1 ARCH Models - Introduction

An ARCH(1) Process has the following properties:

$$r_t = \sigma_t e_t$$
$$e_t = \text{white noise}(0,1)$$
$$\sigma_t = \sqrt{\omega + \alpha_1 r_{t-1}^2}$$

The conditional mean (given the past) of r_t is:

$$\begin{aligned} E(r_t|r_{t-1}, r_{t-2}, ...) &= E(\sigma_t\, e_t|r_{t-1}, r_{t-2}, ...) && (1)\\ &= \sigma_t\, E(e_t\, |r_{t-1}, r_{t-2}, ...) && (2)\\ &= \sigma_t * 0 = 0 && (3) \end{aligned}$$

Applying the law of iterated expectations (LIE), the unconditional mean is:

$$\begin{aligned} E(r_t) \;&=\; E[E(r_t\,|\, r_{t-1}, r_{t-2}, ...)] && (4)\\ &= E(0) = 0 && (5) \end{aligned}$$

Based off equation (5) we can see the ARCH(1) process has a 0 mean.

Demonstrate the ARCH(1) process is serially uncorrelated:

$$\begin{aligned} E(r_t\; r_{t-1}) &= E[E(r_t, r_{t-1}\,|\, r_{t-1}, r_{t-2}, ...)] && (6)\\ &= E[r_{t-1} E(r_{t-1}|r_{t-1}, r_{t-2}, ...)] && (7)\\ &= E(r_{t-1} * 0) = 0 && (8) \end{aligned}$$

Therefore, the covariance between r_t and r_{t-1} is:

$$\begin{aligned} cov(r_t, r_{t-1}) &= E(r_t * r_{t-1}) - E(r)\, E(r_{t-1}) && (9)\\ &= 0 && (10) \end{aligned}$$

In equation (9), we're simply applying the definition of covariance.

$$cov(X,Y) = E(XY) - E(X)\, E(Y)$$

Because the covariance is zero, r_t can't be predicted using its history (i.e. $r_{t-1}, r_{t-2}, ...$), which is evidence for the EMH. But, r^2 can be predicted. Let's show the conditional variance of r_t.

$$\begin{aligned} var(r_t\,|\, r_{t-1}, r_{t-2}, ..) &= E(r_t^2|r_{t-1}, r_{t-2}, ..) && (11)\\ &= E(\sigma^2\, e_t^2|r_{t-1}, r_{t-2}, ..) && (12)\\ &= \sigma_t^2 E(e_t^2|r_{t-1}, r_{t-2}, ..) && (13)\\ &= \sigma_t^2 * 1 = \sigma_t^2 && (14) \end{aligned}$$

- In equation (11), we applied $r_t = \sigma_t \, e_t$
- In equation (12), we pulled out σ^2 since it's dependent on historical data
- In equation (13), we applied $var(e_t) \sim (0, 1)$

So, σ_t^2, represents the conditional variance, which by definition is a function of history. Recall:

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 \tag{15}$$

$$\sigma_t = \sqrt{\omega + \alpha_1 r_{t-1}^2} \tag{16}$$

$$\tag{17}$$

By rewriting the left-hand side of equation (11) and applying equation (15), we have :

$$E(r_t^2 | r_{t-1}, r_{t-2}, ...) = \omega + \alpha_1 r_{t-1}^2 \tag{18}$$

Equation (18) implies we can estimate ω and α_1 by regressing r_t^2 onto an intercept term and r_{t-1}^2. It also implies that r_t^2 follows an ARCH(1) process.

Now, let's look at unconditional variance (why we're looking at this will make sense in the next part).

$$\begin{aligned} var(r_t) &= E(r_t^2) - [E(r_t)]^2 = E(r_t^2) && (19) \\ &= E[E(r_t^2 | r_{t-1}, r_{t-2}, ...)] && (20) \\ &= E(\omega + \alpha_1 r_{t-1}^2) && (21) \\ &= \omega + \alpha_1 E(r_{t-1}^2) && (22) \\ & && (23) \end{aligned}$$

In equation (19), that is the definition of variance. Therefore, we can write:

$$\begin{aligned} \sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 && (24) \\ &= \omega + \alpha_1 \sigma_t^2 && (25) \\ \omega &= \sigma^2 (1 - \alpha_1) && (26) \\ \sigma^2 &= \frac{\omega}{1 - \alpha_1} && (27) \\ & && (28) \end{aligned}$$

- Equation (21), applied the LIE
- Equation (22), applied equation (18)
- Equation (24), applied the result of equation (14)

- Equation (25), substituted σ_t^2 for σ_{t-1}^2 because this is the unconditional variance
- Equation (26) factored σ^2 out of $\sigma^2 - \alpha_1\sigma^2$
- Note, equation (27) holds if $0 < a < 1$

Rewriting equation (27) we have $\omega = \sigma^2(1 - \alpha_1)$. Plugging this into equation (18), $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2$, we have:

$$\sigma_t^2 = \sigma^2 + \alpha_1(r_{t-1}^2 - \sigma^2) \tag{29}$$

So, conditional variance is a combination of unconditional variance and deviation of squared error from its average value. We obtain an ARCH(p) process if r_t^2 follows an AR(p) process: $\sigma^2 = \omega + \sum_{i=1}^{p} \alpha_1 r_{t-i}^2$