Part III – Topics in Set Theory

Based on lectures by Professor B. Löwe

Notes taken by Bhavik Mehta

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0 Introduction

Lecture 1 The main 'topic in set theory' covered in this course will be one of the most important: solving the Continuum Problem. A priori, set theory does not seem intrinsically related to logic, but the continuum hypothesis showed that logic was a very important tool in set theory. In contrast to many other disciplines of mathematics, in set theory we typically try to prove things are *impossible*, rather than showing what is possible.

The second international congress of mathematicians in 1900 was in Paris, where Hilbert spoke. At that time, Hilbert was a 'universal' mathematician, and had worked in every major field of mathematics. He gave a list of problems for the century, the 23 Hilbert Problems. The first on this list was the Continuum Problem.

0.1 Continuum Hypothesis

Here is Hilbert's formulation of the Continuum hypothesis (CH): Every set of infinitely many real numbers is either equinumerous with the set of natural numbers or the set of real numbers. More formally, we might write

$$\forall X \subseteq \mathbb{R}, (X \text{ is infinite} \Rightarrow X \sim \mathbb{N} \text{ or } X \sim \mathbb{R})$$

In more modern terms, we write this as the claim $2^{\aleph_0} = \aleph_1$. These two statements are equivalent (in ZFC).

Assume that $2^{\aleph_0} > \aleph_1$, in particular $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \to \mathbb{R}$. Consider $X := i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i|_{\aleph_1}$ is a bijection between $\aleph_1$ and X, so $X \sim \aleph_1$. But $\aleph_1 \nsim \mathbb{N}$ and $\aleph_1 \nsim \mathbb{R}$. Thus X refutes CH (in its earlier formulation). So: $2^{\aleph_0} \neq \aleph_1 \implies \neg\text{CH}$.

If $2^{\aleph_0} = \aleph_1$. Let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \to \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either $\aleph_0$, i.e. $X \sim \mathbb{N}$ or $\aleph_1$, i.e. $X \sim \mathbb{R}$. So, $2^{\aleph_0} = \aleph_1 \implies \text{CH}$.

0.2 History of CH

- 1938, Gödel: ZFC does not prove $\neg$CH.
- 1963, Cohen: ZFC does not prove CH.

Gödel's proof used the technique of inner models; Cohen's proof used forcing, sometimes referred to as outer models.

Gödel's Completeness Theorem:

$$\text{Cons}(T) \iff \exists (M, E)(M, E) \vDash T$$

From this, we might guess that Gödel's and Cohen's proof will show there is a model of ZFC + CH, and a model of ZFC + $\neg$CH, but by the incompleteness phenomenon, we cannot prove there is a model of ZFC! So, we are not going to be able to prove Cons(ZFC+CH), but instead

$$\text{Cons}(\mathsf{ZFC}) \to \text{Cons}(\mathsf{ZFC}+\text{CH})$$

Or, equivalently,

$$\text{if } M \vDash \mathsf{ZFC}, \text{ then there is } N \vDash \mathsf{ZFC} + \text{CH}.$$

1 Model theory of set theory

Let's assume for a moment that

$$(M, \in) \vDash \mathsf{ZFC}.$$

We refer to the canonical objects in M by the usual symbols, e.g., $0, 1, 2, 3, 4, \ldots, \omega, \omega + 1, \ldots$

What would an "inner model" be? Take $A \subseteq M$, and consider $(A, \in)$. This is a substructure of $(M, \in)$. Note: the language of set theory has no function or constant symbols. But we write down

$$X = \varnothing,\ X = \{Y\},\ X = \{Y, Z\},\ X = \bigcup Z,\ X = \mathcal{P}(Z)$$

which appear to use function or constant symbols. These are technically not part of the language of set theory; they are abbreviations:

$X = \varnothing$	abbreviates	$\forall w\ (\neg w \in X)$
$X = \{Y\}$	abbreviates	$\forall w\ (w \in X \leftrightarrow w = Y)$
$X \subseteq Y$	abbreviates	$\forall w\ (w \in X \to w \in Y)$

and so on.

Definition. If φ is a formula in n free variables. We say

(1) φ is **upwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (A, \in) \vDash \varphi(a_1, \ldots, a_n) \implies (M, \in) \vDash \varphi(a_1, \ldots, a_n)$$

(2) φ is **downwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (M, \in) \vDash \varphi(a_1, \ldots, a_n) \implies (A, \in) \vDash \varphi(a_1, \ldots, a_n)$$

(3) φ is **absolute** between A and M if it is upwards absolute and downwards absolute.

Definition. We say that a formula is Σ_1 if it is of the form

$$\exists x_1 \ldots \exists x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free}$$

or Π_1 if it is of the form

$$\forall x_1 \ldots \forall x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free.}$$

Remark.

(a) If φ is quantifier-free, then φ is absolute between A and M.

(b) If φ is Π_1, then it's downward absolute

(c) If φ is Σ_1, then it's upward absolute

Lecture 2 Under our assumption that $(M, \in) \vDash \mathsf{ZFC}$, which subsets $A \subseteq M$ give a model of ZFC? Using standard model theory, we observed that if φ is quantifier-free, then φ is absolute between $(A, \in)$ and $(M, \in)$, but hardly anything is quantifier-free:

$$x = \varnothing \iff \forall w\ (w \notin x) =: \Phi_0(x)$$

For instance, suppose $A \coloneqq M \setminus \{1\}$ (recall $0, 1, 2, \dots$ refer to the ordinals in M). In A, we have $0, 2, \{1\}$. Clearly $(M, \in) \vDash \Phi_0(0)$. $\Phi_0(x)$ is a Π_1 formula, so by Π_1-downwards absoluteness, $(A, \in) \vDash \Phi_0(0)$.

In reality, $2 = \{0, 1\}$, but 1 is not in A, so informally in A, the object 2 has only one element. Similarly, in A, $\{1\}$ has no elements, since 1 is missing from A. Thus

$$(A, \in) \vDash \Phi_0(\{1\}).$$

Clearly $(M, \in) \nvDash \Phi_0(\{1\})$, so Φ_0 is not absolute between A and M. As a corollary, we get $(A, \in) \nvDash$ Extensionality, since 0 and $\{1\}$ have the same elements in A, but are not equal.

(Remark: We could go on, defining formulas $\Phi_1(x), \Phi_2(x)$ etc. to analyse which of the elements correspond to the natural numbers in A.)

Definition. We call A **transitive** in M, if for all $a \in A$ and $x \in M$ such that $(M, \in) \vDash x \in a$, we have $x \in A$.

Proposition. If A is transitive, then Φ_0 is absolute between A and M.

Proof. Since Φ_0 is Π_1, we only need to show upwards absoluteness. Suppose $a \in A$, such that $(A, \in) \vDash \Phi_0(a)$. Suppose $a \neq 0$. Thus there is some $x \in a$. By transitivity, $x \in A$. So $(A, \in) \vDash x \in a$ and so $(A, \in) \nvDash \Phi_0(a)$, contradiction. □

(Similarly, if Φ_n is the formula describing the natural number n, and there is $a \in A$ such that $(A, \in) \vDash \Phi_n(a)$ and A is transitive, then $a = n$.)

Proposition. If A is transitive in M, then

$$(A, \in) \vDash \mathsf{Ext}.$$

Proof. Take $a, b \in A$ with $a \neq b$. By Extensionality in $(M, \in)$, find without loss of generality some $c \in a \setminus b$. Since $c \in a \in A$, by transitivity, $c \in A$. Thus

$$(A, \in) \vDash c \in a$$
$$(A, \in) \vDash c \notin b,$$

so a and b do not satisfy the assumptions of Extensionality. □

Consider now $A \coloneqq \omega + 2 \subseteq M$, the ordinal consisting of $\{0, 1, 2, \dots, \omega, \omega + 1\}$. This is a transitive subset of M (since it's an ordinal). So

$$(A, \in) \vDash \mathsf{Ext}.$$

Consider the formula $x = \mathcal{P}(y)$, which we can informally define as $x = \{z \mid z \subseteq y\}$, but this is not good enough. More properly, we try

$$x = \mathcal{P}(y) \iff \forall w\ (w \in x \leftrightarrow w \subseteq y).$$

This still includes the symbol $\subseteq$, so still needs improving.

$$x = \mathcal{P}(y) \iff \forall w\ (w \in x \leftrightarrow (\forall v\ (v \in w \to v \in y)))$$

In A, what is $\mathcal{P}(0)$?

$$(A, \in) \vDash \omega + 1 = \mathcal{P}(\omega)$$

1.1 Bounded quantification

We define

$$\exists (v \in w)\ \varphi :\Longleftrightarrow \exists v\ (v \in w \wedge \varphi)$$
$$\forall (v \in w)\ \varphi :\Longleftrightarrow \forall v\ (v \in w \to \varphi).$$

Definition. A formula φ is called Δ_0 if it is in the smallest set of formulas with the following properties

1. All quantifier-free formulas are in S.
2. If $\varphi, \psi \in S$ then so are
   (a) $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$
   (b) $\neg\varphi$
   (c) $\exists (v \in w)\ \varphi$, $\forall (v \in w)\ \varphi$.

Theorem. If φ is Δ_0 and A is transitive, then φ is absolute between A and M.

Proof. We already knew that quantifier free formulas are absolute. Absoluteness is obviously preserved under propositional connectives. So, let's deal with (2c): Let's just do

$$\varphi \mapsto \exists (v \in w)\ \varphi = \exists v\ (v \in w \wedge \varphi).$$

So suppose φ is absolute. We need to deal with downwards absoluteness.

$$(M, \in) \models \exists (v \in a)\ \varphi(v, a) \quad \text{for some } a \in A$$
$$(M, \in) \models \exists v\ (v \in a \wedge \varphi(v, a)).$$

Let's find $m \in M$ such that

$$(M, \in) \models m \in a \wedge \varphi(m, a).$$

Transitivity gives $m \in A$. By absoluteness of φ, we get

$$(A, \in) \models m \in a \wedge \varphi(m, a) \implies (A, \in) \models \exists (v \in a)\ \varphi(v, a).$$ □

Definition. Let T be any 'set theory'. Then we say that φ is Δ_0^T if there is a Δ_0 formula ψ such that $T \vdash \phi \leftrightarrow \psi$.

- φ is called Σ_1^T if it is T-equivalent to $\exists v_1 \dots \exists v_n\ \psi$ where ψ is Δ_0.
- φ is called Π_1^T if it is T-equivalent to $\forall v_1 \dots \forall v_n\ \psi$ where ψ is Δ_0.

Corollary. If A is transitive in M and both $(M, \in)$ and $(A, \in)$ are models of T, then Δ_0^T formulas are absolute between A and M, and Σ_1^T, (Π_1^T) formulas are upwards (downwards) absolute between A and M.

Lecture 3 **Definition.** A formula is called Δ_1^T if it is both Σ_1^T and Π_1^T.

Corollary. If A is transitive, $A, M \models T$ and φ is Δ_1^T, then φ is absolute between A and M.

1.2 'Set theory'

What do we mean by a 'set theory'? The usual theories we care about are:

FST_0	Extensionality Pairing Union PowerSet Separation (Aussonderung)	FST	FST_0 + Foundation (Regularity)
Z_0	FST_0 + Infinity	Z	Z_0 + Foundation
ZF_0	Z_0 + Replacement (Ersetzung)	ZF	ZF_0 + Foundation
ZFC_0	ZF_0 + Choice	ZFC	ZFC_0 + Foundation

The subscript 0 denotes the absence of Foundation.

1.3 A long list of Δ_0^T formulas

We noted earlier that there are very few Δ_0 formulas, so can we find any Δ_0^T formulas?

1. $x \in y$ (in fact, Δ_0)
2. $x = y$ (in fact, Δ_0)
3. $x \subseteq y$. This is an abbreviation, so we have to define what it means:

$$\forall w \ (w \in x \to w \in y).$$

   But note this is $\forall (w \in x) \ (w \in y)$, so Δ_0.

4.

$$\begin{aligned} \Phi_0(t) :&\iff \forall w \ (w \notin x) \\ &\iff \forall w \ (\neg w \in x) \\ &\iff \forall w \ (w \in x \to \neg x = x) \end{aligned}$$

   so it's Δ_0 in predicate logic.

Definition. We say that an **operation** $x_1, \ldots, x_k \mapsto F(x_1, \ldots, x_n)$ is defined by a formula in class Γ (where Γ is any class of formulas) in the theory T if there is a formula $\Phi \in \Gamma$ such that

(1) $T \vdash \forall x_1 \cdots \forall x_n \ \exists y \ \Phi(x_1, \ldots, x_n, y)$

(2) $T \vdash \forall x_1 \cdots \forall x_n \ \forall y, z \ \Phi(x_1, \ldots, x_n, y) \wedge \Phi(x_1, \ldots, x_n, z) \to y = z$

(3) $\Phi(x_1, \ldots, x_n, y)$ if and only if $y = F(x_1, \ldots, x_n)$.

Example.

$$\begin{aligned} x &\mapsto \{x\} \\ x, y &\mapsto \{x, y\} \end{aligned}$$

These are operations in FST_0!

5. $x \mapsto \{x\}$. The formula to express this is

$$\begin{aligned}
\text{`}z = \{x\}\text{'} &\iff \Phi(x, z) \\
&\iff \forall w \ (w \in z \leftrightarrow w = x) \\
&\iff \forall w \ ((w \in z \to w = x) \wedge (w = x \to w \in z)) \\
&\iff \exists (w \in z) \ (w = w) \wedge \forall (w \in z) \ (w = x)
\end{aligned}$$

   So this is Δ_0 in some weak set theory.

6. $x, y \mapsto \{x, y\}$
7. $x, y \mapsto x \cup y$
8. $x, y \mapsto x \cap y$
9. $x, y \mapsto x \setminus y$
10. $x, y \mapsto (x, y)$, where $(x, y) = \{\{x\}, \{x, y\}\}$ which is the combination of earlier formulas

If two operations f, g_1, g_2 are defined by Δ_0^T-formulas, then so is the operation

$$x_1, \dots, x_n \mapsto f(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n))$$

11. $x \mapsto x \cup \{x\} =: S(x)$. By the previous fact from 5. and 7.
12. $x \mapsto \bigcup x$
13. the formula describing 'x is transitive'
14. the formula describing 'x is an ordered pair' (the quantifiers for the two components of x are bounded by $\bigcup x$)
15. $x, y \mapsto x \times y$
16. the formula 'x is a binary relation' (again, the quantifiers can be made bounded)
17. $x \mapsto \mathrm{dom}(x) := \{y \mid \exists p \in x; (p \text{ is an ordered pair}, \ p = (v, w), \ y = v)\}$
18. $x \mapsto \mathrm{ran}(x) := \{y \mid \exists p \in x; (p \text{ is an ordered pair}, \ p = (v, y))\}$
19. the formula 'x is a function'
20. the formula 'x is injective'
21. the formula 'x is function from A to B'
22. the formula 'x is a surjection from A to B'
23. the formula 'x is a bijection from A to B'

What is an ordinal?

Definition. α is an **ordinal** if α is a transitive set well-ordered by $\in$, i.e. it is totally ordered (several axioms, all Δ_0^T) and well-founded:

$$\forall X \ (X \subseteq \alpha \to X \text{ has a} \in \text{-least element}).$$

Observe '(X, R) is a well-founded relation' is not obviously absolute since the bound for the $\forall Z\ (Z \subseteq X \dots)$ quantifier (above, $X \subseteq \alpha$) is the power set, so this is Π_1. However, in models with the Axiom of Foundation, well-foundedness is automatic, so we can say α is an ordinal iff α is transitive and totally ordered by $\in$, which is Δ_0^T.

