Part III – Combinatorics (Ongoing course, rough)

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0 Introduction

Lecture 1 Combinatorics tends to have problems which are easy to state and hard to prove. One of the reasons for this is that it is often unclear where to start - for instance in linear algebra we can often start by picking a basis. Proofs tend to have the property that they seem to take thousands of years to come up with, and a single line to write down. In this course, we learn some techniques which make problems sounding very hard become very easy.

We start with set systems, which builds on the idea of subsets and containment. Next, we study isoperimetric inequalities. In the continuous case, in the plane, a typical problem is to find the maximum area one can enclose with a fixed perimeter, which is solved by a circle. Similarly, a soap bubble will minimise its surface area for a fixed volume. Here, in the discrete case, we will try to understand 'how tightly' we can pack subsets. Finally, we look at continuous projections. For instance, given a subset of space, suppose we know the z-coordinate of all points is between 0 and 1, and the projection to the xy-plane has area A, we know the total volume is bounded by A. We generalise this result into higher dimensions and the box result, which has applications in combinatorics.

While all examinable proofs will be included in lectures, relevant books for this course are:

1. *Combinatorics*, Bollobás, C.U.P., 1996. This matches chapter 1 excellently and parts of chapter 2. It is a gentle read and includes other developments in combinatorics.

2. *Combinatorics of finite sets*, Anderson, O.U.P., 1987. It is a simple and clear study on chapter 1.

1 Set systems

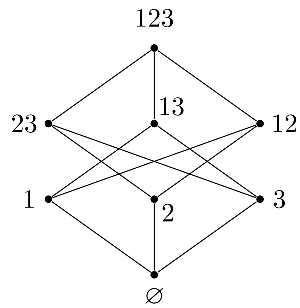
Definition (Set system)**.** Let X be a set. A **set system** on X (or **family of subsets** of X) is a family $\mathcal{A} \subseteq \mathcal{P}(X)$.

For instance, we write $X^{(r)} = \{ A \subseteq X \mid |A| = r \}$, so $X^{(r)}$ is a set system on X. Unless otherwise stated,

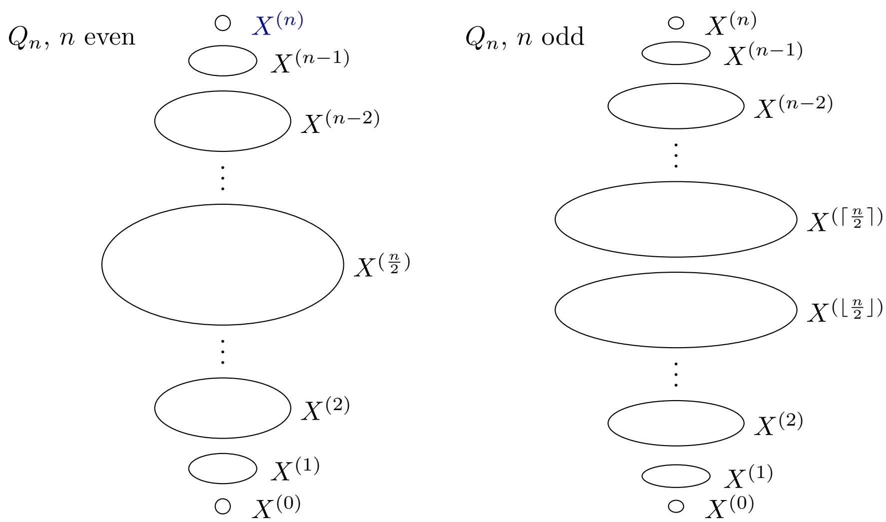
$$X = [n] \coloneqq \{1, 2, \ldots, n\},$$

e.g. $|X^{(r)}| = \binom{n}{r}$. Thus $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$, so $|[4]^{(2)}| = 6$.

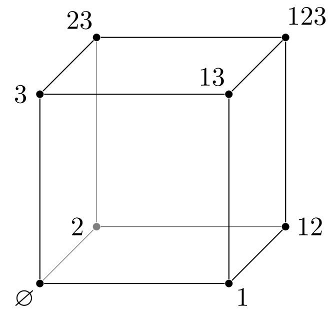
Often, we make $\mathcal{P}(X)$ into a graph, called Q_n, by joining A to B if $|A \triangle B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$ (or vice versa). For instance, here is a picture of Q_3:



As we know, the picture gets 'thicker' in the middle. But, for odd n, is it not clear where exactly the middle is, so in the odd case we have two equally sized large blobs in the middle.



If we identify a set $A \subseteq X$ with a 0-1 sequence of length n, e.g. $134 \longleftrightarrow 1011000\ldots0$, via $A \longleftrightarrow 1_A$ or χ_A, the characteristic function, then Q_3 looks like



Definition (Hypercube)**.** Q_n is often called the **hypercube** or **discrete cube** or n**-cube**.

