# Part III – Model Theory

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## Contents

# 0 Introduction

Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: 'how powerful is our description of these objects to pin them down'? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

# 1 Languages and structures

**Definition 1.1** (Language)**.** A **language** $L$ consists of

(i) a set $\mathscr{F}$ of function symbols, and for each $f \in \mathscr{F}$ a positive integer $m_f$ the **arity** of $f$.

(ii) a set $\mathscr{R}$ of relation symbols, and for each $R \in \mathscr{R}$, a positive integer $m_R$.

(iii) a set $\mathscr{C}$ of constant symbols.

Note: each of $\mathscr{F}, \mathscr{R}$ and $\mathscr{C}$ can be empty.

**Example.** Take $L = \{\{\cdot, ^{-1}\}, \{1\}\}$, for $\cdot$ a binary function and $^{-1}$ an unary function, 1 a constant. This is the language of groups, call it $L_{\text{gp}}$. Also, $L_{\text{lo}} = \{<\}$ a single binary relation, for linear orders.

**Definition 1.2** ($L$-structure)**.** Given a language $L$, say, an $L$**-structure** consists of

(i) a non-empty set $M$, the **domain**,

(ii) for each $f \in \mathscr{F}$, a function $f^{\mathcal{M}} : M^{m_f} \to M$,

(iii) for each $R \in \mathscr{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$,

(iv) for each $c \in \mathscr{C}$, an element $c^{\mathcal{M}} \in M$.

$f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ are the **interpretations** of $f, R, c$ respectively.

**Remark 1.3.** We often fail to distinguish between the symbols in $L$ and their interpretations in a structure, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathscr{F}, \mathscr{R}, \mathscr{C} \rangle$.

**Example 1.4.**

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1 \rangle$ is an $L_{\text{gp}}$-structure.

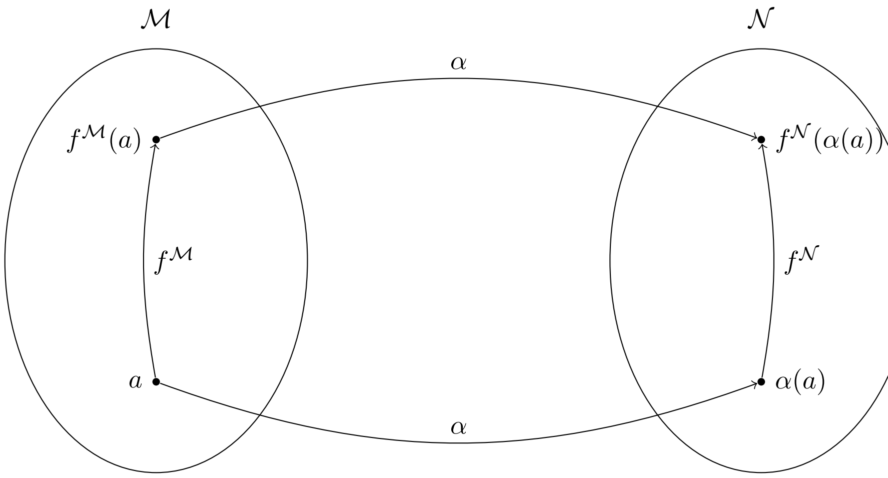
(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an $L_{\text{gp}}$-structure.

(c) $\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an $L_{\text{lo}}$-structure.

**Definition 1.5** (Embedding)**.** Let $L$ be a language, let $\mathcal{M}, \mathcal{N}$ be $L$-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is an injective mapping $\alpha : M \to N$ such that

(i) for all $f \in \mathscr{F}$, and $a_1, \dots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \dots, a_{m_f})) = f^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_{m_f}))$$




(ii) for all $R \in \mathscr{R}$, and $a_1, \ldots, a_{m_R} \in M$

$$(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \ldots, \alpha(a_{m_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathscr{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of $\mathcal{M}$ into $\mathcal{N}$ is a surjective embedding (onto), written $\mathcal{M} \simeq \mathcal{N}$.

**Exercise 1.6.** Let $G_1, G_2$ be groups, regarded as $L_{\text{gp}}$-structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an isomorphism $\alpha : G_1 \to G_2$ in the sense of Definition 1.5.

## 2 Review: Terms, formulae and their interpretations

In addition to the symbols of $L$, we also have

(i) infinitely many variables $\{x_i\}_{i\in I}$

(ii) logical connectives $\wedge, \neg$ (also expresses $\vee$, $\implies$, $\iff$)

(iii) quantifier $\exists$ (also expresses $\forall$)

(iv) ( , )

(v) equality symbol $=$

**Definition 2.1** ($L$-terms)**.** ***L*-terms** are defined recursively as follows:

- any variable $x_i$ is a term
- any constant symbol is a term
- for any $f \in \mathscr{F}$, $f(t_1, \dots, t_{m_f})$ for any terms $t_1, \dots, t_{m_f}$ is a term
- nothing else is a term

Notation: we write $t(x_1, \dots, x_m)$ to mean that the variables appearing in $t$ are among $x_1, \dots, x_m$.

**Example.** Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1\rangle$. Then $\cdot(\cdot(x_1, x_2), x_3)$ is a term, usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a term, written $(1 \cdot x)^{-1}$

**Definition 2.2.** If $\mathcal{M}$ is an $L$-structure, to each $L$-term $t(x_1, \dots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \to M$ defined as follows:

(i) If $t = x_i$, $t^{\mathcal{M}}(a_1, \dots, a_k) = a_i$

(ii) If $t = c$, $t^{\mathcal{M}}(a_1, \dots, a_k) = c^{\mathcal{M}}$.

(iii) If $t = f(t_1(x_1, \dots, x_k), \dots, t_{m_f}(x_1, \dots, x_k))$, then

$$t^{\mathcal{M}}(a_1, \dots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_k), \dots, t_{m_f}^{\mathcal{M}}(a_1, \dots, a_k)).$$

Notice in $L_{\text{gp}}$, the term $x_2 \cdot x_3$ can be described as $t_1(x_1, x_2, x_3)$ or $t_2(x_1, x_2, x_3, x_4)$, or infinitely many other ways. In these cases, $t_1$ is assigned to $t_1^{\mathcal{M}} : M^3 \to M$, with $(a_1, a_2, a_3) \mapsto (a_2, a_3)$, and $t_2$ is assigned to $t_2^{\mathcal{M}} : M^4 \to M$, with $(a_1, a_2, a_3, a_4) \mapsto a_2 \cdot a_3$.

**Fact 2.3.** Let $\mathcal{M}, \mathcal{N}$ be $L$-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any $L$-term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$ we have

$$\alpha(t^{\mathcal{M}}(a_1, \dots, a_k)) = t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k))$$

*Proof.* By induction on the complexity of $t$. Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_k)$. Then

(i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, and $t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds.

(ii) if $t = c$ a constant, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, and $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$, as required.

(iii) if $t = f(t_1(\bar{x}), \dots, t_{m_f}(\bar{x}))$, then

$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{m_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \dots, \alpha(t_{m_f}^{\mathcal{M}}(\bar{a})))$$

since $\alpha$ is an embedding. $t_1(\bar{x}), \dots, t_{m_f}(\bar{x})$ have lower complexity than $t$, so inductive hypothesis applies. □

**Exercise 2.4.** Conclude the proof of Fact 2.3.

**Definition 2.5** (Atomic formula)**.** The set of **atomic formulas** of $L$ is defined as follows

(i) if $t_1, t_2$ are $L$-terms, then $t_1 = t_2$ is an atomic formula

(ii) if $R$ is a relation symbol and $t_1, \dots, t_{m_R}$ are terms, then $R(t_1, \dots, t_{m_R})$ is an atomic formula

(iii) nothing else is an atomic formula.

**Definition 2.6** (Formula)**.** The set of **$L$-formulas** is defined as follows

(i) any atomic formula is an $L$-formula

(ii) if $\varphi$ is an $L$-formula, then so is $\neg\varphi$

(iii) if $\varphi$ and $\psi$ are $L$-formulas, then so is $\varphi \wedge \psi$

(iv) if $\varphi$ is an $L$-formula, for any $i \geq 1$, $\exists x_i\ \varphi$ is an $L$-formula

(v) nothing else is an $L$-formula

**Example.** In $L_{\text{gp}}$, $x_1 \cdot x_1 = x_2$ and $x_1 \cdot x_2 = 1$ are atomic formulas, and $\exists x_1\ (x_1 \cdot x_2) = 1$ is an $L_{\text{gp}}$-formula.

A variable occurs freely in a formula if it does not occur within the scope of a quantifier $\exists$ (the variable is **free**). Otherwise the variable is **bound**. For instance, in $\exists x_1\ (x_1 \cdot x_2) = 1$, $x_1$ is bound and $x_2$ is free.

**Important convention:** no variable occurs both freely and as a bound variable in the same formula.

A **sentence** is a formula with no free variables.

$$\exists x_1 \exists x_2\ (x_1 \cdot x_2 = 1)$$

is an $L_{\text{gp}}$-sentence. Notation: $\varphi(x_1, \dots, x_k)$ means that the free variables in $\varphi$ are among $x_1, \dots, x_k$.

**Definition 2.7** ($\vDash$)**.** Let $\varphi(x_1, \dots, x_k)$ be an $L$-formula, let $\mathcal{M}$ be an $L$-structure, and let $\bar{a} = (a_1, \dots, a_k)$ be elements of $M$. We define $\mathcal{M} \vDash \varphi(\bar{a})$ recursively as follows.

(i) if $\varphi$ is $t_1 = t_2$, then $\mathcal{M} \vDash \varphi(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

(ii) if $\varphi$ is $R(t_1, \dots, t_{m_k})$ then $\mathcal{M} \vDash \varphi(\bar{a})$ iff

$$(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{m_k}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}.$$

(iii) if $\varphi$ is $\psi \wedge \chi$, then $\mathcal{M} \vDash \varphi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$.

(iv) if $\varphi = \neg\psi$ then $\mathcal{M} \vDash \varphi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$. (this is well-defined since $\psi(\bar{a})$ is shorter than $\varphi(\bar{a})$)

(v) if $\varphi$ is $\exists x_j\ \chi(x_1, \ldots, x_k, x_j)$ (where $x_j \neq x_i$ for $i = 1, \ldots, k$). Then $\mathcal{M} \models \varphi(\bar{a})$ iff there is $b \in \mathcal{M}$ such that $\mathcal{M} \models \chi(a_1, \ldots, a_k, b)$.

**Example.** For $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1\rangle$, if $\varphi(x_1) = \exists x_2\ (x_2 \cdot x_2) = x_1$ then $\mathcal{R} \models \varphi(1)$ but $\mathcal{R} \not\models \varphi(-1)$.

**Notation 2.8** (Useful abbreviations)**.** We write

- $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \to \psi$ for $\neg\varphi \vee \psi$
- $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$
- $\forall x_i\ \varphi$ for $\neg\exists x_i\ (\neg\varphi)$

**Proposition 2.9.** Let $\mathcal{M}, \mathcal{N}$ be $L$-structures, let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. Let $\varphi(\bar{x})$ be atomic and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{N} \models \varphi(\alpha(\bar{a})).$$

Question: If $\varphi$ is an $L$-formula, not necessarily atomic, does Proposition 2.9 hold?

*Proof of Proposition 2.9.* Cases:

(i) $\varphi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where $t_1, t_2$ are terms. (Exercise: complete this case, using Fact 2.3)

(ii) $\varphi(\bar{x})$ is of the form $R(t_1(\bar{x}), \ldots, t_{m_R}(\bar{x}))$. Then $\mathcal{M} \models R(t_1(\bar{a}), \ldots, t_{m_R}(\bar{a}))$ if and only if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$. Apply Fact 2.3.

□

**Exercise 2.10.** Show that Proposition 2.9 holds if $\varphi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

**Example 2.11.** Do embeddings preserve *all* formulas? No. Take $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ an $L_{\mathrm{lo}}$-structure. Then $\alpha : \mathbb{Z} \to \mathbb{Q}$ (inclusion) is an embedding, but

$$\varphi(x_1, x_2) = \exists x_3\,(x_1 < x_3 \wedge x_3 < x_2).$$
$$\mathcal{Q} \models \varphi(1, 2) \text{ but } \mathcal{Z} \not\models \varphi(1, 2).$$

**Fact 2.12.** Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\varphi(\bar{x})$ is an $L$-formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{M} \models \varphi(\alpha(\bar{a})).$$

*Proof.* Exercise. □

## 3 Theories and elementarity

Throughout, $L$ is a language, $\mathcal{M}, \mathcal{N}$ are $L$-structures.

**Definition 3.1** ($L$-theory)**.** An **$L$-theory** $T$ is a set of $L$-sentences. $\mathcal{M}$ is a **model** of $T$ if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$. The class of all the models of $T$ is written $\mathrm{Mod}(T)$. The theory of $\mathcal{M}$ is the set

$$\mathrm{Th}(\mathcal{M}) = \{ \sigma \mid \sigma \text{ is an } L\text{-sentence and } \mathcal{M} \vDash \sigma \}.$$

**Example 3.2.** Let $T_{\mathrm{gp}}$ be the set of $L_{\mathrm{gp}}$-sentences

(i) $\forall x_1 x_2 x_3\, (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$

(ii) $\forall x_1\, (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1\, (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly for a group $G$, $G \vDash T_{\mathrm{gp}}$. For a specific $G$, clearly $\mathrm{Th}(G)$ is larger than $T_{\mathrm{gp}}$!

**Definition 3.3** (Elementarily equivalent)**.** Say $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$.

Clearly if $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$ but if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as $L_{\mathrm{lo}}$-structures.

**Definition 3.4** (Elementary substructure)**.**

(i) an embedding $\beta : \mathcal{M} \to \mathcal{N}$ is **elementary** if for all formulas $\varphi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \varphi(\bar{a}) \iff \mathcal{N} \vDash \varphi(\beta(\bar{a})).$$

(ii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is said to be a **substructure** of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$.

(iii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is said to be an **elementary substructure** of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$.

**Example 3.5.** Consider $\mathcal{M} = [0,1] \subseteq \mathbb{R}$, an $L_{\mathrm{lo}}$-structure, where $<$ is the usual order, and $\mathcal{N} = [0,2] \subseteq \mathbb{R}$ in the same way. Then $\mathcal{M} \simeq \mathcal{N}$ as $L_{\mathrm{lo}}$-structures.

Is $\mathcal{M} \equiv \mathcal{N}$? Yes: they are isomorphic!

Is $\mathcal{M} \subseteq \mathcal{N}$? Yes (the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$.)

But $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if $\varphi(x) = \exists y\ (x < y)$, then

$$\mathcal{N} \vDash \varphi(1) \quad \text{and} \quad \mathcal{M} \nvDash \varphi(1).$$

**Definition 3.6** (Parameter)**.** Let $\mathcal{M}$ be an $L$-structure, $A \subseteq M$, then define

$$L(A) \coloneqq L \cup \{ c_a \mid a \in A \}$$

for $c_a$ each constant symbols. An interpretation of $\mathcal{M}$ as an $L$-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way ($c_a^{\mathcal{M}} = a$). The elements of $A$ are called **parameters**. If $\mathcal{M}, \mathcal{N}$ are $L$-structures and $A \subseteq M \cap N$, then we write $\mathcal{M} \equiv_A \mathcal{N}$ when $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$-sentences.

**Exercise 3.7.** $\mathcal{M} \preccurlyeq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where $M$ is the domain of $\mathcal{M}$).

**Lemma 3.8** (Tarski-Vaught test)**.** Let $\mathcal{N}$ be an $L$-structure, let $A \subseteq N$. The following are equivalent:

(i) $A$ is the domain of a structure $\mathcal{M}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$.

(ii) for every $L(A)$-formula $\varphi(x)$ with one free variable, if $\mathcal{N} \vDash \exists x\ \varphi(x)$, then $\mathcal{N} \vDash \varphi(b)$ for some $b \in A$.

*Proof.*

**(i) $\Rightarrow$ (ii)** Suppose $\mathcal{N} \vDash \exists x\ \varphi(x)$. Then by elementarity, $\mathcal{M} \vDash \exists x\ \varphi(x)$, and so $\mathcal{M} \vDash \varphi(b)$ for some $b \in \mathcal{M}$, so again by elementarity $\mathcal{N} \vDash \varphi(b)$.

**(ii) $\Rightarrow$ (i)** First we prove that $A$ is the domain of a substructure $\mathcal{M} \subseteq \mathcal{N}$. By an exercise[1] on examples sheet 1, it is enough to check:

(a) for each constant $c$, $c^{\mathcal{N}} \in A$.

(b) for each function symbol $f$, $f^{\mathcal{N}}(\bar{a}) \in A$ (for all $\bar{a} \in A^{m_f}$).

For (a), use property (ii) with $\exists x\ (x = c)$. For (b) use property (ii) with $\exists x\ (f(\bar{a}) = x)$.

So we now have $\mathcal{M} \subseteq \mathcal{N}$, and the domain of $\mathcal{M}$ is $A$. It remains to verify elementarity. Let $\chi(\bar{x})$ be an $L$-formula. We show that for $\bar{a} \in A^{|\bar{x}|}$,

$$\mathcal{M} \vDash \chi(\bar{a}) \iff \mathcal{N} \vDash \chi(\bar{a}). \tag{$*$}$$

By induction on the complexity of $\chi(\bar{x})$:

- if $\chi(\bar{x})$ is atomic ($*$) follows from $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M}$ is a substructure), cf. Proposition 2.9.
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$: straightforward induction.
- if $\chi(\bar{x}) = \exists y\ \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an $L$-formula, suppose that $\mathcal{M} \vDash \chi(\bar{a})$. Then $\mathcal{M} \vDash \exists y\ \psi(\bar{a}, y)$, hence $\mathcal{M} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom} \mathcal{M}$. But then $\mathcal{N} \vDash \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \vDash \chi(\bar{a})$.

