Part II – Graph Theory

Based on lectures by Prof. P. Russell
Notes taken by Bhavik Mehta

Michaelmas 2017

0 Introduction

0.1 Preliminary

0.2 Informal definitions

0.3 Where do such structures arise?

1 Ramsey Theory

Definition (Graph)**.** A **graph** is an ordered pair $(V, E) = G$ where V is a finite set and E is a set of unordered pairs of distinct elements of V. We call elements of V **vertices** of G and elements of E **edges**. We often write $v \in G$ to mean $v \in V$ and sometimes, where clear, $e \in G$ to mean $e \in E$. Often denote $\{u, v\} \in E$ by uv. Note $uv = vu$.

Definition (Isomorphism)**.** Let $G = (V, E)$ and $G' = (V', E')$ be graphs. An **isomorphism** from G to G' is a bijection $\phi : V \to V'$ such that for all $u, v \in V$, we have $\phi(u)\phi(v) \in E'$ if and only if $uv \in E$. If such an isomorphism exists, we say G is **isomorphic** to G'.

Definition (Subgraph)**.** Suppose also $H = (W, F)$ is a graph. We say H is a **subgraph** of G and write $H \subset G$ if $W \subset V$ and $F \subset E$. Often, we say 'H is a subgraph of G' to mean 'H is isomorphic to a subgraph of G'.

Definition (Complete graph of order n)**.** The **complete graph of order** n, K_n has n vertices with every pair forming an edge.

Definition (Ramsey number)**.** Let $s, t \geq 2$. The Ramsey number $R(s, t)$ is the least n such that whenever K_n has edges coloured red/green there must be a red K_s or a green K_t (if such an n exists). We also write $R(s) = R(s, s)$.

Definition (Infinite graph)**.** An **infinite graph** is an ordered pair $G = (V, E)$ where V is an infinite set and E is a set of unordered pairs of elements of V. Note, in our terminology, an infinite graph is not a graph.

Definition ((Possibly infinite) graph)**.** A **(possibly infinite) graph** is a graph or an infinite graph.

Definition (Infinite complete graph)**.** K_∞, the **infinite complete graph**, is the infinite graph with a countably infinite vertex set and every pair of vertices forming an edge.

1.1 Basic Terminology

Definition (Neighbourhood)**.** Let $v \in G$. Then **neighbourhood** of v is the set

$$\Gamma(v) = \{\, w \in G \mid vw \in E(G) \,\}$$

If $w \in \Gamma(v)$, then w is a **neighbour** of v, or w is **adjacent** to v, we write $w \sim v$.

Definition (Degree)**.** The **degree** of v is $d(v) = |\Gamma(v)|$, the number of vertices adjacent to v.

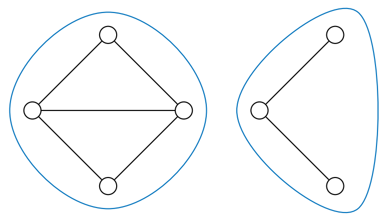
The **maximum degree** of G is $\Delta(G) = \max_{v \in G} d(v)$
The **minimum degree** of G is $\delta(G) = \min_{v \in G} d(v)$
The **average degree** of G is $\frac{1}{|G|} \sum_{v \in G} d(v)$

Definition (Regular)**.** If every vertex in G has the same degree, we say G is **regular**. If this degree is r, say G is **r-regular**.

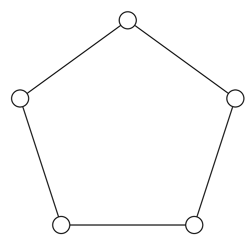
Definition (Path)**.** Let G be a graph. A **path** in G is a finite sequence $v_0, v_1, \ldots, v_l$ of distinct vertices of G with $v_{i-1} \sim v_i$ for $1 \leq i \leq l$.

We say this path has length l and goes from v_0 to v_l. If $v, w \in G$ we write $v \to w$ to mean there is a path from v to w.

