# Part II – Graph Theory

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# 0 Introduction

## 0.1 Preliminary

## 0.2 Informal definitions

## 0.3 Where do such structures arise?

**Theorem** (Schur's Theorem)**.** Let $n$ be a positive integer. Then if $p$ is a sufficiently large positive integer, whenever $\{1, 2, \ldots, p\}$ is partitioned into $n$ parts, we can solve $a + b = c$ with $a, b, c$ all in some part.

# 1 Ramsey Theory

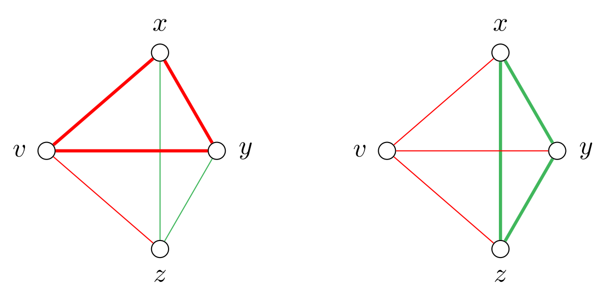
**Theorem** (Schur's Theorem reformulated)**.** Let $k$ be a positive integer. Then there is a positive integer $n$ such that if the set $[n] = \{1, 2, \ldots, n\}$ is coloured with $k$ colours, we can find $a, b, c$ with $a + b = c$ and $a, b, c$ the same colour.

*Proof.* ($k = 2$ of Schur's Theorem, improved) Suppose $[5] = \{1, 2, 3, 4, 5\}$ are coloured red and green. Then some three of these are the same colour, and without loss of generality $i < j < k$ are red. If $j - i$ is red we are done, since $i + (j - i) = j$. Similarly if $k - i$ or $k - j$ is red, we are done. If not, all of $j - i$, $k - i$, $k - j$ are green, but then $k - i = (j - i) + (k - j)$, and we are done. □

*Proof.* (Schur's theorem, $k = 3$) Suppose $[16]$ are coloured red, green and blue. By the pigeonhole principle, some six numbers are the same colour, without loss of generality $x_1 < x_2 < \cdots < x_6$ are red. If $x_j - x_i$ is red for any $i < j$ then we are done: $x_i + (x_j - x_i) = x_j$. So assume all $x_j - x_i$ are blue or green. Consider the five numbers $x_2 - x_1$, $x_3 - x_1$, $x_4 - x_1$, $x_5 - x_1$, $x_6 - x_1$. By the pigeonhole principle, some three of these are the same colour: say $x_i - x_1$, $x_j - x_1$, $x_k - x_1$ are green, for $i < j < k$.

If $x_j - x_i$ is green, we are done: $(x_i - x_1) + (x_j - x_i) = x_j - x_1$, similarly if $x_k - x_i$ or $x_k - x_j$ is green. Otherwise, all of $x_j - x_i$, $x_k - x_i$, $x_k - x_j$ are blue, and we have $(x_j - x_i) + (x_k - x_j) = x_k - x_i$, so we are done. □

*Proof.* Pick $v \in K_6$. $v$ is in five edges, so some three are the same colour, without loss of generality call them $vx, vy, vz$ and say they are red. If any of $xy, xz, yz$ is red, we have a red triangle with $v$. If not, $xyz$ is a green triangle.



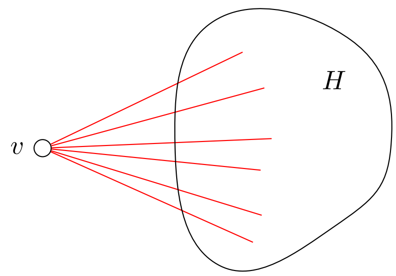

□

**Proposition 1.** Let $k$ be a positive integer. Then there is a positive integer $n$ such that whenever the edges of $K_n$ are coloured with $k$ colours we can find a monochromatic triangle.

*Proof.* Induction on $k$. For $k = 1$, $n = 3$ works, so consider $k > 1$.

By the induction hypothesis, there exists $m$ such that if $K_m$ is $(k - 1)$-coloured, then there is a monochromatic triangle. Let $n = k(m - 1) + 2$.

Now $k$-colour the edges of $K_n$. Pick vertex $v$. The number of edges containing $v$ is $n - 1 = k(m - 1) + 1$. So some $m$ of them are the same colour, without loss of generality red. Let $H$ be a $K_m$ joined to $v$ by red edges. If $H$ contains a red edge, it makes a red triangle with $v$. If not, $H$ is a $(k - 1)$-coloured $K_m$ so by definition of $m$, it contains a monochromatic triangle.




$\square$

**Theorem 2** (Ramsey's Theorem)**.** $R(s,t)$ exists for all $s,t \geq 2$. Moreover, if $s,t > 2$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

*Proof.* Induction on $s+t$.

For $s = 2$, we have $R(2,t) = t$: If all edges of a $K_t$ are green, then we have a green $K_t$, otherwise there is a red edge, which is exactly a $K_2$. Similarly $R(s,2) = s$.

In the case $s,t > 2$, let $a = R(s-1,t)$ and $b = R(s,t-1)$ which exist by the induction hypothesis. Let $n = a+b$. Suppose $K_n$ has edges coloured red/green. Pick $v \in K_n$, which is in $a+b-1$ edges, so either $v$ is in $a$ red edges, or it is in $b$ green edges.

- If $v$ is in $a$ red edges, let $H$ be the $K_a$ joined to $v$ by red edges. Now $a = R(s-1,t)$, so either $H$ has a red $K_{s-1}$, making a red $K_s$ with $v$, or $H$ has a green $K_t$, so done.

- If $v$ is in $b$ green edges, and we can make the same argument with colours reversed. $\square$

**Corollary 3.** For all $s,t \geq 2$, $R(s,t) \leq 2^{s+t}$, so $R(s) \leq 4^s$.

*Proof.* Induction on $s+t$. Base cases $s = 2$, $R(2,t) = t \leq 2^{2+t}$ and similarly for $t = 2$, $R(s,2) = s \leq 2^{s+2}$. For $s,t \geq 2$

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq 2^{s-1+t} + 2^{s+1-t} \\ &= 2^{s+t}. \end{aligned}$$

$\square$

**Theorem 4** (Multicolour Ramsey Theorem)**.** Let $k \geq 1$ and $s \geq 2$. Then there exists some $n$ such that whenever the edges of $K_n$ are coloured with $k$ colours, we can find a monochromatic $K_s$.

*Proof.* Induction on $k$. For the base case $k = 1$, we can take $n = s$.

For $k > 1$, by the induction hypothesis we can find $m$ such that if $K_m$ is $(k-1)$-coloured, then there is a monochromatic $K_s$. Let $n = R(s,m)$ and colour $K_n$ with $k$ colours, including red but not green. Re-colour by turning all non-red edges green. By definition of $n$, we have either

- A red $K_s$, so done

- A green $K_m$. Then in the original colouring, this $K_m$ was $(k-1)$-coloured, so by definition of $m$, it contains a monochromatic $K_s$. $\square$

**Theorem 5** (Infinite Ramsey Theorem)**.** Let $k \geq 1$. Whenever the edges of $K_\infty$ are $k$-coloured, we have a monochromatic $K_\infty$ subgraph.

*Proof.* Take $v_1 \in K_\infty$. The vertex $v_1$ is in infinitely many edges, so infinitely many edges from $v_1$ are the same colour. Let $A_1$ be an infinite set of vertices of $K_\infty$ such that for all $u \in A_1$, $v_1u$ has colour $c_1$. Now pick $v_2 \in A_1$. Similarly, we can find an infinite $A_2 \subset A_1$ such that all edges $v_2u$ for $(u \in A_2)$ have colour $c_2$. Keep going. We get infinite sequences $v_1, v_2, v_3, \ldots$ of vertices, $c_1, c_2, c_3, \ldots$ of colours and $A_1 \supset A_2 \supset A_3 \supset \ldots$ such that

- for $i \geq 2$, $v_i \in A_{i-1}$
- for $i \geq 1$, for all $u \in A_i$, $v_iu$ is an edge of colour $c_i$.