  Now let $\mathcal{N} \vDash \chi(\bar{a})$, i.e. $\mathcal{N} \vDash \exists y\ \psi(\bar{a}, y)$. By property (ii), $\mathcal{N} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom}(\mathcal{M})$. By inductive hypothesis, $\mathcal{M} \vDash \psi(\bar{a}, b)$ and so $\mathcal{M} \vDash \chi(\bar{a})$. $\square$

**Remark 3.9.** We define the cardinality $|L|$ of the language $L$ to be

$$|\{\varphi(\bar{x}) \mid \varphi \text{ is a } L\text{-formula}\}|$$

Assume that the set of variables is countably infinite. Then informally, $|L| = |L| + \omega$, so $|L|$ is at least $\omega$. In other words, if you start with a finite set of symbols, you still get an at least countably infinite set of formulas. Furthermore, for a set of parameter $A$, $|L(A)| = |L| + |A|$.

**Definition 3.10** (Chain)**.** Let $\lambda$ be an ordinal. Then **a chain of length** $\lambda$ of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$. A **chain of $L$-structures** is a sequence $\langle \mathcal{M}_i : i < \lambda \rangle$ such that $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i \leq j < \lambda$.

The **union** of this chain is the $L$-structure $\mathcal{M}$ is defined as follows:

- the domain of $\mathcal{M}$ is $\bigcup_{i<\lambda} M_i$

[1] For an $L$-structure $\mathcal{N}$ and a subset $A \subseteq N$ of the domain, $A$ is the domain of a substructure if and only if for every constant $c$ of $L$, $c^{\mathcal{N}} \in A$, and for every function $f$ of $L$ and $\bar{a} \in A^{n_f}$, $f^{\mathcal{N}}(\bar{a}) \in A$.

– $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$ ($c$ is a constant).

– if $f$ is a function symbol, $\bar{a} \in M^{m_f}$, $f^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}_i}(\bar{a})$ where $i$ is such that $\bar{a} \in M_i^{m_f}$.

– if $R$ is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i<\lambda} R^{\mathcal{M}_i}$

**Theorem 3.11** (Downward Löwenheim-Skolem)**.** Let $\mathcal{N}$ be an $L$-structure, and $|N| \geq |L| + \omega$. Let $A \subseteq N$. Then for any cardinal $\lambda$ such that $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preccurlyeq \mathcal{N}$ such that

(i) $A \subseteq M$, and

(ii) $|\mathcal{M}| = \lambda$.

It helps to think about the case $|L| \leq \omega$, $|A| = \omega$ and $|N|$ is uncountable. For instance, think of $(\mathbb{C}, +, \cdot, -, ^{-1}, 0, 1)$ as a field. Then $\mathbb{Q} \subseteq \mathbb{C}$: it is a subset and a substructure. In particular, the property of being algebraically closed is in the theory of $\mathbb{C}$. Thus Theorem 3.11 gives a algebraically closed field, which is countable and contains $\mathbb{Q}$. A possibility is the algebraic closure of $\mathbb{Q}$.

*Proof.* We inductively build a chain $\langle A_i : i < \omega \rangle$, with $A_i \subseteq N$, such that $|A_i| = \lambda$. Our goal is to define $M = \bigcup_{i<\omega} A_i$.

Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$. At stage $i+1$, assume that $A_i$ has been built, with $|A_i| = \lambda$. Let $\langle \varphi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$-formulas such that $\mathcal{N} \vDash \exists x\, \varphi_k(x)$. Observe there are no more than $\lambda$, since $|L(A)| = |L| + |A| + \omega \leq \lambda$. Let $a_k$ be such that $\mathcal{N} \vDash \varphi_k(a_k)$ and let $A_{i+1} = A_i \cup \{ a_k : k < \lambda \}$. Then $|A_{i+1}| = \lambda$.

Now let $M = \bigcup_{i<\omega} A_i$. We use the Tarski-Vaught test to show that $M$ is the domain of a structure $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|M| = \lambda$:

Let $\mathcal{N} \vDash \exists x\, \psi(x, \bar{a})$, where $\bar{a}$ is a tuple in $M$. Then $\bar{a}$ is a *finite* tuple, so there is an $i$ such that $\bar{a}$ is in $A_i$. Then $A_{i+1}$, by construction, contains $b$ such that $\mathcal{N} \vDash \varphi(b, \bar{a})$. But $A_{i+1} \subseteq M$, so $b \in M$. □

# 4 Two relational structures

## 4.1 Dense linear orders

**Definition 4.1** (Dense linear orders)**.** A **linear order** is an $L_{\mathrm{lo}} = \{<\}$-structure such that

(i) $\forall x\, \neg(x < x)$,

(ii) $\forall xyz\, ((x < y \wedge y < z) \to x < z)$,

(iii) $\forall xy\, ((x < y) \vee (y < x) \vee (x = y))$.

A linear order is **dense** if it also satisfies

(iv) $\exists xy\, (x < y)$,

(v) $\forall xy\, (x < y \to \exists z\, (x < z < y))$ (density).

A linear order has **no endpoints** if

(vi) $\forall x\, (\exists y\, (x < y) \wedge \exists z\, (z < x))$.

$T_{\mathrm{dlo}}$ is the theory that includes axioms (i) to (vi), $T_{\mathrm{lo}}$ is the theory that includes axioms (i) to (iii) only.

Remark: (iv) and (v) imply that if $\mathcal{M} \models T_{\mathrm{dlo}}$ then $|\mathcal{M}| \geq \omega$. Any model of $T_{\mathrm{dlo}}$ must be infinite.

**Definition 4.2** ((Finite) Partial embedding, partially isomorphic)**.** If $\mathcal{M}, \mathcal{N} \models T_{\mathrm{lo}}$, then an injective map $p : A \subseteq M \to N$ is called a **partial embedding** if for all $a, b \in A$,

$$\mathcal{M} \models a < b \iff \mathcal{N} \models p(a) < p(b).$$

If $|\mathrm{dom}(p)| < \omega$, then $p$ is a **finite partial embedding**.

The structures $\mathcal{M}$ and $\mathcal{N}$ are said to be **partially isomorphic** if there is a collection $I$ of partial embeddings such that

(i) if $p \in I$ and $a \in M$, then there is $\hat{p} \in I$ such that $p \subseteq \hat{p}$ and $a \in \mathrm{dom}\,\hat{p}$,

(ii) if $p \in I$ and $b \in N$, then there is $\hat{p} \in I$ such that $p \subseteq \hat{p}$ and $b \in \mathrm{img}\,\hat{p}$.

**Lemma 4.3** (Back and Forth)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be $L$-structures that are countable and partially isomorphic. Then $M \simeq N$.

*Proof.* Enumerate $M = \langle a_i \mid i < \omega \rangle$ and $N = \langle b_i \mid i < \omega \rangle$. We define a chain of partial embeddings $\langle p_i \mid i < \omega \rangle$ such that $a_{i-1} \in \mathrm{dom}\, p_i$, $b_{i-1} \in \mathrm{img}\, p_i$ and $p_i \in I$, the collection that makes $\mathcal{M}$ and $\mathcal{N}$ partially isomorphic.

Let $p_0 \in I$ be any partial embedding. At stage $i+1$, let $p_i$ be given. Use property (i) to extend $p_i$ to $\hat{p}$ such that $a_i \in \mathrm{dom}\,\hat{p}$, and property (ii) to extend $\hat{p}$ to $p_{i+1} \supseteq p_i$ such that $b_i \in \mathrm{img}\, p_{i+1}$. Then $\bigcup_{i<\omega}$ is the required isomorphism. □

**Lemma 4.4** (Extension lemma for dense linear orders)**.** Suppose $\mathcal{M} \models T_{\mathrm{dlo}}$, $\mathcal{N} \models T_{\mathrm{dlo}}$, let $p : \mathrm{dom}\, p \subseteq M \to N$ be a finite partial embedding. Then if $c \in M$, there is a finite partial embedding $\hat{p}$ such that $p \subseteq \hat{p}$ and $c \in \mathrm{dom}(\hat{p})$.

*Proof.* Enumerate $\mathrm{dom}\, p = \langle a_i \mid i < n+1 \rangle$ and let $c \in M$ such that $c \notin \mathrm{dom}\, p$. Split into three cases:

1. $c < a_0$. Use axiom (vi), which asserts that there are no endpoints, to find $d \in N$ such that $d < p(a_0)$ and let $\hat{p} \coloneqq p \cup \{\langle c, d\rangle\}$.

2. $a_i < c < a_{i+1}$ for some $a_i, a_{i+1} \in \operatorname{dom}(p)$. Then $\mathcal{N} \vDash p(a_i) < p(a_{i+1})$, so by density, there exists $d$ such that $\mathcal{N} \vDash p(a_i) < d < p(a_{i+1})$.

3. $c > a_n$. Similar to case 1. $\square$

**Theorem 4.5.** Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$ such that $|M| = |N| = \omega$. Then $\mathcal{M} \simeq \mathcal{N}$.

*Proof.* By Lemma 4.4, the collection $I$ of finite partial embeddings satisfies Items (i) and (ii) in Definition 4.2. Since $\varnothing : \mathcal{M} \to \mathcal{N}$ is a finite partial embedding, $I \neq \varnothing$ and Lemma 4.3 applies. $\square$

**Definition 4.6** (Consistent, complete, $\vdash$)**.** An $L$-theory $T$ is **consistent** if there is $\mathcal{M}$ such that $\mathcal{M} \vDash T$. If $T$ is a theory in $L$ and $\varphi$ is an $L$-sentence, then we write $T \vdash \varphi$ if for all $\mathcal{M}$ such that $\mathcal{M} \vDash T$, we also have $\mathcal{M} \vDash \varphi$. We say $T$ **entails** $\varphi$. An $L$-theory $T$ is **complete** if for all $L$-sentences $\varphi$, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Is $T_{\text{dlo}}$ complete?

**Definition 4.7** ($\omega$-categorical)**.** A theory $T$ in a countable language with a countably infinite model is called $\omega$-categorical if any two countable models of $T$ are isomorphic.

**Corollary 4.8** (of Theorem 4.5)**.** $T_{\text{dlo}}$ is $\omega$-categorical.

*Proof.* Say $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$, and $|\mathcal{M}| = |\mathcal{N}| = \omega$. Then $\varnothing$ (the empty map) is a finite partial embedding. By Theorem 4.5, $\mathcal{M} \simeq \mathcal{N}$. Instead of the empty map, we can also use any $\{\langle a, b\rangle\}$ where $a \in \mathcal{M}, b \in \mathcal{N}$ as initial finite partial embedding. $\square$

**Theorem 4.9.** If $T$ is an $\omega$-categorical theory in a countable language, and $T$ has no finite models then $T$ is complete.

*Proof.* Let $\mathcal{M} \vDash T$ and $\varphi$ be an $L$-sentence.

If $\mathcal{M} \vDash \varphi$, suppose $\mathcal{N} \vDash T$. Then by Downward Löwenheim-Skolem, there are $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$ such that $|\mathcal{M}'| = |\mathcal{N}'| = \omega$. By $\omega$-categoricity, $\mathcal{M}' \simeq \mathcal{N}'$ , so in particular $\mathcal{M}' \equiv \mathcal{N}'$ and so $\mathcal{N}' \vDash \varphi$.

If $\mathcal{M} \vDash \neg\varphi$, similar. $\square$

**Corollary 4.10.** $T_{\text{dlo}}$ is complete.

**Definition 4.11** ((Partial) elementary map)**.** If $\mathcal{M}$, $\mathcal{N}$ are $L$-structures, an injective map $f$ such that $\operatorname{dom} f \subseteq M$ and $\operatorname{img} f \subseteq N$ is called a **(partial) elementary map** if for all $L$-formulae $\varphi(\bar{x})$ and $\bar{a} \in (\operatorname{dom} f)^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \varphi(\bar{a}) \iff \mathcal{N} \vDash \varphi(f(\bar{a})).$$

**Remark 4.12.** A map $f$ is elementary iff every finite restriction of $f$ is elementary.

*Proof.*

$\Leftarrow$ Suppose $f$ is not elementary. Then there are $\varphi(\bar{x})$ and $\bar{a} \in (\operatorname{dom} f)^{|\bar{x}|}$ such that

$$\mathcal{M} \vDash \varphi(\bar{a}) \not\Longleftrightarrow \mathcal{N} \vDash \varphi(f(\bar{a})).$$

Then $f|_{\bar{a}}$ is a finite restriction of $f$ that is not elementary.

$\Rightarrow$ Clear. $\square$

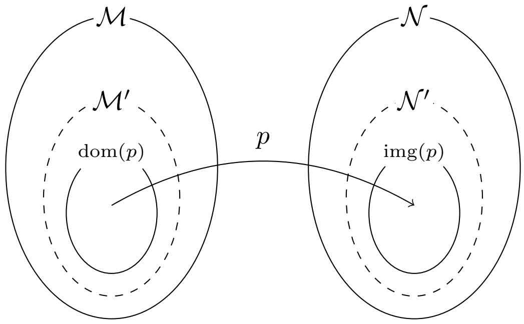
**Proposition 4.13.** Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$ and let $p : A \subseteq M \to N$ be a partial embedding. Then $p$ is elementary.

*Proof.* By Remark 4.12 and the proof of Lemma 4.3, it suffices to consider $p$ finite. By Downward Löwenheim-Skolem, we choose $\mathcal{M}', \mathcal{N}'$ such that

(i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.

(ii) $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$

(iii) $\text{dom}(p) \subseteq \mathcal{M}', \text{img}(p) \subseteq \mathcal{N}'$




Now $p$ is a finite partial embedding between countable models, so $p$ extends to an isomorphism $\pi : \mathcal{M}' \to \mathcal{N}'$ by Lemma 4.4. In particular, $\pi$ is an elementary map between $\mathcal{M}$ and $\mathcal{N}$. □

**Corollary 4.14.** $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

*Proof.* Use Proposition 4.13 with $\text{id} : \mathbb{Q} \to \mathbb{R}$. □

## 4.2 Random graphs

**Definition 4.15** (Random graph)**.** Let $L_{\text{gph}} = \{R\}$, a binary relation symbol. An $L_{\text{gph}}$-structure is a **graph** if

(i) $\forall x \, \neg R(x, x)$,

(ii) $\forall xy \, (R(x, y) \leftrightarrow R(y, x))$.

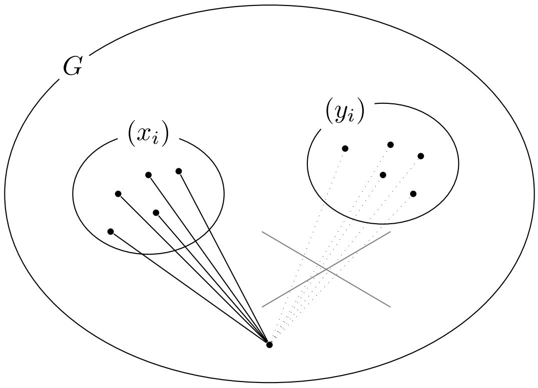
In other words, a graph contains no loops and is undirected.

An $L_{\text{gph}}$-structure is a **random graph** if it is a graph such that, for all $n \in \omega$, axiom $(r_n)$ holds:

$$\forall x_0 \ldots x_n, y_0 \ldots y_n \left( \bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left( \bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii) $\exists xy \, (x \neq y)$.

Axiom $(r_n)$ effectively says that for disjoint subsets $(x_i)$ and $(y_i)$ each of size $n$, there is a (different) node $z$ connected to each $x_i$ and none of the $y_i$.




**Remark.** A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

**Fact 4.16.** There is a random graph.

*Proof.* Let the domain be $\omega$, let $i, j \in \omega$ such that $i < j$. Write $j$ as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff $2^i$ appears in the sum. □

**Exercise.** Prove that $\omega$ with this definition of $R$ is a random graph.

**Remark.** Fact 4.16 shows that $T_{\mathrm{rg}}$ is consistent.

**Definition 4.17** (Graph theories, partial embedding)**.** $T_{\mathrm{gph}}$ consists of the axioms (i),(ii) above, and $T_{\mathrm{rg}} = T_{\mathrm{gph}} \cup \{(\mathrm{iii}), (r_n) : n \in \omega\}$. If $\mathcal{M}, \mathcal{N} \vDash T_{\mathrm{gph}}$, a **partial embedding** is an injective map $p : A \subseteq M$ to $N$ such that

$$\mathcal{M} \vDash R(a, b) \iff \mathcal{N} \vDash R(p(a), p(b))$$

for all $a, b$ in the domain. Just as before, if $|\mathrm{dom}(p)| < \omega$ then $p$ is called a **finite partial embedding**.

**Lemma 4.18** (Extension lemma for random graphs)**.** Let $\mathcal{M} \vDash T_{\mathrm{gph}}$, $\mathcal{N} \vDash T_{\mathrm{rg}}$, let $p : A \subseteq M \to N$ be a finite partial embedding, and let $c \in M$. Then there is a finite partial embedding $\hat{p} : \hat{A} \subseteq M \to N$ such that, $c \in \mathrm{dom}(\hat{p})$, and $p \subseteq \hat{p}$.

