Part IV – Connections between Model Theory and Combinatorics (Unfinished course)

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0 Introduction

Suppose we have a model $\mathscr{M} \vDash T$, then $\varphi(x,y)$ is said to have the k-OP if $\exists a_1, \dots, a_k, b_1, \dots, b_k$ such that $\vDash \varphi(a_i, b_j)$ iff $i \leq j$. In the theory of abelian groups $\langle G, +, -, 0, A\rangle$ with the formula $\varphi(x,y) =$ '$x + y \in A$'. Then $H \leq G$ are 2-stable, and $\bigcup_{i=1}^{k}(H + x_i)$ is $(k+1)$-stable.

Exercise. Show that if $A \subseteq G$ is k-stable, then so is $A + g$ for any $g \in G$. Moreover, A^c is $(k+1)$-stable.

Lemma. Suppose $A_0, A_1 \subseteq G$ are l-stable and k-stable, respectively. Then $A_0 \cup A_1$ is $h(k,l)$-stable, where $h(k,l) = (k+l)2^{k+l}$.

Proof. Suppose not. Then $\exists a_1, \dots, a_{h(k,l)}, b_1, \dots, b_{h(k,l)}$ such that $a_i + b_j \in A_0 \cup A$ iff $i \leq j$.

$$\begin{array}{cc} a_1 & a_1 \\ a_2 & a_2 \\ a_3 & a_3 \\ a_4 & a_4 \\ \\ a_{h(k,l)} & a_{h(k,l)} \end{array}$$

Since $a_i + b_j \in A_0 \cup A_1$ $\forall 1 \leq j \leq h(k,l)$ $\exists i_1 \in \{0,1\}$ and $D_1 = \{j : a_1 + b_j \in A_{i_1}\}$ with $|D_1| \leq h(k,l)/2$. Label D_1 as $j_1 < j_2 < \dots < j_{D_1}$ and define new sequences

$$\begin{aligned} a'_1, \dots, a'_{|D_1|} &= a_1, a_{j_2}, \dots, a_{j_{|D_1|}} \\ b'_1, \dots, b'_{|D_1|} &= b_{j_1}, b_{j_2}, \dots, b_{j_{|D_1|}}. \end{aligned}$$

By pigeonhole, $\exists i_2 \in \{0,1\}$ and $D_2 = \{j | a'_2 + b'_j \in A_{i_2}\}$. Label D_2 as $s_1 < s_2 < \dots < s_{|D_2|}$ and define new sequences

$$\begin{aligned} a^2_1, \dots, a^2_{|D_2|} &= a'_1, a'_2, a'_{S_3}, \dots, a'_{S_{|D_2|}} \\ b^2_1, \dots, b^2_{|D_2|} &= b'_{S_1}, b'_{S_2}, b'_{S_3}, \dots, b'_{S_{|D_2|}} \end{aligned}$$

After $k+1$ steps, we will have sequences

$$a^{k+l}_1, \dots, a^{k+l}_t b^{k+l}_1, \dots, b^{k+l}_t$$

with $t \geq \frac{h(k,l)}{2^{k+l}} = k+l$ such that $\forall 1 \leq j < s \leq t$, $a^{k+1}_s + b^{k+l}_j \notin A_0 \cup A_1$ and $\forall 1 \leq s \leq j \leq t$, $a^{k+l}_s + b^{k+l}_j \in A_{i_s}$.

By pigeonhole again, either $|\{s \mid i_s = 0\}| \geq l$ or $|\{s \mid i_s = 1\}| \geq k$ contradicting the fact that A_0 (A_1) was l (k)-stable. □

The typical model theoretic way of working with this is

Definition. A formula $\varphi(x, y)$ is said to have the order property (OP) if there are sequences $(a_i)_{i<\omega}$, $(b_j)_{i<\omega}$ such that $\vDash \varphi(a_i, b_j)$ iff $i < j$.

Exercise. Show that any Boolean combination of stable formulas is stable.

Definition. A theory has the OP if some formula in some model of the theory has the OP. A theory is stable if it does not have the OP.