In particular, if $i < j$ then $v_iv_j$ has colour $c_i$. Now, infinitely many of the $c_i$ are the same. Say $i_1 < i_2 < i_3 < \ldots$ such that $c_{i_1} = c_{i_2} = c_{i_3} = \ldots$. Consider $v_{i_1}, v_{i_2}, v_{i_3}, \ldots$. Any edge between two of these vertices has colour $c_{i_1}$. So we have a monochromatic $K_\infty$. □

**Corollary 6.** Any bounded sequence has a convergent subsequence.

*Proof.* Any bounded monotone sequence converges, so it is enough to show that any real sequence $(x_n)_{n\geq 1}$ has a monotone subsequence. Let $G$ be a $K_\infty$ with vertex set $\{1, 2, 3, \ldots\}$. Colour $ij$, with $i < j$ red if $x_i < x_j$ and green if not.

By infinite Ramsey, there is a monochromatic subgraph $H \cong K_\infty$. Let the vertices of $H$ be $n_1 < n_2 < n_3 < \ldots$. If $H$ is red, then $(x_{n_j})_{j\geq 1}$ is decreasing, whereas if $H$ is green then $(x_{n_j})_{j\geq 1}$ is increasing. □

## 1.1 Basic Terminology

# 2 Extremal Graph Theory

## 2.1 Forbidden Subgraph Problem

### 2.1.1 Triangles

**Theorem 7.** A graph is bipartite iff it contains no odd cycles.

*Proof.* May return to this later, an exercise for now. $\square$

**Theorem 8** (Mantel's Theorem)**.** Let $n \geq 3$. Suppose $|G| = n$, $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$ and $\triangle \not\subset G$. Then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

*Proof.* Induction on $n$. Start with the $n = 3$ case: Consider $K_{2,1}$, it satisfies $|G| = 3$, $e(G) \geq 2$, $\triangle \not\subset G$, as required.

$n > 3$: Let $|G| = n$, $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$, $\triangle \not\subset G$. First, remove edges from $G$ if necessary to get $H$ with $|H| = n$, $e(H) = \lfloor \frac{n^2}{4} \rfloor$. Clearly $\triangle \not\subset H$. Let $v \in H$ with $d(v) = \delta(H)$ and let $K = H - v$ (i.e. $H$ with vertex $v$ and all edges including $v$ removed). Now, $|K| = n - 1$, $\triangle \not\subset K$ and $e(K) = \lfloor \frac{n^2}{4} \rfloor - \delta(H)$.

Suppose $n$ is even. Then $\delta(H) \leq$ average degree of $H = \frac{2e(H)}{H} = \frac{n^2/2}{n} = \frac{n}{2}$. Hence

$$e(K) \geq \frac{n^2}{4} - \frac{n}{2} = \frac{n^2 - 2n}{4} = \frac{n^2 - 2n + 1}{4} - \frac{1}{4} = \frac{(n-1)^2}{4} - \frac{1}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

Similarly if $n$ odd, also get $e(K) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.

Hence by the induction hypothesis, $K \cong K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$. Also, $d(v) = e(H) - e(K)$. If $n$ even, $d(v) = \frac{n^2}{4} - \frac{n^2 - 2n}{4} = \frac{n}{2}$. $H$ is formed by adding a vertex $v$ to $K \cong K_{\frac{n}{2}, \frac{n-2}{2}}$ and joining $v$ to $\frac{n}{2}$ vertices of $K$, without creating a triangle.

If $K$ has bipartition $(X, Y)$, $v$ cannot be joined both to a vertex in $X$ and a vertex in $Y$. So $v$ must be joined to all vertices in the larger of $X$, $Y$. Thus $H \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and similarly if $n$ is odd. We recover $G$ by adding edges to $H$ without making a $\triangle$. But any new edge creates a $\triangle$, so $G \cong H$. $\square$

### 2.1.2 Complete graphs

**Theorem 9** (Turán's Theorem)**.** Let $r \geq 2$ and $|G| = n \geq r + 1$. If $e(G) \geq t_r(n)$ and $K_{r+1} \not\subset G$ then $G \cong T_r(n)$.

*Proof.* Induction on $n$. Take first $n = r + 1$. $T_r(r + 1)$ has one class of 2 vertices, rest with 1 vertex each. So $T_r(r + 1)$ is $K_{r+1}$ with 1 edge removed. So $G \cong T_r(r + 1)$.

Next consider $n > r + 1$. First delete edges from $G$ to form subgraph $H$ with $|H| = n$, $e(H) = t_r(n)$ and $K_{r+1} \not\subset H$. Let $v \in H$ have minimal degree and let $K = H - v$. We know $|H| = |T_r(n)|$ and $e(H) = e(T_r(n))$ so $H$ and $T_r(n)$ have same average degree. But degrees in $T_r(n)$ are as equal as possible by property 4 earlier.

So $\delta(H) \leq \delta(T_r(n))$. Thus $|K| = n - 1$, $K_{r+1} \not\subset K$ and

$$e(K) = e(H) - \delta(H) \geq e(H) - \delta(T_r(n)) = t_r(n) - \delta(T_r(n)) = t_r(n - 1)$$

by 5. So by induction hypothesis, $K \cong T_r(n-1)$. And $d(v) = e(H) - e(K) = t_r(n) - t_r(n-1)$. To recover $H$, we must add a vertex and $t_r(n) - t_r(n - 1)$ edges to $K$ without creating a

$K_{r+1}$. So by 6, $H \cong T_r(n)$. To recover $G$, add edges to $H$ without creating a $K_{r+1}$. But by property 1 we can't add any edges. So, $G \cong T_r(n)$. $\square$

**Corollary 10.** Let $r \geq 2$. As $n \to \infty$, $\mathrm{ex}(n; K_{r+1}) \sim (1 - \frac{1}{r})\binom{n}{2}$.

### 2.1.3 Bipartite graphs

**Theorem 11.** Let $t \geq 2$. Then $\mathrm{ex}(n; K_{t,t}) = \mathcal{O}(n^{2-\frac{1}{t}})$.

*Proof.* Let $|G| = n$, $e(G) = m = \mathrm{ex}(n; K_{t,t})$ and $K_{t,t} \not\subset G$. How many $t$-fans are there in $G$? Each $v \in G$ is in $\binom{d(v)}{t}$ $t$-fans. This total number is $\sum_{v \in G} \binom{d(v)}{t}$.

Given $W \subset V(G)$ with $|W| = t$, $W$ is in at most $t-1$ $t$-fans (as $K_{t,t} \not\subset G$). So the total number of $t$-fans is $\leq \binom{n}{t}(t-1)$. Hence

$$\begin{aligned}\binom{n}{t}(t-1) &\geq \sum_{v \in G} \binom{d(v)}{t} \\ &\geq n\binom{\frac{1}{n}\sum_{v \in G} d(v)}{t} \qquad \text{by Jensen} \\ &= n\binom{\frac{2m}{n}}{t}\end{aligned}$$

So $\frac{n^t}{t!}(t-1) \geq \frac{n}{t!}(\frac{2m}{n} - t)^t$, taking the highest factor from the LHS and lowest factor from the RHS so

$$n^t(t-1) \geq n\left(\frac{2m}{n} - t\right)^t.$$

If $n$ is sufficiently large (as we may assume) then $m \geq nt$ and so $\frac{m}{n} \geq t$ and so $\frac{2m}{n} - t \geq \frac{m}{n}$. Thus

$$\begin{aligned}n^t(t-1) &\geq n\left(\frac{m}{n}\right)^t \\ m^t &\leq n^{2t-1}(t-1) \\ m &\leq (t-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} = \mathcal{O}(n^{2-\frac{1}{t}}).\end{aligned}$$ $\square$

**Theorem 12.** Let $t \geq 2$. Then $z(n,t) = \mathcal{O}(n^{2-\frac{1}{t}})$.

*Proof.* Let $G$ be bipartite with classes $X$, $Y$ with $|X| = |Y| = n$, $e(G) = m = z(n,t)$ and $K_{t,t} \not\subset G$. Count number of $t$-fans with vertex in $X$ and set in $Y$. Similar to the proof of Theorem 11,

$$\binom{n}{t}(t-1) \geq \sum_{v \in X} \binom{d(v)}{t}$$

Now $\sum_{v \in X} d(v) = m$, so a similar calculation to Theorem 11 gives $m = \mathcal{O}(n^{2-\frac{1}{t}})$. $\square$

### 2.1.4 General graphs

**Proposition 13.** Let $H$ be a graph with at least one edge, and for $n \geq |H|$, let $x_n = \frac{\mathrm{ex}(n;H)}{\binom{n}{2}}$. Then $(x_n)$ converges.