*Proof.* Assume that $c \notin \mathrm{dom}\, p$ and let

$$\begin{aligned} U &:= \{a \in \mathrm{dom}\, p \mid R(a, c)\}, \\ V &:= \{b \in \mathrm{dom}\, p \mid \neg R(b, c)\}. \end{aligned}$$

Since $p$ is finite, $U$ and $V$ are finite. Since $N$ is a random graph, we can find $d \in N \setminus (p(U) \cup p(V))$ such that

(i) $R(d, p(a))$ for all $a \in U$,

(ii) $\neg R(d, p(b))$ for all $b \in V$.

Then $\hat{p} := p \cup \langle c, d \rangle$ is the desired extension. □

**Theorem 4.19.** Let $\mathcal{M}, \mathcal{N} \vDash T_{\mathrm{rg}}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $p : A \subset M \to N$ a finite partial embedding. Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends $p$.

*Proof.* Same as proof of Theorem 4.5, but with Lemma 4.18 instead of Lemma 4.3. □

**Corollary 4.20.** $T_{\mathrm{rg}}$ is $\omega$-categorical and complete. Moreover, every finite partial embedding between models of $T_{\mathrm{rg}}$ is an elementary map.

**Remark 4.21.** The unique (up to isomorphism) countable model of $T_{\mathrm{rg}}$ is *the* countable random graph, or the **Rado graph**. It is universal with respect to finite and countable graphs (i.e. it embeds them all). It is **ultrahomogeneous** i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.

Further information about random graphs can be found in the work of Peter Cameron.

# 5 Compactness

Section 11 contains an alternative version of the first part of this chapter.

**Definition 5.1** (Filter)**.** Let $I$ be a set. Then a **filter** $F$ on $I$ is a subset of the power set $2^I$ such that

(i) $I \in F$,

(ii) if $X, Y \in F$, then $X \cap Y \in F$,

(iii) if $X \in F$ and $X \subseteq Y \subseteq I$, then $Y \in F$.

A filter is **proper** if $F \neq 2^I$, or equivalently if $\varnothing \notin F$. An **ultrafilter** $U$ on $I$ is a proper filter such that for all $X \subseteq I$ either $X \in U$ or $I \setminus X \in U$.

**Remark.** Informally, a filter tells whether a set is large. The entire set $I$ is large, the intersection of two large sets is large, and all supersets of large sets are large. An ultrafilter decides for every set whether its large or small, in which case its complement is large.

**Fact 5.2.** The following are equivalent for a proper filter $U$ on $I$.

(i) $U$ is an ultrafilter,

(ii) $U$ is maximal among the proper filters on $I$,

(iii) if $X \cup Y \in U$ then either $X \in U$ or $Y \in U$.

*Proof.* Exercise. □

**Definition 5.3** (Direct product)**.** Let $\langle \mathcal{M}_i \mid i \in I \rangle$ be a collection of $L$-structures for a not necessarily ordered index set $I$. The **direct product** $\prod_{i\in I} \mathcal{M}_i$ is the set

$$\left\{ f : I \to \bigcup_{i \in I} M_i \;\middle|\; \forall i \in I : f(i) \in M_i \right\}.$$

We will write $X$ for $\prod_{i\in I} \mathcal{M}_i$ when the $\mathcal{M}_i$ and $I$ are understood. We write $a = \langle a(i) \mid i \in I \rangle$. Let $U$ be an ultrafilter on $I$. Define a relation $\sim_U$ on $X$ as follows,

$$a \sim_U b \iff \{i \in I \mid a(i) = b(i)\} \in U.$$

**Fact 5.4.** For every filter $U$, the relation $\sim_U$ is an equivalence relation.

*Proof.* Reflexivity and symmetry are clear. Let $a \sim_U b$ and $b \sim_U c$. Furthermore let $A := \{i \in I \mid a(i) = b(i)\}$ and similarly $B := \{i \in I \mid b(i) = c(i)\}$. By assumption, $A, B \in U$. Then $A \cap B \subseteq \{i \in I \mid a(i) = c(i)\}$ which implies $\{i \in I \mid a(i) = c(i)\} \in U$, so $a \sim_U c$. □

**Remark.** For Fact 5.4, $U$ did not need to be an ultrafilter.

Write $a_U$ for the equivalence class of $a \in X$ under $\sim_U$. We aim at making $X/{\sim_U}$ into an $L$-structure. Call $X_U := \prod_{i\in I} \mathcal{M}_i/{\sim_U}$ the **ultraproduct** of the $\mathcal{M}_i$.

**Fact 5.5.** Let $a^k, b^k \in X$ for $k = 1, \ldots, n$ be such that $a^k \sim_U b^k$ for all $k$. Then

(i) if $f$ is an $n$-ary function symbol, then

$$\left\langle f^{\mathcal{M}_i}(a^1(i), \ldots, a^n(i)) \;\middle|\; i \in I \right\rangle_U = \left\langle f^{\mathcal{M}_i}(b^1(i), \ldots, b^n(i)) \;\middle|\; i \in I \right\rangle_U.$$

(ii) if $R$ is an $n$-ary relation symbol, then

$$\left\{i \in I \;\middle|\; (a^1(i), \ldots, a^n(i)) \in R^{\mathcal{M}_i}\right\} \in U \iff \left\{i \in I \;\middle|\; (b^1(i), \ldots, b^n(i)) \in R^{\mathcal{M}_i}\right\} \in U.$$

*Sketch of the Proof of Fact 5.5.* For Item (i), consider for an ease of notation the case $n = 1$. Let $a, b \in X$ such that $a \sim_U b$. Let $A := \{i \in I \mid a(i) = b(i)\}$ and $C := \left\{i \in I \;\middle|\; f^{\mathcal{M}_i}(a(i)) = f^{\mathcal{M}_i}(b(i))\right\}$. Clearly, $A \subseteq C$, so if $A \in U$, then $C \in U$.

Item (ii) similar. $\square$

**Definition 5.6.** Let $\langle \mathcal{M}_i \mid i \in I\rangle$ as in Definition 5.3. Let $U$ be an ultrafilter on $I$. Then $X_U$ is an $L$-structure where

(i) if $c$ is a constant, then $c^{X_U} := \left\langle c^{\mathcal{M}_i} \;\middle|\; i \in I\right\rangle_U$,

(ii) if $f$ is an $n$-ary function symbol, then for $a^1_U, \ldots, a^n_U$,

$$f^{X_U}(a^1_U, \ldots, a^n_U) := \left\langle f^{\mathcal{M}_i}(a^1(i), \ldots, a^n(i)) \;\middle|\; i \in I\right\rangle_U,$$

(iii) if $R$ is an $n$-ary relation symbol, then for $a^1_U, \ldots, a^n_U$,

$$(a^1_U, \ldots, a^n_U) \in R^{X_U} \iff \left\{i \in I \;\middle|\; (a^1(i), \ldots, a^n(i)) \in R^{\mathcal{M}_i}\right\} \in U.$$

**Remark.** Fact 5.5 ensures that $f^{X_U}$ and $R^{X_U}$ are well-defined.

**Theorem 5.7** (Łoś)**.** Let $\langle \mathcal{M}_i \mid i \in I\rangle$ as in Definition 5.3, $U$ an ultrafilter. Then,

(i) for all $L$-terms $t(x_1, \ldots, x_n)$ and $a^1_U, \ldots, a^n_U \in X_U$,

$$t^{X_U}(a^1_U, \ldots, a^n_U) = \left\langle t^{\mathcal{M}_i}(a^1(i), \ldots, a^n(i)) \;\middle|\; i \in I\right\rangle_U,$$

(ii) for all $L$-formulas $\varphi(x_1, \ldots, x_n)$ and $a^1_U, \ldots, a^n_U \in X_U$,

$$X_U \vDash \varphi(a^1_U, \ldots, a^n_U) \iff \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \varphi(a^1(i), \ldots, a^n(i))\right\} \in U,$$

(iii) for all $L$-sentences $\sigma$,

$$X_U \vDash \sigma \iff \{i \in I \mid \mathcal{M}_i \vDash \sigma\} \in U.$$

*Proof.* Item (i) is an easy induction on the complexity of $t$, Item (iii) is an immediate consequence of Item (ii). We prove Item (ii).

The case of $\varphi$ atomic is an easy induction. Let $\psi = \neg\chi$ for an $L$-formula $\chi(x_1, \ldots, x_n)$ and let $A_\chi := \{i \in I \mid \mathcal{M}_i \vDash \chi(a^1(i), \ldots, a^n(i))\}$. By the induction hypothesis, $X_U \vDash \chi(a^1_U, \ldots, a^n_U)$ if and only if $A_\chi \in U$. Equivalently, $X_U \nvDash \chi(a^1_U, \ldots, a^n_U)$ iff $A_\chi \notin U$. But $U$ is an ultrafilter, so $A_\chi \notin U$ implies $I \setminus A_\chi \in U$.

If $\varphi = \chi \wedge \psi$, let $A_\varphi$, $A_\psi$ and $A_\chi$ as before. Then $A_\varphi = A_\psi \cap A_\chi$ and since $U$ is a filter, $A_\varphi \in U$ iff $A\chi \in U$ and $A_\psi \in U$. The required result follows from the inductive hypothesis on $\psi$ and $\chi$.

If $\varphi = \exists y\ \psi(\bar{x}, y)$, define $A_\varphi$ as usual. Suppose there is $b_U \in X_U$ such that $X_U \vDash \psi(a^1_U, \ldots, a^n_U, b_U)$. We have $\{i \in I \mid \mathcal{M}_i \vDash \psi(a^1(i), \ldots, a^n(i), b(i))\} \subseteq A_\varphi$. By inductive hypothesis, this set is in $U$ and so is $A_\varphi$. For the other implication, suppose $A_\varphi \in U$. For $i \in A_\varphi$ find $b_i \in M_i$ such that $\mathcal{M}_i \vDash \psi(a^1(i), \ldots, a^n(i), b_i)$ and for $i \notin A_\varphi$ let $b_i$ be arbitrary in $M_i$. Define $b \in X_U$ by $b(i) := b_i$. Define $A\psi = \{i \in I \mid \mathcal{M}_i \vDash \psi(a^1(i), \ldots, a^n(i), b(i))\}$. Then $A_\varphi \subseteq A_\psi$, and so $A_\psi \in U$. By the inductive hypothesis, $X_U \vDash \psi(a^1, \ldots, a^n, b)$, so $X_U \vDash \exists y\ \psi(\bar{a}, y)$. $\square$

**Definition 5.8** (Finite intersection property)**.** A subset $S \subseteq 2^I$ has the **finite intersection property** if for all $n \in \omega$ and $A_1, \ldots, A_n \in S$, it holds that,

$$\bigcap_{i=1}^{n} A_i \neq \varnothing.$$

**Remark.** Proper filters have the finite intersection property.

**Lemma 5.9.** (i) If $S \subseteq 2^I$ has the finite intersection property, then it can be extended to a proper filter $F \supseteq S$.

(ii) A proper filter can always be extended to an ultrafilter assuming the axiom of choice.

*Proof.* For Item (i), let $F$ be the extension of $S$ defined as follows,

$$F \coloneqq \{X \subseteq I \mid X \text{ contains a finite intersection of elements from } S\}.$$

For Item (ii), if $F$ is a proper filter, let

$$\mathcal{F} \coloneqq \left\{G \subseteq 2^I \;\middle|\; G \supseteq F \text{ and } G \text{ is a proper filter}\right\}.$$

$\mathcal{F}$ is partially ordered by inclusion. Check that the union of a chain in $\mathcal{F}$ is in $\mathcal{F}$ and apply Zorn's lemma. By Fact 5.2, the maximal element is the required ultrafilter. □

**Definition 5.10.** A theory $T$ is said to be

(i) **consistent** or **satisfiable** if it has a model,

(ii) **finitely consistent** or **finitely satisfiable** if every finite subset of $T$ has a model.

**Theorem 5.11** (Compactness)**.** An $L$-theory $T$ is consistent if and only if it is finitely consistent.

*Proof.* $\Rightarrow$ immediate.

$\Leftarrow$ Let $S \subset T$ be finite. Let $\mathcal{M}_S \vDash S$ be a model of $S$. Let $I = \{S \subseteq T \mid |S| < \omega\}$. The idea is to define an ultrafilter $U$ on $I$ such that $\prod_{S \in I} \mathcal{M}_S/{\sim_U} \vDash T$. By Theorem 5.7, it is enough to find $U$ such that for all $\varphi \in T$, $\{S \in I \mid \varphi \in S\} \in U$. Let $\varphi \in T$ and define $A_\varphi = \{S \in I \mid \varphi \in S\}$.

We claim that $\{A_\varphi \mid \varphi \in T\}$ has the finite intersection property. Let $\varphi_1, \ldots, \varphi_n \in T$. Then $\{\varphi_1, \ldots, \varphi_n\} \in I$ and $\{\varphi_1, \ldots, \varphi_n\} \subseteq \bigcap_{i=1}^{n} A_{\varphi_i} \neq \varnothing$. Then by Lemma 5.9, $\{A_\varphi \mid \varphi \in T\}$ extends to an ultrafilter $U$. By Theorem 5.7, $\prod_{S \in I} \mathcal{M}_S/{\sim_U} \vDash \varphi$ iff $\{S \in I \mid \mathcal{M}_S \vDash \varphi\} \in U$. But $A_\varphi \in U$ and $A_\varphi \subseteq \{S \in I \mid \mathcal{M}_S \vDash \varphi\} \in U$.

□

**Definition 5.12** (Type)**.** Let $L$ be a language. An $L$**-type** $p(\bar{x})$ is a set of $L$-formulas whose free variables are in $\bar{x}$ (and $\bar{x} = \langle x_i : i < \lambda \rangle$).

A type $p(\bar{x})$ is said to be

(i) **satisfiable in an $L$-structure $\mathcal{M}$** if there is $\bar{a} \in M^{|\bar{x}|}$ such that $M \vDash \varphi(\bar{a})$ for all $\varphi(\bar{x}) \in p(\bar{x})$,

(ii) **satisfiable** if it is satisfiable in some $L$-structure $\mathcal{M}$,

(iii) **finitely satisfiable in $\mathcal{M}$** if all its finite subsets are satisfiable in $\mathcal{M}$,

(iv) **finitely satisfiable** if all its finite subsets are satisfiable in some (possibly different) $\mathcal{M}$.

If $p(\bar{x})$ is satisfied in $\mathcal{M}$ by a tuple $\bar{a}$, write $M \vDash p(\bar{a})$ or $M,\ \bar{a} \vDash p(\bar{x})$. We say $\bar{a}$ **realizes** $p(\bar{x})$ in $\mathcal{M}$. Some authors use *consistent* instead of *satisfiable.*

**Remark.** An $L$-type may be finitely satisfiable in $\mathcal{M}$ (i.e. every finite subset is satisfiable in $\mathcal{M}$) but not satisfiable in $\mathcal{M}$.

**Example.** Take $\mathcal{M} = (\mathbb{N}, <)$. Let $\varphi_n(x)$ say 'there are at least $n$ elements less than $x$'.

$$p(x) \coloneqq \{\, \varphi_n(x) \mid n < \omega \,\}$$

Is $p(x)$ finitely satisfiable in $\mathcal{M}$? Yes. But $p(x)$ is not satisfiable in $\mathcal{M}$.

**Theorem 5.13** (Compactness theorem for types)**.** Every finitely satisfiable $L$-type $p(\bar{x})$ is satisfiable.

*Proof.* The idea is to turn the type $p(\bar{x})$ into a theory by substituting fresh constant symbols for the variables $\bar{x}$. After having done this, Theorem 5.11 can be applied.

Let $\bar{x} = \langle x_i : i < \lambda \rangle$, let $\langle c_i : i < \lambda \rangle$ be new constants (not in $L$). Expand $L$ to $L' = L \cup \{c_i : i < \lambda\}$. Then $p(\bar{c})$ is a finitely satisfiable $L'$-theory and Theorem 5.11 applied to $p(\bar{c})$ gives an $L'$-structure $\mathcal{M}'$ such that $\mathcal{M}' \vDash p(\bar{c})$. But $\mathcal{M}'$ reduces to an $L$ structure $\mathcal{M}$, so $\mathcal{M}, \bar{c}^{\mathcal{M}'} \vDash p(\bar{x})$. □

**Lemma 5.14.** Let $\mathcal{M}$ be a structure, let $\bar{a} = \langle a_i : i < \lambda \rangle$ an enumeration of $\mathcal{M}$. Let

$$q(\bar{x}) = \{\, \varphi(\bar{x}) \mid \mathcal{M} \vDash \varphi(\bar{a}) \,\},$$

where $|\bar{x}| < \lambda$. Then $q(\bar{x})$ is satisfiable in $\mathcal{N}$ iff there is $\beta : \mathcal{M} \to \mathcal{N}$ that is an elementary embedding.

