Topology: The Interesting Parts

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This is a math book for mathematically inclined people who don't read much math. It covers Topology, the study of open sets. When I started studying it, I was surprised by how often I was surprised. The story begins with a very broad definition of *open set*, just shy of *whatever you want to call an open set*, but from that seemingly overbroad starting point, all sorts of constructions emerge almost magically. There will be stretchy doughnuts and fractals, but the bulk of the book will be an inquiry into one of the key philosophical problems of mathematics: how an infinite number of points cohere into a line or a surface. Topology has a number of distinct answers to that question, none of which depend on formalizing some concept of infinitesimal distances, revealing that it is connection, not closeness, which is vital. It is a field filled with eccentric counterexamples, each a glimpse into a world much like ours, but strangely different.

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Chapter 1

Introduction: Interesting

Which is bigger, the positive integers $\{1, 2, 3, 4, \ldots\}$ or the even positive integers, $\{2, 4, 6, 8, \ldots\}$? The intuitive answer is that the first is larger than the second, because it consists of the evens plus the odds, while the second is just the even half. But if we think about it more, we see that they are in fact the same size, because we can write a one-to-one mapping, $1 \to 2$, $2 \to 4$, $3 \to 6$, $N \to 2N$, so that every element of the first set has exactly one corresponding element in the second, and a vice-versa one-to-one mapping in the other direction from even integers to all integers, $2 \to 1$, $4 \to 2$, $6 \to 3$, $N \to N/2$.

Or recall Zeno's Paradox: a tortoise crawling from zero to one meter will never fully cross the one meter mark, because first the tortoise's center of gravity must reach half a meter, then must reach $\frac{3}{4}$ meters, then $\frac{7}{8}$ meters, then $\frac{15}{16}$ meters, and to reach an infinite number of checkpoints, the tortoise's center of gravity will need infinite time. That Zeno's Paradox gives us any pause indicates that our intuition about the infinitely small is just as bad as our intuition about the infinitely large. They are in many ways the same problem, as $\{2, 3, 4, 5, \ldots\}$ easily turns into $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$.

David Foster Wallace, in *Everything and More: A Compact History of Infinity* [Wallace, 2003], pitches Zeno's Paradox as one of the core conundrums of mathematics from antiquity to the mid-1900s. The Calculus you learned in high school or college, about infinitely small differentials, clearly depends on all those infinitely small amounts adding up to something finite and definite. Wallace teased Topology in his book's title: Compactness (Chapter 8) will be a key tool by which topologists understand Zeno's paradox, and therefore a key tool in understanding the infinities in our lives.

Topology is the study of open sets. This immediately engenders the question: What is an open set? And the answer is: whatever set of points we choose. Topology is a world-building exercise in which we take a set of points, maybe the real number line, maybe the numbers $\{3, 7, 14\}$, maybe the members of the FC Barcelona futbol team, and we write down a list of groupings—whatever we want—which we declare to be open sets. The character of the world we build will depend on how we choose to invent the concept of open sets. Some will be much like the world we live in now, and others will be strangely different.

The difference between an open set and any other, like a closed set, is that the infinite union of open sets is guaranteed to also be an open set. That infinite union rule will be the (seemingly insufficient) hook we use to ask more questions of the infinitely large and infinitely small.

Those of you familiar with topology as presented in popular media will note that the question of confronting infinities and how smooth surfaces are built from points ("point-set topology") has little to do with doughnuts made of rubber sheets ("rubber-sheet topology"). There will be a limited sampling of stretchy doughnuts, in Chapter 6. But I personally find it more interesting to contemplate where those rubber sheets came from. I mean, have a look at your skin. You know from your Biology classes that it is made from individual cells; you know from your Physics classes that those cells are made from atoms linked to form chemical compounds, linked to form cells. When you look at your skin, you see none of that. Somehow, by some miracle, those individual parts cohere into a surface. Individual cells are so close that they become, for our purposes, a unit. We won't cover anything so applied as biology, chemistry, or topography (the study of the features of the Earth's surface). But the same thing happens with theoretical points, as we take points at $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, and everything in between, and we get a line, a thing we see as a coherent entity and not just an unstructured pile of points. The turtle does cross the finish line.

1.1 Surprise

Putting *The interesting parts* in the title of this book immediately raises the question of what *interesting* means. The ever-quipping

Richard Feynman had some snark about how all mathematical results, once discovered and proven, are obvious. They take the assumptions, apply the rules, and get the result.

But there's still room for surprise.

Martin Gardner was an author who had a column in Scientific American on recreational math. One of his books, with kid-friendly cartoon illustrations, is entitled *The Unexpected Hanging* [Gardner, 1991]. In the lead story, the guard tells the prisoner that he's going to hang him at dawn some day between today and Sunday, and the exact day will be a surprise. *I won't get hung on Sunday, then,* the prisoner thinks, *because it wouldn't be a surprise. But if Sunday is impossible, and the call comes in on Saturday, I won't be surprised—Saturday would be the only option left.* If Saturday is as impossible as Sunday, repeat the logic, and Friday is impossible, because on Thursday night he'll know that it's Friday morning. Repeat, and every day is eliminated!

The guard shows up Thursday morning and the prisoner is completely surprised.

There are different arguments to make about what went wrong with the impeccable logic. But the logic chain set an expectation—nothing—and "surprise" means an expectation defied.

Interesting math breaks our expectations. If you have a reasonable idea of what an open set looks like (I picture a blob with fuzzy edges), then it's not surprising that an open set on the real number line has no maximal element. Remember those ϵ-δ (epsilon-delta) proofs from college calculus, defining a continuous function? Of course you don't, it's a mess: "A function is continuous iff, for every ϵ, there exists a δ such that if $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$." I had to memorize this—in fact, you couldn't graduate from the college I went to without knowing this incantation (and knowing how to swim). It won't be until Chapter 4 that $f^{-1}(U)$ will be fully explicated, but you can already appreciate how much simpler the equivalent statement is via topology: "a function is continuous iff, when U is an open set, then $f^{-1}(U)$ is an open set." I was flabbergasted when I first saw this. How did it happen that all that complicated mess just fell away? When I was memorizing where the ϵ goes I hadn't contemplated that such a simplification could be possible.

A mathematician to whom I showed this book defended the ϵ-δ form, saying that it encapsulates a simple concept: if there is a small change in x, there is a small change in $f(x)$, and formalizing that in algebraic instead of topological terms just requires more machinery. Even this points to potential interesting: we have one definition about open sets, one about small changes, and somehow they are (given a metric as in §2.2) equivalent. How did that happen? And if the two are equivalent, but the story is simpler without infinitessimal distances, maybe the story isn't about distances at all.

I've been reading recreational math for as long as I can remember. In lieu of day care, I'd sometimes get dumped at the University of Illinois's math library, Altgeld Hall, where I'd wander the upper tiers of the stacks. The lights ran on timers that would occasionally shut off, leaving me to find my way via the light coming up through the glass-block floors from below. Sometimes I'd find copies of The Journal of Recreational Math and read the shortest article. Or maybe I'd find a collection of Martin Gardner's Scientific American columns and read about whatever he found to present this week. Math, when there isn't a test coming, can be fun. It can be full of toys to play with in your mind.

I myself have published one column in Scientific American (on the curse of dimensionality). In my pitch to the editor, I mentioned recreational math—not sure if I referenced Martin Gardner by name—and the editor was a little annoyed and focused on all the other parts of the pitch. But even though Mr Gardner took the term *recreational math* to the grave with him, recreational math books still exist, under various guises. They always hit a limit, though, which this book hopes to blow past: they talk about math, but halt early in doing math. I want a book that tells me interesting things, that gives me surprising toys to play with, but which trusts me to be capable of confronting the real thing. So I did what everybody else who wrote a math book did: I wrote the book I wanted to read.

Math textbooks assume you are in training to do more math, but we're here to have a good time. If you're learning to play the violin, you're spending a lot of time practicing simple scales. If you're listening for the purpose of music appreciation, you're waiting for the warm-up scales to be over so you can get to the surprising and engaging bits. That is, the pacing and focus of this book will be very different from the pacing and focus of the textbooks by authors who hope you are reading on the way to becoming a full-time mathematician.

On People

An expectation-setting aside: I think of recreational math as about math, not the colorful characters who write the math. There are bookshelves of people-heavy books—I'm in the middle of Ellenberg [2022] right now and am learning fun things about a lot of people (and a decent amount of math).

I am not going to talk about people here. First, I want to focus on mathematical objects. Second, ever since writing an article about James Sakoda, who developed an influential computational model and method in 1949, and Thomas Schelling, who re-developed the method in 1971 and claimed full credit for it, I've been sour on allocating credit to a single person who "discovered" a given mathematical object [Landau and Klemens, 2023]. In knot theory, there is the Homfly polynomial, which is named for the six(!) authors who simultaneously and independently developed it (Hoste, Ocneanu, Millet, Freyd, Lickorish, Yetter), which sounds like a victory for giving full credit, but the acronym *still* left off two people who had published on the other side of the Iron Curtain (Josef Przytycki and Pawel Traczyk), and numerous Russians who had developed it but didn't think it important enough to publish [Sossinsky, 2002, p 71]. I respect that Hausdorff did much to develop T_2-separability (§7.1.4), but if he hadn't, it is deeply implausible that nobody else would have thought of it. I will avoid naming math objects after people, but will put the names of famous mathematicians in parens, to help you know what other people are talking about when they say a space is Hausdorff.

1.2 Objects and their texture

There's a thread of mathematics that breaks concepts down into fine detail, to determine precisely what you need and don't need for any given property to hold. You don't have to memorize this incidental example, but a *magma* is a combination of a set S and a function f which takes an element of S and returns an element of S. A *monoid* is a magma with an identity element named id for which $f(id, x) = x$, for all x (also, f is associative). Think positive integers and the + operation, with the identity element of zero, as $0 + x = x$ for all x. A *Group* is a monoid where every element has an inverse: for all x there exists an x_{inv} such that $f(x, x_{inv}) = id$. In our example, add in the negative numbers, where $3 + (-3) = 0$.

You can do a lot with monoids which are not a group—just think of how much of our world runs on positive integers and addition—and there's something to be said about thinking hard about exactly what properties are needed when. But I want something completely different. How do spaces which are monoids but not groups feel different—not just like groups that don't do as much, but their own thing that will never look like arithmetic?

Topology gave me oddly behaving systems in ways I had never imagined. Chapter 7 presents a sequence of nested classifications, like how a magma is a type of monoid is a type of group, but with a much more dull naming scheme. The topology which satisfies the $T_{2\frac{1}{2}}$ property but not T_3 is an infinite bloom of mushrooms from the number line, where their stem is a single point and their head is a half-sphere that comes infinitely close to but does not touch the number line itself (see §7.1.9). My friend, that is the sort of weirdness I am looking for from mathematics. And the hierarchy itself, though badly named, was unexpected to me. Which is more common, topologies where every point is a closed set, or topologies where every pair of disjoint closed sets are wrapped in a corresponding pair of disjoint open sets? Maybe that question is incoherent to you now, but by the end of Chapter 7 you'll have a solid answer.

