1 Separation of Variables

The method separation of variables relies upon the assumption that a function fo the form:

$$u(x,t) = \phi(x)\,G(t) \tag{1}$$

will be a solution to a linear homogeneous PDE. Let's look at an example.

Example 1

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad u(x,o) = f(x), \quad u(0,t) = u(L,t) = 0 \tag{2}$$

Let's plug (1) into the given equation.

$$\frac{\partial}{\partial t}\phi(x)\,G(t) = K\frac{\partial^2}{\partial x^2}\,\phi(x)G(t) \tag{3}$$

$$\phi(x)\frac{d}{dt}\,G(t) = KG(t)\frac{d^2}{dx^2}\,\phi(x) \tag{4}$$

In equation (4), we factored out terms not relvant to the derivative, transforming it from partial derivatives to normal derivatives.

At this point, we want all the t's on one side and all the x's on the other side. Dividing by $\phi(x)\,G(t)$, produces the desired result.

$$\frac{1}{G}\,\frac{dG}{dt} = K\frac{1}{\phi}\,\frac{d^2\phi}{dx^2} \tag{5}$$

$$\frac{1}{K\,G}\,\frac{dG}{dt} = \frac{1}{\phi}\,\frac{d^2\phi}{dx^2} \tag{6}$$

In equation (6), we're diving by K, this is done for convenience down the road.

For these to be equal, given they have different variables, they must both equal the same constant. Therefore, we can write:

$$\frac{1}{KG}\,\frac{dG}{dt} = \frac{1}{\phi}\,\frac{d^2\phi}{dx^2} = -\lambda \tag{7}$$

Let's rewrite (7) as two ODEs:

$$\frac{dG}{dt} = -\lambda KG \tag{8}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{9}$$

Let's apply the left BC: $u(0,t) = 0$

$$u(0,t) = \phi(0)\, G(t) = 0 \tag{10}$$

Recall the trivial solution is when $u(x,t) = 0$. To avoid this trivial solution, we have to set $\phi(0) = 0$. Otherwise, $G(t)$ would have to be 0 for all t, resulting in the trivial solution.

Apply the right BC: $u(L,t) = 0$

$$u(L,t) = \phi(L)\, G(t) = 0 \tag{11}$$

Using similar logic as for the left BC to avoid the trivial solution, we set $\phi(L) = 0$.

Recapping what we've done so far, we've transformed the given problem into two ODEs. One ODE is to solve for the time derivative, and the other ODE is to solve the spatial derivative. Additionally, we were able to infer that both boundary conditions equal 0.

Example 2

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad u(x,o) = f(x), \quad \frac{\partial u}{\partial x}(0,t) \;=\; \frac{\partial u}{\partial x}(L,t) = 0 \tag{12}$$

This is an example with no sources and perfectly insulated boundaries. So, let's assume a solution of the following form.

$$u(x,t) = \phi(x)\, G(t) \tag{13}$$

Similar to the prior problem, let's plug the given equation into (13).

$$\frac{\partial}{\partial t}\phi(x)G(t) = K\frac{\partial^2}{\partial x^2}\,\phi(x)G(t) \tag{14}$$

$$\phi(x)\frac{d}{dt}G(t) = K\, G(T)\,\frac{d^2}{dx^2}\,\phi(x) \tag{15}$$

$$\frac{1}{K\, G}\frac{d}{dt}G(t) = \frac{1}{\phi(x)}\,\frac{d^2}{dx^2}\,\phi(x) \tag{16}$$

$$= -\lambda \tag{17}$$

In equation (15) we factored out the constant so we could rewrite it as a normal derivative. In equation (17), we divided by $\phi(x)G(t)$ and K for conveneience, and set it equal to a constant $-\lambda$.

Let's rewrite (17) as two ODEs.

$$\frac{d}{dt}G = -\lambda K G \tag{18}$$

$$\frac{d^2}{dx^2}\phi = -\lambda\phi \tag{19}$$

Let's check the BCs, starting with the left BC

$$\frac{\partial(G(t)\,\phi(x))}{\partial x}(0,t) = 0 \tag{20}$$

$$\tag{21}$$

To avoid the trivial solution, we'll set $\frac{d}{dx}\phi(0) = 0$.

Apply the right BC:

$$\frac{\partial(G(t)\,\phi(x))}{\partial x}(L,t) = 0 \tag{22}$$

$$G(t)\,\frac{d}{dx}\,\phi(L) = 0 \tag{23}$$

To avoid the trivial solution, we'll set $\dfrac{d}{dx}\phi(L) = 0$.

Let's recap what we found. We transformed the PDE to two ODEs, and found the derivatives of both BCs are equal to 0.

$$\frac{d}{dx}\phi(0) = 0 \tag{24}$$

$$\frac{d}{dx}\phi(L) = 0 \tag{25}$$

2 Solving the Heat Equation

Let's look at the given problem.

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad u(x,o) = f(x), \quad u(0,t) = u(L,t) = 0 \tag{26}$$

We've seen this problem in the prior section. The two ODEs we need are:

$$\frac{dG}{dt} = -K\lambda G \tag{27}$$

$$\frac{d^2\phi}{dx^2} + \phi\lambda = 0 \tag{28}$$

and the BCs are: $\phi(0) = 0$ and $\phi(L) = 0$.

