Part III – Advanced Probability (Incomplete)

Based on lectures by Dr. S. Andres
Notes taken by Bhavik Mehta

Michaelmas 2019

Contents

Lecture 1

1 Conditional Expectations

Lecture 2 Take a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$, meaning $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a probability measure, with $\mathbb{P}(\Omega) = 1$. We use the term '**almost surely**' (or a.s.) to mean almost everywhere.

Take X to be a random variable, i.e. $X : \Omega \to \mathbb{R}$ which is $\mathcal{F}$-measurable and write

$$\mathbb{E}[X] = \int X \, d\mathbb{P}$$

for the **expectation** of X. We write also

$$\mathbb{E}[X \mathbb{1}_A] = \int_A X \, d\mathbb{P}$$

for $A \in \mathcal{F}$.

Definition 1.1. Let $B \in \mathcal{F}$ with $\mathbb{P}[B] > 0$. We know

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]},$$

the **conditional probability** of A given B. Similarly,

$$\mathbb{E}[X|B] = \frac{\mathbb{E}[X \mathbb{1}_B]}{\mathbb{P}[B]}$$

the **conditional expectation** of X given B.

There is a significant restriction to this definition: that $\mathbb{P}[B] > 0$. By the end of this lecture, we will generalise this definition to any σ-algebra of events, rather than just one.

Aim. Improve the prediction of X if additional information (given as a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$) is available.

1.1 Discrete case

Take $B_1, B_2, \ldots \in \mathcal{F}$ a disjoint decomposition of Ω. We take

$$\mathcal{G} = \sigma(B_1, B_2, \ldots) = \left\{ \bigcup_{i \in J} B_i : J \subseteq \mathbb{N} \right\} \subseteq \mathcal{F}.$$

That is, the 'extra information' of $\mathcal{G}$ is that we know which of the disjoint events B_i we fall into.

Then,

$$\mathbb{E}[X|\mathcal{G}](\omega) \coloneqq \sum_{i : \mathbb{P}[B_i] > 0} \mathbb{E}[X|B_i] \mathbb{1}_{B_i}(\omega)$$

is the conditional expectation of X given $\mathcal{G}$.

It is easy to see that $\mathbb{E}[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable, and

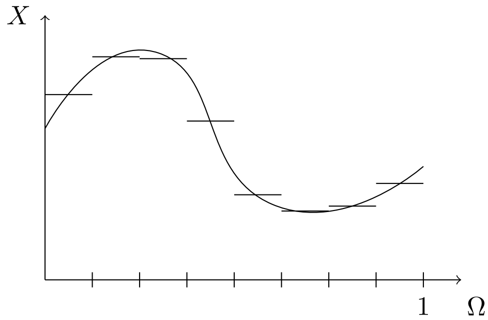
$$\mathbb{E}[\mathbb{1}_A X] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[X|\mathcal{G}]] \quad \forall A \in \mathcal{G}.$$

Example.

(i) Take now $\Omega = (0, 1]$, and $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathbb{P}$ to be Lebesgue measure. Use X as shown below, and use

$$\mathcal{G} = \sigma\left(\left(\tfrac{k}{m}, \tfrac{k+1}{m}\right] : k = 0, \dots, m-1\right).$$

In the picture, we take $m = 8$, and the conditional expectation $\mathbb{E}(X|\mathcal{G})$ is shown.



(ii) Take a random variable $Z : \Omega \to \{z_1, z_2, \dots\} \subseteq \mathbb{R}$, and use $\mathcal{G} = \sigma(Z) = \sigma(\{Z = z_i\} : i = 1, 2, \dots)$. Then,

$$\begin{aligned}\mathbb{E}[X|Z] &\coloneqq \mathbb{E}[X|\sigma(Z)] \\ &= \sum_{i:\mathbb{P}[Z=z_i]>0} \mathbb{E}[X|Z = z_i]\mathbb{1}_{\{Z=z_i\}}.\end{aligned}$$

This is not satisfactory quite yet: if Z has an absolutely continuous distribution (eg $\mathcal{N}(0,1)$), i.e. $\mathcal{P}[Z = z] = 0$ for every z, then $\mathbb{E}[X|Z]$ is not defined yet!

1.2 General case

Definition 1.2. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra. A random variable Y is called (a version of) the **conditional expectation** of X given $\mathcal{G}$ if

(i) Y is $\mathcal{G}$-measurable

(ii) $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$ for all $A \in \mathcal{G}$.

