Part II – Riemann Surfaces (Incomplete)

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0 Complex analysis and complex log

Definition. A smooth function $f : U \to \mathbb{C}$ (where U is a domain in $\mathbb{C}$) is **holomorphic** or **analytic** if either of the following equivalent statements hold:

(1) f is differentiable at all points of U, where differentiability is defined by limits, and checked by the Cauchy-Riemann equations

(2) $\forall a \in U$, f has a power series expansion on a neighbourhood of a:

$$f(z) = \sum_{n \geq 0} a_n (z-a)^n$$

and the series converges on some disk about a with positive radius.

Sketch of proof of equivalence.
($\Rightarrow$) Use the Cauchy Integral Formula to construct a_n, and convergence.
($\Leftarrow$) Show directly that the term-by-term derivative exists, and that it agrees with the limit definition of the derivative. $\square$

Note that a power series tells you about local behaviour. If $f(z)$ is not identically zero near $a \in U$, there exists some minimal n_0 such that $a_{n_0} \neq 0$. We can write f locally as

$$\begin{aligned} f(z) &= a_{n_0}(z-a)^{n_0} + \sum_{n \geq n_0} a_n (z-a)^n \\ &= a_{n_0}(z-a)^{n_0} \left(1 + \sum_{n > n_0} \frac{a_n}{a_{n_0}} (z-a)^{n-n_0} \right) \end{aligned}$$

As $z \to a$, the sum $\to 0$ so the quantity in the brackets tends to 1. So, $f(z) = a_{n_0}(z-a)^{n_0} g(z)$, where g is analytic and nonzero on a neighbourhood of a. Consequently, we have the *principle of isolated zeros*: for f analytic on domain U, then for all $a \in U$ such that $f(a) = 0$, either f is identically 0 on a neighbourhood of a, or f is never 0 on a punctured disk centred at a. But recall a domain refers to an open *and connected* set. So, if f is identically 0 (improve this writing) on a neighbourhood of a, call it D_a. If $f \neq 0$ on a punctured disk about a, call it P_a^*. Construct

$$V = \bigcup_{\substack{a \in U \text{such that} \\ f \equiv 0 \text{on a neighbourhood of} a}} D_a \tag{1}$$

V and W are open, disjoint and $V \cup W = U$. Since U is connected, $U = V$ or $U = W$. So either $f \equiv 0$ on U, or f has only isolated zeros. This will both be referred to as the principle of isolated zeros.

(corollary) (identity principle) If f and g are analytic on U, then either $f \equiv g$ on U, or $z \in U : f(z) = g(z)$ consists of isolated points. Proof (clear)

(definition) If f is analytic on a punctured disk $\mathbb{D}^*(a, r)$, then we say a is an isolated singularity of f. If so, then there exists a Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ near a, which come in three types.

I. Removable singularity. f extends to an analytic function on $\mathbb{D}^*a, r$. Phrased in terms of Laurent expansions, $a_n = 0 \forall n < 0$. (thm) (Removable singularities theorem) f has a removable singularity at a if and only if f is bounded on a punctured neighbourhood about a. (proof sketch) (=¿) follows from continuity of analytic functions (¡=) Cauchy's theorem and integral formula still hold for punctured neighbourhoods so long as $f(z)(z-a) \to 0$ as $z \to a$. With a small circle about a, we can show directly that $a_n = 0$ for $n < 0$.

II. Poles: f has a pole at a if $a_n = 0$ for all $n < n_0$ for some n_0. Locally, this occurs if and only if $|f(z)| \to \infty$ as $z \to a$ (using the Laurent series).

III. Essential singularity: $a_n \neq 0$ for finitely many $n < 0$, f has an essential singularity at a. thm [Casorati-Weierstrass] If f has an essential singularity at a, then the image of f on any punctured neighbourhood of a is dense in $\mathbb{C}$. (proof sketch) Examine $\frac{1}{f(z)-\gamma}$ if the image of f misses a neighbourhood of γ.

Examples $f(z) = \frac{1}{e^z - 1}$ has poles wherever $e^z = 1$. At ∞, we also have an isolated singularity, recalling that punctured neighbourhoods of ∞ are $\mathbb{C}$
$\mathbb{D}(0, R)$. Since e^z takes all nonzero values or strips of $\mathbb{C}$, we cannot have $e^z \to 1$ as $z \to \infty$, hence f cannot have a pole. On the other hand, there exists arbitrarily large solutions (in modulus) to $e^z = 1$, and f cannot have a removable singularity at ∞. Hence, this singularity is essential.