*Proof.*

($\Rightarrow$) If $q(\bar{x})$ is satisfiable in $\mathcal{N}$, there is $\bar{b} \in N^{|\bar{x}|}$ such that

$$\mathcal{N} \vDash \varphi(\bar{b}) \quad \forall \varphi(\bar{x}) \in q(\bar{x}).$$

Then $\beta : a_i \mapsto b_i$ for $i < \lambda$ is an elementary embedding. ($\beta$ preserves, for example, atomic formulas of the form $f(a_{i_1}, \ldots, a_{i_n}) = a_{i_{n+1}}$). More generally, for any $\varphi(\bar{x})$ an $L$-formula,

$$\mathcal{M} \vDash \varphi(\bar{a}) \iff \mathcal{N} \vDash \varphi(\bar{b})$$

but $\beta(\bar{a}) = \bar{b}$ so we have elementarity.

($\Leftarrow$) If $\beta : \mathcal{M} \to \mathcal{N}$ is elementary, then $\beta(\bar{a})$ satisfies $q(\bar{x})$ in $\mathcal{N}$. □

This lemma is sometimes also called the **Diagram Lemma**, and stated as: Suppose $\mathrm{Th}(\mathcal{M}_M)$ is a theory in $L(M)$. Then if $\mathcal{N} \vDash \mathrm{Th}(\mathcal{M}_M)$, then $\mathcal{M}$ embeds elementarily in $\mathcal{N}$.

**Remark 5.15.** We can consider types in $L(A)$, where $A \subseteq M$. In particular, we can have $M = A$.

Types of this kind are said to have **parameters in** $A$ (or to be over $A$). If $p(\bar{x})$ is a type over $M$, then there is $\bar{a}$, an enumeration of $M$, and a type $p'(\bar{x}, \bar{z})$ in $L$ where the $\bar{z}$ are new constants, $|\bar{z}| = |\bar{a}|$, and $p(\bar{x}) = p'(\bar{x}, \bar{a})$.

**Theorem 5.16.** If $\mathcal{M}$ is a structure, and $p(\bar{x})$ is a type in $L(M)$ that is finitely satisfiable in $\mathcal{M}$, then $p(\bar{x})$ is satisfiable in some $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$.

**Example.** Take $\mathcal{M} = (\mathbb{Q}, <)$, and let $\langle a_i : i < \omega \rangle$ a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ from below, and let $\langle b_i : i < \omega \rangle \subseteq \mathbb{Q}$ tend to $\sqrt{2}$ from above. Set $\varphi_n(x) \coloneqq a_n < x < b_n$. Then let $p(x) = \{ \varphi_n(x) \mid n < \omega \}$. Then $p(x)$ is an $L(\mathbb{Q})$-type which is finitely satisfiable in $\mathbb{Q}$. But $p(x)$ is not satisfiable in $\mathcal{M}$. It is, however, satisfiable in $(\mathbb{R}, <) \succcurlyeq (\mathbb{Q}, <)$.

**Example.** Take the interval $(0,1) \subseteq \mathbb{Q}$ and let $\mathcal{M} = ((0,1), <)$. Let $a_n = 1 - 1/n$ for $n \in \omega \setminus \{0\}$. Let $\varphi_n(x) = (a_n < x)$. Then $p(x) = \{\varphi_n(x) \mid n \in \omega \setminus \{0\}\}$ is finitely satisfiable in $\mathcal{M}$ but not satisfiable. However, $p(x)$ is satisfiable in $(\mathbb{R}, <) \succcurlyeq \mathcal{M}$.

*Proof of Theorem 5.16.* Let $\langle a_i : i < \lambda \rangle$ enumerate $\mathcal{M}$, let

$$q(\bar{z}) \coloneqq \{ \varphi(\bar{z}) \mid \mathcal{M} \vDash \varphi(\bar{a}) \}$$

where $|\bar{z}| = \lambda$ and the $z_i$ are new variables (so not among the $\bar{x}$). Write $p(\bar{x})$ as $p'(\bar{x}, \bar{a})$ for some $p'(\bar{x}, \bar{z})$ (an $L$-type).

**Claim**: $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is finitely satisfiable in $\mathcal{M}$.

**Proof:** $p'(\bar{x}, \bar{a})$ is finitely satisfiable by hypothesis and $q(\bar{z})$ is realized by $\bar{a}$.

Then, by Compactness theorem for types, $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is satisfiable. That is, there is $\mathcal{N}$ and $\bar{b} \in \mathcal{N}^{|\bar{z}|}$ and $\bar{c} \in \mathcal{N}^{|\bar{x}|}$ such that

$$\mathcal{N} \vDash p'(\bar{c}, \bar{b}) \cup q(\bar{b}).$$

In particular, $\mathcal{N} \vDash q(\bar{b})$, then by Lemma 5.14, $\beta : a_i \mapsto b_i$ is an elementary embedding. □

**Theorem 5.17** (Upward Löwenheim-Skolem)**.** Let $\mathcal{M}$ be such that $|\mathcal{M}| \geq \omega$. Then for any $\lambda \geq |\mathcal{M}| + |L|$, there is $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|\mathcal{N}| = \lambda$.

*Proof.* Let $\bar{x} = \langle x_i : i < \lambda \rangle$ a tuple of distinct variables. Let

$$p(\bar{x}) = \{ x_i \neq x_j \mid i < j < \lambda \}.$$

Then $p(\bar{x})$ is finitely consistent in $\mathcal{M}$. By Theorem 5.16, $p(\bar{x})$ is realized in some $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|\mathcal{N}| \geq \lambda$. By Downward Löwenheim-Skolem, we may assume $|\mathcal{N}| = \lambda$. □

# 6 Saturation

Anything that might happen does happen.

**Definition 6.1** (Saturated)**.** Let $\lambda$ be an infinite cardinal, let $|\mathcal{M}| \geq \omega$. Then $\mathcal{M}$ is **$\lambda$-saturated** if $\mathcal{M}$ realizes every type $p(x)$ with one free variable such that

(i) $p(x)$ has parameters in $A \subseteq M$ and $|A| < \lambda$.

(ii) $p(x)$ is finitely consistent in $\mathcal{M}$.

$\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated.

Can $\mathcal{M}$ be $\lambda$-saturated if $\lambda > |\mathcal{M}|$? If so, $\mathcal{M}$ would satisfy finitely satisfiable types in $L(M)$. For example,

$$p(x) = \{ x \neq a_i \mid i <\mid \mathcal{M} \mid \}$$

where $\langle a_i : i < |\mathcal{M}| \rangle$ enumerates $\mathcal{M}$. $p(x)$ is finitely satisfiable, but not satisfied in $\mathcal{M}$.

**Definition 6.2** (Type of tuple)**.** Let $\mathcal{M}$ be an $L$-structure, $A \subseteq M$, $\bar{b}$ a tuple in $M$ (possibly infinite). The **type of $\bar{b}$ over** $A$ is the following $L(A)$-type:

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) := \{ \varphi(\bar{x}) \in L(A) \mid \mathcal{M} \vDash \varphi(\bar{b}) \}.$$

The subscript $\mathcal{M}$ is often omitted if clear from context.

**Remark 6.3.**

(i) $\mathrm{tp}_{\mathcal{M}}(\bar{b}/A)$ is complete, i.e. for every $L(A)$-formula $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$ or $\neg\varphi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$.

(ii) If $\mathcal{M} \preccurlyeq \mathcal{N}$, then for $A \subseteq M$, $\bar{b}$ a tuple:

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A).$$

**Fact 6.4.**

(i) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is a (partial) elementary map, then in particular $f$ preserves $L$-sentences, so $\mathcal{M} \equiv \mathcal{N}$.

(ii) If $\mathcal{M} \equiv \mathcal{N}$, then $\varnothing$, the empty map, is an elementary map, as it preserves sentences.

(iii) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is elementary, and $\bar{a}$ is an enumeration of $A = \mathrm{dom}(f)$, then

$$\mathrm{tp}_{\mathcal{M}}(\bar{a}/\varnothing) = \mathrm{tp}_{\mathcal{N}}(f(\bar{a})/\varnothing).$$

More generally, if $f : \mathcal{M} \to \mathcal{N}$ is (partial) elementary and there is $A \subseteq M \cap N$ such that $A \subseteq \mathrm{dom}\, f$, $f|_A = \mathrm{id}$, then for every $\bar{b}$, a tuple in $\mathrm{dom}(f)$,

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) = \mathrm{tp}_{\mathcal{N}}(f(\bar{b})/A).$$

(iv) Let $\bar{a}$ enumerate $A \subseteq M$, $A = \mathrm{dom}(f)$ where $f : \mathcal{M} \to \mathcal{N}$ is elementary. Let $p(\bar{x}, \bar{a})$ be a type in $L(A)$ that is finitely satisfiable in $\mathcal{M}$. Then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in $\mathcal{N}$:

Let

$$\{\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a}).$$

By finite satisfiability of $p(\bar{x}, \bar{a})$,

$$\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{a}).$$

Then

$$\mathcal{N} \models \exists x \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, f(\bar{a}))$$

by elementarity of $f$. (Does $p(\bar{x}, \bar{a})$ satisfiable in $\mathcal{M}$ imply $p(\bar{x}, f(\bar{a}))$ satisfiable in $\mathcal{N}$? No.)

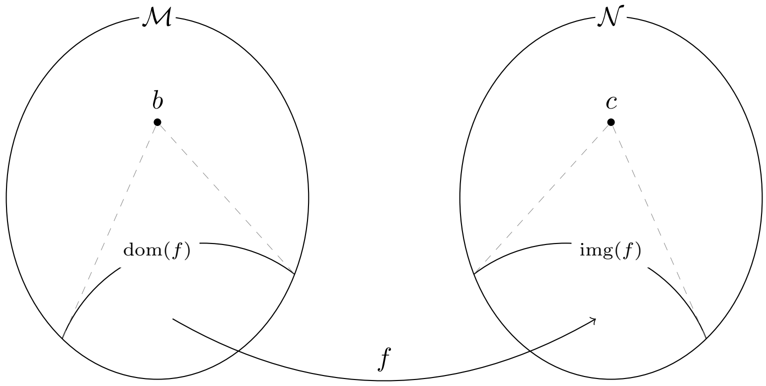
**Theorem 6.5.** Let $\mathcal{N}$ be such that $|\mathcal{N}| \geq \lambda \geq |L|$. The following are equivalent:

(i) $\mathcal{N}$ is $\lambda$-saturated.

(ii) if $b \in M$ and $f : \mathcal{M} \to \mathcal{N}$ partial elementary map such that $|f| < \lambda$, so in particular $\mathcal{M} \equiv \mathcal{N}$, then there is a partial elementary $\hat{f} \supseteq f$ and such that $b \in \operatorname{dom}(\hat{f})$.

(iii) If $p(\bar{z})$ is an $L(A)$-type where $|\bar{z}| \leq \lambda$, $A \subseteq N$, $|A| < \lambda$ and $p(\bar{z})$ is finitely satisfiable in $\mathcal{N}$, then $p(\bar{z})$ is satisfiable in $\mathcal{N}$.

*Proof.* (i) $\Rightarrow$ (ii). Let $f : \mathcal{M} \to \mathcal{N}$ be as in (ii), let $b \in M$. Let $\bar{a}$ be an enumeration of $\operatorname{dom}(f)$, so $|\bar{a}| < \lambda$. Let $p(x, \bar{a}) \coloneqq \operatorname{tp}_{\mathcal{M}}(b/\bar{a})$.




Then $p(x, \bar{a})$ is finitely satisfiable in $\mathcal{M}$, hence $\operatorname{tp}_{\mathcal{N}}(x/f(\bar{a}))$ is finitely satisfiable in $\mathcal{N}$ (by Fact 6.4(iv)). Since $|f(\bar{a})| < \lambda$ and $\mathcal{N}$ is $\lambda$-saturated, $\operatorname{tp}(x/f(\bar{a}))$ is realized in $\mathcal{N}$ by some $c$. Then $f \cup \{\langle b, c\rangle\}$ is the required extension of $f$:

$$\mathcal{M} \models \varphi(b, \bar{a}) \iff \mathcal{N} \models \varphi(c, f(\bar{a})).$$

(ii) $\Rightarrow$ (iii) Let $p(\bar{z})$ be as in (iii). Then by Theorem 5.16, $p(\bar{z})$ is realized by $\bar{b}$ in some $\mathcal{M} \succcurlyeq \mathcal{N}$ where $|\bar{b}| = |\bar{z}|$. Then $\operatorname{id}_A : \mathcal{M} \to \mathcal{N}$ is an elementary map. Idea: extend $\operatorname{id}_A$ to $f : \mathcal{M} \to \mathcal{N}$ such that $\bar{b} \in \operatorname{dom} f$. We build $f$ in stages using (ii). Let $f_0 \coloneqq \operatorname{id}_A$. At stage $i+1$, use (ii) to put $b_i$ in $\operatorname{dom} f_{i+1}$. At limit stages $\mu < \lambda$, let $f_\mu = \bigcup_{i<\mu} f_i$. We get that $\bigcup_{i<\lambda} f_i$ is elementary and $\bar{b} \in \operatorname{dom} f$. Then $f(\bar{b}) \in N$ and $\mathcal{N} \models p(f(\bar{b}))$.

(iii) $\Rightarrow$ (i) is trivial. $\square$

**Corollary 6.6.** If $\mathcal{M}$ and $\mathcal{N}$ are saturated and $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}|$ then any elementary $f : \mathcal{M} \to \mathcal{N}$ such that $|f| < |\mathcal{M}|$ extends to an isomorphism (in particular $\mathcal{M} \simeq \mathcal{N}$).

*Proof.* Use Theorem 6.5(ii) to extend $f : \mathcal{M} \to \mathcal{N}$ to an isomorphism by back-and-forth (take unions at limit stages). Since $\mathcal{M} \equiv \mathcal{N}$, $\varnothing$ is elementary. $\square$

**Corollary 6.7.** Models of of $T_{\text{dlo}}$ and $T_{\text{rg}}$ are $\omega$-saturated.

*Proof.* By Theorem 6.5 and Lemma 4.4 and Remark 4.12 for $T_{\text{dlo}}$ and Theorem 4.19 for $T_{\text{rg}}$. $\square$

So $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\omega$-saturated. Is $(\mathbb{R}, <)$ $\omega_1$-saturated? No, it does not realize

$$p(x) \coloneqq \{ x > q \mid q \in \mathbb{Q} \} .$$

**Definition 6.8** (Automorphism)**.** An isomorphism $\alpha : \mathcal{N} \to \mathcal{N}$ is called an **automorphism**. The automorphisms of $\mathcal{N}$ form a group denoted by $\operatorname{Aut}(\mathcal{N})$. If $A \subseteq N$, then

$$\operatorname{Aut}(\mathcal{N}/A) \coloneqq \{\alpha \in \operatorname{Aut}(\mathcal{M}) \mid \alpha|_A = \mathrm{id}\} .$$

**Definition 6.9** (Universality, homogeneity)**.**

(i) An $L$-structure $\mathcal{N}$ is **$\lambda$-universal** if for every $\mathcal{M} \equiv \mathcal{N}$ such that $|\mathcal{M}| \leq \lambda$ there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$.

(ii) $\mathcal{N}$ is **$\lambda$-homogeneous** if every elementary map $f : \mathcal{N} \to \mathcal{N}$ such that $|f| < \lambda$ extends to an isomorphism of $\mathcal{N}$.

An $L$-structure $\mathcal{N}$ is said to be **homogeneous** if it is $|\mathcal{N}|$-homogeneous. $\mathcal{N}$ is **universal** if it is $|\mathcal{N}|$-universal.

**Remark.** The difference between $\leq$ and $<$ in the assertions is crucial. Some authors refer to universality as $\lambda^+$-universality and to homogeneity as strong homogeneity.

Countable models of $T_{\text{rg}}$ and $T_{\text{dlo}}$ are universal and homogeneous.

**Theorem 6.10.** Let $\mathcal{N}$ be such that $|\mathcal{N}| \geq |L|$. The following are equivalent

(i) $\mathcal{N}$ is saturated,

(ii) $\mathcal{N}$ is universal and homogeneous.

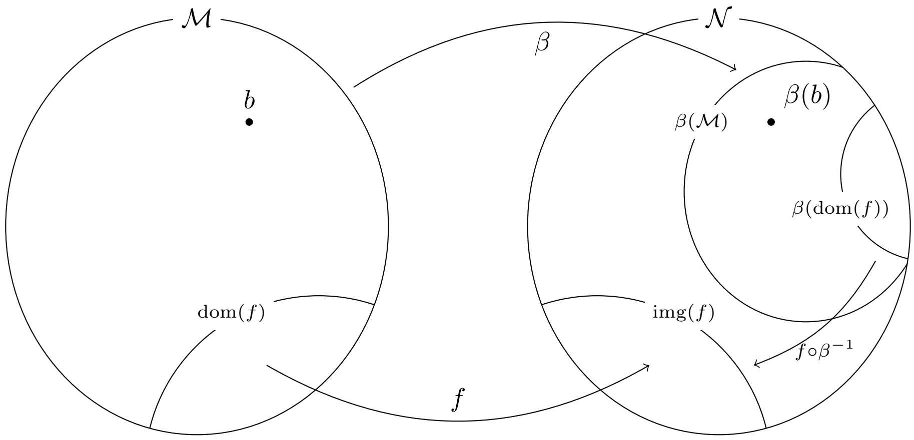
*Proof.* (i) $\Rightarrow$ (ii). Assume $\mathcal{N}$ is saturated, and $\mathcal{M} \equiv \mathcal{N}$ is such that $|\mathcal{M}| \leq |\mathcal{N}|$. Then let $\bar{a}$ enumerate $\mathcal{M}$, let $p(\bar{x}) = \operatorname{tp}_{\mathcal{M}}(\bar{a}/\varnothing)$. Then $p(\bar{x})$ is finitely satisfiable in $\mathcal{M}$.

