1. Set up an initial-boundary value problem (IBVP) that models the temperature in the bar, in following scenario. Do not solve!

A metal bar, of length 1 meter and thermal diffusivity 2, is taken out of a 100°C oven and then fully insulated except for one end, which is placed into a large ice bucket at 0°C.

2. The Dirichlet IBVP for the cable equation is given by

$$
\begin{aligned}
v_t = \alpha^2 v_{xx} - \beta v, \qquad 0 < x < L,\ \beta > 0 \\
v\left(x,0\right) = f\left(x\right), \qquad 0 \le x \le L, \\
v\left(0,t\right) = v\left(L,t\right) = 0 \qquad t > 0,
\end{aligned}
$$

(a) By setting the time derivative in the differential equation to zero, solve for the steady state solution.

(b) Solve the IBVP and compare the solution as $t \to \infty$ to that in part (a)

Solutions:
Part a:

Step 1: Find the roots

$$
\begin{aligned}
0 &= a^2\, r^2 - b && (1)\\
r^2 &= \frac{b}{a^2} && (2)\\
r &= \pm\sqrt{\frac{b}{a}} && (3)
\end{aligned}
$$

Step 2: Find the general solution
The general solution for real, distinct roots is:

$$
\begin{aligned}
y(x) &= c_1\, e^{r_1\, x} \;+\; c_2 e^{r_2\, x} && (4)\\
&= c_1\, e^{\sqrt{\frac{b}{a}}\, x} + c_2\, e^{-\sqrt{\frac{b}{a}}\, x} && (5)
\end{aligned}
$$

Now, let's plug-in both BCs to equation (5).
Step 3: Plug-in the first BC: $u(0,t)$

$$
\begin{aligned}
0 = y(0) &= c_1 + c_2 && (6)\\
\implies\; c_2 &= -c_1 && (7)
\end{aligned}
$$

Step 4: Plug-in the second BC: $u(L,t) = 0$

$$
\begin{aligned}
0 = y(L) &= c_1\, e^{\sqrt{\frac{b}{a}}\, L} + c_2\, e^{-\sqrt{\frac{b}{a}}\, L} && (8)\\
&= c_1\, e^{\sqrt{\frac{b}{a}}} + c_2\, e^{-\sqrt{\frac{b}{a}}\, L} && (9)\\
&= c_1\, (e^{\sqrt{\frac{b}{a}}\, L} - e^{-\sqrt{\frac{b}{a}}\, L}) && (10)
\end{aligned}
$$

It's not possible for $e^{\sqrt{\frac{b}{a}}L} - e^{-\sqrt{\frac{b}{a}}L}$ to equal zero; therefore, c_1 must equal 0.

$$\boxed{\hat{v}(x) = 0} \tag{11}$$

Part b:
Step 1: Assume the solution has the following form:

$$v = T(t)\,X(x) \tag{12}$$

Step 2: Substitute (12) into the given equation ($V_t = A^2\,V_{xx} - bv$)

$$T_t\,X = a^2\,T\,X_{xx} - bTX \tag{13}$$

$$\frac{T_t}{Ta^2} = \frac{X_{xx}}{x} - \frac{b}{a^2} \tag{14}$$

$$\frac{T_t}{T\,a^2} + \frac{b}{a^2} = \frac{X_{xx}}{X} \tag{15}$$

$$\frac{T_t + T\,b}{Ta^2} = \frac{X_{xx}}{X} = C \tag{16}$$

In equation (14) we divided by XTa^2. In equation (16), we factored out an a^2 in the numerator and the denominator on the LHS. Also, notice the LHS is a function of T and the RHS is a function of X. The only way for those two equations, which are functions of different variables, is for both to equal the same constant.

Step 3: Write out both ODEs
At this point, we have two separate ODEs both equal to the same constant. Let's write out both ODEs, and we'll replace C with $-\lambda^2$ for convenience.

