Part II – Algebraic Geometry

Based on lectures by Prof. I. Grojnowski

Notes taken by Bhavik Mehta

Lent 2017

Consider $E = \{ (x, y) \in \mathbb{C}^2 \mid y^2 = x^3 - x \}$. Let's first draw this when $(x, y) \in \mathbb{R}^2$. If $y \in \mathbb{R}$, $y^2 \geq 0$, so if $x \in \mathbb{R}$, $x^3 - x = x(x^2 - 1) \geq 0$ so $x \geq 1$ or $-1 \leq x \leq 0$.

Now consider $(x, y) \in \mathbb{C}$. In general, this is tricky. Here, define $p : E \to \mathbb{C}$ given by $(x, y) \mapsto x$ most of the time ($x \notin \{0, 1, -1\}$), $p^{-1}(x)$ is two points. This doesn't help us visualise.

$$\Gamma = \{ (x, y) \in \mathbb{C}^2 \mid y \in \mathbb{R}, x \in [-1, 0] \cup [1, \infty) \}$$

Claim: $E \setminus \Gamma$ is disconnected and has two pieces. Proof: Exercise.

So, $E \setminus \Gamma$ is two copies of glued together. To glue, turn one of the pieces over (this ruins the representation as a double cover, but is the right gluing). Think of (pic) by adding a point at ∞, so it lives on the Riemann surface.

Take another copy, flip it over and glue back.

1 Dictionary between algebra and geometry

1.1 Basic notions

Definition (Affine space)**. Affine n-space** is $\mathbb{A}^n = \mathbb{A}^n(k) \coloneqq k^n$ for k a field.

Write $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ polynomials in n variables. Any $f \in k[\mathbb{A}^n]$ defines a function $f : \mathbb{A}^n = k^n \to k$ by $(\lambda_1, \ldots, \lambda_n) \mapsto f(\lambda_1, \ldots, \lambda_n)$ by evaluation.

Let $S \subseteq k[x_1, \ldots, x_n]$ be any subset of polynomials.

Definition (Affine variety)**.**

$$Z(S) = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in k^n \mid f(\lambda_1, \ldots, \lambda_n) = 0 \text{ for all } f \in S \}$$

is called the **affine variety defined by** S, the simultaneous zeros of all functions in S. $Z(S)$ is called an affine subvariety of $\mathbb{A}^n$.

Example.

(i) $\mathbb{A}^n = Z(0)$.

(ii) On $\mathbb{A}^1$, $Z(x) = \{0\}$, $Z(x - 7) = \{7\}$. If $f(x) = (x - \lambda_1) \ldots (x - \lambda_n)$, $Z(f(x)) = \{\lambda_1, \ldots, \lambda_n\}$. Affine subvarieties of $\mathbb{A}^1$ are $\mathbb{A}^1$ and finite subsets of $\mathbb{A}^1$.

(iii) in $\mathbb{A}^2$, $E = Z(y^2 - x^3 + x)$ we have sketched when $k = \mathbb{C}$ and $k = \mathbb{R}$.

Remark. If $f \in k[\mathbb{A}^n]$ then $Z(f)$ is called a **hypersurface**.

Observe that if J is the ideal generated by S

$$J = \left\{ \sum a_i f_i \;\middle|\; a_i \in k[x_1, \ldots, x_n], f_i \in S \right\}$$

then $Z(J) = Z(S)$. Hence,

Theorem. If $Z(S)$ is an affine subvariety of $\mathbb{A}^n$, there is a finite set $f_1, \ldots, f_r$ of polynomials with $Z(S) = Z(f_1, \ldots, f_r)$.

Proof. $J = \langle f_1, \ldots, f_r \rangle$ for some $f_1, \ldots, f_r$ by Hilbert basis theorem. □

Lemma.

(i) if $I \subseteq J$, $Z(J) \subseteq Z(I)$

(ii) $Z(0) = \mathbb{A}^n$, $Z(k[x_1, \ldots, x_n]) = \varnothing$.