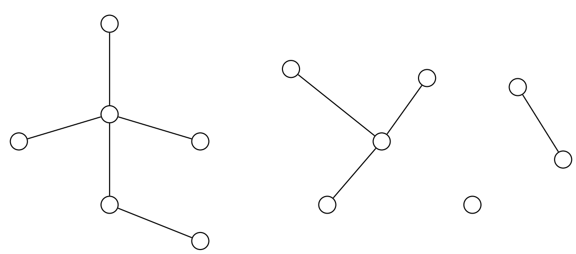
Definition (Components)**.** The equivalence classes of $\to$ are called the **components** of G. If G has only one component, we say G is **connected**.



Definition (Cycle)**.** A **cycle** is a sequence $v_0, v_1, \ldots, v_l$ of vertices of G with $v_0, \ldots, v_{l-1}$ distinct, $v_l = v_0$, $v_{i-1} \sim v_i$ for $1 \leq i \leq l$ and $l \leq 3$. We say that the length of the cycle is l.



Definition (Forest)**.** A graph with no cycles is called a **forest**. A **tree** is a connected forest. Each component of a forest is a tree.



Definition (Disjoint union)**.** Suppose G, H are graphs with $V(G) \cap V(H) = \varnothing$. The **disjoint union** of G, H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

We often write $G \cup H$ even if $V(G) \cap V(H) \neq \varnothing$, this means take graphs G', H' with $G' \cong G$, $H' \cong H$, $V(G') \cap V(H') = \varnothing$ then take $G' \cup H'$.

Definition (Induced subgraph)**.** Let $G = (V, E)$ be a graph, and let $W \subset V$. The **induced subgraph** on W is the graph $G[W]$ with $V(G[W]) = W$ and, for $x, y \in W$, $xy \in E(G[W]) \iff xy \in G$.

Definition (Complement)**.** Let $G = (V, E)$ be a graph. The **complement** of G is the graph $\overline{G}$ with $V(\overline{G}) = V$, and for distinct $x, y \in V$, $xy \in E(\overline{G}) \iff xy \notin E$.

2 Extremal Graph Theory

2.1 Forbidden Subgraph Problem

Definition (Extremal number)**.** Define

$$\mathrm{ex}(n; H) = \max \{ e(G) \mid |G| = n, H \not\subset G \}$$

2.1.1 Triangles

Definition (Bipartite graph)**.** A graph G is **bipartite** (with bipartition (X, Y)) if $V(G)$ can be partitioned as $X \cup Y$ in such a way that if $e \in E(G)$ then $e = xy$ for some $x \in X$, $y \in Y$.

Definition (Complete bipartite graph)**.** Let $s, t \geq 1$. The **complete bipartite graph** $K_{s,t}$ has bipartition (X, Y) with $|X| = s$, $|Y| = t$ and $xy \in E(K_{s,t})\ \forall x \in X, y \in Y$.

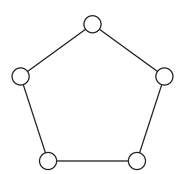
2.1.2 Complete graphs

Definition (r-partite graph)**.** A graph G is **r-partite** if we can partition $V(G) = X_1 \cup \cdots \cup X_r$ in such a way that if $xy \in E(G)$ then $x \in X_i, y \in X_j$ for some $i \neq j$. We say G is **complete r-partite** if whenever $x \in X_i$, $y \in X_j$ with $i \neq j$ then $xy \in E(G)$.

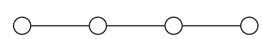
Definition (Turán graph)**.** The **Turán graph** $T_r(n)$ is the complete r-partite graph with n vertices and vertex-classes as equal as possible. Write $t_r(n) = e(T_r(n))$.

2.1.3 Bipartite graphs

Definition (Cyclic graph)**.** The **cyclic graph of order** n, is the cycle of length n, called C_n.



Definition (Path graph)**.** The **path graph** of order n is the path of length n, called P_n.



Definition (t-fan)**.** A **t-fan** in a graph G is an ordered pair (v, W) where $v \in V(G)$, $W \subset V(G)$, $|W| = t$ and $\forall w \in W$, $v \sim w$.