(edit: not isolated as every neighbourhood of infinity contains a singularity)

0.1 Complex logarithm

The complex logarithm is an example of a multivalued function, which arises as the inverse of an analytic function. Given nonzero z, if $e^w = z$, $z = re^{i\theta}$, then $w = \log r + (2\pi n + \theta i)$ for some $n \in \mathbb{Z}$. We cannot make a continuous choice of n on all of $\mathbb{C}$, so we define the complex log on domains like $\mathbb{C}$
$\mathbb{R}_{\leq 0} =: U$. We have for each n, a choice of logarithm which can be analytically defined on U. Recall a *continuous* inverse of an analytic function is analytic.

Let $U_1 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

Proposition. For $n \in \mathbb{Z}$, define $h(z)$ on U, by

$$h(z) = \int_{-1}^{z} \frac{dw}{w} + (2n+1)\pi i$$

with integral along straight line joining -1 and z.

Then h is analytic on U_1 and is the inverse to the exponential function on U_1.

Proof. Let $z \in U_1$. $\tau \in \mathbb{C}$ with $|\tau|$ sufficiently small, such that the triangle is entirely in the domain.

Then claim $\frac{h(z+\tau)-h(z)}{\tau} = \frac{1}{\tau}\int_z^{z+\tau}\frac{dw}{w} \to \frac{1}{z}$. The first equality follows from Cauchy's theorem, since h is continuous in the triangle.

$$\begin{aligned} \left|\frac{1}{\tau}\int_z^{z+\tau}\frac{dw}{w} - \frac{1}{2}\right| &= \left|\frac{1}{\tau}\int_z^{z+\tau}\frac{z-w}{zw}dw\right| \\ &\leq C\tau \to 0 \end{aligned}$$

Thus h is analytic on U_1, with $h'(z) = \frac{1}{z}$.

Look at $g(z) = \frac{e^{h(z)}}{z}$, so $g'(z) = \frac{zh'(z)e^{h(z)}-e^{h(z)}}{z^2} = 0$, thus g is constant. But, we still need to find out what it's value us, so consider $g(-1)$. $g(-1) = \frac{e^{h(-1)}}{-1} = -e^{(2n+1)\pi i} = 1$. Thus, $e^{h(z)} \equiv z$ on U_1, so h is the inverse to the exponential. $\square$

Remark. We can't extend the function H continuously across the positive real axis.

0.2 Analytic continuation

Fix a domain $U \in \mathbb{C}$ which is path connected.

Definition (Direct Analytic Continuation)**.** A **function element** (or **function germ**) on U is a pair, (f, D) where f is analytic on the domain $D \subseteq U$.

Two function elements (f, D) and $g, E)$ are **equivalent** if $D \cap E \neq \varnothing$ and $f = g$ on $D \cap E$. In this case, we say (g, E) is a **direct analytic continuation** of (f, D).

Remark. This is not an equivalence relation. In the diagram, (f_1, D_1) and (f_3, D_3) are not equivalent since $D_1 \cap D_3 = \varnothing$.

Definition (Analytic continuation along a path)**.** We say (g, E) is an analytic continuation of (f, D) along a path $\gamma : [0, 1] \to U$, written as $(f, D) \sim_\gamma (g, E)$. If there exists $(f_1, D_1), \ldots, (f_n, D_n)$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ with $\gamma([t_{i-1}, t_i]) \subseteq D_i$ for $1 \leq i \leq n$ and $(f_1, D_1) = (f, D)$, $(f_n, D_n) = (g, E)$ and $(F_{i-1}, D_{i-1}) \sim (F_i, D_i)$, that is (f_i, D_i) is a direct analytic continuation of f_{i-1}, D_{i-1}.

Definition (Analytic continuation)**.** We say (g, E) is an analytic continuation of (f, D) if there exists a path γ with $(f, D) \sim_\gamma (g, E)$. We write $(f, D) \approx (g, E)$.

Remark. $\approx$ is an equivalence relation. Reflexivity and symmetry are easy, and transitivity can be seen from the diagram.

Definition. An equivalence class $\mathcal{F}$ under $\approx$ is a **complete analytic function**.

Example. Set $U = \mathbb{C}^* = \mathbb{C} \setminus 0$. Fix $(\alpha, \beta) \subseteq \mathbb{R}$, with $|\beta - \alpha| \leq 2\pi$, and define

$$E_{\alpha,\beta} = \{\, z = re^{i\theta} \mid \alpha < \theta < \beta\,,\ r > 0\,\}$$

So we can see $U_1 = E(0, 2\pi)$. Define $f_{(\alpha,\beta)} : E_{(\alpha,\beta)} \to \mathbb{C}$ by $f_{(\alpha,\beta)}(re^{i\theta}) = \log r + i\theta$, $\theta \in (\alpha, \beta)$.

