Part III – Category Theory (Ongoing course, rough)

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Michaelmas 2018

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0 Introduction

Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

1 Definitions and Examples

1.1 Definition (Category)**.** A **category** $\mathscr{C}$ consists of

(a) a collection $\mathscr{C}$ of **objects** $A, B, C, \ldots$

(b) a collection $\operatorname{mor}\mathscr{C}$ of **morphism** $f, g, h, \ldots$

(c) two operations $\operatorname{dom}, \operatorname{cod}$ assigning to each $f \in \operatorname{mor}\mathscr{C}$ a pair of objects, its **domain** and **codomain**. We write $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'.

(d) an operation assigning to each $A \in \operatorname{ob}\mathscr{C}$ a morphism $A \xrightarrow{1_A} A$, called its **identity**.

(e) a partial binary operation **composition** $(f, g) \mapsto fg$ on morphisms, such that fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod} f$ if fg is defined.

satisfying

(f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$

(g) $(fg)h = f(gh)$ whenever fg and gh are defined

1.2 Remark.

(a) This definition is independent of a model of set theory. If we're given a particular model of set theory, we call the category $\mathscr{C}$ **small** if $\operatorname{ob}\mathscr{C}$ and $\operatorname{mor}\mathscr{C}$ are sets.

(b) Some texts say fg means 'f followed by g', i.e. fg defined $\iff$ $\operatorname{cod} f = \operatorname{dom} g$.

(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

1.3 Examples.

(a) The category **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms $A \to B$ are pairs (f, B) where f is a set-theoretic function.)

(b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, $\mathbf{Mod}_R$ the category of R-modules.

(c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.

(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given $\mathscr{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathscr{C}$ a **congruence** if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathscr{C}/\simeq$ with the same objects as $\mathscr{C}$, but congruence classes as morphisms.

(e) Given $\mathscr{C}$, the **opposite category** $\mathscr{C}^{op}$ has the same objects and morphisms as $\mathscr{C}$, but dom and cod are interchanged, and fg in $\mathscr{C}^{op}$ is gf in $\mathscr{C}$. This leads to the **Duality principle** if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

(f) A small category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (i.e. for all f, $\exists g$ such that fg and gf are identities).

(g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space X, the fundamental groupoid $\pi(X)$ has all points of X as objects and morphisms $x \to y$ are homotopy classes rel $\{0,1\}$ of paths $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$. (If you know how to prove that the fundamental group is a group, you can prove that $\pi(X)$ is a groupoid.)

(h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category $\mathscr{C}$ in which, for any pair (A, B) there is at most 1 morphism $A \to B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as **Set**, but morphisms $A \to B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define

$$S \circ R = \{\, (a,c) \in A \times C \mid (\exists b \in B)((a,b) \in R \wedge (b,c) \in S) \,\}.$$

The identity $1_A : A \to A$ is $\{\, (a,a) \mid a \in A \,\}$.

Similarly, the category **Part** of sets and partial functions (i.e. relations such that $(a,b) \in R, (a,b') \in R \implies b = b'$).

(j) Let K be a field. The category $\mathbf{Mat}_K$ has natural numbers as objects, and morphisms $n \to p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.

1.4 Definition (Functor)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \to \mathscr{D}$ consists of

(a) a mapping $A \mapsto FA$ from $\operatorname{ob}\mathscr{C}$ to $\operatorname{ob}\mathscr{D}$

(b) a mapping $f \mapsto Ff$ from $\operatorname{mor}\mathscr{C}$ to $\operatorname{mor}\mathscr{D}$

such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$ and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

1.3 Examples (*Continued*)**.**

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them.

1.5 Examples.

(a) We have **forgetful** functors $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, $\mathbf{Rng} \to \mathbf{Set}$, $\mathbf{Top} \to \mathbf{Set}$, $\mathbf{Rng} \to \mathbf{AbGp}$ (forgetting $\times$), $\mathbf{Rng} \to \mathbf{Mon}$ (forgetting $+$).

(b) Given a set A, the free group FA has the property: given any group G and any function $A \xrightarrow{f} UG$, there's a unique homomorphism $FA \xrightarrow{f} G$ extending f. F is a functor $\mathbf{Set} \to \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$.

Functoriality follows from uniqueness: given $B \xrightarrow{g} C$, $F(gf)$ and $(Fg)(Ff)$ are both homoms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$.

(c) Given a set A, we write PA for the set of all subsets of A. We can make P into a functor $\mathbf{Set} \to \mathbf{Set}$: given $A \xrightarrow{f} B$, we define $Pf(A') = \{\, f(a) \mid a \in A' \,\}$ for $A' \subseteq A$. But we also have a functor $P^* : \mathbf{Set} \to \mathbf{Set}^{op}$ defined on objects by P, but $P^*f(B') = \{\, a \in A \mid f(a) \in B' \,\}$ for $B' \subseteq B$.

By a **contravariant** functor $\mathscr{C} \to \mathscr{D}$. we mean a functor $\mathscr{C} \to \mathscr{D}^{op}$ (or $\mathscr{C}^{op} \to \mathscr{D}$). (A **covariant** functor is one that doesn't reverse arrows).

(d) Let K be a field. We have a functor $* : \mathbf{Mod}_K \to \mathbf{Mod}_K^{op}$ defined by $V^* = \{\text{linear maps } V \to K\}$ and if $V \xrightarrow{f} W$, $f^*(\theta : W \to K) = \theta f$.

(e) We have a functor $op : \mathbf{Cat} \to \mathbf{Cat}$ which is the 'identity' on morphisms. (Note that this is covariant).

(f) A functor between monoids is a monoid homomorphism.