(iii) $Z\left(\bigcup J_i\right) = Z(\sum J_i) = \bigcap Z(J_i)$ for any possibly infinite family of ideals

(iv) $Z(I \cap J) = Z(I) \cup Z(J)$ if I, J ideals

Proof. (i), (ii), (iii) are clear. (iv): $\supseteq$ holds by (i). Conversely, if $x \notin Z(I)$ then $\exists f_1 \in I$ such that $f_1(x) \neq 0$. So if $x \notin Z(J)$ also, $\exists f_2 \in J$ with $f_2(x) \neq 0$ also. Hence $f_1 f_2(x) = f_1(x) f_2(x) \neq 0$, so $x \notin Z(f_1 f_2)$. But $f_1 f_2 \in I \cap J$, as I, J ideals so $x \notin Z(I \cap J)$. □

Looking at these results, $Z(I)$ form closed subsets of a topology on $\mathbb{A}^n$, called the 'Zariski topology'.

If $Z \subset \mathbb{A}^n$ is any subset, let $I(Z) = \{ f \in k[\mathbb{A}^n] \mid f(p) = 0, \forall p \in \mathbb{Z} \}$. Observe that $I(Z)$ is an ideal: if $g \in k[\mathbb{A}^n]$, $f(p) = 0$ then $(gf)(p) = 0$.

Lemma.

(i) $Z \subseteq Z' \implies I(Z') \subseteq I(Z)$

(ii) for any $Y \subseteq \mathbb{A}^n$, $Y \subseteq Z(I(Y))$,

(iii) if $V = Z(J)$ is a subvariety of $\mathbb{A}^n$, then $V = Z(I(V))$.

(iv) if $J \lhd k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ an ideal, then $J \subseteq I(Z(J))$.

Proof. (i), (ii), (iv) are clear. For (iii), first show $\supseteq$. $I(V) = I(Z(J)) \supseteq J$ by (iv) so $Z(I(V)) \subseteq Z(J) = V$ by (i). $\subseteq$ follows by (iv). □

Hence (ii) and (iii) show that $Z(I(Y))$ is the smallest affine subvariety of $\mathbb{A}^n$ containing Y, i.e. it is the closure of Y in the Zariski topology.

Take $\mathbb{Z} \subseteq \mathbb{C} = \mathbb{A}^1$, $k = \mathbb{C}$ the closure of $\mathbb{Z}$ in Zariski topology is $\mathbb{C}$. $I(\mathbb{Z}) = \{0\}$ as if a poly vanishhes at every integer it is 0. Note if $k = \mathbb{C}$, $f \in \mathbb{C}[x_1, \ldots, x_n]$, then f is continuous in the usual topology, so

$$Z(J) = \bigcap_{f \in J} Z(f) = \bigcap_{f \in J} f^{-1}(0)$$

is a closed set in the usual topology, i.e. Zariski closed $Z \implies$ clsoed in the usual topology. So,

$$\{\text{Zariski closed subvarieties of}\mathbb{A}^n\} \qquad \{\text{ideals in } k[x_1, \ldots, x_n]\}$$

But this is not a bijection $Z(X) = Z(X^2) = Z(X^3) = \ldots = \{0\} \subseteq \mathbb{A}^1$. $Z(\langle f_1^{a_1}, \ldots, f_r^{a_r}\rangle) = Z(f_1, f_2, \ldots, f_r)$. but it turns out this kind of thing is the only problem

Definition. An affine variety Y is **reducible** if $\exists$ affine varieties Y_1, Y_2, $Y_i \neq Y$ with $Y = Y_1 \cup Y_2$, and irreducible otherwise, and disconnected if $Y_1 \cap Y_2 = \varnothing$.

So $Z(xy) = Z(x) \cup Z(y)$, reducible. $Z(y(y-1), x(y-1)) = Z(xy) \cup Z(y-1)$ reducible and disconnected.

Proposition. Any affine variety is a finite union of irreducible affine varieties.

Remark. This is very different from usual manifolds.

