Part II – Algebraic Topology

Based on lectures by Dr H. Wilton
Notes taken by Bhavik Mehta

Michaelmas 2017

1 Introduction

Question 1 Is the trefoil really a knot?

How do we make this precise? We can think of the trefoil and the unknot as continuous embeddings

$$f_i : S^1 \hookrightarrow \mathbb{R}^3$$

where f_0 corresponds to the trefoil, and f_1 corresponds to the unknot.

Question 1 (precise) Is there a continuous map

$$F : S^1 \times [0, 1] \to \mathbb{R}^3$$

such that

(i) $F(\theta, 0) = f_0(\theta)$, $F(\theta, 1) = f_1(\theta)$ $\forall \theta \in S^1$

(ii) $F(\cdot, t_0) : S^1 \to \mathbb{R}^3$ is injective, $\forall t_0 \in [0, 1]$

We phrased unknotting as an *extension problem*. We can think about other, easier to state extension problems.

Question 2 Consider $S^2 = \{\, \mathbf{x} \in \mathbb{R}^3 \mid \|x\| = 1 \,\}$, and $D^3 = \left\{\, \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \|x\| \leq 1 \,\right\}$ along with the natural inclusion $i : S^2 \hookrightarrow D^3$. Does there exist a continuous $f : D^3 \to S^2$ such that $f \circ i = \mathrm{id}_{S^2}$?

This seems like a hard question! So, let's instead consider this analogous question in algebra. We take S^2 analogous to $\mathbb{Z}$, and D^3 analogous to $\{0\}$, the trivial group. From here, we get another question:

Question 3 Does there exist a group homomorphism $f : \{0\} \to \mathbb{Z}$ such that $\mathrm{id}_{\mathbb{Z}} = f \circ 0$?

This question seems much easier to solve! This is the essence of algebraic topology - we turn difficult questions in topology into easy questions about algebra.

1.1 Examples and conventions

Zoo of examples

- Point $*$
- Circle S^1
- The circle generalises to the n-sphere, $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$
- But it also can be used to produce the torus $T^2 = S^1 \times S^1$.
- Which itself generalises into the genus-g surface, Σ_g
- The torus can alternatively be found by identifying edges of a square, but by identifying them differently we can make a Klein bottle K
- Or the real projective plane $\mathbb{RP}^2$

In this course, the term map always refers to a continuous map. To check that a map is continuous, we'll almost always use

Lemma (The Gluing Lemma)**.** If $X = C_1 \cup C_2$, with C_i closed, and $f : X \to Y$ is a function such that $f|_{C_1}$ and $f|_{C_2}$ are both continuous, then f is continuous.

Proof. Let $D \subseteq Y$ be closed. Then

$$f^{-1}(D) = (f|_{C_1})^{-1}(D) \cup (f|_{C_2})^{-1}(D)$$

is a finite union of closed sets, hence is closed. Therefore, f is continuous. $\square$

1.2 Cell complexes

There are the spaces we can build by gluing. The *basic operation* we use is to attach an n-dimensional disc, as shown in the non-existent diagram. Formally, if $f : S^{n-1} \to X$ is a continuous map, then we produce

$$(X \sqcup D^n)/\sim \tag{1}$$

where $\sim$ is an equivalence relation: $Y \sim f(y)$, for $y \in S^{n-1}$ and $f(y) \in X$.

Definition (Cell complexes)**.** We define cell complexes by induction:

(i) A zero-dimensional cell complex is a finite discrete topological space

(ii) An n-dimensional cell complex is constructed from an $(n-1)$ dimensional cell complex X by attaching finitely many n-dimensional discs to X