Part III – Combinatorics

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Contents

0 Introduction

Lecture 1 Combinatorics tends to have problems which are easy to state and hard to prove. One of the reasons for this is that it is often unclear where to start - for instance in linear algebra we can often start by picking a basis. Proofs tend to have the property that they seem to take thousands of years to come up with, and a single line to write down. In this course, we learn some techniques which make problems sounding very hard become very easy.

We start with set systems, which builds on the idea of subsets and containment. Next, we study isoperimetric inequalities. In the continuous case, in the plane, a typical problem is to find the maximum area one can enclose with a fixed perimeter, which is solved by a circle. Similarly, a soap bubble will minimise its surface area for a fixed volume. Here, in the discrete case, we will try to understand 'how tightly' we can pack subsets. Finally, we look at continuous projections. For instance, given a subset of space, suppose we know the z-coordinate of all points is between 0 and 1, and the projection to the xy-plane has area A, we know the total volume is bounded by A. We generalise this result into higher dimensions and the box result, which has applications in combinatorics.

While all examinable proofs will be included in lectures, relevant books for this course are:

1. *Combinatorics*, Bollobás, C.U.P., 1996. This matches chapter 1 excellently and parts of chapter 2. It is a gentle read and includes other developments in combinatorics.

2. *Combinatorics of finite sets*, Anderson, O.U.P., 1987. It is a simple and clear study on chapter 1.

1 Set systems

Definition (Set system)**.** Let X be a set. A **set system** on X (or **family** of subsets of X) is a family $\mathcal{A} \subseteq \mathcal{P}(X)$.

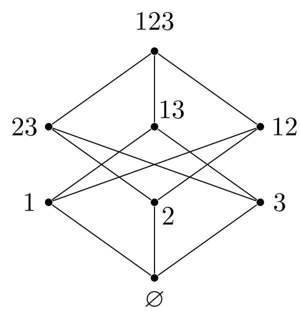
For instance, we write $X^{(r)} = \{ A \subseteq X \mid |A| = r \}$, so $X^{(r)}$ is a set system on X. We often call refer to sets in $X^{(r)}$ as r**-sets**.

Unless otherwise stated,

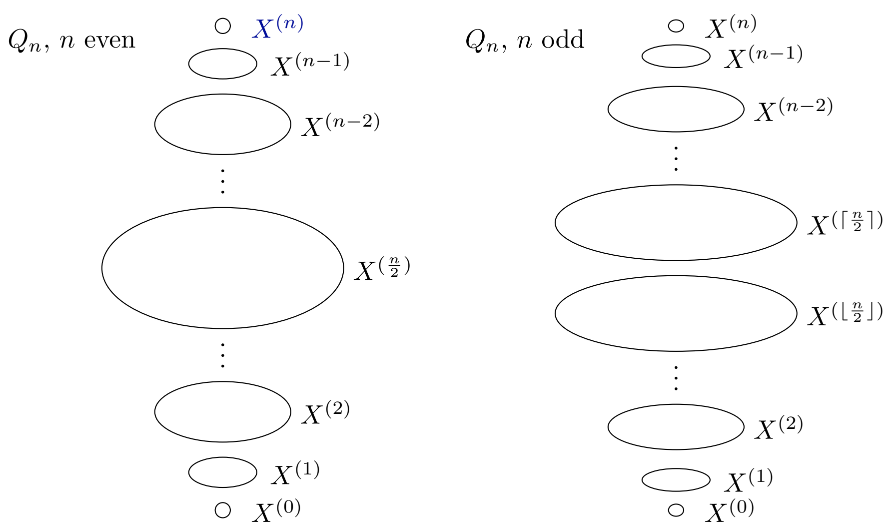
$$X = [n] \coloneqq \{1, 2, \ldots, n\},$$

e.g. $|X^{(r)}| = \binom{n}{r}$. Thus $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$, so $|[4]^{(2)}| = 6$.

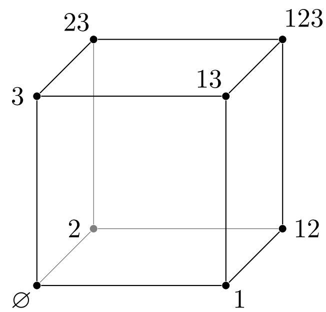
Often, we make $\mathcal{P}(X)$ into a graph, called Q_n, by joining A to B if $|A \triangle B| = 1$, i.e. if $A = B \cup \{i\}$ for some $i \notin B$ (or vice versa). For instance, here is a picture of Q_3:



As we know, the picture gets 'thicker' in the middle. But, for odd n, is it not clear where exactly the middle is, so in the odd case we have two equally sized large blobs in the middle.



If we identify a set $A \subseteq X$ with a 0-1 sequence of length n, e.g. $134 \longleftrightarrow 1011000\ldots0$, via $A \longleftrightarrow 1_A$ or χ_A, the characteristic function, then Q_3 looks like



Definition (Hypercube)**.** Q_n is often called the **hypercube** or **discrete cube** or n**-cube**.

It is important to keep *both* these pictures in mind: for induction the cube image is more instructive, but when thinking about layers the earlier image is more helpful.

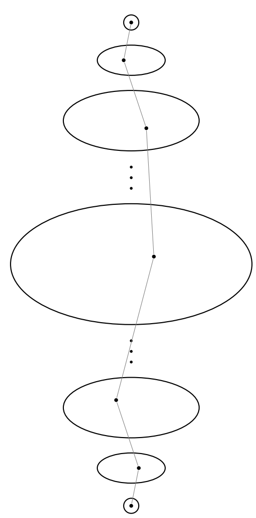
1.1 Chains and antichains

Definition (Chain)**.** A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a **chain** if $\forall A, B \in \mathcal{A}$, we have $A \subseteq B$ or $B \subseteq A$.

Definition (Antichain)**.** A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an **antichain** if $\forall A, B \in \mathcal{A}$, $A \neq B \Rightarrow A \nsubseteq B$.

Example. For instance, $\{12, 125, 123589\}$ is an chain, and $\{1, 467, 2456\}$ is an antichain. (We write 125 to refer to the set $\{1, 2, 5\}$ informally, with the promise that each element of the ground set has one digit.)

In this course, we ask questions like, how large can a chain be? We can achieve $|\mathcal{A}| = n+1$, e.g. $\{\varnothing, 1, 12, 123, \ldots, [n]\}$. It is easy to visualise this by picking 'one per level':



We cannot exceed $n + 1$, since a chain must meet each 'level' $X^{(r)}$ $(0 \leq r \leq n)$ in at most one place.

How large can an antichain be? We could achieve $|\mathcal{A}| = n$, e.g. $\mathcal{A} = \{1, 2, 3, \dots, n\}$ (and this is maximal). Indeed, we could take $\mathcal{A} = X^{(r)}$ for any r, so we can achieve $|\mathcal{A}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Can we beat this?

Aim. Prove this is the winner.

Lecture 2 Inspired by 'each chain meets each level $X^{(r)}$ in at most one place' for chains, we try to decompose Q_n into chains to find large antichains.

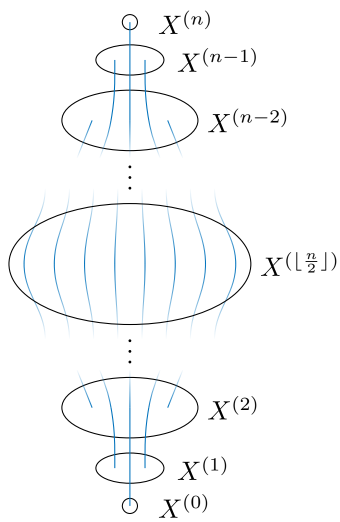
Theorem 1.1 (Sperner's Lemma)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then $|\mathcal{A}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. It is sufficient to partition $\mathcal{P}(X)$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains. For this, it is sufficient to show

(i) $\forall r < \frac{n}{2}$, there is a matching from $X^{(r)}$ to $X^{(r+1)}$

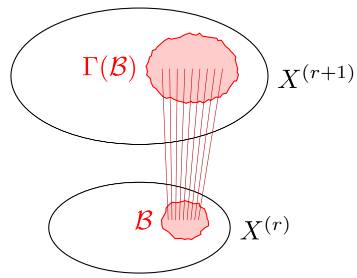
(ii) $\forall r > \frac{n}{2}$, there is a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then put these matchings together to form chains, each passing through $X^{(\lfloor \frac{n}{2} \rfloor)}$, so there are $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of them. (Recall we have a natural graph structure on Q_n)



By taking complements, it is sufficient to prove (i). Consider the subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. It is bipartite. For any $\mathcal{B} \subseteq X^{(r)}$, we have that

- The number of edges from $\mathcal{B}$ to $\Gamma(\mathcal{B})$ is $|\mathcal{B}|(n-r)$ (each point in $X^{(r)}$ has degree $n-r$).
- The number of edges from $\mathcal{B}$ to $\Gamma(\mathcal{B})$ is at most $|\Gamma(\mathcal{B})|(r+1)$ (each point in $X^{(r+1)}$ has degree $r+1$).



Thus

$$\begin{aligned}|\Gamma(\mathcal{B})| &\geq |\mathcal{B}|\frac{n-r}{r+1} \\ &\geq |\mathcal{B}|\end{aligned}$$

as $r < \frac{n}{2}$. Hence by Hall's theorem, there is a matching. $\square$

Remark. Recall $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ was achievable, for example $\mathcal{A} = X^{(\lfloor \frac{n}{2} \rfloor)}$. But this proof says nothing about extremal cases - which antichains have size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$?

Aim. For $\mathcal{A}$ an antichain,

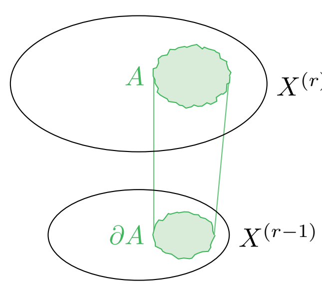
$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

This trivially implies Sperner's Lemma.

Definition (Shadow)**.** Let $\mathcal{A} \subseteq X^{(r)}$, for some $1 \leq r \leq n$. The **shadow** or **corner shadow** of $\mathcal{A}$ is

$$\partial \mathcal{A} \equiv \partial^- \mathcal{A} \coloneqq \{\, A - \{i\} \mid A \in \mathcal{A},\ i \in A \,\}$$

so $\partial \mathcal{A} \subseteq X^{(r-1)}$.



For example, if $\mathcal{A} = \{123, 124, 134, 135\} \subseteq X^{(3)}$, then

$$\partial \mathcal{A} = \{12, 13, 23, 24, 34, 15, 35\} \subseteq X^{(2)}.$$

Lemma 1.2 (Local LYM)**.** Let $\mathcal{A} \subseteq X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\partial \mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

'The fraction of the layer occupied increases when we take the shadow'.

Proof. Counting from above, there are $r|\mathcal{A}|$ edges $\mathcal{A}$ to $\partial\mathcal{A}$. Counting from below, the number of edges $\mathcal{A}$ to $\partial\mathcal{A}$ is at most $(n-r+1)|\partial\mathcal{A}|$, so

$$\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \ge \frac{r}{n-r+1}.$$

But

$$\frac{\binom{n}{r-1}}{\binom{n}{r}} = \frac{r}{n-r+1}. \qquad \square$$

When do we get equality in Local LYM? We'd need $(A - \{i\}) \cup \{j\} \in \mathcal{A} \ \ \forall A \in \mathcal{A},\ i \in A,\ j \notin A$. Hence $\mathcal{A} = X^r$ or $\varnothing$.

The LYM in Local LYM stands for 'Lubell–Yamamoto–Meshalkin'. We can use Local LYM to prove:

Theorem 1.3 (LYM)**.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \le 1.$$

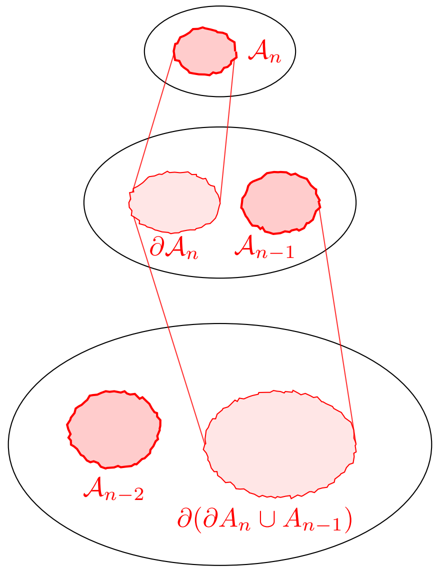
Proof 1: 'Bubble down with Local LYM'. Write $\mathcal{A}_r = \mathcal{A} \cap X^{(r)}$. We have $\frac{|\mathcal{A}_n|}{\binom{n}{n}} \le 1$. Also, $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are distinct since $\mathcal{A}$ was an antichain, so

$$\begin{aligned}
\frac{|\partial A_n|}{\binom{n}{n-1}} + \frac{|A_{n-1}|}{\binom{n}{n-1}} &= \frac{|\partial A_n \cup A_{n-1}|}{\binom{n}{n-1}} \\
\implies \frac{|A_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} &\le 1
\end{aligned}$$

by Local LYM.