Lecture 4

24. 'x is an ordinal' is Δ_0^T (with the right choice of T)
25. 'x is a successor ordinal' $\iff$ 'x is an ordinal' +
$$\exists (y \in x)\ (y \text{ is the } \in\text{-largest element of } x)$$
26. 'x is a limit ordinal' (is an ordinal, not 0 and not a successor)
27. '$x = \omega$' (is the $\in$-minimal limit ordinal), similarly, $x = \omega + \omega$, $x = \omega + 1$, $x = \omega + \omega + 1$, $x = \omega^2$, $x = \omega^3$, $x = \omega^4$, $x = \omega^\omega$

1.4 Absoluteness of well-foundedness

If (X, R) is well-founded, we can define a rank function
$$\mathrm{rk} : X \to \alpha$$
where α is some ordinal such that rk is order-preserving between (X, R) and $(\alpha, \in)$.

This theorem is proved using the right instances of Replacement. In particular, ZF proves:
$$\underbrace{(X, R) \text{ is well founded}}_{\Pi_1^{\mathsf{ZF}}} \iff \underbrace{\begin{array}{c}\exists \alpha\, \exists f : \alpha \text{ is an ordinal and} \\ f \text{ is an order-preserving function} \\ \text{from } (X, R) \text{ to } (\alpha, \in)\end{array}}_{\Sigma_1^{\mathsf{ZF}}}$$
But note that the left hand side is Π_1^{ZF}, while the right hand side is Σ_1^{ZF}. Thus, for sufficiently strong T, '(X, R) is wellfounded' is Δ_1^T and hence absolute for transitive models of T. We can generalise this to concepts defined by transfinite recursion.

Recall the method of **transfinite recursion**: Let (X, R) be well-founded. Let F be 'functional', so for every x there is unique y such that $y = F(x)$. Then there is a unique f with domain X and for all $x \in X$,
$$f(x) = F(f \restriction \mathrm{IS}_R(x))$$
where $\mathrm{IS}_R(x) := \{z \in X \mid zRx\}$

Proposition. Let T be a set theory that is strong enough to prove the transfinite recursion theorem for F. Let F be absolute for transitive models of T. Let (X, R) be in A.

Then f defined by transfinite recursion is absolute between A and M.

Example. Let $\mathscr{L}$ be any first-order language whose symbols are all in A. Then the set of $\mathscr{L}$-formulas and the set of $\mathscr{L}$-sentences are in A. (Assumptions on A are suppressed, it needs to be transitive and strong enough to prove the relevant cases of transfinite recursion and have natural numbers, e.g. ZF).

The relation $S \vDash \varphi$ is defined by recursion and thus is absolute between A and M. So: If S is an $\mathscr{L}$-structure with $S \in A$ then
$$(A, \in) \vDash \text{“}S \vDash \varphi\text{”} \iff (M, \in) \vDash \text{“}S \vDash \varphi\text{”}$$
Furthermore, this shows that consistency is absolute, and can be formulated by saying 'there is no proof of $\perp$'.

1.4.1 Gödel's Incompleteness Theorem

Gödel's Incompleteness Theorem says: For a theory T, if T is consistent, then $T \nvdash \mathrm{Cons}(T)$. Of course, this requires some restrictions on T. Examples for strong enough T include PA, Z, ZF, ZFC, $\mathsf{ZFC} + \varphi$. In particular,

$$\mathsf{ZFC*} \coloneqq \mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC})$$

cannot prove its own consistency.

By Gödel's Completeness Theorem,

$$\mathrm{Cons}(T) \iff \exists M \ (M \vDash T).$$

Now write β for 'there is a transitive set A such that $(A, \in) \vDash \mathsf{ZFC}$'. Clearly, $\beta \Rightarrow \mathrm{Cons}(\mathsf{ZFC})$.

Theorem. If $\mathsf{ZFC*}$ is consistent, then $\mathsf{ZFC*} \nvdash \beta$.

Proof. Let $(M, \in) \vDash \mathsf{ZFC*}$. Suppose $\mathsf{ZFC*} \vdash \beta$. So $(M, \in) \vDash \beta$. Thus in M, find A transitive such that $(A, \in) \vDash \mathsf{ZFC}$. By assumption, $(M, \in) \vDash \mathrm{Cons}(\mathsf{ZFC})$. By absoluteness of consistency, $(A, \in) \vDash \mathrm{Cons}(\mathsf{ZFC})$. Thus, $(A, \in) \vDash \mathsf{ZFC*}$. So we proved $\mathrm{Cons}(\mathsf{ZFC*})$, contradiction. $\square$

In particular, it is stronger to say that there are transitive models of ZFC than that ZFC is consistent. Assuming β is not an obvious assumption, so we need to study under which (natural) assumptions β is true.

So, let us investigate transitive models A inside M.

1.5 Concrete transitive models of ZFC

The two most basic constructions:

1. von Neumann hierarchy (cumulative hierarchy)
2. hereditarily small sets

1.5.1 von Neumann hierarchy

Lecture 5 We define

$$\begin{aligned} V_0 &\coloneqq \varnothing \\ V_{\alpha+1} &\coloneqq \mathcal{P}(V_\alpha) \\ V_\lambda &\coloneqq \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Proposition. V_α is transitive for all α.

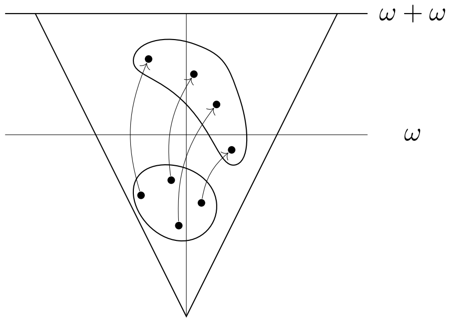
Proof sketch. Induction, the key lemma is: If X is transitive, then $\mathcal{P}(X)$ is transitive. $\square$

Does V_α give a model of ZFC? We know:

1. If λ is a limit ordinal, then $V_\lambda \vDash \mathsf{FST}$.
2. If $\lambda > \omega$ and λ a limit ordinal, $V_\lambda \vDash \mathsf{Z}$ (on example sheet 1).

The critical axiom here is Replacement. Take, as an example, $\lambda = \omega + \omega$. Replacement says: if $F : V_{\omega+\omega} \to V_{\omega+\omega}$ is a function definable in $V_{\omega+\omega}$ and $x \in V_{\omega+\omega}$, then

$$\{F(y) \mid y \in x\} \in V_{\omega+\omega}.$$

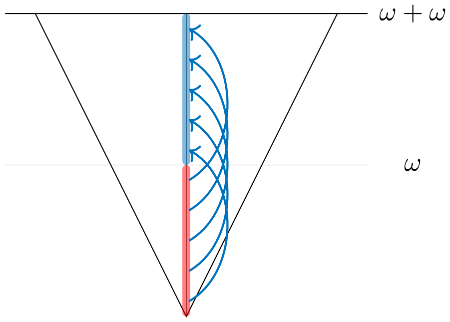


Idea. Take $x = \omega$, and

$$F : \begin{cases} n \mapsto \omega + n \\ y \mapsto 0 & \text{if } y \notin \omega \end{cases}$$

Note $\omega + n$ is definable (in Z): it is the unique ordinal which contains ω and $n - 1$ elements above ω.

Let $Y \coloneqq \{F(n) \mid n \in \omega\}$. Then $Y \subseteq V_{\omega+\omega}$, but $Y \notin V_{\omega+\omega}$: it is not bounded in $V_{\omega+\omega}$.



This example shows concretely that

$$V_{\omega+\omega} \vDash \neg\mathsf{Repl}.$$

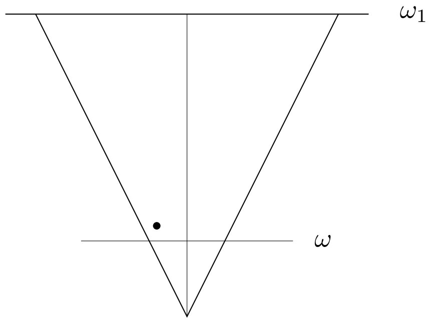
Similarly, if α is any ordinal such that there is a definable function $f : \omega \to \alpha$ such that the range of f is unbounded in α, then $V_\alpha \vDash \neg\mathsf{Repl}$. Even more generally, if $\beta < \alpha$ and a definable function $f : \beta \to \alpha$ with unbounded image, then $V_\alpha \vDash \neg\mathsf{Repl}$.

Definition. We call a cardinal κ **regular** if there is no partition

$$\kappa = \bigcup_{i \in I} A_i$$

such that $|I|, |A_i| < \kappa$ for all $i \in I$. Equivalently, for every $\alpha < \kappa$, there is no unbounded function $f : \alpha \to \kappa$.

We know, e.g. that $\aleph_1$ is regular. Moreover, for any α, $\aleph_{\alpha+1}$ is regular. So our next candidate is $\alpha = \aleph_1$.



Note $\mathcal{P}(\omega) \in V_{\omega+2}$, $\mathcal{P}(\omega) \subseteq V_{\omega+1}$. Clearly, there is a surjection

$$s : \mathcal{P}(\omega) \to \omega_1.$$

so the range of s is unbounded in ω_1. Thus: $V_{\omega_1} \models \neg\mathsf{Repl}$.

Definition. A cardinal κ is called **inaccessible** if

(a) κ is regular

(b) $\forall \lambda < \kappa$, $|\mathcal{P}(\lambda)| < \kappa$ (**strong limit**).

That is, just take the two problems we had, negate them and make a definition.

Remark. We know every successor cardinal is regular, and our simple examples of limit cardinals are all not regular: they were defined as unions. So, we can ask: 'Are there regular limit cardinals?' (Such cardinals are sometimes called **weakly inaccessible**). A partial answer: Under the Generalized Continuum Hypothesis ($\forall \kappa\ 2^\kappa = k^+$), we have:

$$\kappa \text{ is inaccessible} \iff \kappa \text{ is a regular limit cardinal.}$$

Now, let's assume that $\kappa > \aleph_0$ is inaccessible.

Lemma.
$$\forall \lambda < \kappa \quad |V_\lambda| < \kappa.$$

Proof. Clearly $|V_\omega| = \aleph_0$, so $|V_\omega| < \kappa$. Proceed by induction. Suppose $|V_\lambda| < \kappa$. Then $V_{\lambda+1} = \mathcal{P}(V_\lambda)$.

$$|V_{\lambda+1}| = |\mathcal{P}(V_\lambda)| < \kappa$$

by (b). Now let $\lambda < \kappa$ be a limit ordinal. Then

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

Suppose for contradiction that $|V_\lambda| = \kappa$. But $|V_\alpha| < \kappa$ for all $\alpha < \kappa$, so you can write κ as a union of λ many things of smaller cardinality. This contradicts regularity. □

Definition (Mirimanoff rank)**.** Define

$$\varrho(x) := \text{least } \alpha \text{ such that } x \in V_{\alpha+1} \setminus V_\alpha.$$

We prove useful properties about the Mirimanoff rank on question 4 on Example Sheet 1.

Theorem. If κ is inaccessible, then $V_\kappa \models \mathsf{Repl}$.

Proof. Take any $F : V_\kappa \to V_\kappa$ and any $x \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$. Thus, find $\alpha < \kappa$ such that $x \in V_\alpha$. Since V_α is transitive, $x \subseteq V_\alpha$. So $|x| \le |V_\alpha| < \kappa$ (by the lemma).

Now consider $X := \{F(y) \mid y \in x\}$. For each $y \in x$, we have $\varrho(F(y)) < \kappa$ by assumption. Consider

$$R := \{\varrho(F(y)) \mid y \in x\},$$

then $|R| \le |x| < \kappa$. By regularity, $\alpha := \bigcup R < \kappa$. But then, $\forall y \in x\ F(y) \in V_{\alpha+1}$. So $X \subseteq V_{\alpha+1}$, $X \in V_{\alpha+2}$. This proves Replacement. □

Note we didn't even use that F was definable: we showed a statement stronger than Replacement. As a consequence, we get that the existence of inaccessible cardinals cannot be proved in ZFC:

Lecture 6 Write IC for the axiom 'there is an inaccessible cardinal'. If κ is inaccessible, then $V_\kappa \models \mathsf{ZFC}$. V_κ is a transitive model of ZFC, so,

$$\mathsf{ZFC} + \mathsf{IC} \vdash \underbrace{\text{'there is a transitive set that is a model of } \mathsf{ZFC}\text{'}}_{\beta}$$

Recall that $\mathsf{ZFC} + \mathrm{Cons}\,\mathsf{ZFC} \nvdash \beta$ from earlier, so $\mathsf{ZFC} + \mathrm{Cons}\,\mathsf{ZFC} \nvdash \mathsf{IC}$.

Now, recall from model theory:

1. **Löwenheim-Skolem Theorem**: If S is any structure in some countable first-order language $\mathcal{L}$ and $X \subseteq S$ is any subset, then there is a **Skolem hull** of X in S, with $X \subseteq \mathcal{H}^S(X) \subseteq S$ such that

   (a) $\mathcal{H}^S(X) \preccurlyeq S$. Recall $\preccurlyeq$ means **elementary substructure**, meaning that

   $$\forall \varphi\ \forall h_1, \dots, h_n \in \mathcal{H}^S(X),$$
   $$\mathcal{H}^S(X) \models \varphi(h_1, \dots, h_n) \iff S \models \varphi(h_1, \dots, h_n)$$

   (b) $|\mathcal{H}^S(X)| \le \max(\aleph_0, |X|)$

   Proof sketch. The key ingredient for this theorem is the **Tarski-Vaught criterion**, which says that for $Z \subseteq S$, we have $Z \preccurlyeq S$ if and only if for every φ and all $z_1, \dots, z_n$,

   $$S \models \exists x\ \varphi(x, z_1, \dots, z_n) \implies Z \models \exists x\ \varphi(x, z_1, \dots, z_n).$$

   Observe there are $\max(\aleph_0, |X|)$ many possible $\varphi(x, z_1, \dots, z_n)$, so for each formula which we need to satisfy, take a witness in S and add it into X. But this introduces new z_i, so we need to add more witnesses, so repeat this process and take a union. Specifically,

   $$\begin{aligned} Z_0 &:= X \\ Z_1 &:= Z_0 \cup \text{the witnesses for all tuples } \varphi, z_1, \dots, z_m \text{ where } z_1, \dots, z_m \in Z_0 \\ Z_{n+1} &:= Z_n \cup \text{the witnesses for all tuples } \varphi, z_1, \dots, z_m \text{ where } z_1, \dots, z_m \in Z_n \\ Z &:= \bigcup_{n \in \mathbb{N}} Z_n. \end{aligned}$$

   Z is the required model. □

Now, work in $\mathsf{ZFC} + \mathsf{IC}$. Suppose $(M, \in) \models \mathsf{ZFC} + \mathsf{IC}$, which contains $V_\kappa \models \mathsf{ZFC}$ ($V_\kappa \subseteq M$). Apply Löwenheim-Skolem to V_κ with $X := \varnothing$. Then

$$H := \mathcal{H}^{V_\kappa}(\varnothing) \preccurlyeq V_\kappa$$

and $\mathcal{H}^{V_\kappa}(\varnothing)$ has cardinality $\le \aleph_0$, and $H \vDash \mathsf{ZFC}$.

There is a formula φ such that $\varphi(x)$ iff x is the least uncountable cardinal. We have $V_\kappa \vDash \exists x\ \varphi(x)$, since this is a theorem of ZFC, but the only element that satisfies φ in V_κ is $\aleph_1$. So in the notation of Skolem hull construction,

$$\aleph_1 \in Z_1 \subseteq H$$

This implies H can't be transitive, since $\aleph_1$ has uncountably many elements, but H has only countably many. So, try to collapse to a transitive model.

2. **Mostowski Collapse Theorem**: If X is any set and $R \subseteq X \times X$ such that R is well-founded and extensional, then there is a transitive set T such that $(T, \in) \cong (X, R)$.

   Consider $(H, \in) \vDash \mathsf{ZFC}$. Since $(H, \in) \vDash \mathsf{ZFC}$, we must have that $\in$ is extensional on H. Since $\in$ (in M) is well-founded, $\in$ is well-founded on H. So, let T be the Mostowski collapse of H: T is transitive and

$$(T, \in) \cong (H, \in).$$

   But this is an isomorphism, so $(T, \in) \vDash \mathsf{ZFC}$. It is a bijection also, so $|T| = |H| \le \aleph_0$.