Write $L_{(\alpha,\beta)}$ for the function element $(f_{(\alpha,\beta)}, E_{(\alpha,\beta)})$.

Consider the three function elements $L_{-\frac{\pi}{2},\frac{\pi}{2}}$, $L_{\frac{\pi}{6},\frac{7\pi}{6}}$, $L_{\frac{5\pi}{6},\frac{11\pi}{6}}$ We can see the direct analytic continuations, but we do not have direct analytic continuation (), but ().

Aim. Construct a surface on which log is well-defined by gluing together a bunch of domains on which it is well-defined.

For each $n \in \mathbb{Z}$, take a copy of $U_1 = E_{(0,2\pi)}$. Each has a well-defined choice of logarithm: $f_{(2\pi n, 2\pi(n+1))}$, $re^{i\theta} \mapsto \log r + (\theta + 2\pi n)i$, for $\theta \in (0, 2\pi)$.

We can glue together these copies of U, so that the functions $f_{(2\pi n, 2\pi(n+1))}$ glue to give a continuous function $L : \tilde{U} \to \mathbb{C}$.

Definition (Covering map)**.** A **covering map** of a topological space X is is (sic) a continuous map $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ and X are Hausdorff and path connected, and π is a local homeomorphism. Specifically, $\forall \widetilde{x} \in \widetilde{X}$, $\exists$ an open neighbourhood $\widetilde{N}$ of $\widetilde{x}$ on which π restricts to a homeomorphism. We say that X is the **base space** of π.

Note that this is likely weaker than the definition used in Algebraic Topology, which requires that $\forall x \in X$, $\exists$ an open neighbourhood N of x such that $\pi^{-1}(N)$ is a disjoint union of open sets which are mapped homeomorphically by π to N. We will call this a **regular** covering map.

Example. Non-regular covering map: Consider $\pi(z) = e^z$ on the domain

$$\{\, z \mid 0 < \mathrm{Im}(z) < 4\pi \,\} \tag{2}$$

Let $x = 1$. From the diagram, we have a disjoint union of open neighbourhoods, but π does not map them homeomorphically.

Define $R \coloneqq \sqcup_{n\in\mathbb{Z}} \mathbb{C}^* \times n$, and equip R with the topology from basis with elements of the two forms:

1. for $(\eta, n) \in R$ with $\eta \notin \mathbb{R}_{\leq 0}$ and any $r > 0$, such that $\mathbb{D}(\eta, r) \cap \mathbb{R}_{\leq 0} = \varnothing$, define an open set $D((\eta, n), r)$ to be the disk of radius r about η in the nth sheet:
$$D((\eta, n), r) \coloneqq \{\, (z, n) \mid |z - \eta| < r \,\} \tag{3}$$
2. For (η, n) with $\eta \in \mathbb{R}_{\leq 0}$, define
$$A((\eta, n), r) \coloneqq \{\, (z, n) \mid |z - \eta| < r,\ \mathrm{Im}\, z \geq 0 \,\} \sqcup (z, n-1) | \, |z - \eta| < r,\ \mathrm{Im}\, z < 0 \tag{4}$$
where $r < |\eta|$

This defines a path-connected, Hausdorff topology on R. The map $\pi : \mathbb{R} \to \mathbb{C}^*$ given by $\pi((z, n)) = z$ is a (regular) covering map. Define $f : R \to \mathbb{C}$ by $f(re^{i\theta}, n) = \log r + i(2\pi n + \theta)$ where $\theta \in [0, 2\pi)$. By construction, f is continuous, f is also a bijection, and (diagram). In this sense, f is a logarithm.

Note we can similarly construct a gluing spae for $z^{1/n}$ as a multivalued function, $z^{1/n} = r^{1/n} e^{i\theta/n} e^{2\pi k i/n}$ so the nth sheet glues to the first. This is a regular covering map, but only because 0 is not included.

0.3 Power series and continuation

Recall a power series is absolutely and uniformly convergent on any closed disk inside its radius of convergence. If that radius of convergence is not ∞, what can we say about how far the series can be analytically continued? Without loss of generality assume we are working on the unit disk $\mathbb{D}$, that is a series about zero and radius of convergence 1. We denote $\mathbb{T} \coloneqq \partial\mathbb{D}$ and write $f(z) = \sum_{k\geq 0} a_k z^k$.

Definition. We say that $z \in \mathbb{T}$ is **regular** if $\exists$ an open disk N about z and an analytic function g on N such that $f = g$ on $N \cap \mathbb{D}$. If z is not regular, it is **singular**.