Also, $\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})$ is disjoint from $\mathcal{A}_{n-2}$, again since $\mathcal{A}$ is an antichain so

$$\begin{aligned}
\frac{|\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1, \\
\implies \frac{|(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-1}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1, \\
\implies \frac{\mathcal{A}_n}{\binom{n}{n}} + \frac{\mathcal{A}_{n-1}}{\binom{n}{n-1}} + \frac{\mathcal{A}_{n-2}}{\binom{n}{n-2}} &\le 1.
\end{aligned}$$



Keep going, we obtain
$$\frac{\mathcal{A}_n}{\binom{n}{n}} + \frac{\mathcal{A}_{n-1}}{\binom{n}{n-1}} + \cdots + \frac{\mathcal{A}_0}{\binom{n}{0}} \leq 1. \qquad \square$$

When do we get equality in LYM? We must have had equality in each use of Local LYM. So the first (greatest) r with $\mathcal{A}_r \neq \varnothing$ must have $\mathcal{A}_r = X^{(r)}$ thus $\mathcal{A} = X^{(r)}$. Hence equality in Sperner's Lemma $\Leftrightarrow \mathcal{A} = X^{(\frac{n}{2})}$ for n even and $\mathcal{A} = X^{(\lfloor \frac{n}{2} \rfloor)}$ or $X^{(\lceil \frac{n}{2} \rceil)}$ for n odd.

Lecture 3 *Proof 2.* Choose uniformly at random, a maximal chain $\mathcal{C}$ (i.e. $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_n$) with $|\mathcal{C}_i| = i \ \forall i$. For a given r-set A,

$$\begin{aligned}
\mathbb{P}(A \in \mathcal{C}) &= \frac{1}{\binom{n}{r}} && \text{(all } r\text{-sets are equally likely)} \\
\mathbb{P}(\mathcal{A}_r \text{ meets } \mathcal{C}) &= \frac{|\mathcal{A}_r|}{\binom{n}{r}} && \text{(events are disjoint)} \\
\implies \mathbb{P}(\mathcal{A} \text{ meets } \mathcal{C}) &= \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}. \\
\implies \sum_{i=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} &\leq 1. && \square
\end{aligned}$$

Remark. Equivalently, the number of maximal chains is $n!$, and the number containing a given r-set is $r!(n-r)!$, so $\sum |\mathcal{A}_r| r!(n-r)! \leq n!$.

1.2 Shadows

For $\mathcal{A} \subseteq X^{(r)}$, we know $|\partial A| \geq |A| \frac{r}{n-r+1}$ - but equality is rare (only for $\mathcal{A} = \varnothing$ or $\mathcal{A} = X^{(r)}$).

Given $|\mathcal{A}|$, how should we choose $\mathcal{A} \subseteq X^{(r)}$ to minimise $|\partial \mathcal{A}|$? (Ultimately, we are asking 'how tightly can we pack some r-sets?') If $|\mathcal{A}| = \binom{k}{r}$, it is believable that we'd take $\mathcal{A} = [k]^{(r)}$

- giving $\partial\mathcal{A} = [k]^{(r-1)}$. What if $\binom{k}{r} < |\mathcal{A}| < \binom{k+1}{r}$? Believable that we'd take $[k]^{(r)}$ and some other r-sets from $[k+1]^{(r)}$. For instance, if $\mathcal{A} \subseteq X^{(3)}$ with $|\mathcal{A}| = \binom{7}{3} + \binom{4}{2}$, we'd take

$$\mathcal{A} = [7]^{(3)} \cup \{\, A \cup \{8\} \mid A \in [4]^{(2)} \,\}\,.$$

1.2.1 Two total orderings on $X^{(r)}$

Given $A, B \in X^{(r)}$, say $A = a_1 \cdots a_r$, $B = b_1 \cdots b_r$.

Definition (Lexicographic order)**.** Say $A < B$ in the **lexicographic order** or **lex order** if for some i we have $a_i < b_i$ and $a_j = b_j$ $\forall j < i$. Equivalently, $a_i < b_i$, where $i = \min\{\, j \mid a_j \neq b_j \,\}$.

'Use small numbers', dictionary order.

Example.

- The lex order on $[4]^{(2)}$ is
$$12, 13, 14, 23, 24, 34.$$
- The lex order on $[6]^{(3)}$ is
$$\begin{gathered}123, 124, 125, 126, 134, 135, 136, 145, 146, 156,\\ 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.\end{gathered}$$

Definition (Colexicographic order)**.** Say $A < B$ in the **colexicographic order** or **colex order** if for some i have $a_i < b_i$ and $a_j = b_j$ $\forall j > i$. Equivalently, $a_i < b_i$ where $i = \max\{\, j \mid a_j \neq b_j \,\}$.

'Avoid large numbers'. Equivalently, $A < B$ if $\sum_{i \in A} 2^i < \sum_{i \in B} 2^i$.

Example.

- Colex on $[4]^{(2)}$ is
$$12, 13, 23, 14, 24, 34$$
- Colex on $[6]^{(3)}$ is
$$\begin{gathered}123, 124, 134, 234, 125, 135, 235, 145, 245, 345,\\ 126, 136, 236, 146, 246, 346, 156, 256, 356, 456.\end{gathered}$$

Note: In colex, $[k]^{(r)}$ is an initial segment of $[k+1]^{(r+1)}$, so we could view colex as an enumeration of $\mathbb{N}^{(r)}$ (but this is false for lex).

Aim. Initial segments of colex minimise ∂, i.e. if $\mathcal{A} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the first $|\mathcal{A}|$ r-sets in colex, then $|\partial\mathcal{A}| \geq |\partial\mathcal{C}|$.

This is known as the Kruskal-Katona theorem. In particular,

$$|\mathcal{A}| = \binom{k}{r} \implies |\partial A| \geq \binom{k}{r-1}.$$

1.3 Compressions

Idea. We want to 'replace' $\mathcal{A} \subseteq X^{(r)}$ with some $\mathcal{A}' \subseteq X^{(r)}$ such that

(i) $|\mathcal{A}'| = |\mathcal{A}|$

(ii) $|\partial\mathcal{A}'| \le |\partial\mathcal{A}|$

(iii) $\mathcal{A}'$ 'looks more like $\mathcal{C}$' than $\mathcal{A}$ did.

Ideally, we'd compress $\mathcal{A} \to \mathcal{A}' \to \mathcal{A}'' \to \cdots \to \mathcal{B}$ where either $\mathcal{B} = \mathcal{C}$ or $\mathcal{B}$ is so similar to $\mathcal{C}$ that we can see directly that $|\partial\mathcal{B}| \ge |\partial\mathcal{C}|$.

Lecture 4 Use the idea that 'colex prefers 1 to 2' to inspire:

Definition (ij-compression)**.** For $1 \le i < j \le n$, the ij**-compression** C_{ij} defined by: for $A \subseteq X$,

$$C_{ij}(A) = \begin{cases} A - \{j\} \cup \{i\} & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$$

and for $\mathcal{A} \subseteq \mathcal{P}(X)$,

$$C_{i,j}(\mathcal{A}) = \{C_{ij}(A) \mid A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \mid C_{ij}(A) \in \mathcal{A}\}.$$

Say $\mathcal{A}$ is ij**-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$.

Example. If $\mathcal{A} = \{123, 134, 234, 235, 247\}$, then

$$C_{12}(\mathcal{A}) = \{123, 134, 234, 135, 147\}.$$

So $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$.

Proposition 1.4. Let $\mathcal{A} \subseteq X^{(r)}$, $1 \le i < j \le n$. Then

$$|\partial C_{ij}(\mathcal{A})| \le |\partial\mathcal{A}|.$$

Proof. Write $\mathcal{A}'$ for $C_{ij}(\mathcal{A})$. We'll show that if $B \in \partial\mathcal{A}' - \partial\mathcal{A}$ then $i \in B, j \notin B$ and $B \cup \{j\} - \{i\} \in \partial\mathcal{A} - \partial\mathcal{A}'$, then done, since this gives an injection.

We have $B \cup \{x\} \in \mathcal{A}'$ for some $x \notin B$, and $B \cup \{x\} \notin \mathcal{A}$ (as $B \notin \partial\mathcal{A}$). Hence $i \in B \cup \{x\}$, $j \notin B \cup \{x\}$ and $(B \cup \{x\}) \cup \{j\} - \{i\} \in \mathcal{A}$. Note that $x \ne i$, else $B \cup \{j\} \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Certainly $B \cup \{j\} - \{i\} \in \partial\mathcal{A}$.

Claim: $B \cup \{j\} - \{i\} \notin \partial\mathcal{A}'$.

Proof of claim: Suppose $(B \cup \{j\} - \{i\}) \cup \{y\} \in \mathcal{A}'$. We cannot have $y = i$, else $B \cup \{j\} \in \mathcal{A}'$, whence $B \cup \{j\} \in \mathcal{A}$ as $j \in B \cup \{j\}$, a contradiction.

Thus

$$\begin{gathered} j \in (B \cup \{j\} - \{i\}) \cup \{y\} \\ i \notin (B \cup \{j\} - \{i\}) \cup \{y\} \end{gathered}$$

so

$$(B \cup \{j\} - \{i\}) \cup \{y\} \in \mathcal{A}$$

and $B \cup \{y\} \in \mathcal{A}$ (definition of C_{ij}), contradicting the assumption that $B \in \partial\mathcal{A}' - \partial\mathcal{A}$. □

Remark. We actually showed
$$\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial A).$$

Definition (Left-compressed)**.** Say $\mathcal{A} \subseteq X^{(r)}$ is **left-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}\ \forall i < j$.

Proposition 1.5. Let $\mathcal{A} \subseteq X^{(r)}$. Then $\exists$ left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{A}|$.

Proof. Among all $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \le |\partial\mathcal{A}|$, choose one with $\sum_{A\in\mathcal{B}}\sum_{x\in A} x$ minimal. Then $\mathcal{B}$ left-compressed, as if $C_{ij}(\mathcal{B}) \ne \mathcal{B}$ we contradict minimality. □

Remark. Or apply a C_{ij}, then another, and so on - this must terminate. In fact, can apply each C_{ij} at most once, if you chose a sensible order.

Certainly initial segments of colex are left-compressed. The converse is false - e.g. $\mathcal{A} = \{123, 124, 125, 126, 127\}$.

'Colex prefers 23 to 14' inspires:

Definition (UV-compression)**.** For $U, V \subseteq X$ with $|U| = |V|$ and $U \cap V = \varnothing$, the **UV-compression** C_{UV} is defined as follows: For $A \subseteq X$,

$$C_{UV}(X) = \begin{cases} A \cup U - V & \text{if } V \subseteq A, U \cap A = \varnothing \\ A & \text{otherwise} \end{cases}$$

and if $\mathcal{A} \subseteq X^{(r)}$,

$$C_{UV}(\mathcal{A}) = \{C_{UV}(A) \mid A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \mid C_{UV}(A) \in \mathcal{A}\}.$$

Say $\mathcal{A}$ is **UV-compressed** if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Example. If $\mathcal{A} = \{123, 134, 235, 145, 146, 157\}$ then

$$C_{23,14}(\mathcal{A}) = \{123, 134, 235, 145, 236, 157\}.$$

Note that $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$. Sadly, C_{UV} need not decrease ∂ - e.g. $\mathcal{A} = \{146, 467\}$ has $|\partial\mathcal{A}| = 5$, but $C_{23,14}(\mathcal{A}) = \{236, 467\}$ has $|\partial C_{23,14}(\mathcal{A})| = 6$. $C_{23,14}$ moved some things a long way. However:

Proposition 1.6. Let $\mathcal{A} \subseteq X^{(r)}$ and $U, V \subseteq X$ with $|U| = |V|$ and $U \cap V = \varnothing$. Suppose $\forall x \in U\ \exists y \in V$ such that $\mathcal{A}$ is $(U - x, V - y)$-compressed. Then $|\partial C_{UV}(\mathcal{A})| \le |\partial\mathcal{A}|$.

Lecture 5 *Proof.* Write $\mathcal{A}'$ for $C_{UV}(\mathcal{A})$. Given $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we'll show that $U \subseteq B$, $V \cap B = \varnothing$, and $B \cup V - U \in \partial\mathcal{A} - \partial\mathcal{A}'$ (then done).

We have $B \cup \{x\} \in \mathcal{A}'$ for some $x \in B$, with $B \cup \{x\} \notin \mathcal{A}$.

- So $U \subseteq B \cup \{x\}$, $V \cap (B \cup \{x\}) = \varnothing$, and $(B \cup \{x\}) \cup V - U \in \mathcal{A}$. Thus certainly $V \cap B = \varnothing$.
- If $x \in U$: We have $\mathcal{A}$ is $(U - x, V - y)$-compressed for some $y \in V$. So from $(B \cup \{x\}) \cup V - U \in \mathcal{A}$, we obtain $B \cup \{y\} \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Hence $x \notin U$, and so $U \subseteq B$.
- We have $B \cup V - U \in \partial\mathcal{A}$ (as $(B \cup \{x\}) \cup V - U \in \mathcal{A}$). Suppose $B \cup V - U \in \partial\mathcal{A}'$. Then $(B \cup V - U) \cup \{w\} \in \mathcal{A}'$ for some w.

– If $w \notin U$: Then $V \subseteq (B \cup V - U) \cup \{w\}$ and $U \cap ((B \cup V - U) \cup \{w\}) = \varnothing$, so from $(B \cup V - U) \cup \{w\} \in \mathcal{A}'$ we can conclude that both $(B \cup V - U) \cup \{w\} \in \mathcal{A}$ and $B \cup \{w\} \in \mathcal{A}$, contradicting $B \notin \partial A$.