John Conway, who cemented his position in the pantheon of recreational mathematicians by his development of the Game of Life (and surreal numbers like omega; see box in §7.3.3), took a firm stance that mathematical objects are as real as anything:

> There's no doubt that they do exist but you can't poke and prod them except by thinking about them. It's quite astonishing and I still don't understand it, despite having been a mathematician all my life. How can things be there without actually being there? There's no doubt that 2 is there or 3 is there or the square root of omega. They're very real things. I still don't know the sense in which mathematical objects exist, but they do. Of course, it's hard to say in what sense a cat is out there, too, but we know it is, very definitely. Cats have a stubborn reality but maybe numbers are stubborner still. You can't push a cat in a direction it doesn't want to go. You can't push a number, either. [Roberts, 2021, p 2]

Much of this book is about the process of poking and prodding objects too strange to exist in this world. Some examples are via a classic in the field named *Counterexamples in Topology* [Steen and Seebach, 1978], and its modernization, `topology.pi-base.org` [pi Base Community]. If the thought of a book full of objects that break assumptions sounds interesting, you're thinking like a topologist.

Once you're done inspecting the object for its own sake, you can build things with it. Stand-up comics use the problem of building furniture as a reliable go-to when they need something people have trouble with or dislike doing. First, that's not necessarily accurate, as more than enough people actually enjoy the process of taking a set of minimal parts like flat boards and nails, and combining them to form something complex and beautiful, like a bookcase. There's something a little miraculous in it. Kids build things with building blocks for the inherent joy of building, and it is only the most mean-spirited who would ask why a child wasted their time building something fanciful but impractical.

Second, mathematical constructions lack much of the tædium of physical construction. As a klutz, hammering a nail straight has never been my forté, and though I do like the process of watching a pile of boards become a bookcase, the process is stressful to me, as I know that I might derail everything by bending a nail at any moment. Of course, mathematical constructions have no such issues. I know many find the process of hammering a row of twenty nails soothing, but I get bored, and appreciate that mathematical constructions show you how one step is done, then tell you to assume that step is repeated an infinite number of times.

You'll see constructions in this book typographically distinguished from other chains of logic by being a sequence of numbered steps. The constructions are typically longer than non-constructive chains of logic, but I hope you find them to be correspondingly more interesting, as a metaphorical bookcase forms before your eyes from parts you might not expect add up to a bookcase.

1.3 Albert Einstein thought he was bad at Topology.

This is a math book for mathematically curious people who don't usually read proper math books. Expect segments at the level of chatting about math like this intro, but also look forward to segments in which we take up mathematical objects and follow through the implications of their definitions in step-by-step detail. If you lead a life steeped in mathematical notation, I expect you should be able to read this book (and may enjoy it more) if you are lightly inebriated.

Many people with a healthy curiosity don't read math because they're afraid of it. Not infrequently, I'll tell somebody that I do computational social science, and they will all but interrupt me to inform me that they are bad at math, frequently adding that they grew to fear math after a teacher somewhere along the line was deeply confusing or condescending. Being bad at math somehow became a part of their identity.

There are enough systems in the world designed to make us feel bad about ourselves. Math shouldn't be one of them.

I'm not your therapist and can't exorcise your middle-school demons, but if the sight of mathematical symbols causes you to freeze, I can at least cheer you on and encourage you to slow down but keep reading, to lean into the passages presenting a symbol-laden logical chain instead of succumbing to the urge to skip ahead to parts that feel safer. Take it one bullet point at a time. This is a book about thinking through weird inventions, not juggling numbers, of which there will be fewer than you might be expecting (the number eleven only appears as a page number and in this parenthetical). We won't take the log of anything until the final pages of the book, and you'll be able to gloss over the two or three steps that use them if you don't remember how logs work. As language vocab lists go, the list of symbols in the mathematical language of this book is not much, and as you'll see from the list below, all of it can be read as plain English.

Mathematical writing is often characterized as an inherently alienating foreign language. The author of the Tony- and Pulitzer-winning play *Proof*, David Auburn, went to the same college I did, so we know he had to learn how to do an ϵ-δ proof. At key points in the play, he parodies the staccato rhythm of how mathematicians write, and/or uses it to express non-mathematical truths: "Let X equal the quantity of all quantities of X. Let X equal the cold. . . . The number of books approaches infinity as the number of months of cold approaches four. I will never be as cold now as I will in the future. The future of cold is infinite. . . . " [Auburn, 2001]

Such parody is common; the cynical go even further and say that that mathematical notation is designed to intimidate and exclude. But I wholeheartedly reject those accusations. The standard notation of today evolved after an unbelievable amount of thought as the means of writing which most clarifies difficult work. Check your local math history archives for an edition of Euclid or any other pre-algebraic author who wrote math using only words in complete sentences. It's maddening, and leaves one longing for xs and ys.

On the line-by-line level, you will hit walls and get lost—if you don't, it's time to find a more challenging Topology text where you will. Programmers have a joking-but-serious technique called *rubber duck debugging*, in which you explain a bug you're stuck on to an inanimate object, and in slowly breaking down the steps during the one-sided conversation, you realize the problem and its solution. When stuck on a line of math, you can readily do the same. Imagine explaining what you're not getting to a loved one, or your actual therapist, and what questions they would ask. Honestly, those lines where you get stuck and then untie the knot are the best parts, because that's where you see new things.

On the conceptual level, people with math anxiety often set the bar for fully understanding a concept as just beyond their current level of understanding. I invite you to go back to the first paragraph of this book, where we proved that there are as many even integers as there are integers. If you're a normal human, you feel uneasy about it and will never understand it the way you understand how a pair of scissors works. That's great—the use of formal methods to push ourselves past day-to-day intuition is the entire point, why we're here. Please enjoy it.

1.4 What do mathematicians do?

The core grind of the working mathematician is DTP: definition, theorem, proof, the same format Euclid and many others around the world used a few thousand years ago. Here's a sample, using a proof by contradiction:

Definition 1.1. Canonical open set
The canonical open set is the set of all real numbers between zero and one, excluding zero and one themselves. Usually written $(0, 1)$.

Theorem 1.1. No maximal element
The canonical open set has no maximal element.

Proof: Assume the canonical open set has a maximal element, x. There is, between x and 1, another value x_2 defined as the midpoint between x and one: $x_2 = (x + 1)/2$. Because x_2 is less than one but greater than x, it is in the canonical open set, but $x_2 > x$, contradicting our assumption that x was the maximal element in that set. If, for any claimed maximal element, we can construct a larger element, then there can be no maximal element. ■

Let's break down the three steps.

Definition: I've grown to appreciate that the definition step can sometimes be a more important—and interesting—step than the theorem part.

We could philosophy-of-science this and ask whether the canonical open set was invented or discovered. Like the mineral bauxite, it was always lying deep under the Earth, waiting to be separated from close friends like the set of numbers between zero and one but including both zero and one, for which the maximal element is simply one. It could be a simple invention, but if it is, then to steal a line from Eugene Wigner, it's an unreasonably effective invention, which does more than the close cousins to which we pay no heed. Bauxite, if you don't know it, is a rock from which aluminum is extracted. Over the centuries, we collectively worked out that you can take this ugly rock and pull one segment from it and use only that segment to make soda cans. Aluminum, an element, was discovered, but it had to be refined by human effort to be recognizably useful to other humans.

I used to focus on the theorems, but understand more now that it's all a unit. One of my professors in grad school, Jeff Banks, pointed out that when proving a theorem, you set the rules, so you might as well make it easy for yourself. Find the definitions that make the theorem work.

Theorem: Theorem 1.1 has more-or-less the same form as every other theorem: *now that we took the time to define an object, that object has certain useful properties, or a certain relationship to another carefully defined object.* Interestingness comes from this interplay between invented objects, and the properties about them we can discover. Math is built from an æsthetics of parsimony, that if you can get a lot of results about a simple concept, that's a win. That's also a win for interesting. The definition of *open set* we'll see below is obnoxiously simple: it's a list you defined yourself, plus the intersections and unions of those items. You made up that list. But like a magician asking you to pick a card, any list you invent that meets the rules will correctly behave the way the canonical open set behaves in appropriate situations. Like, wow.

Kids' books on math are filled with these sorts of tricks. Try this: pick any number between one and fifty. Got one? OK, then:

- add 25,
- divide by five then round to the nearest multiple of ten,
- divide by ten,
- multiply by six,

- add thirty,
- divide by three,
- round up to the next multiple of ten,
- then divide by four.

You will get five.

It feels like magic, though we know from the context of this being a math book that it's simply a question of an equation which, when written down, is baroque yet easily reduced to five. Nonfiction authors will sometimes set up your expectations wrong: it's more interesting to write "You may have thought that recreational math died with Martin Gardner, but in fact, many authors are still writing it under a different guise" than "Recreational math is still written, but not by that name." The minimalist school of writing far prefers the latter, but the former creates interesting. The weird expression I'd walked you through is equivalent to the expression "5", but one is succinct and one creates interesting.

In the mathematical context, the most interesting events are when the left and right sides of the relations are from different genres of math. Infinite sequences are hard to work with, but can we say anything about them by equating them with groups of open sets? Yes, we can; see §5.2.1.

Proof: William Thurston, a working mathematician, stresses that proofs are a social construct. "When I started as a graduate student at Berkeley, I had trouble imagining how I could 'prove' a new and interesting mathematical theorem. I didn't really understand what a 'proof' was. . . . Mathematical knowledge and understanding were embedded in the minds and in the social fabric of the community of people thinking about a particular topic. This knowledge was supported by written documents, but the written documents were not really primary. . . . " [Thurston, 1994]

Thurston is stressing a point that it's human understanding that matters. There are proofbots that will invent objects beyond human comprehension and prove all their properties; nobody cares. "We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from [computer-programmable] formal proofs."

The format of a proof is going to remain a sequence of steps, each of which is a logical conclusion from the prior, in this book just like you did it in high school geometry. But don't forget in all the mechanics that the point is not to be convinced that the mechanism works, but to understand how and why.

1.5 Logistics

OK, here's the Topology book I always wanted, which will tell you about all the components you need to build Topology yourself, but at a pacing to linger on the interesting stuff.

The mathematical prerequisites for this book are not large, especially given that none of the topics in this book are computationally intensive. You already made it through a definition-theorem-proof sequence, and if you followed it, you are probably at the level I'm expecting.

My apologies to any topological specialists whose specialization I left out of the book. This book covers point-set topology, and is a subset of what you might cover in an undergraduate intro to Topology class.