The first ODE (the time ODE) is:

$$0 = T_T + Tb + \lambda^2\,T\,a^2 \tag{17}$$

$$= T_T + T(b + \lambda^2\,Ta^2 \tag{18}$$

$$\tag{19}$$

This is simply the LHS of (16) rewritten.

And now the second ODE (the spatial ODE) is:

$$0 = X_{xx} + \lambda^2\,X \tag{20}$$

Step 4: Solve both ODEs
Let's solve the spatial ODE first. We'll start by finding the characteristic equation.

$$0 = r^2 + \lambda^2 \tag{21}$$
$$r^2 = -\lambda^2 \tag{22}$$
$$r = \lambda i \tag{23}$$

In equation (23) we used the following fact: $\sqrt{-\lambda^2} = \sqrt{i^2\lambda^2} = \lambda\, i$

This is a complex root. Recall, the general solution for complex roots is:

$$y(x) = c_1\, e^{ax} cos(ux) \,+\, c_2\, e^{ax} sin(ux) \tag{24}$$

where we have: $r_{1,2} = a \pm ui$.
So, for our characteristic equation the λ corresponds to the u in the general equation. And the a in the general solution corresponds to 0 in our characteristic equation, because our characteristic equation is technically $0 \pm \lambda\, i$. Writing out the general equation we have:

$$y(x) = c_1 e^{0\,x} cos(\lambda\, x) \,+\, c_2\, e^{0\,x} sin(\lambda\, x) \tag{25}$$

From here, we can see the exponential terms fall away because they're raised to 0.

Step 5: Plug-in Boundary conditions
Let's plug-in the first boundary condition, $v(0, t) = 0$, to the general equation.

$$0 = y(0) = c_1 cos(0) + c_2 sin(0) \tag{26}$$
$$\implies c_1 = 0 \tag{27}$$

Let's plug-in the second BC, $v(L, t) = 0$, to the general equation.

$$0 = y(L) = c_2 sin(\lambda L) \tag{28}$$

At this point, we want to avoid the trivial solution (i.e. $u(x, t) = 0$). Since c_1 already equals 0, we don't want to set $c_2 = 0$ because that'd be trivial. So, we need $sin(\lambda\, L) = 0$, which can be achieve by seetting $\lambda\, L = \pi$.

$$\lambda\, L = \pi \tag{29}$$
$$\lambda = \frac{n\pi}{L} \quad \forall\, n = 1, 2, 3, ... \tag{30}$$

The n is because all integer multiples of π are 0. Therefore, the eigenvalues are $\lambda = \frac{n\pi}{L}$ and the eigenfunction is $sin(\lambda_n \pi)$, where $\lambda_n = \frac{n\pi}{L}$, and we plugged that into the general solution (equation (25)).

Now, let's solve the time ODE.

$$0 = T_t + T(b + a^2\,\lambda^2) \tag{31}$$

This is a first order, separable, ODE. The solution is:

$$T = e^{(-b-c^2a^2)+c_1} \tag{32}$$

Recall, $c_1 = 0$ so we can drop that. Also, since the solution above is periodic, we excluded the c_2 term in the eigensolution (i.e. $sin(\lambda_n \pi)$) because we're absorbig it into $T(t)$. Thus we have the following:

$$T_n(t) = T_n(0)\,e^{-(b+a^2\,\lambda_n^2)t} \tag{33}$$

Remember, $\lambda = \frac{n\pi}{L}\ \ \forall\ \ n = 1, 2, 3, ...$

Step 6: Find the coefficients by applying the initial condition (IC)
Let's apply the IC to find T_n.

$$v(x,t) = f(x) = \sum_{n=1}^{\infty} T_n(0)\,sin\left(\frac{n\pi x}{L}\right) \tag{34}$$

$$\implies T_n(0) = \frac{2}{L}\int_0^L f(x)\,sin\left(\frac{n\pi x}{L}\right)dx \tag{35}$$

In equation (34) the $e^{-(b+a^2\lambda_n^2)}$ term falls away becasue $t = 0$. And we found equation (35) via definition of the Fourier Sine Series.