– If $w \in U$: We know $\mathcal{A}$ is $(U - w, V - z)$-compressed for some $z \in V$. So from $(B \cup V - U) \cup \{w\} \in \mathcal{A}$ (as it is in $\mathcal{A}'$ and contains V, so could not have moved).

We deduce $B \cup \{z\} \in \mathcal{A}$, contradicting $B \notin \partial \mathcal{A}$. □

Remark. Actually showed $\partial C_{UV}(\mathcal{A}) \subseteq C_{UV}(\partial \mathcal{A})$.

Theorem 1.7 (Kruskal-Katona theorem)**.** Let $\mathcal{A} \subseteq X^{(r)}$ (for $1 \leq r \leq n$) and let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial \mathcal{A}| \geq |\partial \mathcal{C}|$. In particular,

$$|\mathcal{A}| = \binom{k}{r} \implies |\partial \mathcal{A}| \geq \binom{k}{r-1}.$$

Proof. Let

$$\Gamma = \{(U, V) \mid U, V \subseteq X, |U| = |V| > 0, U \cap V = \varnothing, \max U < \max V\}.$$

Define a sequence of set systems $\mathcal{A}_0, \mathcal{A}_1, \dots$ in $X^{(r)}$ as follows. Put $\mathcal{A}_0 = \mathcal{A}$.

Having defined $\mathcal{A}_k$, if $\mathcal{A}_k$ is (U, V)-compressed $\forall (U, V) \in \Gamma$ then stop the sequence with $\mathcal{A}_k$. If not, choose $(U, V) \in \Gamma$ such that $\mathcal{A}_k$ is not (U, V)-compressed with $|U|$ minimal.

Set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$. Note that $\forall x \in U$ we have $(U - \{x\}, V - \{y\}) \in \Gamma \cup \{(\varnothing, \varnothing)\}$ for $y = \min V$. So by Proposition 1.6, $|\partial \mathcal{A}_{n+1}| \leq |\partial \mathcal{A}_n|$. Continue.

The sequence must terminate, e.g. as $\sum_{A \in \mathcal{A}_n} \sum_{i \in A} 2^i$ is decreasing in n. The final system $\mathcal{B} = \mathcal{A}_k$ satisfies $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial \mathcal{B}| \leq |\partial \mathcal{A}|$, and is (U, V)-compressed $\forall (U, V) \in \Gamma$.

Claim: $\mathcal{B} = \mathcal{C}$.

Proof of claim: Suppose $\mathcal{B}$ is not an initial segment of colex. Then $\exists A < B$ in colex with $A \notin \mathcal{B}$ and $B \in \mathcal{B}$, but then $U = A - B$ and $V = B - A$ have $(U, V) \in \Gamma$ and $C_{UV}(B) = A$, a contradiction. □

Remark.

1. Equivalently: If $\mathcal{A} \subseteq X^{(r)}$ with

$$|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s},$$

where $k_r > k_{r-1} > \cdots > k_s$ and $s > 0$, then

$$|\partial \mathcal{A}| = \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$

2. In the proof of the Kruskal-Katona theorem, we used only Proposition 1.6, not Proposition 1.4 or Proposition 1.5.

3. Uniqueness? Can check that if $|\partial \mathcal{A}| = |\partial \mathcal{C}|$ and $|\mathcal{A}| = \binom{k}{r}$, then $\mathcal{A} = Y^{(r)}$, for some k-set Y (i.e. uniqueness).

   But, in general, it is not true that $|\partial \mathcal{A}| = |\partial \mathcal{C}| \implies \mathcal{A}$ isomorphic to $|\mathcal{C}|$ (where $\mathcal{A}, \mathcal{B}$ are **isomorphic** if $\exists$ a permutation of X sending $\mathcal{A}$ to $\mathcal{B}$).

Definition (Upper shadow)**.** For $A \subseteq X^{(r)}$ $(0 \le r \le n-1)$, the **upper shadow** of $\mathcal{A}$ is

$$\partial^+ \mathcal{A} = \{A \cup \{x\} \mid A \in \mathcal{A}, x \notin A\}.$$

Note also $A < B$ in colex $\iff$ $A^c < B^c$ in lex with ground-set order reversed.

Corollary 1.8. Let $\mathcal{A} \subseteq X^{(r)}$ $(0 \le r \le n-1)$ and let $\mathcal{C}$ be the initial segment of lex with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^+ \mathcal{A}| \ge |\partial^+ \mathcal{C}|$.

Proof. Take complements. □

Also, the shadow of an initial segment of colex (in $X^{(r)}$) is again an initial segment of colex in $X^{(r-1)}$. Indeed, if

$$\mathcal{C} = \{A \in X^{(r)} \mid A \le a_1 a_2 \dots a_r\}$$

then

$$\partial \mathcal{C} = \{B \in X^{(r-1)} \mid B \le a_2 \dots a_r\}.$$

Corollary 1.9. Let $\mathcal{A} \subseteq X^{(r)}$ and let $\mathcal{C} \subseteq X^{(r)}$ be the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^t \mathcal{A}| \ge |\partial^t \mathcal{C}|$ $\forall 1 \le t \le r$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t \mathcal{A}| \ge \binom{k}{r-t}$.

Proof. If $|\partial^t \mathcal{A}| \ge |\partial^t \mathcal{C}|$ then $|\partial^{t+1} \mathcal{A}| \ge |\partial^{t+1} \mathcal{C}|$ by Kruskal-Katona theorem. □

1.4 Intersecting families

Lecture 6 **Definition** (Intersecting)**.** A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is **intersecting** if $A \cap B \ne \varnothing$ $\forall A, B \in \mathcal{A}$.

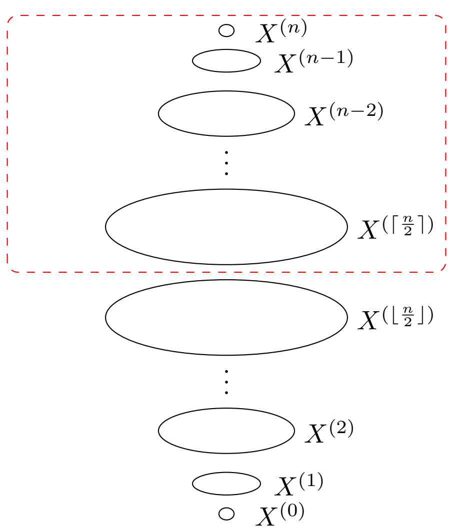
How large can $|\mathcal{A}|$ be? Can achieve $|\mathcal{A}| = 2^{n-1}$, by taking, e.g. $\mathcal{A} = \{A \subseteq X \mid 1 \in A\}$.

Proposition 1.10. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be intersecting. Then $|\mathcal{A}| \le 2^{n-1}$.

Proof. For each $A \subseteq X$, can have at most 1 of A, A^c in $\mathcal{A}$. □

Note: there are many examples with $|\mathcal{A}| = 2^{n-1}$, e.g. for n odd we can take

$$\left\{A \subseteq X \;\middle|\; |A| > \tfrac{n}{2}\right\}.$$



What if we insist $\mathcal{A} \subseteq X^{(r)}$?

If $r > \frac{n}{2}$, this is a silly question, as we can take $\mathcal{A} = X^{(r)}$. If $r = \frac{n}{2}$, the maximum is $\frac{1}{2}\binom{n}{r}$ - just choose one of A, A^c for each $A \in X^{(r)}$. So assume $r < \frac{n}{2}$.

Taking

$$\mathcal{A} = \{A \in X^{(r)} \mid 1 \in A\}$$

gives $|\mathcal{A}| = \binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$. Could also try, e.g.

$$\mathcal{B} = \left\{A \in X^{(r)} \,\middle|\, |A \cap \{1,2,3\}| \ge 2\right\}.$$

Try both on $[8]^{(3)}$:

$$|\mathcal{A}| = \binom{7}{2} = 21$$

$$|\mathcal{B}| = 1 + \binom{3}{2}\binom{5}{1} = 16 < 21.$$

where the first term counts the number of $|A \cap \{1,2,3\}| = 3$, and the second term counts $|A \cap \{1,2,3\}| = 2$.

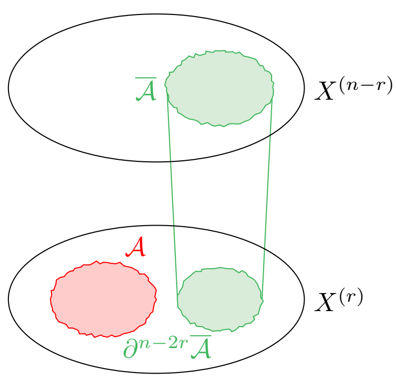
In fact, the size from $\mathcal{A}$ is optimal:

Theorem 1.11 (Erdős-Ko-Rado)**.** Let $r < \frac{n}{2}$, and let $\mathcal{A} \subseteq X^{(r)}$ be intersecting. Then $|\mathcal{A}| \le \binom{n-1}{r-1}$.

Proof 1: 'Bubble down with Kruskal-Katona theorem'. For $A, B \in \mathcal{A}$, have $A \cap B \neq \varnothing$, i.e. $A \not\subseteq B^c$. Writing

$$\overline{\mathcal{A}} := \{A^c \mid A \in \mathcal{A}\} \subseteq X^{(n-r)},$$

this says that $\partial^{n-2r}\overline{\mathcal{A}}$ is disjoint from $\mathcal{A}$.



Suppose $|\mathcal{A}| > \binom{n-1}{r-1}$. Then $|\overline{\mathcal{A}}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$, so by Kruskal-Katona theorem (as given in Corollary 1.9), have $|\partial^{n-2r}\overline{\mathcal{A}}| \ge \binom{n-1}{r}$. But

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

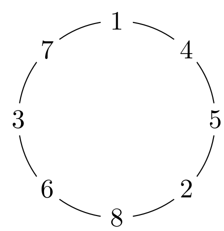
i.e.

$$|\partial^{n-2r}\overline{\mathcal{A}}| + |\mathcal{A}| > |X^{(r)}|.$$

a contradiction. $\square$

Remark. The numbers *had* to work, as we get equality for $\mathcal{A} = \{A \in X^{(r)} \mid 1 \in A\}$.

Proof 2. Consider a cyclic ordering of $[n]$, i.e. a bijection $C : [n] \to \mathbb{Z}_n$, e.g.



How many $A \in \mathcal{A}$ are intervals (sets of r consecutive elements) in our ordering? Answer: $\leq r$. Indeed, suppose $c_1 \dots c_r \in \mathcal{A}$. Then, for each $1 \leq i \leq r-1$, at most one of the two intervals (of length r) $\dots c_{i-1}c_i$ and $c_{i+1}c_{i+2}\dots$ can belong to $\mathcal{A}$.

Also, a given r-set A is an interval in exactly $nr!(n-r)!$ of the $n!$ cyclic orderings. Hence $|\mathcal{A}|nr!(n-r)! \leq n!r$, i.e.

$$|\mathcal{A}| \leq \frac{(n-1)!}{(r-1)!(n-r)!} = \binom{n-1}{r-1}. \qquad \square$$

Remark.

1. Equivalently, we are double-counting the edges in the bipartite graph, with vertex classes $\mathcal{A}$ and all cyclic orderings, in which A is joined to C if A is an interval in C.

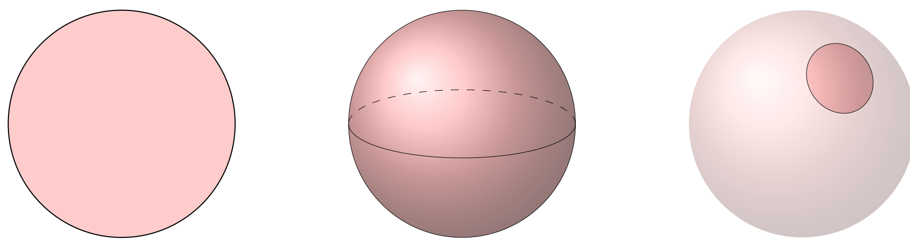
2. This method is called **averaging**, or Katona's method.

Equality in Erdős-Ko-Rado? Can check equality holds $\iff$ $\mathcal{A} = \{\, A \in X^{(r)} \mid i \in A \,\}$ for some i. This follows (proof 1) from equality case of Kruskal-Katona theorem, or (proof 2) by considering changing the cyclic ordering bit by bit.

2 Isoperimetric inequalities

Lecture 7 'How tightly can we pack a subset of given size in a space?' We are familiar with this kind of inequality in the continuous sense:

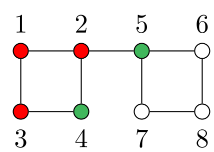
- Among subsets of $\mathbb{R}^2$ of given area, the disc has smallest perimeter.
- Among subsets of $\mathbb{R}^3$ of given volume, the sphere has smallest surface area.
- Among subsets of S^2 of given area, the cap has smallest perimeter.



Definition (Boundary)**.** For a set A of vertices in a graph G, the **boundary** is

$$b(A) = \{x \in V(G) \mid x \notin A,\ xy \in E \text{ for some } y \in A\}.$$

For example, in the picture, if $A = \{1, 2, 3\}$, then $b(A) = \{4, 5\}$.