Proof. If not, Y is not irreducible, so $Y = Y_1 \cup Y_1'$ and one of Y_1, Y_1', (say Y_1) is not the finite union of irreducible affine varieties, so

$$Y_1 = Y_2 \cup Y_2', \ldots$$

and so we get an infinite chain of affine varities $Y \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots$. But each $Y_i = Z(I_i)$ for some ideal I_l. Let $W = \bigcap Y_l = Z(\sum I_i) = Z(I)$. $I = \sum I_i$ is an ideal. As the ideal I is finitely generated $I = \langle f_1, \ldots, f_r\rangle$ for some f_i. $f_i \in I_{a_i}$ for some $a_1, \ldots, a_r$ so $I = I_{a_1} + \cdots + I_{a_r}$, $W = Y_{i_1} \cap \cdots \cap Y_{i_r}$ contradicting $Y_N \subsetneq Y_{a_1} \cap \cdots \cap Y_{a_r}$ if $N > r$. □

Exercise. If Y is a subvariety of $\mathbb{A}^{\mathbb{N}}$, $Y = Y_1 \cup \cdots \cup Y_r$ with Y_i irreducible, and r minimal is unique up to reordering. Call the Y_l the irreducible components of Y.

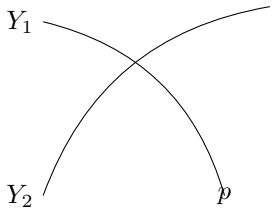
Proposition. Y is irreducible $\iff$ $I(Y)$ is a prime ideal in $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$.

Example. (i) (xy) is not a prime ideal.

(ii) Exercise: Let R be a UFD, $f \in R$, $f \neq 0$, f irreducible $\iff$ (f) a prime ideal.

(iii) Exercise: $k[x_1, \ldots, x_n]$ is a UFD. hence $Z(y^2 - x^3 + x)$ is irreducible, $Z(y - x^2)$ is irreducible.

Proof. If $Y = Y_1 \cup Y_2$ is reducible, $\exists p \in Y_1 \setminus Y_2$ so $\exists f \in I(Y_2)$ such that $f(p) \neq 0$ and similarly, $\exists q \in Y_2 \setminus Y_1$ so $\exists g \in I(Y_1)$ such that $g(q) \neq 0$. Then $fg \in I(Y_1) \cap I(Y_2) = I(Y)$. But $f \notin I(Y)$, $g \notin I(Y)$ so not prime.



Conversely, if $I(Y)$ is not prime $\exists f_1 f_2 \in k[\mathbb{A}^n]$ such that $f_1, f_2 \notin I(Y)$ but $f_1 f_2 \in I(Y)$. Let $Y_i = Y_n \cap Z(f_i) = \{\, p \in Y \mid f_i(p) = 0 \,\}$. $Y_1 \cup Y_2 = Y$, as $p \in Y \implies f_1 f_2(p) = 0 \implies f_1(p) = 0$ or $f_2(p) = 0$. $Y_i \neq Y$ as $f_i \notin I(Y)$ (i.e. $\exists p_l \in Y$ such that $f_i(p_i) \neq 0$ so $p_i \notin Y_i$). □

Lemma. X irreducible affine subvariety of $\mathbb{A}^n$, $\mathcal{U} \subseteq X$ open and non-empty $\implies \overline{\mathcal{U}} = X$.

Proof. Let $Y = X - \mathcal{U}$, closed. Then $\overline{\mathcal{U}} \cup Y = X$, and $\mathcal{U} \neq \varnothing \implies Y \neq X$. But X is irreducible, so $\overline{\mathcal{U}} = X$. □

Application: Cayley-Hamilton Theorem $A \in \mathrm{Mat}_n(k)$, an $n \times n$ matrix, with

$$\mathrm{char}_A(x) = \det(xI - A) \in k[x]$$

the characteristic polynomial. This gives a function $\mathrm{char}_A : \mathrm{Mat}_n(k) \to \mathrm{Mat}_n(k)$ $B \mapsto \mathrm{char}_A(B)$. Cayley-Hamilton theorem says that $\forall A \in \mathrm{Mat}_n(k)$, $\mathrm{char}_A(A) = 0$. Notice this is an equality of matrices, so it is n^2 equations.

