Gravitational Waves

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1.

CALCULATION OF ENERGY AND ω OF ROTATING OBJECTS

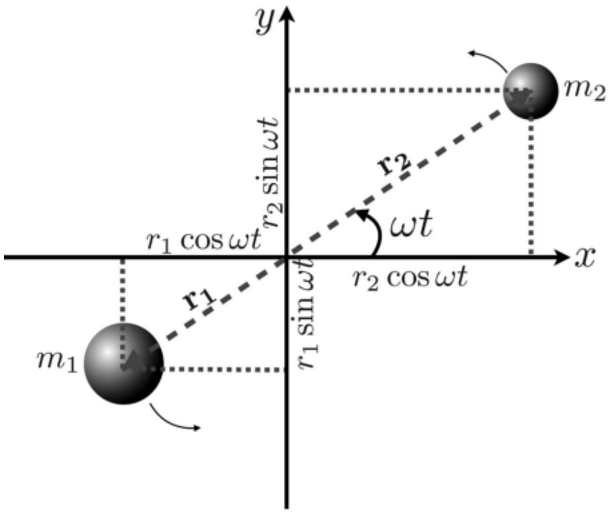

Figure 1: Figure 1:Rotating Mass

Masses of the objects are m_1 and m_2 and their respective distances from the center are r_1 and r_2 respectively.

$$r_1 = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad r_2 = \frac{m_1}{m_1 + m_2} \tag{1}$$

As the centripetal force is provided by the gravitational force. By equating, we get:

$$\frac{Gm_1m_2}{r^2} = m_1\omega^2 r_1 \tag{2}$$

$$\omega^2 = \frac{Gm_2}{r_1 r^2}$$

by substituting from equation 1

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3} \tag{3}$$

Moment of Inertia of the system is

$$I = m_1 r_1^2 + m_2 r_2^2 \tag{4}$$

$$I = \frac{m_1 m_2 r^2}{m_1 + m_2} \tag{5}$$

TOTAL ENERGY=KINETIC ENERGY +POTENTIAL ENERGY

$$\text{TOTAL ENERGY} = \frac{I\omega^2}{2} - \frac{Gm_1m_2}{r}$$

FROM 5 AND 3 WE GET

$$\text{TOTAL ENERGY} = -\frac{Gm_1m_2}{2r} \tag{6}$$

2.

RADIATED GRAVITATIONAL POWER
By analogy to electromagnetic dipole radiation we expect that the gravitational radiation field should be proportional to the quadrupole moment that sources it and hence the radiated power$\propto$ I*I . This is because the energy density in a gravitational wave depends on the square of the field much as the energy density in an electromagnetic wave depends on the squares of the electric and magnetic fields. We also expect the radiated power to depend on the frequency ω at which the radiating system is oscillating and on the fundamental constants G and c.The value of power radiated will be depended on speed of light because gravitational radiation travels at the speed of light.As a result, the emission of gravitational waves and the associated loss of energy occur at a rate determined by the speed at which these waves propagate through spacetime.Hence on dimensional grounds we expect.In summary, the radiated gravitational power depends on the frequency of rotation of masses because higher frequencies of rotation lead to faster-changing quadrupole moments, resulting in a higher rate of emission of gravitational waves and thus a greater power radiated away from the system.

$$P = \alpha \mathcal{I}^2 \omega^a G^b c^d \tag{7}$$

By dimensional analysis we will get the following linear equations,

$$2 - b = 1 \tag{8a}$$

$$d + 3b = -2 \tag{8b}$$

$$d + 2b + a = 3 \tag{8c}$$

By solving this system of linear equations, we get $b = 1$, $d = -5$ and $a = 6$. Putting $\alpha = \frac{32}{5}$ we get,

$$P = \frac{32}{5} \frac{\mathcal{I}^2 \omega^6 G^b}{c^5} \tag{9}$$

Suppose the rotating system lies in the x-y plane and rotates about the z-axis. After a half rotation the components of the the quadrupole tensor will have returned to their original values. As far as the quadrupole moment goes the radiating system is back to its original state after just half a rotation; hence the frequency of the radiation is twice the frequency of rotation

3.

POWER RADIATED IN TERMS OF ANGULAR FREQUENCY FROM EQUATION 3 AND 5 WE GET

$$\mathcal{I} = \frac{m_1 m_2 G^{\frac{2}{3}}}{(m_1 + m2)^{\frac{1}{3}} \omega^{\frac{4}{3}}} \tag{10}$$

Substituting in equation 9 we get

$$P_{\text{rad}} = \frac{32 G^{\frac{7}{3}} (m_1 + m_2)^{-\frac{2}{3}} m_1^2 m_2^2 \omega^{\frac{10}{3}}}{5c^5} \tag{11}$$

4.

CHIRP MASS

$$E_{\text{Tot}} = -\frac{1}{2} \frac{G^{\frac{2}{3}} m_1 m_2}{(m_1 + m_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}} \tag{12}$$

Differentiating total energy wrt time we get,

$$-\frac{dE_{\text{Tot}}}{dt} = \frac{1}{3} G^{\frac{2}{3}} \frac{m_1 m_2}{(m_1 + m_2)^{\frac{1}{3}}} \omega^{-\frac{1}{3}} \frac{d\omega}{dt} \tag{13}$$

Equating this with P_{rad} we get,

$$\frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1 + m_2)^{\frac{1}{5}}} = \frac{c^3}{G} \left(\frac{5}{96} \omega^{-\frac{11}{3}} \frac{d\omega}{dt} \right)^{\frac{3}{5}} \tag{14}$$

5.

CHIRP MASS IN TERMS OF FREQUENCY OF RADIATED GRAVITATIONAL POWER
Equating 2π*f to 2*ω because frequency of radiation is twice to that of orbital frequency , we have that $\omega = \pi$f. Making this substitution in eq (14) we obtain

$$M_c = \frac{c^3}{G} \left(\frac{1}{3\alpha} \pi^{-\frac{8}{3}} f^{-\frac{11}{3}} \frac{df}{dt} \right)^{\frac{3}{5}}$$

(15)