Claim: $p(\bar{x})$ is finitely satisfiable in $\mathcal{N}$. Indeed, let $\{\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\mathcal{M} \vDash \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x})$, and so $\mathcal{N} \vDash \exists x \bigwedge \varphi_i(\bar{x})$ since $\mathcal{M} \equiv \mathcal{N}$.

Since $|\bar{x}| \leq |\mathcal{N}|$, $\mathcal{N}$ realizes $p(\bar{x})$ by saturation (Theorem 6.5), $\mathcal{N} \vDash p(\bar{b})$ say. Then $a_i \mapsto b_i$ is an elementary embedding. Homogeneity follows from Corollary 6.6 with $\mathcal{N} = \mathcal{M}$.

(ii) $\Rightarrow$ (i). We show that if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$, $f : \mathcal{M} \to \mathcal{N}$ elementary such that $|f| < |\mathcal{N}|$ then there is $\hat{f} \supseteq f$ elementary defined on $b$, cf. Theorem 6.5.

By working in $\mathcal{M}' \preccurlyeq \mathcal{M}$ such that $\operatorname{dom}(f) \cup \{b\} \subseteq \mathcal{M}'$ if necessary (using Theorem 3.11), we may assume $|\mathcal{M}| \leq |\mathcal{N}|$. Since $\mathcal{M} \equiv \mathcal{N}$, by universality there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$. Then $\beta(\mathcal{M}) \preccurlyeq \mathcal{N}$.




Then the map $f \circ \beta^{-1} : \beta(\mathrm{dom}(f)) \to \mathrm{img}(f)$ is as composition of elementary maps elementary. By homogeneity, there is $\alpha \in \mathrm{Aut}(\mathcal{N})$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b))\}$ is elementary (it is a restriction of $\alpha \circ \beta$). $\square$

**Definition 6.11** (Orbit, defined set)**.** Let $\bar{a}$ be a tuple in $\mathcal{N}$ and $A \subseteq N$. The **orbit** of $\bar{a}$ over $A$ is the set

$$O_{\mathcal{N}}(\bar{a}/A) \coloneqq \{\, \alpha(\bar{a}) \mid \alpha \in \mathrm{Aut}(\mathcal{N}/A) \,\}.$$

If $\varphi(\bar{x})$ is an $L(A)$-formula, then

$$\varphi(\mathcal{N}) \coloneqq \left\{ \bar{a} \in N^{|\bar{x}|} \;\middle|\; \mathcal{N} \vDash \varphi(\bar{a}) \right\}$$

is the **set defined by** $\varphi(\bar{x})$. A set is **definable** over $A$ if it is defined by some $L(A)$-formula. There are analogous notions of a type defining a set, and a set being type-definable.

**Remark 6.12.** If $\bar{a}$, $\bar{b}$ are tuples in $\mathcal{N}$ of the same length, and $A \subseteq N$, then the following are equivalent.

(i) $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$,

(ii) $\{\, a_i \mapsto b_i : i < |\bar{a}| \,\} \cup \mathrm{id}_A$ is an elementary map from $\mathcal{N}$ to $\mathcal{N}$.

**Proposition 6.13.** Let $\mathcal{N}$ be $\lambda$-homogeneous, $A \subseteq N$, with $|A| < \lambda$ and let $\bar{a}$ a tuple in $\mathcal{N}$ such that $|\bar{a}| < \lambda$. Then

$$O_{\mathcal{N}}(\bar{a}/A) = p(\mathcal{N})$$

where $p(\bar{x}) = \mathrm{tp}_{\mathcal{N}}(\bar{a}/A)$.

*Proof.* If $\alpha(\bar{a}) = \bar{b}$, where $\alpha \in \mathrm{Aut}(\mathcal{N}/A)$, then $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$.

If $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$, then $\{\, \langle a_i, b_i \rangle \mid i < |\bar{a}| \,\} \cup \mathrm{id}_A$ is elementary and has cardinality less than $\lambda$. By homogeneity it extends to $\alpha \in \mathrm{Aut}(\mathcal{N})$, and in particular $\alpha \in \mathrm{Aut}(\mathcal{N}/A)$. Thus, $\bar{b} \in O(\bar{a}/A)$. $\square$

# 7 The Monster Model

Given a complete theory $T$ with an infinite model, we work in a saturated structure $\mathcal{U}$ (sometimes denoted $\mathbb{M}$) that is a model of $T$, which is sufficiently large such that any other model of $T$ we might be interested in is an elementary substructure of $\mathcal{U}$. $\mathcal{U}$ is an expository device. See Tent/Ziegler for more details, also Marker.

Establishing the existence of the monster $\mathcal{U}$ requires set-theoretic assumptions or the use of specific properties of $T$.

**Definition 7.1** (Terminology and conventions)**.** When working in $\mathcal{U}$, we say

- '$\varphi(\bar{x})$ **holds**' to mean that $\mathcal{U} \vDash \forall \bar{x}\, \varphi(\bar{x})$,
- '$\varphi(\bar{x})$ is **consistent**' to mean $\mathcal{U} \vDash \exists \bar{x}\, \varphi(\bar{x})$,
- 'the type $p(\bar{x})$ is **consistent/satisfiable**' to mean $\mathcal{U} \vDash \exists \bar{x}\, p(\bar{x})$, write $\mathcal{U} \vDash p(\bar{a})$ if $\bar{a}$ witnesses $p(\bar{x})$ in $\mathcal{U}$,
- a cardinality $\lambda$ is **small** if $\lambda < |U|$ (usually denote $|U|$ by $\kappa$,
- a **model** is some $\mathcal{M} \preccurlyeq \mathcal{U}$ such that $|M|$ is small.

We establish the following conventions:

- all tuples assumed to have small length, unless specified otherwise,
- formulas have parameters in $U$, unless specified otherwise,
- types have parameters in small sets, otherwise they are *global types* and not relevant in this course,
- definable sets have the form $\varphi(U)$ for some $L(U)$-formula $\varphi(\bar{x})$,
- type definable sets have the form $p(U)$ for some type $p(\bar{x}, A)$ where $|A| < \kappa$,
- Orbits and types of tuples are within $\mathcal{U}$, so $\mathrm{tp}(\bar{a}/A)$ means $\mathrm{tp}_{\mathcal{U}}(\bar{a}/A)$,
$$O(\bar{a}/A) = O_{\mathcal{U}}(\bar{a}/A),$$
- If $p(\bar{x})$, $q(\bar{x})$ are types, we write $p(\bar{x}) \to q(\bar{x})$ to mean $p(\mathcal{N}) \subseteq q(\mathcal{N})$ (think of $p(\bar{x})$ as an infinite conjunction of formulas),
- small subsets of $U$ will be denoted by letters $A, B, C$, etc.,
- sets of arbitrary cardinality will be denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc.

**Fact 7.2.** Let $p(\bar{x})$ be a satisfiable $L(A)$-type, and $q(\bar{x})$ a satisfiable $L(B)$-type, such that
$$p(\bar{x}) \to \neg q(\bar{x}).$$
Or explicitly, $p(\bar{x})$ and $q(\bar{x})$ have no common realisations. Then there are $\varphi_i(\bar{x}) \in p(\bar{x})$ and $\psi_i(\bar{x}) \in q(\bar{x})$ such that
$$\bigwedge_{i=1}^{n} \varphi_i(\bar{x}) \to \neg \left( \bigwedge_{i=1}^{m} \psi_i(\bar{x}) \right).$$

*Proof.* $p(\bar{x}) \cup q(\bar{x})$ is not realized in $\mathcal{U}$. By saturation of $\mathcal{U}$, $p(\bar{x}) \cup q(\bar{x})$ is not finitely satisfiable, hence there exist finite subsets $\{\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\{\psi_1(\bar{x}), \ldots, \psi_n(\bar{x})\} \subseteq q(\bar{x})$ such that their union is not satisfiable. Then
$$\bigwedge \varphi_i(\bar{x}) \to \neg \left( \bigwedge \psi_i(\bar{x}) \right).$$
□

**Remark 7.3.** Let $\varphi(\mathcal{U}, \bar{b})$ be such that $\varphi(\bar{x}, \bar{z})$ is an $L$-formula, $\bar{b} \in \mathcal{U}^{|\bar{z}|}$. If $\alpha \in \operatorname{Aut}(\mathcal{U})$, then

$$\begin{aligned} \alpha[\varphi(\mathcal{U}, \bar{b})] &= \{\, \alpha(\bar{a}) \mid \varphi(\bar{a}, \bar{b}), \bar{a} \in \mathcal{U}^{|\bar{x}|} \,\} \\ &= \{\, \alpha(\bar{a}) \mid \varphi(\alpha(\bar{a}), \alpha(\bar{b})), \bar{a} \in \mathcal{U}^{|\bar{x}|} \,\} \\ &= \varphi(\mathcal{U}, \alpha(\bar{b})) \end{aligned}$$

So $\operatorname{Aut}(\mathcal{U})$ acts on the definable sets in a natural way. (Similarly for the type-definable sets).

Also if $p(\bar{x}, \bar{z})$ is a type in $L$ and $\bar{b} \in \mathcal{U}^{|\bar{z}|}$, then $\alpha(p(\mathcal{U}, \bar{b})) = p(\mathcal{U}, \alpha(\bar{b}))$.

**Definition 7.4** (Invariance)**.** An arbitrary large subset $\mathcal{D} \subseteq \mathcal{U}^\lambda$ for a small cardinal $\lambda$ is **invariant** under $\operatorname{Aut}(\mathcal{U}/A)$ (**invariant over** $A$) if $\alpha(\mathcal{D}) = \mathcal{D}$ for every $\alpha \in \operatorname{Aut}(\mathcal{U}/A)$.

Equivalently, for all $\bar{a} \in \mathcal{D}$, $O(\bar{a}/A) \subseteq \mathcal{D}$.

If $\bar{a} \in \mathcal{D}$, $q(\bar{x}) = \operatorname{tp}(\bar{a}/A)$ and $\bar{b} \models q(\bar{x})$, then $\bar{b} \in \mathcal{D}$. ($\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{a}/A)$, so there is $\alpha \in \operatorname{Aut}(\mathcal{U}/A)$ s.t. $\alpha(\bar{a}) = \bar{b}$ by homogeneity of $\mathcal{U}$). Hence we could also define invariance over $A$ as

$$\forall \bar{a} \in \mathcal{D}, \qquad \bar{b} \equiv_A \bar{a} \implies \bar{b} \in \mathcal{D}.$$

**Proposition 7.5.** Let $\varphi(\bar{x})$ be an $L(U)$-formula and let $A \subseteq U$, then the following are equivalent:

(i) $\varphi(\bar{x})$ is equivalent to some $L(A)$-formula $\psi(\bar{x})$, i.e. $\forall \bar{x}\ (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$,

(ii) $\varphi(\mathcal{U})$ is invariant over $A$.

*Proof.* (i) $\Rightarrow$ (ii) is clear by Remark 7.3 since $\alpha \in \operatorname{Aut}(\mathcal{U}/A)$ fixes $\psi(\mathcal{U})$ setwise.

(ii) $\Rightarrow$ (i): Let $\varphi(\bar{x}, \bar{z})$ be an $L$-formula such that $\varphi(\mathcal{U}, \bar{b})$ is invariant over $A$, for suitable $\bar{b} \in U^{|\bar{z}|}$.

Let $q(\bar{z})$ be the type $\operatorname{tp}(\bar{b}/A)$. If $\bar{c} \models q(\bar{z})$, then there is $\alpha \in \operatorname{Aut}(\mathcal{U}/A)$ such that $\alpha(\bar{b}) = \bar{c}$. Then

$$\begin{aligned} \varphi(\mathcal{U}, \bar{c}) &= \alpha(\varphi(\mathcal{U}, \bar{b})) && \text{by Remark 7.3} \\ &= \varphi(\mathcal{U}, \bar{b}) && \text{by invariance} \end{aligned}$$

Hence

$$q(\bar{z}) \to \forall \bar{x}\ (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b})).$$

By an argument similar to Fact 7.2, there is $\theta(\bar{z}) \in q(\bar{z})$ such that $\theta(\bar{z}) \to \forall \bar{x}\ (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b}))$. Then $\theta(\bar{z})$ is an $L(A)$-formula and $\exists z\ [\theta(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})]$ defines $\varphi(\mathcal{U}, \bar{b})$. □

**Definition 7.6.** An injective map $p : A \subseteq \mathcal{M} \to \mathcal{N}$ is a **partial embedding** if for all tuples in $A = \operatorname{dom}(p)$, $p$ satisfies conditions (i), (ii), (iii) in Definition 1.5.

Idea: a partial embedding preserves quantifier-free formulas.

**Proposition 7.7.** Let $\varphi(\bar{x})$ be an $L$-formula. The following are equivalent:

(i) there is $\psi(\bar{x})$, a quantifier-free $L$-formula such that

$$\forall \bar{x}\ [\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})].$$

(ii) for all partial embeddings $p : \mathcal{U} \to \mathcal{U}$, for all $\bar{a} \in \operatorname{dom}(\bar{p})$,

$$\varphi(\bar{a}) \leftrightarrow \varphi(p(\bar{a})).$$

*Proof.* (i) $\Rightarrow$ (ii): clear since partial embeddings preserve the truth of quantifier-free formulas.

(ii) $\Rightarrow$ (i). For $\bar{a} \in U$, set

$$\operatorname{qftp}(\bar{a}) := \{\, \psi(\bar{x}) \in \operatorname{tp}(\bar{a}/\varnothing) \mid \psi(\bar{x}) \text{ is quantifier free} \,\},$$

the type consisting of all quantifier-free formulas that are satisfied by $\bar{a}$. Let

$$D := \{\, q(\bar{x}) \mid q(\bar{x}) = \operatorname{qftp}(\bar{a}) \text{ for some } \bar{a} \text{ such that } \varphi(\bar{a}) \,\}$$

be the collection of these types for $\bar{a}$ which satisfy $\varphi(\bar{x})$. We claim that

$$\varphi(\mathcal{U}) = \bigcup_{q(\bar{x}) \in D} q(\mathcal{U}).$$

It is clear that $\varphi(\mathcal{U}) \subseteq \bigcup_{q(\bar{x}) \in D} q(\mathcal{U})$. For $\supseteq$, let $q(\bar{x}) \in D$, say $q(\bar{x}) = \operatorname{qftp}(\bar{a})$. Let $\bar{b} \models q(\bar{x})$. Then $a_i \mapsto b_i$ is a partial embedding, so by (ii), $\varphi(\bar{b})$ holds in $\mathcal{U}$. Thus, $\bar{b} \in \varphi(\mathcal{U})$. So, $q(\mathcal{U}) \subseteq \varphi(\mathcal{U})$, and hence $\varphi(\mathcal{U}) = \bigcup_{q(\bar{x}) \in D} q(\mathcal{U})$.

This shows that $q(\bar{x}) \to \varphi(\bar{x})$. By (an argument similar to) Fact 7.2, there is $\theta_q(\bar{x})$ in $q(\bar{x})$ a finite conjunction of formulas such that $\theta_q(\bar{x}) \to \varphi(x)$. So we have

$$\varphi(\bar{x}) \leftrightarrow \bigvee_{q(\bar{x}) \in D} \{\theta_q(\bar{x})\}.$$

By Fact 7.2, there are $\psi_{q_1}(\bar{x}), \ldots, \psi_{q_m}(\bar{x})$ such that

$$\varphi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(\bar{x}).$$

So $\bigvee \psi_{q_i}(\bar{x})$ is the required quantifier-free formula. □

**Definition 7.8.** An $L$-theory $T$ has **quantifier elimination** if for every $L$-formula $\varphi(\bar{x})$ there is $\psi(\bar{x})$ quantifier free such that

$$T \vdash \forall \bar{x}\, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

**Remark.** Usually, quantifier elimination with additional assumptions is used to show that a theory is complete.

**Theorem 7.9.** Let $T$ be a complete theory with an infinite model. Then the following are equivalent:

(i) $T$ has quantifier elimination,

(ii) every $p : \mathcal{U} \to \mathcal{U}$ partial embedding is elementary,

(iii) If $p : \mathcal{U} \to \mathcal{U}$ is partial embedding and $|\operatorname{dom} p| < |\mathcal{U}|$ and $b \in \mathcal{U}$, then there is a partial embedding $\hat{p} \supseteq p$ such that $b \in \operatorname{dom} \hat{p}$.

*Proof.* (i) $\Leftrightarrow$ (ii). Follows from Proposition 7.7.

(ii) $\Rightarrow$ (iii). If $p : \mathcal{U} \to \mathcal{U}$ is a partial embedding, then it is elementary. Let $b \in \mathcal{U}$. By homogeneity of $\mathcal{U}$, there is $\alpha \in \operatorname{Aut}(\mathcal{U})$ such that $p \subseteq \alpha$, and so $p \cup \{\langle b, \alpha(b) \rangle\}$ is the required extension of $p$.

(iii) $\Rightarrow$ (ii). Let $p : \mathcal{U} \to \mathcal{U}$ be a partial embedding. Consider $p_0 \subseteq p$, $p_0$ finite or small. Use property (iii) and saturation to extend $p_0$ to $\alpha \in \operatorname{Aut}(U)$ by back and forth. So $p_0$ is elementary. The assumption on smallness is needed for the back-and-forth argument. □

**Remark.** There is a fourth condition equivalent to (i), (ii), (iii):

(iv) for every finite partial embedding $p : \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$ there is $\hat{p} \supseteq p$, a partial embedding such that $b \in \operatorname{dom}(\hat{p})$.