*Proof.* Let $n > |H|$. Let $|G| = n$, $e(G) = \mathrm{ex}(n; H) = x_n\binom{n}{2}$ and $H \not\subset G$. For any $v \in G$, $|G - v| = n - 1$ and $H \not\subset G - v$, so

$$e(G - v) \leq \mathrm{ex}(n - 1; H) = x_{n-1}\binom{n-1}{2}.$$

Each edge $xy \in E(G)$ is in $G - v$ for all $v \neq x, y$. Hence

$$(n-2)x_n\binom{n}{2} = (n-2)e(G) = \sum_{v \in G} e(G - v) \leq n x_{n-1}\binom{n-1}{2}.$$

So $x_n \leq x_{n-1}$, so $(x_n)$ is decreasing and bounded below by zero. □

**Theorem 14** (Erdős-Stone Theorem)**.** Let $r, t \geq 1$ be integers, and let $\epsilon > 0$ be real. Then $\exists n_0$ such that $\forall n \geq n_0$,

$$|G| = n,\ e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} \implies K_{r+1}(t) \subset G.$$

*Proof.* Coming up in Section 2.1.5, non-examinable. □

**Corollary 15.** Let $H$ be a graph with at least one edge. Then

$$\mathrm{ex}(H) = 1 - \frac{1}{\chi(H) - 1}.$$

*Proof.* Let $r = \chi(H) - 1$. Then $H$ is $(r+1)$-partite so we can find $t$ such that $H \subset K_{r+1}(t)$ (for instance $t = |H|$ suffices). Let $\epsilon > 0$, and take $n_0$ as in Theorem 14. Then for all $n \geq n_0$,

$$\begin{aligned} |G| = n,\ e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} &\implies K_{r+1}(t) \subset G \\ &\implies H \subset G. \end{aligned}$$

So, for all $n \geq n_0$, $\mathrm{ex}(n; H) < (1 + \frac{1}{r} + \epsilon)\binom{n}{2}$. Hence

$$\mathrm{ex}(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(n; H)}{\binom{n}{2}} \leq 1 - \frac{1}{r} + \epsilon.$$

But $\epsilon > 0$ was arbitrary, so $\mathrm{ex}(H) \leq 1 - \frac{1}{r}$.

On the other hand, for all $n$, $H \not\subset T_r(n)$ as $H$ is not $r$-partite. So $\mathrm{ex}(n, H) \geq t_r(n)$, and

$$\frac{t_r(n)}{\binom{n}{2}} \to 1 - \frac{1}{r} \quad \text{so} \quad \mathrm{ex}(H) \geq 1 - \frac{1}{r}.$$ □

**Corollary 16.** For any infinite graph $G$,

$$\mathrm{ud}(G) \in \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}.$$

*Proof.* It is enough to show that for $r = 1, 2, 3, \dots$,

$$\mathrm{ud}(G) > 1 - \frac{1}{r} \implies \mathrm{ud}(G) \geq 1 - \frac{1}{r+1}.$$

Suppose $\mathrm{ud}(G) > 1 - \frac{1}{r}$. Pick $\alpha$ such that $\mathrm{ud}(G) > \alpha > 1 - \frac{1}{r}$ and fix $n$. For sufficiently large $N$, $G$ has subgraphs of order $N$ and density $\geq \alpha > 1 - \frac{1}{r} = \mathrm{ex}(T_{r+1}(n))$. So $T_{r+1}(n) \subset G$, but $D(T_{r+1}(n)) \to 1 - \frac{1}{r+1}$ as $n \to \infty$. We can do this for every $n$, hence $\mathrm{ud}(G) \geq 1 - \frac{1}{r+1}$. □

### 2.1.5 Proof of Erdős-Stone (non-examinable)

*Proof.* Induction on $r$. Fix $T$ such that

$$T > \left(\frac{2}{r\epsilon}\right)^t (t-1)$$

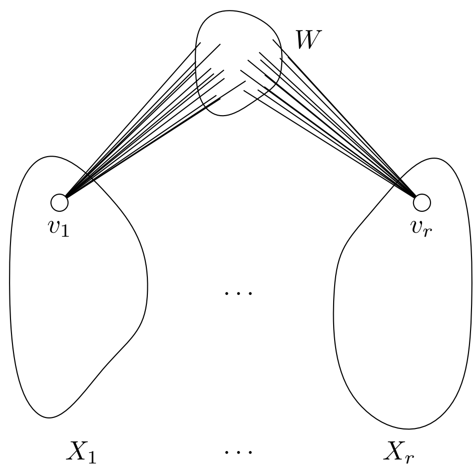
Then choose $n_0$ such that for all $n \geq n_0$,

$$|G| = n, \quad \delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n \implies K_r(T) \subset G.$$

(How? For $r = 1$, $K_1(T) \subset G \iff |G| \geq T$. For $r > 1$, $1 - \frac{1}{r} > 1 - \frac{1}{r-1}$ so it follows by inductive hypothesis.)

Suppose the result is not true. Then we can find arbitrarily large $n$ and graphs $G$ with $|G| = n$ and $\delta(G) \geq (1 - \frac{1}{r} + \epsilon)n$, $K_{r+1}(t) \not\subset G$. Pick such an $n$, $G$ with $n \geq n_0$ and also $n \geq \frac{2t}{r\epsilon}$.

Then we can find $K_r(T) \subset G$, say with vertex classes $X_1, \ldots, X_r$.




Let

$$A = \{\, (W, v_1, \ldots, v_r) \mid W \subset V(G), |W| = t, \forall i \; v_i \in X_i, \forall i \forall w \in W : v_i \sim w \,\}$$

What can we say about $|A|$? First, given $v_1 \in X, \ldots, v_r \in X_r$, we can check from the minimum degree condition that

$$|\Gamma(v_1) \cap \ldots \cap \Gamma(v_r)| \geq r\epsilon n$$

So there are at least $\binom{r\epsilon n}{t}$ choices for $W$. Hence $|A| \geq T^r \binom{r\epsilon n}{t}$.

On the other hand given the set $W$, as $K_{r+1}(t) \not\subset G$, we know that there is some $X_i$ containing at most $t-1$ vertices joined to all of $W$. Hence $|A| \leq \binom{n}{t}(t-1)T^{r-1}$, thus

$$T^r \binom{r\epsilon n}{t} \leq \binom{n}{t}(t-1)T^{r-1}.$$

Now,

$$\text{RHS} \le \frac{n^t}{t!}(t-1)T^{r-1}$$

while

$$\text{LHS} \ge T^r \frac{1}{t!}(r\epsilon n - t)^t \ge T^r \frac{1}{t!}\left(\frac{r\epsilon n}{2}\right)^t.$$

Combining, we get

$$T^r \left(\frac{r\epsilon n}{2}\right)^t \le n^t (t-1) T^{r-1}$$

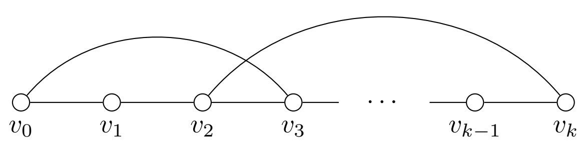
hence $T \le (\frac{2}{r\epsilon})^t(t-1)$, a contradiction. $\square$

## 2.2 Hamiltonian graphs

**Theorem 17** (Dirac's Theorem)**.** Let $|G| = n \ge 3$ and $\delta(G) \ge \frac{n}{2}$. Then $G$ is Hamiltonian.

*Proof.* First, observe $G$ is connected. Indeed, if $x \ne y$, $x \nsim y$, then $|\Gamma(x) \cup \Gamma(y)| \le n-2$, but $|\Gamma(x)| + |\Gamma(y)| \ge \frac{n}{2} + \frac{n}{2} = n$ so $\Gamma(x) \cap \Gamma(y) \ne \varnothing$.

Let $v_0, v_1, \dots, v_k$ be a path of maximal length in $G$, say length $k \le n-1$. By maximality, $\Gamma(v_0) \subset \{v_1, \dots, v_k\}$. Similarly, $\Gamma(v_k) \subset \{v_0, \dots, v_{k-1}\}$.