Definition (Isoperimetric inequality)**.** An **isoperimetric inequality** on G is an inequality of the form

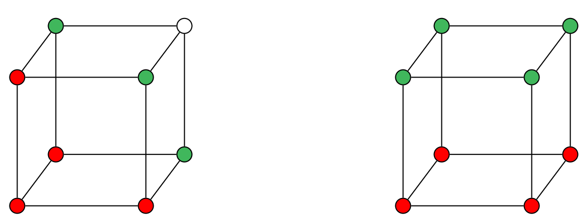
$$|b(A)| \geq f(|A|) \quad \forall A \subseteq V(G).$$

Definition (Neighbourhood)**.** Equivalently, minimise the **neighbourhood** of A:

$$N(A) = A \cup b(A) = \{\, x \in G \mid d(x, A) \leq 1 \,\}$$

where d denotes the usual graph distance.

A natural guess to minimise neighbourhood is often $B(x, r) = \{y \in G \mid d(x, y) \leq r\}$. What happens in Q_n? For example, try $|A| = 4$ in Q_3:



$|b(A)| = 3$ $\qquad$ $|b(A)| = 4$

Guess: 'Balls are best', i.e.

$$|A| = |X^{(\leq r)}| \implies |N(A)| \geq |X^{(\leq r+1)}|$$

(where $X^{(\leq r)}$ is shorthand for $X^{(0)} \cup \cdots \cup X^{(r)}$).

But what if $|A|$ is strictly between $\sum_{i=0}^{r} \binom{n}{i}$ and $\sum_{i=0}^{r+1} \binom{n}{i}$? Guess: Take

$$A = X^{(\leq r)} \cup B,$$

for some $B \subseteq X^{(r+1)}$ (called a 'Hamming Ball'). If we knew this, then

$$N(A) = X^{(\leq r+1)} \cup \partial^+ B,$$

so by Kruskal-Katona theorem we'd take B to be an initial segment of lex.

Definition (Simplicial order)**.** Define the **simplicial order** on Q_n by $x < y$ if either $|x| < |y|$ or $|x| = |y|$ and $x < y$ in lex.

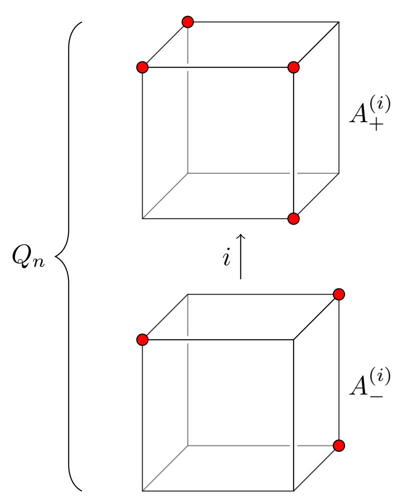
The slogan for the simplicial order is 'Go up in levels, and use lex inside a level.'

Aim. Initial segments of simplicial order are best for minimising $N(A)$.

Definition (Sections)**.** Given $A \subseteq Q_n$ and $1 \leq i \leq n$, the ***i*-sections** are the set systems $A_+^{(i)}, A_-^{(i)} \subseteq \mathcal{P}(X - \{i\})$ given by

$$A_-^{(i)} = \{x \in A \mid i \notin x\} \subseteq \mathcal{P}(X - \{i\})$$
$$A_+^{(i)} = \{x - \{i\} \mid x \in A, \, i \in x\} \subseteq \mathcal{P}(X - \{i\}).$$

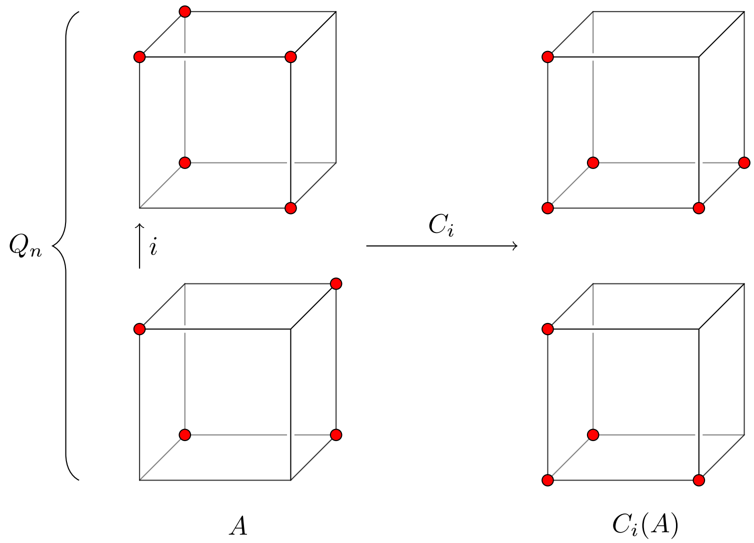
Note $\mathcal{P}(X - \{i\})$ is isomorphic to Q_{n-1}.



Definition (Compression)**.** Define the ***i*-compression** $C_i(A)$ of A by giving its i-sections.

- $C_i(A)_+^i$ is the initial segment (in simplicial) of Q_{n-1} of size $|A_+^{(i)}|$
- $C_i(A)_-^i$ is the initial segment (in simplicial) of Q_{n-1} of size $|A_-^{(i)}|$

Say A is **i-compressed** if $C_i(A) = A$.



Certainly $|C_i(A)| = |A|$. Also, $C_i(A)$ 'looks more like' an initial segment of simplicial than A did.

Theorem 2.1 (Harper's theorem)**.** Let $A \subseteq Q_n$, and let C be the initial segment of the simplicial order with $|C| = |A|$. Then $|N(A)| \ge |N(C)|$. In particular,

$$|A| \ge \sum_{i=0}^{r} \binom{n}{i} \implies |N(A)| \ge \sum_{i=0}^{r+1} \binom{n}{i}.$$

Remark.

1. If we know A is a Hamming ball, then done by Kruskal-Katona theorem.

2. Harper's theorem $\Rightarrow$ Kruskal-Katona theorem: Given $B \subseteq X^{(r)}$, apply Harper's theorem to $A = B \cup X^{(\le r-1)}$

Proof. Induction on n. $n = 1$ works. Given $A \subseteq Q_n$ for $n > 1$, fix $1 \le i \le n$.
Claim: $|N(C_i(A))| \le |N(A)|$.
Proof of claim: Write $B = C_i(A)$. We have

$$\begin{aligned} |N(A)| &= |A_+ \cup N(A_-)| + |A_- \cup N(A_+)| \\ |N(B)| &= |B_+ \cup N(B_-)| + |B_- \cup N(B_+)| \end{aligned}$$

where the first term is 'downstairs', and the second term is 'upstairs'. Now, $|B_+| = |A_+|$ and $|N(B_-)| \le |N(A_-)|$ by induction.

But $N(B_-)$ is an initial segment of simplicial on Q_{n-1} as is B_+. So $N(B_-)$ and B_+ are nested (in some direction). Hence

$$|B_+ \cup N(B_-)| \le |A_+ \cup N(A_-)|$$

and similarly

$$|B_- \cup N(B_+)| \le |A_- \cup N(A_+)|$$

Thus $|N(B)| \le |N(A)|$. ■

Lecture 8 Among all $B \subseteq Q_n$ with $|B| = |A|$ and $|N(B)| \le |N(A)|$, choose one with

$$\sum_{x \in B} f(x)$$

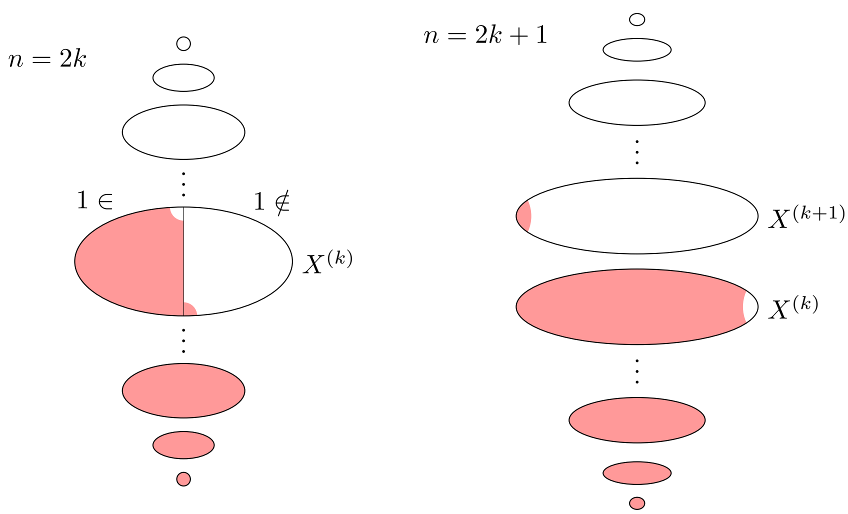
minimal, where $f(x)$ = the position of x in the simplicial order on Q_n. Then B is i-compressed $\forall i$. Must such a B be an initial segment of simplicial? Unfortunately, no - e.g. $B = \{\varnothing, 1, 2, 12\} \subseteq Q_3$. However,

Lemma 2.2. Let $B \subseteq Q_n$ be i-compressed $\forall i$ but not an initial segment of simplicial order. Then if n is odd, say $n = 2k + 1$, we have

$$\begin{aligned} B &= X^{(\le k)} - \{\text{last } k\text{-set}\} \cup \{\text{first } k+1\text{-set}\} \\ &= X^{(\le k)} - \{(k+2)\dots(2k+1)\} \cup \{12\dots k(k+1)\} \end{aligned}$$

while if n is even, say $n = 2k$, we have

$$\begin{aligned} B &= X^{(\le k-1)} \cup \{\, x \in X^{(k)} \mid 1 \in x \,\} - \{\text{last } k\text{-set with } 1\} \cup \{\text{first } k\text{-set without } 1\} \\ &= X^{(\le k-1)} \cup \{\, x \in X^{(k)} \mid 1 \in x \,\} - \{1(k+2)\dots(2k)\} \cup \{234\dots k(k+1)\} \end{aligned}$$



Once we have proved this, we are done, as in each case $|N(B)| \ge |N(C)|$.

Proof. Suppose $x \notin B, y \in B$ for some x, y with $x < y$ in simplicial. We cannot have $i \in x, i \in y$ (as B is i-compressed), and cannot have $i \notin x, i \notin y$ (as B is i-compressed). So, for each i, i belongs to exactly one of x, y. Thus $y = x^c$.

Hence for each $y \in B$, at most one $x < y$ has $x \notin B$ (namely y^c) and for each $x \notin B$, at most $y > x$ has $y \in B$ (namely x^c).

Hence $B = \{\, z \mid z \le y \,\} - \{x\}$ where x is the immediate predecessor of y and $x = y^c$. But then x = last k-set (if $n = 2k+1$) or the last k-set containing 1 (if $n = 2k$) by definition of simplicial ordering. □

$\square$

Remark.

1. Can also prove Harper's theorem by UV-compressions.
2. Can also use these 'codimension 1' compressions to prove Kruskal-Katona theorem.

Definition (t-neighbourhood)**.** For $A \subseteq Q_n$, the ***t*-neighbourhood** of A is

$$A_{(t)} \coloneqq N^t(A) = \{x \in Q_n \mid d(x, A) \leq t\}.$$

Corollary 2.3. Let $A \subseteq Q_n$ with $|A| = |X^{(\leq r)}|$. Then, for $1 \leq t \leq n - r$, have

$$|A_{(t)}| \geq |X^{(\leq r+t)}|.$$

Proof. Harper's theorem and induction. $\square$

To get a feel for what Corollary 2.3 is saying, we'll need some estimates for things like $\sum_{i=0}^{r} \binom{n}{i}$.

Proposition 2.4. Let $0 \leq \epsilon < \frac{1}{4}$, then

$$\sum_{i=0}^{\lfloor (\frac{1}{2} - \epsilon)n \rfloor} \binom{n}{i} < \frac{1}{\epsilon} e^{(-\epsilon^2 \frac{n}{2})} 2^n$$

(an exponentially small fraction of 2^n for ϵ fixed).

Roughly we are going around $\epsilon\sqrt{n}$ standard deviations from the mean.

Proof. For $i \leq (\frac{1}{2} - \epsilon)n$:

$$\frac{\binom{n}{i-1}}{\binom{n}{i}} = \frac{i}{n-i+1} \leq \frac{\frac{1}{2} - \epsilon}{\frac{1}{2} + \epsilon} = 1 - \frac{2\epsilon}{\frac{1}{2} + \epsilon} \leq 1 - 2\epsilon.$$

Hence

$$\sum_{i=0}^{\lfloor (\frac{1}{2} - \epsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\epsilon} \binom{n}{\lfloor (\frac{1}{2} - \epsilon)n \rfloor}$$

as a geometric progression. Similarly, replacing ϵ with $\frac{\epsilon}{2}$,

$$\begin{aligned}
\binom{n}{\lfloor (\frac{1}{2} - \epsilon)n \rfloor} &\leq (1 - \epsilon)^{\epsilon \frac{n}{2} - 1} \binom{n}{\lfloor (\frac{1}{2} - \frac{\epsilon}{2})n \rfloor} \\
&\leq 2(1 - \epsilon)^{\frac{\epsilon}{2}} 2^n \\
&\leq 2e^{-\epsilon \cdot \epsilon \frac{n}{2}} 2^n,
\end{aligned}$$

using $e^{-x} \geq 1 - x$ in the final step. Thus

$$\sum_{i=0}^{\lfloor (\frac{1}{2} - \epsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\epsilon} 2e^{-\epsilon^2 \frac{n}{2}} \cdot 2^n.$$

$\square$

Lecture 9 **Theorem 2.5.** Let $A \subseteq Q_n$ with $\frac{|A|}{2^n} \ge \frac{1}{2}$, and $0 < \epsilon < \frac{1}{4}$. Then

$$\frac{|A_{(\epsilon n)}|}{2^n} \ge 1 - \frac{2}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}.$$

'$\frac{1}{2}$-sized sets have exponentially large ϵn-neighbourhoods'

Proof. Enough to show that if ϵn an integer, then

$$\frac{|A_{(\epsilon n)}|}{2^n} \ge 1 - \frac{1}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}.$$

We have

$$|A| \ge \sum_{i=0}^{\lceil \frac{n}{2} - 1 \rceil} \binom{n}{i}$$

So by Harper's theorem, we have

$$|A_{(\epsilon n)}| \ge \sum_{i=0}^{\lceil n(\frac{1}{2} + \epsilon) - 1 \rceil} \binom{n}{i}$$

i.e.,

$$|{A_{(\epsilon n)}}^c| \le \sum_{i=\lceil n(\frac{1}{2}+\epsilon) \rceil}^{n} \binom{n}{i} = \sum_{i=0}^{\lfloor n(\frac{1}{2} - \epsilon) \rfloor} \binom{n}{i} \le \frac{1}{\epsilon} e^{-\epsilon^2 \frac{n}{2}} \cdot 2^n. \qquad \square$$

2.1 Concentration of measure

Definition (Lipschitz)**.** Say $f : Q_n \to \mathbb{R}$ is **Lipschitz** if $|f(x) - f(y)| \le 1$ for all $x, y \in Q_n$ adjacent.