Note the collection of regular points is open, and the collection of singular points is closed.

Warning: regularity is independent of series convergence!

(1) $f(z) \coloneqq \sum_{k \geq 0} z^k$. This doesn't converge anywhere on $\mathbb{T}$, but as it agrees with $\frac{1}{1-z}$, all points except $z = 1$ are regular.

(2)
$$g(z) \coloneqq \sum_{k=2}^{\infty} \frac{z^k}{k(k-1)} \tag{5}$$
converges on $\mathbb{T}$ but 1 is a singular point, as if g extends analytically to a neighbourhood of 1, so g'' does also, a contradiction.

Proposition. If $f(z) = \sum a_k z^k$ has radius of convergence 1, then $\exists$ singular point on $\mathbb{T}$.

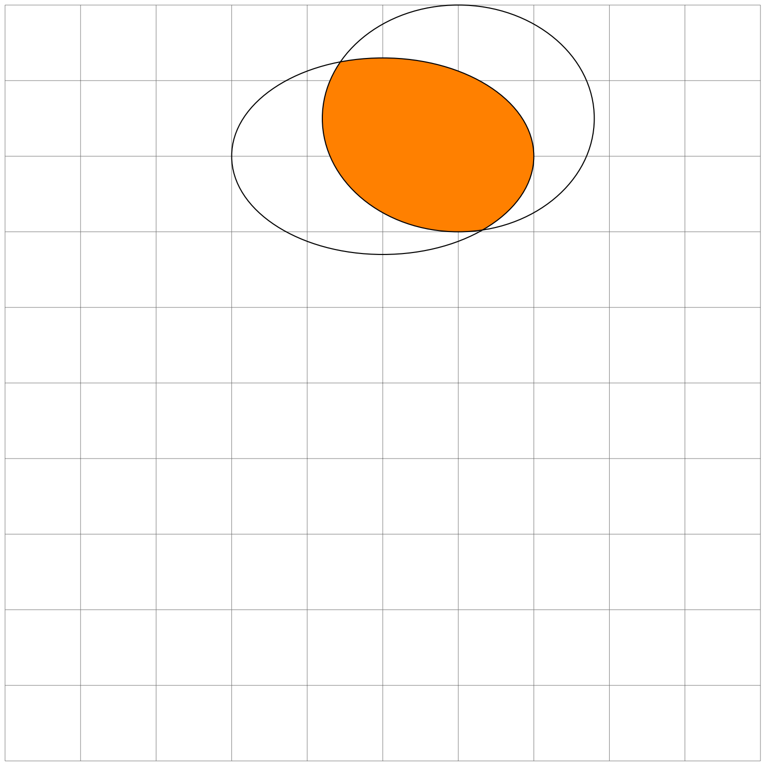
Proof. Suppose not, then for each $z \in \mathbb{T}$, $\exists D_z$ and analytic g_z on D_z with $f = g_z$ on $D_z \cap \mathbb{D}$. Given $z_1 \neq z_2$ on $\mathbb{T}$ with $D_{z_!} \cap D_{z_2} \neq \varnothing$, then since these are disks centered at points of $\mathbb{T}$, $D_{z_1} \cap D_{z_2} \cap \mathbb{D} \neq \varnothing$, so $f = g_{z_1} = g_{z_2}$ on $\mathbb{D} \cap D_{z_1} \cap D_{z_2}$. By the identity principle $g_{z_1} = g_{z_2}$ on $D_{z_1} \cap D_{z_2}$, as $\mathbb{T}$ is compact, it many be covered by finitely many of these disks, so f can be extended to the union $\mathbb{D}$ with this finite collection of D_z, and that union contains $\mathbb{D}(0, 1+\delta)$ for some $\delta > 0$, contradiction. □

Definition. We say $\mathbb{T}$ is a **natural boundary** of f if every point $z \in \mathbb{T}$ is singular.

Example. $f(z) \coloneqq \sum_{k=0}^{\infty} z^{k!}$. Let ω be a root of unity, $\omega = e^{2\pi i p/q}$. Then

$$\begin{aligned} f(re^{2\pi i p/q}) &= \sum_{k=0}^{\infty} r^{k!} (e^{2\pi i p/q})^{k!} \\ &= \underbrace{\sum_{k=0}^{q-1} r^{k!} (e^{2\pi i p/q})^{k!}}_{\text{bounded as } r \to 1} + \underbrace{\sum_{k=q}^{\infty} r^{k!}}_{\text{approaches } \infty \text{ as } r \to 1} \end{aligned}$$

But



Definition. A **Riemann surface** is a connected, Hausdorff topological space R, together with a collection of open subsets $\mathcal{U}_\alpha \subset \mathbb{R}$ and homeomorphisms $\phi_\alpha : \mathcal{U}_\alpha \to D_\alpha$, where D_α is an open subset of $\mathbb{C}$ satisfying

1. $\bigcup_\alpha \mathcal{U}_\alpha = R$
2. for any α, β with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$, then the map
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to D_\alpha$$
is analytic as a map of open sets in $\mathbb{C}$.