If we have some situation like in the diagram, we get a cycle. To be more precise, if

$$A = \{\, i \in [k] \mid v_0 \sim v_i \,\} \quad \text{and} \quad B = \{\, i \in [k] \mid v_k \sim v_{i-1} \,\}$$

then $A \cap B \ne \varnothing \implies$ we have a cycle. But $|A \cup B| \le k < n$ while $|A| + |B| \ge \frac{n}{2} + \frac{n}{2} = n$. Hence $\exists i \in A \cap B$ so we have a cycle $C = v_0 v_1 \dots v_{i-1} v_k v_{k-1} \dots v_i v_0$ of length $k+1$.

If $k = n-1$, we have a Hamiltonian cycle as required. If $k < n-1$, relabel the cycle $C = u_0 u_1 \dots u_k u_0$. By connectedness, we have some $u_i \in C$ and $w \notin C$ with $w \sim u_i$. Then $w u_i u_{i+1} \dots u_k u_0 \dots u_{i-1}$ is a path of length $k+1$, contradicting maximality. $\square$

**Proposition 18.** Let $G$ be a connected graph. Then

$$G \text{ Eulerian if and only if } \forall v \in G,\ d(v) \text{ is even.}$$

*Proof.* ($\Rightarrow$) is obvious: an Eulerian circuit must go in and out of a given vertex the same number of times.

($\Leftarrow$): use induction on $e(G)$. For $e(G) = 0$ it is clearly true.

Consider $e(G) > 0$. Let $v_0 v_1 \dots v_k = C$ be a circuit in $G$ of maximal length. If $C$ uses all edges of $G$ then we are done. If not, delete all edges used in $C$ from $G$ to form $H$. In $H$, every vertex still has even degree. Let $H_1$ be a component of $H$ with at least one edge.

By induction hypothesis, $H_1$ has an Euler circuit $D$. Certainly $C, D$ meet at some vertex $v$. Join them at $v$ to produce a longer circuit in $G$, a contradiction. (Walk along $C$ until we get to $v$, then walk all round $D$ starting/ending at $v$, then walk along the rest of $C$). $\square$

# 3 Graph Colouring

## 3.1 Planar Graphs

**Theorem 19** (Kuratowski's Theorem)**.** Let $G$ be a graph. Then $G$ planar iff $G$ contains no subdivision of $K_5$ or $K_{3,3}$.

**Proposition 20.** Every tree of order at least 2 has a leaf.

*Proof.* Let $T$ be a tree $|T| \geq 2$ and let $v_0v_1 \ldots v_k$ be a path of maximum length in $T$. Now $v_k \sim v_{k-1}$, but $v_k$ has no other neighbours in the path (as $T$ acyclic) and $v_k$ has no neighbours outside path (by maximality). Hence $v_k$ is a leaf. ☐

**Proposition 21.** Let $T$ be a tree, $|T| = n \geq 1$. Then $e(T) = n - 1$.

*Proof.* Use induction on $n$. For $n = 1$, $e(T) = 0$ as required.

For $n > 1$, let $v$ be a leaf. Then $T - v$ is a tree with $|T - v| = n - 1$ so by the induction hypothesis, $e(T - v) = n - 2$, so $e(T) = n - 1$. ☐

**Proposition 22.** Every tree is planar.

*Proof.* Let $T$ be a tree, $|T| = n$ and use induction on $n$. For $n = 0, 1$ we are done immediately.

For $n > 1$, let $v \in T$ be a leaf. By the induction hypothesis, $T - v$ can be drawn. Let $u \in T$ be the neighbour of $v$. Take a small circle around $u$ in the drawing; so the circle contains only the centre and some radii in the drawing. Hence, we can easily add $v$ and $uv$ to the drawing. ☐

**Theorem 23** (Euler's Formula)**.** Take $G$ connected and planar. Take $|G| = n \geq 1$, $e(G) = m$ with $l$ faces. Then $n - m + l = 2$.

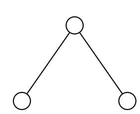
*Proof.* Induction on $m$. If $G$ is a tree: $m = n - 1$, $l = 1$, and we are done.

Otherwise, $G$ has a cycle. Pick an edge $e$ in the cycle and consider $G - e$. Then $|G - e| = n$, $e(G - e) = m - 1$. Moreover, in our drawing, removing $e$ combines two faces so we have drawn $G - e$ with $l - 1$ faces. So, by the induction hypothesis $n - (m - 1) + (l - 1) = 2$, so $n - m + l = 2$. ☐

**Corollary 24.** Let $G$ be planar, $|G| = n \geq 3$. Then $e(G) \leq 3n - 6$.

*Proof.* Let $e(G) = m$. Draw $G$ with $l$ faces. Without loss of generality, $G$ is connected (if not, add edges to make it so).

We have a special case where

$G =$ 

with $n = 3$, $m = 2$ and $3n - 6 = 3 \geq 2$.

Otherwise, we know $n - m + l = 2$ and each face has at least 3 edges in its boundary, and each edge is in the boundary of at most 2 faces. So, $l \leq \frac{2m}{3}$. Thus, $n - m + \frac{2}{3}m \geq 2$ so $\frac{1}{3}m \geq n - 2$ so $m \geq 3n - 6$. ☐

**Proposition 25** (Six colour theorem)**.** Any planar graph is 6-colourable.

*Proof.* Let $G$ be planar, $|G| = n$. Induction on $n$. For $n \leq 6$, give every vertex a different colour, as required.

For $n > 6$, by Corollary 24, $e(G) \leq 3n - 6$. Hence

$$\delta(G) \leq \frac{2e(G)}{|G|} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6.$$

So, $\delta(G) \leq 5$.

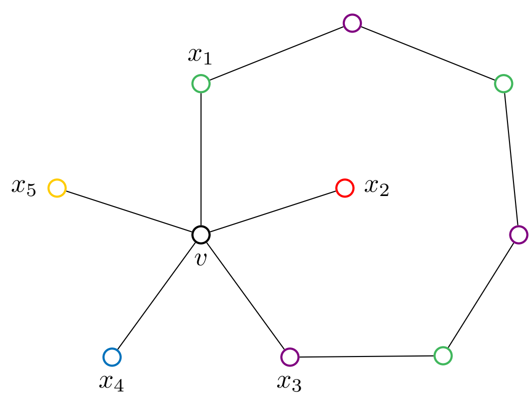
Pick $v \in G$ with $d(v) \leq 5$. By inductive hypothesis, $G - v$ can be 6-coloured. Some colour is missing from $\Gamma(v)$, use this colour to colour $v$. □

**Theorem 26** (Five colour theorem)**.** Any planar graph is 5-colourable.

*Proof.* Let $G$ be planar, $|G| = n$. Use induction on $n$. The base case $n \leq 5$ is trivial, so take $n > 5$. As in Proposition 25, we can find $v \in G$ with $\delta(v) \leq 5$ and 5-colour $G - v$. If there is a colour missing from $\Gamma(v)$, use that colour at $v$. Otherwise, draw $G$; WLOG $v$ has neighbours $x_1, \dots, x_5$ in clockwise order around $v$ with colours $1, 2, 3, 4, 5$ respectively.

Call a path in $G - v$ an $ij$**-path** if all its vertices have colour $i$ or $j$. Given $x \in G - v$, the $ij$**-component** of $x$ consists of those vertices reachable from $x$ along $ij$-paths.

If $x_1, x_3$ are in different 13 components, then swap the colours of $1, 3$ on the 13 component of $x_1$ and give $v$ colour 1.



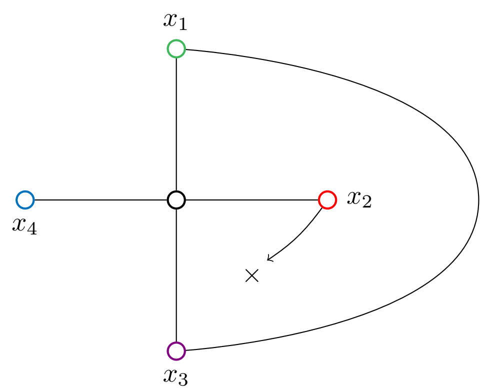

If not, then $x_2, x_4$ are in different 24 components as we see in the diagram, so swap colours $2, 4$ on the 24 component of $x_2$ and give $v$ colour 2. □

**Theorem 27** (Four colour theorem)**.** Any planar graph is 4-colourable.

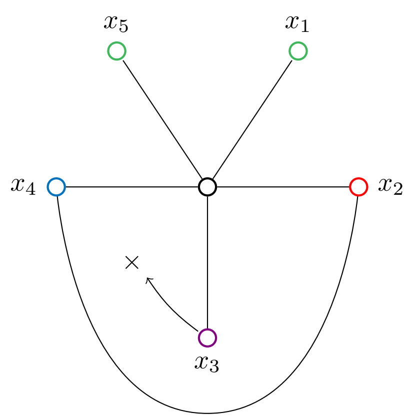
*Proof.* Let $G$ be planar, $|G| = n$ and use induction on $n$. For $n = 4$, we are immediately done.