Definition (Median)**.** Say $M \in \mathbb{R}$ is a **median** or **Lévy mean** of f if

$$|\{x \mid f(x) \le M\}|, \ |\{x \mid f(x) \ge M\}| \ge \frac{1}{2} \cdot 2^n.$$

Now we are ready to show 'every well-behaved function on Q_n is roughly constant nearly everywhere'.

Theorem 2.6. Let $f : Q_n \to \mathbb{R}$ be Lipschitz with median M, and $0 < \epsilon < \frac{1}{4}$. Then

$$\frac{|\{x \mid |f(x) - M| \le \epsilon n\}|}{2} \ge 1 - \frac{4}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}.$$

Proof. Let $A = \{x \mid f(x) \le M\}$. Then $\frac{|A|}{2^n} \ge \frac{1}{2}$, so

$$\frac{|A_{(\epsilon n)}|}{2^n} \ge 1 - \frac{2}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}.$$

But $x \in A_{(\epsilon n)} \implies f(x) \le M + \epsilon n$ (as f is Lipschitz), so

$$\frac{|\{x \mid f(x) \le M + \epsilon n\}|}{2^n} \ge 1 - \frac{2}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}.$$

Similarly,

$$\frac{|\{x \mid f(x) \le M - \epsilon n\}|}{2^n} \ge 1 - \frac{2}{\epsilon} e^{-\epsilon^2 \frac{n}{2}}$$

and intersect these. □

Remark. This is the '**concentration of measure** phenomenon'.

Let G be a graph of diameter D (recall the diameter is $\max\{d(x,y) \mid x, y \in G\}$). Let

$$\alpha(G, \epsilon) = \max \left\{ 1 - \frac{|A_{(\epsilon D)}|}{|G|} \;\middle|\; A \subseteq G, \frac{|A|}{|G|} \ge \frac{1}{2} \right\}.$$

So $\alpha(G, \epsilon)$ *small* says '$\frac{1}{2}$-sized sets have big ϵD-neighbourhoods.'

Definition (Lévy family)**.** Say a sequence of graphs $G_1, G_2, \dots$ is a **Lévy family** if $\alpha(G_n, \epsilon) \to 0$ as $n \to \infty$ (for each fixed ϵ).

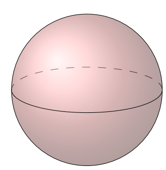
Thus Theorem 2.5 says $Q_1, Q_2, Q_3, \dots$ forms a Lévy family and even a **normal** Lévy family, meaning $\alpha(G_n, \epsilon)$ exponentially small in n (for each fixed ϵ). Thus we have concentration of measure as in (Theorem 2.6) for any Lévy family.

Many natural families of graphs form Lévy families, e.g. symmetric group S_n - made into a graph by joining x, y if xy^{-1} is a transposition.

Similarly, we can define $\alpha(X, \epsilon)$ for X any metric measure space (of finite diameter and finite measure) - so again have concentration of measure for any Lévy family.

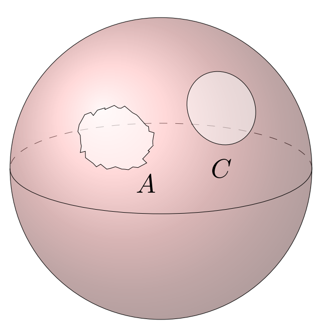
Example. Take the sphere $S^n \subseteq \mathbb{R}^{n+1}$.

$S^2 \subseteq \mathbb{R}^3$

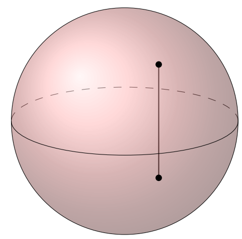


To show '$\frac{1}{2}$-sized sets have big ϵ-neighbourhoods', we need two ingredients:

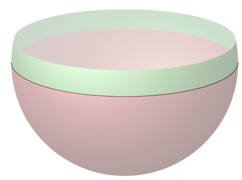
1) An isoperimetric inequality: if $A \subseteq S^n$ and $C \subseteq S^n$ is the circular cap with $|C| = |A|$, then $|A_{(\epsilon)}| \ge |C_{(\epsilon)}|$.



Compression is 'stamp on your set', i.e. we always move a point above the equator to corresponding point below the equator, called 2-point symmetrisation.



2) An estimate: Let C be the circular cap with $|C| = \frac{1}{2}$, i.e. angle $= \frac{\pi}{2}$.



Then $C_{(\epsilon)}$ is the circular cap of angle $\frac{\pi}{2} + \epsilon$. But the remaining volume is proportional to $\int_\epsilon^{\frac{\pi}{2}} (\cos\theta)^n \, d\theta$, which $\to 0$ (exponentially fast) as $n \to \infty$.

Lecture 10 We deduced concentration of measure from isoperimetric inequalities. Conversely,

Proposition 2.7. Let G be a graph such that for any Lipschitz function $G \to \mathbb{R}$ with median M, we have

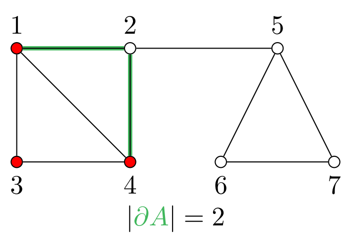
$$\frac{|\{x \in G \mid |f(x) - M| \le t\}|}{|G|} \ge 1 - \alpha$$

for some given t, α. Then for $A \subseteq G$ with $\frac{|A|}{|G|} \ge \frac{1}{2}$, we have $\frac{|A_{(t)}|}{|G|} \ge 1 - \alpha$.

Proof. The function $f(x) = d(x, A)$ is Lipschitz and has 0 as a median (as $|A| \ge \frac{1}{2}|G|$). $\square$

2.2 Edge-isoperimetric inequalities

Definition (Edge boundary)**.** For $A \subseteq G$ (G a graph), the **edge-boundary** of A is $\partial A = \{xy \in E(G) \mid x \in A, y \notin A\}$.



In Q_n how should we choose A (with $|A|$ given) to minimise $|\partial A|$? Take for example $|A| = 4$ in Q_3:

$|\partial A| = 6$ $\qquad$ $|\partial A| = 4$

This suggests that perhaps subcubes are best. But what if $2^k < |A| < 2^{k+1}$? Maybe fill up all of Q_k, then add in subcubes.

Definition (Binary ordering)**.** Say $x < y$ in the **binary order** on Q_n if $\max(x \triangle y) \in y$.

Equivalently $x < y$ if $\sum_{i \in x} 2^i < \sum_{i \in y} 2^i$: 'go up in subcubes'.

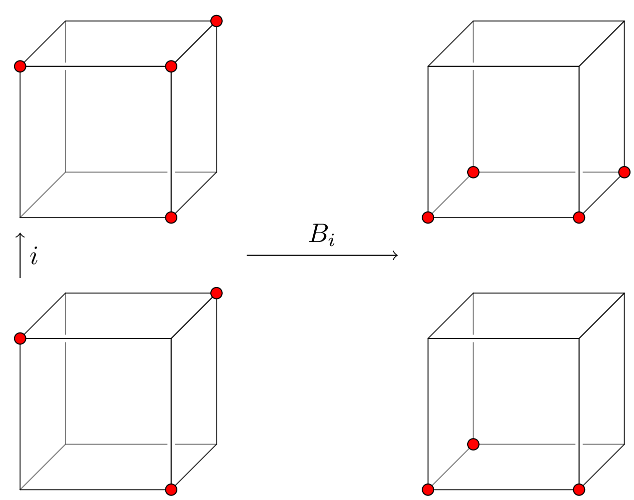
Aim. Initial segments of binary are best for (minimise) ∂A. In particular,

$$|A| = 2^k \implies |\partial A| \ge 2^k(n-k).$$

Definition (Binary compression)**.** For $A \subseteq Q_n$ and $1 \le i \le n$, the i**-binary compression** $B_i(A)$ is defined by giving its i-sections:

$$B_i(A)_+^{(i)} = \text{first } |A_+^{(i)}| \text{ elements of } \mathcal{P}(X - \{i\}) \text{ in binary}$$
$$B_i(A)_-^{(i)} = \text{first } |A_-^{(i)}| \text{ elements of } \mathcal{P}(X - \{i\}) \text{ in binary}$$

Note that $|B_i(A)| = |A|$. Say A is i-**binary compressed** if $B_i(A) = A$.



A $\qquad$ $B_i(A)$

Theorem 2.8 (Edge-isoperimetric inequality in the cube)**.** Let $A \subseteq Q_n$, and let C be the initial segment of the binary ordering with $|C| = |A|$. Then $|\partial A| \ge |\partial C|$. In particular,

$$|A| = 2^k \implies |\partial A| \ge 2^k(n-k).$$

Remark. Sometimes called the theorem of Harper, Lindsey, Bernstein and Hart.

Proof. Induction on n: $n = 1$ is immediate. Given $A \subseteq Q_n$ for $n > 1$ and $1 \le i \le n$: **Claim** $|\partial B_i(A)| \le |\partial A|$.

Proof of claim: Write B for $B_i(A)$. We have

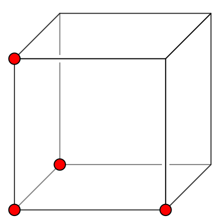
$$|\partial A| = |\partial(A_-)| + |\partial(A_+)| + |A_i \triangle A_+|$$
$$|\partial B| = |\partial(B_-)| + |\partial(B_+)| + |B_i \triangle B_+|.$$

Now, $|\partial(B_-)| \le |\partial(A_-)|$ and $|\partial(B_+)| \le |\partial(A_+)|$ by induction. Also, $|A_-| = |B_-|$, $|A_+| = |B_+|$ and B_-, B_+ nested (as both initial segments of binary). Whence $|B_- \triangle B_+| \le |A_i \triangle A_+|$. Among all $B \subseteq Q_n$ with $|B| = |A|$ and $|\partial B| \le |\partial A|$, choose one with

$$\sum_{x \in B} (\text{position of } x \text{ in binary order})$$

minimal, completing the claim. ■

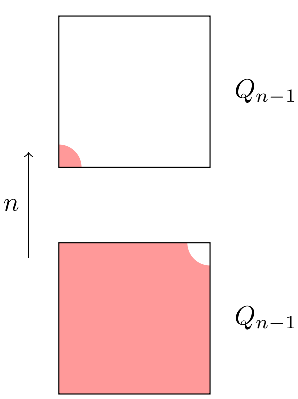
Then B is i-binary compressed $\forall i$ by claim. This B need not be an initial segment of binary, e.g.



However, **claim:** If $B < Q_n$ is i-binary compressed $\forall i$ but not an initial segment of binary, then

$$B = \mathcal{P}([n-1]) \cup \{n\} - \{123\cdots(n-1)\}.$$

(Bottom half with last point removed, first point of top half added).



Then done, as clearly $|\partial B| \ge |\partial C|$.

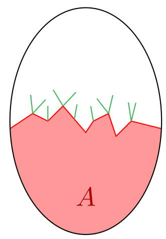
Proof of claim: Suppose $x < y$ with $x \notin B$ and $y \in B$. Then for any i, cannot have $i \notin x, y$ and cannot have $i \in x, y$ as $B_i(B) = B$. So $x = y^c$. Thus for each $y \in B$, there is at most 1 $x < y$ with $x \notin B$ (namely x^c), and for each $x \notin B$, there is at most one $y > x$ with $y \in B$ (namely y^c). Hence $B = \{\, z \mid z \le y \,\} - \{x\}$ where x is the immediate predecessor of y and $x = y^c$. Hence $y = \{n\}$. ■ □

Remark. In the proofs of Theorem 2.1 and Theorem 2.8, it was vital that the extremal sets in dimension $n-1$ were nested, i.e. were the initial segments of some ordering.

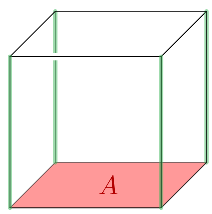
Lecture 11 **Definition** (Isoperimetric number)**.** The **isoperimetric number** of a graph G is

$$i(G) := \left\{ \frac{|\partial A|}{|A|} \;\middle|\; A \subseteq G, |A| \le \frac{1}{2}|G| \right\}.$$

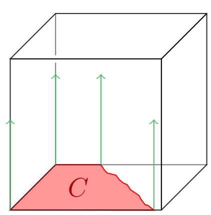
'How small can the average out-degree be?'