It is important to keep *both* these pictures in mind: for induction the cube image is more instructive, but when thinking about layers the earlier image is more helpful.

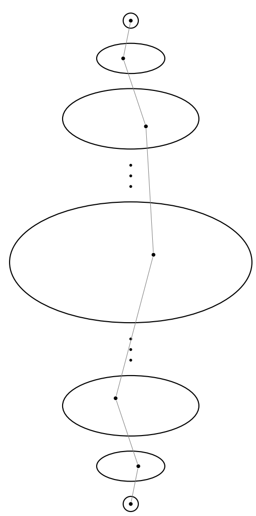
1.1 Chains and antichains

Definition (Chain)**.** A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{A}$, we have $A \subseteq B$ or $B \subseteq A$.

Definition (Antichain)**.** A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an **antichain** if $\forall A, B \in \mathcal{A}$ with $A \neq B \Rightarrow A \nsubseteq B$.

Example. For instance, $\{12, 125, 123589\}$ is an chain, and $\{1, 467, 2456\}$ is an antichain.

In this course, we ask questions like, how large can a chain be? We can achieve $|\mathcal{A}| = n+1$, e.g. $\{\varnothing, 1, 12, 123, \ldots, [n]\}$. It is easy to visualise this by picking 'one per level':



We cannot exceed $n+1$, since a chain must meet each 'level' $X^{(r)}$ $(0 \leq r \leq n)$ in at most one place.

How large can an antichain be? We could achieve $\mathcal{A} = n$, e.g. $\mathcal{A} = \{1, 2, 3, \ldots, n\}$ (and this is maximal). Indeed, we could take $\mathcal{A} = X^{(r)}$ for any r, so we can achieve $|\mathcal{A}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Can we beat this? Aim: Prove this is the winner.

Lecture 2 Inspired by 'each chain meets each level $X^{(r)}$ in at most one place' for chains, we try to decompose Q_n into chains to find large antichains.

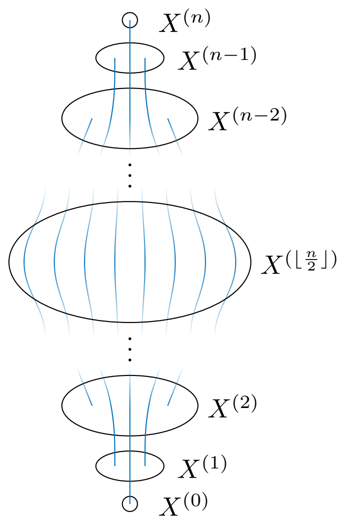
Theorem 1 (Sperner's Lemma)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then $|\mathcal{A}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. It is sufficient to partition $\mathcal{P}(X)$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains. For this, it is sufficient to show

(i) $\forall r < \frac{n}{2}$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$

(ii) $\forall r > \frac{n}{2}$, there is a matching from $X^{(r)}$ to $X^{(r-1)}$.

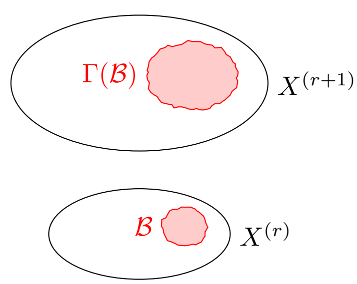
Then put these matchings together to form chains, each passing through $X^{(\lfloor \frac{n}{2} \rfloor)}$, so there are $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of them. (Recall we have a natural graph structure on Q_n)



By taking complements, it is sufficient to prove (i).

Consider the subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. It is bipartite. For any $\mathcal{B} \subseteq X^{(r)}$, we have that

- The number of edges from $\mathcal{B}$ to $\Gamma(\mathcal{B})$ is $|\mathcal{B}|(n-r)$ (each point in $X^{(r)}$ has degree $n-r$).
- The number of edges from $\mathcal{B}$ to $\Gamma(\mathcal{B})$ is at most $|\Gamma(\mathcal{B})|(r+1)$ (each point in $X^{(r+1)}$ has degree $r+1$).



Thus

$$\begin{aligned}|\Gamma(\mathcal{B})| &\geq |\mathcal{B}|\frac{n-r}{r+1} \\ &\geq |\mathcal{B}|\end{aligned}$$

as $r < \frac{n}{2}$. Hence by Hall's theorem, there is a matching. $\square$

Remark. Recall $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ was achievable, for example $\mathcal{A} = X^{(\lfloor \frac{n}{2} \rfloor)}$. This proof says nothing about extremal cases - which antichains have size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$?