For $n > 4$, pick $v \in G$ with $d(v) \leq 5$, then draw $G$ and 4-colour $G - v$. If some colour is missing on $\Gamma(v)$ then we are done. If not, there are three cases.

Case 1: $d(v) = 4$. There cannot be both a 13-path from $x_1$ to $x_3$ and a 24 path from $x_2$ to $x_4$, so done as in the proof of Theorem 26.

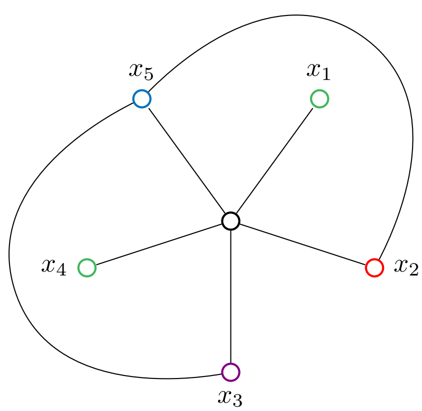



Case 2: $d(v) = 5$, $v$ has neighbours $x_1, \ldots, x_5$ clockwise with colours 1, 2, 3, 4, 1.




Then there is either no 24 path from $x_2$ to $x_4$ or no 13 path from $x_3$ to $x_1$, so again we are done.

Case 3: $d(v) = 5$, colours 1, 2, 3, 1, 4.



If there is no 24 path from $x_2$ to $x_5$, we are done. If there is no 34 path from $x_3$ to $x_5$, we are done.

Otherwise, swap colours $1, 3$ on the 13 component of $x_1$ and swap colours $1, 2$ on the 12 component of $x_4$. Then, use colour 1 at $v$, but this is false. □

## 3.2 General Graphs

**Theorem 28** (Brooks' theorem)**.** Let $G$ be a connected graph that is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

*Proof.* Induction on $|G|$. Write $\Delta = \Delta(G)$. We cannot have $\Delta = 0, 1$ as $G \ncong K_1, K_2$. If $\Delta = 2$ then $G$ is a path or an even cycle, so $\chi(G) = 2$. So assume $\Delta \geq 3$. Pick $v \in G$ and let $H$ be a component of $G - v$.

**Either** $\Delta(H) < \Delta$, in which case by greedy algorithm bound (**??**) we have

$$\chi(H) \leq \Delta(H) + 1 \leq \Delta.$$

**Or** $\Delta(H) = \Delta$. Then $H$ is connected and not an odd cycle (as $\Delta \geq 3$). Moreover, $\exists u \in H$ with $u \sim v$ in $G$. In $G$, $d(u) \leq \Delta$ so in $H$, $d(u) \leq \Delta - 1$. So $H$ is not regular, so not complete. Hence by induction hypothesis, $\chi(H) \leq \Delta$.

Do this for each component of $G - v$ to obtain a $\Delta$-colouring $c$ of $G - v$. If there is a colour missing from $\Gamma(v)$, then use that colour at $v$, as required. So assume that is not the case.

So have $\Gamma(v) = \{x_1, \ldots, x_\Delta\}$ with $\forall i, c(x_i) = i$. We can also assume:

(i) if $i \neq j$ then there is an $ij$-path $P_{ij}$ from $x_i$ to $x_j$,

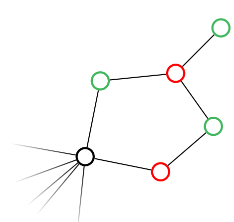
(ii) if $i \neq j$ then $P_{ij}$ is entire $ij$-component containing $x_i, x_j$, and

(iii) if $i, j, k$ distinct then $P_{ij}, P_{ik}$ meet only at $x_i$.

(Why? If any of these fail then it is easy to check that the colouring $c$ can be modified to change the colour of $x_i$ and allowing us to use colour $i$ at $v$).

(i) is as in previous proofs.

(ii) if $P_{ij}$ is not the entire $ij$-component, then at some point in the path we get a point (wlog colour $j$) which 'branches off'.



Then this vertex has 3 green neighbours, so there are at most $\Delta - 3$ colours used in the rest of its neighbours, so there is a colour not in the colours of its neighbours$\cup\{i, j\}$. We can swap it to that, breaking the chain $P_{ij}$.

(iii) if $P_{ij}, P_{ik}$ meet at some point which must have colour $i$, then that point has 2 $j$ neighbours and 2 $k$ neighbours, so as before there is a colour other than $i$ that none of its neighbours have, which we can swap it to, breaking $P_{ij}$ and $P_{ik}$.

As $G \not\cong K_{\Delta+1}$, there are some $i, j$ with $i \neq j$, $x_i \nsim x_j$. As $\Delta \geq 3$, pick $k \in [\Delta] \setminus \{i, j\}$. Let $u$ be the neighbour of $x_i$ of colour $j$. Swap colours $i, k$ on the $ik$-component of $x_i$ (i.e. on $P_{ik}$). This gives a new colouring $c'$, with $c'(x_i) = k$, $c'(x_k) = i$.

Also, if $w \in P_{ij}$ with $w \neq x_i$ then $c'(w) = c(w)$ so $c'(x_j) = c'(u) = j$. $c'$ must satisfy conditions (i),(ii),(iii) as before. Then by (i) there is a $kj$-path from $x_i$ to $x_j$ , $P'_{kj}$ . By (ii), $u \in P'_{kj}$.

By (i), there is a $ji$-path $P'_{ji}$ from $x_j$ to $x_k$. By (ii), $u \in P'_{ji}$. But now $P'_{kj}$ and $P'_{ji}$ meet at $u$, contradicting (iii). □

## 3.3 Graphs on surfaces

**Theorem 29** (Heawood's Theorem)**.** Let $S$ be a closed boundaryless surface of Euler characteristic $E \leq 1$. Then

$$\chi(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

*Proof.* Let $\chi = \chi(S)$. Let $G$ be a minimal $\chi$-colourable graph that can be drawn on $S$ - i.e. $\chi(G) = \chi$ but

$$H \subset G, H \neq G \implies \chi(G) \leq \chi - 1.$$

Clearly $G$ is connected and $|G| \geq \chi$.

Let $|G| = n \geq \chi$, $e(G) = m$ and draw $G$ on $S$ with $l$ faces. Then by Euler-Poincaré, $n - m + l \geq E$. As before, $l \leq \frac{2}{3}m$ so $n - \frac{1}{3}m \geq E$ so $m \leq 3n - 3E$. Hence

$$\delta(G) \leq \frac{2m}{n} \leq \frac{6n - 6E}{n} = 6 - \frac{6E}{n}. \qquad (*)$$

On the other hand, if $v \in G$ then $G - v$ is $(\chi - 1)$-colourable and this colouring does not extend to a $(\chi - 1)$-colouring of $G$ so $d(v) \geq \chi - 1$. Hence $\delta(G) \geq \chi - 1$.

Combining this with $(*)$: If $E \leq 0$: $\chi - 1 \leq \delta(G) \leq 6 - \frac{6E}{n} \leq 6 - \frac{6E}{\chi}$ as $n \geq \chi$. Hence $\chi^2 - 7\chi + 6E \leq 0$ and so $\chi \leq \frac{7+\sqrt{49+24E}}{2}$.

If $E = 1$, $\delta(G) \leq 6 - \frac{6}{n} < 6$ so $\delta(G) \leq 5$. Hence $\chi - 1 = 5$ so $\chi \leq 6 = \frac{7+\sqrt{49+24}}{2}$. □

## 3.4 Edge Colouring

**Theorem 30** (Vizing's theorem)**.** Let $G$ be a graph. Then

$$\chi'(G) \leq \Delta(G) + 1$$

*Proof.* Induction on $e(G)$. The $e(G) = 0$ case is immediate.