1.5.1 The notation

Although a vocab list is not the most exciting start, here it is, to help you read math like the pros. It's long-ish because I wanted to go beyond a terse list of definitions and include some discussion of the math you're about to encounter. Think of this as the flyover glimpsing pieces of the world you are about to delve into. If you are already familiar with all this, it will just take a minute to clarify the conventions used in this book, like what I mean by $\widetilde{S}$.

- p: A point. Generally the zero-dimensional dot on a number line or plane, but you'll see that the spaces points live in are often just a convenience to easily express how an infinite number of sets of points fit together. Nothing breaks if *a point* is a number in no underlying space, or a non-numeric but indivisible item like a member of your family tree.
- $\{\ldots\}$: curly braces indicate a set of things; e.g., $\{1, 2, 3\}$ is the set consisting of one, two, and three. Mathematical philosophers have come to the consensus that everybody knows what a set is, and it doesn't need to be explicated further than your intuition.
  - The point p and the one-element set $\{p\}$ are closely related but conceptually distinct.
  - We will deal heavily with sets of sets, like how $\{\{p_1\}, \{p_1, p_2, p_3\}\}$ is a two-element set consisting of a one-element set and a three-element set.
- $\emptyset$: the empty set, $\{\}$. Sometimes read as *null*.

- (a, b): as a set, all points in the real number line between a and b, excluding a and b themselves. These will be referred to as u-open (the usual open) sets throughout, to contrast with all the other strange topologies we'll encounter. As per Theorem 1.1, the edges of such open sets verge on embodying Zeno's paradox, as $(0, 1)$ contains 0.9, 0.99, 0.999, ..., but not 1 itself.
- $[a, b]$: The set of all points between a and b, including a and b themselves. Referred to as u-closed when needed. We will sometimes have half-open intervals, like $[0, 1)$, including zero but excluding one.
- $A|B$: read as *A given B* or *A such that B*, which is incoherent by itself but fits into various contexts. Statisticians might write $P(A|B)$ to express the probability that A occurs given that B has occurred.
- $\{x|x < 3\}$: Referred to as *set builder notation*, this is an amalgamation of notations to this point. This expression reads as *the set of* x*s such that* $x < 3$, which we might otherwise write as $(-\infty, 3)$. Before everybody had a keyboard with a pipe (|) key, this was (and often still is) written as $\{x : x < 3\}$.
- $\mathbb{Q}$: Rational numbers. All the numbers expressible as an integer divided by another nonzero integer. The noun *ratio* is embedded in the adjective *rational*.
- $\mathbb{R}$: Real numbers. All the rational numbers plus all the irrational numbers.
- $\mathbb{R}^+$: Positive reals. Zero is not positive.
- $\mathbb{R}^2$: two-dimensional reals, points on the (x, y) plane. Yes, parens are also used to indicate two-dimensional points as well as one-dimensional intervals as above; you'll work it out from context. Higher-level mathematicians don't like assuming the horizontal axis is named x and the vertical y. But when I was in high school, any time math would come up, my mother would dismiss it by saying "All I know is there's an x and a y," and then laugh. If even my math-phobic mother has internalized this notation, we might as well use it.
- $f : X \to Y$: When introducing a function, it is polite to mention the spaces in which its inputs and outputs live. Here the inputs are in the X space and outputs in the Y space. In your algebra class, functions were all of the form $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R}^2 \to \mathbb{R}$, so it wasn't worth belaboring, but here things will be much more general. We won't see functions until Chapter 4, though.
- $A \cup B$: The union of A and B, the set of points found in either

one or the other (or both). Notice that it is a stylized *U*.

- $A \cap B$: The intersection, the set of points in both A and B. You can remember it as the one that isn't a stylized *U*.
- $\bigcup_{x=1}^{\infty}(x, x+1)$: the big $\cup$ with a subscript and a superscript is a compact means of writing the union of a sequence. The format for each set in the sequence is given, in this example $(x, x+1)$, and we're specifying that x begins at 1 and counts to infinity, giving us the sequence $(1,2) \cup (2,3) \cup (3,4), \ldots$, adding up to the real number line past one, excluding the integers. Similarly for the intersection of a sequence, $\bigcap_{x=1}^{\infty}(-\frac{1}{x}, \frac{1}{x})$ (which is $\{0\}$).
- $\Sigma_{i=2}^{\infty} \frac{1}{2^i}$: an infinite sum, with the same indexing custom of counting i from the lower bound to the upper. In this case $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which approaches one as $i \to \infty$ (as i goes to infinity). Read Σ as *sigma*.
- $A \subset B$: A *is a subset of* B. The above two items, $\cup$ and $\cap$, took nouns and created compound nouns; the same stylized *U* sideways is an assertion. It looks a little like $A < B$, which is what we write when the smaller number A fits into the larger number B.
- $A \subseteq B$: Whether $A \subset A$ is a point of nomenclature debate. If an author wants to be clear, though, they'll say that A is a *strict* or *proper subset* of B if they want $A = B$ to mean $A \not\subset B$, and will use the version that sort-of incorporates an equals sign, $A \subseteq B$, if they want to allow the set to be a subset of itself. Looks somewhat like $A \leq B$.
- $A - x$: The minus sign does what you expect with sets and points. This means the set A excluding x (typically a point).
- $\widetilde{A}$: The complement of A. If the entire space is named X, then $\widetilde{A}$ is $X - A$. This will be heavily used because of Definition 3.1.
  - E.g., if $X = \{1,2,3,4,5\}$ and $A = \{1,3,5\}$, then $\widetilde{A} = \{2,4\}$.
  - Sometimes written as $X \backslash A$, though in most cases it's easier to not explicitly write the overall space.
  - In logician circles, $\sim$ is sometimes read as *not*, as if our sets were Venn diagrams and the area outside the circle A represents the group of things that lack property A.
- ϵ: The Greek letter *epsilon*. The common English equivalent is "an iota", indicating a very small amount. Not a variable, but a fixed amount in any given context; maybe think of it as 10^{-100}.
- *iff*: an English word you will find in your favorite dictionary,

defined as *if and only if*. For example, $a < b$ iff $b - a > 0$.

- $\equiv$: *is defined as*. Though $\equiv$ implies to the reader that a new association is being declared and *iff* is usually reserved for inferences from other statements, the rules of logic would allow us to replace instances of $\equiv$ with *iff*. If something is some kind of object iff it has some clever property, mathematicians sometimes find benefit in using that property to define the object. For example, the condition I gave in the example for *iff* could be used as a definition: $(a < b) \equiv (b - a > 0)$. Or, observing that integers are in one-to-one correspondence with even integers, Rudin [1976] defines: infinite set $\equiv$ a set which is in one-to-one correspondence with a proper subset of itself.
- $A \Rightarrow B$: A *implies* B, or *if* A, *then* B. For example, $(A \Rightarrow B$ and $B \Rightarrow A) \Rightarrow A \equiv B$.
- $\{\forall, \exists, \in, \ni\}$: {*For all*, *There exists*, *In*, *Such that*}. I'm putting these all here on one line because they're often used together. E.g., we could explain that $(0,1)$ has no maximum value by stating that $\forall x \in (0,1), \exists y \in (0,1) \ni x < y$, i.e., for all x in $(0,1)$, there exists a y in $(0,1)$ such that $x < y$. Whether such expressions are written out in English or compressed with symbols depends on the author's mood.
  - *Such that* is sometimes abbreviated to *s.t.*.
  - If you have a point p contained in a set S, use $p \in S$. If you have a set S_{sub} which is a subset of S, use $S_{sub} \subset S$.

1.5.2 Thanks

Thanks to Peter Gedeck, John Kaufhold, Elizabeth Landau, Aaron Schumacher, and Josh Tokle for comments and suggestions.

Chapter 2

Topologies

To define Topology, the field of study, let us define *a topology*, the structure:

Definition 2.1. Topology
A set of points X with a list of open sets of those points.

1. The full set X is on the list, as is $\emptyset$.
2. Any intersection of a finite number of open sets is also an open set.
3. Any union of open sets is also an open set.

If referring to a generic space, the whole of it will usually be referred to as X, and the list of open sets τ_x (read as *tau sub x*); then a topology is a combination (X, τ_x). Sometimes the space is taken as given and *the topology* will refer to just τ_x; you'll pick it up from the context.

Let's pause to admire how stupid simple this definition is. You make up any list of groupings—whatever you want—and we'll complete it by adding finite intersections and arbitrary unions of the elements of your list. As a technicality, we add *everything* and *nothing* to your list. This is the magician's *pick a card, any card* setup for the tricks our topologists will pull for the remainder of the text.

You have some idea of open sets from the math you've already learned, where the usual open sets are the intervals between a and b (with a and b real numbers and $a < b$), formally the set $\{x|a < x < b\}$, or more succinctly written as (a, b). The usual closed interval $\{x|a \leq$

$x \leq b\}$ is written as $[a,b]$. Following Mansfield [1963], I'm going to refer to (a,b) as u-open, the usual open sets. But the definition of *open set* implicit in the definition of topology is much, much more broad and accommodates a great deal of unusual. I'll stress this by stating a direct definition of *open set*:

Definition 2.2. Open set
Whatever set of points we want. The intersection of any finite number of open sets is also an open set, and the possibly infinite union of any open sets is an open set, and $\emptyset$ and the entire space are open sets.

A main goal is to develop definitions around the generalization of open sets which match your intuition about u-open sets. The next several chapters will basically follow that thread, asking questions of what certain expected objects will be and behave like under this definition. Infinite sequences have limits, in the case of Zeno's paradox from §1 the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \to 1$, and this will tie in with our open sets. We have some intuition that b, sitting just at the edge of (a,b) is a sort of limit point to the set, and often refer to $[a,b]$ as the closure of (a,b). If you give me any list of sets and declare that to be the open sets, can I find concepts of limit and closure that match (Chapter 3)?

From there we'll be able to hit what was to some extent the goal of a large portion of pre-1900s mathematics: defining continuity (Chapter 4). The real number line is just a list of individual points, yet we have some intuition of a universe where there is a smooth line between the mark at $\frac{3}{4}$ meters and the mark at $\frac{7}{8}$ meters. By the definition of a topology, you can see that the goal is to define continuity not in terms of real numbers, but in terms of open sets. Mansfield [1963, p 11] characterizes the game of topology as the problem of describing how things can stretch and bend without tearing. Continuity "is the key to the kingdom. The notions of 'stretching and bending' can be mathematically expressed in terms of functions. The notion of 'without tearing' can be expressed in terms of the continuity of these functions."

Or we can recast the question of whether the real numbers are a bag of points or a smooth line by asking whether it is "connected". Chapter 5 will present three different ways of defining that term.

The last few chapters cover an organization of topologies into an eight-level hierarchy (Chapter 7), another way of understanding infinite sequences (Chapter 8), and an application of the same tool we used for infinite sequences to measure things in fractional dimensions (Chapter 9).