Aim: For $\mathcal{A}$ an antichain,

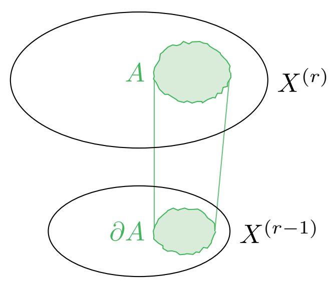
$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

This trivially implies Sperner's Lemma.

Definition (Shadow)**.** Let $\mathcal{A} \subseteq X^{(r)}$, for some $1 \leq r \leq n$. The **shadow** or **corner shadow** of $\mathcal{A}$ is

$$\partial\mathcal{A} \equiv \partial^{-}\mathcal{A} \coloneqq \{\, A - \{i\} \mid A \in \mathcal{A},\ i \in A \,\}$$

so $\partial\mathcal{A} \subseteq X^{(r-1)}$.



For example, if $\mathcal{A} = \{123, 124, 134, 135\} \subseteq X^{(3)}$, then

$$\partial\mathcal{A} = \{12, 13, 23, 24, 34, 15, 35\} \subseteq X^{(2)}.$$

Lemma 2 (Local LYM)**.** Let $\mathcal{A} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

'The fraction of the layer occupied increases when we take the shadow'.

Proof. Counting from above, there are $r|\mathcal{A}|$ edges $\mathcal{A}$ to $\partial\mathcal{A}$. Counting from below, the number of edges $\mathcal{A}$ to $\partial\mathcal{A}$ is at most $(n-r+1)|\partial\mathcal{A}|$, so

$$\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1}.$$

But

$$\frac{\binom{n}{r-1}}{\binom{n}{r}} = \frac{r}{n-r+1}.$$

$\square$

When do we get equality in Local LYM? We'd need $(A - \{i\}) \cup \{j\} \in \mathcal{A}\ \ \forall A \in \mathcal{A},\ i \in A,\ j \notin A$. Hence $\mathcal{A} = X^r$ or $\varnothing$.

The LYM in Local LYM stands for 'Lubell–Yamamoto–Meshalkin'. We can use Local LYM to prove:

Theorem 3 (LYM)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \le 1.$$

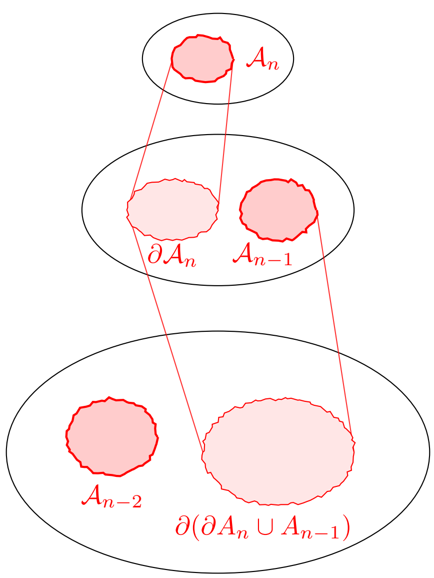
Proof 1: 'Bubble down with Local LYM'. Write $\mathcal{A}_r = \mathcal{A} \cap X^{(r)}$. We have $\frac{|\mathcal{A}_n|}{\binom{n}{n}} \le 1$. Also, $\partial \mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are distinct since $\mathcal{A}$ was an antichain, so

$$\begin{aligned} \frac{|\partial A_n|}{\binom{n}{n-1}} + \frac{|A_{n-1}|}{\binom{n}{n-1}} &= \frac{|\partial A_n \cup A_{n-1}|}{\binom{n}{n-1}} \\ \implies \frac{|A_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} &\le 1 \end{aligned}$$

by Local LYM.

Also, $\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})$ is disjoint from $\mathcal{A}_{n-2}$, again since $\mathcal{A}$ is an antichain so

$$\begin{aligned} \frac{|\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1, \\ \implies \frac{|(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-1}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1, \\ \implies \frac{\mathcal{A}_n}{\binom{n}{n}} + \frac{\mathcal{A}_{n-1}}{\binom{n}{n-1}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1. \end{aligned}$$



Keep going, we obtain

$$\frac{\mathcal{A}_n}{\binom{n}{n}} + \frac{\mathcal{A}_{n-1}}{\binom{n}{n-1}} + \cdots + \frac{\mathcal{A}_0}{\binom{n}{0}} \leq 1. \qquad \square$$

When do we get equality in LYM? We must have had equality in each use of Local LYM. So the first (greatest) r with $\mathcal{A}_r \neq \varnothing$ must have $\mathcal{A}_r = X^{(r)}$ thus $\mathcal{A} = X^{(r)}$. Hence equality in Sperner's Lemma $\iff$ $\mathcal{A} = X^{(\frac{n}{2})}$ for n even and $\mathcal{A} = X^{(\lfloor \frac{n}{2} \rfloor)}$ or $X^{(\lceil \frac{n}{2} \rceil)}$ for n odd.