0.1 Characterisation in terms of trees

A tree in the set theoretic sense is simply a partial order $(P, \lhd)$ such that $\forall p \in P$, $\{ q \in P : q \lhd p \}$ is a well-order.

Notation.

$$2^{<n} = \bigcup_{i<n} \{0, 1\}^i$$

$$\{0, 1\}^0 = \langle\rangle \text{ the empty string}$$

$$2^i = \{0, 1\}^i$$

The set $2^{<n}$ has a natural tree structure: $\rho \unlhd \eta$ iff $(\rho = \langle\rangle)$ or ρ is an initial segment of η. If $\eta = \langle \eta_1, \ldots, \eta_i \rangle$, $j \in \{0, 1\}$ then $\eta \wedge j = \langle n_1, \ldots, n_i, j \rangle$.

Definition. Given a graph $\Gamma = \langle V, E \rangle$, the tree bound $d(\Gamma)$ is the least integer d such that there do not exist sequences $(a_\eta)_{\eta \in 2^d}, (b_\rho)_{\rho \in 2^{<d}}$ of elements of V with the property that for each $\eta \in 2^d, \rho \in 2^{<d}$, if $\rho \lhd \eta$, then $a_\eta b_\rho \in E$ iff $\rho \wedge 1 \unlhd \eta$.

Example. A graph has tree bound 2 if it does not contain the following:

Theorem (Shelah 1978, Hodges 1996, Alon et al 2018)**.** For each k, $\exists d = d(k)$ such that if Γ is a k-stable graph, then $d(\Gamma) \leq d$. (We will get $d(k) = 2^k + 1$, Hodges gives $2^{k+2} - 2$).

Conversely, if Γ contains the 2^k-OP, then it contains a tree of height k. $k = 2$,

More generally, a formula φ admits a tree of height d if $\exists (a_\eta)_{\eta \in 2^d}, (b_\rho)_{\rho \in 2^{<d}} \in M$ such that if $\rho \lhd \eta$, then $\vDash \varphi(a_\eta, b_\rho) \leftrightarrow \rho \wedge 1 \unlhd \eta$.

If G has the 2^k-OP, then it admits a tree of height k.

Want to show: If G admits a tree of height 2^k+1, then G has the k-OP. Unfortuately, the $k = 2$ case doesn't immediately generalise.

Exercise (Ramsey lemma)**.** Suppose p, q are positive integers and T is a tree of height $p + q - 1$ whose internal nodes are coloured red and blue. Then there is a subtree of height p all of whose internal nodes are red or a subtree of height q all of whose internal nodes are blue.

Proof. Induction on k, where the induction statement is that the result is true, with one class a subset of leaves and the other class a subset of internal nodes. Assume $I(k)$, and we want $I(k + 1)$. Given a leaf y, colour an internal node red if it is connected to y by an edge in G, and blue otherwise. Use the Ramsey lemma on our tree, which has height $2(2^k + 1) - 1$, giving two cases

- Case 1: there is a leaf y such that we get a red subtree T' of height $2^k + 1$, say its root is x (include the leaves too). Let T'' be the subtree of T' rooted at the left child of x. Note that T'' has height 2^k

  Let X', Y' be the set of leaves, nodes of T'', respectively Note that no element of Y' connects to η in G. By the inductive hypothesis, we find $X_0 \subseteq X'$, $Y_0 \subseteq Y'$ that give a half graph of height k.

Observe $y \in Y_0$. Let $X = X_0 \cup \{x\}$, $Y = Y_0 \cup \{y\}$. y is connected to everything in X_0 but x is connected to nothing in Y_0, giving the required halfgraph.

- Case 2: Suppose no leaf y produces a red subtree of height $2^k + 1$. Say x is the root of T, and say T' is the subtree rooted at the right child of x, and consider only the leaves of T'.

  Pick a leaf of T'. This, by assumption, induces a blue subtree T'' in T' of height 2^k in T'.

  By the inductive hypothesis, there are X_0, Y_0 which give a halfgraph of height k using $X = \{x\} \cup X_0$, $Y = \{y\} \cup Y_0$ (attached to the front). □

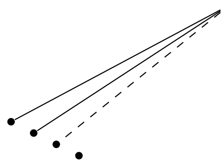
Exercise. Show that the theory of the random graph is unstable.

