• Heat Eq'n Derivation

↳ Energy Balance:

Rate of E Accumulation = Flow of E in − Flow of E out + Rate of E generation

Flow of energy in x-direction = $q_x|_x$ ⇐ q is simply the flow of heat

Flow of energy out in x-direction = $q_x|_{x+dx}$

Question: how do we relate $q_x|_x$ to $q_x|_{x+dx}$

↳ We can use Taylor Series, only look at first-order term

$$q_x|_{x+dx} = q_x|_x + \frac{\partial q_x}{\partial x} dx \quad ①$$

↳ Now, we can start thinking about the net flow of energy, which is the Flow of E in − Flow of E out, and can be defined as:

$$q_x|_x - q_x|_{x+dx} \quad ②$$

$$= q_x|_x - \left(q_x|_x + \frac{\partial q_x}{\partial x} dx\right)$$ ← Plugging ① into ②

$$= -\frac{\partial q_x}{\partial x} dx$$ ← Net flow of energy in/out in the x direction

↳ Now, we can look at energy generation

Rate of E generation = $\dot{q} \cdot$ volume = $\dot{q}\, \underbrace{dx\,dy\,dz}_{\text{length of each side of box}}$

$\dot{q}$ = energy per unit volume

↳ Let's look at Rate of E Accumulation

$$\frac{du}{dt} \cdot \text{mass} \quad ③$$

$\frac{du}{dt}$ ↑ rate at which energy changes over time

← Note on u:
u = internal energy per unit mass; therefore, u equals:

$$u = \underset{\substack{\uparrow \\ \text{heat capcity}}}{C_p} \underbrace{(T - T_{ref})}_{\text{Temp. diff. from some reference state}}$$

↳ Given the above, we write

$$\frac{du}{dt} = C_p \frac{dT}{dt}$$

and mass equals:

$$\text{mass} = \underset{\substack{\uparrow \\ \text{density}}}{\rho}\, dx\,dy\,dz$$

↳ Now, we can rewrite ③ as;

$$\frac{du}{dt} \cdot mass = \rho C_p \frac{dT}{dt} dx\, dy\, dz$$

↳ Let's go back to our "energy balance" eq'n (top of previous pg) + review each component

$$\underset{\rho C_p \frac{dT}{dt} dxdydz}{\overset{\text{Rate of E Accumulation}}{\|}} = \underbrace{\text{Flow of E in} - \text{Flow of E out}}_{-\frac{\partial q_x}{\partial x} dx \text{ ⑤}} + \underbrace{\text{Rate of E generation}}_{\dot{q}\, dxdydz} \quad \text{⑥}$$

← All the eq'ns below the words

- At this point, we haven't explicitly addressed heat, only flow. We need to connect heat flux to temperature
  ↳ heat flux is the "Flow of E in - Flow of E out" $\left(-\frac{\partial q_x}{\partial x} dx\right)$
    ↳ Heat flux is the amount of heat transferred per unit area per unit of time

$$q_x = -K(dydz)\frac{\partial T}{\partial x} \quad \text{④}$$

Recall this is the flow of energy into the left side of the cube

Temp. gradient in x direction

↳ Interpretation of the eq'n: The negative sign states that heat will flow down hill.
  ↳ If temp. is getting smaller as we move from left to right that means $\frac{\partial T}{\partial x}$ is negative, making q_x a positive value meaning is flowing from left to right (i.e the positive x direction)

- Now, we've connected heat flux to temperature. Let's plug ④ into ⑤

$$-\frac{\partial q_x}{\partial x} dx = \frac{-\partial}{\partial x}\underbrace{\left(-K\, dydz\, \frac{\partial T}{\partial x}\right)}_{q_x} dx$$

$$= \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) dx\, dy\, dz$$ ← This can be written for y + z axis too

- Final step: Rewrite ⑥ using ⑤

$$\rho C_p \frac{dT}{dt} dxdydz = \left[\frac{\partial}{\partial x} K \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} K \frac{\partial T}{\partial y} + \frac{\partial}{\partial z} K \frac{\partial T}{\partial z} + \dot{q}\right] dxdydz$$

$$\Leftrightarrow \rho C_p \frac{\partial T}{dt} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) + \dot{q}$$