1 Fourier Sine Series

Let's start off by assuming some function, $f(x)$, is odd (i.e. $f(-x) = -f(x)$). Because $f(x)$ is odd it makes sense that we can write a series represntation in terms of *sine* only (recall *sine* is odd - look at Taylor Series for it). What we'll try to do is write $f(x)$ as the following series representation, called a Fourier Sine Series on $-L \leq x \leq L$.

$$f(x) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

where $f(x)$ is an odd function and can be approximated by the Fourier Sine Series (i.e. the RHS of (1)).

The question now is how to find the coefficient, b_n? The following details how to solve for b_n.

Step 1: Multiply both sides by $sin\left(\frac{m\pi x}{L}\right)$

$$f(x)sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) \tag{2}$$

Step 2: Integrate and factor out b_n

$$\int_{-L}^{L} f(x)sin\left(\frac{m\pi x}{L}\right) dx = b_n \sum_{n=1}^{\infty} \int_{-L}^{L} sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) dx \tag{3}$$

Step 3: Orthogonality of sin, integral of RHS from equation (3)

$$\int_{-L}^{L} sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L, \text{if} \quad n = m \\ 0, \text{if } n \quad \neq m \end{cases} \tag{4}$$

Step 4: Rewrite Step 2 replacing n with m

$$\int_{-L}^{L} f(x)sin\left(\frac{m\pi x}{L}\right) dx = b_n \tag{5}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\left(\frac{m\pi x}{L}\right) dx \tag{6}$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)sin\left(\frac{m\pi x}{L}\right) dx \tag{7}$$

In equation (7) we incorporated the fact about odd integrals (i.e. changing the limits on the integrations). Also, we swapped m for n since we only have a non-zero result when $m = n$. Let's look at an example.

Example 1:
Find the Fourier Sine Series of:

$$f(x) = x \quad \text{on} \quad -L \leq x \leq L \tag{8}$$

Plug x into our formula for b_n

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L x sin\left(\frac{n\pi}{L}\right) dx &(9)\\
&= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left[L\, sin\left(\frac{n\pi}{L}\right) - n\pi x\ cos\left(\frac{n\pi}{L}\right)\right]\Bigg|_0^L &(10)\\
&= \frac{2}{n^2\pi^2}\left[\left(L\, sin(n\pi) - n\pi L\ cos\left(\frac{n\pi L}{L}\right) - (0-0)\right)\right] &(11)\\
&= \frac{2}{n^2\pi^2}\left[(L\ sin(n\pi) - n\pi L\ cos(n\pi))\right] &(12)
\end{aligned}$$

We know n is an integer, consequently $cos(n\pi) = -1^n$; thereore, we can write:

$$\begin{aligned}
b_n &= \frac{2}{n^2\pi^2}(-n\pi L)(-1)^n &(13)\\
&= \frac{2L}{n\pi}(-1)^{n+1} &(14)
\end{aligned}$$

The Fourier Sine Series is then

$$\begin{aligned}
X &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi}\ \ sin\left(\frac{n\pi x}{L}\right) &(15)\\
&= \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\ \ sin\left(\frac{n\pi x}{L}\right) &(16)
\end{aligned}$$

2 Fourier Cosine series

Let's start by assuming the function $f(x)$ is even (i.e. $f(-x) = f(x)$), and we want to write a series representation for this function on $-L \leq x \leq L$ in terms of *cosines*. In other words, we're looking for this:

$$f(x) = \sum_{n=0}^{\infty} A_n \, cos\left(\frac{n\pi x}{L} i\right) \tag{17}$$

Note that we're starting with $n = 0$, compared to the case with *sine* since $sin(0) = 0$.

Before diving into the details, let's recall the following fact:

$$\int_{-L}^{L} cos\left(\frac{n\pi x}{L}\right) \, cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 2L & \text{if} \quad n = m = 0 \\ L & \text{if} \quad n = m \neq 0 \\ 0 & \text{if} \quad n \neq m \end{cases} \tag{18}$$

To derive the coefficients, A_n, let's multiply (17) by $cos\left(\frac{m\pi x}{L}\right)$.

$$f(x) \, cos\left(\frac{m\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \, cos\left(\frac{n\pi x}{L}\right) cos\left(\frac{m\pi x}{L}\right) \tag{19}$$

Integrate and factor the A_n term

$$\int_{-L}^{L} f(x) \, cos\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \, cos\left(\frac{n\pi x}{L}\right) \, cos\left(\frac{m\pi x}{L}\right) dx \tag{20}$$

We know that if $m \neq n$ then the integral equals 0. Now, there are 2 cases to consider: i) $n = m = 0$ and ii) $n = m \neq 0$.
Case 1: $n = m = 0$

$$\int_{-L}^{L} f(x) dx = A_0(2L) \tag{21}$$

$$\implies A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \tag{22}$$

Case 2: $n = m \neq 0$

$$\int_{-L}^{L} f(x) cos\left(\frac{m\pi x}{L}\right) dx = A_m \, L \tag{23}$$

$$\implies A_m = \frac{1}{L} \int_{-L}^{L} f(x) cos\left(\frac{m\pi x}{L}\right) dx \tag{24}$$

To summarize, we can write the following

$$f(x) = \sum_{n=0}^{\infty} A_n cos\left(\frac{n\pi x}{L}\right) \tag{25}$$

$$\implies A_n = \begin{cases} \frac{1}{2L} \int_{-L}^{L} f(x)dx & \text{if } n = m = 0 \\ \frac{1}{L} \int_{-L}^{L} f(x)cos\left(\frac{n\pi x}{L}\right)dx & \text{if } n = m \neq 0 \end{cases} \tag{26}$$

3 Fourier Series coefficients

The goal in this section is to demonstrate how to calculate the coefficients in the Fourier Series (A_0, A_n, b_n).

