1 Complex Definitions of Sine and Cosine

The goal of this section will be to write *cosine* and *sine* as functions of e and i, where i is an imaginary number.

Let's begin by defining Euler's formula,

$$e^{i\theta} = cos(\theta) + i\,sin(\theta) \tag{1}$$

Step 1: rewrite (1) twice.
We'll rewrite it once by plugging-in z for θ and a second time by plugging-in$-z$ for θ.

$$e^{iz} = cos(z) + i\,sin(z) \tag{2}$$

$$e^{-iz} = cos(-z) + i\,sin(-z) \tag{3}$$

Now we can solve for $cos(-z)$ and $sin(-z)$.

Step 2: Solve for cos(-z)
We can solve for $cos(-z)$ by rewriting (2) and (3).

$$cos(z) + i\,sin(z) = e^{iz} \tag{4}$$

$$cos(z) - i\,sin(z) = e^{-iz} \tag{5}$$

To clarify what happened, (4) is simply (2) rewritten. Equation (5) is simply (3) rewritten, using the following facts: $cos(-z) = cos(z)$ and $sin(-z) = -sin(z)$.

Step 3: Add (4) and (5) together.

$$2\,cos(z) = e^{iz} + e^{-iz} \tag{6}$$

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{7}$$

Equation (7) is the complex definition of *cosine*.

Step 4: Let's solve for sin(z)

$$cos(z) + i\,sin(z) = e^{iz} \tag{8}$$

$$-cos(z) + i\,sin(z) = -e^{-iz} \tag{9}$$

At this point, (8) is simply (4), and (9) is (5) multiplied by -1. Let's add (8) and (9).

$$2\,i\,sin(z) \;=\; e^{iz} + e^{-iz} \tag{10}$$

$$sin(z) = \frac{e^{iz} + e^{-iz}}{2i} \tag{11}$$

Equation (11) is the definition of complex *cosine*.

2 Taylor Series for Cosine and Sine

The goal of this section to show how *cosine* and *sine* are odd and even functions, respectively.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ \dots \ \frac{f^n(a)}{n!}(x-a)^n \tag{12}$$

First, we'll look at sin(x)
Let's take the derivatives of $sin(x)$ at $x = 0$.

$$f(0) = sin(0) = 0, \quad f'(0) = cos(0) = 1, \quad f''(0) = -sin(0) = 0$$
$$f'''(0) = -cos(0) = -1, \quad f^{(4)}(0) = sin(0) = 0, f^{(5)}(0) = cos(0) = 1$$

We can see the pattern that clearly emerges. Let's plug these derivatives into the Taylor Series formula and set $x = 0$ and $x = a$.

$$f(x) = 0 + 1 * (x-0) + 0 - \frac{1\,(x-0)^3}{3!} + 0 + \frac{1\,(x-0)^5}{5!} \tag{13}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \tag{14}$$

From here we can see that *sin* is an odd function.

Let's look at cos(x)
Similar to the process for $sin(x)$, let's take the derivatives and plug-in $x = 0$.

$$f(0) = cos(0) = 1, \ \ f'(0) = -sin(0) = 0, \ \ f''(0) = -cos(0) = -1$$
$$f'''(0) = sin(0) = 0, \ \ f^{(4)}(0) = cos(0) = 1$$

Let's plug the derivatives into the Taylor Series formula.

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{15}$$

Remember that the $x - a$ terms in the Taylor Series fall away becaus we set $x = a$.

3 Integral of e^{ikx} from $-\pi$ to π

$$\int_{-\pi}^{\pi} e^{i\,k\,x}\,dx$$

There are two cases to consider:

1. $k = 0$
2. $k \neq 0$

Let's begin by looking at case 1.

$$\begin{aligned}
&\int_{-\pi}^{\pi} e^{i\,0\,x}\,dx && (16)\\
&= \int_{-\pi}^{\pi} e^{0}\,dx && (17)\\
&= \pi - (-\pi) && (18)\\
&= 2\pi && (19)
\end{aligned}$$

Now, let's look at case 2.