Together: there is a countable transitive model of ZFC.

Notice that

$$\begin{aligned} \varphi(x) &:= \text{`}x\text{ is countable'} \\ &= \exists f\ (\underbrace{f : x \to \mathbb{N}}_{\Delta_0^{\mathsf{ZFC}}}, \underbrace{f\text{ is injective}}_{\Delta_0^{\mathsf{ZFC}}}) \end{aligned}$$

is Σ_1^{ZFC}, so is upwards absolute. But this formula is not downwards absolute: If $\alpha \in \mathrm{Ord}$, $\alpha \in T$ then $V_\kappa \vDash \alpha$ is countable. But since $(T, \in) \vDash \mathsf{ZFC}$, there is some $\alpha \in T$ such that $(T, \in) \vDash \alpha$ is uncountable, so V_κ and T disagree about the truth value of $\varphi(\alpha)$.

Consider now

$$\begin{aligned} \psi(x) &:= \text{`}x\text{ is a cardinal'} \\ &= \forall \alpha\ (\alpha < x \to \text{there is no injection from } x \text{ to } \alpha). \end{aligned}$$

This is Π_1^{ZFC}. In $(T, \in)$, take α least such that $(T, \in) \vDash \neg\varphi(\alpha)$. Then $(T, \in) \vDash$ 'α is a cardinal'. Clearly, $V_\kappa \vDash$ 'α is not a cardinal'.

Note that if λ is an uncountable cardinal in V_κ, then $\lambda \notin T$, so the downwards absoluteness of ψ is not very interesting.

So, try to ensure the cardinal is in T: Instead of building $\mathcal{H}^{V_\kappa}(\varnothing)$, build $H^* := \mathcal{H}^{V_\kappa}(\omega_1 + 1)$. Clearly $\omega_1 \in H^*$ and $\omega_1 \subseteq H^*$, so $\omega_1 \subseteq T^*$ and $\omega_1 \in T^*$. We have $|H^*| = \aleph_1$.

Now we have $V_\kappa \vDash \omega_1$ is a cardinal, so by downwards absoluteness of ψ, so $T^* \vDash \omega_1$ is a cardinal. But there may be other cardinals below ω_1 in T^*, so it may not be the case that $T^* \vDash \omega_1 = \aleph_1$.

Lecture 7 In our goal to prove the Continuum Hypothesis, we have

1. Decided to go for transitive models
2. Looked at 'inner models'
3. In particular, models of type V_α

4. Seen if α is inaccessible then $V_\alpha \models \mathsf{ZFC}$
5. Found countable transitive submodels $T \subseteq V$ such that $T \models \mathsf{ZFC}$.

But this isn't particularly helpful, since it is not going to change the truth value of CH. If CH is true in our original model $(M, \in)$, then there is a bijection between $\mathbb{R}$ (or $\mathcal{P}(\mathbb{N})$) and ω_1. Certainly, $\mathbb{R} \in V_{\omega+20}$, and also $\omega_1 \in V_{\omega_1+1}$, and both of these are in V_κ. So, the bijection is in V_{ω_1+20}, certainly in V_κ. This means

$$(M, \in) \models \mathrm{CH} \iff (V_\kappa, \in) \models \mathrm{CH}$$

By Löwenheim-Skolem, we found countable $H \preccurlyeq V_\kappa$, so

$$(H, \in) \models \mathrm{CH} \iff (V_\kappa, \in) \models \mathrm{CH}.$$

And Mostowski had $(T, \in) \cong (H, \in)$ so

$$(T, \in) \models \mathrm{CH} \iff (H, \in) \models \mathrm{CH}.$$

In summary, the method of finding countable transitive elementary submodels of V_κ is not going to change the value of CH.

1.5.2 Models of hereditarily small sets

Definition. Let κ be a regular cardinal, e.g. $\kappa = \omega, \kappa = \omega_1$. Then x is called **hereditarily of size** $< \kappa$ if $|\operatorname{tcl}(x)| < \kappa$. Recall the **transitive closure**

$$\operatorname{tcl}(x) := \bigcup_{m \in \mathbb{N}} t_n(x)$$

where

$$t_0(x) := x$$
$$t_{n+1}(x) := \bigcup t_n(x).$$

The definition of $|\operatorname{tcl}(x)| < \kappa$ captures the intuition of 'x has size $< \kappa$, all elements of x have size $< \kappa$, all elements of elements of x have size $< \kappa$, etc.'

Side remark: If κ is not regular, this intuition might not work: Let $\kappa = \aleph_\omega$. Then define

$$\begin{aligned} x_n^0 &:= \aleph_n \\ x_n^{k+1} &:= \{x_n^k\} \\ X &:= \{x_0^0, x_1^1, x_2^2, \dots\} \\ &= \{\aleph_0, \{\aleph_1\}, \{\{\aleph_2\}\}, \{\{\{\aleph_3\}\}\}, \dots\} \end{aligned}$$

X is countable. In the notation from above,

$$|t_{n+1}(X)| = \aleph_n$$

But $|\operatorname{tcl}(X)| = \aleph_\omega$.

Definition. Take H_κ to be the sets which are hereditarily of size $< \kappa$.

Observe

1. H_κ is transitive

2. If $X \subseteq H_\kappa$ and $|X| < \kappa$, then $X \in H_\kappa$. (Follows directly from regularity of κ and the definition).

Take as our first example $H_{\aleph_0} =: HF$ (hereditarily finite).

Proposition. $HF = V_\omega$.

Proof. First show $V_\omega \subseteq HF$. Need to show $V_n \subseteq HF$ for all natural n.

- Clearly $V_0 = \varnothing \subseteq HF$.
- If $V_n \subseteq HF$ and $Z \subseteq V_n$, then by Observation 2, $Z \in HF$ so $\mathcal{P}(V_n) = V_{n+1} \subseteq HF$.

Next, we show $HF \subseteq V_\omega$. Suppose not, then there are $x \in HF \setminus V_\omega$. Take such an x with minimal rank α. Minimality says: If $y \in HF$ with $\varrho(y) < \alpha$, then there is $k \in \mathbb{N}$ such that $\varrho(y) = k$.

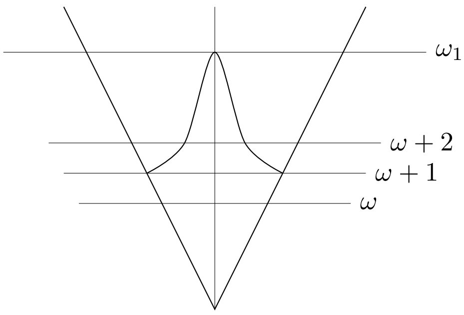
We have $x \in V_{\alpha+1} \setminus V_\alpha$, so $x \in \mathcal{P}(V_\alpha)$, $x \subseteq V_\alpha$. Also, $x \in HF$, so x is finite, say $x = \{x_1, \dots, x_n\}$. Using minimality, each x_i is in V_{k_i} for some natural k_i. So

$$x \in V_{\max(k_1,\dots,k_n)+1} \subseteq V_\omega$$

a contradiction. □

So we know $HF \vDash \mathsf{FST}$.

Our next example of a regular cardinal was $\aleph_1$, so now try $H_{\aleph_1} =: HC$ (hereditarily countable).



- The ordinals in HC are exactly ω_1
- $V_{\omega+2} \setminus HC \neq \varnothing$ (take, say, $\mathcal{P}(\mathbb{N})$)
- $V_{\omega+1} \subseteq HC$

Which axioms are true in HC?

Pair Take $x, y \in HC$. $\{x, y\} \subseteq HC$, $|\{x, y\}| < \aleph_1$, so $\{x, y\} \in HC$.

Separation and Union are similarly easy. Foundation follows trivially from foundation of the universe, and Extensionality follows by transitivity. Replacement is harder:

Replacement Let $F : HC \to HC$ and $x \in HC$. Consider

$$R := \{F(y) \mid y \in x\} \subseteq HC$$

then $|R| \leq |x| < \aleph_1$, and $R \subseteq HC$ so by observation 2, $R \in HC$.

Power Set We know that $\mathbb{N} \in HC$ and that $\mathcal{P}(\mathbb{N}) \notin HC$. But this is not enough to show $HC \vDash \neg\mathsf{Pow}$.

Instead, we need to show that for all $A \in HC$,

$$HC \vDash \text{`}A \text{ is not the power set of } \mathbb{N}\text{'}$$

i.e.

$$HC \vDash \exists X\ (X \subseteq \mathbb{N} \wedge X \notin A).$$

Fix $A \in HC$ and presume that this might be the HC-powerset of $\mathbb{N}$. Thus, if $X \subseteq \mathbb{N}$ and $X \in HC$ then $X \in A$.

But if $X \subseteq \mathbb{N}$, then $X \subseteq HC$ and $|X| < \aleph_1$, so $X \in HC$. So if A is any set such that $\forall X\ (X \subseteq \mathbb{N}$ and $X \in HC \to X \in A)$, then A is uncountable, so $A \notin HC$. Thus $HC \vDash \neg\mathsf{Pow}$.

Lecture 8 Using a similar result, we now have that H_κ is a model of all of ZFC without Power Set if κ is regular.

Proposition. If κ is a strong limit cardinal, then $H_\kappa \vDash \mathsf{Pow}$.

Proof. We'll show: if $x \in H_\kappa$, then $\mathcal{P}(x) \in H_\kappa$. This is much stronger than the Power Set axiom.

$$\mathrm{tcl}(\mathcal{P}(x)) = \underbrace{\mathcal{P}(x)}_{\text{size } 2^{|x|} < \kappa} \cup \underbrace{\mathrm{tcl}(x)}_{\text{size} < \kappa}$$

Together, $|\,\mathrm{tcl}(\mathcal{P}(x))| < \kappa$, so $\mathcal{P}(x) \in H_\kappa$. □

1.6 The Constructible Universe

So far, all of our proofs were fairly crude: instead of forming the 'internal' power set, we just used the external power set and showed it works. Similarly for Replacement, we showed a much stronger statement.

So a more general idea would be to build inner models using 'definability' properties, using the internal logic of a model. The problem with this is that definability is not definable.

Theorem (Tarski, Undefinability of truth)**.** Let $(M, \in) \vDash \mathsf{ZFC}$. We assume that the language of set theory $\mathscr{L}_\in \subseteq M$. Consider the set S of sentences in $\mathscr{L}_\in$ and the set U of unary predicates ($\mathscr{L}_\in$-formulas in one free variable). A **truth predicate** would be a formula $T(x)$ in $\mathscr{L}_\in$ (i.e. $T \in U$) such that

$$(M, \in) \vDash \varphi \iff (M, \in) \vDash T(\varphi).$$

There can be no such truth predicate.

Contrast this with our previous result: If M is a set, then '$(M, \in) \vDash \varphi$' is Δ_0.

Proof. The key idea is diagonalisation. If $\varphi(x) \in U$, then we can ask whether

$$\underbrace{\varphi(\varphi)}_{\in S} \text{ is true.}$$

Let's assume that T is a truth predicate, and define

$$\delta(x) := \neg T(x(x))$$

where $x(x)$ is x applied to x if $x \in U$ and $\varnothing$ otherwise. Now apply δ to δ, and note $\delta(\delta) \in S$.

$$M \vDash \delta(\delta) \iff M \vDash T(\delta(\delta))$$

since T was a truth predicate, but also

$$M \vDash \delta(\delta) \iff M \vDash \neg T(\delta(\delta)),$$

a contradiction. $\square$

Definition. Again, let $M \vDash \mathsf{ZFC}$. We say $x \in M$ is **definable** if there is a formula φ such that

$$\forall y \in M \quad x = y \iff M \vDash \varphi(y).$$

We say that a formula D is called a **definition of definability** if

$$\forall x \in M \quad x \text{ is definable} \iff M \vDash D(x).$$

Theorem (Undefinability of definability)**.** There is no formula D that is a definition of definability.

Proof. Assume that D is a definition of definability. Consider

$$\alpha := \min\{\beta \mid \beta \text{ is not definable but } \forall \gamma < \beta\ \exists \gamma'\ \gamma < \gamma' < \beta \text{ and } \gamma' \text{ is definable}\},$$

the supremum of the definable ordinals. This defined by the formula:

$$y = \alpha \longleftrightarrow \begin{array}{c} y \text{ is an ordinal and } \neg D(y) \text{ and} \\ \forall \gamma\ (\gamma < y \rightarrow \exists \gamma'\ (\gamma < \gamma' < y \wedge D(\gamma'))) \end{array}$$

This proves that α is definable, so

$$(M, \in) \vDash D(\alpha).$$

But one of the conjuncts in the definition implies $M \vDash \neg D(\alpha)$. Contradiction. $\square$

We now learned that 'definability' is not going to work without keeping track of parameters. So, we need to define 'definability' with direct reference to what parameters are allowed.

Definition. Fix A and $m \in \mathbb{N}$. We're going to define by recursion what it means to be a **definable subset** of A^n:

$$\begin{aligned}
\mathrm{Diag}_{\in}(A, n, i, j) &:= \{s \in A^n \mid s_i \in s_j\} \\
\mathrm{Diag}_{=}(A, n, i, j) &:= \{s \in A^n \mid s_i = s_j\} \\
\mathrm{Proj}(A, R, n) &:= \{s \in A^n \mid \exists t \in R\ (t \restriction n = s)\} \\
\mathrm{Def}(0, A, n) &:= \{\mathrm{Diag}_{\in}(A, n, i, j) \mid i, j < n\} \cup \{\mathrm{Diag}_{=}(A, n, i, j) \mid i, j < n\} \\
\mathrm{Def}(k+1, A, n) &:= \mathrm{Def}(k, A, n) \cup \{R \cap S \mid R, S \in \mathrm{Def}(k, A, n)\} \\
&\qquad \cup \{A^n \setminus S \mid S \in \mathrm{Def}(k, A, n)\} \\
&\qquad \cup \{\mathrm{Proj}(A, R, n) \mid R \in \mathrm{Def}(k, A, n+1)\} \\
\mathrm{Def}(A, n) &:= \bigcup_{k \in \mathbb{N}} \mathrm{Def}(k, A, n).
\end{aligned}$$

Observe the definitions of $\mathrm{Def}(k+1, A, n)$ and $\mathrm{Def}(0, A, n)$ are Δ_0, so the definition of $\mathrm{Def}(A, n)$ is a recursive definition based on absolute notions and thus absolute for transitive models (containing A).

Next, use $\mathrm{Def}(A, n)$ to define the 'definable power set'. Note that $\mathrm{Def}(A, 1)$ will be countable, so isn't a reasonable definition of the power set. After that, define a 'definable von Neumann hierarchy'.

Lecture 9 Our goal is to construct an inner model $L \subseteq M$ that is based on definability. As we saw, this had problems, connected to Tarski's Undefinability of Truth. The 'definable' fragment of truth is that where we fix the scope of existential quantifiers in advance.

We recursively defined $\mathrm{Def}(A, n)$, the definable subsets of A^n where definable is interpreted in A.

Lemma. If $X \subseteq A^n$ such that there is a formula φ such that

$$(x_1, \ldots, x_n) \in X \iff (A, \in) \vDash \varphi(x_1, \ldots, x_n)$$

then $X \in \mathrm{Def}(A, n)$.

Proof. Simple induction over complexity of φ. □

Lemma. In M, we have that $\mathrm{Def}(A, n)$ is countable. (We even have a concrete surjection $\mathbb{N} \twoheadrightarrow \mathrm{Def}(A, n)$)

Remark. In the definition of $\mathrm{Def}(k, A, n)$, we only used notions absolute for transitive models of ZF. So, since $\mathrm{Def}(A, n)$ was defined by recursion over $\mathrm{Def}(k, A, n)$, also $\mathrm{Def}(A, n)$ is absolute.

Aim. Our goal is to find a definable power set.