To solve the original problem, we plug-in the ODE solutions to (12). The solutions to our two ODEs are:

$$X = sin(\lambda_n \pi) \tag{36}$$

$$T = T_n(0) \tag{37}$$

Step 7: Plug our ODE solutions into our assumed solution (i.e. $V = T(t)X(x)$)
Plugging this into (12) we have:

$$v(x,t) = \sum_{n=1}^{\infty} T_n(0)\,e^{-(b+a^2\lambda^2)t}\,sin\left(\frac{n\pi x}{L}\right) \tag{38}$$

with

$$T_n(0) = \frac{2}{L}\int_0^L f(x)\,sin\left(\frac{n\pi x}{L}\right)\,dx \tag{39}$$

3. The following IBVP :

$$
\begin{aligned}
u_{tt} = c^2 u_{xx}, \quad & 0 < x < L,\ t > 0 \\
u(x,0) = f(x),\ u_t(x,0) = 0 \quad & 0 \le x \le L, \\
u(0,t) = u(L,t) = 0 \quad & t > 0, \\
f(x) = \begin{cases} 2hx/L & 0 \le x \le L/2 \\ 2h(L-x)/L & L/2 \le x \le L, \end{cases}
\end{aligned}
$$

represents a string (both ends fixed) with the mid-point pulled aside a distance h and released from rest. Solve for the displacement $u(x,t)$ using method of separation of variables.

Solution:
First, let's note that since there is a second derivative with respect to t in the PDE, we need two initial conditions (ICs). There will be undetermined constants in the solution if we don't specify the second IC.

We can think of $u(x,0) = f(x)$ as the initial displacement, and $u_t(x,0) = 0$ as the shape of the initial velocity (in this case there is no shape since it equals 0).

Step 1: Assume the PDE has a Product Solution
Let's begin solving this problem via separation of variables. We assume the solution has the following form:

$$u(x,t) = T(t)\,X(x) \tag{40}$$

Let's plug (40) into the given equation.

$$T_{tt}\,X = c^2\,X_{xx} \tag{41}$$

$$\frac{T_{tt}}{c^2\,T} = \frac{X_{xx}}{X} = -\lambda^2 \tag{42}$$

In (42) we're dividing by XTc^2, and the $-\lambda^2$ is the separation constant.

Step 2: Write the Original PDE has Two ODEs
Let's write out both ODEs.

$$0 = X_{xx} + \lambda^2\,X \tag{43}$$

$$0 = T_{tt} + \lambda^2 c^2 T \tag{44}$$

Step 3: Solve the first ODE
Let's solve the first ODE, equation :

$$
\begin{aligned}
0 &= r^2\lambda^2 && (45)\\
r &= \sqrt{i^2\lambda^2} && (46)\\
&= \lambda\, i && (47)
\end{aligned}
$$

Recall, the general solution for complex roots is:

$$y(x) \;=\; c_1 e^{a\,x}\, cos(ux) \;+\; c_2\, e^{a\,x} sin(ux) \tag{48}$$

where

$$r_{1,2} = a \pm u\, i \tag{49}$$

Let's plug equation (47) into the general solution:

$$
\begin{aligned}
y(x) &= c_1\, e^{0x}\, cos(\lambda x) \;+\; c_2\, e^{0x}\, sin(\lambda\, x) && (50)\\
&= c_1\, cos(\lambda x) \;+\; c_2\, sin(\lambda x) && (51)
\end{aligned}
$$

Now, let's apply the left BC to (51)

$$
\begin{aligned}
0 = v(0) &= c_1 && (52)\\
&\implies c_1 = 0 && (53)
\end{aligned}
$$