That's the book, and this entire exposition is chasing down the implications and applications of the obnoxiously simple whatever-you-want Definition 2.2. You could may be argue that *mapping* and *preimage*, introduced in Chapter 4 are new and distinct concepts, but the rest of what is to come—closed sets, limit points, density, continuity, three concepts of connectivity, nine interlocking classes of separation properties, coverings and compactness and their implications for convergence, the torus and M-band and K-bottle—are specific configurations or variants of open sets as per Definiton 2.2. It's all in there waiting to be unfurled.

But first, let's write down some examples of topologies as per Definition 2.1 and see where they take us.

2.1 Some basic topologies

My bookshelves are full, and my junk drawer overflows, but I still go to stores and browse. Apologies to those of you who work retail, but I rarely buy anything, just kill time picking things up, inspecting them, and putting them back down. Camp stores are my favorite, where people have come up with so many clever ways to make things that fold into themselves, familiar things like forks and stools and jackets in shapes I'd never considered before. This chapter is like that, but you don't have to make shelf space for any of the objects. They cram lines of infinite length into a one-by-one box, or shave off 100% of the weight of the $[0, 1]$ interval and yet still map one-to-one to that same interval, or break our expectations entirely by defining closed sets as open sets. Nonetheless, every definition of some characteristic of a topology in later chapters (like limits or closed sets) must apply to even the weirdest topologies on this list.

As above, the rules are simple: we specify any list of sets we want, but then we have to add all finite intersections and possibly infinite unions, repeating until we have a complete, self-contained list. Though I'll sometimes omit this detail in the definitions to follow, don't forget that the entire space and $\emptyset$ must be open sets in all topologies.

As promised in the introduction, these are very different from each other and will get weird. But you live with multiple topologies now. You'll see the Manhattan metric below (§2.2); it's also called the *taxicab metric*, but most Manhattanites take the subway. What is the set of points five minutes from the World Trade Center? It includes 7th Ave & 14th and 7th & 34th Streets (take the 2 or 3 line), but not 7th & 18th or 23rd street (you'd have to transfer to the 1, and good luck doing that within five minutes), some parts of Brooklyn across the East River but no part of the East River itself, a portion of New Jersey (take the Path), but very little else on the island of Manhattan itself.

Wherever you live, and whatever the quality of your nearest subway system, you maintain different concepts of how points relate beyond the strictest tape-measure distance. You have friend circles which include some people and exclude others, whom you categorize into other circles. You might be the sort of person who goes straight to building a topology when you meet somebody new at a party: *So who do you know here? How are you linked?* The points of interest in your area and the people at the party don't change, but what sets they lie in and how those sets intersect is malleable and something you can define and redefine.

The rest of this chapter will introduce various topologies, in three batches. The first are primarily a chance to get familiar with all the subtleties in the definition of an open set (Definition 2.2), like why they took pains to allow topologies to include possibly infinite unions but only finite intersections. The second batch are those generated by metrics (aka distances) and clarifies what people mean when they say that distance doesn't matter in Topology. The third batch is when things start getting weird, with topologies where the set of points is something unusual, or where the set of open sets is different enough to have significant implications, ending with my personal favorite, the seemingly simple but somehow perverse Lexicographic Topology. There will be more throughout; I doodled a little symbol, , to mark the 24 different points in this book when new topologies are introduced.

The first several in this section are not going to be especially surprising, beyond a few little twists. The definition of an open set (Definition 2.2) is a straightforward definition with subtle implications. Given that I defined *Topology* as the study of open sets, it's worth considering those subtleties in detail.

2.1.1 **A bundle of points**

This won't happen often in this book, but there's a certain back-and-forth in developing a topology that lends itself to the style of a Platonic dialogue. Here, Socrates is teaching a talking doughnut how to build a topology.

SOCRATES: If we have a space of four points, $\{p_1, p_2, p_3, p_4\}$, what kind of topologies could we generate?

DONUT: OK, a topology is whatever I want, so the set $\{p_1\}$ is my topology. Have a nice day.

SOCRATES: Does it conform to all the rules? It looks like some things may be missing.

DONUT: You told me I can be the world-building doughnut, and an open set is whatever I want it to be. I want $\{p_1\}$ to be the only open set.

SOCRATES: But there are other conditions you have to meet. Are there missing elements from your topology?

DONUT: I guess you want the entire space—

SOCRATES: We usually notate that as X.

DONUT: —OK, X and the empty set to be in the list of open sets? So, $\{p_1, X, \emptyset\}$, is that a good topology? I think it's good enough.

SOCRATES: The definition also requires unions and intersections of all the sets in the topology.

DONUT: I only have three sets, and $\emptyset$ is barely even a set.

SOCRATES: So it'll be easy for you to tell me about its unions and intersections.

DONUT: The intersection of $\emptyset$ and anything is $\emptyset$, so that's already on my list. And if you give me *any* set S, the union of $\emptyset$ and S is just S. So intersections and unions with $\emptyset$ don't add anything.

SOCRATES: Good, now tell me about X

DONUT: The union of X and anything is X, and the intersection of X and S is just S. It doesn't do anything either. This feels pedantic.

SOCRATES: I'm glad you took the time to check unions and intersections with X and $\emptyset$ now, because what you're saying applies to any topology. We've checked them once and for all.

DONUT: I'm relieved. And that means I *am* done: if I define my topology at $\tau_1 \equiv \{p_1, X, \emptyset\}$, then I have whatever sets I want, plus the mandatory *everything* and *nothing* sets, plus the intersections and unions.

SOCRATES: Good job.

DONUT: OK, that's my topology.

SOCRATES: Formally, a topology consists of a space paired with a list of open sets, so let's define the topology as (X, τ_1).

DONUT: Let's go bake cookies.

SOCRATES: You built one topology, but you know there are others. And. . . you eat cookies?

DONUT: You take balls of cookie dough, and you can watch the little spheres flatten down to cookies. It's fun.

Note that $\{p_1, p_2\}$, for example, is not on our talking doughnut's list, so that set is not in the topology. Yes, $\{p_1, p_2\} \subset X$, and X is always on the list of open sets, but sets can't slide into our list of open sets by being a subset of other open sets. The invitation to a topology is non-transferable.

SOCRATES: Let's add one item, so now $\tau_2 \equiv \{X, \emptyset, \{p_1\}, \{p_2\}\}$. How does that look as a topology?

DONUT: Fine, I guess. You said I already talked about how intersections and unions with X and $\emptyset$ don't add any sets that aren't already on my list and I never have to check it again.

SOCRATES: Thank you for the reminder. But what about intersections and unions between $\{p_1\}$ and $\{p_2\}$?

DONUT: They don't intersect: $\{p_1\} \cap \{p_2\} = \emptyset$.

SOCRATES: Good, so we don't have to think about the intersection. The union?

DONUT: The union is another set, $\{p_1, p_2\}$, so OK, $\tau_2 \equiv \{X, \emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$.

SOCRATES: Are we done?

DONUT: I think we're done. The intersection of the new set $\{p_1, p_2\}$ with $\{p_1\}$ is just $\{p_1\}$, which we have, and the union of the new set with $\{p_1\}$ is the new set. Same story with $\{p_2\}$.

SOCRATES: Nice, now you have two topologies on this simple four-point space. What do you think of $\tau_3 \equiv \{X, \emptyset, \{p_1, p_2, p_4\}\}$?

DONUT: I can't eat it, but if there's only one set that isn't X or $\emptyset$—

SOCRATES: We sometimes call X and $\emptyset$ *trivial open sets*—

DONUT: That sounds apropos. They feel like they're just there as some sort of trivia. If there's only one nontrivial set, then there aren't intersections or unions to worry about, and you're done.

SOCRATES: You've built another world.

DONUT: To the extent that one nontrivial set is a world.

SOCRATES: We're just getting started, my doughnut friend. And if one set isn't enough to allow interesting unions and intersections, what do you think of $\tau_k \equiv \{X, \emptyset, \{p_1\}, \{p_2\}, \{p_3\}, \{p_4\}\}$,

DONUT: The union of any two of those is a pair, like $\{p_1, p_2\}$, $\{p_1, p_3\}$, $\{p_1, p_4\}$, $\{p_2, p_3\}$, all those have to be in the topology. How many of those are there?

SOCRATES: Six: three with p_1 first, two with p_2 first, one with p_3 first.

DONUT: Wait, does order matter with these sets?

SOCRATES: Not at all. Every set is just an unordered bundle, but it's nice to keep things ordered so we can keep track. Otherwise you might count $\{p_1, p_2\}$ and $\{p_2, p_1\}$ as separate sets, even though they're identical.

DONUT: OK, makes sense. Then there are four triplets, $\{p_2, p_3, p_4\}$, $\{p_1, p_3, p_4\}, \ldots$.

SOCRATES: You can keep track of those because the first is missing p_1, the second is missing p_2, and so on.

DONUT: Yeah, that's what I was doing.

SOCRATES: Is there anything else we need?

DONUT: That seems to be everything: all the singletons, pairs, triplets. The four-element set is already in my topology, it's called X.

We started with what the mathematicians might call a kernel, the six-set list τ_k, but from that kernel sprouted all the pairs and all the triplets, and the full topology is all of those (16 sets total). We also need to check for intersections, like $\{p_1, p_2\} \cap \{p_2, p_3\} = \{p_2\}$, but given that we already have every possible combination, checking intersections won't add more sets.

Definition 2.3. Basis
A set of sets B such that every open set in a topology is a union or intersection of sets in B.

- The process of finding all the possible outcomes from a process is usually referred to as *finding the closure*, a phrase I like because of how close it is in both sound and sense to *achieving a sense of closure*. To give an example from an entirely different context, say we have the operation + and the number 1. From that, you could construct $1+1 = 2$, from which you could construct $2 + 1 = 3$ or $2 + 2 = 4$. Following to the logical conclusion, the closure of the + operator and the kernel 1 is the set of all positive integers.

- Unfortunately, the word *closure* is decidedly taken in the Topological context; see Definition 3.3.
- We can assume the elements of the basis are disjoint. If you have two basis elements B_1 and B_2 where $B_1 \cap B_2$ is nonempty, then call those the pre-basis, and set up a new basis with three disjoint elements: $B_a \equiv B_1 \cap B_2$, $B_b \equiv B_1 - B_a$, and $B_c \equiv B_2 - B_a$.
- This is what the typical textbook definition of a basis does. Then we achieve closure by adding to the basis all unions of basis elements. My treatment here is a little different because the most natural basis is not necessarily the one where all basis elements are disjoint.
- You should also know that some authors also refer to a *point* having a basis. This is an entirely different concept, typically an infinite sequence of nested sets. This won't appear in this book until §9.1, where I'll call it a point-basis.