The information $(\mathcal{U}_\alpha, \phi_\alpha)$ is called a **chart** of R. The compositions $\phi_\alpha \circ \phi_\beta^{-1}$ are the **transition functions** of R. The collection of charts $\{U_\alpha, \phi_\alpha\}$ is the **atlas** of R.

Remark.

1. As transitions are invertible with analytic inverses, they are conformal equivalences.
2. R is path connected as it is connected and locally path connected

   Fix $z_0 \in R$. Define $U = \{\, z \in R \mid \exists \text{path from } z_0 \text{ to } z \,\}$. U is open, as it its complement. As R is connected, U^c is empty.
3. Occasionally, 'connected' will not be included in the definition, but this can be annoying without limiting the number of connected components.

Example. $\mathbb{C}$ as a topological space, for instance $(\mathbb{C}, \phi(z) = z)$, $(\mathbb{C}, \phi(z) = z+1)$, $(\mathbb{C}, \phi(z) = \bar{z})$.

Definition. Let R be a Riemann surface. Two atlases $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$, $\{(\mathcal{U}_\beta, \psi_\beta)\}$ are **equivalent** if their refinement $\{(\mathcal{U}_\beta, \phi_\beta)\} \cup \{(\mathcal{U}_\beta, \psi_\beta)\}$ is an atlas. In other words,

$$\psi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha \cap V_\beta) \to \psi_\beta(\mathcal{U}_\alpha \cap V_\beta) \tag{6}$$

is analytic (and similarly for $\phi_\alpha \circ \psi_\beta^{-1}$).

This defines an equivalence relation, as we will soon see.

Example. $(\mathbb{C}, \phi(z) = z)$, $(\mathbb{C}, \phi(z) = z+1)$: the refinement has transition maps: identity, $z \mapsto z+1$, $z \mapsto z-1$, so these are equivalent atlases. Non-example: $(\mathbb{C}, \phi(z) = z)$, $(\mathbb{C}, \phi(z) = \bar{z})$, the transition functions of the refinement are identity and complex conjugation, so these are *not* equivalent atlases.

Definition. We call an equivalence class of atlases a **conformal structure** on R.

Remark.
1. We could have defined a Riemann surface as a connected Hausdorff topological space which admits a conformal structure.
2. If $S \subset R$ is open, connected then R a Riemann surface implies S a Riemann surface via restriction of charts.

Definition. Let R, S be Riemann surfaces with atlases $\{(\mathcal{U}_\alpha, \phi_\alpha)\}, \{(\mathcal{U}_\beta, \psi_\beta)\}$ respectively, We say a map $f : R \to S$ is analytic if it is continuous and if for any $\mathcal{U}_\alpha cap f^{-1}(V\beta) \neq \varnothing$, the map

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(\mathcal{U}_\alpha \cap f^{-1}(V_\beta))) \tag{7}$$

is analytic.

Definition. An analytic map $f : R \to S$ of Riemann surfaces is a **conformal equivalence** (biholomorphism or analytic isomorphism) if $\exists$analytic inverse $g : S \to R$ of f.

Example. We saw that $(\mathbb{C}, \phi(z) = z)$ and $(\mathbb{C}, \psi(z) = \bar{z})$ are inequivalent atlases, ie, define different conformal structures on $\mathcal{C}$. However, $f : (\mathbb{C}, \phi(z) = z) \to (\mathbb{C}, \psi(z) = \bar{z})$ given by $z \mapsto \bar{z}$ is a conformal equivalence of these two Riemann surfaces as the functions $(\psi \circ f \circ \phi^{-1})(z) = \bar{\bar{z}} = z$ are conformal isomorphisms.

Lemma. The composition of analytic maps $f : R \to S$ and $g : S \to T$ of Riemann surfaces is analytic.

We need for any γ, α with $\mathcal{U}_\alpha \cap h^{-1}(W_\gamma)$ that $\theta_\gamma \circ h \circ \phi_\alpha^{-1}$ is analytic on $\phi_\alpha(\mathcal{U}_\alpha \cap h^{-1}(W_\gamma))$. In this set, analyticity is local, so it suffices to show that for any β with $f^{-1}(V_\beta) \cap \mathcal{U}_\alpha \cap h^{-1}(W_\gamma) \neq \varnothing$, we have $\theta_\gamma \circ h \circ \phi_\alpha^{-1} |_{\phi_\alpha(\mathcal{U}_\alpha \cap h^{-1}(W_\gamma))}$ analytic on $\phi_\alpha()$.

