Part II – Linear Analysis

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0 Introduction

As the name suggests, Linear Analysis is the study of linear spaces of functions, mostly infinite dimensional. In particular, properties like convexity, completeness, closedness are of interest here. Like any pure course, we start with a lot of definitions which come out of nowhere, and then clear them up, but functional analysis is not devoid of functional analysis. In particular, in the field of differential equations both ordinary and partial it is often useful to view the differential operators as a linear operator on a space of functions. Markov processes can also be seen using a transition operator, and dynamical processes are given by a measure preserving map, all fitting into the realm of linear maps. Quantum mechanics to a certain extent is the study of the spectrum of certain self-adjoint linear operators on a Hilbert space, and so requires functional analysis. As much as possible, examples of applications will be given briefly.

1 Normed vector spaces

Unless stated, vector spaces will be either over the real numbers or the complex numbers, denoted by $\mathbb{K}$ to represent $\mathbb{R}$ or $\mathbb{C}$.

1.1 Topology of vector spaces

Definition (Normed vector space)**.** A **normed vector space** is a vector space V with a norm $\|\cdot\| : V \to \mathbb{R}$ satisfying

i. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if $v = 0$. (Positive definite)

ii. $\|\lambda v\| = |\lambda| \, \|v\|$ for every $v \in V$ and $\lambda \in K$. (Positive homogeneous)

iii. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$. (Triangle inequality)

In particular, a metric on V is defined by $d(v, w) = \|v - w\|$.

Fact. The vector space operations of scalar multiplication and vector addition are continuous.

$$\begin{aligned} K \times V &\to V & \qquad (\lambda, v) &\mapsto \lambda v \\ V \times V &\to V & \qquad (v, w) &\mapsto v + w \end{aligned}$$

Proof. We only check that scalar multiplication is continuous. Since K and V are metric spaces, it suffices to show that $\lambda_j \to \lambda$ and $v_j \to v$ implies $\lambda_j v_j \to \lambda v$. But

$$\begin{aligned}\|\lambda_j v_j - \lambda v\| &= \|(\lambda_j - \lambda)v_j + \lambda(v_j - v)\| \\ &\leq \underbrace{|\lambda_j - \lambda|}_{\to 0} \underbrace{\|v_j\|}_{\text{bounded}} + |\lambda| \underbrace{\|v_j - v\|}_{\to 0}\end{aligned}$$

as required. □

Corollary. Translations ($v \mapsto v + v_0$) and dilations ($v \mapsto \lambda v, \lambda \neq 0$) are homomorphisms.

Definition (Topological vector space)**.** A **topological vector space** is a vector space together with a topology that makes the vector space operations continuous and in which points are closed.

Notation. For a subset C of a vector space V over $\mathbb{K}$ and $t \in \mathbb{K}$, we write tC for the following subset:

$$tC \coloneqq \{ tv \mid v \in C \}$$

Definition (Convex subset)**.** Let V be a vector space and $C \subset V$ a subset. We say that C is **convex** iff $tC + (1-t)C \subset C$ for all $t \in [0,1]$. Specifically, this means $tv + (1-t)w \in C$ for all $v, w \in C$ and $t \in [0,1]$.

Fact. Let V be a normed vector space. Then $B_1(0)$ is convex.

Fact. If C is convex, then $v + \lambda C$ is convex for all $\lambda \in K$ and $v \in V$.

Definition (Locally convex space)**.** A topological vector space is **locally convex** if its topology has a basis of convex sets.

Definition (Bounded subset)**.** Let V be a topological vector space and $B \subset V$. We say that B is **bounded** if for every open neighbourhood U of 0, there exists $t > 0$ such that $sU \supset B$ for all $s \geq t$.

Definition (Balanced subset)**.** Let V be a vector space, and $C \subset V$ a subset. Call C **balanced** if for all $|\lambda| \leq 1$, we have $\lambda C \subset C$.