We'll begin by defining the Fourier Series

$$f(x) = A_0 + A_1\,cos(1x) + A_2\,cos(2x) + A_3\,cos(3x) + \ldots + b_1\,sin(1x) + b_2\,sin(2x) + b_3\,sin(3x) + \ldots \tag{27}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n\,cos(nx) + \sum_{n=1}^{\infty} b_n\,sin(nx) \tag{28}$$

At this point, the formula is straightforward, but we have three coefficient terms (i.e. A_0, A_n, b_n) which are not defined. Let's proceed to define those three terms.

Step 1: Solve for A_0
To solve this let's integrate both sides of (28) from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx \;=\; \int_{-\pi}^{\pi} \underbrace{A_0}_{\text{Term 1}}\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{A_n\,cos(nx)}_{\text{Term 2}}\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{b_n\,sin(nx)}_{\text{Term 3}}\,dx \tag{29}$$

As a quick preview for how this will play out, Terms 2 and 3 will both reduce to zero leaving us with the integral of Term 1 and the LHS of the equation.

Integrate Term 1.

$$\int_{-\pi}^{\pi} A_0\,dx = A_0 X|_{-\pi}^{\pi} = 2\pi\,A_0 \tag{30}$$

Integrate Term 2.

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} cos(nx)dx = \sum_{n=1}^{\infty} sin(nx)|_{-\pi}^{\pi} \tag{31}$$

$$= \sum_{n=1}^{\infty} sin(n\pi) + \sum_{n=1}^{\infty} sin(n\pi) \tag{32}$$

$$= 0 + 0\; = 0 \tag{33}$$

Integrate Term 3.

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\, sin(nx)dx = \sum_{n=1}^{\infty} -cos(nx)|_{-\pi}^{\pi} \tag{34}$$
$$= -\pi - (-\pi) \tag{35}$$
$$= 0 \tag{36}$$

Now, Terms 2 and 3 both reduced to 0; so, we're left with the following:

$$\int_{-\pi}^{\pi} f(x) = \int_{-\pi}^{\pi} A_0 dx \tag{37}$$
$$= 2\pi A_0 \tag{38}$$

Isolating the A_0 term results in:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \tag{39}$$

Step 2: Let's find the A_n term
We'll begin by multiplying each term in (28) by $cos(mx)$, where m is just some constant.

$$\int_{-\pi}^{\pi} \underbrace{f(x)cos(mx)\, dx}_{\text{Term 1}} \tag{40}$$
$$= \int_{-\pi}^{\pi} \underbrace{A_0 cos(mx)\, dx}_{\text{Term 2}} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{A_n cos(nx)\, cos(mx)\, dx}_{\text{Term 3}} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{b_n sin(nx) cos(mx)\, dx}_{\text{Term 4}} \tag{41}$$

Let's begin by integrating Term 2.

$$\int_{-\pi}^{\pi} cos(mx)dx = 0 \tag{42}$$

And then integrating Term 3.

$$A_n \int_{-\pi}^{\pi} cos(nx)\, cos(mx)\, dx$$
$$= \begin{cases} 0 \ \text{ if } \ n \neq m \\ A_m \pi \ \text{ if } \ n = m \end{cases}$$

Notice, in the case where $n = m$ we wrote A_m instead of A_n - this is because in the case where $n \neq m$ every term equals zero.

Finally integrating Term 4.

$$\int_{-\pi}^{\pi} sin(nx)cos(mx)$$
$$= \begin{cases} 0 \text{ if } m \neq n \\ 0 \text{ if } m = n \end{cases}$$

Putting this all together we have:

$$A_m\pi = \int_{-\pi}^{\pi} f(x)\, cos(mx)\, dx \tag{43}$$
$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx) \tag{44}$$

Notice in (44) that A_m is rewrote to be A_n – we're able to do this since we're assuming $m = n$. Recall, if $n \neq m$ then the integral is zero.

Step 3: Let's find the b_n terms

To find the b_n terms we'll multiply (28) by $sin(mx)$, where m is just some constant. Rewriting (28) provides:

$$\int_{-\pi}^{\pi} \underbrace{f(x)sin(mx)\, dx}_{\text{Term 1}} \tag{45}$$
$$= \int_{-\pi}^{\pi} \underbrace{A_0 sin(mx)\, dx}_{\text{Term 2}} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{A_n cos(nx)\, sin(mx)\, dx}_{\text{Term 3}} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \underbrace{b_n sin(nx)\, sin(mx)\, dx}_{\text{Term 4}} \tag{46}$$

Applying similar logic as when we found the A_n terms, we can write the following.

$$\text{Term 2} = 0$$
$$\text{Term 3} = 0$$

And integrating Term 4 results in:

$$\int_{-\pi}^{\pi} sin(nx)\, sin(mx)\, dx = \tag{47}$$
$$= \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases} \tag{48}$$

Putting this all together we have:

$$b_n\,\pi = \int_{-\pi}^{\pi} f(x)\,sin(mx)\,dx \tag{49}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,sin(nx)\,dx \tag{50}$$

Again, notice that we wrote $sin(mx)$ as $sin(nx)$ because we're assuming that $m = n$.

Let's recap all the coefficients.

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,sin(nx)\,dx$$