Corollary. Equivalence of atlases is an equivalence relation.

Proof. Atlases a_1 and a_2 are equivalent by definition if the identity map $(R, a_1) \xrightarrow{\text{id}} (R, a_2)$ is analytic. Reflexivity and symmetry are immediate, and transitivity follows from the previous lemma. □

Proposition. Let R be a Riemann Surface and $\pi : \widetilde{R} \to R$ a covering map. Then $\exists!$conformal structure on $\widetilde{R}$ for which π is analytic.

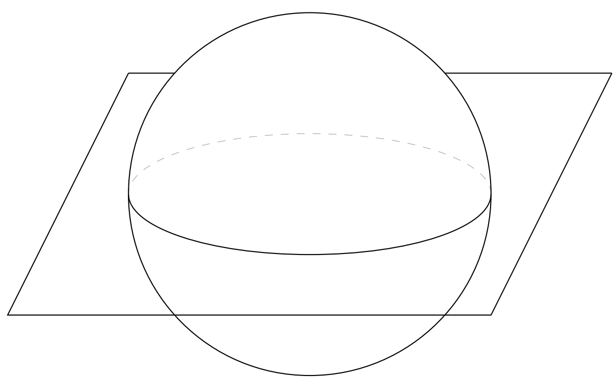
Proof. Given $z \in \widetilde{R}$, $\exists$neighbourhood $\widetilde{N_z}$ of z such that π is homeomorphic on $\widetilde{N_z}$. $\pi(z) \in U$ for some chart neighbourhood $\mathcal{U}$; $\pi(\widetilde{N}) \cap \mathcal{U}$ is open in R, so define $V := \pi^{-1}(U) \cap \widetilde{N}$, an open set in $\widetilde{R}$. Define $\psi : V \to \mathbb{C}$ to be $\phi \circ \pi$, we obtain an atlas on $\widetilde{R}$, π is analytic as the composition functions are just the transitions of the atlas on R. So $\exists$conformal structure on $\widetilde{R}$ for which π is analytic, call this atlas a. Suppose $\exists a^*$ on $\widetilde{R}$ for which π is analytic; we will show these atlases are equivalent.

Say (W, θ) is a chart of a^* and $z \in W$, find (V, ψ) (and $\widetilde{N}$ and $\mathcal{U}$) as above. We assumed that π is analytic for this atlas. As π is analytic, $\phi \circ \pi \circ \theta^{-1}$ is analytic; it is also a homeomorphism, so its inverse is also analytic. So both types of transitions are analytic and the atlases are equivalent. $\square$

$$
\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{C} \arrow[dl, "\exp"] \\
\mathbb{C}^* &
\end{tikzcd}
$$

As a corollary, we see that the gluing surface R we constructed for log gives a conformal structure on R for which π is analytic. Note f is continuous by the open mapping theorem. It follows that f is analytic (looking locally) but f is a bijection. So f has an analytic inverse, because the inverse is continuous. So R is conformally equivalent to $\mathbb{C}$.

Example. The Riemann sphere. Let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, equipped with the topology whose open sets are of the form: open subset of $\mathbb{C}$ or $\{\infty\} \cup \mathbb{C} \setminus K$ where $K \subseteq \mathbb{C}$ is compact. With this topology, $\mathbb{C}_\infty$ is homeomorphic to S^2 via stereographic projection and $\pi((0,0,1)) = \infty$. C_∞ is connected, Hausdorff and compact. Define the atlas via two charts: $(\mathbb{C} : \phi(z) = z)$ and $(\mathbb{C}_\infty \setminus \{0\}, \phi(z) = \frac{1}{z})$. The transitions $\frac{1}{z}$ are analytic on $\mathbb{C}^*$.



Definition. We define the Riemann Sphere as the above surface.

Definition. If R is a Riemann surface, an analytic map $R \to \mathbb{C}$ is an analytic function: in terms of charts, if $(\mathcal{U}, \phi)$ is a chart for R, this required $f \circ \phi^{-1}$ is analytic.

Theorem (Inverse function theorem)**.** Given g analytic on an open set $V \subseteq \mathbb{C}$, and $a \in V$ with $g'(a) \neq 0$, $\exists$neighbourhood N of a, $N \subseteq V$ such that $g|_N : N \to g(N)$ is a conformal equivalence.