Proof. Let $X = \mathbb{A}^{n^2} = \mathrm{Mat}_n(k)$, affine space, hence irreducible algebraic variety. Consider $CH = \{\, A \in \mathrm{Mat}_n(k) \mid \mathrm{char}_A(A) = 0 \,\}$. Claim: this is a Zariski closed subvariety of $\mathbb{A}^{n^2}$, cut out by n^2 equations, $\mathrm{char}_A(A)_y = 0$. We must check that these equations are polynomials in the matrix coefficients of A.

Consider $\mathrm{char}_A(x) \in k[\mathbb{A}^{n^2+1}] = \det(xI - A)$, a polynomial in x and in the matrix coefficients of A.

$$\mathrm{char}_{\begin{pmatrix} a & b//c \\ & d \end{pmatrix}}(x) = \det \begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} = x^2 - (a+d)x + (ad - bc)$$

The ijth coefficient of A^r is also a polynomial (of deg r) in the matrix coefficients of A, eg

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & \dots \\ \vdots & \ddots \end{pmatrix}$$

hence $\mathrm{char}_A(A)_y = 0$ is a poly in the matrix coefficients of A, proving the claim.

Now, it is enough to prove the theorem when when $k = \overline{k}$, as $\mathrm{Mat}_n(k) \subseteq \mathrm{Mat}_n(\overline{k})$. Next, notice that $\mathrm{char}_A(x) = \mathrm{char}_{gAg^{-1}}(x)$, for $g \in \mathrm{GL}_n$. and $\mathrm{char}_A(gBg^{-1}) = g\,\mathrm{char}_A(B)g^{-1}$ for $g \in \mathrm{GL}_n$. Hence $\mathrm{char}_A(A) = 0 \iff \mathrm{char}_{gAg^{-1}}(gAg^{-1}) = 0$, so $A \in CH \iff gAg^{-1} \in CH$. Now, let $\mathcal{U} = \{\, A \in \mathrm{Mat}_n(k) \mid A \text{ has distinct eigenvalues} \,\}$. As $k = \overline{k}$, $A \in \mathcal{U} \implies \exists g \in \mathrm{GL}_n$ with

$$gAg^{-1} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and it is clear that $gAg^{-1} \in CH$. As $k = \overline{k}$, $\#k$ is infinite, so $\mathcal{U}$ is non-empty so

$$\varnothing \neq \mathcal{U} \subseteq CH \subseteq \mathbb{A}^{n^2} = X$$

hence if we show that $\mathcal{U}$ is Zariski open in X then $\mathcal{U} = X$, as X is irreducible. But CH is closed, so $\mathcal{U} \subseteq CH$, so $CH = X$.

Finally, we must show $\mathcal{U}$ is Zariski open. Observe $A \in \mathcal{U} \iff \mathrm{char}_A(x) \in k[x]$ has distinct roots. Now recall from Galois theory, if $f(x)$ is a polynomial, $\exists$ poly $D(f)$ in the coefficients of the poly f such that f has distinct roots $\iff D(f) \neq 0$.

So $A \in \mathcal{U} \iff D(\mathrm{char}_A(x)) \neq 0$ is a polynomial in matrix coefficients of A. □

1.2 Nullstellensatz

Suppose $Y \subseteq \mathbb{A}^n$ is a subvariety, let $I(Y) = \{ f \in k[x_1, \ldots, x_n] \mid f(Y) = 0 \}$. Recall we have maps

$$\begin{array}{ccc} k[\mathbb{A}^n] & \longrightarrow & \{\text{functions from } k^n = \mathbb{A}^n \to k\} \\ & \searrow & \downarrow \\ & & \{\text{functions from } Y \to k\} \end{array}$$

where the composite is constructed by restricting a function from $\mathbb{A}^n \to k$ to $Y \to k$. Also note that the top map is injective if $\#k = \infty$.

Definition (Polynomial functions on subvariety)**.** Let $k[Y] = k[x_1, \ldots, x_n]/I(Y)$ by the **polynomial functions on** Y, also called **regular functions**.