Apply the right BC

$$0 = v(L) = c_2\, sin(\lambda\, L) \tag{54}$$

In this equation, $c_1\, cos(\lambda x)$ falls away because $c_1 = 0$. To avoid the trivial solution (i.e. $u(x,t) = 0$) we need $c_2 \neq 0$. Therefore, we need the following:

$$
\begin{aligned}
sin(\lambda L) &= 0 && (55)\\
&\implies \lambda L = n\pi \quad \forall\, n = 1, 2, 3, ... && (56)\\
\lambda &= \frac{n\pi}{L} && (57)
\end{aligned}
$$

Plugging this into (51) we get

$$X = c_2\, sin\left(\frac{n\pi x}{L}\right) \tag{58}$$

Step 4: Solve the Second ODE
Now, we can solve the second ODE (equation (44)). The characteristic equation is:

$$
\begin{aligned}
0 &= r^2 \,+ \lambda^2\, c^2 && (59)\\
r^2 &= -\lambda^2 c^2 && (60)\\
r &= \pm\lambda\, c\, i && (61)
\end{aligned}
$$

Plug this into the general solution for complex numbers (equation (51)) we get:

$$T = c_1\, e^{0t} cos(\lambda ct) \;+\; c_2\, e^{0t}\, sin(\lambda\, c\, t) \tag{62}$$
$$= c_1 cos(w_n t) + c_2 sin(w_n t) \tag{63}$$

Where $w = \lambda_n c = \frac{cn\pi}{L}$. We denote λ with a subscript n because the solution is periodic and has infinite solutions.

Step 5: Plug the ODE Solutions into the Product Solution
At this point, we can put it all together.

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\, X_n(x) \tag{64}$$
$$= \sum_{n=1}^{\infty} \left[A_n sin(w_n t) + B_n cos(w_n t)\right]\; sin\left(\frac{n\pi x}{L}\right) \tag{65}$$

Step 6: Find the Coefficients by Applying the ICs
Now, we need to find the coefficients by applying the IC. Given equation (65), let's solve for A_n by applying the first IC to (65).

$$u(x,0) \;= f(x)\, \sum_{n=1}^{\infty} B_n\, sin\left(\frac{n\pi x}{L}\right) \tag{66}$$

And now let's apply the second IC to the derivative of (65).

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} w_n\, A_n\, cos(w_n\, t)\, sin\left(\frac{n\pi x}{L}\right) \tag{67}$$
$$= \sum_{n=1}^{\infty} \frac{cn\pi}{L}\, A_n\, sin\left(\frac{n\pi x}{L}\right) \tag{68}$$

where $w_n = \frac{cn\pi}{L}$ and $cos(w_n, 0) = 1$.

Step 7: Appy Fourier Sine Series
Notice, that (66) and (68) is a Fourier Sine Series. Recall a Fourier Sine Series, if we have a function f(x):

$$f(x) = \sum_{n=1}^{\infty} B_n\, sin\left(\frac{n\pi x}{L}\right) \tag{69}$$

then we can define B_n via the following:

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right)\, dx \quad \forall \;\; n = 1, 2, 3, ... \tag{70}$$

Applying (70) to (68) we get the following:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L u_t(x,0)\, sin\left(\frac{n\pi x}{L}\right) dx \qquad (71)\\ &= 0 \qquad (72)\end{aligned}$$

In (72) we applied the fact that $u_t(x, 0) = 0$.

And now let's apply (70) to (66):

