1 Eigenvalues and Eigenfunction

Recall, for a given square matrix, A, with an eigenvalue of λ and eigenvector of $\overrightarrow{x}$, we can write:

$$A\overrightarrow{x} = \lambda\overrightarrow{x} \tag{1}$$

In order for λ to be an eigenvalue then we had to be able to find non-zero solutions to the equation.

This raises the question, how do eigenvalues connect to boundary value problems (BVPs)? Values of λ that produce a non-trivial solution will be referred to as eigenvalues, and the non-trivial solutions wil be called eigenfunctions.

Let's consider an equation of the following form:

$$y'' + \lambda y = 0 \tag{2}$$

For those values of λ that produce a non-trivial solution we will call that an eigenvalue, and the solution itself will be referred to as the eigensolution.

Let's look at a quick example.

Example 1:
Solve the BVP

$$y'' + 4y = 0 \quad y(0) = 0, \quad y(\lambda\pi) = 0 \tag{3}$$

The characteristic equation is:

$$0 = r^2 + 4 \tag{4}$$

$$r = 2i \tag{5}$$

The general solution to complex roots is:

$$y(x) = c_1 e^{\lambda x} cos(ux) + c_2 e^{\lambda x} sin(ux) \tag{6}$$

where the roots are defined as $r_{1,2} = \lambda \pm u\,i$; so, $\lambda = 0$ and $u = 2$ (because $r = 0 \pm 2\,i$). Thefore, we can write:

$$y(x) \ = \ c_1\, cos(2x) + c_2\, sin(2x) \tag{7}$$

Let's find c_1 and c_2 by plugging-in the boundary conditions (BCs).

Apply the first BC: $y(0) = 0$

$$\begin{aligned} 0 = y(0) &= c_1\, cos(0) + c_2\, sin(0) & (8)\\ &= c_1\, 1 + c_2\, 0 & (9)\\ &\implies c_1 = 0 & (10) \end{aligned}$$

Apply the second BC: $y(2\pi) = 0$

$$\begin{aligned} 0 = y(2\pi) &= c_1\, cos(4\pi) + c_2\, sin(4\pi) & (11)\\ &= c_1 + 0 & (12)\\ &\implies c_1 = 0 & (13) \end{aligned}$$

Therefore, c_2 is arbitrary and the solution is:

$$y(x) = c_2\, sin(2x) \tag{14}$$

In this example, $\lambda = 4$, and we found a non-trivial solution. Therefore, the eigenvalue is 4, and the eigensolution is $y(x) = c_2\, sin(2x)$.

Example 2:
Find the eigenvalues and eigensolutions for the BVP.

$$y'' \ + \ 3y = 0 \qquad y(0) = 0, \ \ y(2\pi) = 0 \tag{15}$$

The characteristic equation is:

$$\begin{aligned} 0 &= r^2 \ + \ 3 & (16)\\ r &= \sqrt{3}\, i & (17) \end{aligned}$$

The general solution is:

$$\begin{aligned} y(x) \ &= \ c_1 e^{\lambda x} cos(ux) \ + \ c_2 e^{\lambda x} sin(ux) & (18)\\ &= c_1 cos(\sqrt{3}\, x) + c_2 sin(\sqrt{3}\, t) & (19) \end{aligned}$$

Let's apply the BCs.

Apply the first BC: $y(0) = 0$

$$0 = y(0) = c_1 \tag{20}$$
$$\implies c_1 = 0 \tag{21}$$

Apply the second BC: $y(2\pi) = 0$

$$0 = y(2\pi) = 0 + c_2 sin(\sqrt{3}\, 2\pi) \tag{22}$$
$$= 0 + c_2(-0.9936) \tag{23}$$
$$\implies c_2 = 0 \tag{24}$$

Since $c_1 = c_2 = 0$ the solution is $y(x) = 0$, which is obviously trivial. Therefore, there are no eigenvalues or eigensolutions to this equation.

At this point, we've seen two examples, from the same generalized formula: $y'' + \lambda y = 0$ with the same BCs. Let's find all the eigenvalues and eigensolutions to this formula.