2.1.4 General graphs

Definition (Asymptotic extremal number)**.** Write

$$\mathrm{ex}(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(n; H)}{\binom{n}{2}}$$

which exists by **??**.

Definition (Complete r-partite graph)**.** Write $K_r(t)$ for the **complete r-partite graph** with t vertices in each class (so $K_r(t) = T_r(rt)$).

Definition (Chromatic number)**.** If H is a graph, the **chromatic number** of H, denoted $\chi(H)$, is the least r such that H is r-partite.

Definition (Density)**.** We can define the **density** of a graph G to be

$$D(G) = \frac{e(G)}{\binom{|G|}{2}} \in [0, 1].$$

Definition (Upper density)**.** The upper density of an infinite graph G is

$$\mathrm{ud}(G) = \lim_{n\to\infty} \sup \{ D(H) \mid H \subset G, |H| = n \} .$$

2.1.5 Proof of Erdős-Stone (non-examinable)

2.2 Hamiltonian graphs

Definition (Hamiltonian)**.** A **Hamiltonian cycle** in a graph G is a cycle of length G, i.e. going through all vertices of G. If G has a Hamiltonian cycle, we say G is **Hamiltonian**.

Definition (Euler circuit)**.** A **circuit** of a graph G is a sequence $v_0 v_1 \dots v_n$ of vertices of G, not necessarily distinct with $v_0 = v_n$, where if $1 \leq i \leq k$ then $v_{i-1} \sim v_i$ and if $1 \leq i < j \leq k$ then edges $v_{i-1}v_i$ and $v_{j-1}v_j$ are distinct. It is an **Euler circuit** if for every $e \in E(G)$, there is some i with $e = v_{i-1}v_i$. If G has an Euler circuit we say G is **Eulerian**.

3 Graph Colouring

Definition (Colouring)**.** A k-**colouring** of a graph G is a function $c : V(G) \to [k]$. In proofs we often say 'red', 'green' for 1,2, etc.

3.1 Planar Graphs

Definition (Graph drawing)**.** A **drawing** of G is an ordered pair (f, γ) where $f : V \to \mathbb{R}^2$ is an injection and $\gamma : E \to C([0,1], \mathbb{R}^2)$ such that

(i) If $uv \in E$ then $\{\gamma(uv)(0), \gamma(uv)(1)\} = \{f(u), f(v)\}$.

(ii) If $e, e' \in E$ with $e \neq e'$ then $\gamma(e)((0,1)) \cap \gamma(e')((0,1)) = \varnothing$.

(iii) If $e \in E$ then $\gamma(e)$ is injective.

(iv) If $e \in E$ and $v \in V$ then $f(v) \notin \gamma(e)((0,1))$

That is,

$$\begin{aligned} \text{vertices} &\longleftrightarrow \text{points} \\ \text{edges} &\longleftrightarrow \text{continuous curves between end vertices,} \end{aligned}$$

with no unnecessary intersections. If G has a drawing, we say G is **planar**.

Definition (Subdivision)**.** Let G be a graph. A **subdivision** of G is a graph formed by repeatedly selecting $vw \in E(G)$, removing vw and adding vertex u and edges uv, uw.

Definition (Leaf)**.** A **leaf** of a tree is a vertex of order 1.

Definition (Faces)**.** If we have a drawing of a graph, it divides the plane into connected regions called **faces**. Precisely one of these regions, the **infinite face** is unbounded.

3.2 General Graphs

3.3 Graphs on surfaces

Definition (Chromatic number of surface)**.** Given a surface S, the **chromatic number of** S is

$$\chi(S) = \max \{\, \chi(G) \mid G \text{ can be drawn on } S \,\}$$

3.4 Edge Colouring

Definition (Edge colouring)**.** A k-edge colouring of a graph $G = (V, E)$ is a function $\varphi : E \to [k]$ such that if $e, e' \in E$ with precisely one common vertex then $\varphi(e) \neq \varphi(e')$.