Example.

(i) Balanced sets in $\mathbb{R}$ are sets of the form $[-t, t]$, $(-t, t)$, or $\{0\}$ and all of $\mathbb{R}$, and $\mathbb{Q}$ is not balanced in $\mathbb{R}$.

(ii) In $\mathbb{C}$, the only balanced sets are $\{0\}$, $\mathbb{C}$, and the open or closed balls centred at 0.

(iii) There are more interesting examples of balanced sets in $\mathbb{R}^2$, for instance the open disk or any ellipse centred at 0.

Lemma. Let V be a topological vector space and $C \subset V$ be a bounded, convex neighbourhood of 0. Then there exists a bounded, balanced, convex neighbourhood $\widetilde{C}$ of 0.

Proof. Exercise (on example sheet). □

Proposition. Let V be a topological vector space and $C \subset V$ be a bounded, convex neighbourhood of 0. Then the topology on V is induced by a norm.

Proof. Use the previous lemma to construct $\widetilde{C}$. Let

$$\mu_{\widetilde{C}}(v) = \inf \{ t > 0 \mid v \in t\widetilde{C} \}$$

referred to as the Minkowski functional of $\widetilde{C}$. We claim that $\|v\| = \mu_{\widetilde{C}}(v)$ is a norm on V and that the topology induced by it is the same as the original topology. Check the norm axioms in turn:

i. We clearly have positivity, and $\mu_{\widetilde{C}}(v) = 0$ iff $v = 0$ since $\widetilde{C}$ is bounded.

ii. Since $\widetilde{C}$ is balanced,

$$\begin{aligned}\mu_{\widetilde{C}}(\lambda v) &= \inf \left\{ t > 0 \,\middle|\, \lambda v \in t\widetilde{C} \right\} \\ &= \inf \left\{ t > 0 \,\middle|\, v \in \frac{t}{|\lambda|}\widetilde{C} \right\} \\ &= \inf \left\{ |\lambda| \frac{t}{|\lambda|} > 0 \,\middle|\, v \in \frac{t}{|\lambda|}\widetilde{C} \right\} \\ &= |\lambda| \, \mu_{\widetilde{C}}(v)\end{aligned}$$

iii. Given $v, w \in V$, write $v = \lambda v_0$ and $w = \mu w_0$ with $\lambda, \mu > 0$, $v_0, w_0 \in \widetilde{C}$. Since $\widetilde{C}$ is convex,

$$\begin{aligned} &\frac{\lambda v_0 + \mu w_0}{\lambda + \mu} \in \widetilde{C} \\ \implies\ &\mu_{\widetilde{C}}\left(\frac{\lambda v_0 + \mu w_0}{\lambda + \mu}\right) \leq 1\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_{\widetilde{C}}(v + w) &= (\lambda + \mu)\ \mu_{\widetilde{C}}\left(\frac{\lambda v_0 + \mu w_0}{\lambda + \mu}\right) \\ &\leq \lambda + \mu \\ &\leq \mu_{\widetilde{C}}(v) + \mu_{\widetilde{C}}(w)\end{aligned}$$

□

Corollary. A topological vector space is normable iff it is locally convex and locally bounded (that is, there exists a bounded convex neighbourhood of 0).

Definition (Banach space)**.** A **Banach** space is a normed vector space that is *complete* as a metric space, that is, any Cauchy sequence converges.

Example.

(i) Any finite dimensional vector space is a Banach space (with any norm).

(ii) Let X be a set, and $\mathcal{B}(X)$ the set of bounded $\mathbb{K}$-valued functions on X. Then $\mathcal{B}(X)$ is a Banach space with norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \,, \quad f \in \mathcal{B}(X)$$

(iii) Let X be a *compact* Hausdorff space (eg. $X = [0, 1]$) and $\mathcal{C}(X)$ be the space of continuous functions on X. Then $\mathcal{C}(X) \subset \mathcal{B}(X)$ since every continuous function on a compact space is bounded. In addition, $\mathcal{C}(X)$ is a Banach space, as the uniform limit of a sequence of continuous functions is continuous.