We just observed that $k[Y] \to \{\text{all functions from } Y \to k\}$ is injective if $\#k = \infty$. We've seen Y irreducible $\iff$ $I(Y)$ is prime $\iff$ $k[Y]$ is an integral domain. Now let $p \in Y$. We have a map $k[Y] \to k$, given by $f \mapsto f(p)$. This is an algebra homomorphism, so the kernel

$$m_p = \{ f \in k[Y] \mid f(p) = 0 \}$$

is an ideal. (The homomorphism is surjective as constants go to constants). This is a maximal ideal, as R/M a field $\iff$ M is a maximal ideal in R and we have $k[Y]/m_p = k$.

A natural question to ask now is whether or not there are any other maximal ideals in $k[Y]$? In particular, what are the possible surjetive algebra homomorphisms

$$k[x_1, \ldots, x_n] \twoheadrightarrow L, \quad k \subseteq L, L \text{ field.}$$

For example, suppose $Y = Z(x^2 + 1)$ and $k = \mathbb{R}$. Then $k[Y] = \frac{\mathbb{R}[x]}{x^2+1}$ is not of the above form, since it is $\mathbb{C}$ instead of $\mathbb{R}$.

Claim: This is the only issue. If $k = \overline{k}$, there are no other algebra homomorphisms $k[Y] \to k$ other than evaluating at points $p \in Y$, and if $k \neq \overline{k}$ you just get for L algebraic extensions of k, as in the above example.

Theorem (Nullstellensatz, v1)**.** Let $m \subseteq k[x_1, \ldots, x_n]$ be a maximal ideal, and $A = k[x_1, \ldots, x_n]/m$. Then A is finite dimensional over k.

Remark. A is finite dimensional over k $\iff$ every $a \in A$ is algebraic over k. (Proof: $\Rightarrow$ clear, as $1, a, a^2, \ldots$ can't all be linearly independent over k. $\Leftarrow$ image of $x_1, \ldots, x_n$ in A each satisfy an algebraic relation over k and they generate A).

Corollary. If k is algebraically closed, then $k \hookrightarrow A$ is an iso, ie $A \cong k$, that is, every maximal ideal is of the form $M = (x_1 - p_1, \ldots, x_n - p_n)$ for $p \in k^n$.

Proof. M a maximal ideal $\implies$ A a field, but if $k \subseteq \overline{k}$ that means $k = \overline{k}$ algebraic over k. Now let a_i be the image of x_i in A, and M is as stated. So if $k = \overline{k}$, solutions of equations $I \longleftrightarrow$ max ideal $M \subseteq k[Y] \longleftrightarrow$ alg homomorphisms $k[Y] \to k$ and if $k \neq \overline{k}$, then they are 'galois orbits of solutions over bigger fields'. $\square$

We can interpret this in the case $k \neq \overline{k}$ as saying: to study solutions of algebraic equations over K, i.e. simultaneous zero of an ideal I, it is necessary to study their solutions over fields bigger than k, such as $\overline{k}$.

Proof. When k is uncountable: If the result is not true, $\exists t \in L \setminus k$ with t transcendental over k. In particular, $k(t) \subseteq L$. SO $\frac{1}{t-\lambda} \in L, \forall \lambda \in k$. But L has countable dimension over k (let V_d be the k-vector space which is the image of $\{ f \in k[x_1, \ldots, x_n] \mid \deg f \leq d \}$, V_d is finite dimensional, $\bigcup V_d = L$). Now consider $\frac{1}{t-\lambda}, \ldots, \frac{1}{t-\lambda_r}$ for $\lambda_1, \ldots, \lambda_r \in k$ distinct. If these are linearly dependent over k, i.e. $\exists a_i \in k$ with $\sum \frac{a_i}{t-\lambda_i} = 0$, then clearing denominators gives a poly relation in t, contradicting t is transcendental. So they are linearly independent, but there are uncountably many $\lambda \in k$, a contradiction. □

Corollary. If $k = \overline{k}$, take $I \leq k[x_1, \ldots, x_n]$ an ideal. Then $Z(I) \neq \varnothing \iff I \neq k[x_1, \ldots, x_n]$. More generally, $I \leq k[Y]$, $Z(I) \neq \varnothing \iff I \neq k[Y]$.

Note if $k \neq \overline{k}$, this is obviously false.