$\mathrm{Def}(A, 1)$ is not a good candidate, since if A is uncountable, then there is $a \in A$ such that $\{a\} \notin \mathrm{Def}(A, 1)$ by the second lemma.

Definition. We define

$$\mathcal{D}(A) := \left\{ X \subseteq A \;\middle|\; \begin{array}{c} \exists n\ \exists s \in A^n\ \exists R \in \mathrm{Def}(A, n+1) \\ \text{s.t. } X = \{a \in A \mid (a, s_0, \ldots, s_{n-1}) \in R\} \end{array} \right\}$$

We call this the **definable power set**.

Remark.

1. If X is informally definable with parameters from A, i.e.

$$X = \{a \in A \mid (A, \in) \vDash \varphi(a, \bar{p})\}$$

with $\bar{p} \in A$, then $X \in \mathcal{D}(A)$.

2. As $\mathrm{Def}(A, n)$ was absolute and the quantifiers in the definition of $\mathcal{D}(A)$ are all bounded, $\mathcal{D}(A)$ is absolute for transitive models.

Proposition. If A is transitive, then $\mathcal{D}(A)$ is transitive.

Proof. Suppose $x \in X \in \mathcal{D}(A)$. Then $x \in X \subseteq A$, so by transitivity of A, $x \in A$. But since $x \in A$, we can define x as a subset by the formula $v \in x = \varphi(v)$,

$$x = \{z \in A \mid (A, \in) \vDash \varphi(z)\}.$$

□

Definition. We can now define the **constructible hierarchy**.

$$L_0 := \varnothing$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

and we refer to the class $\bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$ as 'L' or the '**constructible universe**'.

1. If $\alpha \le \omega$, $V_\alpha = L_\alpha$.
2. For every α, L_α is transitive (follows from the proposition by induction).
3. If $\alpha \le \beta$ then $L_\alpha \subseteq L_\beta$.
4. $\mathrm{Ord} \cap L_\alpha = \alpha$.
5. If $\alpha \ge \omega$ then $|L_\alpha| = |\alpha|$:

   Proof. Induction. $\alpha = \omega$: $|L_\omega| = |V_\omega| = \aleph_0 = |\omega|$. Suppose for all $\beta < \alpha$ we have $|L_\beta| \le |\beta|$. Show that $|L_\alpha| < |\alpha|^+$.

   - If $\alpha = \beta + 1$ is a successor, $L_\alpha = L_{\beta+1} = \mathcal{D}(L_\beta)$. Then there is a surjection from $\aleph_0 \times \bigcup_{n\in\mathbb{N}} L_\beta^n$ onto $\mathcal{D}(L_\beta)$, but $\aleph_0 \times \bigcup_{n\in\mathbb{N}} L_\beta^n$ has size $|L_\beta|$.
   - If α is a limit, let $\pi_\beta : \alpha \to L_\beta$ be a surjection. Then we find surjection from $\alpha \times \alpha \twoheadrightarrow L_\alpha$, with $(\gamma, \gamma') \mapsto \pi_\gamma \gamma'$.

   $|L_\alpha| \ge |\alpha|$ follows from 4. □

6. What about $V_{\omega+1}$ and $L_{\omega+1}$? $V_{\omega+1}$ is uncountable and $L_{\omega+1}$ is countable. $L_{\omega+1} \subseteq V_{\omega+1}$, and $V_{\omega+1} \setminus L_{\omega+1} \neq \varnothing$.

Definition. If x is constructible, $x \in L$, then

$$\varrho_L(x) := \min\{\alpha \mid x \in L_{\alpha+1}\}.$$

Definition. V=L is the **axiom of constructibility**, and refers to

$$\forall x\ \exists \alpha\ (x \in L_\alpha).$$

This is a sentence in $\mathcal{L}_\in$.

Note that the formula describing '$x \in L_\alpha$' and '$x = L_\alpha$' are both absolute for transitive models of set theory.

Proposition. If A is a transitive set model of ZFC + V=L, then there is a limit ordinal λ such that $A = L_\lambda$.

Lecture 10

Proof. Consider $\lambda := \mathrm{Ord} \cap A$. Clearly λ is a limit ordinal (if not, A thinks there is a largest ordinal, but $A \vDash$ ZFC). Claim that $A = L_\lambda$.

First show that $A \subseteq L_\lambda$. If $x \in A$, then by $(A, \in) \vDash$ V=L, find β such that $(A, \in) \vDash x \in L_\beta$. By absoluteness, $x \in L_\beta \subseteq L_\lambda$.

Next show $L_\lambda \subseteq A$: If $x \in L_\lambda$, since λ is a limit, $L_\lambda = \bigcup_{\beta<\lambda} L_\beta$ so find $\beta < \lambda$ and $x \in L_\beta$. Since $(A, \in) \vDash$ ZFC, we know that there is X such that $(A, \in) \vDash X = L_\beta$. By absoluteness, $X = L_\beta$. Therefore $L_\beta \in A$. $x \in L_\beta$, so by transitivity, $x \in A$. □

Our next major goal is to show

Theorem (Gödel 1938)**.** If κ is inaccessible, then $L_\kappa \vDash \mathsf{ZFC} + \mathsf{V{=}L}$.

Corollary. If κ is inaccessible, then there is a countable α such that

$$L_\alpha \vDash \mathsf{ZFC} + \mathsf{V{=}L}.$$

Proof. Take $H \vDash \mathcal{H}^{L_\kappa}(\varnothing) \preccurlyeq L_\kappa$. H is countable. Take $T \cong H$ the Mostowski collapse, $(T, \in) \equiv (L_\kappa, \in)$, so we have

$$(T, \in) \vDash \mathsf{ZFC} + \mathsf{V{=}L}$$

where T is countable transitive.

By the Proposition, $T = L_\alpha$ for some ordinal α. By one of the earlier properties of L, we have $|L_\alpha| = |\alpha|$, so α is countable. $\square$

Contrast this with: If $V_\alpha \vDash \mathsf{ZFC}$, then α cannot be countable: If α is countable, there is a code for a surjection $f : \mathbb{N} \twoheadrightarrow \alpha$ in $V_{\omega+1} \subseteq V_\alpha$, so $V_\alpha \vDash$ 'α is countable', contradicting $V_\alpha \vDash \mathsf{ZFC}$.

Proof of Gödel's Theorem.

Extensionality Follows from transitivity.

Pair $x, y \in L_\kappa$, find α such that $x, y \in L_\alpha$. $\{x, y\} \subseteq L_\alpha$. Clearly the formula $\varphi(z, x, y) :\equiv z = x \wedge z = y$ defines the pair $\{x, y\}$, so by our Lemma earlier, the pair lies in $\mathcal{D}(L_\alpha) = L_{\alpha+1} \subseteq L_\kappa$.

The same proof takes care of, say, Union.

Power Set Consider $x \in L_\kappa$. As before, $\alpha < \kappa$ with $x \in L_\alpha$. Since L_α is transitive, $x \subseteq L_\alpha$. Then $|x| \le |L_\alpha| < \kappa$, because κ was inaccessible.

Consider $\mathcal{P}(x)$ in M, $|\mathcal{P}(x)| = 2^{|x|} < \kappa$. Now $|L_\kappa \cap \mathcal{P}(x)| \le |\mathcal{P}(x)| < \kappa$.

Since L_κ, $\mathcal{P}(x)$ are sets in M, $L_\kappa \cap \mathcal{P}(x)$ is a set in V_κ and it is definable in V_κ by the formula

$$z \in L_\kappa \cap \mathcal{P}(x) \iff z \in L_\kappa \wedge z \subseteq x.$$

But that's not good enough to prove that $L_\kappa \cap \mathcal{P}(x) \in L_\kappa$.

For each $z \in L_\kappa \cap \mathcal{P}(x)$ find $\alpha_z := \varrho_L(z) < \kappa$. Consider

$$\{\alpha_z \mid z \in L_\kappa \cap \mathcal{P}(x)\} \subseteq \kappa,$$

of size $< \kappa$. By the regularity of κ, find a bound $\beta < \kappa$ such that

$$\{\alpha_z \mid z \in L_\kappa \cap \mathcal{P}(x)\} \subseteq \beta$$

so $L_\kappa \cap \mathcal{P}(x) \subseteq L_\beta$.

Define $\mathcal{P} := \{z \mid (L_\beta, \in) \vDash z \subseteq x\}$. By our lemma, $\mathcal{P} \subseteq \mathcal{D}(L_\beta) = L_{\beta+1} \subseteq L_\kappa$. But $L_\kappa \vDash \forall z\ (z \in \mathcal{P} \iff z \subseteq x)$ so $L_\kappa \vDash \mathsf{Pow}$.

Lecture 11

Separation Let $x \in L_\kappa$, φ a formula and take $a_1, \ldots, a_n \in L_\kappa$. Separation says that (informally) $\{z \in x \mid \varphi(z, a_1, \ldots, a_n)\}$ exists. More formally, we need to make the set

$$\{z \in x \mid L_k \vDash \varphi(z, a_1, \ldots, a_n)\}.$$

For each $1 \le i \le n$, find $\alpha_i < \kappa$ such that $a_i \in L_{\alpha_i}$ and find $\alpha < \kappa$ such that $x \in L_\alpha$. Define $\beta := \max\{\alpha, \alpha_1, \ldots, \alpha_n\}$. So for any $z \in x$, we have $z, a_1, \ldots, a_n \in L_\beta$ by transitivity of L_β. So

$$\{z \in x \mid L_\beta \vDash \varphi(z, a_1, \ldots, a_n)\} \in \mathcal{D}(L_\beta) = L_{\beta+1}.$$

However, in general,

$$\{z \in x \mid L_\beta \vDash \varphi(z, a_1, \dots, a_n)\} \neq \{z \in x \mid L_\kappa \vDash \varphi(z, a_1, \dots, a_n)\}$$

Run the 'modified Skolem hull construction' from Example Sheet 1, Question 12. Set $\alpha_0 \coloneqq \mathrm{Ord} \cap L_\beta$, and α_{n+1} as the least γ such that L_γ contains L_{α_n} and a witness for each existential statement true in L_κ with parameters in L_{α_n}. Then

$$\bar{\alpha} \coloneqq \bigcup_{n \in \mathbb{N}} \alpha_n$$
$$L_{\bar{\alpha}} \preccurlyeq L_\kappa \quad \text{with } \bar{\alpha} < \kappa.$$

Define the set via φ over $L_{\bar{\alpha}}$:

$$\begin{aligned} &\{z \in x \mid L_{\bar{\alpha}} \vDash \varphi(z, a_1, \dots, a_n)\} \\ = &\{z \in x \mid L_\kappa \vDash \varphi(z, a_1, \dots, a_n)\}. \end{aligned}$$

The rest of the axioms are similar. □

Theorem (Condensation Lemma)**.** If κ is inaccessible and $x, y \in L_\kappa$, $y \subseteq x$ then there is an $\alpha < \kappa$ with

$$|\alpha| \le |\operatorname{tcl}(x)|$$

such that $y \in L_\alpha$.

Proof. Let $t \coloneqq \operatorname{tcl}(x \cup \{y\})$. Clearly $|t| = |\operatorname{tcl}(x)|$. Form $\mathcal{H}^{L_\kappa}(t) \preccurlyeq L_\kappa$. Mostowski collapse the Skolem hull to

$$T \xrightarrow{\pi} \mathcal{H}^{L_\kappa}(t) \preccurlyeq L_\kappa.$$

Now $|T| = |\mathcal{H}^{L_\kappa}(t)| = |t| = |\operatorname{tcl}(x)|$. π is the identity on t, so $\pi(y) = y$ and $\forall z \in x$, $\pi(z) = z$ and $\pi(x) = x$.

By our lemma, find β such that $T = L_\beta$, which we can do since $L_\kappa \vDash \mathsf{ZFC} + \mathsf{V{=}L}$, and $L_\beta \equiv L_\kappa$. Now y can be defined over L_β with parameters in L_β (viz. y) and $|T| = |L_\beta| = |\beta|$. □

Lecture 12 **Corollary.** If $x = \mathbb{N}$ and $y \subseteq \mathbb{N}$, then there is $\alpha < \omega_1$ such that $y \in L_\alpha$.

Corollary.

(a) $\mathcal{P}(\mathbb{N}) \cap L_\kappa \subseteq L_{\omega_1}$. Observe that $\mathcal{P}(\mathbb{N}) \cap L_\kappa = \mathcal{P}^{L_\kappa}(\mathbb{N})$ where $\mathcal{P}^{L_\kappa}(\mathbb{N})$ refers to the unique $p \in L_\kappa$ such that $L_\kappa \vDash p = \mathcal{P}(\mathbb{N})$.

(b) $\mathcal{P}^{L_\kappa}(\mathbb{N}) \subseteq L_{\omega_1}$

(c) $|\mathcal{P}^{L_\kappa}(\mathbb{N})| \le |L_{\omega_1}| = |\omega_1| = \aleph_1$.

[A one-line proof of the Condensation Lemma:

$$y \in L_\alpha = T \cong \mathcal{H}^{L_\kappa}(\operatorname{tcl}(x) \cup \{y\}) \preccurlyeq L_\kappa.$$

] Let's improve this to show $L_\kappa \vDash \mathrm{CH}$. The key idea is that $L_\kappa \vDash \mathsf{ZFC}$, then the Condensation Lemma is a theorem of ZFC, so apply Condensation Lemma inside L_κ. But this is a lie! Our Condensation Lemma is a theorem of ZFC+IC.

Remark. So if you assume ZFC + $\mathsf{2IC}$, then this argument gets you that $L_\kappa \vDash \mathrm{CH}$ for κ the second inaccessible cardinal. This feels a bit odd. Let's try to do this without the second inaccessible.

Work in L_κ. We know that $\mathcal{P}^{L_\kappa}(\mathbb{N}) \subseteq L_{\omega_1}$ where ω_1 refers to the ω_1 in M. Note that $\omega_1 < \kappa$, $L_{\omega_1} \subseteq L_\kappa$.

$$\mathcal{P}^{L_\kappa}(\mathbb{N}) \subseteq L_\beta = T \cong \mathcal{H}^{L_\kappa}(L_{\omega_1}) \preccurlyeq L_\kappa.$$

By the standard argument, get

$$\beta < \aleph_2 < \kappa.$$

But now $L_\beta \vDash \mathsf{ZFC} + \mathsf{V{=}L}$. Run the Condensation Lemma proof for V_κ as M and L_β as L_κ:

$$y \in L_\alpha = T \cong \mathcal{H}^{L_\beta}(\mathbb{N} \cup \{y\}) \preccurlyeq L_\beta.$$

Now $L_\kappa \vDash$ 'a is countable'. So $L_\kappa \vDash 2^{\aleph_0} \le \aleph_1$, hence $L_\kappa \vDash \mathsf{CH}$.

Theorem. $\mathrm{Cons}(\mathsf{ZFC} + \mathsf{IC}) \implies \mathrm{Cons}(\mathsf{ZFC} + \mathsf{CH})$.

Remark.

(1) The same argument with $x = \lambda$ for some L_κ-cardinal λ gives us $L_\kappa \vDash 2^\lambda \le \lambda^+$. So GCH holds: $\forall \lambda, 2^\lambda = \lambda^+$.

(2) What about the inaccessible?

   (a) If we have a transitive set model of ZFC then we can mimic this proof.

   (b) There is a way of getting around that assumption as well: Use the Lévy Reflection Theorem (Example sheet 2): Fix in advance some finite list Φ of sentenes you wish to preserve and find sufficiently large α such that Φ is absolute for V_α.
   Go through all needed absoluteness results and lemmas and collect for each of them φ the finite set Φ_ϕ of axioms of ZFC needed to prove them. Form $\Phi = \bigcup_\phi \Phi_\phi$, a finite union over all the relevant ϕ. Apply Lévy to Φ and run the previous proof to get a model of $\Phi + \mathrm{CH}$. Now consider all finite subsets Ψ such that $\Phi \subseteq \Psi \subseteq \mathsf{ZFC}$ and get models of ψ+CH. Compactness gives a model of ZFC+CH.