0.2 Characterisation of stability in terms of types

Definition. Let $\mathscr{M} \vDash T$, $A \subseteq M$ a set of parameters, $\varphi(x, y)$ a formula. Then a (partial) φ-type over A is a collection of formulas of the form $\varphi(x, a)$, $\neg\varphi(x, a)$ for some $a \in A$.

Definition. A complete φ-type over A is a maximal consistent partial type over A (i.e. $\forall a \in A$, either $\varphi(x, a)$ or $\neg\varphi(x, a)$ is in the type). Let $S\varphi(A)$ denote the space of complete φ-types over A.

Example. $G = \langle V, E\rangle$, $A \subseteq V$, $\varphi(x, y) = E(x, y)$. Suppose $A = \{a_1, a_2, a_3, a_4\}$ Then a possible type is $p(x) = \{E(x, a_1), E(x, a_2), \neg E(x, a_3)\}$, and the type defines the set of vertices which connect to a_1 and a_2 but not to a_3.



$p(x)$ is not complete, but if, say, $\neg E(x, a_4)$ were added, then it is a complete E-type over A.

Definition. Let $b \in M$, $A \subseteq M$. Then the type of b over A is the collecitno of all formulas with parameters in A that are satisfied by b:

$$\mathrm{tp}_\varphi(b/A) \coloneqq \{\varphi(x, a) \mid a \in A, \vDash \varphi(b, a)\}.$$

We have

$$S\varphi(A) \supseteq \{\mathrm{tp}_\varphi(b/A) \mid b \in M\}.$$

Exercise.

(i) Prove the Erdős-Makkai theorem: Let A be an infinite set and let $\mathcal{F} \subseteq \mathcal{P}(A)$ such that $|\mathcal{F}| > |A|$. Then there are sequences $(a_i)_{i<\omega}$, $a_i \in A$, $(F_j)_{j<\omega}$, $F_j \in \mathcal{F}$ such that either

$$\text{either } a_i \in F_j \leftrightarrow j < i \ \forall i, j \in \omega \text{or } a_i \in F_j \qquad \leftrightarrow i < j \ \forall i, j \in \omega$$

(ii) Deduce that if $|S\varphi(A)| > |A|$, then $\varphi(x, y)$ is unstable.

Theorem. Let $G = (V, E)$ be an infinite graph. Suppose $\exists$ countable $A \subseteq V$ such that

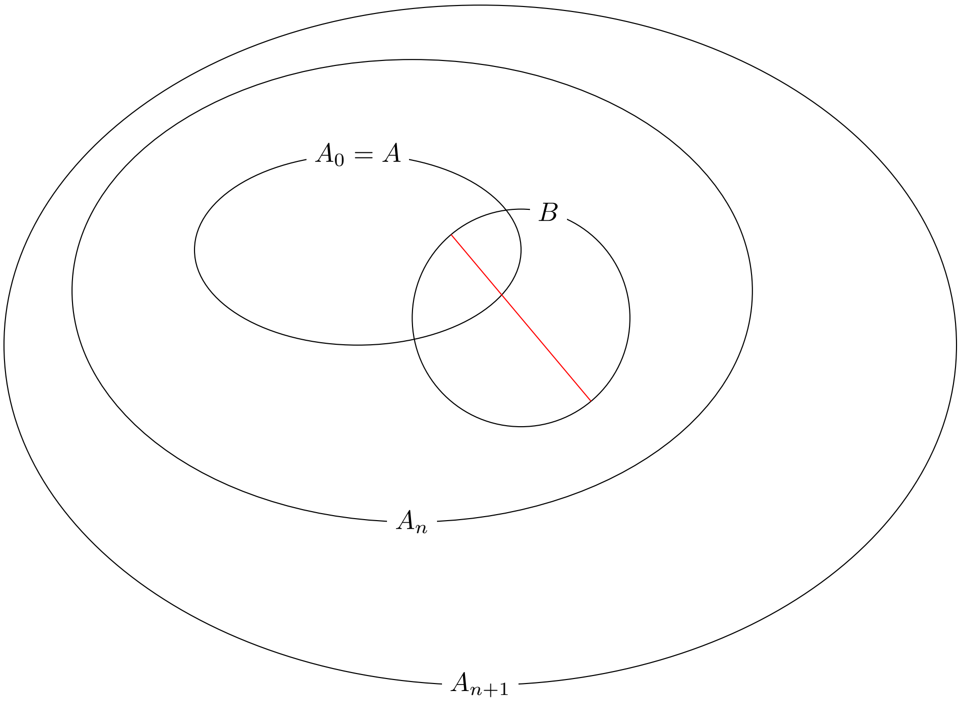
$$|\{\{a \in A \mid E(a, b)\} \mid b \in V\}| > \aleph_0.$$