2.1.2 The Usual

The u-open sets are the basis for a topology over the real numbers. Consider how completing the topology with all requisite unions and intersections would work:

SOCRATES: We start with familiar open intervals like $(1,5)$ and $(4,7)$. How could those form unions or intersections?

DONUT: I already worked out how intersections and unions with X and $\emptyset$ are handled.

SOCRATES: Thanks for the reminder that those need to be included.

DONUT: Well, it matters, because if you have intervals that don't overlap like $(1,4)$ and $(5,7)$, then their intersection is $\emptyset$.

SOCRATES: True. We didn't even need to separately specify that it would be on the list.

DONUT: But your example, it doesn't need special handling either. If two intervals overlap, the overlap is an interval: $(1,5) \cap (4,7) = (4,5)$. Their union is also an interval: $(1,5) \cup (4,7) = (1,7)$. The intervals cover it.

SOCRATES: Do they? What if you have disjoint intervals, and you want their union, like $(1,4) \cup (5,7)$?

DONUT: That's not an interval, that's just two intervals taped together.

SOCRATES: Is it in our topology?

DONUT: I don't want it to be.

SOCRATES: But what do the rules say?

DONUT: OK, yeah, that this two-part union counts as an open set.

SOCRATES: This is also our first space with an infinite number of open sets, so we also have to think about infinite unions.

DONUT: Isn't that just the same thing, where you get either bigger intervals or an infinite number of intervals taped together?

SOCRATES: There are two ways it could be a little more interesting. If we take the union of $(0,1)$, $(0,2)$, $(0,3)$, $\ldots$, written more succinctly as $\bigcup_{i=1}^{\infty}(0,i)$—

DONUT: That's an interval with infinity as an endpoint, like $(0,\infty)$. Is that valid?

SOCRATES: Think of it as a nice way to write an interval with no actual end point at all. And if a sequence of endpoints doesn't diverge, then it converges, like the union $\bigcup_{i=1}^{\infty}(\frac{1}{i},1)$.

DONUT: As $i \to \infty$, $\frac{1}{i} \to 0$, yeah, so isn't the infinite union just $(0,1)$? That's already on my list.

SOCRATES: Sounds like we're done then.

DONUT: I have so many topologies!

The easiest two-dimensional analogue to the $(0,1)$ interval is the set of points at a distance less than one away from the origin, meaning everything inside a circle of radius one (but excluding the circle itself). Under the presumption that these are usually small and the custom that small values are written as ϵ, these open sets are frequently referred to as ϵ-balls. We'll have much more to say about them when we get to metrics below, but for now just bear in mind that the Usual Topology can easily be in higher dimensions, where those straight-line intervals become circles (or in 3-D, balls, or in 4-D, hyper-balls).

2.1.3 Subspace topologies

Within the space X, pick any subset Z. The main space's topology τ_x consists of a list of open sets, $\{s_1, s_2, s_3, \ldots\}$. Then $(Y, \{s_1 \cap Z, s_2 \cap Z, s_3 \cap Z, \ldots\})$ is a valid topology. In Figure 2.1, the open sets are the usual circles, but when restricted to a square subspace, the open sets are balls cut off wherever the square fell, leaving odd shapes, a mix of round and sharp edges.

Or you could even restrict the open sets in the 2-D topology to the line $y = 2x$. Then the Usual Topology induces on that line a set of open sets that look much like you'd expect: one-dimensional open intervals, centered around any point on the line, with any size. It's not complicated.

But bear in mind that we define open within the context of the subspace. If our host topology is the Usual Topology and our subset is $Z \equiv [0,1]$, then within the context of that subspace, an interval like $[0,0.3)$ is open, because it is formed by $(-0.3,0.3) \cap Z$. If nothing else, the entire space is always part of the topology's list of open sets, so $[0,1]$ itself must be open in Z's topology, though it's obviously not open in the parent topology.

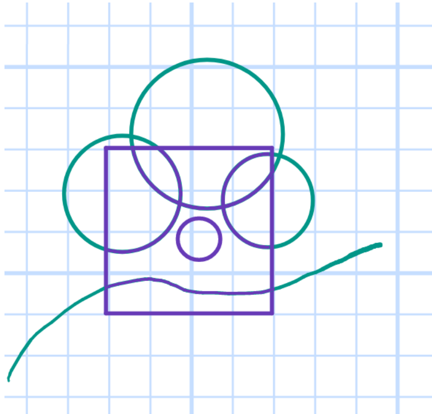

Figure 2.1: If we select a subset of the main space, the topology uses the same open sets as the parent space, but cut off within the subspace.

2.1.4 Closed intervals et al.

DONUT: You said I can make anything my open intervals?

SOCRATES: Yes, whatever you want.

DONUT: OK, the closed intervals, like $[0,1]$. I want those to be open intervals. [*Donut stares defiantly.*]

SOCRATES: You want me to be uncomfortable with this, but this is perfectly fine. But we have to think through the unions and intersections.

DONUT: It's not gonna be different from the open intervals, like how the union or intersection of overlapping closed intervals are also closed intervals.

SOCRATES: True. What about something like $[a,b]$ and $[b,c]$. What is the intersection of those two?

DONUT: It's a singleton, like $[a,b] \cap [b,c] = \{b\}$. That's just another closed interval, of zero length.

SOCRATES: This is different from the Usual Topology, where singletons weren't in the list of open sets.

DONUT: So what? It's a different topology.

SOCRATES: Let's remember this. It might be important later. Tell me about the infinite union $\bigcup_{x=3}^{\infty}\left[\frac{1}{x}, 1-\frac{1}{x}\right]$, so the union of $\left[\frac{1}{3}, \frac{2}{3}\right]$, $\left[\frac{1}{4}, \frac{3}{4}\right]$, $\left[\frac{1}{5}, \frac{4}{5}\right]$, $\left[\frac{1}{6}, \frac{5}{6}\right]$, ...

DONUT: Looks like that would cover every point between zero and one, eventually.

SOCRATES: Does it cover zero or one themselves?

DONUT: No, those are outside of every open set in the infinite sequence.

SOCRATES: So what does the infinite union look like?

DONUT: Wait, it's just the canonical open set, $(0, 1)$? I have to have open sets in my topology anyway? The whole point was to not.

SOCRATES: We're just chasing down the implications. This is where it got us. . . so far.

2.1.5 The Discrete Topology

Define every point to be its own open set. Adding all unions, all combinations of points are also open sets, meaning the Discrete topology could also be given the title of the complete topology.

- We require infinite unions, but only finite intersections. If we had required infinite intersections, then the Usual Topology would have to include $\bigcap_{x=1}^{\infty}(-\frac{1}{x}, \frac{1}{x}) = (-1, 1)\cap(-\frac{1}{2}, \frac{1}{2})\cap(-\frac{1}{3}, \frac{1}{3})\cap\cdots$, which is a single point $\{0\}$. Do the same for every other point up and down the real line, and the Usual Topology would then become the Discrete Topology.

- When completing the closed-intervals topology above, we found that that singleton points are on the list of open sets, meaning that the closed-intervals topology we were trying to construct is actually the Discrete Topology, and every possible combination of points is an open set. You're welcome to be a contrarian and declare the u-closed sets to be open sets in your new topology, but you have to accept the consequences.

That little distinction in the definition of open sets between intersections (finite only) and unions (infinite or finite) turns out to be a key distinction between open and closed sets.

2.1.6 The Indiscrete Topology

The opposite extreme from the Discrete Topology, its list of open sets has only two elements: everything and $\emptyset$. It's a valid topology, and it's sometimes worth bearing in mind the extreme cases, but there is absolutely nothing you can do with it. All the other topologies first state the space they act over and then the open sets in that space, but the indiscrete topology over $\mathbb{R}$, the indiscrete topology over $\{1, 2, 3\}$, the indiscrete topology over the unit circle in the complex plane are all identical from a topological perspective.

2.1.7 Half-closed intervals

The point set is the real line again, just like the Usual Topology, where the basic interval was of the form (a, b) for all a and b where $a < b$. Now, we'll use the same real line, but the base of the topology is the half-closed intervals $[a, b)$. Call this half-closed topology τ_{hc}. Half-closed intervals are nice because they are the smallest change we can make from the u-open intervals we grew up with and still have a topology that doesn't reduce to the Discrete everything-goes topology.

- For closed sets $[a, b]$ and $[b, c]$, their intersection is the single-point set $\{b\}$, and now you've got the Discrete Topology. But we just dodge that with half-open intervals: $[a, b) \cap [b, c) = \emptyset$.
- The infinite union of $\left[\frac{1}{2}, b\right), \left[\frac{1}{4}, b\right), \left[\frac{1}{8}, b\right), \ldots$ has a lower bound which approaches zero, but zero is never in any of the elements of the union. That is, the infinite union adds up to $(0, b)$, and the u-open sets are thus part of τ_{hc}.

Levels of infinity

The first paragraph of this book presented the chestnut that the even integers are in one-to-one correspondence with all the integers.

Rational numbers, numbers produced by dividing one integer by another, are also in one-to-one correspondence with the positive integers:

- Write each positive rational as N/D (numerator over denominator). Bundle them into groups based on $N + D$.
- There is only one rational with $N + D = 2$, which is $\frac{1}{1}$.
- There are two with $N + D = 3$: $\frac{1}{2}$ and $\frac{2}{1}$.
- There are three with $N + D = 4$: $\frac{1}{3}$, $\frac{2}{2}$, and $\frac{3}{1}$.
- There are four with $N + D = 5$: $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, and $\frac{4}{1}$.

- You get the pattern. If there is one item for $N + D = 2$, two for $N + D = 3$, three for $N + D = 4$, ..., then we can look up the standard formula for sums of *triangular numbers* and find that if $N + D = k$, there are $(k - 2)(k - 1)/2$ items with $N + D < k$.
- Within the $N+D = k$ group, we can order by the numerator as we did above, with $\frac{1}{3}$ first, $\frac{2}{2}$ second, and $\frac{3}{1}$ third.
- Then N/D is the $(N + D - 2)(N + D - 1)/2 + N$-th rational number in this ordering. We did it: we assigned a single integer to every rational number.
- There's the annoyance that many numbers are repeated $\left(\frac{2}{3} = \frac{4}{6}\right)$, which we could fix by a rule that every number not in lowest-denominator form be stricken. It messes up the formula, though it's conceptually simple. After that fix, we have a one-to-one correspondence between rationals and integers.

All these sets which are one-to-one with integers are referred to as *countably infinite* or $\aleph_0$ (pronounced aleph zero or aleph nought).

Above $\aleph_0$ there must be an $\aleph_1$, and that is the real numbers: numbers expressible by a possibly infinite decimal. All the rationals are reals, but there are far, far more real numbers. In contrast to $\aleph_0$, this is often described as uncountably infinite.