At this point, we have two ODEs to solve.

Let's solve the spatial differential equation first. Notice, the spatial differential equation is very similar to the first example in the eigenvalues and eigenfunctions section (i.e. $y'' + \lambda y = 0$). So, following a similar method from that section, we'll solve for three different cases of λ: $\lambda > 0, \lambda = 0$, and $\lambda < 0$.

Case 1: $\lambda > 0$

$$r^2 + \lambda = 0 \tag{29}$$

$$r_{1,2} = \pm\sqrt{\lambda}\, i \tag{30}$$

The general solution for complex roots is:

$$\phi(x) = c_1 e^{a\,x}\, cos(ux) + c_2 e^{a\,x}\, sin(ux) \tag{31}$$

Where the roots are written as: $r_{1,2} = a \pm u\,i$. So, in this case, the $e^{a\,x}$ term falls away because $a = 0$.

Apply the first BC: $u(0,t) = 0$

$$\phi(0) = c_1\, cos(0) + c_2 sin(0) \tag{32}$$

$$\implies 0 = c_1 \tag{33}$$

Apply the second BC: $u(L,t) = 0$

$$\phi(L) = c_2\, sin(\sqrt{\lambda}\, L) \tag{34}$$

In the above equation, $c_1 cos(ut)$ falls away because $c_1 = 0$. Since we're after the non-trivial solution, we want:

$$sin(\sqrt{\lambda}\, L) = 0 \tag{35}$$

$$\implies \sqrt{\lambda}\, L = n\pi \quad \forall \;\; n = 1, 2, 3, ... \tag{36}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \tag{37}$$

Plugging this into the formla for the general solution, we get the following:

$$\phi(x) = sin\left(\frac{n\pi x}{L}\right) \qquad \forall\, n = 1, 2, 3, ... \tag{38}$$

Notice the c_2 term is not written explicity - that's because we'll absorb it into another constant later.

Case 2: $\lambda = 0$

$$0 = r^2 + \lambda \tag{39}$$

$$r^2 = 0 \tag{40}$$

Integrating twice results in(we're integrating because we need the value of r, not r^2):

$$r = c_1 + c_2 x \tag{41}$$

Applying the first BC: $\phi(0) = 0$

$$0 = \phi(0) = c_1 \tag{42}$$

Apply the second BC: $\phi(L) = 0$

$$0 = \phi(L) = c_2 L \tag{43}$$

$$\implies c_2 = 0 \tag{44}$$

Takeaway: only the trivial solution is possible in this case

Case 3: $\lambda < 0$

$$0 = r^2 - \lambda \tag{45}$$

$$r = \pm\sqrt{\lambda} \tag{46}$$

Recall the usual general solution for real distinct roots is:

$$\phi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{47}$$

but we can rewrite this in hyperbolic form:

$$\phi(x) = c_1 cosh(a\, x) + c_2 sinh(a\, x) \tag{48}$$

where $r_1 = a$ and $r_2 = a$. Therefore, we can write the following:

$$\phi(x) = c_1\, cosh(\sqrt{-\lambda} x) + c_2 sinh(\sqrt{-\lambda} x) \tag{49}$$

Let's apply the first BC:

$$0 = \phi(0) = c_1\, cos(0) + 0 \tag{50}$$
$$\implies c_1 = 0 \tag{51}$$

Applying the second BC:

$$0 = \phi(L) = c_2\, sinh(\sqrt{-\lambda}\, L) \tag{52}$$

To avoid the trivial solution we need to avoid setting $c_2 = 0$. We are already assuming that $\lambda < 0$ and that means $L\sqrt{-\lambda} \neq 0$, meaning $sinh(L\sqrt{-\lambda}) \neq 0$. Therefore, we must have $c_2 = 0$, resulting in the trivial solution.

Finally, the complete list of eigenvalues and eigenfunctions for the spatial differnetial equation are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{53}$$
$$\phi(x) = sin\left(\frac{n\pi x}{L}\right) \quad \forall\, n = 1, 2, 3, ... \tag{54}$$

Now, let's solve the time differential equation.

$$\frac{dG}{dt} = -K\lambda_n G \tag{55}$$

This is a first order, linear, separable differential equation with the following solution:

$$G(t) = ce^{-K\lambda_n t)} \tag{56}$$
$$= ce^{-K(\frac{n\pi}{L})^2\, t} \tag{57}$$

In the last equation, we're plugging in $\lambda = (\frac{n\pi}{L})^2$

We've solved both the differential equations so we can write down a solution.

$$u_n(x, t) = \phi(x)\, G(t) \tag{58}$$
$$= B_n\, sin\left(\frac{n\pi x}{L}\right)\, e^{-K\,(\frac{n\pi}{L})^2\, t} \quad \forall\, n = 1, 2, 3, ... \tag{59}$$

We changed c in the solution to the time differential equation to b_n to denote it will probably be different for each value of n.