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L u(x,0) sin\left(\frac{n\pi x}{L}\right)\, dx \qquad (73)\\ &= \frac{2}{L}\int_0^L f(x)\, sin\left(\frac{n\pi x}{L}\right)\, dx \qquad (74)\\ &= \frac{2}{L}\left[\int_0^{L/2} \frac{2hx}{L}\, sin\left(\frac{n\pi x}{L}\right)\, dx + \int_{L/2}^{L} \frac{2h(L-X)}{L}\, sin\left(\frac{n\pi x}{L}\right)\, dx\right] \qquad (75)\\ &= \frac{4h}{L^2}\left[\int_0^{L/2} x sin\left(\frac{n\pi x}{L}\right)\, L\, dx + \int_{L/2}^{L} (L-X)\, sin\left(\frac{n\pi x}{L}\right)\, dx\right] \qquad (76)\\ &= \frac{4h}{L^2}\left[-\frac{-L^2}{2n\pi}\, cos(n\pi) + \frac{L^2}{n^2\pi^2}\, sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^{L/2} + ...\\ &...\left[\frac{L^2}{2n\pi} cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\, sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{L/2}^{L} \qquad (77)\\ &= \frac{8h}{n^2\pi^2}\, sin\left(\frac{n\pi}{2}\right) \qquad (78)\\ &= \begin{cases} 0 \quad \text{if} \quad n \quad \text{is even} \\ \dfrac{8h(-1)^{\frac{n-1}{2}}}{n^2\pi^2} \qquad \text{if} \quad n \;\; \text{is odd} \end{cases} \qquad (79)\end{aligned}$$

Let's recap what exactly happened. In equation, (75) we replaced the $f(x)$ term with its piecewise equivalent, which was provided in the problem. In equation (76) we rewrote the previous equation by factoring out the constants (i.e. $2h$ and L). To get to (77) we integrated by parts.

Step 8: Plug-in A_n and B_n terms to find solution
Finally, we'll plug our A_n and B_n terms into equation (65).

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \left[A_n\, sin(w_n\, t) + B_n cos(w_n t)\right]\, sin\left(\frac{n\pi x}{L}\right) &\quad (80)\\
&= \sum_{n=1}^{\infty} \left[0 * sin(w_n t) + \frac{8h(-1)^{\frac{n-1}{2}}}{n^2\pi^2} cos(w_n t)\right] sin\left(\frac{n\pi x}{L}\right) &\quad (81)\\
&= \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2}\, cos(w_n t)\, sin\left(\frac{n\pi x}{L}\right) &\quad (82)
\end{aligned}
$$

Recall that $w_n = \lambda_n c = \frac{cn\pi}{L}$.

4. A uniform insulated metal bar 1 meter long is stored at room temperature of 20° C. An experimenter places one end of the bar in boiling water and the other end in an ice bucket.

(a) Set up an IBVP that models the temperature in the bar.

(b) Solve the IBVP (hint: read Chapter 3, section 3.2 of Professor Tung's lecture notes).

Solution:

Part a:

$$u_t = a^2\, u_{xx} \quad 0 < x < 1, \quad t > 0 \tag{83}$$
$$u(x, 0) = 20 \quad 0 \leq x \leq 1 \tag{84}$$
$$u(0, t) = 0, \quad u(1, t) = 100, \quad t > 0 \tag{85}$$

Part b:
We need to make the problem homogeneous - we can do this by definining a new term $v(x, t)$.

$$v(x, t) = u(x, t) - \left[T_1 + \frac{X}{L}(T_2 - T_1)\right] \tag{86}$$

Step 1: Rewrite the PDE so it's homogeneous
Plugging in our given numbers we get:

$$v(x, t) = u(x, t) - \left[0 + \frac{x}{1}(100 - 0)\right] \tag{87}$$
$$= u(x, t) - 100x \tag{88}$$

which gives the following reformulated PDE:

$$v_t = a^2\, v_{xx} \quad 0 < x < 1 \quad t > 0 \tag{89}$$
$$v(0, t) = v(L, t) = 0 \quad t > 0 \tag{90}$$
$$v(x, 0) = u(x, 0) - [T_1 + \frac{X}{L}(T_2 - T_1)] \tag{91}$$
$$= 20 - 100x \tag{92}$$

Step 2: Assume the PDE has a Product Solution
Let's solve the PDE by assuming it has the following form:

$$v = T(t)\,X(x) \tag{93}$$

Let's plug (93) into (89).

$$T_t X = a^2\,T X_{xx} \tag{94}$$
$$\frac{T_t}{a^2 T} = \frac{X_x x}{x} = -\lambda^2 \tag{95}$$

In (95) we divided by a^2TX and set the equation equal to a constant ($-\lambda^2$).