Rationals can always be expressed as a decimal which eventually repeats. For example, $\frac{1}{7} = 0.142857142857\ldots$, which we'll write using an overline (the "vinculum") to indicate repetition: $\frac{1}{7} = 0.\overline{142857}$. This pattern even holds for fractions with nicer decimal versions if we accept trailing zeros, like $\frac{1}{2} = 0.5\overline{000}$.

In our day-to-day we almost entirely deal with rational numbers, with occasional appearances by celebrity irrationals like $\sqrt{2}$ or π. That is a consequence of convenience, our frequent need to divide integer values, and our mortal abilities. If you had a scale of truly infinite precision and measured the weight of an object, would you really expect the readout to end in a nicely repeating pattern?

Irrationals vastly outnumber the rationals: there are an infinite number of ways to modify the zeros in $0.75\overline{000}$, and similarly for every other rational number. If every rational has an infinite number of irrational variants, there's no way to get a one-to-one mapping between rationals and irrationals, so we have two types of infinity, $\aleph_0$ and $\aleph_1$, where the second is infinitely more infinite than the first.

- Say we met a contrarian who insists that they could write down an enumerated list of real numbers, giving a one-to-one correspondence between reals, each with an infinite number of decimals, and the integers.
- I would look at the first decimal in the first item in our contrarian friend's list, and being more of a contrarian, write down one plus that number. Say the first number on the list is $0.1234321234\ldots$; I would write a 2 on my notepad.
- I would look at the second decimal in the second item in the list and write down one plus that number. Say the second number is $0.1234321245\ldots$; I would write a 3 on my notepad.
- It would take me a while, but I would write down the wrong thing for the N^{th} digit of the N^{th} number all the way down the list, thus generating a new decimal, which starts 0.23 and continues out an infinite number of decimals.
- The thing I wrote down doesn't match the first number on the list, because the first decimal is wrong. It doesn't match the second number, because the second decimal is wrong. And so on to infinity—our contrarian gave us an ostensibly comprehensive list, and we gave back a number that isn't on it.

This is the "diagonalization" argument.

Among finite, quotidian sets, if you have N items, there are 2^N ways of picking a subset of them.

- Write down a 1 if you want to keep the first item, 0 if not. Repeat for the second, third, , N^{th} items.
- More on binary below, but you have just written down a binary number. Every in-and-out combination is another binary number.
- How many binary numbers are there between 0000 and 1111, where we have N zeros or ones? There are 2^N.
- Or, turn a number like 1001 into a binary decimal by putting a dot in front: 0.1001. It expresses the same thinking (keep the first and fourth items, leave out the second and third), but it is now a number between zero and one.

- If you forgot how binary decimals work, we'll review below, but all we need for now is that every real number between zero and one can be expressed as a binary decimal; the binary decimals cover the real numbers in $[0,1]$.
- That is, we've formed another one-to-one correspondence, this time between in-and-out selections of integers and the real numbers $\in [0,1]$.

By analogy to there being 2^4 in-and-out selections of four numbers, we write the real numbers as $2^{\aleph_0} \equiv \aleph_1$, because they correspond to the set of in-and-out selections of all integers. This book won't need them, but we can keep counting up the chain: the count of sets of real numbers is $2^{\aleph_1} \equiv \aleph_2$, the count of sets of those sets of real numbers is $2^{\aleph_2} \equiv \aleph_3$, and so on to a (countable) infinity of infinities.

Or here's another one: given a real number like $0.1234567892\ldots$, I can generate two numbers by reading all odd digits as a number, then reading all even digits as another number. From the above starting number, the first would be all the odds, 0.13579 and the second all the evens 0.24682. That pair of numbers is a point on the real plane, $\mathbb{R}^2$, and the mapping is entirely reversible, like how $(0.1234\ldots, 0.9876\ldots)$ would interleave to form $0.19283746\ldots$. So, we have a one-to-one correspondence between points in $\mathbb{R}$ and pairs of points in $\mathbb{R}^2$. Free your mind of expectations that correspondences have to all have the same dimension.

As a final example, how many ways are there to draw a curve from the origin point in $\mathbb{R}^2$ of $(0,0)$ to the point $(0,1)$, i.e., how many continuous functions $f(x)$ are there for $x \in [0,1]$?

- We've decided that $f(0) = 0$, but $f(\frac{1}{2})$ could be any real number; call it x_1.
- One could work out a smooth line from $f(0) = 0$ to $f(\frac{1}{2}) = x_1$ to $f(1) = 0$.
- Keep subdividing: $f(\frac{1}{4}) \equiv x_2$ could be any real number, as could $f(\frac{3}{4}) \equiv x_3$. Smoothing the curve to hit $0, x_2, x_1, x_3, 1$ is still not a complicated task.
- Keep subdividing: we eventually have an infinite sequence of $x_1, x_2, \ldots$, where every single point could be any real number.

That is, we've put the set of curves into correspondence with the set of all combinations of real numbers: $2^{\aleph_1} \equiv \aleph_2$.

2.2 Metrics

The concept of distance doesn't exist in what we've been looking at so far, not for sets, and not even for individual points. There are several ways to invent such a thing.

- On a grid, you are very familiar with simple Euclidian distance, the shortest line between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, formally $d(p_1, p_2) \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ in two dimensions, and appropriately generalized in more. We have a Usual Topology; maybe we could call this the Usual Metric. The circle in Figure 2.2 is everything Usual-distance one from the origin.
- The *Manhattan metric* is $d(p_1, p_2) \equiv |x_1 - x_2| + |y_1 - y_2|$, where the vertical bars represent absolute value, and the overall gist is that we first have to walk along the East-West streets, then along the North-South avenues to get from point 1 to point 2, without the aid of the diagonals the Euclidian metric would measure along (that is, stay between 14$^{\text{th}}$ and 59$^{\text{th}}$ and assume Broadway does not exist). The diamond in Figure 2.2 is everything Manhattan-distance one from the origin.
- In the *Discrete metric*, $d(a, a) = 0$ for any point a, and for any two distinct points, $d(a, b) = 1$. It's the lowest-possible-effort metric, asking whether two points are different, then refusing to do any further calculations.

The rules for a metric are three:

1. The distance from A to A is zero: $d(p_1, p_1) = 0, \forall p_1$.
2. Symmetry: $d(p_1, p_2) = d(p_2, p_1)$
3. The triangle inequality: $d(p_1, p_2) + d(p_2, p_3) \geq d(p_1, p_3)$. If you go from point A to point C directly, it's less of a walk than if you go from A to B then B to C—at best, it's the same distance.

Maybe pause here and verify that the three metrics so far (standard Euclidian, Manhattan, Discrete), all meet the rules.

Symmetry may seem pedantic, but it's a frequently broken condition. If p_1 is at the base of a hill and p_2 at the peak, then the

Calories you would expend on a bike are greater going from $p_1 \to p_2$ than $p_2 \to p_1$. If you drive an automobile in actual Manhattan and use minutes of drive time as your metric, one-way streets and traffic congestion mean there is no reason to presume that the drive times $d(p_1, p_2) = d(p_2, p_1)$ for any pair (p_1, p_2) anywhere in the city. My personal favorite function which fails the symmetry condition is K-L divergence. Given two probability distributions, it measures the distance from one to the other, but in a manner such that $A \to B \neq B \to A$.

2.2.1 Metric topologies

But this is a topology book, so let's use the metrics to generate topologies. We do so by defining an ϵ-ball around a center point c as the set of points p such that $d(c, p) < \epsilon$. With real numbers and the standard Euclidian metric, we've just generated the Usual Topology. In $\mathbb{R}$, those are intervals around a center point like $(c - \epsilon, c + \epsilon)$, in $\mathbb{R}^2$ those are the interior of circles, in $\mathbb{R}^3$ the interior of spheres. At a single point, there are an infinite number of such balls, from size approaching zero to size approaching infinite, and such balls exist around every point.

- For the Manhattan metric in $\mathbb{R}^2$, our ϵ-balls look like fuzzy-edged diamonds, not spheres like Euclidian ϵ-balls, though we still call them balls, and we still have an infinite number of them centered around every point.
- Or more generally, given a point p and a distance ϵ, define the set $\{x | d(p, x) < x\}$. That's the ϵ-ball in the context of whatever distance function you might be working with. Declare all of those, for all p and all ϵ, to be open sets, and you have induced a topology from a metric.
- Well, you almost have. Don't forget to include all unions and intersections. We tend to focus on the simple circles and spheres, but if you have a clump of six or seven simple open balls in some odd arrangement—maybe a balloon animal or looped into a necklace—that amalgam is also an open set.

Now that we have a topology, we can ignore the source metrics and focus on the open sets. Consider the set of open sets generated by the Euclidians and the set of open sets generated by Manhattanites.

- Pick an open set in the Manhattan-metric topology, such as every point (p_x, p_y) around a focal point (x, y) such that $|x - p_x| +$

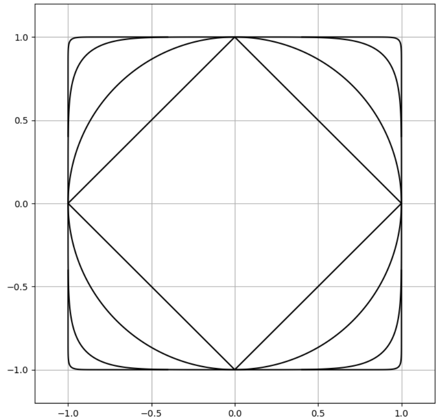

Figure 2.2: Some ℓ-norms: The unit ball in the Manhattan metric (ℓ_1) is the diamond. In the Euclidian metric (ℓ_2) the unit ball is the circle. The ℓ_6 norm is a rectangle with rounded corners, and the ℓ_{40} norm approaches a simple square.

$|y - p_y| < 3$. That's a definite set of points, though do note the use of *less than* instead of *less than or equal*, giving us soft-edged boundaries just as with the canonical open set.

- This would not be the case if we had a hard boundary, but as it is, there is always a *Euclidian* ϵ-ball around any point in the Manhattanite set small enough that it is entirely within the set.
- If every point in a set is part of a Euclidian ϵ-ball, then you could construct the set from an infinite union of open sets in the Euclidian topology, then that set is in the Euclidian topology.

So the open-diamond topology is a subset of the open-circle topology. All the same logic holds if we start with a Euclidian-topology open set, call it E: every point is inside a Manhattan-topology open set within E, so E is part of the Manhattan topology. Dropping down to basic logic: When you have two sets A and B, and $A \subseteq B$ and $B \subseteq A$, then $A = B$. Applying that here, both metrics generate the same topology.

Both Euclidian and Manhattan metrics are in the same family, meaninglessly named the ℓ-norm family. Everything in the family has the form $d(p_1, p_2) = (|x_1 - x_2|^b + |y_1 - y_2|^b)^{\frac{1}{b}}$, including the Euclidian metric when $b = 2$, and Manhattan when $b = 1$. See Figure 2.2 for some examples. For $b \to \infty$, you get the *sup norm*, where $d(p_1, p_2) = \max(|x_1 - x_2|, |y_1 - y_2|)$.