(3) Consider

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha \subseteq M.$$

Our proof does not say what axioms hold in L, but using the Lévy Reflection Theorem, you can prove that $V \vDash \mathsf{ZFC}$, then $L \vDash \mathsf{ZFC} + \mathsf{V{=}L}$.

Recall our question about regular limit cardinals: We have two notions: regular/singular and successor/limit.

	Reg	Sing
succ	✓	✗
limit	?	✓

If we strengthen 'limit' ($\forall \lambda < \kappa\ (\lambda^+ < \kappa)$) to 'strong limit' ($\forall \lambda < \kappa\ (2^\lambda < \kappa)$), then we showed that ZFC cannot prove the existence of regular strong limits = inaccessible cardinals. Clearly ZFC+GCH gives that every limit is a strong limit. So, ZFC+GCH gives that every regular limit is an inaccessible cardinal.

Proof. Assume that $M \vDash \mathsf{ZFC}$ and that $\mathsf{ZFC} \vdash$ there are regular limits. Towards a contradiction with Gödel's Incompleteness Theorem, prove $M \vDash \mathrm{Cons}(\mathsf{ZFC})$. Consider $L \subseteq M$. Then by remark (3), $L \vDash \mathsf{ZFC}$+GCH. By $\mathsf{ZFC} \vdash \exists$ regular limit, we get $L \vDash \mathsf{ZFC} + \mathrm{GCH} + \exists\kappa$ regular limit. Thus, $L \vDash \mathsf{ZFC} + \mathsf{IC}$. Then $L \vDash \exists\kappa(L_\kappa \vDash \mathsf{ZFC})$, so $L \vDash \mathrm{Cons}(\mathsf{ZFC})$ thus $M \vDash \mathrm{Cons}(\mathsf{ZFC})$ by absoluteness, a contradiction. □

1.7 The limitations of the method of inner models

Lecture 13 **Definition.** If $(M, \in) \vDash \mathsf{ZFC}$ and $N \subseteq M$, we say that N is an **inner model** of M if

(a) $(N, \in) \vDash \mathsf{ZFC}$

(b) $\mathrm{Ord} \cap N = \mathrm{Ord} \cap M$

(c) N is transitive in M.

Theorem (Minimality Theorem)**.** If $M \vDash \mathsf{ZFC} + \mathsf{V{=}L}$ and N is an inner model then $N = M$.

Proof. We know that $M = \bigcup_{\alpha \in \mathrm{Ord} \cap M} L_\alpha^M$ where L_α^M is the set L_α (interpreted in M). This follows directly from $\mathsf{V{=}L}$ in M. So in order to show $N = M$, it's enough to show $L_\alpha^M \subseteq N$ for all $\alpha \in \mathrm{Ord} \cap M$. An easy induction shows $L_\alpha^N \subseteq N$ for $\alpha \in \mathrm{Ord} \cap N$. By absoluteness $L_\alpha^N = L_\alpha^M$ □

Remark.

1. If you drop (b), you still get $N = L_\Omega^M$ where $\Omega = \mathrm{Ord} \cap N$.
2. Of course, we don't even need full ZFC for this.

The 'technique of inner models' is then:

- Want to show $\mathrm{Cons}(\mathsf{ZFC} + \varphi)$
- Start with $M \vDash \mathsf{ZFC} + \neg\varphi$
- Go to an inner model $N \subseteq M$, where $N \vDash \mathsf{ZFC} + \varphi$.

Definition. A **definable inner model** is an $L_\in$-formula Φ with one free variable with the property: if $(M, \in) \vDash \mathsf{ZFC}$, then define

$$N := \{x \in M \mid M \vDash \Phi(x)\}.$$

Then N is an inner model of M.

Example. L 'is' such an inner model. We mean $\Phi(x) :\leftrightarrow \exists\alpha\ (x \in L_\alpha)$.

Now we can define what we mean by 'the consistency of φ can be shown by inner models' This means: We find an inner model Φ such that for all $M \vDash \mathsf{ZFC}$ and

$$N := \{x \in M \mid M \vDash \Phi(x)\},$$

$N \vDash \mathsf{ZFC} + \varphi$.

Corollary. There is no inner models proof of the consistency of $\neg\mathrm{CH}$.

Proof. Suppose otherwise, so let Φ be an inner model that proves consistency of $\neg\mathrm{CH}$. Take an arbitrary $M \vDash \mathsf{ZFC}$. Build L^M and form

$$N^* := \{x \in L^M \mid L^M \vDash \varphi(x)\}.$$

This is an inner model of L^M, thus by minimality $N^* = L^M$. So, $N^* \vDash \neg\mathrm{CH} \wedge \mathrm{CH}$. Contradiction. □

2 Outer models

So, try outer models. As an illustration, take $\mathcal{L}$ the language of arithmetic $+, \cdot, 0, 1$. Take FLD, the axioms of fields, and Φ_0 the axiom (schema) for characteristic zero, and $\mathrm{FLD}_0 := \mathrm{FLD} + \Phi_0$, fields of characteristic 0.

Each characteristic has a prime field $\mathbb{Q}$. $\mathbb{Q}$ is minimal in the sense that it has no proper subfields.

$$\mathrm{NSRT} := \forall x\ (x \cdot x \neq 1 + 1)$$

We know $Q \models \mathrm{NSRT}$ (there is No Square Root of Two). In analogy to the discussion of inner models, the technique of submodels cannot show $\mathrm{Cons}(\mathrm{FLD}_0 + \neg\mathrm{NSRT})$ So, we use outer models. Find $X \notin \mathbb{Q}$ (from the surrounding meta-universe), with $X^2 = 2$. We can't just take $\mathbb{Q} \cup \{X\}$, since it may not be a model of FLD_0, so we close under the field axioms: we need $X + X$, $X + X + X$, $q \cdot X$, X^3 and so on. Algebra has various techniques that allow us to do constructions and obtain $\mathbb{Q}(X) \models \mathrm{FLD}_0 + \neg\mathrm{NSRT}$.

Back to set theory: Suppose we have a countable transitive $M \models \mathsf{ZFC} + \mathsf{CH}$. Then all of its elements are countable: $\mathbb{R}^M, \aleph_1^M, \aleph_2^M, \aleph_3^M$ and so on. Since $\mathbb{R}^M$ is countable, there are lots of reals not in M.

In particular, there is an injection $i : \aleph_2^M \to \mathbb{R}$ such that the range of i is disjoint from M. Now form $M(i)$, the smallest ZFC-model containing M as a subset and i as an element (we have not yet formally defined this, but pretend for now that we have a technique to do so). Now $M(i) \models \mathsf{ZFC} + (|\mathbb{R}| \geq |\aleph_2^M|)$.

Unfortunately, $\neg\mathsf{CH}$ would need $|\mathbb{R}| \geq |\aleph_2^{M(i)}|$. So, we additionally need $\aleph_1^M = \aleph_1^{M(i)}$ and $\aleph_2^M = \aleph_2^{M(i)}$ in order to get this.

Thus our proof components are:

1. Find a construction of $M(i) \models \mathsf{ZFC}$
2. Preservation theorems for cardinals

Our only preservation theorem so far: $\mathbb{R}^M = \mathbb{R}^N \Rightarrow \aleph_1^M = \aleph_1^N$ (Example Sheet question 10).

General problem: If x codes a well order on $\mathbb{N}$ of order type $\aleph_1^M$ and $i(\gamma) = x$ then $M(i) \models \aleph_1$ is countable.

Lecture 14 So, in the meta-universe, we have $i : \aleph_2^M \to \mathbb{R}$, and we would like

1. $M[i] \models \mathsf{ZFC}$ transitive, with $M \subseteq M[i]$ and $i \in M[i]$.
2. $\aleph_2^{M[i]} = \aleph_2^M$. This is stronger than $\aleph_1^{M[i]} = \aleph_1^M$.

We had the issue that if x is in the range of i, then $x \in M[i]$. If x is a code for the countability of $\aleph_1^M$, then $M[i] \not\models \aleph_2^{M[i]} = \aleph_2^M$.

Even worse: if such an x can be constructed in ZFC from i, then it will be in $M[i]$. So, we need to guarantee that no such objects can be constructed.

With this goal, Paul Cohen described **Forcing** in 1963.

Definition.

- As usual, we call $(\mathbb{P}, \leq, \mathbb{1})$ a **partial order**/forcing/forcing partial order if $\mathbb{P}$ is a set, $\leq$ is a reflexive, transitive, antisymmetric relation, and $\mathbb{1}$ is the largest element.
- Elements of $\mathbb{P}$ are called **conditions**

$$p \leq q \text{ is read as `}p\text{ is stronger than }q\text{'}$$

- As usual, $C \subseteq \mathbb{P}$ is called a **chain** if $(C, \leq)$ is a total order.
- If $p, q \in \mathbb{P}$, say that p and q are **incompatible**, written $p \perp q$ if there is no $r \leq p$, $r \leq q$.
- $A \subseteq \mathbb{P}$ is called an **antichain** if

$$\forall p, q \in A \text{ if } p \neq q \to p \perp q.$$

- We say that $\mathbb{P}$ has the **countable chain condition** (ccc) if every antichain $\mathbb{P}$ is countable.
- If $D \subseteq \mathbb{P}$, we say D is **dense** if

$$\forall p \in \mathbb{P}\ \exists q \in D\ q \leq p$$

- If $F \subseteq \mathbb{P}$, we say F is a **filter** if
  (a) $\forall p \in F\ \forall q\ (q \geq p \to q \in F)$
  (b) $\forall p, q \in F\ \exists r \in F\ r \leq p, q$.
- We say that $\mathbb{P}$ is **splitting** if for all $p \in \mathbb{P}$

$$\exists q_1, q_2 \in \mathbb{P}\ q_1, q_2 \leq p \text{ and } q_1 \perp q_2.$$

- If $\mathcal{D}$ is a set of dense sets and $G \subseteq \mathbb{P}$, we say that G is **$\mathcal{D}$-generic** if $\forall D \in \mathcal{D}$, $D \cap G \neq \varnothing$.

Example (Cohen forcing)**.**

$$\begin{aligned} \mathbb{P} &:= \{p \mid p \text{ is a partial function from } \mathbb{N} \text{ to } 2 \text{ with finite domain}\} \\ p \leq q &:\Leftrightarrow p \supseteq q \\ \mathbb{1} &:= \varnothing \end{aligned}$$

Note that if $p \perp q$, then there is $n \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ such that $p(n) \neq q(n)$. If F is a filter in $\mathbb{P}$, then $\bigcup F$ is a partial function from $\mathbb{N}$ into 2. Consider $D_n := \{p : n \in \mathrm{dom}(p)\}$. This is dense in $\mathbb{P}$. Set $\mathcal{D} := \{D_n : n \in \mathbb{N}\}$. If F is $\mathcal{D}$-generic, then $\bigcup F : \mathbb{N} \to 2$.

Example. Take now

$$\mathbb{P}_X := \{p \mid p \text{ is a partial function from } \mathbb{N} \text{ to } X \text{ with finite domain}\}$$

for X any set. As before, if F is a filter, $\bigcup F$ is a partial function. If F is $\mathcal{D}$-generic, $\bigcup F : \mathbb{N} \to X$.

Consider $E_x := \{p \mid x \in \mathrm{ran}(p)\}$ for every $x \in X$.

$$\mathcal{D}^* := \mathcal{D} \cup \{E_x \mid x \in X\}.$$

Suppose G is a $\mathcal{D}^*$-generic filter. By the above, $\bigcup G : \mathbb{N} \to X$. For every $x \in X$, $E_x \cap G \neq \varnothing$, so there is $p \in G$ with $x \in \mathrm{ran}(p)$. So $X = \mathrm{ran}(\bigcup G)$. Thus $\bigcup G$ is a surjection from $\mathbb{N}$ to X.

Definition. If M is a transitive model of ZFC and $\mathbb{P} \in M$ a partial order. We say that a filter G (not necessarily an element of M) is **generic over** M if it is $\mathcal{D}$-generic for $\mathcal{D} := \{D \in M \mid D \text{ is dense in } \mathbb{P}\}$.

Lemma. Suppose $\mathbb{P}$ is splitting, $\mathbb{P} \in M$ and M is a transitive model of ZFC. Suppose G is a $\mathbb{P}$-generic filter over M. Then $G \notin M$.

Proof. Suppose $G \in M$. Then $D := \mathbb{P} \setminus G \in M$. We claim that D is dense. Take $p \in \mathbb{P}$ arbitrary. $\mathbb{P}$ is splitting, so we find $q_1, q_2 \le p$ with $q_1 \perp q_2$. Since G is a filter, not both $q_1, q_2 \in G$, so at least one is in D. So by definition $G \cap D = G \cap \mathbb{P} \setminus G \neq \varnothing$, a contradiction. □

Lecture 15 **Lemma.** If $\mathcal{D}$ is a countable set of dense sets and $p \in \mathbb{P}$, then there is a $\mathcal{D}$-generic filter G in $\mathbb{P}$ such that $p \in G$.

Proof. Let $\mathcal{D} = \{D_n \mid n \in \mathbb{N}\}$. Define $p_0 := p$. Suppose $p_0 \ge \cdots \ge p_n$ are already defined. By definition, D_n has an element q such that $q \in D_n$, $q \le p_n$. Then $p_{n+1} := q$.

Consider $X := \{p_n \mid n \in \mathbb{N}\} \subseteq \mathbb{P}$. Note that if $p_n, p_k \in X$ then $p_{\max(n,k)} \le p_n, p_k$. So $G := \{p \mid \exists n\ p_n \le p\}$ is a filter on $\mathbb{P}$. Clearly, for every n, $G \cap D_n \neq \varnothing$, as required. □

Corollary. If M is a countable transitive model of set theory, $p \in \mathbb{P} \in M$. Then there is a $\mathbb{P}$-generic filter G over M with $p \in G$.

Proof. $\mathbb{P} \in M$, then $\mathbb{P}$ is countable.

$$\{D \subseteq \mathbb{P} \mid D \text{ is dense, } D \in M\} \subseteq \mathcal{P}^M(\mathbb{P})$$

so this is countable as well. □

2.1 Forcing language

Definition. Fix M a transitive model of set theory, $\mathbb{P} \in M$.

$$\begin{aligned} &\mathrm{Name}_0(M, \mathbb{P}) := \varnothing \\ \lambda > 0 \quad &\mathrm{Name}_\lambda(M, \mathbb{P}) := \left\{ \tau \;\middle|\; \begin{array}{c} \text{each element of } \tau \text{ is an ordered pair } (\sigma, p) \\ \text{where } \sigma \in \mathrm{Name}_\alpha(M, \mathbb{P}) \text{ for some } \alpha < \lambda \text{ and } p \in \mathbb{P} \end{array} \right\} \end{aligned}$$

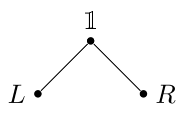
The elements of

$$M^{\mathbb{P}} := \bigcup_{\lambda \in \mathrm{Ord} \cap M} \mathrm{Name}_\lambda(M, \mathbb{P})$$

are called $\mathbb{P}$-**names**.

Example (Silliest example)**.** Take $\mathbb{P} = \{1\}$. Then $\varnothing$ is a name, so $\{(\varnothing, \mathbb{1})\}$ is a name. This results in (an isomorphic copy) of the von Neumann hierarchy inside M.

Example (Silly example)**.** Take the following partial order:



- What are the dense sets? $\{\mathbb{1}, L, R\}, \{L, R\}$.
- What is a filter? $\{\mathbb{1}\}, \{\mathbb{1}, L\}$ and $\{\mathbb{1}, R\}$.
- What does generic mean? Meets every dense set, so contains L or R. Hence $\{\mathbb{1}, L\}$, $\{\mathbb{1}, R\}$ are the generic filters.