Proof. For $I \leq k[Y] = k[x_1, \ldots, x_n]/I(Y)$, replace I by its inverse image in $k[x_1, \ldots, x_n]$ to see it suffices to prove the specific case instead of the general case.

If $I \neq k[x_1, \ldots, x_n]$, then $I \subseteq m \subsetneq k[x_1, \ldots, x_n]$ for m a maximal ideal. I is contained in some maximal ideal. But Nullstellensatz gives $Z(m) = \{p\}$ for some $p \in k^n$. So $Z(I) \supseteq Z(m) = \{p\} \neq 0$. □

Remark. This means, any ideal of equations which aren't all the equations have a simultaneous solutions. This is equivalent to the Nullstellensatz.

Definition (Radical ideal)**.** Take R a ring, $J \lhd R$ an ideal. The **radical** is

$$\sqrt{J} \coloneqq \{ f \in R \mid \exists n \geq 1, f^n \in J \} \supseteq J$$

Lemma. $\sqrt{J}$ is an ideal.

Proof. If $\gamma \in R$, $f \in \sqrt{J}$, then $(\gamma f)^n = \gamma^n f^n \in J$ if $f^n \in J$. If $f, g \in \sqrt{J}$ with $f^n \in J$, $g^m \in J$ for some n, m then $(f+g)^{n+m} = \sum \binom{n+m}{i} f^i g^{n+m-i}$. Either $i \geq n$ so $f^i \in J$ or $n + m - i \geq m$ then $g^{n+m-i} \in J$, so $f + g \in J$. □

Example. (1) $\sqrt{(x^n)} = (x)$ in $k[x]$.

(2) if J is a prime ideal, $\sqrt{J} = J$.

(3) if $f \in k[x_1, \ldots, x_n]$ is irreducible, then (f) is prime as $k[x_1, \ldots, x_n]$ is a UFD, so $\sqrt{(f)} = (f)$.

Observe $Z(\sqrt{J}) = Z(J)$.

Theorem (Nullstellensatz, v2)**.** If $k = \overline{k}$, $I(Z(J)) = \sqrt{J}$.

Proof. Let $f \in I(Z(J))$, i.e. $f(p) = 0 \forall p \in Z(J)$. We must show that $\exists n$ such that $f^n \in J$. Consider $k[x_1, \ldots, x_n, t]/tf - 1 \coloneqq k[x_1, \ldots, x_n, \frac{1}{f}]$. Let i be the ideal of this, generated by the image of J. Claim: $Z(I) = \varnothing$. Proof: If not, let $p \in Z(I)$. As $J \subseteq I$, we have $p \in Z(J)$ and so $f(p) = 0$. But $p = (p_1, \ldots, p_n, p_t)$ with $p_t \cdot f(p_1, \ldots, p_n) = 1$, so $f(p) \neq 0$, contradiction. But now the corollary to Nullstellensatz version 1 gives $I = k[x_1, \ldots, x_n, \frac{1}{f}]$. So, $1 \in I$. But I is generated by J, so this says $1 = \sum_1^N \gamma_i / f^i$ for some $\lambda_i \in J$, $\gamma_N \neq 0$ for some N. Clear denominators and we get

$$f^N = \sum \tilde{\gamma_i}, \tilde{\gamma_i} \in J, i.e. f^N \in J.$$

□

Remark. This proof uses $k[x_1, \ldots, x_n, t]/tf - 1 \twoheadleftarrow k[\mathbb{A}^{n+1}]$. This is $k[Y]$, where $Y = Z(tf - 1) \subseteq \mathbb{A}^{n+1}$ and $Z(tf - 1) = \{ (p, t_0) \mid f(p)t_0 = 1 \}$. Clearly $Y \xrightarrow{\sim} \{ p \in \mathbb{A}^n \mid f(p) \neq 0 \} = \mathbb{A}^n \setminus Z(f)$.

We will return to this, but first lets deduce some consequences of Nullstellensatz version 2.

Corollary. If $k = \overline{k}$, $Z(I) = Z(J) \iff I(Z(I)) = I(Z(J)) \iff \sqrt{I} = \sqrt{J}$. So we have a bijection