Theorem 2.1. All ℓ norms are alike
All topologies generated by an ℓ-norm are equivalent to each other.

You've probably read about how topologists don't care about distances, or about how everything can be freely stretched and adjusted, and this theorem gets to the core of what that means. It's not that one metric transforms into another via appropriate transformation, but that each generates a truly identical set of open sets.

The Discrete metric produces one of two topologies: if $0 < \epsilon < 1$ you get the Discrete Topology, because if $d(a, b) = 1$ for all $a \neq b$ then no ϵ-ball spans two points; if $\epsilon > 1$ you get the Indiscrete Topology, because if $d(a, b)$ is either zero or one, an ϵ-ball of size 1.1 around a reaches all points in the whole space.

No, metrics didn't get us far in inventing new topologies, but it's worth taking the time to note that the concept of distance is somewhat separate from the open sets issue we want to talk about here. Your intuition might be that the best way to resolve Zeno's paradox is some means of clarifying what small distances mean (that's my intuition, at least), but that's not what the topologists will be doing at all: Which ϵ-balls overlap depends on closeness in the usual metric sense, but once we generate those open sets, we can otherwise throw away the metric.

In the opposite direction, given some arbitrary topology, is there some metric that generates it? Could you pin the points in the topology to points in $\mathbb{R}^N$ such that the open sets generated by the metric match those in the arbitrary topology? Chapter 9 will ask that question and give a straightforward answer: yes, if the topology is T_2, second-countable, and completely regular. The text between here and there will put you in a position to understand what that's supposed to mean.

2.3 Some less basic topologies

One reader of this book was saddened by how many topologies are over the space of plain old real numbers, which on the one hand shows some lack of imagination, given all the things in our world which could be grouped and subsetted, but on the other hand shows incredible imagination, that the same numbers can behave remarkably differently depending on the groupings we impose on them.

This section presents a few more topologies which go much further afield from the open intervals in the introductory list of basic topologies above. The first few in this section define "points" differently: a point is an entire graph or an infinite sequence. The last goes back to $\mathbb{R}$, but does surprisingly unusual things with it.

2.3.1 Graphs

The pedantic will recognize that lines along an x and y axis are *plots*, while a *graph* is a network of nodes and edges. One could build one incrementally, starting with a graph consisting of a single node, then adding another node and an edge between them, then adding another node and edge to produce a chain of 1—2—3. There are no choices to be made up to this point, but the next step could be adding just an edge linking nodes 1 and 3 to form a triangle; adding node 4 and an edge to node 3, producing a 1—2—3—4 chain; or adding node 4 and an edge to node 2, forming a three-pointed star centered at node 2. A *point* in this space will be a single graph, with all its nodes and edges. Then a topology over this space will be a list of sets of graphs.

Define an open set to be one graph plus all graphs which could be formed by adding an edge (possibly with a node), or subtracting an edge (possibly removing a node) from that base graph. Figure 2.3 shows the sorts of open sets that result.

- Almost all sets as defined here therefore include one graph, say it has n edges, and some number of graphs with $n+1$ and $n-1$ edges.
- But only *almost all*, because if the starting point is a one-node graph, there are no edges to remove. There is only one other graph in the set starting from the trivial one-node graph, with two nodes linked by one edge.
- The triangle at left is a complete graph: every edge between the three nodes is present. If we don't have labels on the nodes, a complete graph can expand in only one way, by adding a node and an edge anywhere, and can contract in only one way, by subtracting an edge. A graph with 30 nodes has a capacity of 435 edges, given 30 source nodes which could point

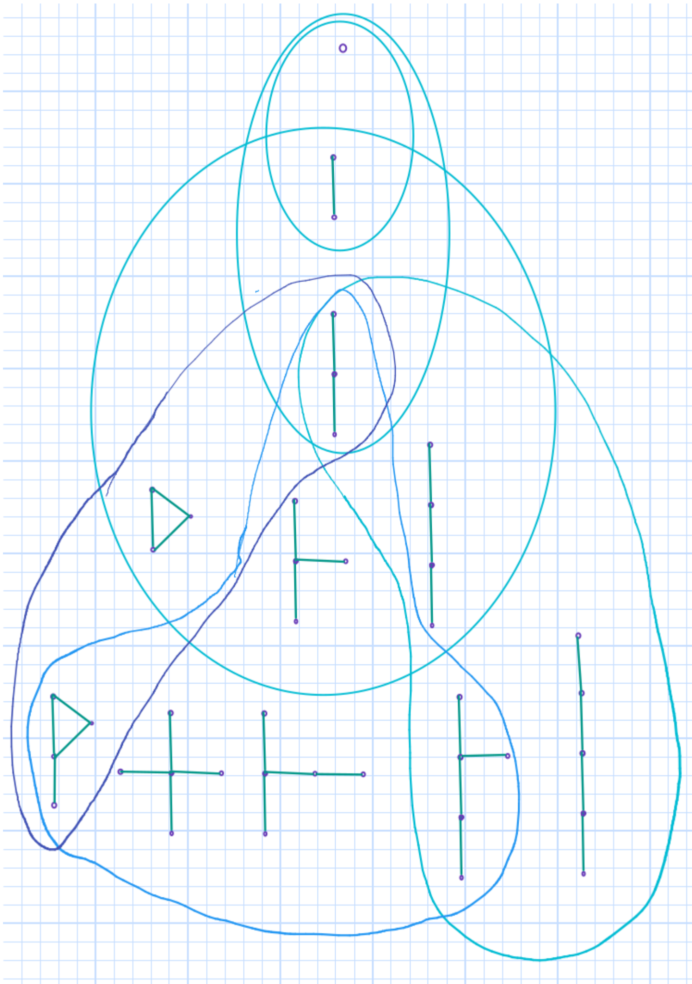

Figure 2.3: Graphs, and the open sets they lie in. Each graph is the center of one open set, including all graphs that can be built from adding or removing one edge—except the trivial graph of one node and no edges, which has no smaller graph. Sets centered around the graphs in the last row are no shown.

to 29 destinations. If complete, even that has only one way to expand and one way to subtract.

- Other graphs have far more neighbors. The four-node sideways-T-shaped graph next to the triangle can expand four different ways.
- I'd described building a graph via a stepwise process, and the topology records how that building happens. You can trace a path from the root graph of just one node to the next graph with two nodes, which are in the same set, to the next graph with three nodes, which is in the same set as that two-node graph. Between any two graphs, there is a chain of open sets.

We've converted questions about growing networks into topology questions. Other topologies would encode very different information. Start over: define open set n as the set of all graphs with n nodes. This is a valid topology, but much less interesting, as there is no overlap between any open sets.

2.3.2 Arithmetic sequences

The space for this topology will be $\mathbb{Z}$, the integers $\dots, -2, -1, 0, 1, 2, \dots$. Let a and b be integers, and let $N_{a,b}$ be the set of points $\{a + xb | x \in \mathbb{Z}\}$. For example, $N_{2,5} = \{2 + 5x | x \in \mathbb{Z}\} = \{\dots, -13, -8, -3, 2, 7, 12, 17, \dots\}$. Then each $N_{a,b}$, $\forall a, b \in \mathbb{Z}$, is an open set in our topology.

- If you have a few offsets a with the same b, you can cover the space. For example, $N_{0,3} = \{\dots, -6, -3, 0, 3, 6, \dots\}$, and $N_{1,3} = \{\dots, -5, -2, 1, 4, 7, \dots\}$, and $N_{2,3} = \{\dots, -4, -1, 2, 5, 8, \dots\}$, and all the integers are in one or the other. The easiest such case is $N_{1,2}$, which is all the odd integers, and $N_{2,2}$, which is all the evens.
- It might be amusing for you to verify that the intersection and the union of $N_{a,b}$ and $N_{c,d}$ is a sequence also of the form $N_{e,f}$, by solving for e and f in the cases of union and of intersection. Of course, for cases like odds and evens in the prior bullet point, the union is $\mathbb{Z}$ and the intersection is $\emptyset$. But multiples of three alternate even and odd, so $N_{0,3}$ intersects $N_{0,2}$ in a definite pattern.

We've imposed a shape and texture to the monotone number line. If you want to walk from odds to evens, they're disjoint, so you have to go through some other sequence, like walking from evens (the set $N_{0,2}$) to multiples of three (the set $N_{0,3}$) to odds (the

Figure 2.4: The Outer-thirds Set: start with $[0, 1]$, then remove the middle third. Next step: of the two segments remaining, remove their middle thirds. Next step: of the four segments remaining, remove all middle thirds. Repeat infinitely.

set $N_{1,2}$). The sequence $0 + 6n$ and the sequence $1 + 6n$ are also disjoint, but you can't get from one to the other via multiples of three: $0 + 3n$ will never meet $1 + 6n$ and $0 + 6n$ is entirely a subset within $0+3n$; and $1+3n$ is a superset of $1+6n$, and $1+3n$ will never meet $0 + 6n$. Where a regular person might say *Six is three times two*, a topologist could say *for $i = 0, 1$, $N_{i,6}$ is a subset of $N_{j,2}$ and of $N_{j,3}$ for $i = j$ and is disjoint from those sets when $i \neq j$.*

If you were a number theorist, this is how you'd start asking questions of numeric sequences using the topological tools you'll see over the course of this book. Later, we'll use this topology to prove that there are an infinite number of primes.

2.3.3 The Outer-thirds Set

I had an actual dream while ideating about this book, about Trinity Island. There was an East and a West part with a valley in between. It was the sort of place where the coffee shops made ordering complicated. Small, medium, large? Unprocessed, half-caffeinated, decaf? No milk, half, or something more milk than coffee? Oat, almond, or soy milk? The cashier would write your decisions on your cup, like 0220. The apparel shops were all well-stocked, with sections for men, women, and everybody. Even within the men's section, there were male-leaning, uncoded, and female-leaning apparel. We went back years later, and climate change had left the valley permanently flooded, leaving an East island and a West island. People were gruff and polarized. The coffee shop had reduced your options so they could move people faster: small or large, caffeinated or no, milk or no, though my order still turned out to be 0220. The weirdest part was that the locals didn't notice a difference. Even after the losses it still felt like a lot of latitude to move about.

1. Start with the interval $[0, 1]$, then remove the middle third, leaving two intervals, $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. The top of Figure 2.3 is

the whole interval, then the next row down is with the middle third removed.

2. Repeat, taking out the middle thirds again, so writing $\frac{1}{3}$ as $\frac{3}{9}$ and $\frac{2}{3}$ as $\frac{6}{9}$, the interval $[0, \frac{3}{9}]$ gets cut into $[0, \frac{1}{9}]$ and $[\frac{2}{9}, \frac{3}{9}]$, and $[\frac{6}{9}, 1]$ gets cut into $[\frac{6}{9}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$.
3. Take those four intervals, $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{3}{9}]$, $[\frac{6}{9}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$, and remove the middle thirds to produce eight intervals. Repeat, infinitely.