Proof: Later, exercise.

This gives quantifier elimination for $T_{\mathrm{rg}}$ and $T_{\mathrm{dlo}}$.

**Remark.** If $T$ has quantifier elimination and $\mathcal{M} \models T$, any substructure of $\mathcal{M}$ is an elementary substructure ($T$ is 'model-complete'). Under certain conditions, model completeness implies completeness.

**Definition 7.10.** An element $a \in \mathcal{U}$ is **definable** over a small set $A \subseteq U$ if there is an $L(A)$-formula $\varphi(x)$ such that $\varphi(U) = \{a\}$. (In particular, any element of $A$ is definable over $A$; $x = a$ for $a \in A$).

An element $a \in \mathcal{U}$ is **algebraic** over $A \subseteq U$ if there is an $L(A)$-formula $\varphi(x)$ such that $|\varphi(U)| < \omega$ and $a \in \varphi(\mathcal{U})$.

The **definable closure** of $A$ is

$$\operatorname{dcl}(A) = \{\, a \in \mathcal{U} \mid a \text{ definable over } A \,\}$$

and the **algebraic closure** of $A$ is

$$\operatorname{acl}(A) = \{\, a \in \mathcal{U} \mid a \text{ algebraic over } A \,\}.$$

$A$ is **algebraically closed** if $\operatorname{acl}(A) = A$.

**Proposition 7.11.** For $a \in \mathcal{U}$ and $A \subseteq \mathcal{U}$, the following are equivalent

(i) $a \in \operatorname{dcl}(A)$,

(ii) $O(a/A) = \{a\}$.

*Proof.* For (i) $\Rightarrow$ (ii), let $\varphi(x)$ define $a$ over $A$. Then $\varphi(\mathcal{U})$ is invariant over $A$, so $O(a/A) \subseteq \varphi(\mathcal{U})$ and so $O(a/A) = \{a\}$. For (ii) $\Rightarrow$ (i), we use that $O(a/A)$ is invariant over $A$, so by Proposition 7.5 there is an $L(A)$-formula that defines $O(a/A) = \{a\}$. □

**Theorem 7.12.** Let $A \subseteq \mathcal{U}$, $a \in \mathcal{U}$, the following are equivalent:

(i) $a \in \operatorname{acl}(A)$,

(ii) $|O(a/A)| < \omega$,

(iii) $a \in \mathcal{M}$ for any model[2] $\mathcal{M}$ which contains $A$.

*Proof.* (i) $\Rightarrow$ (ii). If $a \in \operatorname{acl}(A)$, then there is an $L(A)$-formula $\varphi(x)$ such that $\varphi(a)$ holds and $|\varphi(\mathcal{U})| < \omega$. But $\varphi(\mathcal{U})$ is invariant over $A$, and so $O(a/A) \subseteq \varphi(\mathcal{U})$, and so $|O(a/A)| < \omega$. ´ (ii) $\Rightarrow$ (i). If $|O(a/A)| < \omega$, then $O(a/A)$ is definable by $\bigvee_{i=1}^{n}(x = a_i)$ where $O(a/A) = \{a_1, \ldots, a_n\}$. Also $O(a/A)$ is invariant over $A$, so by Proposition 7.5, there is an $L(A)$-formula $\varphi(x)$ that defines $O(a/A)$.

(i) $\Rightarrow$ (iii). $a \in \operatorname{acl}(A)$, so there is $\varphi(x)$, an $L(A)$-formula such that there is $n \in \omega \setminus \{0\}$ with

$$\varphi(a) \wedge \exists^{=n} x\, \varphi(x),$$

i.e. there exist precisely $n$ witnesses for $\varphi(x)$. Then by elementarity, $\varphi(a) \wedge \exists^{=n} x\, \varphi(x)$ holds in every $\mathcal{M} \supseteq A$, and the $n$ realizations of $\varphi(x)$ in $\mathcal{U}$ must coincide with the realizations in $\mathcal{M}$. Therefore $a \in \mathcal{M}$.

[2] In the setup of this section, $\mathcal{M}$ is a small elementary substructure of $\mathcal{U}$.

(iii) $\Rightarrow$ (i). Suppose $a \notin \mathrm{acl}(A)$, let $p(x) = \mathrm{tp}(a/A)$. Then for all $\varphi(x) \in p(x)$, $|\varphi(\mathcal{U})| \geq \omega$. Then from the second examples sheet[3], $|p(\mathcal{U})| \geq \omega$. And we can show[4] that $|p(\mathcal{U})| = |\mathcal{U}|$.

Let $\mathcal{M} \supseteq A$, then $p(\mathcal{U}) \setminus \mathcal{M} \neq \varnothing$ by cardinality considerations. So there is $b \in p(\mathcal{U}) \setminus \mathcal{M}$. Since $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$, there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(b) = a$ by homogeneity. But then $\alpha(\mathcal{M})$ is a model that contains $A$, but $a \notin \alpha(\mathcal{M})$ while $a = \alpha(b)$. □

**Proposition 7.13.** Let $a \in \mathcal{U}$, $A \subseteq \mathcal{U}$. Then:

(i) if $a \in \mathrm{acl}(A)$, then there is finite $A_0 \subseteq A$ such that $a \in \mathrm{acl}(A_0)$.

(ii) if $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.

(iii) $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$

(iv) $A \subseteq \mathrm{acl}(A)$.

(v) $\mathrm{acl}(A) = \bigcap_{A \subseteq \mathcal{M}} \mathcal{M}$ where $\mathcal{M}$ is a small elementary substructure of $\mathcal{U}$.

*Proof.*

(i) $A_0$ contains the parameters of the formula that make $a$ algebraic.

(iv) $a \in A$ is definable over $A$, hence algebraic.

(iii) $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$ by monotonicity. For $\supseteq$, let $a \in \mathrm{acl}(\mathrm{acl}(A))$. By Theorem 7.12, $a \in \mathcal{M}$ for every $\mathcal{M} \supseteq \mathrm{acl}(A)$. But $\mathrm{acl}(A) \subseteq \mathcal{M} \iff A \subseteq \mathcal{M}$, so $a \in \mathcal{M}$ for every $\mathcal{M} \supseteq A$, i.e. $a \in \mathrm{acl}(A)$.

(v) follows from Theorem 7.12. □

**Remark.** Proposition 7.13(i)-(iv) show that acl in the monster is a closure operator of finite character.

**Proposition 7.14.** If $\beta \in \mathrm{Aut}(\mathcal{U})$, $A \subseteq \mathcal{U}$ is small, then $\beta(\mathrm{acl}(A)) = \mathrm{acl}(\beta(A))$.

*Proof.* $\subseteq$: Let $a \in \mathrm{acl}(A)$, let $\varphi(x, \bar{z})$ be an $L$-formula such that $\varphi(a, \bar{b})$ holds for $\bar{b}$ in $A$ and $\left|\varphi(\mathcal{U}, \bar{b})\right| < \omega$. Then $\varphi(\beta(a), \beta(\bar{b}))$ holds, $\left|\varphi(\mathcal{U}, \beta(\bar{b}))\right| < \omega$, and so $\beta(a)$ is algebraic over $\beta(\bar{b})$.

The same proof with $\beta^{-1}$ in place of $\beta$ and $\beta(A)$ in place of $A$ shows $\supseteq$. □

[3] Question 6 on examples sheet 2: Let $\mathcal{N}$ be saturated and let $p(x)$ be a type with parameters in $A \subseteq N$ such that $|A| < |N|$. Suppose $p(x)$ is closed under conjunction, i.e. if $\varphi(x)$ and $\psi(x)$ are in $p(x)$, then also $\varphi(x) \wedge \psi(x)$. Then the following are equivalent:

(i) $p(x)$ has finitely many realizations,

(ii) $p(x)$ contains a formula with finitely many realizations.

[4] Question 5 on examples sheet 2: Let $\mathcal{N}$ be a saturated $L$-structure, and let $p(x)$ be a type (with one free variable) in $L(A)$, where $A \subseteq N$ and $|A| < |N|$. Let $p(\mathcal{N}) = \{a \in N \mid \mathcal{N} \vDash p(a)\}$. Then the following are equivalent:

(i) $p(\mathcal{N})$ is infinite,

(ii) $|p(\mathcal{N})| = |\mathcal{N}|$.

# 8 Strongly Minimal Theories

We continue to work in the monster $\mathcal{U}$.

**Definition 8.1** (Cofinite)**.** For $\mathcal{M}$ a structure, $A \subseteq M$ is **cofinite** if $M \setminus A$ is finite.

**Remark 8.2.** Finite and cofinite sets are definable in every structure.

In this chapter, we'll look at structures where these are the only definable sets.

**Definition 8.3** (Minimality, strong minimality)**.** A structure $\mathcal{M}$ is **minimal** if all its definable subsets are finite or cofinite. $\mathcal{M}$ is **strongly minimal** if it is minimal and all its elementary extensions are minimal.

If $T$ is a consistent theory without finite models, $T$ is **strongly minimal** if for every formula $\varphi(x, \bar{z})$ there is $n \in \omega \setminus \{0\}$ such that

$$T \vdash \forall \bar{z} \, [\exists^{\leq n} x \; \varphi(x, \bar{z}) \vee \exists^{\leq n} x \; \neg\varphi(x, \bar{z})].$$

**Example.** Take $L = \{E\}$, a binary relation, let $\mathcal{M}$ be the $L$-structure where $E$ is an equivalence relation with exactly one class of size $n$ for all $n \in \omega$ and no infinite classes. Then one can show that $\mathcal{M}$ is minimal (can only say things like '$x$ is in the same class as a').

But, there is $\mathcal{N} \succcurlyeq \mathcal{M}$ where $\mathcal{N}$ has an infinite class. Then if the equivalence class of $a \in \mathcal{N}$ is infinite, the set defined by $E(x, a)$ is infinite/coinfinite, so $\mathcal{M}$ is not strongly minimal.

Note that strongly minimal theories have monster models. Let from now on $T$ be strongly minimal, complete such that it possesses an infinite model.

**Definition 8.4** (Independence)**.** Let $a \in \mathcal{U}$, $B \subseteq \mathcal{U}$. Then $a$ is **independent** from $B$ if $a \notin \operatorname{acl}(B)$. The set $B$ is **independent** if for all $a \in B$, $a \notin \operatorname{acl}(B \setminus \{a\})$.

**Example.**

- Vector spaces. Fix an infinite field $K$, and use $L = \{+, -, \mathbf{0}, \{\lambda\}_{\lambda \in K}\}$, where $\lambda$ are unary functions (for scalar multiplication). The theory of vector spaces over $K$, $T_{VSK}$ includes:
  - axioms in $\{+, -, \mathbf{0}\}$ for abelian group
  - axiom schemata for scalar multiplication:
    * $\forall xy \, [\lambda(x + y) = \lambda x + \lambda y]$ for each $\lambda \in K$, $\lambda x$ means $\lambda(x)$.
    * $\vdots$
    * $\forall x \, [1x = x]$ (since $1 \in K$).
    * $\exists x \, (x \neq \mathbf{0})$.

  Then it can be shown $T_{VSK}$ is complete and has quantifier elimination.

  Atomic formulas express equality of linear combinations, any atomic formula in one variable and with parameters is equivalent to '$\lambda x = a$', so atomic formulas in one variable define singletons. Quantifier-free formulas in one variable and with parameters define sets that are either finite or cofinite.

  By quantifier elimination, $T_{VSK}$ is strongly minimal. Also, $\operatorname{acl}(A) = \langle A \rangle$, the linear span, and $a$ is independent from $A$ if $a$ is linearly independent from $A$, and $A$ is independent if it is linearly independent.

- Fields. Take $L_{\text{ring}} = \{+, \cdot, -, 0, 1\}$. Then $ACF$ is the theory that includes

- axioms for abelian group in $\{+, -, 0\}$
- axioms for multiplicative monoids in $\{\cdot, 1\}$
- $\forall xyz\ [x \cdot (y+z) = x \cdot y + x \cdot z]$
- $\forall x\ [x = 0 \vee \exists y\ (x \cdot y) = 1]$
- $0 \neq 1$
- axioms for algebraic closure: for all $n$,

$$\forall x_0 \cdots x_n\ \exists y\ [x_n y^n + \cdots + x_1 y + x_0 = 0].$$

If

$$\chi_p \equiv \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0,$$

for $p$ prime, then $ACF \cup \{\chi_p\} =: ACF_p$, which can be shown to be complete and have quantifier elimination. By adding $\{\neg\chi_n \mid n \in \omega\}$ to $ACF$, get $ACF_0$ (also complete with quantifier elimination).

Now, atomic formulas with parameters are polynomial equations. An atomic formula with one variable (and parameters in $A$) is equivalent to $p(x) = 0$, where $p(x)$ is a polynomial in the subfield generated by $A$. So such atomic formulas define finite sets, and quantifier free formulas define finite or cofinite sets, and so by quantifier elimination, $ACF_p$ ($ACF_0$) is strongly minimal. If $a \in \mathcal{M} \vDash ACF_p$, $A \subseteq \mathcal{M}$, then $a \in \operatorname{acl}(A)$ if $a$ is algebraic over the field generated by $A$.

**Notation.** We write $\operatorname{acl}(a, B)$ for $\operatorname{acl}(\{a\} \cup B)$ and $\operatorname{acl}(B \setminus a)$ for $\operatorname{acl}(B \setminus \{a\})$.

**Theorem 8.5** (Exchange lemma)**.** Let $B \subseteq \mathcal{U}$, and $a, b \notin \operatorname{acl}(B)$, i.e. $a, b \in \mathcal{U} \setminus \operatorname{acl}(B)$. Then

$$b \in \operatorname{acl}(a, B) \iff a \in \operatorname{acl}(b, B).$$

*Proof.* Let $a, b \notin \operatorname{acl}(B)$. Assume $b \notin \operatorname{acl}(a, B)$ and $a \in \operatorname{acl}(b, B)$. Let $\varphi(x, y)$ be an $L(B)$-formula such that for some $n$,

$$\varphi(a, b) \wedge \exists^{\leq n} x\ \varphi(x, b).$$

Since $b \notin \operatorname{acl}(a, B)$

$$\psi(a, y) := \varphi(a, y) \wedge \exists^{\leq n} x\ \varphi(x, y)$$

is such that $|\psi(a, \mathcal{U})| \geq \omega$. By question 5, examples sheet 2, cf. Footnote 4, $|\psi(a, U)| = |\mathcal{U}|$. By strong minimality, $|\neg\psi(a, U)| < \omega$. By cardinality considerations, if $\mathcal{M} \supseteq B$, then $\mathcal{M}$ contains $c$ such that $\psi(a, c)$. But then $a \in \operatorname{acl}(c, B)$, so $a \in \mathcal{M}$. Therefore $a$ is in all models that contain $B$, so $a \in \operatorname{acl}(B)$ by Theorem 7.12, a contradiction. □

**Definition 8.6** (Basis)**.** Let $B \subseteq C \subseteq \mathcal{U}$. Then $B$ is a **basis** of $C$ if

(i) $B$ is independent,

(ii) $C \subseteq \operatorname{acl}(B)$ (or equivalently, $\operatorname{acl}(B) = \operatorname{acl}(C)$).

**Lemma 8.7.** If $B$ is independent and $a \notin \operatorname{acl}(B)$, then $\{a\} \cup B$ is independent.

*Proof.* Let $a \notin \operatorname{acl}(B)$, and suppose (for contradiction) that $\{a\} \cup B$ is not independent. Then there is $b \in B$ such that $b \in \operatorname{acl}(a, B \setminus b)$. But $b \notin \operatorname{acl}(B \setminus b)$. Since $a \notin \operatorname{acl}(B \setminus b)$, by Theorem 8.5 we have

$$a \in \operatorname{acl}(b, B \setminus b) = \operatorname{acl}(B),$$

a contradiction. □

**Corollary 8.8.** If $B \subseteq C$, the following are equivalent:

(i) $B$ is a basis of $C$,

(ii) if $B \subseteq B' \subset C$ and $B'$ is independent, then $B = B'$.

*Proof.* By Lemma 8.7. □

**Theorem 8.9.** Let $C \subseteq \mathcal{U}$, then

(i) every independent subset $B \subseteq C$ can be extended to a basis,

(ii) if $A, B$ are bases of $C$, then $|A| = |B|$.

*Proof.*

(i) If $\langle B_i : i < \lambda \rangle$ is a chain of independent sets containing $B$, then $\bigcup_{i<\lambda} B_i$ is independent (by Proposition 7.13(i)). By Zorn's lemma, there is a maximal independent subset of $C$ that contains $B$. By Corollary 8.8, that maximal subset is a basis of $C$.

(ii) Let $|B| \geq \omega$, assume (for contradiction) that $|A| < |B|$. Then $a \in A$ is also in $\mathrm{acl}(B)$. Let $D_a \subseteq B$ be finite such that $a \in \mathrm{acl}(D_a)$. Let $D = \bigcup_{a \in A} D_a$. Then $A \subseteq \mathrm{acl}(D)$ and $C \subseteq \mathrm{acl}(D)$, but $|D| < |B|$ contradicting the independence of $B$.