Then G contains an infinite halfgraph.

Proof. Pick an uncountable sequence $(c_i)_{i<\omega_1}$, of distinct elements of V, each inducing a different partition of A. By induction on $n < \omega$, we define an increasing sequence of countable sets $A_n \subseteq V$ as follows:

- $A_0 \coloneqq A$
- having constructed A_n for some $n \geq 0$, we choose $A_{n+1} \supseteq A_n$ such that $\forall$ finite $B \subseteq A_n$, every partition of B which is induced by a vertex in V is already induced by a vertex in A_{n+1}.

Remark: A_{n+1} is countable, since there are countably many B, and we only need to add in one vertex each.



Claim: $\exists i < \omega_1$ such that $\forall n < \omega$, $\forall$ finite $B \subseteq A_n$, we can find two elements $v = v_{n,B}$ and $w = w_{n,B}$ in $A_{n+1} \setminus \{c_i\}$ such that v and w induce the same partition on B but $E(c_i, v)$ and $\neg E(c_i, w)$.

For now, assume this claim, and construct the halfgraph. Fix $c_* = c_i$ for $i < \omega$ in the claim. We will construct three vertex classes, and use a Ramsey argument to give the halfgraph. Construct sequences $(a_n)_{n<\omega}, (b_n)_{n<\omega}, (c_n)_{n<\omega}$ with $a_n, b_n, c_n \in A_{2n+2}$. Having completed step $n-1$, let

$$B_n = \bigcup_{m<n} \{a_m, b_m, c_m\}.$$

Note that $B_n \subseteq A_{2(n-1)+2} = A_{2n}$. By choice of c_*, $\exists a_n, b_n \in A_{2n+1} \setminus \{c_*\}$ such that $E(c_*, a_n), \neg E(c_*, b_n)$ and a_n and b_n induce the same partition on B_n.

To complete step n, choose $c_n \in A_{2n+2}$ such that it induces the same partition of $B_n \cup \{a_n, b_n\}$ as c^*. (Note c^* and c_n induce the same partition on B_n, but this doesn't have to be the same partition that a_n and b_n induce on B_n).

Observe:

- if $m > n$, then a_m and b_m relate to c_n in the same way: $E(a_m, c_n) \leftrightarrow E(b_m, c_n)$.

- if $m \leq n$, c_* and c_n relate to a_m and b_m in the same way, and $\forall m$, $E(a_m, c^*)$ and $\neg E(b_m, c^*)$. So $E(a_m, c_n)$ and $\neg E(b_m, c_n)$ for all $m \leq n$.

If $E(a_m, c_n)$ on an infinite subsequence, $E(b_m, c_n) \leftrightarrow n < m$. If not, $E(a_m, c_n) \leftrightarrow m \leq n$.

Finally, it remains to prove the claim. Suppose the conclusion fails. Then $\forall i < \omega_1$, $\exists n < \omega$, $\exists$ finite $B \subseteq A_n$ such that whether or not c_i connects to $v \in A_{n+1} \setminus \{c_i\}$ is entirely determined by the partition of B induced by v.

Replacing $(c_i)_{i<\omega_1}$ by a subsequence, may assume that n is constant and B is constant. Fix n, B. By construction, since B is finite, $\exists$ finite $C \subseteq A_{n+1}$ such that every partition of B is already induced by an element of C.