Lecture 3

Proof 2. Choose uniformly at random a maximal chain $\mathcal{C}$ (i.e. $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_n$) with $|\mathcal{C}_i| = i \ \forall i$. For a given r-set A,

$$\begin{aligned} \mathbb{P}(A \in \mathcal{C}) &= \frac{1}{\binom{n}{r}} && \text{(all r-sets are equally likely)} \\ \mathbb{P}(\mathcal{A}_r \text{ meets } \mathcal{C}) &= \frac{|\mathcal{A}_r|}{\binom{n}{r}} && \text{(events are disjoint)} \\ \implies \mathbb{P}(\mathcal{A} \text{ meets } \mathcal{C}) &= \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}. \\ \implies \sum_{i=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} &\leq 1. \end{aligned} \qquad \square$$

Remark. Equivalently, the number of maximal chains is $n!$, and the number containing a given r-set is $r!(n-r)!$, so $\sum |\mathcal{A}_r| r!(n-r)! \leq n!$.

1.2 Shadows

For $\mathcal{A} \subseteq X^{(r)}$, we know $|\partial A| \geq |A| \frac{r}{n-r+1}$ - but equality is rare (only for $\mathcal{A} = \varnothing$ or $\mathcal{A} = X^{(r)}$).

Given $|\mathcal{A}|$, how should we choose $\mathcal{A} \subseteq X^{(r)}$ to minimise $|\partial A|$? (Ultimately, we are asking 'how tightly can we pack some r-sets?') If $|\mathcal{A}| = \binom{k}{r}$, it is believable that we'd take $\mathcal{A} = [k]^{(r)}$ - giving $\partial \mathcal{A} = [k]^{(r-1)}$. What if $\binom{k}{r} < |\mathcal{A}| < \binom{k+1}{r}$? Believable that we'd take $[k]^{(r)}$ and some other r-sets from $[k+1]^{(r)}$. For instance, if $\mathcal{A} \subseteq X^{(3)}$ with $|\mathcal{A}| = \binom{7}{3} + \binom{4}{2}$, we'd take $\mathcal{A} = [7]^{(3)} \cup \{ A \cup \{8\} \mid A \in [4]^{(2)} \}$

1.2.1 Two total orderings on $X^{(r)}$

Given $A, B \in X^{(r)}$, say $A = a_1 \cdots a_r$, $B = b_1 \cdots b_r$.

Definition (Lexicographic order)**.** Say $A < B$ in the **lexicographic order** or **lex order** if for some i we have $a_i < b_i$ and $a_j = b_j \ \forall j < i$.

Equivalently, $a_i < b_i$, where $i = \min \{ j \mid a_j \neq b_j \}$. 'Use small numbers', dictionary order.

Example.

- The lex order on $[4]^{(2)}$ is

$$12, 13, 14, 23, 24, 34.$$

- The lex order on $[6]^{(3)}$ is

$$123, 124, 125, 126, 134, 135, 136, 145, 146, 156,$$
$$234, 235, 236, 245, 246, 256, 345, 346, 356, 456.$$

Definition (Colexicographic order)**.** Say $A < B$ in the **colexicographic order** or **colex order** if for some i have $a_i < b_i$ and $a_j = b_j \; \forall j > i$.

Equivalently, $a_i < b_i$ where $i = \max \{ j \mid a_j \neq b_j \}$. 'Avoid large numbers'. Equivalently, $A < B$ if $\sum_{i \in A} 2^i < \sum_{i \in B} 2^i$.

Example.

- Colex on $[4]^{(2)}$ is

$$12, 13, 23, 14, 24, 34$$

- Colex on $[6]^{(3)}$ is

$$123, 124, 134, 234, 125, 135, 235, 145, 245, 345,$$
$$126, 136, 236, 146, 246, 346, 156, 256, 356, 456.$$

Note: In colex, $[k]^{(r)}$ is an initial segment of $[k+1]^{(r+1)}$, so we could view colex as an enumeration of $\mathbb{N}^{(r)}$ (but this is false for lex).

Aim. Initial segments of colex minimise ∂, i.e. if $\mathcal{A} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the first $|\mathcal{A}|$ r-sets in colex, then $|\partial\mathcal{A}| \geq |\partial\mathcal{C}|$.

This is known as the Kruskal-Katona theorem. In particular,

$$|\mathcal{A}| = \binom{k}{r} \implies |\partial A| \geq \binom{k}{r-1}.$$

1.3 Compressions

Idea: We want to 'replace' $\mathcal{A} \subseteq X^{(r)}$ with some $\mathcal{A}' \subseteq X^{(r)}$ such that

(i) $|\mathcal{A}'| = |\mathcal{A}|$

(ii) $|\partial\mathcal{A}'| \leq |\partial\mathcal{A}|$

(iii) $\mathcal{A}'$ 'looks more like $\mathcal{C}$' than $\mathcal{A}$ did.