What's left is the Outer-thirds Set. As a historic note, the development of the Outer-thirds Set was a turning point in Western mathematics, because of the properties discussed below. But after some digressions about measures, we're going to use it as an eccentric topology. It will reappear to cause trouble throughout.

- The Outer-thirds Set is usually called *The Cantor Set*.
- *Stigler's Law*, first articulated in the academic literature by Robert K Merton, states that everything named after a person is named after the wrong person.
- Henry J Stephen Smith, in an 1874 paper on integrability of discontinuous functions, offers a construction where $[0, 1]$ is divided into m segments, then the first $m - 1$ segments are divided into m segments again, and, with $m = 3$, we wind up with a sequence of divisions that looks much like the middle-third sequence of Cantor's Set (published 1883), albeit with the unbroken thirds at the end and not the middle. Even the most revolutionary ideas have precedent somewhere.

Binary and trinary

Let's take a beat to remember how binary, decimal, and other base-n systems work.

- We do this instinctively, but a number like 283 breaks down to 200 + 80 + 3, whose pattern is most clear if we spell out the powers of ten: $2 \cdot 10^2 + 8 \cdot 10^1 + 3 \cdot 10^0$.
- This also works after the decimal point, bearing in mind that $10^{-1} = \frac{1}{10}$ and $10^{-2} = \frac{1}{100}$. E.g., the number $23.76 = 2 \cdot 10^1 + 3 \cdot 10^0 + 7 \cdot 10^{-1} + 6 \cdot 10^{-2}$.
- Binary works by replacing all the 10 bases with 2 bases. For example, $101.1 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} = 5\frac{1}{2}$.

- Since the digital revolution, we've all gotten used to binary, but one could do this with any base, like trinary, where $12.02 = 1 \cdot 3^1 + 2 \cdot 3^0 + 0 \cdot 3^{-1} + 2 \cdot 3^{-2} = 5 + \frac{1}{9}$.

Now that you're fluent in trinary, we'll be looking at the set of points between zero and one, inclusive, expressible in trinary using only zeros and twos.

- Putting that set of numbers into one-to-one correspondence with binary numbers couldn't be easier: just replace all the twos with ones, like 0.2202 becomes 0.1101.
- When you've done so, you've now got all binary decimals between zero and one, which means all of $[0, 1]$.
- But on the trinary side, there are holes, because all those numbers that need a one to be expressed, like 0.12, were excluded. The range from $0.1\overline{000}$ to $0.1\overline{222}$ is the range between $\frac{1}{3}$ and $\frac{2}{3}$. I.e., it's the part that we threw out in the first step of generating the Outer-third set.
- At the next level, anything between 0.01 and 0.02 will need a one to be expressed, by which I mean everything between $\frac{1}{9}$ and $\frac{2}{9}$. Same with anything between 0.21 and 0.22, meaning everything between $\frac{7}{9}$ and $\frac{8}{9}$.
- As a relevant aside, you know how $0.\overline{999}$ is another way to write 1?
  - If $x \equiv 0.\overline{999}$, then $10x = 9.\overline{999}$.
  - Subtracting $10x - x$, everything after the decimal cancels out: $10x - x = 9.\overline{999} - 0.\overline{999} = 9$.
  - If $10x - x = 9x = 9$, then $x = 1$.
  - The same holds for later digits, like how $0.72\overline{999} = 0.73$.
- You can do the same in trinary, where $0.\overline{222} = 1$. That means elements like $\frac{1}{9} = 0.01$ really are in our set, because we can write them as $0.00\overline{222}$.
- But maybe you see where this is going: every point in the Outer-thirds Set, as constructed by removing middle thirds from ever-shorter lines, can be expressed as a trinary decimal using only zeros and twos, and vice versa.

Measure theory can get weird, but it starts simple: the range $[0, 1]$ has measure (total length) one. So does the open interval

$(0, 1)$. If you subtract one from the other, you are left with two points, 0 and 1, and there is no trickery in measuring them: we took a length of measure one and subtracted a length of measure one, leaving two points with measure zero.

With the Outer-thirds Set, we started with $[0, 1]$ and removed

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i}.$$

I won't present a proof, but that's a convergent sum, which totals one. From a line segment of measure one, we removed segments with total measure one, meaning that the Outer-thirds Set has measure zero. This is how Cantor blew up mathematics. We have the same number of points, but in one configuration the set has length one, and in another it is a bunch of disconnected points with length zero. Instead of a world where some sets are bundles of points and some are lines, we have something in between—Chapter 9.4 finds a compromise between calling it zero-dimensional like a bundle of points and one-dimensional like a line by a calculation which concludes that it is 0.631-dimensional.

We've constructed a countably infinite set of points, with an easy one-to-one correspondence with all points in $[0, 1]$ (changing all 2s to 1s, like 0.2202 to 0.1101), but which itself has measure zero. This is itself a great feat of interesting. But this is a topology textbook, so we're going to add it to our list of topologies as our first subspace topology: take the Usual Topology, then restrict the set of points to the Outer-thirds Set, and the set of open sets to the intersection of the usual open sets with the Outer-thirds Set.

- One more interesting bit: As in our usual base-10 decimals, a number ending in a repeating pattern, possibly $\overline{000}$, is rational, but there are infinitely more irrationals whose expression never repeats. Because the endpoints are all cuts made at a rational number, you might expect that there are an uncountable number of points which are not endpoints of segments of the Outer-thirds Set, and so in the interior. We'll see below that there are in fact *no* interior points. Chapter 3 will introduce limit points, however, which will allow us to say something about how points in the set clump together.

2.3.4 Lexicographic ordering

When you alphabetize a list, only the first letter matters—unless two words both start with the same letter, in which case only the

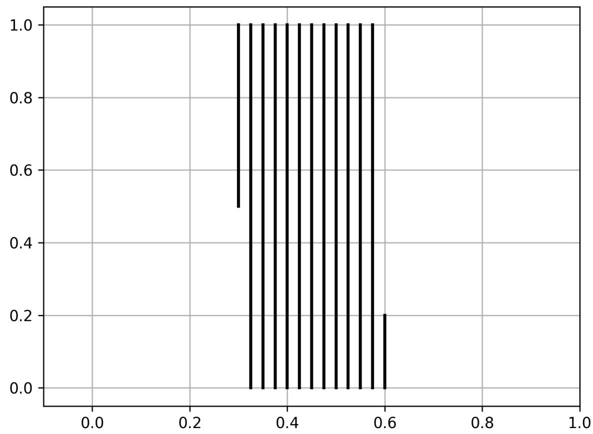

Figure 2.5: A schematic of the open set between $(0.3, 0.5)$ and $(0.6, 0.2)$ in the Lexicographic Topology.

second letter matters.

We could order $\mathbb{R}^2$ like this, first using the x-axis and then the y-axis only to break ties. For two given points, $(x_1, y_1) > (x_2, y_2)$ iff either (1) $x_1 > x_2$, or (2) $x_1 = x_2$ and $y_1 > y_2$. Set the space for the Lexicographic Topology to be the one-by-one box $[0, 1] \times [0, 1]$. It's all very simple, the numeric equivalent of the sort of alphabetization you learned when you were maybe seven years old—what could go wrong?

Each pair of points p_1 and p_2 generates what we'll call a basic open set, $\{q | p_1 < q < p_2\}$. This looks like how open intervals are defined in the Usual Topology, but this is not a subset topology or otherwise related to the Usual—we are effectively redefining what $<$ means, so the open sets will look completely different. We broke the concept of distance entirely:

- The distance from $(1, 1)$ to a point just above it like $(1, 1.1)$ is sensible and not actually broken, and we can call it 0.1.
- Between $p_1 \equiv (1, 1)$ and $p_2 \equiv (1.1, 1)$, though, we have all points of the form $(1.05, y)$, with $y \in [0, 1]$, so by our usual expectations that's length one. We also have all points of the form $(1.01, y)$, $(1.07, y)$, and in fact an infinite number of lines of height one between p_1 and p_2. Add them up, and the distance between p_1

and p_2 is infinite.

So we have a complete order, where for every two points we can say which is greater than the other, and we have a sensible concept of distance on any vertical bar. But if p_1 and p_2 aren't on the same vertical bar, there is infinite distance between them.

For our open sets, instead of balls, visualize vertical stripes, starting at (x_1, y_1), including all values above that point, every point with an x-value between p_1 and p_2 regardless of y-value, then topped off with every point south of (x_2, y_2).

Then, don't forget to take all unions and intersections and include the complete one-by-one box and $\emptyset$ in your topology.

- Put the ϵ-balls in the Usual Topology out of your mind. Figure 2.4 gives a schematic of the open set between $(0.3, 0.5)$ and $(0.6, 0.2)$. It starts at $(0.3, 0.5)$ and included everything above that point with $x = 0.3$, then includes every vertical bar with $0.3 < x < 0.6$, then finishes with everything with $x = 0.6$ but $y < 0.2$. I'm calling the figure a schematic because there are of course an infinite number of vertical bars in the body of the open set, not just the few I've drawn.
- What looks like a straight horizontal line, say from the point $(\frac{1}{2}, \frac{1}{4})$ to the point $(\frac{1}{2}, \frac{3}{4})$, is actually an infinite sequence of disconnected points. In this topology, an ϵ-ball with $\epsilon < 1$ will never contain two of the points in that horizontal line, because it takes a step of one to wrap around from bottom to top.
- What would an ϵ-ball in this world look like? The ball of radius $\frac{1}{4}$ around the center point $c = (0.3, 0.5)$ seems natural: it's the vertical line between $(0.3, 0.25)$ and $(0.3, 0.75)$.
- The ball of radius $\frac{1}{4}$ around a center point $c = (0.4, 0)$ goes up to $(0.4, 0.25)$, and down to. . . nowhere, because any candidate point less than c is on another vertical line, and as above, there are an infinite number of vertical lines of height one between that point and c.

It still feels weird to me. Everything is perfectly ordered, using an ordering scheme you learned in grade school, and yet at regular intervals there is a drop-off where no points seem to be. Some call this topology *the long line*, because we usually visualize fully ordered lists in a line, but this line has infinite distance between any two points where $x_1 \neq x_2$.

This chapter introducing topologies has gone full circle, so to speak. We started with the Usual Topology, which produces what you expect and are familiar with within the subspace $[0,1] \times [0,1]$, and ended with the Lexicographic Topology, which assigns new open sets to exactly the same bunch of points. People who travel around Manhattan by taxis have what is in many ways an entirely different conception of what Manhattan is from the ones who travel by subway. How points of interest connect is vital, and is what Topology is about.

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