(1) By passing to a subsequence, may assume that all $(c_i)_{i<\omega_1}$ induce the same partition on C.

(2) Any two c_is induce distinct partitions on A, so $\exists a_* \in A$ such that $E(c_i, a_*)$ but $\neg E(c_j, a_*)$.

(3) By choice of C, there is $a_{**} \in C$ such that a_* and a_{**} induce the same partition on B.

(4) But B is such that whether or not $v \in A_{n+1} \setminus \{c_i\}$ is connected to c_i is entirely determined by the partition it induces on B.

□

1 Applications of stability

1.1 Stable Ramsey/Erdős-Hajnal

Definition. Let $A \subseteq 2^{<n}$ be closed under initial segments (CUIS) and let G be a graph on n vertices. We say G **is a type tree** on A if there is an indexing $V = \{a_\eta : \eta \in A\}$ such that $\forall \eta \in A$, the following holds.

(1) If $\eta \wedge 0$ is in A, then $\neg E(a_\eta, a_{\eta \wedge 0})$

(2) If $\eta \wedge 1$ is in A, then $E(a_\eta, a_{\eta \wedge 1})$

(3) If $\sigma, \tau \in A$ and $\eta \triangleleft \sigma \triangleleft \tau$, then

$$E(a_\eta, a_\sigma) \leftrightarrow E(a_\eta, a_\tau)$$

A type tree of A has height h if $A \subseteq 2^{<h}$ but $A \not\subseteq 2^{<h-1}$

Lemma. Every graph on n vertices is a type tree on A for some $A \subseteq 2^{<n}$ (CUIS).

Proof. Let $a_{\langle\rangle}$ be an arbitrary element of V. Let $A_0 = \{a_{\langle\rangle}\}$, $X_{\langle\rangle} = V$. Set $X_1 = N_G(a_{\langle\rangle})$, $X_0 = V \setminus (N_G(a_{\langle\rangle}) \cup \{a_{\langle\rangle}\})$ Observe X_0 and X_1 partition $V \setminus A_0$.

Suppose we've constructed $A_0, A_1, \ldots, A_m$ for $m \geq 0$, and that for each $\eta \in A_m$, we have a partition of X_η with the following properties:

1. $\{X_{\eta \wedge i} : \eta \in A_m, i = 0, 1\}$ partition $V \setminus \bigcup_{i=0}^m \{a_\eta : \eta \in A_i\}$
2. $\forall \eta \in A_m$, $X_{\eta \wedge 1} \subseteq N_G(a_\eta)$, $X_{\eta \wedge 0} \subseteq V \setminus (\{a_\eta\} \cup N_G(a_\eta))$.

Now for each $\eta \in A_m$ and $i \in \{0,1\}$ let $a_{\eta\wedge i}$ be an arbitrary element of $X_{\eta\wedge i}$ be an arbitrary element of $X_{\eta\wedge i}$. If $X_{\eta\wedge i} \neq \varnothing$. Let A_{m+1} be the set of all these elements.

For each $\sigma \in A_{m+1}$, $i \in \{0,1\}$,

$$X_{\sigma\wedge 1} = N(a_0) \cap X_0$$
$$X_{\sigma\wedge 0} = (V \setminus (\{a_0\} \cup N_G(a_d))) \cap X_\sigma$$

Check that the set $A = \bigcup_{i=1} A_i$ satisfies the properties of a type tree. □

Definition. Say $G = (V, E)$ contains a type tree of height h if $\exists V' \subseteq V$ such that the induced graph on V' is a type tree of height h on A for some $A \subseteq 2^{<h}$ (CUIS).

The tree height of G, denoted by $h(G)$ is the largest h such that G contains a type tree of height h.

Definition. Say $G = (V, E)$ contains a full binary type tree of height t if $\exists V' \subseteq V$ such that the induced graph on V' is a type tree on the set $2^{<t}$. The tree rank of G, $t(G)$ is the largest full binary type tree of height t.

Observe that $d(G) \leq k$ then $t(G) \leq k$.

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