Ideally, we'd compress $\mathcal{A} \to \mathcal{A}' \to \mathcal{A}'' \to \cdots \to \mathcal{B}$ where either $\mathcal{B} = \mathcal{C}$ or $\mathcal{B}$ is so similar to $\mathcal{C}$ that we can see directly that $|\partial\mathcal{B}| \geq |\partial\mathcal{C}|$.

Lecture 4 Use the idea that 'colex prefers 1 to 2' to inspire:

Definition (ij-compression)**.** For $1 \leq i < j \leq n$, the **ij-compression** C_{ij} defined by: for $A \subseteq X$,

$$C_{ij}(A) = \begin{cases} A - \{j\} \cup \{i\} & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$$

and for $\mathcal{A} \subseteq \mathbb{P}(X)$,

$$C_{i,j}(\mathcal{A}) = \{ C_{ij}(A) \mid A \in \mathcal{A} \} \cup \{ A \in \mathcal{A} \mid C_{ij}(A) \in \mathcal{A} \}.$$

Say $\mathcal{A}$ is **ij-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$.

Example. If $\mathcal{A} = \{123, 134, 234, 235, 247\}$, then

$$C_{12}(\mathcal{A}) = \{\, 123, 134, 234, 135, 147 \,\}\,.$$

So $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$.

Proposition 4. Let $\mathcal{A} \subseteq X^{(r)}$, $1 \le i < j \le n$. Then

$$|\partial C_{ij}(\mathcal{A})| \le |\partial \mathcal{A}|.$$

Proof. Write $\mathcal{A}'$ for $C_{ij}(\mathcal{A})$. We'll show that if $B \in \partial\mathcal{A}' - \partial\mathcal{A}$ then $i \in B, j \notin B$ and $B \cup \{j\} - \{i\} \in \partial\mathcal{A} - \partial\mathcal{A}'$, then done, since this gives an injection.

We have $B \cup \{x\} \in \mathcal{A}'$, for some $x \notin B$, and $B \cup \{x\} \notin \mathcal{A}$ (as $B \notin \partial\mathcal{A}$). Hence $i \in B \cup \{x\}$, $j \notin B \cup \{x\}$ and $(B \cup \{x\}) \cup \{j\} - \{i\} \in \mathcal{A}$. Note that $x \ne i$, else $B \cup \{j\} \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Certainly $B \cup \{j\} - \{i\} \in \partial\mathcal{A}$.

Claim: $B \cup \{j\} - \{i\} \notin \partial\mathcal{A}'$.

Proof of claim: Suppose $(B \cup \{j\} - \{i\}) \cup \{y\} \in \mathcal{A}'$. We cannot have $y = i$, else $B \cup \{j\} \in \mathcal{A}'$, whence $B \cup \{j\} \in \mathcal{A}$ as $j \in B \cup \{j\}$, a contradiction.

Thus

$$\begin{aligned} j &\in (B \cup \{j\} - \{i\}) \cup \{y\} \\ i &\notin (B \cup \{j\} - \{i\}) \cup \{y\} \end{aligned}$$

so

$$(B \cup \{j\} - \{i\}) \cup y \in \mathcal{A}$$

and $B \cup \{y\} \in \mathcal{A}$ (definition of C_{ij}), contradicting the assumption that $B \in \partial\mathcal{A}' - \partial\mathcal{A}$. $\square$

Remark. We actually showed

$$\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial A).$$

Definition (Left-compressed)**.** Say $\mathcal{A} \subseteq X^{(r)}$ is **left-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}\ \forall i < j$.

Proposition 5. Let $\mathcal{A} \subseteq X^{(r)}$. Then $\exists$ left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{A}|$

Proof. Among all $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{A}|$, choose one with $\sum_{A \in \mathcal{B}} \sum_{x \in A} x$ minimal. Then $\mathcal{B}$ left-compressed, as if $C_{ij}(\mathcal{B}) \ne \mathcal{B}$ we contradict minimality. $\square$

Remark. Or apply a C_{ij}, then another, and so on - this must terminate. In fact, can apply each C_{ij} at most once, if you chose a sensible order.

Certainly initial segments of colex are left-compressed. The converse is false - e.g. $\mathcal{A} = \{123, 124, 125, 126, 127\}$.

'Colex prefers 23 to 14' inspires:

Definition (UV-compression)**.** For $U, V \subseteq X$ with $|U| = |V|$ and $U \cap V = \varnothing$, the **UV-compression** C_{UV} is defined by For $A \subseteq X$,

$$\begin{cases} A \cup U - V & \text{if } V \subseteq A, U \cap A = \varnothing \\ A & \text{otherwise} \end{cases}$$

and if $\mathcal{A} \subseteq X^{(r)}$,

$$C_{UV}(\mathcal{A}) = \{\, C_{UV}(A) \mid A \in \mathcal{A} \,\} \cup \{\, A \in \mathcal{A} \mid C_{UV}(A) \in \mathcal{A} \,\}$$

Say $\mathcal{A}$ is **UV-compressed** if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Example. If $\mathcal{A} = \{123, 134, 235, 145, 146, 157\}$ then

$$C_{23,14}(\mathcal{A}) = \{123, 134, 235, 145, 236, 157\}.$$

Note that $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$.