Step 3: Write out both ODEs

$$0 = T_t + \lambda^2 a^2 T \tag{96}$$
$$0 = X_{XX} + \lambda^2 X \tag{97}$$

Step 4: Solve the ODEs
Let's begin by solving (111). The characteristic equation is:

$$0 = r^2 + \lambda^2 \tag{98}$$
$$r = \lambda i \tag{99}$$

The general solution for complex equations is:

$$y(t) = c_1 e^{at}\,cos(ut) + c_2 e^{at}\,sin(ut) \tag{100}$$

where $r_{1,2} = a \pm ui$.

Plugging in our values into the general solution we get:

$$X = c_1\,e^{0t} cos(\lambda t) + c_2\,e^{0t}\,sin(\lambda t) \tag{101}$$

Apply the first BC to : $v(0, t) = 0$

$$0 = c_1 \tag{102}$$

Apply the second BC to labelgen sol: $v(1, 5) = 0$

$$0 = c_2\,sin(\lambda) \tag{103}$$

To avoid the trivial solution $sin(\lambda)$ needs to equal 0. Therefore, we get:

$$0 = sin(\lambda) \tag{104}$$
$$\implies \lambda i = n\pi \quad \forall\, n = 1, 2, 3, ... \tag{105}$$
$$\lambda = n\pi \tag{106}$$

After applying the BCs, we plug our results (i.e. $\lambda n = \pi$ and $c_1 = 0$) into (101), resulting in:

$$X = c_2\, sin(n\pi t) \tag{107}$$

Let's solve the first ODE (equation (96)).

$$0 = T_t + \lambda^2 a^2 T \tag{108}$$

This is a first order, separable ODE with the following solution:

$$T = e^{-\lambda^2 a^2 t + c_1} \tag{109}$$

Now, let's recall that $c_1 = 0$.

Step 5: Plug ODE solutions into Product Solution
Plug both solutions into (93)

$$\begin{aligned} V &= T(t)X(x) && (110)\\ &= \sum_{n=1}^{\infty} A_n e^{-(n\pi a)^2\, t} sin(n\pi x) && (111) \end{aligned}$$

where $\lambda = n\pi$.

Step 6: Solve for the coffecients
Let's solve for A_n by applying the IC.

$$\begin{aligned} v(x,0) &= 20 - 100x && (112)\\ &= \sum_{n=1}^{\infty} A_n\, sin(n\pi x) && (113) \end{aligned}$$

This is a Fourier Sine Series, which can be denoted as:

$$f(x) = \sum_{n=1}^{\infty} B_n sin\left(\frac{n\pi x}{L}\right) \tag{114}$$

where B_n is defined as:

$$B_n = \frac{2}{L}\int_0^L f(x)\, sin\left(\frac{n\pi x}{L}\right) \quad \forall \;\; n = 1, 2, 3, ... \tag{115}$$

Applying the Fourier Sine Series to (113) we get:

$$\begin{aligned} A_n &= \frac{2}{1}\int_0^L (20 - 100x) sin(n\pi x) && (116)\\ &= \frac{40}{n\pi}(4(-1)^n - 1) && (117) \end{aligned}$$

In (117) we integrated by parts.
Plugging this A_n value into (111) we get:

$$V = T(t)X(x) \tag{118}$$
$$= \sum_{n=1}^{\infty} \frac{40}{n\pi}(4(-1)^n - 1)\, e^{-(n\pi a)^2\, t} sin(n\pi x) \tag{119}$$

Step 7: Plug Coefficients into the Solution
Apply this to the final solution which can be written as:

$$u(x,t) = v(x,t) + 100x \tag{120}$$
$$= \left[\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{4(-1)^n - 1}{n}e^{-(n\pi a)^2 t}\, sin(n\pi x)\right] + 100x \tag{121}$$

In equatin (120) we simply rewrote equation (88).