If $A$ and $B$ are finite, show that $|A| \leq |B|$ (and the converse symmetrically) by using: if there is $a \in A \setminus B$, then there is $b \in B \setminus A$ such that $\{b\} \cup (A \setminus \{a\})$ is independent. This holds because if $a \in A \setminus B$, then since $a \in \mathrm{acl}(B)$, we have that $B \not\subseteq \mathrm{acl}(A \setminus \{a\})$ (otherwise $A$ is not independent). So let $b \in B \setminus \mathrm{acl}(A \setminus a)$. Then $\{b\} \cup (A \setminus a)$ is independent by Lemma 8.7.

Use finite induction argument to get $|A| \leq |B|$. □

**Definition 8.10** (Dimension)**.** Let $C \subseteq \mathcal{U}$ be algebraically closed. Then the **dimension** of $C$ is $\dim(C) = |A|$ where $A$ is any basis of $C$.

**Proposition 8.11.** Let $f : \mathcal{U} \to \mathcal{U}$ be (partial) elementary. Let $b \notin \mathrm{acl}(\mathrm{dom}(f))$ and $c \notin \mathrm{acl}(\mathrm{img}(f))$. Then $f \cup \{\langle b, c \rangle\}$ is elementary.

*Proof.* Let $\bar{a}$ enumerate $\mathrm{dom}(f)$, let $\varphi(x, \bar{a})$ be a formula with parameters in $\bar{a}$. The claim is that $\varphi(b, \bar{a})) \leftrightarrow \varphi(c, f(\bar{a}))$. We distinguish cases:

1. $|\varphi(\mathcal{U}, \bar{a})| < \omega$. Then $|\varphi(\mathcal{U}, f(\bar{a}))| < \omega$. Then $b \notin \varphi(\mathcal{U}, \bar{a})$ (because $b \notin \mathrm{acl}(\bar{a})$) and $c \notin \varphi(\mathcal{U}, f(\bar{a}))$. Then it holds in $\mathcal{U}$ that

$$\neg\varphi(b, \bar{a}) \wedge \neg\varphi(c, f(\bar{a})).$$

2. $|\varphi(U, \bar{a})| \geq \omega$. Then by strong minimality, $|\neg\varphi(\mathcal{U}, \bar{a})| < \omega$, and so

$$\varphi(b, \bar{a}) \wedge \varphi(c, f(\bar{a}))$$

holds in $\mathcal{U}$. □

**Corollary 8.12.** Every bijection between small independent subsets of $\mathcal{U}$ is elementary.

*Proof.* Pick $A, B \subseteq U$ independent and let $f : A \to B$ be any bijection. Let $\bar{a}$ enumerate $A$, write $f(a_i) = b_i$. Then $a_0 \notin \mathrm{acl}(\varnothing)$ and $b_0 \notin \mathrm{acl}(\varnothing)$ (otherwise $A, B$ not independent). By Proposition 8.11, $\{\langle a_0, b_0 \rangle\}$ is an elementary map.

At stage $i + 1$, $a_{i+1} \notin \mathrm{acl}(a_0, \ldots, a_i)$ and $b_{i+1} \notin \mathrm{acl}(b_0, \ldots, b_i)$ so use the same argument. □

**Remark 8.13.** If $\mathcal{M} \preccurlyeq \mathcal{U}$ is a model such that $|\mathcal{M}| < |\mathcal{U}|$, then by Proposition 7.13, $\mathcal{M}$ is algebraically closed.

**Theorem 8.14.** Suppose that $\mathcal{M}, \mathcal{N} \preccurlyeq \mathcal{U}$ are such that $\dim(M) = \dim(N)$, then $\mathcal{M} \simeq \mathcal{N}$.

*Proof.* Let $A, B$ be bases of $\mathcal{M}, \mathcal{N}$ respectively. Then a bijection $f : A \to B$ is elementary (by Corollary 8.12). Then there is $\alpha \in \mathrm{Aut}(\mathcal{U})$ such that $f \subseteq \alpha$ by homogeneity of the monster. Then by Proposition 7.14,

$$\alpha(\mathcal{M}) = \alpha(\mathrm{acl}(\mathcal{M})) = \mathrm{acl}(\alpha(A)) = \mathrm{acl}(B) = \mathcal{N}. \qquad \square$$

**Corollary 8.15.** Let $T$ be strongly minimal, let $\lambda > |L|$. Then $T$ is $\lambda$-categorical.

*Proof.* If $A \subseteq \mathcal{U}$, then $|\mathrm{acl}(A)| \leq |L(A)| + \omega$ (there are at most $|L(A)| + \omega$ formulas, each element $m$ in $\mathrm{acl}(A)$ is one of finitely many solutions of one of those formulas). If $|\mathcal{M}| = \lambda$, then a basis of $\mathcal{M}$ must have cardinality $\lambda$. By Theorem 8.14, any two models with equal dimension are isomorphic. $\square$

In $T_{VSK}$, if $K$ is infinite countable, the vector space can have finite dimension ($\omega$-categoricity fails). If $K$ is finite, the vector space must have dimension $\geq \omega$.

# 9 Countable Models

> Any fool can realise a type but it takes a model theorist to omit one. — Gerald Sacks

Let $T$ be a complete countable theory with a monster model $\mathcal{U}$.

**Definition 9.1.** An $L$-formula $\varphi(\bar{x})$ is said to **isolate** a type $p(\bar{x})$ in $L$ if

(i) $\varphi(\bar{x}) \to p(\bar{x})$, i.e. $T \vdash \forall \bar{x}\,(\varphi(\bar{x}) \to \psi(\bar{x}))$ for all $\psi(\bar{x}) \in p(\bar{x})$, and

(ii) $\varphi(\bar{x})$ is consistent, i.e. it is realised in $\mathcal{U}$.

A set of formulas $\Delta$ with variables in $\bar{x}$ **isolates** $p(\bar{x})$ if there is $\delta(\bar{x}) \in \Delta$ that isolates $p(\bar{x})$. When $\Delta$ contains all formulas in $L(A)$ with free variables in $\bar{x}$, we say that $A$ **isolates** $p(\bar{x})$. If $A$ is clear from the context, we say that $p(\bar{x})$ is **isolated** or **principal**. A model $\mathcal{M}$ is said to **omit** $p(\bar{x})$ if $p(\bar{x})$ is not realized in $\mathcal{M}$.

**Remark 9.2.** If $\mathcal{M}$ is a model and $p(\bar{x})$ is a type in $L(M)$, then $\mathcal{M}$ realises $p(\bar{x})$ if and only if $\mathcal{M}$ isolates $p(\bar{x})$.

*Proof.* $\Leftarrow$ Let $\varphi(\bar{x})$ isolate $p(\bar{x})$, so $\mathcal{M} \vDash \varphi(\bar{a})$ for some $\bar{a}$, so $\bar{a} \vDash p(\bar{x})$.

$\Rightarrow$ Let $\bar{a} \vDash p(\bar{x})$. Then $\varphi(\bar{x}) \coloneqq (\bar{x} = \bar{a})$ isolates $p(\bar{x})$. □

In particular, if $A$ isolates $p(\bar{x})$ then every model $\mathcal{M}$ containing $A$ realises $p(\bar{x})$.

**Lemma 9.3.** Let $|L(A)| = \omega$. Let $p(\bar{x})$ be a type in $L(A)$ and suppose that $A$ does not isolate $p(\bar{x})$. Let $\psi(z)$ be a consistent $L(A)$-formula in one free variable. Then there is $a \in U$ sucht that

(i) $\mathcal{U} \vDash \psi(a)$,

(ii) $A \cup \{a\}$ does not isolate $p(\bar{x})$.

Lemma 9.3 will be proven later.

**Theorem 9.4** (Omitting Types)**.** Let $|L(A)| = \omega$ and let $p(\bar{x})$ be a consistent type in $L(A)$ with variables in $\bar{x}$, $|\bar{x}| < \omega$. Then the following are equivalent:

(i) all models $\mathcal{M}$ containing $A$ realise $p(\bar{x})$,

(ii) $A$ isolates $p(\bar{x})$.

*Proof.* (ii) implies (i): Let $\varphi(\bar{x})$ be an $L(A)$-formula that isolates $p(\bar{x})$. Then $\vDash \exists \bar{x}\, \varphi(\bar{x})$, so if $\mathcal{M}$ contains $A$, then there is $\bar{a} \in M$ such that $\mathcal{M} \vDash \varphi(\bar{a})$ and $\bar{a}$ realizes $p(\bar{x})$. This follows from Remark 9.2.

(i) implies (ii): Argue by contraposition and assume $A$ does not isolate $p(\bar{x})$. We build a chain of sets $\langle A_i : i < \omega \rangle$ such that

(i) $A_0 = A$,

(ii) $|A_i| = \omega$ for all $i < \omega$,

(iii) $A_i$ does not isolate $p(\bar{x})$ for any $i < \omega$.

At stage $i+1$, enumerate all consistent $L(A_i)$-formulas with one free variable, say $\langle \psi_k(z) : k < \omega \rangle$. For $k \in \omega$, we find $a_k^i \in U$ such that $\vDash \psi_k(a_k^i)$, and such that $A_i \cup \{a_0^i, \ldots, a_k^i\}$ does not isolate $p(\bar{x})$. This is possible by Lemma 9.3. Let $A_{i+1} \coloneqq A_i \cup \{a_k^i \mid k < \omega\}$. Let $M \coloneqq \bigcup A_i$. We claim that

(a) $\mathcal{M} \preccurlyeq \mathcal{U}$,

(b) $\mathcal{M}$ omits $p(\bar{x})$.

For (a), use Tarski-Vaught Test (Lemma 3.8): if $\psi(z)$ is a consistent $L(M)$-formula, then it has parameters in $A_i$ for some $i$, so by construction, $\mathcal{M}$ contains a witness. Condition (b) is satisfied by construction: none of the $A_i$ isolates $p(\bar{x})$, so $\mathcal{M}$ does not isolate $p(\bar{x})$, so since $\mathcal{M}$ is a model, by Remark 9.2 it does not realise $p(\bar{x})$. $\square$

**Remark.** We need $|L(A)| = \omega$. There are known counterexamples in the uncountable case.

*Proof of Lemma 9.3.* Let $\psi(z)$ be the consistent $L(A)$-formula. We build a sequence $\langle \psi_i(z) : i < \omega \rangle$ starting with $\psi_0 := \psi$ such that

(i) $\psi_i(z)$ is consistent,

(ii) $\psi_{i+1}(z) \to \psi_i(z)$ for all $i < \omega$,

(iii) a realisation of the type $\{\psi_i(z) \mid i < \omega\}$ is the required solution of $\psi(z)$.

Let $\langle \chi_i(\bar{x}, z) : i < \omega \rangle$ be an enumeration of $L(A)$-formula with free variables in $\bar{x} \cup \{z\}$. At stage $i + 1$, if $\chi_i(\bar{x}, z)$ is inconsistent, set $\psi_{i+1}(z) = \psi_i(z)$. If otherwise $\chi_i(\bar{x}, z)$ is consistent, let $\varphi(\bar{x}) \in p(\bar{x})$ such that

$$\psi_i(z) \wedge \exists \bar{x}\, (\chi_i(\bar{x}, z) \wedge \neg\varphi(\bar{x}))$$

is consistent. Let $\psi_{i+1}(z)$ be this conjunction. These conditions guarantee that a realisation of the type $\{\psi_i(z) \mid i < \omega\}$ does not isolate $p(\bar{x})$. We must show that it is possible to find such $\varphi(\bar{x})$. If no such $\varphi(\bar{x})$ existed, then for all $\varphi(\bar{x}) \in p(\bar{x})$,

$$\chi_i(\bar{x}, z) \wedge \psi_i(z) \to \varphi(\bar{x}).$$

But this implies that

$$\exists z\, (\chi_i(\bar{x}, z) \wedge \psi_i(z)) \to p(\bar{x}).$$

This is an $L(A)$-formula which yields a contradiction to the assumption that $A$ does not isolate $p(\bar{x})$. $\square$

**Definition 9.5.** Let $\mathcal{M}$ be a model, $A \subseteq M$. Then

(i) $\mathcal{M}$ is **prime over** $A$ if for every $\mathcal{N} \supseteq A$ there is an elementary embedding $f : \mathcal{M} \to \mathcal{N}$ that fixes $A$ pointwise. If $A = \varnothing$, then $\mathcal{M}$ is a **prime model**.

(ii) $\mathcal{M}$ is **atomic over** $A$ if for all $n \in \omega$ and $\bar{a} \in M^n$, the type $\mathrm{tp}(\bar{a}/A)$ is isolated. When $A = \varnothing$, then $\mathcal{M}$ is said to be **atomic**.

**Fact 9.6.** Let $\bar{a}, \bar{b} \in U^n$ and $A \subseteq U$. Suppose that $\mathrm{tp}(\bar{b}\bar{a}/A)$ is isolated, then $\mathrm{tp}(\bar{b}/A\bar{a})$ and $\mathrm{tp}(\bar{a}/A)$ are isolated.

*Proof (sketch).* Let $p(\bar{x}, \bar{z}) = \mathrm{tp}(\bar{b}\bar{a}/A)$. Then $\mathrm{tp}(\bar{b}/A\bar{a}) = p(\bar{x}, \bar{a})$ and $\mathrm{tp}(\bar{a}/A) = \{\exists \bar{x}\, \varphi(\bar{x}, \bar{z}) \mid \varphi(\bar{x}, \bar{z}) \in p(\bar{x}, \bar{z})\}$. Suppose $\varphi(\bar{x}, \bar{z})$ isolates $p(\bar{x}, \bar{z})$. Then

(i) $\varphi(\bar{x}, \bar{a})$ isolates $\mathrm{tp}(\bar{b}/A\bar{a})$,

(ii) $\exists \bar{x}\, \varphi(\bar{x}, \bar{z})$ isolates $\mathrm{tp}(\bar{a}/A)$. $\square$

**Proposition 9.7.** If $\mathcal{M}$ is atomic over $A$ and $\bar{a} \in M^n$, then $\mathcal{M}$ is atomic over $A \cup \bar{a}$.

*Proof.* Let $\bar{b} \in M^m$. Then $\mathrm{tp}(\bar{b}\bar{a}/A)$ is isolated, so $\mathrm{tp}(\bar{b}/A\bar{a})$ is isolated by Fact 9.6. $\square$

**Proposition 9.8** (Extension Lemma)**.** Let $f : \mathcal{M} \to \mathcal{N}$ be elementary and $\mathcal{M}$ atomic over $\mathrm{dom}(f)$. Then for every $b \in M$ there is $c \in N$ such that $f \cup \{\langle b, c\rangle\}$ is elementary.

*Proof.* Let $\bar{a}$ enumerate $\mathrm{dom}(f)$ and let $p(x, \bar{z}) = \mathrm{tp}(b\bar{a}/\varnothing)$. Let $\varphi(x, \bar{a})$ isolate $p(x, \bar{a})$. Then by elementarity, $\varphi(x, f(\bar{a}))$ isolates $p(x, f(\bar{a}))$. We can find $c \in N$ such that $\mathcal{N} \vDash \varphi(c, f(\bar{a}))$ and this $c$ is the required element. □

**Proposition 9.9.** Any two countable atomic models are isomorphic.

*Proof.* By a back-and-forth argument using Propositions 9.7 and 9.8. □

**Theorem 9.10.** Let $|L(A)| = \omega$. Then for any model $\mathcal{M} \supseteq A$, the following are equivalent:

(i) $\mathcal{M}$ is countable and atomic over $A$,

(ii) $\mathcal{M}$ is prime over $A$.

*Proof.* (i) implies (ii): Let $\mathcal{N} \supseteq A$ be a model. Then $\mathrm{id}_A : \mathcal{M} \to \mathcal{N}$ is elementary and by Propositions 9.7 and 9.8, $\mathrm{id}_A$ can be extended to a elementary embedding $g : \mathcal{M} \to \mathcal{N}$.

(ii) implies (i): By Downward Löwenheim-Skolem (Theorem 3.11), we know that there is a countable model that contains $A$. Then $\mathcal{M}$ embeds into this model and so $|M| = \omega$. Suppose that $\mathcal{M}$ is not atomic over $A$, i.e. that there is $\bar{b} \in M^n$ such that $\mathrm{tp}(\bar{b}/A)$ is not isolated. By the Omitting Type Theorem (Theorem 9.4), there is a countable model $\mathcal{N}$ that omits $\mathrm{tp}(\bar{b}/A)$. Hence, $\mathrm{tp}(\bar{b}/A)$ is realised in $\mathcal{M}$ but not in $\mathcal{N}$. So $\mathcal{M}$ does not embed into $\mathcal{N}$, a contradiction to $\mathcal{M}$ being prime. □

**Definition 9.11.** For $n \in \omega$, $A \subseteq U$, let $S_n(A)$ denote the collection of complete consistent types in $L(A)$ with $n$ free variables. When $A = \varnothing$, we write $S_n(T)$ for $S_n(\varnothing)$.

**Theorem 9.12** (Ryll–Nardzewski–Engeler–Svenonins)**.** For a countable theory $T$, the following are equivalent:

(i) $T$ is $\omega$-categorical,

(ii) for all $n$, every type $p(\bar{x}) \in S_n(T)$ is isolated,

(iii) for all $n$, $|S_n(T)| < \omega$,

(iv) $\mathrm{Aut}(\mathcal{U})$ has finitely many $n$-orbits (i.e., orbits when acting on $n$-tuples) for all $n \in \omega$.