Sadly, C_{UV} need not decrease ∂ - e.g. $\mathcal{A} = \{146, 467\}$ has $|\partial\mathcal{A}| = 5$, but $C_{23,14}(\mathcal{A}) = \{236, 147\}$ has $|\partial C_{23,14}(\mathcal{A})| = 6$. $C_{23,14}$ moved some things a long way. However:

Proposition 6. Let $\mathcal{A} \subseteq X^{(r)}$ and $U, V \subseteq X$ with $|U| = |V|$ and $U \cap V = \varnothing$. Suppose $\forall x \in U\ \exists y \in V$ such that $\mathcal{A}$ is $(U - x, V - y)$-compressed. Then $|\partial C_{UV}(\mathcal{A})| \le |\partial\mathcal{A}|$.

Lecture 5 *Proof.* Write $\mathcal{A}'$ for $C_{UV}(\mathcal{A})$. Given $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we'll show that $U \subseteq B$, $V \cap B = \varnothing$, and $B \cup V - U \in \partial\mathcal{A} - \partial\mathcal{A}'$ (then done).

We have $B \cup \{x\} \in \mathcal{A}'$ for some $x \in B$, with $B \cup \{x\} \notin \mathcal{A}$. So $U \subseteq B \cup \{x\}$, $V \cap (B \cup \{x\}) = \varnothing$, and $(B \cup \{x\}) \cup V - U \in \mathcal{A}$. Thus certainly $V \cap B = \varnothing$.

If $x \in V$: We have $\mathcal{A}$ is $(U - x, V - y)$-compressed for some $y \in V$. So from $(B \cup \{x\}) \cup V - U \in \mathcal{A}$, we obtain $B \cup \{y\} \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Hence $x \notin U$, and so $U \subseteq B$.

Also, $B \cup V - U \in \partial\mathcal{A}$ (as $(B \cup \{x\}) \cup V - U \in \mathcal{A}$). Suppose $B \cup V - U \in \partial\mathcal{A}'$. Then $(B \cup V - U) \cup \{w\} \in \mathcal{A}'$ for some w.

If $w \notin U$: Then $V \subseteq (B \cup V - U) \cup \{w\}$ and $U \cap ((B \cup V - U) \cup \{w\}) = \varnothing$, so from $(B \cup V - U) \cup \{w\} \in \mathcal{A}'$ we can conclude that both $(B \cup V - U) \cup \{w\} \in \mathcal{A}$ and $B \cup \{w\} \in \mathcal{A}$, contradicting $B \notin \partial A$.

If $w \in U$: We know $\mathcal{A}$ is $(U - w, V - z)$-compressed for some $z \in V$. So from $(B \cup V - U) \cup \{w\} \in \mathcal{A}$ (as it is in $\mathcal{A}'$ and contains V, so could not have moved). We deduce $B \cup \{z\} \in \mathcal{A}$, contradicting $B \notin \partial A$. $\square$

Remark. Actually showed $\partial C_{UV}(\mathcal{A}) \subseteq C_{UV}(\partial\mathcal{A})$.

Theorem 7 (Kruskal-Katona Theorem)**.** Let $\mathcal{A} \subseteq X^{(r)}$ (for $1 \le r \le n$) and let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial\mathcal{A}| \ge |\partial\mathcal{C}|$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial\mathcal{A}| \ge \binom{k}{r-1}$.

Proof. Let

$$\Gamma = \{\, (U, V) \mid U, V \subseteq X, |U| = |V| > 0, U \cap V = \varnothing, \max U < \max V \,\}.$$

Define a sequence of set systems $\mathcal{A}_0, \mathcal{A}_1, \dots$ in $X^{(r)}$ as follows. Put $\mathcal{A}_0 = \mathcal{A}$.

Having defined $\mathcal{A}_k$, if $\mathcal{A}_k$ is (U, V)-compressed $\forall (U, V) \in \Gamma$ then stop the sequence with $\mathcal{A}_k$. If not, choose $(U, V) \in \Gamma$ such that $\mathcal{A}_k$ is not (U, V)-compressed with $|U|$ minimal.

Set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$. Note that $\forall x \in U$ we have $(U - \{x\}, V - \{y\}) \in \Gamma \cup \{(\varnothing, \varnothing)\}$ for $y = \min V$. So by Proposition 6, $|\partial\mathcal{A}_{n+1}| \le |\partial\mathcal{A}_n|$. Continue.