(iv) Let $U \subset \mathbb{R}^n$ be an open, bounded subset, and let $\mathbb{C}^{\mathbb{K}}(\bar{U})$ be the space of k–times contiuously differentiable functions $f : \bar{U} \to \mathbb{K}$ with norm defined as follows:

$$D^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f(x)$$

for any multi-index $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}_0^n$, and where $|\alpha| = \sum_{i=1}^n \alpha_i$. From here, we use the norm

$$\|f\|_{C^k(\bar{U})} = \max_{|\alpha| \leq k} \|D^\alpha f\|_\infty$$

Then $C^{\mathbb{K}}(\bar{U})$ is a Banach space.

(v) For a sequence $x = (x_1, x_2, \dots) \subset \mathbb{K}$, define

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

for $p \in [1, \infty)$, and $\|x\|_\infty = \sup_i |x_i|$. Then $\ell^p = \{ x \mid \|x\| < \infty \}$ for $p \in [1, \infty]$ is a Banach space. Note for $p < 1$, this does not generate a norm.

(vi) Let $U \subset \mathbb{R}^n$ be open (not necessarily bounded) and denote by $\mathcal{C}(U)$ the space of continuous functions on U (not necessarily bounded). Then $\mathcal{C}(U)$ is a topological vector space with topology generated as follows.

Let $K_i \subset U$ be compact subsets such that $K_i \subset K_{i+1}$ and $\bigcup_{i=1}^\infty K_i = U$.

Let $V(i, n) = \left\{ f \;\middle|\; \|f\|_{\mathcal{C}(K_i)} < \frac{1}{n} \right\}$ where we use $\|f\|_{\mathcal{C}(K_i)}$ to denote $\sup_{x \in K_i} |f(x)|$.

Then the topology generated by the $V(i, n)$ and their translates makes $\mathcal{C}(U)$ a locally convex topological vector space, and the topology on $\mathcal{C}(U)$ is generated by the metric

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|f - g\|_{\mathcal{C}(K_i)}}{1 + \|f - g\|_{\mathcal{C}(K_i)}}$$

This is not a Banach space, but it is a Frechet space.

(vii) Let $X = \{ f : [0, 1] \to \mathbb{K} \text{ continuous} \}$. Then

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$ is a norm on X. However, X is not complete with this norm.

1.2 Bounded linear maps and the dual space

Fact. In any topological vector spaces V, W, a linear map $T : V \to W$ is continuous if and only if it is continuous at 0.

Proof. Let T be continuous at 0 and $v \in V$. Let $w = Tv$ and $U \subset W$ an open neighbourhood of w. Then $U-w$ is an open neighbourhood of $0 \in W$. Since T is continuous at 0, $T^{-1}(U-w)$ contains an open neighbourhood $U' \subset V$ of 0. By linearity, $T(v + U') = Tv + T(U') \subset Tv+U-w = U$. Since $v+U'$ is an open neighbourhood of v this means that T is continuous at v. □

Definition (Bounded linear map)**.** Let V, W be topological vector spaces and $T : V \to W$ a linear map. Then T is **bounded** if $T(B)$ is bounded for any bounded $B \subset V$.

Fact. If V, W are normed vector spaces, a linear map $T : V \to W$ is bounded iff there is $\lambda > 0$ such that

$$T(B_1(0)) \subset B_\lambda(0) \quad \text{i.e.} \quad \|Tv\| < \lambda \text{ if } \|v\| \leq 1$$

Definition (Operator norm)**.** Let V, W be normed vector spaces. The **operator norm** of a linear map $T : V \to W$ is

$$\|T\| = \sup_{\|v\|=1} \|Tv\| = \sup_{\|v\|\leq 1} \|Tv\|$$