*Proof.* (ii) implies (iii): We have $U^n = \bigcup_{p(\bar{x}) \in S_n(T)} p(\mathcal{U})$. But $p(\bar{x}) \in S_n(T)$ is isolated by a formula $\varphi_p(\bar{x})$, say. And so $U^n = \bigcup_{p(\bar{x}) \in S_n(T)} \varphi_p(\mathcal{U})$. By compactness, $U^n = \bigcup_{i=1}^k \varphi_{p_i}(\mathcal{U})$ for $k$ finite and certain $p_1, \ldots, p_k$. This implies that $|S_n(T)| < \omega$.

(iii) implies (ii): Let $p(\bar{x}) \in S_n(T)$. If $|S_n(T)| < \omega$, then $\mathcal{U} \setminus p(\mathcal{U})$ is a union of finitely many type-definable sets, so there it is type-definable by $q(\bar{x})$, not necessarily complete. By an argument similar to Fact 7.2, there are formulas $\varphi(\bar{x})$ and $\chi(\bar{x})$ such that $\varphi(\mathcal{U}) = p(\mathcal{U})$ and $\chi(\mathcal{U}) = \mathcal{U} \setminus p(\mathcal{U})$. But then $\varphi(\bar{x})$ isolates $p(\bar{x})$.

(iii) and (iv) are equivalent: This follows from the fact that for $\bar{a}, \bar{b} \in U$,

$$\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b}) \iff O(\bar{a}/\varnothing) = O(\bar{b}/\varnothing).$$

(i) and (ii) are equivalent: This is an exercise. It follows from atomicity. □

## 10 Bonus: Existence of saturated models

If $\mathcal{M}$ is saturated, then

- $\mathcal{M}$ is homogeneous.
- $\mathcal{M}$ is universal.

If $\mathcal{M}$ is $\lambda$-saturated, then:

- $\mathcal{M}$ is weakly $\lambda$-homogeneous, i.e. for all $f : \mathcal{M} \to \mathcal{M}$ (partial) elementary such that $|f| < \lambda$, for every $b \in \mathcal{M}$, then $\exists \hat{f} \supseteq f$ elementary and such that $b \in \operatorname{dom} f$.

Can prove: $\lambda$-homogeneous is equivalent to homogeneity when $|\mathcal{M}| = \lambda$.

**Definition** (Cofinality)**.** If $\alpha$ is a limit ordinal $\geq \omega$, $\operatorname{cof}(\alpha)$ (**cofinality** of $\alpha$) is the least $\lambda$ such that there is $f : \lambda \to \alpha$ such that $\operatorname{img}(f)$ is unbounded in $\alpha$.

**Example.**

$$\operatorname{cof}(\omega) = \aleph_0 \qquad \operatorname{cof}(\omega_\omega) = \aleph_0.$$

**Definition** (Regular)**.** A cardinal $\kappa$ is **regular** if $\operatorname{cof}(\kappa) = \kappa$.

**Example.** $\aleph_0$ is regular. Also, every successor cardinal is regular.

Are there any limit cardinals other than $\aleph_0$ that are regular?

**Definition** ($S_1^{\mathcal{M}}$)**.** If $\mathcal{M} \vDash T$, $A \subseteq \mathcal{M}$, then define

$S_1^{\mathcal{M}}(A) \coloneqq \{\, p(x) \mid p(x)$ is a complete type in a single variable with parameters in $A \,\}$

**Lemma.** If $\mathcal{M}$ is such that $|\mathcal{M}| \geq |L| + \omega$, let $\kappa > \aleph_0$. Then there is $\mathcal{M}' \succcurlyeq \mathcal{M}$ such that for all $A \subseteq \mathcal{M}$ with $|A| < \kappa$, if $p(x) \in S_1^{\mathcal{M}}(A)$, then $p(x)$ is realized in $\mathcal{M}'$, $|\mathcal{M}'| \leq |\mathcal{M}|^\kappa$.

*Proof.* First, note

$$|\{\, A \subseteq \mathcal{M} \mid |A| \leq \kappa \,\}| \leq |\mathcal{M}|^\kappa$$
$$\left|S_1^{\mathcal{M}}(A)\right| \leq 2^\kappa.$$

Enumerate $S_1^{\mathcal{M}}(A)$ as $\langle p_\alpha : \alpha < |\mathcal{M}|^\kappa \rangle$. Build $\langle \mathcal{M}_\alpha : \alpha < |\mathcal{M}|^\kappa \rangle$ as follows:

- $\mathcal{M}_0 = \mathcal{M}$
- $\mathcal{M}_\alpha = \bigcup_{\beta < \alpha} \mathcal{M}_\beta$ when $\alpha$ is a limit.
- $\mathcal{M}_\alpha \preccurlyeq \mathcal{M}_{\alpha+1}$ such that $\mathcal{M}_{\alpha+1}$ realizes $p_\alpha(x)$ and $|\mathcal{M}_{\alpha+1}| = |\mathcal{M}_\alpha|$. Then $\bigcup_{\alpha < |\mathcal{M}|^\kappa} \mathcal{M}_\alpha$ realizes all types in $S_1^{\mathcal{M}}(A)$ and

$$\left| \bigcup_{\alpha < |\mathcal{M}|^\kappa} \mathcal{M}_\alpha \right| \leq |\mathcal{M}|^\kappa. \qquad \square$$

**Theorem.** Let $\kappa > \aleph_0$, let $\mathcal{M} \vDash T$. Then there is a $\kappa^+$-saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ such that $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$.

*Proof.* Build an elementary chain $\langle \mathcal{N} : \alpha < \kappa^+ \rangle$ such that

- $\mathcal{N}_0 = \mathcal{M}$

- take unions at limit stages
- Given $\mathcal{N}_\alpha$, find $\mathcal{N}_{\alpha+1} \succcurlyeq \mathcal{N}_\alpha$ such that all types in $S_1^{\mathcal{N}_\alpha}(A)$ with $|A| \leq \kappa$ are realized.

Moreover, $|\mathcal{N}_\alpha| \leq |\mathcal{M}|^\kappa$ (follows from previous result). Let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. Since $\kappa^+ \leq |\mathcal{M}|^\kappa$, $\mathcal{N}$ is the union of at most $|\mathcal{M}|^\kappa$ sets each of size at most $|\mathcal{M}|^\kappa$, hence $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$.

To see that $\mathcal{N}$ is $\kappa^+$ saturated, pick $A \subseteq \mathcal{N}$ such that $|A| \leq \kappa$. By the regularity of $\kappa^+$, there is $\alpha$ such that $A \subseteq \mathcal{N}_\alpha$, hence all types $/A$ with one free variable are realized in $\mathcal{N}$. $\square$

Recap: For arbitrarily large $\kappa$, there is a $\kappa^+$ saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ with $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$. If $\kappa$, $|\mathcal{M}|$ are such that $|\mathcal{M}| \leq 2^\kappa$, then $|\mathcal{M}|^\kappa = 2^\kappa$ so you get a $\kappa^+$-saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ such that $|\mathcal{N}| = 2^\kappa$. So GCH implies saturated models exist.

Alternatively, suppose there are arbitrarily large cardinals $\kappa$ such that

$$\kappa^{<\kappa} = \bigcup \{ \kappa^\alpha \mid \alpha < \kappa \} = \kappa$$

(strongly inaccessible cardinals). Then the chain stabilises, giving the required structure.

**Definition.** Take $T$ a complete theory in a countable language, $\kappa \geq \aleph_0$ a cardinal. Then $T$ is $\kappa$-**stable** if for all $\mathcal{M} \vDash T$, $A \subseteq \mathcal{M}$, $|A| \leq \kappa$, $\forall n \leq \omega$, we have

$$\left|S_n^{\mathcal{M}}(A)\right| \leq \kappa$$

where $S_n^{\mathcal{M}}(A)$ is the set of complete types with $n$ variables and parameters in $A$.

**Theorem.** Let $\kappa$ be a regular cardinal, and $T$ $\kappa$-stable. Then there is a $\mathcal{M} \vDash T$, $|\mathcal{M}| = \kappa$, $\mathcal{M}$ saturated.

*Proof.* We build an elementary chain $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ where $|\mathcal{M}_\alpha| < \kappa$ as follows:

- $\mathcal{M}_0 \vDash T$
- unions at limit stages
- given $\mathcal{M}_\alpha$, $|\mathcal{M}_\alpha| = \kappa \Rightarrow S_1^{\mathcal{M}_\alpha}(\mathcal{M}_\alpha) = \kappa$, there is $\mathcal{M}_{\alpha+1} \succcurlyeq \mathcal{M}_\alpha$ that realizes all types in $S_1^{\mathcal{M}_\alpha}(\mathcal{M}_\alpha)$ and $|\mathcal{M}_{\alpha+1}| = |\mathcal{M}_\alpha|$. Let $\bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$, then $|\bigcup \mathcal{M}_\alpha| = \kappa$ and $\bigcup \mathcal{M}_\alpha$ is $\kappa$-saturated by construction.

Now, $\mathcal{M}$ $\kappa$-saturated, $\kappa$-strongly homogeneous, $|\mathcal{M}| \gg \kappa$. $\square$

# 11 Bonus: An Alternative Proof of Compactness

**Definition 11.1.** Take an $L$-theory $T$.

(i) $T$ is **finitely satisfiable** if every finite subset of sentences in $T$ has a model.

(ii) $T$ is **maximal** if for all $L$-sentences $\sigma$, either $\sigma \in T$ or $\neg\sigma \in T$.

(iii) $T$ has the **witness property** if for all $\varphi(x)$ ($L$-formula with one free variable) there is a constant $c \in \mathscr{C}$ such that

$$((\exists x\ \varphi(x)) \to \varphi(c)) \in T.$$

**Lemma 11.2.** If $T$ is maximal and finitely satisfiable and $\varphi$ is an $L$-sentence, and $\Delta \overset{\text{finite}}{\subseteq} T$ with $\Delta \vdash \varphi$, then $\varphi \in T$.

*Proof.* If $\varphi \notin T$ then $\neg\varphi \in T$ (by maximality). But then $\Delta \cup \{\neg\varphi\}$ is a finite subset of $T$ which does not have a model. □

**Lemma 11.3.** Let $T$ be a maximal, finitely satisfiable theory with the witness property. Then $T$ has a model. Moreover, if $\lambda$ is a cardinal and $|\mathscr{C}| \le \lambda$, then $T$ has a model of size at most $\lambda$.

*Proof.* Let $c, d \in \mathscr{C}$, define $c \sim d$ iff $c = d \in T$.

We claim that $\sim$ is an equivalence relation. For transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$, so $c = e \in T$ (by Lemma 11.2), and so $c \sim e$. Reflexivity follows from maximality, and symmetry is immediate. ■

We denote $[c] \in \mathscr{C}/\sim$ by $c^*$. Now, define a structure $\mathcal{M}$ whose domain is $\mathscr{C}/\sim\, = M$. Clearly, $|M| \le \lambda$ if $|\mathscr{C}| \le \lambda$. We must define interpretations in $\mathcal{M}$ for symbols of $L$.

- If $c \in \mathscr{C}$, then $c^{\mathcal{M}} = c^*$.
- If $R \in \mathscr{R}$, define

$$R^{\mathcal{M}} := \{\, (c_1^*, \ldots, c_{n_R}^*) \mid R(c_1, \ldots, c_n) \in T \,\}.$$

**Claim:** $R^{\mathcal{M}}$ is well defined. **Proof:** Suppose $\bar{c}, \bar{d} \in \mathscr{C}^{n_R}$ and suppose $c_i \sim d_i$. That is, $c_i = d_i \in T$ for $i = 1, \ldots, n_R$ so by Lemma 11.2

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T. \qquad ■$$

- If $f \in \mathscr{F}$, and $\bar{c} \in \mathscr{C}^{n_R}$, then $f\bar{c} = d \in T$ for some $d \in \mathscr{C}$. (This is because $\exists x\ (f(\bar{c}) = x) \in T$ so apply witness property.)

  Then define $f^{\mathcal{M}}(\bar{c}^*) = d^*$. Exercise: Check $f^{\mathcal{M}}(\bar{c}^*)$ is well-defined!

**Claim:** if $t(x_1, \ldots, x_n)$ is an $L$-term and $c_1, \ldots, c_n, d \in \mathscr{C}$, then

$$t(c_1, \ldots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*.$$

**Proof:**

($\Rightarrow$) by induction on the complexity of $t$.

($\Leftarrow$) Assume $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. Then

$$t(c_1, \ldots, c_n) = e \in T$$

for some constant $e$ by witness property and Lemma 11.2. Use ($\Rightarrow$) to get that $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. But then $d^* = e^*$, i.e. $d = e \in T$. Then $t(c_1, \ldots, c_n) = d \in T$.
■

**Claim:** For all $L$-formulas $\varphi(\bar{x})$, and $\bar{c} \in \mathscr{C}^{|\bar{x}|}$,

$$\mathcal{M} \vDash \varphi(\bar{c}) \iff \varphi(\bar{c}) \in T.$$

**Proof:** By induction on $\varphi(\bar{x})$. (Exercise: Fill in the details). ■ This shows $\mathcal{M} \vDash T$. □

**Lemma 11.4.** Let $T$ be a finitely satisfiable $L$-theory. Then there are $L^* \supseteq L$ and a finitely satisfiable $L^*$-theory $T^* \supseteq T$ such that

(i) $|L^*| = |L| + \omega$.

(ii) any $L^*$-theory extending $T^*$ has the witness property.

*Proof.* We define $\langle L_i : i < \omega \rangle$ a chain of languages containing $L$ and such that $|L_i| = |L| + \omega$, and $\langle T_i : i < \omega \rangle$ of finitely satisfiable theories such that $\forall i, T_i$ is an $L_i$-theory and $T_i \supseteq T$.

Set $L_0 = L$ and $T_0 = T$. At stage $i+1$, $L_i$ and $T_i$ are given. List all $L_i$-formulas $\varphi(x)$ (one free variable) and let

$$L_{i+1} = L_i \cup \{ c_\varphi \mid \varphi(x) \text{ an } L_i \text{ formula} \}.$$

For all $\varphi(x)$, an $L_i$ formula in one free variable, let $\Phi_\varphi$ be the $L_{i+1}$-sentence

$$\exists x\ \varphi(x) \to \varphi(c_\varphi).$$

Then let

$$T_{i+1} = T_i \cup \{ \Phi_\varphi \mid \varphi(x) \text{ is an } L_i \text{ formula} \}.$$

**Claim**: $T_{i+1}$ is finitely satisfiable.
**Proof**: Let $\Delta \subseteq T_{i+1}$ be finite. Then

$$\Delta = \Delta_0 \cup \{\Phi_{\varphi_1}, \dots, \Phi_{\varphi_n}\}$$

where $\Delta_0 \subseteq T_i$. Let $\mathcal{M} \vDash \Delta_0$ ($\mathcal{M}$ is an $L_i$ structure; it exists because $T_i$ is finitely satisfiable).

We define an $L_{i+1}$-structure $\mathcal{M}'$ with domain $M$. Define the interpretation of new constants as follows: if $\mathcal{M} \vDash \exists x\ \varphi(x)$, then let $a$ be such that $\mathcal{M} \vDash \varphi(a)$, and set $c_\varphi^{\mathcal{M}'} := a$. Otherwise, $c_\varphi^{\mathcal{M}'}$ is arbitrary. Then $\mathcal{M}' \vDash \Delta$. ■

Let

$$L^* = \bigcup_{i<\omega} L_i, \qquad T^* = \bigcup_{i<\omega} T_i.$$

By construction, any extension of $T^*$ has the witness property (check this!) and $T^*$ is finitely satisfiable. (If $\Delta \overset{\text{finite}}{\subseteq} T^*$ then $\Delta \subseteq T_i$ for some $i$). □

**Lemma 11.5.** If $T$ is finitely satisfiable, there exists a maximal finitely satisfiable $T' \supseteq T$.

*Proof.* Let

$$I := \{ S \mid S \text{ is a finitely satisfiable } L\text{-theory such that } T \subseteq S \}.$$

$I$ is partially ordered by inclusion, and non-empty.

If $\langle C_i : i < \lambda \rangle$ is a chain in $I$, then $\bigcup_{i<\lambda} C_i$ is an upper bound for the chain - it is finitely satisfiable. Then by Zorn's lemma, $I$ has a maximal element (with respect to $\subseteq$).

**Claim:** the maximal element $T'$ of $I$ is the required extension of $T$ (check that for all $L$-sentences $\sigma$, $\sigma \in T'$ or $\neg\sigma \in T'$). □

**Theorem 11.6** (Compactness)**.** If $T$ is a finitely satisfiable $L$-theory and $\lambda \geq |L| + \omega$, then there is $\mathcal{M} \vDash T$ such that $|\mathcal{M}| \leq \lambda$.

*Proof sketch.* Extend $T$ to $T^*$, an $L^*$-theory that is finitely satisfiable and such that any $S \supseteq T^*$ has the witness property (by Lemma 11.4).

By Lemma 11.5, there is $T' \supseteq T^*$, which is maximal and finitely satisfiable. Then $T'$ has the witness property. Then by Lemma 11.3 there is $\mathcal{M} \vDash T'$ with $|\mathcal{M}| \leq \lambda$, and $\mathcal{M} \vDash T$. $\square$

# Index