The sequence must terminate, e.g. as $\sum_{A \in \mathcal{A}_n} \sum_{i \in A} 2^i$ is decreasing in $\mathcal{A}$. The final system $\mathcal{B} = \mathcal{A}_k$ satisfies $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{A}|$, and is (U, V)-compressed $\forall (U, V) \in \Gamma$.

Claim: $\mathcal{B} = \mathcal{C}$.

Proof of claim: Suppose $\mathcal{B}$ is not an initial segment of colex. Then $\exists A < B$ in colex with $A \notin \mathcal{B}$ and $B \in \mathcal{B}$, but then $U = A - B$ and $V = B - A$ have $(U, V) \in \Gamma$ and $C_{UV}(B) = A$, a contradiction. $\square$

Remark.

1. Equivalently: If $\mathcal{A} \subseteq X^{(r)}$ with

$$|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s},$$

where $k_r > k_{r-1} > \cdots > k_s$ and $s > 0$, then

$$|\partial\mathcal{A}| = \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$

2. In the proof of the Kruskal-Katona Theorem, we used only Proposition 6, not Proposition 4 or Proposition 5.

3. Uniqueness? Can check that if $|\partial\mathcal{A}| = |\partial\mathcal{C}|$ and $|\mathcal{A}| = \binom{k}{r}$, then $\mathcal{A} = Y^{(r)}$, for some k-set Y (i.e. uniqueness).

   But, in general, it is not true that $|\partial\mathcal{A}| = |\partial\mathcal{C}| \implies \mathcal{A}$ isomorphic to $|\mathcal{C}|$ (where $\mathcal{A}, \mathcal{B}$ are **isomorphic** if $\exists$ a permutation of X sending $\mathcal{A}$ to $\mathcal{B}$).

Definition (Upper shadow)**.** For $A \subseteq X^{(r)}$ $(0 \le r \le n-1)$, the **upper shadow** of $\mathcal{A}$ is

$$\partial^+\mathcal{A} = \{\, A \cup \{x\} \mid A \in \mathcal{A}, x \notin A \,\}.$$

Note also $A < B$ in colex $\iff$ $A^c < B^c$ in lex with ground-set order reversed.

Corollary 8. Let $\mathcal{A} \subseteq X^{(r)}$ $(0 \le r \le n-1)$ and let $\mathcal{C}$ be the initial segment of lex with $|\mathcal{C}| = |\mathcal{A}$. Then $|\partial^+\mathcal{A}| \ge |\partial^+\mathcal{C}|$.

Proof. Take complements. □

Also, the shadow of an initial segment of colex (in $X^{(r)}$) is again an initial segment of colex in $X^{(r-1)}$. Indeed, if

$$\mathcal{C} = \{\, A \in X^{(r)} \mid A \le a_1 a_2 \dots a_r \,\}$$

then

$$\partial\mathcal{C} = \{\, B \in X^{(r-1)} \mid B \le a_2 \dots a_r \,\}$$

.

Corollary 9. Let $\mathcal{A} \subseteq X^{(r)}$ and let $\mathcal{C} \subseteq X^{(r)}$ be the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^t\mathcal{A}| \ge |\partial^t\mathcal{C}|$ $\forall 1 \le t \le r$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t\mathcal{A}| \ge \binom{k}{r-t}$.

Proof. If $|\partial^t\mathcal{A}| \ge |\partial^t\mathcal{C}|$ then $|\partial^{t+1}\mathcal{A}| \ge |\partial^{t+1}\mathcal{C}|$ by Kruskal-Katona Theorem. □

1.4 Intersecting families

Lecture 6 **Definition** (Intersecting)**.** A family $\mathcal{A} \subseteq \mathbb{P}(X)$ is **intersecting** if $A \cap B \neq \varnothing$ $\forall A, B \in \mathcal{A}$.

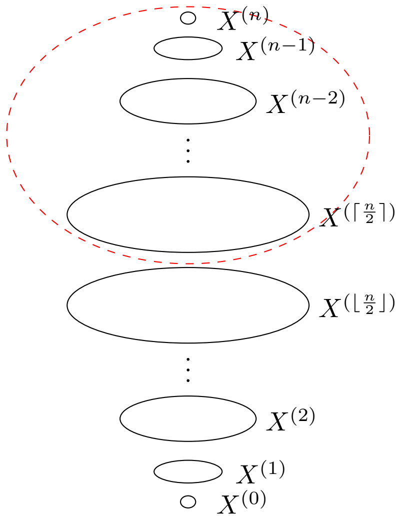
How large can $|\mathcal{A}|$ be? Can achieve $|\mathcal{A}| = 2^{n-1}$, by taking, e.g. $\mathcal{A} = \{\, A \subseteq X \mid 1 \in A \,\}$.

