Part II – Linear Analysis

Based on lectures by Dr R. Bauerschmidt
Notes taken by Bhavik Mehta

Michaelmas 2017

0 Introduction

As the name suggests, Linear Analysis is the study of linear spaces of functions, mostly infinite dimensional. In particular, properties like convexity, completeness, closedness are of interest here. Like any pure course, we start with a lot of definitions which come out of nowhere, and then clear them up, but functional analysis is not devoid of functional analysis. In particular, in the field of differential equations both ordinary and partial it is often useful to view the differential operators as a linear operator on a space of functions. Markov processes can also be seen using a transition operator, and dynamical processes are given by a measure preserving map, all fitting into the realm of linear maps. Quantum mechanics to a certain extent is the study of the spectrum of certain self-adjoint linear operators on a Hilbert space, and so requires functional analysis. As much as possible, examples of applications will be given briefly.

1 Normed vector spaces

Unless stated, vector spaces will be either over the real numbers or the complex numbers, denoted by $\mathbb{K}$ to represent $\mathbb{R}$ or $\mathbb{C}$.

Definition (Normed vector space)**.** A **normed vector space** is a vector space V with a norm $\|\cdot\| : V \to \mathbb{R}$ satisfying

i. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if $v = 0$. (Positive definite)

ii. $\|\lambda v\| = |\lambda| \, \|v\|$ for every $v \in V$ and $\lambda \in K$. (Positive homogeneous)

iii. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$. (Triangle inequality)

In particular, a metric on V is defined by $d(v, w) = \|v - w\|$.

Fact. The vector space operations of scalar multiplication and vector addition are continuous.

$$
\begin{aligned}
K \times V &\to V & (\lambda, v) &\mapsto \lambda v \\
V \times V &\to V & (v, w) &\mapsto v + w
\end{aligned}
$$

Proof. We only check that scalar multiplication is continuous. Since K and V are metric spaces, it suffices to show that $\lambda_j \to \lambda$ and $v_j \to v$ implies $\lambda_j v_j \to \lambda v$. But

$$\begin{aligned}\|\lambda_j v_j - \lambda v\| &= \|(\lambda_j - \lambda)v_j + \lambda(v_j - v)\| \\ &\leq \underbrace{|\lambda_j - \lambda|}_{\to 0} \underbrace{\|v_j\|}_{\text{bounded}} + |\lambda| \underbrace{\|v_j - v\|}_{\to 0}\end{aligned}$$

as required. □

Corollary. Translations ($v \mapsto v + v_0$) and dilations ($v \mapsto \lambda v, \lambda \neq 0$) are homomorphisms.

Definition (Topological vector space)**.** A **topological vector space** is a vector space together with a topology that makes the vector space operations continuous and in which points are closed.

Notation. For a subset C of a vector space V over $\mathbb{K}$ and $t \in \mathbb{K}$, we write tC for the following subset:

$$tC \coloneqq \{\, tv \mid v \in C \,\}$$

Definition (Convex subset)**.** Let V be a vector space and $C \subset V$ a subset. We say that C is **convex** iff $tC + (1-t)C \subset C$ for all $t \in [0,1]$. Specifically, this means $tv + (1-t)w \in C$ for all $v, w \in C$ and $t \in [0,1]$.

Fact. Let V be a normed vector space. Then $B_1(0)$ is convex.

Fact. If C is convex, then $v + \lambda C$ is convex for all $\lambda \in K$ and $v \in V$.

Definition (Locally convex)**.** A topological vector space is **locally convex** if its topology has a basis of convex sets.

Definition (Bounded)**.** Let V be a topological vector space and $B \subset V$. We say that B is **bounded** if for every open neighbourhood U of 0, there exists $t > 0$ such that $sU \supset B$ for all $s \geq t$.

Definition (Balanced)**.** Let V be a vector space, and $C \subset V$ a subset. Call C **balanced** if for all $|\lambda| \leq 1$, we have $\lambda C \subset C$.

Example.

(i) Balanced sets in $\mathbb{R}$ are sets of the form $[-t,t]$, $(-t,t)$, or $\{0\}$ and all of $\mathbb{R}$, and $\mathbb{Q}$ is not balanced in $\mathbb{R}$.

(ii) In $\mathbb{C}$, the only balanced sets are $\{0\}$, $\mathbb{C}$, and the open or closed balls centred at 0.

(iii) There are more interesting examples of balanced sets in $\mathbb{R}^2$, for instance the open disk, but also any ellipse centred at 0, as in [figure]. (BM todo $x^2 - xy + y^2 = 1$).

Lemma. Let V be a topological vector space and $C \subset V$ be a bounded, convex neighbourhood of 0. Then there exists a bounded, balanced, convex neighbourhood $\widetilde{C}$ of 0.

Proof. Exercise (on example sheet). □

Proposition. Let V be a topological vector space and $C \subset V$ be a bounded, convex neighbourhood of 0. Then the topology on V is induced by a norm.

Proof. Use the previous lemma to construct $\widetilde{C}$. Let

$$\mu_{\widetilde{C}}(v) = \inf \{ \, t > 0 \mid v \in t\widetilde{C} \, \}$$

referred to as the Minkowski functional of $\widetilde{C}$. We claim that $\|v\| = \mu_{\widetilde{C}}(v)$ is a norm on V and that the topology induced by it is the same as the original topology. Check the norm axioms in turn:

i. We clearly have positivity, and $\mu_{\widetilde{C}}(v) = 0$ iff $v = 0$ since $\widetilde{C}$ is bounded.

ii. Since $\widetilde{C}$ is balanced,

$$\begin{aligned}
\mu_{\widetilde{C}}(\lambda v) &= \inf \left\{ \, t > 0 \,\middle|\, \lambda v \in t\widetilde{C} \, \right\} \\
&= \inf \left\{ \, t > 0 \,\middle|\, v \in \frac{t}{|\lambda|}\widetilde{C} \, \right\} \\
&= \inf \left\{ \, |\lambda| \, \frac{t}{|\lambda|} > 0 \,\middle|\, v \in \frac{t}{|\lambda|}\widetilde{C} \, \right\} \\
&= |\lambda| \, \mu_{\widetilde{C}}(v)
\end{aligned}$$

iii. Given $v, w \in V$, write $v = \lambda v_0$ and $w = \mu w_0$ with $\lambda, \mu > 0$, $v_0, w_0 \in \widetilde{C}$. Since $\widetilde{C}$ is convex,

$$\begin{aligned}
& \frac{\lambda v_0 + \mu w_0}{\lambda + \mu} \in \widetilde{C} \\
\implies \; & \mu_{\widetilde{C}}\left(\frac{\lambda v_0 + \mu w_0}{\lambda + \mu} \right) \leq 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{\widetilde{C}}(v + w) &= (\lambda + \mu) \; \mu_{\widetilde{C}} \left(\frac{\lambda v_0 + \mu w_0}{\lambda + \mu} \right) \\
&\leq \lambda + \mu \\
&\leq \mu_{\widetilde{C}}(v) + \mu_{\widetilde{C}}(w)
\end{aligned}$$

□