Corollary 2.9. $i(Q_n) = 1$.



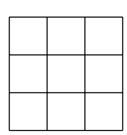
Proof. The set $A = \mathcal{P}([n-1])$ shows $i(Q_n) \le 1$. To show $i(Q_n) \ge 1$, sufficient to show (by Theorem 2.8) that if C is an initial segment of binary with $|C| \le 2^{k-1}$ then $|\partial C| \ge |C|$.



But this is clear because $C \subseteq \mathcal{P}([n-1])$. □

2.3 Inequalities in the grid

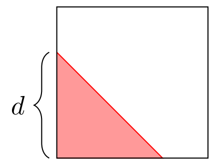
Definition (Grid)**.** The **grid** is the graph on vertex-set $[k]^n = \{1, 2, \ldots, k\}^n$ in which $(x_1, \ldots, x_n)$ is joined to $(y_1, \ldots, y_n)$ if for some i, we have $|x_i - y_i| = 1$ and $x_j = y_j \quad \forall j \ne i$ ('ℓ^1-distance').



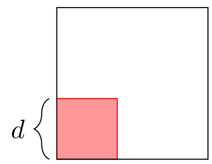
$[4]^2$

For $k = 2$, this is the discrete cube Q_n.

Do Theorem 2.1 and Theorem 2.8 have analogues in $[k]^n$? What is the best vertex-boundary? Take $[k]^2$ as an example:



has $|b(A)| \sim d \sim \sqrt{2|A|}$



has $|b(A)| \sim 2d \sim 2\sqrt{|A|}$

This suggests that sets of the form $\{x \mid |x| \leq r\}$ are best.

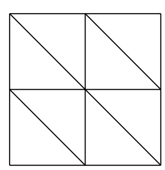
What about sizes in between? For given $|x|$, we'd 'keep x_1 big'

Definition (Simplicial ordering)**.** We define the **simplicial order** on $[k]^n$ by setting $x < y$ if either $|x| < |y|$ or $|x| = |y|$ and $x_i > y_i$, where $i = \min\{\, j \mid x_j \neq y_j \,\}$. Note: for $k = 2$ this agrees with our previous definition.

Example.

- On $[3]^2$, our ordering is

$$(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(3,2),(2,3),(3,3).$$



- On $[4]^3$, we get

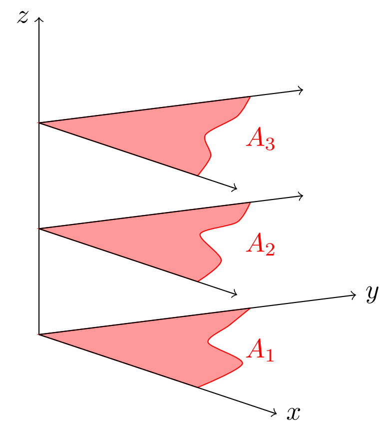
$$111, 211, 121, 112, 311, 221, 212, 131, 122, 113, 411, 321, \ldots$$

Aim. Initial segments of simplicial minimise neighbourhood. In particular,

$$|A| = |\{x \mid |x| \leq r\}| \implies |N(A)| \geq |\{x \mid |x| \leq r+1\}|.$$

Definition (Sections)**.** For $A \subseteq [k]^n$ and $1 \leq i \leq n$, the i**-sections** of A are the sets $A_1, \ldots, A_k$ (or $A_1^{(i)}, \ldots, A_k^{(i)}$) in $[k]^{n-1}$ given by

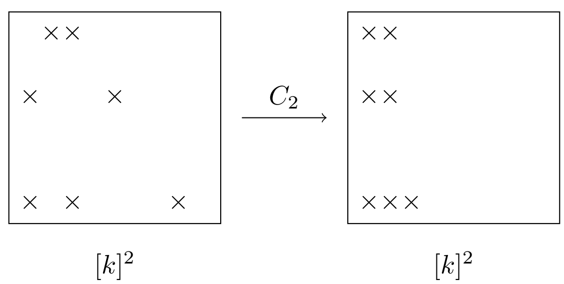
$$A_t = \{(x_1, \ldots, x_{n-1}) \in [k]^{n-1} \mid (x_1, x_2, \ldots, x_{i-1}, t, x_i, x_{i+1}, \ldots, x_{n-1}) \in A\} \quad \forall 1 \leq t \leq k$$



Definition (i-compression)**.** The ***i*-compression** $C_i(A) \subseteq [k]^n$ is defined by giving its i-sections:

$$C_i(A)_t = \text{first } |A_t| \text{ points in simplicial order on } [k]^{n-1}.$$

Certainly $|C_i(A)| \le |A|$. Say A is ***i*-compressed** if $C_i(A) = A$.



Theorem 2.10 (Vertex-isoperimetric inequality in the grid)**.** Let $A \subseteq [k]^n$ and let C be the initial segment of the simplicial order on $[k]^n$ with $|C| = |A|$. Then $|N(A)| \ge |N(C)|$. In particular,

$$|A| \ge |\{x \mid |x| \le r\}| \implies |N(A)| \ge |\{x \mid |x| \le r+1\}|$$

Proof. Induction on n: For $n = 1$, if $A \subseteq [k]^1$ with $A \ne 0, [k]^1$ then $|N(A)| \ge |A| + 1 = |N(C)|$, as required.

Given $A \subseteq [k]^n$ (for $n > 1$) and $1 \le t \le n$, **claim** $|N(C_t(A))| \le |N(A)|$.

Proof of claim: For $1 \le t \le k$,

$$N(A)_t = \overset{\text{from level } t}{\overset{\downarrow}{N(A_t)}} \cup \underset{\text{from below}}{\underset{\uparrow}{A_{t-1}}} \cup \overset{\text{from above}}{\overset{\downarrow}{A_{t+1}}}$$

(where $A_0 = A_{k+1} = \varnothing$) and similarly

$$N(B)_t = N(B_t) \cup B_{t-1} \cup B_{t+1}.$$

Now, $|B_{t-1}| = |A_{t-1}|$ and $|B_{t+1}| = |A_{t+1}|$. Also, $|N(B_t)| \le |N(A_t)|$ (induction).

But the sets B_{t-1}, B_{t+1}, $N(B_t)$ are nested (as each is an initial segment), so $|N(B)_t| \le |N(A)_t|$. This holds for each $1 \le t \le k$. ■

Among all $B \subseteq [k]^n$ with $|B| = |A|$ and $|N(B)| \le |N(A)|$, choose one with minimal

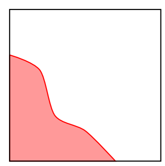
$$\sum_{x \in B} (\text{position of } x \text{ in simplicial}).$$

Then B is i-compressed $\forall i$ (else $C_i(B)$ contradicts our discussion of B).

Lecture 12

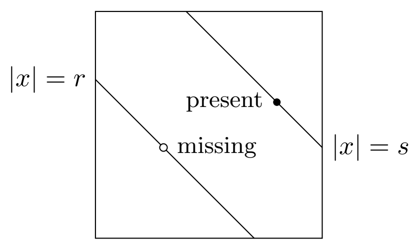
We want to show $N(B) \ge N(C)$.

- **Case 1**: $n = 2$. B is i-compressed $\forall i$ if and only if B is a **down-set** (if $x_i \le y_i$ $\forall i$ and $y \in B$) then $x \in B$). That is, 'going down or left, we stay in B'.



Suppose $B \ne C$. Let

$$\begin{aligned} r &= \min\{|x| \mid x \notin B\} \\ s &= \max\{|x| \mid x \in B\}. \end{aligned}$$



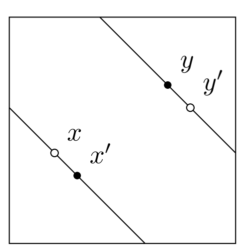
We must have $r \le s$, since $r > s \Rightarrow B = C$, as B would be an exact ball.

If $r = s$, have

$$\{x \mid |x| \le r - 1\} \subseteq B \subseteq \{x \mid |x| \le r\},$$

so clearly $|N(B)| \ge |N(C)|$.

If $r < s$, we cannot have $\{x \mid |x| = r\}$ disjoint from B (as B is a down-set and $\exists x \in B$ with $|x| = s$). Similarly cannot have $\{x \mid |x| = s\} \subseteq B$ (as B a down-set and $\exists x \notin B$ with $|x| = r$). So $\exists x, x'$ on level r with $x \notin B$, $x' \in B$ and $x' = x \pm (e_1 - e_2)$, and $\exists y, y'$ on level s with $y \in B$, $y' \notin B$ and $y' = y \pm (e_1 - e_2)$.



Now let $B' = B \cup \{x\} - \{y\}$. Then $N(B') \le N(B)$ (we lose ≥ 1 point from level $s+1$ and gain ≤ 1 point from level $r+1$), contradicting choice of B.

- **Case 2**: $n \ge 3$. If $x \in B$ then must have $x - e_n + e_i \in B$ (for any $1 \le i \le n-1$, $x_n > 1$, $x_i \le k$), because B is j-compressed for any $j \ne n, i$ (as $n \ge 3$). So $N(B_t) \subseteq B_{t-1}$. We had $N(B)_t = N(B_t) \cup B_{t-1} \cup B_{t+1}$, so $N(B)_t = B_{t-1}$. Thus

$$|N(B)| = |B_{k-1}| + |B_{k-2}| + \cdots + |B_1| + |N(B_1)| = |B| - |B_k| + |N(B_1)|$$

where B_1 is in level 2, and $N(B_1)$ is in level 1. Similarly,

$$|N(C)| = |C| - |C_k| + |N(C_1)|.$$

Thus it is sufficient to show that $|B_k| \le |C_k|$ and $|B_1| \ge |C_1|$. Focus on the former first: Define $D \subseteq [k]^n$ by

$$\begin{aligned} D_k &= B_k \\ D_t &= N(D_{t+1}), \quad t = k-1, k-2, \dots, 1 \end{aligned}$$

Then D is an initial segment of simplicial, and $D \subseteq B$, so $|D| \le |B| = |C|$, whence $D \subseteq C$ (as D, C nested as initial segments of simplicial). Hence $D_k \subseteq C_k$.

Now aim to show $|B_1| \ge |C_1|$: Define $E \subseteq [k]^n$ by

$$\begin{aligned} E_1 &= B_1 \\ E_t &= \{\, x \in [k]^{n-1} \mid N(\{x\}) \subseteq E_{t-1} \,\}, \quad t = 2, 3, \dots, k. \end{aligned}$$

Then E is an initial segment of simplicial and $E \supset B$, so $|E| \ge |B| = |C|$, whence $E \supset C$ (as E, C nested). Hence $E_1 \supset C_1$. This completes the claim, and thus completes the induction. $\square$

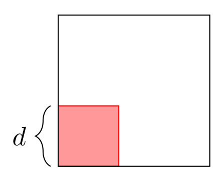
Corollary 2.11. Let $A \subseteq [k]^n$ with $|A| = |\{x \mid |x| \le r\}|$. Then $|A_{(t)}| \ge |\{x \mid |x| \le r + t\}$.

Proof. Induction on t. $\square$

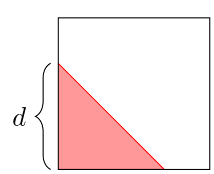
Remark. Can check from this that, for fixed k, the sequence $[k]^1, [k]^2, [k]^3, \dots$ is a normal Lévy family.

2.4 Edge-isoperimetric inequalities in the grid

We aim to minimise $|\partial A|$ in $[k]^m$. For example, in $[k]^2$:

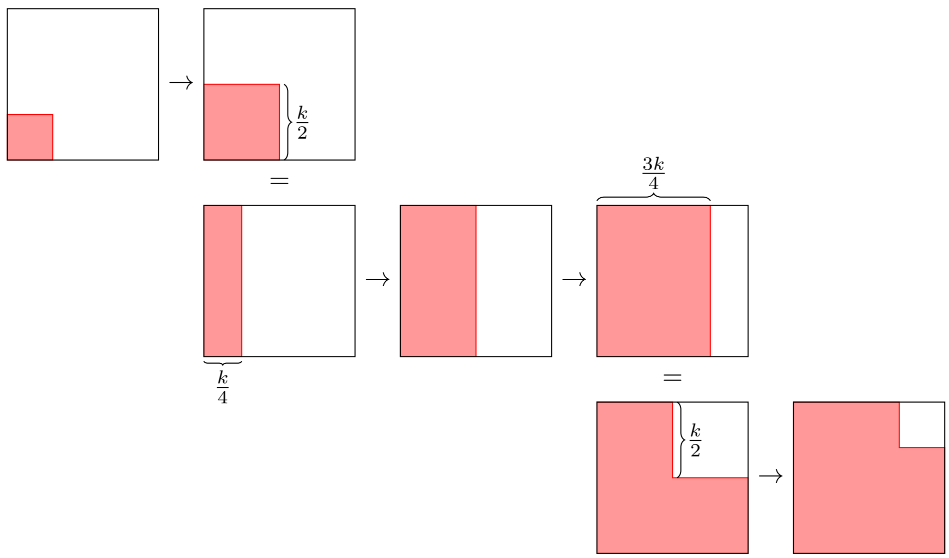


$|\partial A| \sim 2d \sim 2\sqrt{|A|}$



$|\partial A| \sim 2d \sim 2\sqrt{2|A|}$

This suggests squares are best. But,



so have 'phase transitions' at $|A| = \frac{k^2}{4}$ and $|A| = \frac{3k^2}{4}$. The extremal sets are not nested! (So cannot compress, $\nexists$ an ordering, etc.)