Proposition 10. Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be intersecting. Then $|\mathcal{A}| \le 2^{n-1}$.

Proof. For each $A \subseteq X$, can have ≤ 1 of A, A^c in $\mathcal{A}$. □

Note: there are many examples with $|\mathcal{A}| = 2^{n-1}$, e.g. for n odd we can take

$$\left\{ A \subseteq X \;\middle|\; |A| > \frac{n}{2} \right\}.$$



What if we insist $A \subseteq X^{(r)}$?

If $r > \frac{n}{2}$, this is a silly question, as we can take $\mathcal{A} = X^{(r)}$. If $r = \frac{n}{2}$, the maximum is $\frac{1}{2}\binom{n}{r}$ - just choose one of A, A^c for each $A \in X^{(r)}$. So assume $r < \frac{n}{2}$.

Taking $\mathcal{A} = \{\, A \in X^{(r)} \mid 1 \in A \,\}$ gives $|\mathcal{A}| = \binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$.

Could also try, e.g. $\mathcal{B} = \{\, A \in X^{(r)} \mid |A \cap \{1, 2, 3\}| \ge 2 \,\}$.

Try both on $[8]^{(3)}$:

$$|\mathcal{A}| = \binom{7}{2} = 21$$
$$|\mathcal{B}| = 1 + \binom{3}{2}\binom{5}{1} = 16 < 21.$$

where the first term counts the number of $|A \cap \{1, 2, 3\}| = 3$, and the second term counts $|A \cap \{1, 2, 3\}| = 2$.

In fact, the size from $\mathcal{A}$ is optimal:

Theorem 11 (Erdős-Ko-Rado)**.** Let $r < \frac{n}{2}$, and let $\mathcal{A} \subseteq X^{(r)}$ be intersecting. Then $|\mathcal{A}| \le \binom{n-1}{r-1}$.

Proof 1: 'Bubble down with Kruskal-Katona Theorem'. For $A, B \in \mathcal{A}$, have $A \cap B \ne 0$, i.e. $A \not\subseteq B^c$. Writing $\bar{\mathcal{A}}$ for $\{\, A^c \mid A \in \mathcal{A} \,\} \subseteq X^{(n-r)}$, this says that $\partial^{n-2r}\bar{\mathcal{A}}$ is disjoint from $\mathcal{A}$.

Suppose $|\mathcal{A}| > \binom{n-1}{r-1}$. Then $|\mathcal{A}^c| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$, so by Kruskal-Katona Theorem (as given in Corollary 9), have $|\partial^{n-2r}\bar{\mathcal{A}}| \ge \binom{n-1}{r}$. But

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

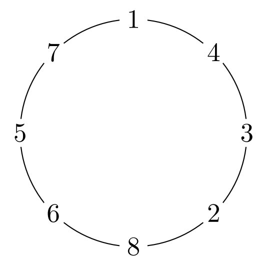
i.e.

$$|\partial^{n-2r}\bar{\mathcal{A}}| + |\mathcal{A}| > |X^{(r)}|.$$

a contradiction. $\square$

Remark. The numbers *had* to work, as we get equality for $\mathcal{A} = \{\, A \in X^{(r)} \mid 1 \in A \,\}$.

Proof 2. Consider a cyclic ordering of $[n]$, i.e. a bijection $C : [n] \to \mathbb{Z}_n$, e.g.



How many $A \in \mathcal{A}$ are intervals (sets of r consecutive elements) in our ordering? Answer: $\leq r$. Indeed, suppose $c_1 \dots c_r \in \mathcal{A}$. Then, for each $1 \leq i \leq r-1$, at most one of the two intervals $\dots c_{i-1}c_i$ and $c_{i+1}c_{i+2}\dots$ can belong to $\mathcal{A}$.

Also, a given r-set A is an interval in exactly $nr!(n-r)!$ of the $n!$ cyclic orderings. Hence $|\mathcal{A}|nr!(n-r)! \leq n!r$, i.e.

$$|\mathcal{A}| \leq \frac{(n-1)!}{(r-1)!(n-r)!} = \binom{n-1}{r-1}.$$ $\square$

Remark.

1. Equivalently, we are double-counting the edges in the bipartite graph, vertex classes $\mathcal{A}$ and all cyclic orderings, in which A is joined to C if A is an interval in C.

2. This method is called **averaging**, or Katona's method.

Equality in Erdős-Ko-Rado? Can check equality holds $\iff$ $\mathcal{A} = \{\, A \in X^{(r)} \mid i \in A \,\}$ for some i. This follows (proof 1) from equality case of Kruskal-Katona Theorem, or (proof 2) by considering changing the cyclic ordering bit by bit.

2 Isoperimetric inequalities

'How tightly can we pack a subset of given size in a space?'

3 Projections

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