Now, $\varnothing$ is a name. On the next level, the relevant ordered pairs are $(\varnothing, \mathbb{1})$, $(\varnothing, L)$, $(\varnothing, R)$. So, the new names at level 1 are

$$\begin{gathered}\{(\varnothing,\mathbb{1})\},\{(\varnothing,L)\},\{(\varnothing,R)\}\\ \{(\varnothing,\mathbb{1}),(\varnothing,L)\},\{(\varnothing,\mathbb{1}),(\varnothing,R)\},\{(\varnothing,L),(\varnothing,R)\}\\ \{(\varnothing,\mathbb{1}),(\varnothing,L),(\varnothing,R)\}\end{gathered}$$

Now that we have these names, we would like to interpret them.

Definition. Let $\mathbb{P}, M$ be as before and let G be $\mathbb{P}$-generic over M, $G \subseteq \mathbb{P}$. If $\tau \in \mathrm{Name}(M, \mathbb{P})$, I define the G-**value** of τ:

$$\mathrm{val}(\tau, G) \coloneqq \{\mathrm{val}(\sigma, G) \mid \exists p \in G, (\sigma, p) \in \tau\}.$$

Define also

$$M[G] \coloneqq \{\mathrm{val}(\tau, G) \mid \tau \in \mathrm{Name}(M, \mathbb{P})\}.$$

Observation. If N is a transitive model of ZFC such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.

(Preview: If we knew $M[G] \vDash$ ZFC, this would be a minimality theorem.)
Returning to our silly example:

$$\mathrm{val}(\varnothing, G) = \varnothing$$

independent of what G is. If τ is any of the other seven names and $(\sigma, p) \in \tau$, then $\sigma = \varnothing$. So

$$\mathrm{val}(\tau, G) = \begin{cases}\varnothing \\ \{\varnothing\}\end{cases}$$

Consider $\tau_L = \{(\varnothing, L)\}$, $\tau_R = \{\varnothing, R\}$. Consider $G_L = \{\mathbb{1}, L\}$ and $G_R = \{\mathbb{1}, R\}$. Then

$$\begin{gathered}\mathrm{val}(\tau_L, G_L) = \{\varnothing\} = \mathrm{val}(\tau_R, G_R)\\ \mathrm{val}(\tau_L, G_R) = \varnothing = \mathrm{val}(\tau_R, G_L).\end{gathered}$$

Definition. Let $x \in M$. We define the **canonical name** for x by recursion:

$$\check{x} \coloneqq \{(\check{y}, \mathbb{1}) \mid y \in x\}.$$

Proposition. For any G such that $\mathbb{1} \in G$, we have $\mathrm{val}(\check{x}, G) = x$.

Proof sketch. By $\in$-induction. (Note we only need $\mathbb{1} \in G$, not full $\mathbb{P}$-genericity). □

Corollary. $M \subseteq M[G]$.

Lecture 16 We now try to embed G in $M[G]$. We might try

$$\check{G} \coloneqq \{(\check{p}, \mathbb{1}) \mid p \in G\}$$

but this only works if $G \in M$ already, which is not helpful.

Definition.

$$\Gamma \coloneqq \{(\check{p}, p) \mid p \in \mathbb{P}\}.$$

Lemma. $\mathrm{val}(\Gamma, G) = G$.

Proof. Suppose $x \in \mathrm{val}(\Gamma, G)$. By definition, $x = \mathrm{val}(\check{p}, G) = p$ for some $p \in \mathbb{P}$ with $p \in G$. So $x \in G$.

Suppose $x \in G$. Then $(\check{x}, x) \in \Gamma$, but then since $x \in G$, $x = \mathrm{val}(\check{x}, G) \in \mathrm{val}(\Gamma, G)$. So $x \in \mathrm{val}(\Gamma, G)$. $\square$

Corollary. If M is transitive, $G \in M[G]$.

The important question remaining is: Does $M[G] \vDash \mathsf{ZFC}$?

Lemma. $M[G]$ is transitive.

Proof. We need to show $x \in y \in M[G] \implies x \in M[G]$. $y \in M[G]$ means there is τ with $y = \mathrm{val}(\tau, G)$. $x \in y$ means $\exists \sigma, p$ with $p \in G$, $(\sigma, p) \in \tau$ and $x = \mathrm{val}(\sigma, G)$. Thus $x \in M[G]$. $\square$

Corollary. $M[G] \vDash \mathsf{Ext}$ and $M[G] \vDash \mathsf{Found}$ (by foundation of the universe).

Let's now look at Pairing. Take $x, y \in M[G]$. Suppose that $x = \mathrm{val}(\tau, G)$ and $y = \mathrm{val}(\sigma, G)$. Write down the name

$$\mu_{\sigma,\tau} = \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}.$$

Now,

$$\begin{aligned} \mathrm{val}(\mu_{\sigma,\tau}, G) &= \{\mathrm{val}(\sigma, G), \mathrm{val}(\tau, G)\} \quad \text{because } \mathbb{1} \in G \\ &= \{x, y\}. \end{aligned}$$

Proposition. $M[G] \vDash \mathsf{Pair}$.

Note this name is not unique. On Example Sheet 3, we will show $M[G] \vDash \mathsf{Union}$.

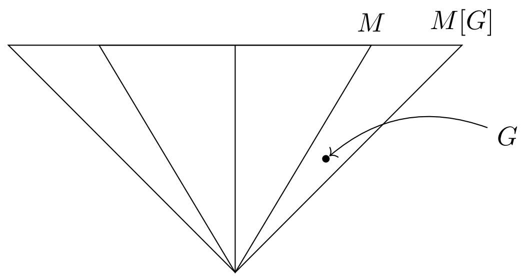
Corollary. $\check{\omega} \in M^{\mathbb{P}}$, and $\mathrm{val}(\check{\omega}, G) = \omega$ so $M[G] \vDash \mathsf{Inf}$.

Lemma. If $\tau \in M^{\mathbb{P}}$, then

$$\varrho(\mathrm{val}(\tau, G)) \leq \varrho(\tau).$$

Proof. Simple induction. $\square$

Corollary. $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[G]$.



It would be nice to be able to talk about whether a name σ is a name for a subset of $\mathrm{val}(\tau, G)$ without referring to the precise set-theoretic make-up of τ and σ.

Definition. The **forcing language**, written $\mathcal{L}(M^{\mathbb{P}})$, is just $\mathcal{L}_{\in}$ augmented with one constant symbol for each $\tau \in M^{\mathbb{P}}$. If G is $\mathbb{P}$-generic over M, there is a canonical interpretation of $\mathcal{L}(M^{\mathbb{P}})$ in $M[G]$.

$$M[G] \vDash \varphi(\tau_1, \ldots, \tau_n) :\iff M[G] \vDash \varphi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G)).$$

If φ is a sentence of $\mathcal{L}(M^{\mathbb{P}})$ and $p \in \mathbb{P}$, the **forcing relation** is

$$p \Vdash_M \varphi :\iff \text{for every } \mathbb{P}\text{-generic filter } G \text{ over } M \text{ such that } p \in G, \text{ we have } M[G] \vDash \varphi.$$

2.2 Forcing relations

We will eventually prove:

Theorem (Forcing theorem)**.** The following are equivalent:

1. $M[G] \vDash \varphi$
2. $\exists p \in G\ (M \vDash p \Vdash \varphi)$.

Theorem. The forcing relation is definable in M: there is a definable relation $p \Vdash^* \varphi$ such that

$$\forall p, \varphi \quad p \Vdash \varphi \iff p \Vdash^* \varphi.$$

Lecture 17 Technically, the meaning of $M[G] \vDash \varphi$ does not just depend on $M[G]$, but on G itself, from the definition above. Better notation would look like $M[G], G \vDash \varphi$.

Similarly, we should more precisely write $\Vdash_M$ for the forcing relation. A priori, $\Vdash$ is not definable in M, but we'll show that it is. We shall give a different definition of a relation $p \Vdash^* \varphi$ which is definable in M and show $p \Vdash^* \varphi$ iff $p \Vdash \varphi$. We call $\Vdash^*$ the **syntactic forcing relation** and $\Vdash$ the **semantic forcing relation**.

Let's restate the forcing theorem more formally.

Theorem (Forcing theorem)**.** Let M be a countable transitive model of set theory and $\mathbb{P} \in M$. Let φ be a sentence in the forcing language. Then the following are equivalent.

(i) $M[G] \vDash \varphi$

(ii) there is a $p \in G$ such that $M \vDash p \Vdash^* \varphi$

(Note that we haven't defined $\Vdash^*$ yet). Let's prove the equivalence of $\Vdash^*$ and $\Vdash$ from the Forcing theorem.

Proposition. Let M be countable transitive model, $\mathbb{P} \in M$, $p \in \mathbb{P}$, φ a sentence of $\mathcal{L}(M^{\mathbb{P}})$. Then $p \Vdash \varphi \iff p \Vdash^* \varphi$.

Proof. Suppose $p \Vdash_M \varphi$. For every G which is $\mathbb{P}$-generic over M with $p \in G$, $M[G] \vDash \varphi$. By the Forcing theorem, find $q \in G$ such that $M \vDash q \Vdash^* \varphi$.

Closure properties of $\Vdash$:

1. If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.
2. If $p \Vdash \varphi$ and $p \Vdash \psi$, then $p \Vdash \varphi \wedge \psi$.

We know that $p, q \in G$, so there is some $r \in G$ such that $r \leq p, q$. By 1, we get that $r \Vdash \varphi$. Stuck... □

Let's postpone this until we have actually defined $\Vdash^*$.

Definition. We say that D is **dense below** p if

$$\forall q \leq p\ \exists r \leq q\ r \in D.$$

Definition. Now we define $\Vdash^*$ by recursion on the rank of the names involved and the complexity of φ. It suffices to:

1. define for $\tau_1 \in \tau_2$ where τ_1, τ_2 are names.
2. define for $\tau_1 = \tau_2$ where τ_1, τ_2 are names.
3. conjunctions: from φ, ψ define for $\varphi \wedge \psi$.

4. negation: from φ define for $\neg\varphi$.
5. existential: from φ define for $\exists x\ \varphi$.

In particular, we set

3. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$. (Remark: We have no choice here due to property 2 in the failed proof).
4. $p \Vdash^* \neg\varphi$ iff $\forall q \le p$, $q \nVdash^* \varphi$.
5. $p \Vdash^* \exists x\ \varphi(x, \tau_1, \ldots, \tau_n)$ iff
$$\{r \mid \text{there is a } \mathbb{P}\text{-name } \sigma \text{ such that } r \Vdash^* \varphi(\sigma, \tau_1, \ldots, \tau_n)\}$$
is dense below p.
1. $p \Vdash^* \tau_1 \in \tau_2$ iff
$$\{q \mid \text{there is } (\pi, s) \in \tau_2 \text{ such that } q \le s \text{ and } q \Vdash^* \pi = \tau_1\}$$
is dense below p.
2. $p \Vdash^* \tau_1 = \tau_2$ iff for all $(\pi_1, s_1) \in \tau_1$,
$$\{q \mid q \le s_1 \to \exists(\pi_2, s_2) \in \tau_2 \text{ such that } q \le s_2 \text{ and } q \Vdash^* \pi_1 = \pi_2\}$$
is dense below p and for all $(\pi_2, s_2) \in \tau_2$,
$$\{q \mid q \le s_2 \to \exists(\pi_1, s_1) \in \tau_1 \text{ such that } q \le s_1 \text{ and } q \Vdash^* \pi_1 = \pi_2\}$$
is dense below p.

Note that only 5 is not absolute.

Lecture 18 From Example Sheet 3, exercise 23, we have:

(1) If D is dense below p and $r \le p$, then D is dense below r.

(2) If $\{r \mid D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

Lemma. The following are equivalent:

(i) $p \Vdash^* \varphi$

(ii) $\forall r \le p\ (r \Vdash^* \varphi)$

(iii) $\{r \mid r \Vdash^* \varphi\}$ is dense below p.

Proof. Clearly (ii) $\Rightarrow$ (i), (iii). Let's show (i) $\Rightarrow$ (ii). Proof by induction on the definition of $\Vdash^*$. Of the five parts of the recursion, 1,2,5 are formulated in terms of 'dense below p':
$$p \Vdash^* \varphi \iff X_\varphi \text{ is dense below } p$$
By Exercise 23 as quoted above, if $r \le p$ and X_φ is dense below p, X_φ is dense below r. Observe that cases 3 and 4 of (i) $\Rightarrow$ (ii) follow directly from definition.

Similarly, (iii) $\Rightarrow$ (i) goes by recursion via 1.-5. using item (2) of the exercise rather than item (1). □

Corollary.
$$p \Vdash_M \varphi \iff M \vDash p \Vdash^* \varphi.$$
(assuming the forcing theorem and the lemma).

Proof. Assume $p \Vdash_M \varphi$. So for any G such that $p \in G$, $M[G] \vDash \varphi$. The Forcing Theorem says $\exists q \in G$ such that $M \vDash q \Vdash^* \varphi$. We want to show $p \Vdash^* \varphi$. By the lemma, it's enough to show

$$D := \{r \mid r \Vdash^* \varphi\} \text{ is dense below } p$$

Fix any $q' \leq p$. Since M is a countable transitive model of ZFC, there is some H with $q' \in H$ and H is $\mathbb{P}$-generic over M. Since $q' \leq p$, we know $p \in H$. So by $p \Vdash_M \varphi$, we get $M[H] \vDash \varphi$. Apply the Forcing Theorem here and get $q'' \in H$ such that $M \vDash q'' \Vdash^* \varphi$. But then $q'' \in D \cap H$. Since H is a filter, we find $r \leq q', q''$ with $r \in H$. r witnesses that D is dense below p.

Now assume $M \vDash p \Vdash^* \varphi$. Want to show $p \Vdash_M \varphi$. Let G be $\mathbb{P}$-generic over M with $p \in G$. By the Forcing Theorem, $M[G] \vDash \varphi$, as required. □

2.3 Proving the Forcing Theorem

It still remains to prove the Forcing Theorem.

Proof of Forcing Theorem. We'll prove this by induction

(a) rank of names

(b) assume that $=$ is done

(c)-(e) complexity of formulas.

We'll do this in the order 3,4,5,1,2 (increasing difficulty). Induction statement:

$$M[G] \vDash \varphi \iff \exists p \in G,\ M \vDash p \Vdash^* \varphi.$$

3. Assume the induction hypothesis holds for φ and ψ.

   $(\Rightarrow)$

   $$\begin{aligned} M[G] \vDash \varphi \wedge \psi &\Rightarrow M[G] \vDash \varphi \text{ and } M[G] \vDash \psi \\ &\Rightarrow \exists p \in G\ p \Vdash^* \varphi \text{ and } \exists q \in G\ q \Vdash^* \psi \\ &\Rightarrow \exists r \leq p, q,\ r \in G \text{ so by Lemma } r \Vdash^* \varphi \text{ and } r \Vdash^* \psi \\ &\Rightarrow \exists r \in G\ r \Vdash^* \varphi \wedge \psi. \end{aligned}$$

   by definition.

   $(\Leftarrow)$ Assume $p \in G$

   $$\begin{aligned} p \Vdash^* \varphi \wedge \psi &\Rightarrow p \Vdash^* \varphi \text{ and } p \Vdash^* \psi \\ &\Rightarrow M[G] \Vdash \varphi \text{ and } M[G] \Vdash \psi \\ &\Rightarrow M[G] \Vdash \varphi \wedge \psi. \end{aligned}$$

4. $(\Rightarrow)$ We are now considering $M[G] \vDash \neg\varphi$, using the induction hypothesis for φ. Consider

   $$\mathcal{D} := \{p \mid p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}.$$

   Claim: $\mathcal{D}$ is dense. Proof: Obvious from definition of syntactic forcing of negation 4.

   Thus find $p \in \mathcal{D} \cap G$. If $p \Vdash^* \neg\varphi$, done. Suppose $p \Vdash^* \varphi$. Then by induction hypothesis, $M[G] \vDash \varphi$. Contradiction!

