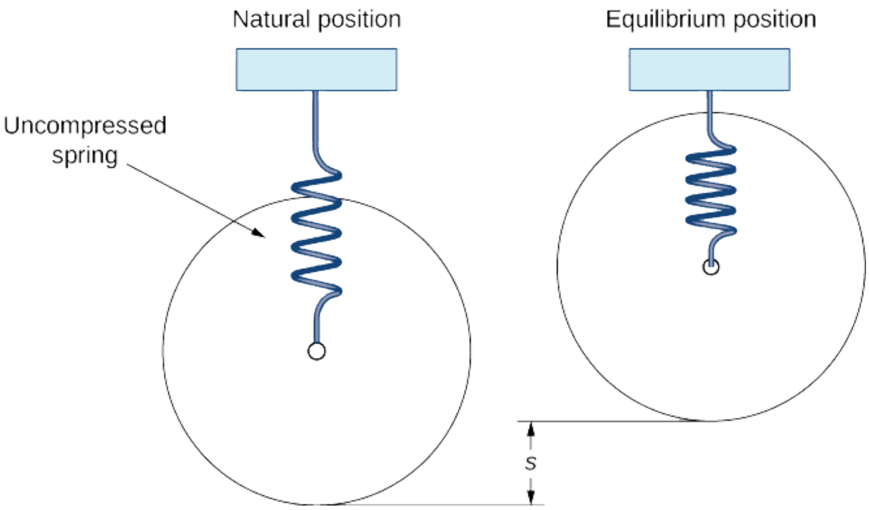

Figure $\mathbf{17.3.9}$***:*** *We can use a spring-mass system to model a motorcycle suspension.*

This system can be modeled using the same differential equation we used before:

$$mx''+bx'+kx=0. \tag{17.3.55}$$

A motocross motorcycle weighs 204 lb, and we assume a rider weight of 180 lb. When the rider mounts the motorcycle, the suspension compresses 4 in., then comes to rest at equilibrium. The suspension system provides damping equal to 240 times the instantaneous vertical velocity of the motorcycle (and rider).

1. Set up the differential equation that models the behavior of the motorcycle suspension system.
2. We are interested in what happens when the motorcycle lands after taking a jump. Let time

$$t=0 \tag{17.3.56}$$

   denote the time when the motorcycle first contacts the ground. If the motorcycle hits the ground with a velocity of 10 ft/sec downward, find the equation of motion of the motorcycle after the jump.
3. Graph the equation of motion over the first second after the motorcycle hits the ground.

Solution

1. We have defined equilibrium to be the point where $mg = ks$, so we have

$$mg = ks \tag{17.3.57}$$
$$384 = k(\frac{1}{3}) \tag{17.3.58}$$
$$k = 1152. \tag{17.3.59}$$

We also have

$$W = mg \tag{17.3.60}$$
$$384 = m(32) \tag{17.3.61}$$
$$m = 12. \tag{17.3.62}$$

Therefore, the differential equation that models the behavior of the motorcycle suspension is

$$12x'' + 240x' + 1152x = 0. \tag{17.3.63}$$

Dividing through by 12, we get

$$x'' + 20x' + 96x = 0. \tag{17.3.64}$$

2. The differential equation found in part a. has the general solution

$$x(t) = c_1 e^{-8t} + c_2 e^{-12t}. \tag{17.3.65}$$

Now, to determine our initial conditions, we consider the position and velocity of the motorcycle wheel when the wheel first contacts the ground. Since the motorcycle was in the air prior to contacting the ground, the wheel was hanging freely and the spring was uncompressed. Therefore the wheel is 4 in. ($\frac{1}{3}$ft) below the equilibrium position (with respect to the motorcycle frame), and we have $x(0) = \frac{1}{3}$. According to the problem statement, the motorcycle has a velocity of 10 ft/sec downward when the motorcycle contacts the ground, so $x'(0) = 10$. Applying these initial conditions, we get $c_1 = \frac{7}{2}$ and $c_2 = -(\frac{19}{6})$,so the equation of motion is

$$x(t) = \frac{7}{2}e^{-8t} - \frac{19}{6}e^{-12t}. \tag{17.3.66}$$

3. The graph is shown in Figure 17.3.10.

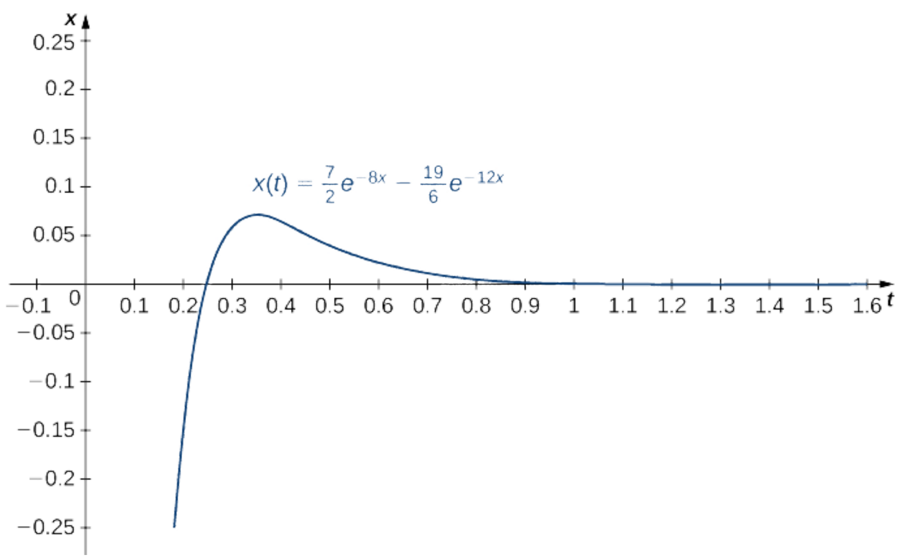

Figure **17.3.10:** Graph of the equation of motion over a time of one second.

NOTE: LANDING VEHICLE

NASA is planning a mission to Mars. To save money, engineers have decided to adapt one of the moon landing vehicles for the new mission. However, they are concerned about how the different gravitational forces will affect the suspension system that cushions the craft when it touches down. The acceleration resulting from gravity on the moon is 1.6 m/sec^2, whereas on Mars it is 3.7 m/sec^2.

The suspension system on the craft can be modeled as a damped spring-mass system. In this case, the spring is below the moon lander, so the spring is slightly compressed at equilibrium, as shown in Figure 17.3.11.