Example 3
Let's generalize the previous two examples by looking at the following formula:

$$y'' + \lambda y = 0 \tag{25}$$

We'll solve this problem by looking at the following three cases: i) $\lambda > 0$, ii) $\lambda = 0$, and iii) $\lambda < 0$.

Case 1: $\lambda > 0$

In this case, the characteristic equation is:

$$0 = r^2 + \lambda \tag{26}$$
$$r_{1,2} = \pm\sqrt{-\lambda} \tag{27}$$

Since we're assuming $\lambda > 0$, we can write:

$$r_{1,2} = \pm\sqrt{\lambda}\, i \tag{28}$$

The general solution is:

$$y(x) = c_1 cos(\sqrt{\lambda}\,x) + c_2 sin(\sqrt{\lambda}\,x) \tag{29}$$
$$\tag{30}$$

Apply the first BC: $y(0) = 0$

$$0 = y(0) = c_1 \tag{31}$$
$$\implies c_1 = 0 \tag{32}$$

Apply the second BC: $y(2\pi) = 0$

$$0 = y(2\pi) = c_2 sin(2\pi\sqrt{\lambda}) \tag{33}$$

This means either $c_2 = 0$ or $sin(2\pi\sqrt{\lambda}) = 0$. Remember, we want non-trivial solutions, and if $c_2 = 0$ then that would be the trivial solution. Therefore, let's assume $c_2 \neq 0$, resulting in:

$$0 = sin(2\pi\sqrt{\lambda}) \tag{34}$$
$$\implies 2\pi\sqrt{\lambda} = n\pi \quad \forall n \;=\; 1, 2, 3, ... \tag{35}$$
$$\implies \lambda = \frac{n^2}{4} \tag{36}$$

We're taking advantage of the fact that we know where $sin \;=\; 0$ and that $\lambda = 0$ results in $2\pi\sqrt{\lambda} > 0$.

The corresponding eigenvalues and eigenfunctions are:

$$\lambda = \frac{n^2}{4} \quad \forall \quad n = 1, 2, 3, ... \tag{37}$$
$$y_n(x) = sin\left(\frac{nx}{2}\right) \tag{38}$$

Case 2: $\lambda = 0$
In this case, the BVP becomes;

$$y'' = 0\ , \quad y(0) = 0\ , \quad y(2\pi) = 0 \tag{39}$$

Integrating twice results in:

$$y' = c_1 \tag{40}$$
$$y = c_1 x + c_2 \tag{41}$$

Apply the first BC: $y(0) = 0$

$$0 = y(0) = c_2 \tag{42}$$

Apply the second BC: $y(2\pi) = 0$

$$\begin{aligned} 0 = y(2\pi) &= c_1 2\pi + c_2 && (43) \\ \implies c_1 &= 0 && (44) \end{aligned}$$

Since c_2 is already 0, then c_1 must be 0 resulting in the trivial solution.

Case 3: $\lambda < 0$
The characteristic equation is:

$$\begin{aligned} 0 &= r^2 - \lambda && (45) \\ r_{1,2} &= \pm\sqrt{\lambda} && (46) \end{aligned}$$

The usual general solutions for a differential equation with two distinct real roots is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{47}$$

but it's more convenient to rewrite that in hyperbolic form:

$$y(x) = c_1 cosh(\lambda x) + c_2 sinh(\lambda x) \tag{48}$$

Apply the first BC: $y(0) = 0$

$$\begin{aligned} 0 = y(0) &= c_1 cosh(0) + c_2 sinh(0) && (49) \\ &= c_1 && (50) \\ \implies c_1 &= 0 && (51) \end{aligned}$$

Apply the second BC: $y(2\pi) = 0$

$$0 = y(2\pi) = c_2 sinh(2\pi\sqrt{\lambda}) \tag{52}$$

Since we know that $sinh(2\pi\sqrt{\lambda}) \neq 0$ we know that $c_2 = 0$.

Takeaway: for this BVP, when $\lambda < 0$, we only have the trivial solution.

Summary: For all three cases, we have the following eigenvalues and eigensolutions:

$$\begin{aligned} \lambda_n &= \frac{n^2}{4} && (53) \\ y_n(x) &= sin\left(\frac{nx}{2}\right) \quad \forall n = 1, 2, 3, ... && (54) \end{aligned}$$