   $(\Leftarrow)$ Now assume $p \in G$ and $p \Vdash^* \neg\varphi$. Suppose $M[G] \vDash \varphi$ for contradiction. By the induction hypothesis, find $q \in G$ with $q \Vdash^* \varphi$. Find $r \leq p, q$ with $r \in G$. By the Lemma, $r \Vdash^* \varphi$ but by definition of $p \Vdash^* \neg\varphi$, $r \nVdash^* \varphi$. Contradiction.

5. (⇒) We have $M[G] \vDash \exists x\ \varphi(x)$. But $\varphi(x)$ is not a sentence, so isn't part of the induction hypothesis. Instead, the induction hypothesis works for $\varphi(\sigma/x)$ for any $\sigma \in M^{\mathbb{P}}$.

$$\begin{aligned}
&\Rightarrow \text{there is } a \in M[G]\ M[G] \vDash \varphi(a/x) \\
&\Rightarrow \text{there is } \sigma \in M^{\mathbb{P}}\ M[G] \vDash \varphi(\sigma/x) \\
&\Rightarrow \text{there is } p \in G \text{ such that } M \vDash p \Vdash^* \varphi(\sigma/x)
\end{aligned}$$

Thus $\{r \mid \text{there is } \sigma \text{ with } \sigma \Vdash^* \varphi(\sigma/x)\}$ is not only dense below P, but is everything below p.

$$\Rightarrow p \Vdash^* \exists x\ \varphi.$$

(⇐) Conversely, assume $p \in G$ with $p \Vdash^* \exists x\ \varphi$. By definition,

$$\mathcal{D} := \{r \mid \text{there is } \sigma \text{ with } r \Vdash^* \varphi(\sigma/x)\}$$

is dense below p. Find $r \in \mathcal{D} \cap G$. Fix a witness σ to the fact that $r \in \mathcal{D}$, so $r \Vdash^* \varphi(\sigma/x)$. By the induction hypothesis,

$$\begin{aligned}
&M[G] \vDash \varphi(\sigma/x) \\
&M[G] \vDash \varphi(\mathrm{val}(\sigma, G)/x) \text{ so } M[G] \vDash \varphi(a/x) \text{ where } a = \mathrm{val}(\sigma, G)
\end{aligned}$$

So: $M[G] \vDash \exists x\ \varphi$.

1. (⇐) Let $p \in G$ such that $p \Vdash^* \tau_1 \in \tau_2$. Induction hypothesis only applies for $=$ now. By definition

$$\mathcal{D} := \{q \mid \exists(\pi, s) \in \tau_2\ (q \le s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p, so find $q \in \mathcal{D} \cap G$. Fix $(\pi, s) \in \tau_2$ such that

$$q \le s \quad \text{and} \quad q \Vdash^* \pi = \tau_1.$$

This gives $s \in G$, so $\mathrm{val}(\pi, G) \in \mathrm{val}(\tau_2, G)$. By the induction hypothesis, we also get $M[G] \vDash \pi = \tau_1$, i.e. $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_1, G)$. This gives $\mathrm{val}(\tau_1, G) \in \mathrm{val}(\tau_2, G)$, so $M[G] \vDash \tau_1 \in \tau_2$.

Lecture 19

(⇒) We have $M[G] \vDash \tau_1 \in \tau_2$, thus $\mathrm{val}(\tau_1, G) \in \mathrm{val}(\tau_2, G)$. Note this doesn't give $\tau_1 \in \tau_2$ as names, only that there is some $(\pi, s) \in \tau_2$ with $s \in G$ and $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_1, G)$. This means $M[G] \vDash \pi = \tau_1$, so by the induction hypothesis find $r \in G$ with $r \Vdash^* \pi = \tau_1$. Find $p \le s, r$ with $p \in G$. Then

$$\{q \le p \mid \exists(\bar{\pi}, \bar{s}) \in \tau_2\ (q \le \bar{s} \wedge q \Vdash^* \bar{\pi} = \tau_1)\}$$

This set, or even the smaller set

$$\{q \le p \mid q \le s \wedge q \Vdash^* \pi = \tau_1\}$$

is everything below p, thus dense.

2. (⇐) Fix $p \in G$ such that $p \Vdash^* \tau_1 = \tau_2$. Our induction hypothesis now only applies to $=$ with names of lower rank. The form of the definition of $p \Vdash^* \tau_1 = \tau_2$ is symmetric in the sense that if we're using the first half to show that $\mathrm{val}(\tau_1, G) \subseteq \mathrm{val}(\tau_2, G)$, then the other half shows $\mathrm{val}(\tau_2, G) \subseteq \mathrm{val}(\tau_1, G)$.

So, we'll show $\mathrm{val}(\tau_1, G) \subseteq \mathrm{val}(\tau_2, G)$. Fix $x \in \mathrm{val}(\tau_1, G)$. Fix a $(\pi_1, s_1) \in \tau_1$ such that $s_1 \in G$ and $\mathrm{val}(\pi_1, G) = x$. Since G is a filter, find $r \leq p, s_1$ with $r \in G$.

$$\mathcal{D} \coloneqq \{q \leq r \mid q \leq s_1 \Rightarrow \exists(\pi_2, s_2) \in \tau_2\ (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below r. Find $q \in \mathcal{D} \cap G$. Since $r \leq s_1$, we know that $q \leq r \leq s_1$, so the antecedent in the definition of $\mathcal{D}$ is true, so there is $(\pi_2, s_2) \in \tau_2$ with $q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2$.

As earlier, we get $s_2 \in G$, thus $\mathrm{val}(\tau_2, G) \in \mathrm{val}(\tau_2, G)$. $\pi_1 \in \tau_1$ and $\pi_2 \in \tau_2$, so both have lower rank, thus by the induction hypothesis, $M[G] \vDash \pi_1 = \pi_2$, i.e. $\mathrm{val}(\pi_1, G) = \mathrm{val}(\pi_2, G)$. Hence, $x = \mathrm{val}(\pi_1, G) \in \mathrm{val}(\tau_2, G)$.

($\Rightarrow$) Assume $M[G] \vDash \tau_1 = \tau_2$, that is $\mathrm{val}(\tau_1, G) = \mathrm{val}(\tau_2, G)$, Consider

$$\begin{aligned}
\mathcal{D}_1 &\coloneqq \{r \mid r \Vdash^* \tau_1 = \tau_2\} \\
\mathcal{D}_2 &\coloneqq \left\{ r \;\middle|\; \begin{array}{c} \text{there is } (\pi_1, s_1) \in \tau_1 \ \big(r \leq s_1 \text{ and for all } (\pi_2, s_2) \in \tau_2 \\ \text{and all } q\ ((q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \to q \perp r)\big) \end{array} \right\} \\
\mathcal{D}_3 &\coloneqq \left\{ r \;\middle|\; \begin{array}{c} \text{there is } (\pi_2, s_2) \in \tau_2 \ \big(r \leq s_2 \text{ and for all } (\pi_1, s_1) \in \tau_1 \\ \text{and all } q\ ((q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \to q \perp r)\big) \end{array} \right\} \\
\mathcal{D} &\coloneqq \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3
\end{aligned}$$

Observation. If $r \in G$, then neither $r \in \mathcal{D}_2$ nor $r \in \mathcal{D}_3$ can hold. By symmetry, it's enough to deal with $\mathcal{D}_2$. Towards a contradiction, assume $r \in G$ and $r \in \mathcal{D}_2$.

Find $(\pi, s_1) \in \tau_1$ with desired properties. Since $r \leq s_1 \Rightarrow s_1 \in G$ so

$$\mathrm{val}(\pi_1, G) \in \mathrm{val}(\tau_1, G) = \mathrm{val}(\tau_2, G).$$

So there is $(\pi_2, s_2) \in \tau_2$ with $s_2 \in G$ and $\mathrm{val}(\pi_2, G) = \mathrm{val}(\pi_1, G)$. By induction hypothesis, find $q' \in G$ with $q' \Vdash^* \pi_1 = \pi_2$. Find $q \leq r, s_1, s_2, q'$ so q contradicts $r \in \mathcal{D}_2$, as required.

So, we're done if we show that $\mathcal{D}$ is dense. Fix $p \in \mathbb{P}$. If $p \Vdash^* \tau_1 = \tau_2$, then $p \in \mathcal{D}$ done. If it doesn't, we'll find $r \leq p$ for $p \in \mathcal{D}_2$ or $\mathcal{D}_3$. Again by symmetry, we'll show: If the first half of the definition of $p \nVdash^* \tau_1 = \tau_2$ fails, then find $r \in \mathcal{D}_2$.

Lecture 20 We showed if $p \in G$ then neither $p \in \mathcal{D}_2$ or $p \in \mathcal{D}_3$ can hold. Thus if $p \in G \cap \mathcal{D}$ then $p \Vdash^* \tau_1 = \tau_2$. So: need to show that $\mathcal{D}$ is dense. Let $p \in \mathbb{P}$ be arbitrary. If $p \Vdash^* \tau_1 = \tau_2$, done. So assume $p \nVdash^* \tau_1 = \tau_2$.

We will show that if the first half of the definition of $p \Vdash^* \tau_1 = \tau_2$ is violated, then we find $r \leq p$ such that $r \in \mathcal{D}_2$. (Similarly, if the second half is violated, then for some $r \leq p$, we have $r \in \mathcal{D}_3$) The first half fails means: there is $(\pi_1, s_1) \in \tau_1$ such that

$$\mathcal{D}' \coloneqq \{q \leq p \mid q \leq s_1 \to \exists(\pi_2, s_2) \in \tau_2\ (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is not dense below p. Fix this $(\pi_1, s_1) \in \tau_1$ and fix $r \leq p$ such that $\mathcal{D}'$ has no element below r. That is,

$$\forall q \leq r\ (q \leq s_1 \wedge \forall(\pi_2, s_2) \in \tau_2\ (q \not\leq s_2 \vee q \nVdash^* \pi_1 = \pi_2)) \tag{$*$}$$

Fix arbitrary $(\pi_2, s_2) \in \tau_2$ and q such that

$$q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2 \tag{$**$}$$

If q is compatible with r, find $q' \leq r, q$ satisfying ($*$). But now $q' \nVdash^* \pi_1 = \pi_2$ by ($*$) and $q' \Vdash^* \pi_1 = \pi_2$ by ($**$) and $q' \leq q$. Contradiction. So $q \perp r$, as required. $\square$

Theorem (Generic Model Theorem)**.** Take M a countable transitive model of ZFC, $\mathbb{P} \in M$, and say G is $\mathbb{P}$-generic over M,

$$M[G] \vDash \mathsf{ZFC}.$$

Corollary. $M[G]$ is the minimal transitive model of ZFC with $M \subseteq M[G]$, $G \in M[G]$.

Observe that if $M = L_\alpha$ for some countable α and $L_\alpha(G)$ is the relativised L-construction as on Example Sheet 2, then by minimality of L (or $L(G)$ for models containing G), $M[G] = L_\alpha(G)$.

Proof of Generic Model Theorem.

Ext Follows by transitivity of $M[G]$

Found Follows by transitivity of $M[G]$

Pair Was proved earlier by hand

Union On Example Sheet 3, by hand

Inf By absoluteness of '$x = \mathbb{N}$', we only need $\mathbb{N} \in M[G]$. But $\mathbb{N} = \mathrm{val}(\check{\mathbb{N}}, G)$

Choice On Example Sheet 4

Repl On Example Sheet 4

Sep Fix a formula φ, parameters $x_1, \dots, x_n \in M[G]$ and $x \in M[G]$. We would like to construct

$$\{z \in x \mid M[G] \vDash \varphi(z, x_1, \dots, x_n)\}.$$

Fix names $\tau_1, \dots, \tau_n$ for $x_1, \dots, x_n$, and σ for x. Set

$$A := \{z \in \mathrm{val}(\sigma, G) \mid M[G] \vDash \varphi(z, \tau_1, \dots, \tau_n)\}$$
$$\rho := \{(\pi, p) \mid p \Vdash^* \pi \in \sigma \wedge \varphi(\pi, \tau_1, \dots, \tau_n)\}$$

Claim: $A = \mathrm{val}(\rho, G)$.

($\supseteq$) Suppose $z \in \mathrm{val}(\rho, G)$. There is $(\pi, p) \in \rho$ with $p \in G$ and $\mathrm{val}(\pi, G) = z$. $(\pi, p) \in \rho$ means that

$$p \Vdash^* \pi \in \sigma \wedge \varphi(\pi, \tau_1, \dots, \tau_n)$$

By the Forcing Theorem, $M[G] \vDash \pi \in \sigma \wedge \varphi(\pi, \tau_1, \dots, \tau_n)$. Thus $M[G] \vDash z \in x \wedge \varphi(z, x_1, \dots, x_n)$ so $z \in A$.

($\subseteq$) Let $z \in A$, so $z \in x$ and $M[G] \vDash \varphi(z, x_1, \dots, x_n)$. Fix π name for z: $z = \mathrm{val}(\pi, G)$. The Forcing Theorem together with $z \in x$ says there is $p \in G$ with $p \Vdash^* \pi \in \sigma$. Similarly, the Forcing Theorem together with $M[G] \vDash \varphi(z, x_1, \dots, x_n)$ gives that there is $q \in G$ with $q \Vdash^* \varphi(\pi, \tau_1, \dots, \tau_n)$. Find $r \leq p, q$ with $r \in G$. Then $r \vDash^* \pi \in \sigma \wedge \varphi(\pi, \tau_1, \dots, \tau_n)$. So by definition, $(\pi, r) \in \rho$. $z = \mathrm{val}(\pi, G) \in \mathrm{val}(\rho, G)$.

Pow Since we have Separation, it's enough to show

$$\forall x \, \exists y \, \forall z \, (z \subseteq x \to z \in y).$$

Fix a name σ for x. Let's write for any name τ:

$$\mathrm{dom}(\tau) := \{\tau' \mid \exists (\tau', p) \in \tau\}$$

We think of names for subsets of x as names whose domain is a subset of the domain of σ.

$$\rho_\sigma := \{(\tau, \mathbb{1}) \mid \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\}$$

Let's prove that $\mathrm{val}(\rho_\sigma, G)$ satisfies $\forall z\ (z \subseteq x \to z \in \mathrm{val}(\rho_\sigma, G))$. Assume that $z \subseteq x$. Fix a name μ for z: $\mathrm{val}(\mu, G) = z$. Define

$$\mu^* := \{(\pi, p) \mid \pi \in \mathrm{dom}(\sigma) \text{ and } p \Vdash^* \pi \in \mu\}.$$

By def, $\mathrm{dom}(\mu^*) \subseteq \mathrm{dom}(\sigma)$. So $(\mu^*, \mathbb{1}) \in \rho_\sigma$. Hence $\mathrm{val}(\mu, G) \in \mathrm{val}(\rho_\sigma, G)$. Remains to show $\mathrm{val}(\mu, G) = \mathrm{val}(\mu^*, G)$.

If $w \in \mathrm{val}(\mu, G)$: Since $\mathrm{val}(\mu, G) \subseteq \mathrm{val}(\sigma, G)$, we find $(\pi, p) \in \sigma$ with $p \in G$ such that $\mathrm{val}(\pi, G) = w$. $M[G] \vDash \pi \in \mu$. By the Forcing Theorem, there is $q \in G$ with $q \Vdash^* \pi \in \mu$. So $(\pi, q) \in \mu^*$. Since $q \in G$, this means $\mathrm{val}(\pi, G) \in \mathrm{val}(\mu^*, G)$.

If $w \in \mathrm{val}(\mu^*, G)$, find $(\pi, p) \in \mu^*$, $p \in G$ with $w = \mathrm{val}(\pi, G)$. By definition, $p \Vdash^* \pi \in \mu$. The Forcing Theorem says $M[G] \vDash \pi \in \mu$, thus $w = \mathrm{val}(\pi, G) \in \mathrm{val}(\mu, G)$. $\square$

2.4 Consistency proofs

Lecture 21 We will show that V≠L is consistent. Suppose M is a countable transitive model of ZFC + V=L. By our work on L, we know that there is $\alpha < \omega_1$ such that $M = L_\alpha$. Let $\mathbb{P} \in L_\alpha$ be any partial order that is splitting. By countability, we get a G which is $\mathbb{P}$-generic over M, and $G \notin M$ by splitting. So $M[G] \neq M$.