In $[k]^3$:

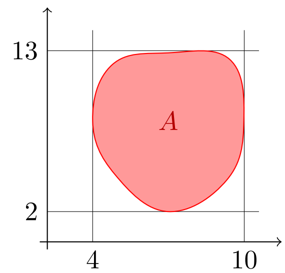
equation

Similarly in higher dimensions. This has been proved: 'edge-isoperimetric inequality in the grid'.

Very few isoperimetric inequalities are known exactly or asymptotically.

3 Projections

'If a set has small projections, must it be small?'



$$|A| \leq 6 \cdot 11$$

Lecture 13 **Definition** (Projection)**.** Let $A \subseteq \mathcal{P}(X)$. For $Y \subseteq X$ the **projection** or **trace** of A on Y is

$$A \mid Y \coloneqq \{\, x \cap Y \mid x \in A \,\}\,.$$

'Project A onto coordinates corresponding to Y'.

Example. If $A = \{14, 25, 26, 127, 128\}$. Then $A \mid \{1, 2\} = \{1, 2, 12\}$, so $A \mid Y \subseteq \mathcal{P}(Y)$.

Definition (Cover)**.** Say A **covers** or **shatters** Y if $A \mid Y = \mathcal{P}(Y)$. The **trace number** or **VC-dimension** of A is

$$\operatorname{tr} A \coloneqq \max\{|Y| \mid A \text{ shatters } Y\}.$$

Given $|A|$, how small can $\operatorname{tr} A$ be? Equivalently, if $\operatorname{tr} A < k$ (A does not shatter any k-set), how large can A be?

Trivially, must have

$$|A| \leq \left(1 - \frac{1}{2^k}\right) 2^n$$

else A shatters every k-set. Could take $A = X^{(<k)}$ - no k-set Y is shattered, as $Y \notin A \mid Y$.

Aim. This is best.

Remark. Very striking, as from each k-projection having size $\leq (1 - \frac{1}{2^k}) \cdot$ total, we are getting a very small (polynomial in n) bound on $|A|$.

Idea. Trivial that $|A| \leq |X^{(<k)}|$ if A is a down-set (if $x \in A$ and $y \subseteq x$ then $y \in A$). Indeed, must have $A \subseteq X^{(<k)}$, since if A contains a set x with $|x| \geq k$ then $A \mid x = \mathcal{P}(x)$. So 'try to make A into a down-set'.

Definition (Down compression)**.** For $A \subseteq \mathcal{P}(X)$ and $1 \leq i \leq n$, the i**-down-compression** of A is defined as follows: For $x \in \mathcal{P}(X)$, set

$$D_i(x) = \begin{cases} x & \text{if } i \notin x \\ x - \{i\} & \text{if } i \in x \end{cases}$$

and set

$$D_i(A) = \{\, D_i(x) \mid x \in A \,\} \cup \{\, x \in A \mid D_i(x) \in A \,\}$$

i.e. remove element i where possible.

Theorem 3.1 (Sauer-Shelah Lemma)**.** Let $A \subseteq \mathcal{P}(X)$ with $\operatorname{tr} A < k$. Then $|A| \le |X^{(<k)}|$

Proof. Given $1 \le i \le n$: Claim $\operatorname{tr}(D_i(A)) \le \operatorname{tr} A$.

Proof: Write $B = D_i(A)$. We'll show that if B shatters Y for some Y, then A shatters Y. If $i \notin Y$ then $B \mid Y = A \mid Y$, so may assume $i \in Y$. Given $z \subseteq Y$ with $i \notin z$, we'll show $z, z \cup \{i\} \in A \mid Y$. Since $z \cup \{i\} \in B \mid Y$, we have $z \cup \{i\} \cup x \in B$, for some $x \subseteq X \setminus Y$. Hence $z \cup x$ and $z \cup \{i\} \cup x \in A$ (by definition of D_i) whence $z, z \cup \{i\} \in A \mid Y$, completing the claim.

Now let $D = D_n(D_{n-1}(\cdots D_1(A)\cdots))$. Then $|D| = |A|$, D is a down-set, and $\operatorname{tr} D \le \operatorname{tr} A < k$. Thus $|D| \le |X^{(<k)}|$. □

Remark. We had 1-dimensional compression.

Have: if all k-dimensional projections have size $\le 2^k - 1$, then A small:

$$|A| \le \sum_{i=0}^{k-1} \binom{n}{k}.$$

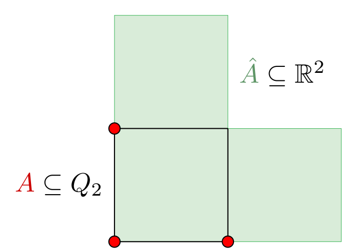
What about other bounds? For example, what if each k-dimensional projection is $\le \frac{1}{2}$-sized: $|A \mid Y| \le 2^{k-1}$?

Definition (Box)**.** A **box** or **brick** in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $a_i \le b_i \ \forall i$.

Definition (Body)**.** A **body** $S \subseteq \mathbb{R}^n$ is a finite union of bricks. Write $|S|$ or $m(S)$ for the volume of S.

Remark.

1. Everything is unchanged if we only assume S compact (or just bounded and measurable).
2. For $A \subseteq \mathcal{P}(X) \leftrightarrow \{0,1\}^n$ we have corresponding body $\hat{A} \subseteq \mathbb{R}^n$ with $m(\hat{A}) = |A|$, namely:
$$\hat{A} = \bigcup_{x \in A} [x_1, x_1 + 1] \times [x_2, x_2 + 1] \times \cdots \times [x_n, x_n + 1].$$



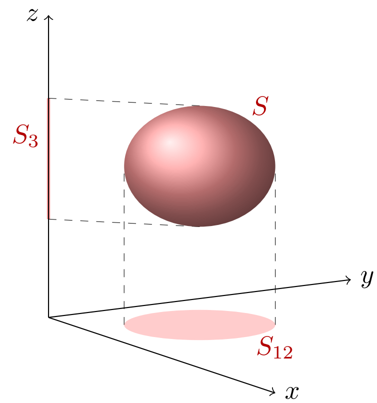
For a body $S \subseteq \mathbb{R}^n$, and $Y \subseteq \{1, \ldots, n\}$, write S_Y for the projection of S onto the subspace spanned by the e_i, $i \in Y$.

Example. For $S \subseteq \mathbb{R}^3$: S_1 is the projection of S onto the x-axis, i.e.

$$S_1 = \{x_1 \mid (x_1, x_2, x_3) \in S, \text{ some } x_2, x_3\}$$

S_{12} is the projection of S onto the xy-plane.

$$S_{12} = \{(x_1, x_2) \mid (x_1, x_2, x_3) \in S, \text{ some } x_3\}$$



Do bounds on some of the $|S_Y|$ give bounds on $|S|$?

Lecture 14 For example, for $S \subseteq \mathbb{R}^3$, we have both $|S| \le |S_1||S_2|\,|S_3|$, as $S \subseteq S_1 \times S_2 \times S_3$, and $|S| \le |S_{12}||S_3|$, as $S \subseteq S_{12} \times S_3$.

But $|S_{12}||S_{13}|$ does not bound $|S|$ - take for instance

$$S = \left[0, \frac{1}{N}\right] \times [0, N] \times [0, N].$$

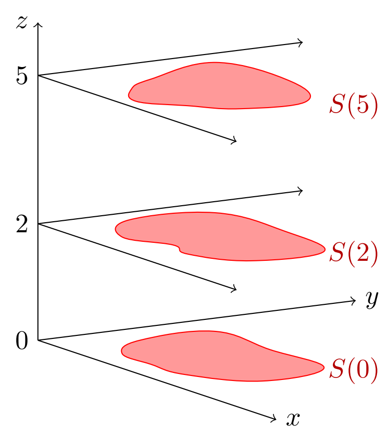
How about $|S_{12}|,\ |S_{13}|,\ |S_{23}|$?

Proposition 3.2. Let S be a body in $\mathbb{R}^3$. Then $|S|^2 \le |S_{12}||S_{13}||S_{23}|$.

Remark.

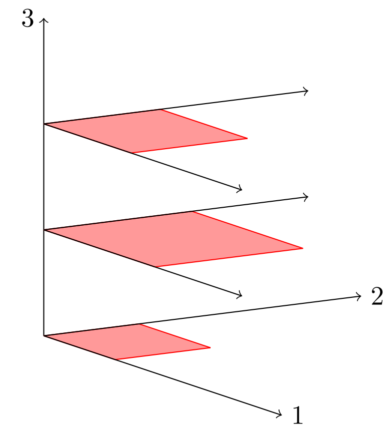
1. Can have equality, e.g. when S is a brick.
2. For $S \subseteq \mathbb{R}^n$, the **sections** of S are the sets $S(x) \subseteq \mathbb{R}^{n-1}$ (for $x \in \mathbb{R}$) given by

$$S(x) = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \ldots, x_{n-1}, x) \in S\}.$$



Proof. Suppose first that every section of S is a square:

$$S(x) = [0, f(x)] \times [0, f(x)]\ \forall x.$$



Then $|S_{12}| = M^2$, where $M = \max f$. Also,

$$|S_{13}| = |S_{23}| = \int f(x)\,dx.$$

So want

$$\left(\int f^2\right)^2 \leq M^2 \left(\int f\right)^2$$

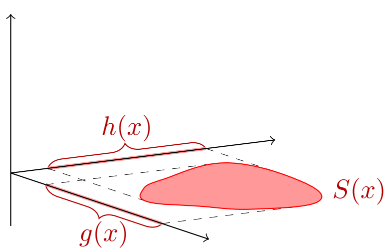
i.e. $\int f^2 \leq M \int f$, which is true because $f(x)^2 \leq M f(x)$ for all x.

For general S, define a body $T \subseteq \mathbb{R}^3$ by giving its sections:

$$T(x) = \left[0, \sqrt{|S(x)|}\right] \times \left[0, \sqrt{|S(x)|}\right].$$

So $|T| = |S|$. Certainly have $|T_{12}| \leq |S_{12}|$ - since $|T_{12}| = \max |T(x)|$. Write

$$\begin{aligned} g(x) &= |S(x)_1| \\ h(x) &= |S(x)_2| \end{aligned} \qquad \text{so} \qquad |S(x)| \leq g(x)h(x).$$



We have $|S_{13}| = \int g(x)\,dx$ and $|S_{23}| = \int h(x)\,dx$. Also,

$$|T_{13}| = |T_{23}| = \int \sqrt{|S(x)|}\,dx \leq \int \sqrt{g(x)h(x)}\,dx.$$

So, we need $(\int \sqrt{gh})^2 \leq (\int g)(\int h)$, i.e. $\int \sqrt{gh} \leq (\int g)^{\frac{1}{2}}(\int h)^{\frac{1}{2}}$, which is Cauchy-Schwarz applied to $g^{\frac{1}{2}}, h^{\frac{1}{2}}$. $\square$

Definition (Cover, uniform cover)**.** Sets $Y_1, \dots, Y_r \subseteq [n]$ **cover** $[n]$ if $Y_1 \cup \dots \cup Y_r = [n]$. They are a k**-uniform cover** if each $i \in [n]$ belongs to exactly k of $Y_1, \dots, Y_r$.

Example.

- $\{1\}, \{2\}, \{3\}$ is a 1-uniform cover of $[3]$.
- $\{1\}, \{2, 3\}$ is a 1-uniform cover of $[3]$.
- $\{1, 2\}, \{1, 3\}$ is not a uniform cover of $[3]$.
- $\{1, 2\}, \{1, 3\}, \{2, 3\}$ is a 2-uniform cover of $[3]$.

Aim. $|S|^k \le |S_{Y_1}| \cdots |S_{Y_r}|$ where $Y_1, \dots, Y_r$ a k-uniform cover of $[n]$.

Let $\mathcal{C} = Y_1, \dots, Y_r$ be a k-uniform cover of $[n]$. This is a multiset, i.e. repetition allowed e.g. $\mathcal{C} = \{1, 1, 2, 3, 23\}$ is a 2-uniform cover of $[3]$. Let

$$\mathcal{C}_- = \{Y \in \mathcal{C} \mid n \notin Y\}, \qquad \mathcal{C}_+ = \{Y - \{n\} \mid Y \in \mathcal{C}, n \in Y\}.$$

So $|\mathcal{C}_+| = k$ and $\mathcal{C}_- \cup \mathcal{C}_+$ is a k-uniform cover of $[n-1]$. Note that for a body $S \subseteq \mathbb{R}^n$, if $n \notin Y$ then $|S_Y| \ge |S(x)_Y|$ $\forall x$, e.g. $|S_1| \ge |S(x)_1|$ $\forall x$ when $S \subseteq \mathbb{R}^3$. Also, if $n \in Y$ then

$$|S_Y| = \int \left|S(x)_{Y - \{n\}}\right| \, dx$$

e.g. $|S_{13}| = \int |S(x)_1| \, dx$.