For $e(G) > 0$, let $\Delta = \Delta(G)$. Pick an edge $xy$. By the induction hypothesis we can find an $(\Delta + 1)$-edge colouring of $G - xy_1$, call it $\varphi$.

As $\Delta(G) < \Delta + 1$, there some colour 'missing' at each vertex. Let $c_1$ be missing at $y_1$. If $c_1$ is also missing at $x$, then colour $xy_1$ with colour $c$, so done.

If not, let $y_2 \in \Gamma(x)$ with $\varphi(xy_2) = c_1$ and let $c_2$ be missing at $y_2$. Continue inductively $(*)$: Given distinct $y_1, \ldots, y_n \in \Gamma(x)$, distinct colours $c_1, \ldots, c_k$ such that $c_i$ is missing at $y_i$

(for $1 \leq i \leq k$) and $\varphi(xy_{i+1}) = c_i$ (for $1 \leq i \leq k-1$). If $c_k$ is missing at $x$, re-colour $xy_i$ in colour $c_i$. Otherwise, let $y_{k+1} \in \Gamma(x)$ with $\varphi(xy_{k+1}) = c_k$. Let $c_{k+1}$ be missing at $y_k$. If $c_{k+1} \notin \{c_1, \dots, c_k\}$, continue as at $(*)$.

Otherwise assume WLOG $c_{k+1} = c_1$. (If instead $c_{k+1} = c_j$ for some $j > 1$, then uncolour $xy_j$, recolour $xy_i$ in colour $c_i$ for $1 \leq i < j$ and relabel $y_j, y_{j+1}, \dots$ as $y_1, y_2, \dots$). Let $c$ be a colour missing at $x$. Consider the $cc_1$ subgraph of $G$, i.e. only edges coloured $c$ or $c_1$. This subgraph has maximum degree $\leq 2$ so each component is a path or a cycle. Moreover, $x, y_1, y_{k+1}$ have degree $\leq 1$ in this subgraph so $x_1, y_1, y_{k+1}$ are not all in the same component.

If $x, y_1$ in different $cc_1$ component, then swap $c, c_1$ on the component of $y_1$ and give $xy_1$ colour $c$. Otherwise $x, y_{k+1}$ in different components. In this case, uncolour $xy_{k+1}$ and recolour $xy_i$ with colour $c_i$ for $1 \leq i \leq k$. Then swap colours $c, c_1$ on the $cc_1$-component of $y_{k+1}$ and give $xy_{k+1}$ colour $c$. □

# 4 Connectivity

## 4.1 The Marriage Problem

**Theorem 31** (Hall's Marriage Theorem)**.** Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ has a matching from $X$ to $Y$ iff $G$ satisfies **Hall's condition**:

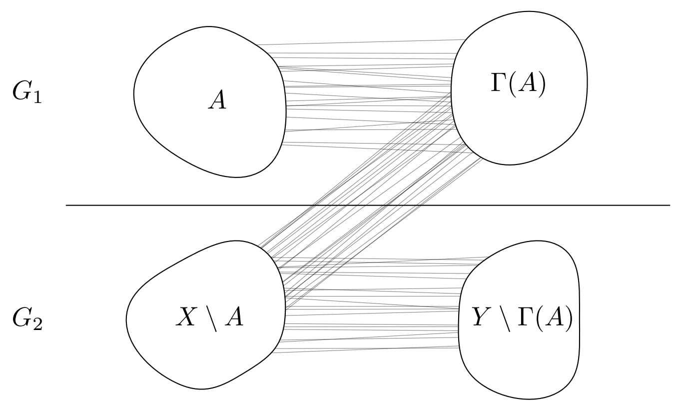
$$\forall A \subset X, |\Gamma(A)| \geq |A|$$

*Proof.* $(\Rightarrow)$ is obvious. For $(\Leftarrow)$, use induction on $|X|$. $|X| = 0, 1$ are immediate.

For $|X| \geq 2$, there are two cases. The easy case has $|\Gamma(A)| > |A|$ for all $A \subset X$ with $A \neq \varnothing, X$. Pick any $x \in X$. $|\Gamma(x)| \geq 1$ so pick $y \in \Gamma(x)$. Look at $G - \{x, y\}$; this graph satisfies Hall's condition, so has a matching from $X - \{x\}$ to $Y - \{y\}$. Add edge $xy$, and done.

The harder case has $|\Gamma(A)| = |A|$ for some $A \neq 0, X$. Let

$$G_1 = G[A \cup \Gamma(A)]$$
$$G_2 = G[(X \setminus A) \cup (Y \setminus \Gamma(A))].$$




First, $G_1$ obviously satisfies Hall's condition so $\exists$ a matching $A$ to $\Gamma(A)$. What about $G_2$? Take $B \subset X \setminus A$. Then

$$|\Gamma_{G_2}(B)| = |\Gamma(A \cup B) \setminus \Gamma(A)| = |\Gamma(A \cup B)| - |\Gamma(A)| \geq |A \cup B| - |A| = |B|.$$

Hence $G_2$ also satisfies Hall and has a matching from $X \setminus A$ to $Y \setminus \Gamma(A)$. Combine the two matchings to get a matching from $X$ to $Y$ in $G$. □

**Corollary 32** (Defect Hall)**.** Let $G$ be a bipartite graph with bipartition $(X, Y)$ and let $d \geq 1$. Then $G$ contains $|X| - d$ independent edges if and only if $\forall A \subset X,\ |\Gamma(A)| \geq |A| - d$.

*Proof.* $(\Rightarrow)$ is immediate. $(\Leftarrow)$. In marriage terminology: Introduce $d$ imaginary perfect men, suitable husbands for all the women. This satisfies Hall's condition so has matching from $X$ to $Y$. In real life, at most $d$ women unmarried. □

**Corollary 33** (Polyandrous Hall)**.** Let $G$ be a bipartite graph, bipartition $(X, Y)$, $d \geq 2$. Then $G$ contains a set of $d|X|$ edges, each vertex in $X$ in precisely $d$ of them, each vertex in $Y$ in at most one $\iff \forall A \subset X, |\Gamma(A)| \geq d|A|$.

*Proof.* (⇒) is immediate. (⇐). In marriage terminology: clone each woman $d-1$ times so there are $d$ copies of each. This satisfies Hall's condition so have matching from $X$ to $Y$. Destroy the clones. □

## 4.2 Connectivity

**Theorem 34.** Let $G$ be a graph and $A, B \subset V(G)$. Let

$$k = \min \{ |W| \mid W \text{ is an } AB\text{-separator} \}.$$

Then $G$ contains $k$ vertex-disjoint $AB$-paths.

*Proof.* Coming soon. □

**Corollary 35** (Menger's Theorem)**.** Let $G$ be an incomplete $k$-connected graph and let $a, b \in V(G)$, $a \neq b$. Then $G$ contains $k$ independent $ab$-paths.

*Proof.* Suppose first $a \nsim b$. Let $A = \Gamma(a)$ and $B = \Gamma(b)$. We have $G$ $k$-connected and $G - A, G - B$ disconnected, so $|A|, |B| \geq k$.

Hence, any $AB$-separator $W$ has $|W| \geq k$, so by Theorem 34, there are $k$ vertex-disjoint $AB$ paths. Extend these to $a$,$b$.

If instead $a \sim b$: $G - ab$ is $(k-1)$-connected, so has $k-1$ independent $ab$ paths by first part. $ab$ is another, as required. □

*Proof of Theorem 34.* Induction on $e(G)$. If $e(G) = 0$, the smallest $AB$-separator is $A \cap B$, and each vertex of $A \cap B$ gives an $AB$-path (of length zero), so done.

If $e(G) > 0$, pick $xy \in E(G)$. Let $W$ be an $AB$-separator of minimum order in $G - xy$. If $|W| \geq k$ then by the induction hypothesis there are $k$ vertex-disjoint $AB$-paths in $G - xy$ and so also in $G$.