Definition (Edge chromatic number)**.** The **edge-chromatic number** of G is

$$\chi'(G) = \min \{\, k \mid G \text{ has a } k\text{-edge colouring} \,\}.$$

4 Connectivity

4.1 The Marriage Problem

Definition (Matching)**.** Let G be a bipartite graph with bipartition (X, Y). A **matching** from X to Y is a set $M \subset E(G)$ such that $\forall x \in X$, $\exists$ unique $e \in M$ with $x \in e$ and for all $y \in Y$ there is at most one $e \in M$ with $y \in e$.

Definition (Independent set)**.** Let $G = (V, E)$ be a graph. A set $F \subset E$ is **independent** if no two edges of F share a vertex.

4.2 Connectivity

Definition (k-connectivity)**.** Let $k \geq 1$. We say a graph G is **k-connected** if whenever $W \subset V(G)$ with $|W| < k$ then $G - W$ is connected.

Definition (Independent paths)**.** Let G be a graph and $a, b \in V$ be distinct. A collection of paths from a to b is **independent** if the paths meet only at a and b.

Definition (AB-path)**.** Let G be a graph and $A, B \subset V(G)$. An **AB-path** is a path that meets A in its first vertex and nowhere else, and meets B in its last vertex and nowhere else. A set $W \subset V(G)$ is an AB-separator if $G - W$ contains no AB-path.

Definition (Connectivity)**.** If G is an incomplete graph, the **connectivity** of G is

$$\kappa(G) \coloneqq \max\left(\{\, k \geq 1 \mid G \text{ is } k\text{-connected} \,\} \cup \{0\}\right).$$

4.3 Edge connectivity

Definition (l-edge connected)**.** Let G be a graph with $|G| \geq 2$ and let $l \geq 1$. We say G is **l-edge connected** if whenever $D \subset E(G)$ with $|D| < l$ we have $G - D$ connected. The **edge-connectivity** of G is

$$\lambda(G) \coloneqq \max\left(\{\, l \geq 1 \mid G \text{ is } l\text{-edge connected} \,\} \cup \{0\}\right).$$

5 Probabilistic Techniques

5.1 The Probabilistic Method

5.2 Modifying a Random Graph

5.3 The Structure of Random Graphs

6 Algebraic Methods

Definition (Distance)**.** Let G be a connected graph, $u, v \in G$. The **distance** from u to v is $d(u, v)$, the length of the shortest path from u to v.

Definition (Diameter)**.** The **diameter** of a connected graph G is

$$\max_{u,v \in G} d(u, v).$$

Definition (Moore graph)**.** A **Moore graph** is a graph G such that for some k, $|G| = k^2+1$, $\Delta(G) = k$, diameter of G is 2.

6.1 The Chromatic Polynomial

Definition (Contraction)**.** Let G be a graph and $e = uv \in E(G)$. The **contraction of G over e** is the graph G/e formed from G by deleting vertices u, v, adding a new vertex e^* with $\Gamma(e^*) = \Gamma(x) \cup \Gamma(y)$.

6.2 Eigenvalues

Definition (Adjacency matrix)**.** Let G be a graph with $V(G) = \{1, 2, \dots, n\}$. The **adjacency matrix** of G is the $n \times n$ matrix A where

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{if } i \nsim j. \end{cases}$$

Definition (Walk)**.** Define a **walk** of length l from u to v to be a sequence

$$u = u_0, u_1, \dots, u_l = v$$

of (not necessarily distinct) vertices with $u_{i-1} \sim u_i$ for $1 \leq i \leq l$.

Definition (Eigenvalues)**.** If G is a graph, the **eigenvalues** of G are the eigenvalues of its adjacency matrix.

6.3 Strongly Regular Graphs

Definition (Strongly regular graph)**.** Let $k, b \geq 1$ and $a \geq 0$. A graph G is (k, a, b)**-strongly regular** if G is k-regular and, for all $x, y \in G$ with $x \neq y$

- $x \sim y \implies |\Gamma(x) \cap \Gamma(y)| = a$,
- $x \nsim y \implies |\Gamma(x) \cap \Gamma(y)| = b$,