Proof. Replace g with $g(z) - g(a)$ to assume without loss of generality that $g(a) = 0$. Take a disk $\mathbb{D}(a, \epsilon)$ with
$mathbbD(a, \epsilon) \subseteq V$ on which a is the only zero of g, assume also $g'(z) \neq 0$ on $\overline{\mathbb{D}(a, \epsilon)}$. The argument principle tells us: if γ is positively oriented disk boundary, then $n(g \circ \gamma, 0) =$ number of zeros of g in the disk $= 1$. $g \circ \gamma$ is compact so closed, so choose a disk Δ centred at 0 with $\Delta \cap (g \circ \gamma) = \varnothing$.

For all $w \in \Delta$, $n(g \circ \gamma, w) = 1$ so let $N = g^{-1}(\Delta)$. Then N is an open neighbourhood of a, and $g|_N : N \to \Delta$ is a bijection. The inverse of g is continuous by the open mapping theorem, and therefore analytic. So, N is as needed. □

Suppose now that $a \in \mathcal{U} \subseteq \mathbb{C}$, and $g \not\equiv 0$ analytic function on $\mathcal{U}$ domain with $g(a) = 0$. We may write $g(z) = (z - a)^r h(z)$ where $h(a) \neq 0$ and h is analytic on $\mathcal{U}$.

Choose $a \in \mathbb{R}$ and a disk $a \in D \subseteq \mathcal{U}$ such that $h(D)$ is disjoint from the ray of angle α (can do by continuity and $h(a) \neq 0$) so we can define an analytic rth root of h by using this ray as a branch cut for log. So we can write

$$g(z) = f(z)^r \quad \text{on } D, \quad \text{where } f(z) = (z - a)l(z) = f(z) = (z - a)h(z)^{\frac{1}{r}} \tag{8}$$

and f has a simple zero at a. Therefore $\exists$open neighbourhood of a on which $f(z)$ is a conformal equivalence by the inverse function theorem.

If $f : R \to \mathbb{C}$ is an analytic function of a Riemann Surface R, locally around $p_0 \in R$, we may find a chart $\phi : \mathcal{U} \to \mathbb{C}$. Without loss of generality $f(p_0) = 0$ and write $a = \phi(p_0)$. On a neighbourhood of a, we have $f \circ \phi^{-1}(z) = g(z)^r$, for some local conformal equivalence g and some r. Define on that neighbourhood a chart with $\psi = g \circ \phi$, we see that

So locally, f is a powering map (for this choice of chart).

Given $\tau_1, \tau_2 \in \mathbb{C}$ nonzero with $\tau_2/\tau_1 \notin \mathbb{R}$ (that is, linearly independent over $\mathbb{R}$)., define $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2 = \{\, m\tau_1 + n\tau_2 \mid m, n \in \mathbb{Z} \,\}$.

Λ is an additive subgroup of $\mathbb{C}$, so form the quotient group $T \coloneqq \mathbb{C}/\Lambda$.

Topology Define $\pi : \mathbb{C} \to T$ to be the projection map and equip T with the quotient topology (ie $\mathcal{U} \subset T$ is open iff $\pi^{-1}(\mathcal{U})$ is open in $\mathbb{C}$), by construction π is continuous. Note then T is connected.

π is open Given $V \subseteq \mathbb{C}$ open, need to show $\pi(V)$ open, ie $\pi^{-1}(\pi(V))$ is open in $\mathbb{C}$. Since $\pi^{-1}(\pi(V)) \coloneqq \bigcup_{\omega \in \Lambda}(w + V)$, a union of open sets, it is open as claimed.

Λ is discrete (ie consists of isolated points). Since Λ is closed under subtraction, if $\exists$limit point of Λ then 0 is a limit point of Λ, ie $\forall k \in \mathbb{N}$, $\exists m_k, n_k \in \mathbb{Z}$ such that $|m_k\tau_1 + n_k\tau_2| < \frac{1}{k}$ ie $\left|\frac{m_k}{n_k} + \frac{\tau_2}{\tau_1}\right| < \frac{1}{kn_k|\tau_1|}$, which $\to 0$ as $k \to \infty$, So, $\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{\tau_2}{\tau_1} \implies -\frac{\tau_2}{\tau_1} \in \mathbb{R}$, a contradiction. So Λ is discrete.

π is a regular covering map given any $z \in \mathbb{C}$, $\exists$disk D_z about z on which π is injective, by discreteness of Λ. So $\pi|_{D_z} : D_z \to \pi(D_z)$ is continuous, open, injective, surjective, hence a homeomorphism. So, π is a covering map. π is regular as for all sufficiently small disk D_z (on which π is injective), $\pi^{-1}(\pi(D_z))$ is a disjoint union of translates (by lattice elements) on D_z.