$$\begin{aligned}
&\int_{-\pi}^{\pi} e^{i\,k\,x}\,dx && (20)\\
&= \frac{1}{ik} e^{i\,k\,x}\bigg|_{-\pi}^{\pi} && (21)\\
&= \frac{1}{i\,k} e^{i\,k\,\pi} - \frac{i}{i\,k} e^{i\,k\,(-\pi)} && (22)\\
&= \frac{1}{i\,k}\Bigg[\underbrace{(cos(\pi\,k) + i\,sin(k\,\pi))}_{\text{Term 1}} - \underbrace{(cos(-k\,\pi) + i\,sin(-k\,\pi))}_{\text{Term 2}}\Bigg] && (23)
\end{aligned}$$

To get to (23) we have to recall Euler's formula.

$$e^{i\theta} = cos(\theta) + i\,sin(\theta) \tag{24}$$

We can replace the $k\pi$ terms in (22) with θ. Notice, Term 1 and Term 2 are just the LHS of (24). By replacing Term 1 and Term 2 in (22) with the RHS of (24), we get to equation (23). In (23), notice the *cosine* terms cancel because *cosine* is an even function (i.e. $cos(-\pi) = cos(\pi)$). The *sine* terms are always 0 bceause they're integer multiples of π.

Therefore, rewriting (23) we have:

$$= \frac{1}{ik}[(0+0)-(0+0)] \tag{25}$$

$$= 0 \tag{26}$$

4 Complex Fourier Series

Let's begin by defining the Fourier Series

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \, cos(nx) + \sum_{n=1}^{\infty} b_n \, sin(nx) \tag{27}$$

Before trying to rewrite this as a complex equation, let's recall the complex definitions of *sine* and *cosine*.

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{28}$$

$$sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{29}$$

$$= \frac{i[-e^{i\theta} + e^{-i\theta}]}{2} \tag{30}$$

We wrote (30) by multiplying the numerator and denominator in (29) by i. Recall $i * i = -1$ so we distributed the negative sign through the numerator.

Let's rewrite (27) using (28) and (30).

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \left[\frac{e^{inx} + e^{-inx}}{2} \right] + \sum_{n=1}^{\infty} b_n \left[\frac{-i \, e^{inx} + i \, e^{-inx}}{2} \right] \tag{31}$$

Notice, we have similar terms. We have two e^{inx} terms and two e^{-inx} terms. Let's collect like terms by removing the negative sign from the second exponential.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \underbrace{\frac{A_n - b_n \, i}{2} e^{inx}}_{\text{Term 1}} + \sum_{n=1}^{\infty} \underbrace{\frac{A_n + b_n \, i}{2} e^{-inx}}_{\text{Term 2}} \tag{32}$$

Notice that Term 1 and Term 2 are the same except for a negative symbol in Term 2. Let's rewrite Term 2 by reindexing it from n to $-n$. Therefore, Term 2 becomes:

$$\sum_{n=-\infty}^{-1} \frac{A_{-n} + b_{-n} \, i}{2} \, e^{inx}$$

Now, we have the same e terms. The summation has $-\infty$ at the bottom of the summation because we always write the smallest term at the bottom. We wrote e^{inx} because the negatives cancel out in e^{-i-nx}.

Now, both Term 1 and Term 2 have the same e^{inx} term, we can combine the summations.

Let's rewrite the terms.

$$f(x) = \underbrace{A_0}_{\text{Term 1}} + \sum_{n=1}^{\infty} \underbrace{\frac{A_n - i\,b_n}{2} e^{inx}}_{\text{Term 2}} + \sum_{n=-\infty}^{-1} \underbrace{\frac{A_{-n} + i\,b_{-n}}{2} e^{inx}}_{\text{Term 3}} \tag{33}$$

Let's set Term 1 $= C_0$ and Terms 2 and 3 equal to C_n. This raises the question, how can we set two seemingly different terms to the same variable? Notice, terms 2 and 3 have n's in all the same places. So, we can write it as the following:

Need to confirm details of how to get from equation 33 to 34

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx} \tag{34}$$

Now, we need to figure out what C_n is. Let's multiply the above equation by e^{-imx} and then integrate.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx &= \sum_{n=-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx & (35) \\ &= 2\pi C_m & (36) \end{aligned}$$

There are two cases for the RHS of (35). If $n \neq m$ then it equals 0, and if $n = m$ then it equals 2π, which is how we arrived at (36). We're playing fast and loose with the subscripts for m and n, which is simply because we're only dealing with the case where $m = n$ because the integral is 0 otherwise.

Therefore, we can isolate the C_n and define it as:

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{37}$$