So assume $|W| < k$. Then $W \cup \{x\}$ is an $AB$-separator in $G$. Hence $|W \cup \{x\}| \geq k$ so $|W| \geq k - 1$ so $|W| = k - 1$. Write $W = \{w_1, w_2, \ldots, w_{k-1}\}$. As $|W| < k$, $G - W$ contains an $AB$-path. This path must include the edge $xy$. Assume WLOG $x$ comes before $y$ in this path (if not, swap $x$ and $y$).

Let $X = W \cup \{x\}$ and $Y = W \cup \{y\}$. Let $U$ be an $AX$-separator in $G - xy$. Then $U$ is an $AB$-separator in $G$, so $|U| \geq k$. So by the induction hypothesis, we have $k$ vertex-disjoint $AX$-paths in $G - xy$, say $P_0, P_1, \ldots, P_{k-1}$ ending at $x, w_1, \ldots, w_{k-1}$ respectively.

Similarly, there are $k$ vertex-disjoint $YB$ paths in $G - xy$, say $Q_0, Q_1, \ldots, Q_{k-1}$ starting at $y, w_1, w_2, \ldots, w_{k-1}$ respectively.

Given path $P = u_0 u_1 \cdots u_l$ and $Q = v_0 v_1 \cdots v_m$ meeting only at $u_l = v_0$, write $P \vee Q$ for the path $u_0 \cdots u_l v_1 \cdots v_m$. Then $P_0 \vee xy \vee Q_)$ and $P_i \vee Q_i$ (for $1 \leq i \leq k-1$) are $k$ vertex-disjoint $AB$-paths in $G$. □

## 4.3 Edge connectivity

**Corollary 36** (Edge Menger)**.** Let $G$ be $l$-edge connected and $a, b \in V(G)$ be distinct. Then $G$ has $l$ edge-disjoint $ab$-paths.

*Proof.* The **line graph** of $G = (V, E)$ is the graph $L(G) = (E, F)$ where

$$F = \{ ee' \mid e, e' \in E,\ e, e' \text{ share precisely one vertex} \}.$$

Then $L(G)$ is $l$-connected. Add extra vertices $a', b'$ to $L(G)$ with

- $a'$ joined to each $e \in E$ with $a \in e$ and
- $b'$ joined to each $e \in E$ with $b \in e$.

By Corollary 35, $L(G)$ has $l$ independent $a'b'$-paths. This yields $l$ edge-disjoint $ab$-paths in $G$. $\square$

# 5 Probabilistic Techniques

## 5.1 The Probabilistic Method

**Theorem 37** (Erdős)**.**
$$R(s) = \Omega(\sqrt{2}^s)$$

*Proof.* Fix $n, s$. Colour each edge of $K_n$ red/green at random, independently, each colour equally likely. Given $H \subset K_n$ with $H \cong K_s$,

$$\mathbb{P}(H \text{ monochromatic}) = 2\mathbb{P}(H \text{ red}) = 2 \times \left(\frac{1}{2}\right)^{\binom{s}{2}}.$$

So

$$\begin{aligned}\mathbb{P}(K_n \text{ has a monochromatic } K_5) &\le \binom{n}{s} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{s}{2}} \\ &\le \frac{n^s}{s!} \cdot 2 \left(\frac{1}{2}\right)^{\binom{s}{2}} \\ &\le n^s \cdot 2^{-\frac{s(s-1)}{2}} \\ &= \left(\frac{n}{\sqrt{2}^{s-1}}\right)^s < 1 \quad \text{if } n < \sqrt{2}^{s-1}.\end{aligned}$$

So if $n < \sqrt{2}^{s-1}$ then there is *some* colouring with no monochromatic $K_5$. So $R(s) \ge \sqrt{2}^{s-1}$. □

## 5.2 Modifying a Random Graph

**Theorem 38.** If $t \ge 2$ then $z(n,t) = \Omega(n^{2-\frac{2}{t+1}})$.

*Proof.* Strategy: Given $n, p$, let $G$ be a random bipartite graph with vertex classes $X, Y$ with $|X| = |Y| = n$, where for each $x \in X$, $y \in Y$ we have $xy \in E(G)$ with probability $p$, independently. Let $A = e(G)$ and $B$ be the number of copies of $K_{t,t}$ in $G$ (so $A$,$B$ are random variables).

Aim: Given $n$, try to choose $p$ such that $\mathbb{E}(A - B)$ is large, specifically $\mathbb{E}(A-B) = \Omega(n^{2-\frac{2}{t+1}})$. Then we can find a specific graph $G$ with $A - B = \Omega(n^{2-\frac{2}{t+1}})$. Remove an edge from each $K_{t,t}$ in $G$ to form a graph $H$ with no $K_{t,t}$ and $e(H) = \Omega(n^{2-\frac{2}{t+1}})$, as required.

Now, $\mathbb{E}(A) = n^2 p$ and $\mathbb{E}(B) = \binom{n}{t}^2 p^{t^2} \le \frac{1}{(t!)^2} n^{2t} p^{t^2}$. So $\mathbb{E}(A - B) \ge n^2 p - \frac{1}{t!^2} n^{2t} p^{t^2}$.

We want $n^2 p = n^{2t} p^{t^2}$ i.e. $p = n^{\frac{2-2t}{t^2-1}} = n^{-\frac{2}{t+1}}$. So, take $p = n^{-\frac{2}{t+1}}$. Then

$$\mathbb{E}(A - B) \ge \left(1 - \frac{1}{(t!)^2}\right) n^{2-\frac{2}{t+1}}.$$ □

**Theorem 39.** Let $g \ge 3$, $k \ge 2$. Then there is a graph $G$ with no cycles of length $\le g$ and $\chi(G) \ge k$.

*Proof.* Strategy: Fix $n$ and $p$. Let $G$ be a random graph with $n$ vertices, each possible edge present independently with probability $p$. Let $X$ be the number of short cycles in $G$. Recall that $\chi(G) \geq \frac{|G|}{\alpha(G)}$ where $\alpha(G)$ is the independence number of $G$. For cycles, by 'short', we mean of length $\leq g$.

Aim: Pick $n$ and $p$ such that

1. $\mathbb{P}(X > \frac{n}{2}) < \frac{1}{2}$ and
2. $\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) > \frac{1}{2}$.

Then $\mathbb{P}(X > \frac{n}{2}$ or $\alpha(G) \geq \frac{n}{2k}) < 1$ so can pick a specific $G$ such that $X \leq \frac{n}{2}$ and $\alpha(G) < \frac{n}{2k}$. Remove a vertex from each short cycle to get $H$ with $|H| \geq \frac{n}{2}$ and $\alpha(H) < \frac{n}{2k}$ so $\chi(H) > \frac{n/2}{n/(2k)} = k$, as required.

1. For $3 \leq i \leq g$, let $X_i$ be the number of cycles of length $i$ appearing as subgraphs of $G$. Then
$$\mathbb{E}(X_i) = \binom{n}{i}\frac{i!}{2i}p^i \leq (np)^i.$$
Now $X = \sum_{i=3}^{g} X_i$ so
$$\mathbb{E}(X) \leq \sum_{i=3}^{g}(np)^i < g(np)^g \text{ as long as } np \geq 1. \tag{$*$}$$
By Markov,
$$\mathbb{P}\left(X > \frac{n}{2}\right) \leq \frac{\mathbb{E}(X)}{n/2} < 2gn^{g-1}p^g \leq \frac{1}{2}$$
if $p \leq (\frac{1}{4g})^{\frac{1}{g}} n^{\frac{1}{g}-1}$. Take $p = (\frac{1}{4g})^{\frac{1}{g}} n^{\frac{1}{g}-1}$. Then $np = \left(\frac{1}{4g}\right)^{\frac{1}{g}} n^{\frac{1}{g}} \geq 1$ if $n$ sufficiently large, satisfying the condition of ($*$).

2. 
$$\begin{aligned}\mathbb{P}\left(\alpha(G) \geq \frac{n}{2k}\right) &\leq \binom{n}{\frac{n}{2k}}(1-p)^{\binom{n/2k}{2}} \\ &\leq n^{\frac{n}{2k}} e^{-p\frac{n^2}{16k^2}} \\ &= \exp\left\{\frac{2}{2k}\log n - \frac{n^2}{16k^2}\left(\frac{1}{4g}\right)^{\frac{1}{g}} n^{\frac{1}{g}-1}\right\} \to 0 \text{ as } n \to \infty.\end{aligned}$$
So if $n$ is sufficiently large, $\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) < \frac{1}{2}$. □

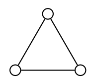
## 5.3 The Structure of Random Graphs

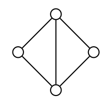
**Proposition 40.** $p = \frac{1}{n}$ is a sharp threshold for $G \in \mathcal{G}(n,p)$ to contain a $\triangle$, in the sense that:

- if $p = o(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has no $\triangle$, whereas
- if $p = \omega(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has a $\triangle$.