(g) A functor between posets is an order-preseving map.

(h) Let G be a group. A functor $F : G \to \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A, i.e. a permutation representation of G. (Use $*$ to refer to the unique object of the group). Similarly a functor $G \to \mathbf{Mod}_K$ is a K-linear representation of G.

(i) The construction of a the fundamental group $\pi_1(X, x)$ of a space X with basepoint x is a functor $\mathbf{Top}_* \to \mathbf{Gp}$ where $\mathbf{Top}_*$ is the set of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \to \mathbf{Gpd}$ where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

1.6 Definition (Natural transformation)**.** Let $\mathscr{C}, \mathscr{D}$ be categories and $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ two functors. A **natural transformation** $\alpha : F \to G$ consists of an assignment $A \mapsto \alpha_A$ from $\operatorname{ob}\mathscr{C}$ to $\operatorname{mor}\mathscr{D}$, such that $\operatorname{dom}\alpha_A = FA$ and $\operatorname{cod}\alpha_A = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathscr{C}$ the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_a} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)\alpha_A$).

1.3 Examples (*Continued*)**.**

(l) Given categories $\mathscr{C}, \mathscr{D}$, we write $[\mathscr{C}, \mathscr{D}]$ for the category whose objects are functors $\mathscr{C} \to \mathscr{D}$, and whose morphisms are natural transformations.

1.7 Examples.

(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \to V^{**}$ given by
$$\alpha_V(v)(\theta) = \theta(v)$$
for $\theta \in V^*$. This is the V-component of a natural transformation
$$1_{\mathbf{Mod}_K} \to ** : \mathbf{Mod}_K \to \mathbf{Mod}_K.$$

(b) For any set A, we have a mapping $\sigma_A : A \to PA$ sending a to $\{a\}$. If $f : A \to B$, then $Pf\{a\} = \{f(a)\}$, so σ is a natural transformation $1_{\mathbf{Set}} \to P$.

(c) Let $F : \mathbf{Set} \to \mathbf{Gp}$ be the free group functor (Examples 1.5(b)) and $U : \mathbf{Gp} \to \mathbf{Set}$ the forgetful functor. The inclusions $A \to UFA$ form a natural transformation $1_{\mathbf{Set}} \to UF$.

(d) Let G, H be groups and $f, g : G \rightrightarrows H$ two homomorphisms. A natural transformation $\alpha : f \to g$ corresponds to an element $h = \alpha_*$ of H such that $h.f(x) = g(x).h$ for all $x \in G$, or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.

(e) Let A, B be two G-sets regarded as funcrtors $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \to B$ is a function f satisfying $f(g.a) = g.f(a)$ for all $a \in A$, i.e. a G-equivariant map.

1.8 Lemma. Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two functors, and $\alpha : F \to G$ a natural transformation. Then α is an isomorphism in $[\mathscr{C}, \mathscr{D}]$ iff each α_A is an isomorphism in $\mathscr{D}$.

Proof.

$\Rightarrow$ trivial

$\Leftarrow$ Suppose each α_A has an inverse β_A. Given $f : A \to B$ in $\mathscr{C}$, we need to show that

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, "\beta_A"] & GB \arrow[d, "\beta_B"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

commutes.

But

$$\begin{aligned} (Ff)\beta_A &= \beta_B\alpha_B(Ff)\beta_A \\ &= \beta_B(Gf)\alpha_A\beta_A \\ &= \beta_B(Gf). \end{aligned}$$

□

1.9 Definition (Equivalent category)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. By an **equivalence** between $\mathscr{C}$ and $\mathscr{D}$, we mean a pair of functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms $\alpha : 1_{\mathscr{C}} \to GF$ and $\beta : FG \to \mathscr{D}$. We write $\mathscr{C} \simeq \mathscr{D}$ if $\mathscr{C}$ and $\mathscr{D}$ are equivalent.

We say a property P of categories is a **categorical property** if whenever $\mathscr{C}$ has P and $\mathscr{C} \simeq \mathscr{D}$, then $\mathscr{D}$ has P.

For instance, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

1.10 Examples.

(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint-preserving functions). We define $F : \mathbf{Set}_* \to \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$ and if $f : (A, a) \to (B, b)$

$$Ff(x) = \begin{cases} f(x) & \text{if } f(x) \neq b \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $G : \mathbf{Part} \to \mathbf{Set}_*$ by $G(A) = A^+ = A \cup \{A\}$ and if $f : A \to B$ is a partial function, we define $Gf : A^+ \to B^+$ by

$$Gf = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

The composite FG is the identity on **Part**, but GF is not the identity, however there's an isomorphism

$$(A, a) \longrightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism $\mathbf{Set}_* \to \mathbf{Part}$ since **Part** has a 1-element isomorphism class $\{\varnothing\}$ and $\mathbf{Set}_*$ doesn't.

(b) The category $\mathbf{FdMod}_K$ of finite-dimensional vector spaces over K is equivalent to $\mathbf{FdMod}_K^{op}$: the functors in both directions are $(-)^*$ and both isomorphisms are the natural transformations of Examples 1.7(a).

(c) $\mathbf{FdMod}_K$ is also equivalent to $\mathbf{Mat}_K$: We define $F : \mathbf{Mat}_K \to \mathbf{FdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A with respect to the standard bases of K^n and K^p.

To define $G : \mathbf{FdMod}_K \to \mathbf{Mat}_K$, choose a basis for each finite dimensional vector space,and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ as the matrix representing f with respect to the chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen basis gives isomorphisms $FG(V) = K^{\dim V} \to V$ for each V, which form a natural isomorphism.

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