T admits a conformal structure Given $a \in T$, find $w_a \in \mathbb{C}$ with $\pi(w_a) = a$, and an open neighbourhood N_a of w_a on which π is a homeomorphism. Define $\mathcal{U}_a := \pi(N_a)$ and $\phi_a := (\pi|_{N_a})^{-1}$. Suppose $\mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing$. Given $z \in \phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$, $\phi_b \circ \phi_a^{-1}(z) = z + \omega_z$ for some $\omega_z \in \Lambda$, by definition of ϕ_a, ϕ_b. Obtain a map $z \mapsto \omega_z$ from $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b) \to \Lambda$, which is continuous. Since $z \mapsto \omega_z$ is continuous with discrete image, it is locally constant. So $\exists$neighbourhood of z on which $\phi_b \circ \phi_a^{-1}$ is translation by some element of Λ and hence analytic.

T **is compact** For any $z \in \mathbb{C}$, define $P_z := \{ z + r\tau_1 + s\tau_2 \mid r, s \in [0,1] \}$. Then $\pi(P_z) = T$ as P_z is compact, so is T. Note topologically this is a torus.

Definition (Complex torus)**.** Any Riemann surface constructed in this way is a **complex torus**.

We will show there are no non-constant analytic functions analytic functions on a complex torus.

Theorem (Open mapping theorem for Riemann Surfaces)**.** Suppose $f : R \to S$ analytic non-constant map of Riemann Surfaces, then f is open.

Proof. Suppose W is open in R and $z \in W$, we want to find an open neighbourhood of $f(z)$ contained in $f(W)$. Find $(\mathcal{U}, \phi)$ a chart of R with $z \in \mathcal{U}$, and (V, ψ) a chart of S with $f(z) \in V$. Choose an open disk D with center $\phi(z)$ such that $\phi^{-1}(D) \subseteq \mathcal{U} \cap f^{-1}(V) \cap W$. $\psi \circ f \circ \phi^{-1}$ is analytic and non-constant by the identity principle for Riemann Surfaces (proved on an example sheet).

So by the open mapping theorem for $\mathbb{C}$, $(\psi \circ f)(\phi^{-1}(D))$ is open in $\psi(f(\mathcal{U} \cap f^{-1}(V) \cap W))$. As ψ is homeomorphic, $f \circ \phi^{-1}(D)$ is open in $f(\mathcal{U} \cap f^{-1}(V) \cap W) \subseteq f(W)$. □

Corollary. If $f : R \to S$ is analytic, non-constant map of Riemann surfaces, and R is compact, then $f(R) = S$ and S is compact.

Proof. $f(R)$ is open by the open mapping theorem, it is also compact and so closed as S is Hausdorff. Since S is connected and $f(R) \neq \varnothing$, $f(R) = S$. By continuity, S is also compact. □

Immediate consequences: there are no non-constant analytic functions on complex tori or the Riemann sphere, as these are compact but $\mathbb{C}$ is not.

Remark. If R is a Riemann surface and $a \in R$ and $f : R \setminus \{a\} \to \mathbb{C}$ is analytic, then looking locally, we see that f has a removable singularity at $a \iff f$ is bounded on a punctured neighbourhood of a. If $R = \mathbb{C}_\infty$ and $a = \infty$, then an analytic function extends to $\infty \iff$ is is bounded on a neighbourhood of ∞, so by Liouville's Theorem, it is constant.

Recall $u : D \to \mathbb{R}$ on domain $D \subseteq \mathbb{C}$ is harmonic if $u \in C^2(D)$ and $\Delta u = u_{xx} + u_{yy} = 0$. By Cauchy-Riemann, if $f = u + iv$ is analytic on D, then u and v are harmonic and if D is a disk the converse holds.

Remark. If $g : \mathcal{U} \to V$ is analytic and $u : V \to \mathbb{R}$ is harmonic, then $u \circ g$ is harmonic, since on any disk in V we can write $u = \operatorname{Re} f$ where f is analytic and $u \circ g$ is the real part of $f \circ g$ on that disk.

Definition. A harmonic function on a Riemann Surface R is a continuous function $h : R \to \mathbb{R}$ such that for any chart $(\mathcal{U}, \phi)$ of R, the function $h \circ \phi^{-1}$ is harmonic on $\phi(\mathcal{U})$.

By previous remark, harmonicity if consistent independent of choice of chart, ie sufficient to check harmonicity on a neighbourhood of each point of R.

Proposition. A non-constant harmonic function h on a RS R is an open map, therefore if R is compact, all harmonic functions on R are constant.