We notate $Y = \mathbb{E}[X|\mathcal{G}]$.

Remark 1.3.

(a) We took $X \in L^1$, but this can be changed to $X \geq 0$ throughout.

(b) If $\mathcal{G} = \sigma(\mathcal{C})$ for some $\mathcal{C} \subseteq \mathcal{F}$, it suffices to check (ii) for all $A \in \mathcal{C}$.

(c) If $\mathcal{G} = \sigma(Z)$ where Z is a random variable, we write $\mathbb{E}[X|Z] \coloneqq \mathbb{E}[X|\sigma(Z)]$. This is $\sigma(Z)$ measurable by (i), so it's of the form $f(Z)$ for some function f. It's then common to define $\mathbb{E}[X|Z = z] = f(z)$.

Theorem 1.4 (Existence and uniqueness)**.** Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra.

(i) $E[X|\mathcal{G}]$ exists

(ii) Any two versions of $\mathbb{E}[X|\mathcal{G}]$ coincide $\mathbb{P}$-almost surely.

Proof.

(ii) Uniqueness. Let Y be as in Definition 1.2, and let Y' satisfy Definition 1.2(i) and (ii) for some $X' \in L^1$ with $X \leq X'$ almost surely. Let $Z = (Y - Y')\mathbb{1}_A$ with $A \coloneqq \{Y \geq Y'\} \in \mathcal{G}$.

$$\mathbb{E}[Y\mathbb{1}_A] = E[X\mathbb{1}_A] \leq \mathbb{E}[X'\mathbb{1}_A] = \mathbb{E}[Y'\mathbb{1}_A] < \infty$$

and note that $\mathbb{E}[X'\mathbb{1}_A] < \infty$, so $\mathbb{E}[Y'\mathbb{1}_A] < \infty$.

By definition of Z, this means $\mathbb{E}[Z] \leq 0$. But $Z \geq 0$ almost surely, so $Z = 0$ a.s. therefore $Y \leq Y'$ a.s. (This shows monotonicity of conditional expectation.) If $X = X'$, we can run the same argument to show that $Y = Y'$ almost surely (using $A = \{Y > Y'\}$ and $A = \{Y < Y'\}$, we see both sets are measure zero).

(i) Existence. Step 1: Assume first $X \in L^2(\mathcal{F})$. Since $L^2(\mathcal{G})$ is a complete subspace of $L^2(\mathcal{F})$, X has an orthogonal projection Y on $L^2(\mathcal{G})$, i.e. there is $Y \in L^2(\mathcal{G})$ such that $\mathbb{E}[(X - Y)Z] = 0$ for every $Z \in L^2(\mathcal{G})$. Choosing $Z = \mathbb{1}_A$ for $A \in \mathcal{G}$ we get $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$ so Y satisfies the conditions of Definition 1.2.

Step 2: Assume $X \geq 0$. Then $X_n = X \wedge n \in L^2(\mathcal{F})$ and $0 \leq X_n \nearrow X$ as $n \to \infty$. By Step 1, we can find $Y_n \in L^2(\mathcal{G})$ such that $\mathbb{E}[X_n\mathbb{1}_A] = \mathbb{E}[Y_n\mathbb{1}_A]$ for all $A \in \mathcal{G}$ and $0 \leq Y_n \leq Y_{n+1}$ almost surely (from the proof of (ii)). Let $Y_\infty = \lim_n Y_n\mathbb{1}_{\Omega_0}$ with

$$\Omega_0 = \{\omega \in \Omega : 0 \leq Y_n(\omega) \leq Y_{n+1}(\omega)\ \forall n\}.$$

Then Y_∞ is a non-negative random variable, is $\mathcal{G}$-measurable as a limit of $\mathcal{G}$-measurable r.v.s and by monotone convergence $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y_\infty\mathbb{1}_A]$ for every $A \in \mathcal{G}$. Taking $A = \Omega$, $\mathbb{E}[Y_\infty] = \mathbb{E}[X] < \infty$, since $X \in L_1$. So $Y_\infty < \infty$ almost surely and $Y \coloneqq Y_\infty\mathbb{1}_{\{Y_\infty < \infty\}}$ satisfies Definition 1.2(i) and (ii).

Step 3: For general $X \in L^1$, apply Step 2 on X^+ and X^- to obtain Y^+ and Y^-. Then $Y = Y^+ - Y^-$ satisfies the conditions of Definition 1.2. $\square$

Index