*Proof.* Let $G \in \mathcal{G}(n,p)$ and let $X$ be the number of $\triangle$s in $G$. Let $\mu = \mathbb{E}X$ and $\sigma^2 = \mathrm{Var}(X)$. Then $\mu = \binom{n}{3}p^3 \sim \frac{1}{6}(np)^3$. Also,

$$\sigma^2 = \binom{n}{3}(p^3 - p^6) + \binom{n}{3} \cdot 3 \cdot (n-3)(p^5 - p^6) \leq n^3p^3 + n^4p^5.$$





Suppose first $p = \wr(\frac{1}{n})$, i.e. $np \to 0$. Then by Markov,

$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \geq 1) \geq 1 - \mu \to 1 \text{ as } n \to \infty.$$

Suppose instead $p = \omega(\frac{1}{n})$ so $np \to \infty$. Then with Chebyshev,

$$\frac{\sigma^2}{\mu^2} \leq \frac{1}{\mu^2}(n^3p^3 + n^4p^5) \sim \frac{36}{n^6p^6}(n^3p^3 + n^4p^5) = \frac{36}{(np)^3} + \frac{36}{n \cdot np} \to 0 \text{ as } n \to \infty. \qquad \square$$

**Theorem 41.** There exists a function $d : \mathbb{N} \to \mathbb{N}$ such that a.e. $G \in \mathcal{G}(n,p)$ has $\omega(G) \in \{d-1, d, d+1\}$ (where $d = d(n)$).

*Proof sketch.* **(Currently missing).** $\square$

**Corollary 42.** Almost every $G \in \mathcal{G}(n,p)$ has

$$\chi(G) \geq (1 + o(1))\frac{n \log \frac{1}{q}}{2 \log n}$$

where $q = 1 - p$.

*Proof.* Let $G \in \mathcal{G}(n,p)$. Then $\overline{G} \in \mathcal{G}(n,q)$. So by Theorem 41, with probability tending to 1 as $n \to \infty$ we have

$$\begin{aligned} \omega(\overline{G}) &\sim \frac{2 \log n}{\log \frac{1}{q}} \\ \implies \alpha(G) &\sim \frac{2 \log n}{\log \frac{1}{q}} \\ \implies \chi(G) &\geq \frac{|G|}{\alpha(G)} \sim \frac{n \log \frac{1}{q}}{2 \log n}. \end{aligned} \qquad \square$$

# 6 Algebraic Methods

## 6.1 The Chromatic Polynomial

**Theorem 43** (Cut-fuse relation)**.** Let $G$ be a graph, $e \in E(G)$, $k \geq 1$. Then $f_G(k) = f_{G-e}(k) - f_{G/e}(k)$.

*Proof.* Let $e = uv$. Let $c$ be a $k$-colouring of $G - e$. If $c(u) \neq c(v)$ then $c$ is a $k$-colouring of $G$ and every $k$-colouring of $G$ arises uniquely like this. If $c(u) = c(v)$ then $c$ yields a $k$-colouring of $G/e$ and every $k$-colouring of $G/e$ arises uniquely like this.

$$f_{G-e}(k) = f_G(k) + f_{G/e}(k).$$ □

**Corollary 44.** Let $G$ be a graph. Then $f_G$ is a polynomial.

*Proof.* Induction on $e(G)$. For $e(G) = 0$, $f_G(k) = k^{|G|}$. For $e(G) > 0$, pick $e \in E(G)$. Then $f_{G-e}$, $f_{G/e}$ are polynomials by the induction hypothesis and hence $f_G = f_{G-e} - f_{G/e}$ is a polynomial. □

**Corollary 45.** If $|G| = n$, $e(G) = m$ then

$$f_G(X) = X^n - mX^{n-1} + \dots$$

*Proof.* Induction on $e(G)$. If $e(G) = 0$, then $f(X) = X^n$, as required. For $e(G) > 0$, pick $e \in E(G)$. Then

$$\begin{aligned} f_G(X) &= f_{G-e}(X) - f_{G/e}(X) = (X^r(m-1)X^{n-1} + \dots) - (X^{n-1} + \dots) \\ &= X^n - mX^{n-1} + \dots \end{aligned}$$ □

## 6.2 Eigenvalues

**Theorem 46.** Let $G$ be a graph, $\Delta(G) = \Delta$, $\lambda$ an eigenvalue of $G$. Then $|\lambda| \leq \Delta$. Moreover if $G$ is connected then $\Delta$ is an eigenvalue $\iff$ $G$ is $\Delta$-regular; in this case $\Delta$ has multiplicity 1 and eigenvector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

*Proof.* Let $A$ be the adjacency matrix and $x$ an eigenvector with eigenvalue $\lambda$. Let $i$ be such that $|x_i|$ is maximal. Without loss of generality, $x_i = 1$ and $\forall j, |x_j| \leq 1$. Then

$$\begin{aligned} |\lambda| = |\lambda x_i| = |(Ax)_i| &= \left| \sum_{j=1}^{n} A_{ij} x_j \right| = \left| \sum_{j \in \Gamma(i)} x_j \right| \\ &\leq \sum_{j \in \Gamma(i)} |x_j| \leq d(i) \leq \Delta. \end{aligned}$$

Assume now $G$ is connected. ($\Leftarrow$). If $G$ is $\Delta$-regular then clearly $(1, \dots, 1)^T$ is an eigenvector with eigenvalue $\Delta$.

($\Rightarrow$). Suppose $\Delta$ is an eigenvalue. Then taking $\lambda = \Delta$ in previous:

$$\Delta = (Ax)_i = \sum_{j \in \Gamma(i)} x_j.$$

Hence $d(i) = \Delta$ and $\forall j \in \Gamma(i), x_j = 1$. Repeat: $\forall j \in \Gamma(i)$ we have $d(j) = \Delta$ and $\forall k \in \Gamma(j)$, $x_k = 1$. Continuing, as $G$ connected, $\forall k, d(k) = \Delta$ and $x_k = 1$. □

### 6.3 Strongly Regular Graphs

**Theorem 47** (Rationality condition)**.** Let $G$ be $(k,a,b)$-strongly regular. Then

$$\frac{1}{2}\left\{(n-1) \pm \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\} \in \mathbb{Z}.$$

*Proof.* Let $A$ be the adjacency matrix of $G$. Let $|G| = n$. By Theorem 46, $k$ is an eigenvalue of multiplicity 1 with eigenvector $(1,1,\ldots,1)^T = v$.

What about other eigenvalues? Let $\lambda \neq k$ be an eigenvalue with eigenvector $x$. Now

$$(A^2)_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i \sim j \\ b & \text{if } i \nsim j \end{cases}$$

Thus $A^2 = kI + aA + b(J - I - A)$ where $J$ is the matrix with a 1 in every place. Applying this to $x$, and noting that $x \perp v$, giving $Jx = 0$, we get $\lambda^2 x = kx + a\lambda x - bx - b\lambda x$. Also $x \neq 0$, so $\lambda^2 + (b-a)\lambda + (b-k) = 0$.

So the remaining eigenvalues are

$$\lambda = \frac{(a-b) + \sqrt{(b-a)^2 - 4(b-k)}}{2} \text{ and } \mu = \frac{(a-b) - \sqrt{(b-a)^2 - 4(b-k)}}{2}$$

with multiplicities $r, s$ say, respectively.

Now $A$ is diagonalizable so

$$r + s + 1 = n \tag{1}$$

and $\operatorname{Tr} A = 0$ so

$$\lambda r + \mu s + k = 0 \tag{2}$$

Now take $\lambda \times (1) - (2)$: $(\lambda - \mu)s = \lambda(n-1) + k$.

We have $\lambda - \mu = \sqrt{(b-a)^2 - 4(b-k)}$ so

$$s = \frac{1}{2}\left\{(n-1) + \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\}$$

and so

$$r = \frac{1}{2}\left\{(n-1) - \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\}. \qquad \square$$