In the proof of Proposition 3.2, we used Cauchy-Schwarz:

$$\int fg \le \left(\int f^2\right)^{\frac{1}{2}} \left(\int g^2\right)^{\frac{1}{2}}.$$

Here we'll need Hölder:

$$\int fg \le \left(\int f^p\right)^{\frac{1}{p}} \left(\int g^q\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Iterating:

$$\int f_1 \cdots f_k \le \left(\int f_1^k\right)^{\frac{1}{k}} \cdots \left(\int f_k^k\right)^{\frac{1}{k}}.$$

Theorem 3.3 (Uniform covers theorem)**.** Let S be a body in $\mathbb{R}^n$, and let $\mathcal{C}$ be a k-uniform cover of $[n]$. Then $|S|^k \le \prod_{Y \in \mathcal{C}} |S_Y|$.

Proof. Induction on n: $n = 1$ case works. Given $S \subseteq \mathbb{R}^n$ for $n \ge 2$:

$$|S| = \int |S(x)| \, dx \le \int \prod_{Y \in \mathcal{C}_-} |S(x)_Y|^{\frac{1}{k}} \prod_{Y \in \mathcal{C}_+} |S(x)_Y|^{\frac{1}{k}}$$

by induction, as $\mathcal{C}_- \cup \mathcal{C}_+$ a k-uniform cover of $[n-1]$

$$\begin{aligned}
&\le \prod_{Y \in \mathcal{C}_-} |S_Y|^{\frac{1}{k}} \int \prod_{Y \in \mathcal{C}_+} |S(x)_Y|^{\frac{1}{k}} \\
&\le \prod_{Y \in \mathcal{C}_-} |S_Y|^{\frac{1}{k}} \prod_{Y \in \mathcal{C}_+} \left(\int |S(x)_Y|\right)^{\frac{1}{k}} \\
&\le \prod_{Y \in \mathcal{C}_-} |S_Y|^{\frac{1}{k}} \prod_{Y \in \mathcal{C}_+} \left|S_{Y \cup \{n\}}\right|^{\frac{1}{k}} = \prod_{Y \in \mathcal{C}} |S_Y|^{\frac{1}{k}}.
\end{aligned}$$

□

Lecture 15 **Corollary 3.4** (Loomis-Whitney theorem)**.** Let S be a body in $\mathbb{R}^n$. Then

$$|S|^{n-1} \le \prod_{i=1}^{n} \left|S_{[n]-\{i\}}\right|.$$

Remark. The $n = 3$ case is Proposition 3.2.

Proof. The sets $[n] - \{i\}$ for $1 \le i \le n$ form an $(n-1)$-uniform cover of $[n]$. □

Corollary 3.5. Let $A \subseteq \mathcal{P}([n])$ and let $\mathcal{C}$ be a k-uniform cover of $[n]$. Then

$$|A|^k \le \prod_{Y \in \mathcal{C}} |A \mid Y|.$$

In particular, if $|A \mid Y| \le (2^{|Y|})^c \ \forall Y \in \mathcal{C}$ then $|A| \le (2^n)^c$.

Proof. First part: identify A with a body $\hat{A} \subseteq \mathbb{R}^n$. Second part:

$$\begin{aligned} |A|^k &\le \prod_{Y \in \mathcal{C}} |A \mid Y| \\ &\le \prod_{Y \in \mathcal{C}} (2^{|Y|})^c = (2^{\sum |Y|})^c \\ &= 2^{knc}. \end{aligned}$$ □

Aim. Bollobás-Thomason Box Theorem: For any $S \subseteq \mathbb{R}^n$, $\exists$ a box B with $|B| = |S|$ and $|B_Y| \le |S_Y| \ \forall Y \subseteq [n]$.

This looks way too strong to be true - e.g. it tells us that, to verify any proposed projection inequality, it suffices to check it on boxes.

Definition (Irreducible cover)**.** A uniform cover $\mathcal{C}$ of $[n]$ is **irreducible** if we cannot write $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ for some uniform covers $\mathcal{C}', \mathcal{C}''$.

Example. If $n = 3$, $\{12, 13, 23\}$ is irreducible, and $\{12, 3, 1, 23\}$ is not.

Lemma 3.6. There are only finitely many irreducible covers of $[n]$.

Proof. Suppose $\mathcal{C}_1, \mathcal{C}_2, \dots$ are irreducible covers. List $\mathcal{P}([n])$ as $E_1, \dots, E_{2^n}$. There are subsequences $\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \dots$ on which the number of occurrences of E_1 is (not strictly) increasing. Find a subsequence of this on which the number of occurrences of E_2 is (not strictly) increasing. Repeating, we get $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}, \dots$ on which $\forall E \subseteq [n]$, the number of occurrences of E is increasing. But then $\mathcal{C}_{j_2}$ contains $\mathcal{C}_{j_1}$, so C_{j_2} is not irreducible, a contradiction. □

Theorem 3.7 (Bollobás-Thomason Box Theorem)**.** Let $S \subseteq \mathbb{R}^n$ be a (non-empty) body. Then $\exists$ a box $B \subseteq \mathbb{R}^n$ with $|B| = |S|$ and $|B_Y| \le |S_Y| \ \forall Y$.

Proof. Without loss of generality, $|S| > 0$ and $n \ge 2$. Take real variables x_Y for each $Y \subseteq [n]$ with $Y \ne \varnothing, [n]$. Consider the inequalities

(i) $0 \le x_Y \le |S_Y| \ \forall Y$.

(ii) $x_Y \le \prod_{i \in Y} x_i$ for each $|Y| \ge 2$

(iii) $|S|^k \leq \prod_{Y \in \mathcal{C}} x_Y$, for each irreducible k-uniform cover $\mathcal{C}$ of $[n]$ and any k, with $\mathcal{C} \neq \{n\}$.

(Our hope is to find such an x_Y with $|S| = x_1 \cdots x_n$ and $x_{12} = x_1 x_2$, etc - then the box is $[0, x_1] \times \cdots \times [0, x_n]$.)

Note that (iii) therefore holds for *all* uniform covers, as it holds for irreducible ones. Now, $\exists$ a solution (e.g. set $x_Y = |S_Y|\ \forall Y$), and the solution set is compact, so $\exists$ a solution (x_Y) with $\sum_Y x_Y$ minimal. Must have $x_Y > 0\ \forall Y$ (as $|S| \leq x_Y x_{Y^c}$, by (iii)).

Claim: For each $1 \leq i \leq n$, x_i occurs on the right hand side of an inequality in (iii) in which equality holds.

Proof: Must have x_i on the right hand side of some inequality in which equality holds otherwise, we could decrease x_i (as the set of inequalities is finite). It cannot be in (i) (as $x_i > 0$). If it is in (iii), we are done. If it is in (ii), we have $x_Y = \prod_{j \in Y} x_j$, for some Y with $i \in Y$.

But x_Y must appear on right hand side of an equality (else could decrease it), which can only be in (iii). So we have a k-uniform cover $\mathcal{C}$ with $Y \in \mathcal{C}$ and $|S|^k = \prod_{z \in \mathcal{C}} x_z$. But then equality also holds for the cover $\mathcal{C}' = \mathcal{C} - \{Y\} \cup \{\,\{j\} \mid j \in Y\,\}$. Now just take an irreducible $\mathcal{C}'' \subseteq \mathcal{C}$ with $\{i\} \in \mathcal{C}''$. ■

So, for each i, have a cover $\mathcal{C}_i$ with $\{i\} \in \mathcal{C}_i$ and equality holding in (iii). Put $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$. Then $\{i\} \in \mathcal{C}\ \forall i$, and have equality for $\mathcal{C}$ in (iii). But $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ for some $\mathcal{C}''$, where $\mathcal{C}' = \{\,\{i\} : 1 \leq i \leq n\,\}$, hence have equality in (iii) for $\mathcal{C}'$, i.e. $|S| = x_1 \cdots x_n$. Now for any Y with $|Y| \geq 2$, must have $x_Y = \prod_{i \in Y} x_i$, because Y, Y^c is a uniform cover, so that

$$|S| \leq x_Y x_{Y^c} \leq \prod_{i \in Y} x_i \prod_{i \notin Y} x_i = x_1 \cdots x_n = |S|. \qquad \square$$

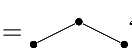
3.1 Intersecting families of graphs

Lecture 16 So far, for intersecting families our objects lived in $[n]$. What if the ground set has some structure? For example, take the ground set as $[n]^{(2)}$, the edges of a graph on $[n]$, equivalently the subgraphs of K_n. There are $2^{\binom{n}{2}}$ possible graphs.

Definition (Intersecting)**.** Let $A \subseteq \mathcal{P}([n]^{(2)})$ be a family of graphs on n vertices. For any fixed graph H, we say A is H**-intersecting** if $\forall G, G' \in A$, $G \cap G'$ contains a copy of H ("$G \cap G' \supseteq H$").

For instance, take $H = P_1$ = a single edge =

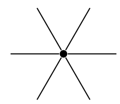
Then A is H-intersecting $\Rightarrow |A| \leq \frac{1}{2} 2^{\binom{n}{2}}$ (as we cannot have $G, G^c \subseteq A$), and can achieve this, e.g. $A = \{\, G \mid 12 \in G \,\}$. (Indeed, for any non-empty H, we have A H-intersecting $\Rightarrow |A| \leq \frac{1}{2} 2^{\binom{n}{2}}$).

What about $H = P_2 =$ ? Obvious guess: the best is $A = \{\, G \mid G \text{ contains } H_0 \,\}$, where H_0 is some fixed copy of P_2. This has size $|A| = \frac{1}{4} 2^{\binom{n}{2}}$.

But can do better, e.g. $A = \{\, G \mid d_G(1) \geq \frac{n}{2} + 1 \,\}$, where $d_G(1)$ is the number of edges out of 1. This has size

$$|A| = 2^{\binom{n}{2}} \left(\frac{1}{2} - \frac{c}{\sqrt{n}} \right) = \left(\frac{1}{2} - o(1) \right) 2^{\binom{n}{2}}.$$

Similarly if H is any star



we have H-intersecting families of size $\left(\frac{1}{2}-o(1)\right)2^{\binom{n}{2}}$.

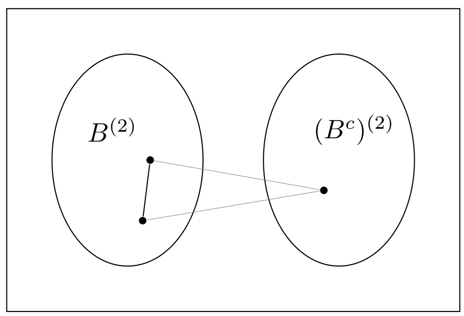
What about $\triangle$-intersecting? Obvious guess is $|A| = \frac{1}{8}2^{\binom{n}{2}}$, achieved by

$$A = \{\, G \mid G \supseteq \text{ fixed triangle} \,\}.$$

Conjecture (Simonovits-Sos)**.** If A is $\triangle$-intersecting then $|A| \le \frac{1}{8}2^{\binom{n}{2}}$.

Theorem 3.8. Let $A \subseteq \mathcal{P}([n]^{(2)})$ be $\triangle$-intersecting. Then $|A| \le \frac{1}{4}2^{\binom{n}{2}}$.

Proof. Say n even. Consider the projection of A onto the edge-set $Y = B^{(2)} \cup (B^c)^{(2)}$, for any $B \subseteq [n]$ with $|B| = \frac{n}{2}$. Then $G, G' \in A \Rightarrow G \cap G'$ must meet Y (because every triangle meets Y).



Then $A \mid Y$ is an intersecting family of sets, so

$$|A \mid Y| \le \frac{1}{2}2^{2\binom{n/2}{2}} = 2^{2\binom{n/2}{2}\cdot\left(1-\frac{1}{2\binom{n/2}{2}}\right)}.$$

But the Y form a uniform cover of $[n]^{(2)}$ (as B varies), so by Corollary 3.5 have

$$|A| \le 2^{\binom{n}{2}\cdot\left(1-\frac{1}{2\binom{n/2}{2}}\right)}.$$

So done if

$$\binom{n}{2}\frac{1}{2\binom{n/2}{2}} \ge 2.$$

But LHS $= \frac{n(n-1)}{2\frac{n}{2}(\frac{n}{2}-1)} = \frac{n-1}{\frac{n}{2}-1} > 2$.

For n odd: same with $|B| = \frac{n-1}{2}$. □

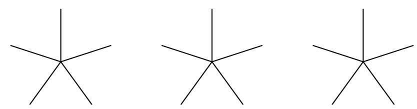
The Simonovits-Sos conjecture was proved in 2010 (Ellis, Filmus, Friedgut).

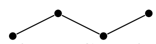
Definition (Common)**.** Say H **common** if

$$\max\{\, |A| \mid A \subseteq \mathcal{P}([n]^{(2)}) \text{ is } H\text{-intersecting} \,\} = \left(\frac{1}{2}-o(1)\right)2^{\binom{n}{2}}.$$

For instance, every star is common, and $\triangle$ not common. Any disjoint union of stars is also common, e.g. take n very large, k large and

$$A = \left\{ G \;\middle|\; \text{at least } \frac{n}{2} + 3 \text{ of vertices } 1, \ldots, k \text{ have degree } \geq \frac{n}{2} + 5 \right\}.$$



Key question: Is $P_3 =$  common? This is open. Easy fact: Every G not a union of stars contains $\triangle$ of P_3. So, if we know P_3 not common, we would know:

Conjecture (Alor's common graphs conjecture)**.** H common $\iff$ H is a union of stars.

But Christofides (2008) gave a P_3-intersecting family with density $\frac{17}{128} > \frac{1}{8}$.

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