Claim: $M[G] \vDash$ V≠L. Suppose not, then $M[G] = L_\beta$ for some β. Then $\mathrm{Ord} \cap M[G] = \mathrm{Ord} \cap M = \alpha$. So $\beta = \alpha$. So $M[G] = L_\beta = L_\alpha = M$, a contradiction.

Note this didn't use any property of $\mathbb{P}$, other than that it is splitting. We might be able to do more if we use specific $\mathbb{P}$s. So, let's remind ourselves of the specific $\mathbb{P}$s we have:

$$\begin{aligned} \mathbb{P} &= \mathrm{Fn}(X, Y) \quad \text{where } X, Y \in M \\ &= \{p \mid p \text{ is a partial function from } X \text{ to } Y \text{ with finite domain}\}. \end{aligned}$$

If G is $\mathbb{P}$-generic over M, then $\bigcup G =: f_G : X \to Y$ surjective. So if $X = \mathbb{N}$, then $M[G] \vDash Y$ is countable. In particular, if $Y = \aleph_1^M$, then forcing with $\mathrm{Fn}(\mathbb{N}, \aleph_1^M)$ will 'collapse' $\aleph_1^M$:

$$M[G] \vDash \aleph_1^M \text{ is a countable ordinal.}$$

Excursion. Suppose $M \vDash 2^{\aleph_0} = \aleph_2$ and $\mathbb{P} = \mathrm{Fn}(\mathbb{N}, \aleph_1^M)$. If $\aleph_1^M$ is no longer a cardinal in $M[G]$, then the original bijection $f : \mathbb{R}^M \to \aleph_2^M$ will be a bijection between $\mathbb{R}^M$ and an ordinal which can be at best $\aleph_1^{M[G]}$. In order to make this into a proof of $M[G] \vDash$ CH, we need:

1. $M[G] \vDash \aleph_2^M$ is a cardinal, thus $M[G] \vDash \aleph_2^M = \aleph_1$.
2. $M[G] \vDash |\mathbb{R}| = |\mathbb{R}^M|$

Since $\aleph_1^M < \aleph_1^{M[G]}$, we know $\mathbb{R}^{M[G]} \supsetneq \mathbb{R}^M$. If we had 1 and 2, we get a consistency proof of ZFC + CH + V≠L.

Back to forcing ¬CH. Instead, take $\mathbb{P} := \mathrm{Fn}(\mathbb{N} \times \aleph_2^M, \mathbb{N})$. This generates

$$f_G = \bigcup G : \mathbb{N} \times \aleph_2^M \to \mathbb{N}.$$

Define in $M[G]$ for $\alpha < \aleph_2^M$:

$$\begin{aligned} f_\alpha(n) &\coloneqq f(n, \alpha) \\ f_\alpha &: \mathbb{N} \to \mathbb{N}. \end{aligned}$$

We'll show that if $\alpha \neq \beta$ then $f_\alpha \neq f_\beta$. In particular, the map

$$\alpha \mapsto f_\alpha$$

gives an injection from $\aleph_2^M$ into $\mathbb{N}^\mathbb{N}$. Then $M[G] \vDash |\aleph_2^M| \le 2^{\aleph_0}$.

Proof of injectivity. Fix $\alpha \neq \beta$ and define

$$D_{\alpha,\beta} \coloneqq \{p \mid \exists n\ p(n, \alpha) \neq p(n, \beta)\}.$$

I claim $D_{\alpha,\beta}$ is dense. Fix $p \in \mathbb{P}$ arbitrarily. Then $\mathrm{dom}(p)$ is finite, so find $m \in \mathbb{N}$ such that both $(n, \alpha), (n, \beta) \notin \mathrm{dom}(p)$

$$q \coloneqq p \cup \{((n, \alpha), 0), ((n, \beta), 1)\}$$

So $q(n, \alpha) \neq q(n, \beta)$, so $q \le p$, $q \in D_{\alpha,\beta}$. Find $p \in G \cap D_{\alpha,\beta}$. Then $f_G = \bigcup G$ with $p \in G$, so there is n such that

$$\begin{aligned} & f_G(n, \alpha) \neq f_G(n, \beta) \\ \Rightarrow\ & f_\alpha(n) \neq f_\beta(n) \\ \Rightarrow\ & f_\alpha \neq f_\beta. \end{aligned}$$

□

So to summarise, if G is $\mathrm{Fn}(\mathbb{N} \times \aleph_2^M, \mathbb{N})$-generic over M, then

$$M[G] \vDash 2^{\aleph_0} \ge |\aleph_2^M|$$

We need therefore:

$$\begin{aligned} \aleph_1^M &= \aleph_1^{M[G]} \\ \aleph_2^M &= \aleph_2^{M[G]} \end{aligned}$$

in order to get $M[G] \vDash 2^{\aleph_0} \ge \aleph_2$.

Definition. Let M a countable transitive model, and $\mathbb{P} \in M$. We say that $\mathbb{P}$ **preserves cardinals** if for all α an ordinal and all G which is $\mathbb{P}$-generic over M, we have

$$M \vDash \alpha \text{ is a cardinal} \iff M[G] \vDash \alpha \text{ is a cardinal}.$$

Note $\Leftarrow$ follows from the fact that 'is a cardinal' is Π_1.

Theorem. If $M \vDash \mathbb{P}$ has the countable chain condition, then $\mathbb{P}$ preserves cardinals.

Putting this all together, from sheet 3 we know that if Y is countable, then $\mathrm{Fn}(X, Y)$ has the countable chain condition (in ZFC), so $\mathrm{Fn}(\mathbb{N} \times \aleph_2^M, \mathbb{N})$ certainly has the countable chain condition in every model M of ZFC.

So we derive that $\aleph_1^M$ and $\aleph_2^M$ are cardinals in $M[G]$. Thus by the fact that being a cardinal is Π_1, $\aleph_1^{M[G]} = \aleph_1^M$ and $\aleph_2^{M[G]} = \aleph_2^M$.

Corollary. With $\mathbb{P}, G, M$ as above, $M[G] \vDash \neg\mathsf{CH}$.

Corollary. If $M \vDash \mathsf{ZFC}$ countable transitive, then there is a countable transitive $M[G] \vDash \mathsf{ZFC} + \neg\mathsf{CH}$.

Lemma. Suppose $M \vDash \mathbb{P}$ has the countable chain condition,

$$A, B \in M,\ f : A \to B,\ f \in M[G].$$

Then there is $F \in M$ with

(a) $\mathrm{dom}(F) = A$

(b) $\forall a \in A\ F(a) \subseteq B$

(c) $\forall a \in A\ M \vDash F(a)$ is countable

(d) $\forall a \in A\ f(a) \in F(a)$

Proof of Theorem from Lemma. Suppose

$$\begin{gathered} M \vDash \lambda \text{ is a cardinal} \\ M[G] \vDash \lambda \text{ is not a cardinal.} \\ \text{Thus } M[G] \vDash \exists \gamma < \lambda\ \exists f : \gamma \to \lambda \text{ bijective .} \end{gathered}$$

Apply the Lemma to $A = \gamma$, $B = \lambda$, $f = f$ to get $F \in M$. Since f is surjective, $\lambda = \bigcup_{\alpha<\gamma} F(\alpha)$. So $M \vDash |\lambda| \le |\omega \times \gamma| < \lambda$. $\square$

Lecture 22 Reviewing where we are: Our goal is to show CH is independent of ZFC. Take

$$\mathbb{P} := \{p \mid p \text{ a partial function from } \aleph_2^M \times \mathbb{N} \text{ into } \mathbb{N} \text{ such that } p \text{ is finite}\}.$$

If G is $\mathbb{P}$-generic over M, then in $M[G]$ there is an injection from $\aleph_2^M$ into $\mathbb{R}$. If $\aleph_1^M = \aleph_1^{M[G]}$ and $\aleph_2^M = \aleph_2^{M[G]}$ then $M[G] \vDash 2^{\aleph_0} \ge \aleph_2$.

Theorem. If $M \vDash \mathbb{P}$ has the countable chain condition, then $\mathbb{P}$ preserves cardinals.

From Example Sheet 3, $M \vDash \mathbb{P}$ has the countable chain condition. We reduced the Theorem to the Lemma above, which we will now prove.

Proof. Fix a name τ for f. Let '$\tau : \check{A} \to \check{B}$' be the sentence in the forcing language that expresses 'τ is a function from $\check{A}$ to $\check{B}$'.

By the Forcing Theorem, find $p \in G$ with $p \Vdash \tau : \check{A} \to \check{B}$.

$$F(a) := \{b \in B \mid \exists q \le p\ q \Vdash \tau(\check{a}) = \check{b}\}$$

(this makes sense because below p, τ is a function, so we have (a)). Formally, $\tau(\check{a}) = \check{b}$ is the sentence in the forcing language expressing 'the value of τ at $\check{a}$ is $\check{b}$'.

Certainly $\forall a \in A$, $F(a) \subseteq B$ giving property (b). Next, let $\bar{b} := f(a) \in B$. Then $M[G] \vDash \bar{b} = f(a)$. That is, $\tau(\check{a}) = \check{\bar{b}}$. By the Forcing Theorem, find $q \in G$ such that

$$q \Vdash \tau(\check{a}) = \check{\bar{b}}.$$

Since G is a filter, find $q' \le q, p$ with $q' \in G$. But now q' witnesses that $f(a) = \bar{b} \in F(a)$, showing (d).

Finally, need to check that each $F(a)$ is countable in M, to give (c). If I pick for each $b \in F(a)$ a q_b witnessing that $b \in F(a)$, i.e. $q_b \Vdash \tau(\check{a}) = \check{b}$, then if $b \neq b'$, $q_b \perp q_{b'}$ Thus $\{q_b \mid b \in F(a)\}$ is an antichain in $\mathbb{P}$. So, it's countable in M. So $F(a)$ is countable in M. $\square$

Corollary. If there is a transitive set model of ZFC then there is

(a) a transitive set model of ZFC+CH

(b) a transitive set model of ZFC+¬CH

Question. What is the size of $2^{\aleph_0}$ in our models?

So far, the only thing we know is that it is $\geq \aleph_2$.

We will show that if $M \vDash \mathsf{CH}$, then $M[G]$ for G $\mathbb{P}$-generic over M with $\mathbb{P} = \mathrm{Fn}(\aleph_2^M \times \mathbb{N}, \mathbb{N})$ is a model of $2^{\aleph_0} = \aleph_2$.

For this, we need to 'count' the names for subsets of $\mathbb{N}$.

Definition. A name τ for a subset of $\mathbb{N}$ is called **nice** if for every $n \in \mathbb{N}$, there is an antichain $A_n \subseteq \mathbb{P}$ and

$$\tau = \{(\check{n}, p) \mid n \in \mathbb{N}\ p \in A_n\}.$$

How many nice names are there? Well, how many antichains are there? $\left|\mathbb{P}^{\aleph_0}\right|$ is an upper bound. So, what's the size of $\mathbb{P}$? $|\mathbb{P}| = \aleph_2$, so there are $\aleph_2^{\aleph_0}$ antichains. But what is $\aleph_2^{\aleph_0}$? Hausdorff's Formula says: $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$, so

$$\begin{aligned} \aleph_{1+1}^{\aleph_0} &= \aleph_2 \cdot \aleph_1^{\aleph_0} \\ &= \aleph_2 \cdot \aleph_{0+1}^{\aleph_0} \\ &= \aleph_2 \cdot \aleph_1 \cdot \aleph_0^{\aleph_0} \\ &= \aleph_2 \cdot 2^{\aleph_0} \end{aligned}$$

In M, this is $\aleph_2 \cdot \aleph_1 = \aleph_2$. This shows that there are at most $\aleph_2$ many antichains. A nice name is a countable collection of antichains, so at most $\aleph_2^{\aleph_0} = \aleph_2$ many.

Theorem. If $x \subseteq \mathbb{N}$ in $M[G]$, then there is a nice name τ such that $x = \mathrm{val}(\tau, G)$.

Proof. Fix a name μ for x. For every $n \in \mathbb{N}$ construct an antichain A_n with the properties

(a) $\forall p \in A_n\ p \Vdash \check{n} \in \mu$

(b) A_n is antichain

(c) A_n is maximal with respect to (a) and (b).

Now set $\tau := \{(\check{n}, p) \mid p \in A_n\}$. This is a nice name. We claim that $M[G] \vDash \tau = \mu$.

($\subseteq$). Let $y \in \mathrm{val}(\tau, G)$. There is some $n \in \mathbb{N}$ such that $y = n$ and find $p \in A_n \cap G$ that witnesses $y \in \mathrm{val}(\tau, G)$. Since $p \in A_n$, $p \Vdash \check{n} = \mu$. So $n \in \mathrm{val}(\mu, G)$.

($\supseteq$). Let $y \in \mathrm{val}(\mu, G)$. Since $y = n \in \mathbb{N}$, $y = \mathrm{val}(\check{n}, G)$ We know that $(\check{n}, p) \in \tau$ for every $p \in A_n$. If there is $p \in A_n \cap G$ then $n \in \mathrm{val}(\tau, G)$ and we're done.

If $A_n \cap G = \varnothing$ (remember example 24), there is $q \in G$ such that $\forall p \in A_n$, $p \perp q$. This is a contradiction to maximality (in the sense of (c)) of A_n. □

2.5 Remaining questions

Lecture 23

1. Possible size of $2^{\aleph_0}$
2. V=L and CH
3. Forcing CH
4. What about 2^κ for $\kappa > \aleph_0$?

1. We've seen that if $M \vDash \mathsf{CH}$, G is $\mathrm{Fn}(\aleph_2^M \times \mathbb{N}, \mathbb{N})$-generic over M, then

$$M[G] \vDash 2^{\aleph_0} = \aleph_2.$$

Proof via nice names: there are $\aleph_2^{\aleph_0}$ many nice names for $\mathrm{Fn}(\aleph_2^M \times \mathbb{N}, \mathbb{N})$. With CH, we can calculate $\aleph_2^{\aleph_0} = \aleph_2 \cdot \aleph_1^{\aleph_0} = \aleph_2 \cdot \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_2$ using Hausdorff's Formula.

If you replace $\aleph_2^M$ by $\aleph_n^M$ $(n > 0)$ in the forcing, we get $\aleph_n^{\aleph_0}$ many nice names: By Hausdorff, this is $\aleph_n$. So the same proof with $\mathbb{P} = \mathrm{Fn}(\aleph_n^M \times \mathbb{N}, \mathbb{N})$ gives

$$\text{if } M \vDash \mathsf{CH}, \text{ then } M[G] \vDash 2^{\aleph_0} = \aleph_1.$$

What about $\aleph_\omega$? There is a ZFC-theorem called König's Lemma: $\kappa^{\mathrm{cf}\kappa} > \kappa$. Let's see how König's Lemma implies $2^{\aleph_0} \neq \aleph_\omega$. Suppose it is:

$$\mathrm{cf}(2^{\aleph_0}) = \mathrm{cf}(\aleph_\omega) = \aleph_0.$$
$$2^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = (2^{\aleph_0})^{\aleph_0} = (2^{\aleph_0})^{\mathrm{cf}(2^{\aleph_0})} > 2^{\aleph_0}.$$

The proof of König's Lemma in this particular case: $\aleph_\omega^{\aleph_0} > \aleph_\omega$. Suppose $F : \aleph_\omega \to \aleph_\omega^{\aleph_0}$. We'll show (by diagonalisation) that this is not a surjection. If $\alpha < \aleph_\omega = \bigcup_{n \in \mathbb{N}} \aleph_n$, then I find a unique n such that $\aleph_n \le \alpha < \aleph_{n+1}$. Consider $F_n : \aleph_\omega \to \aleph_\omega$ given by $\alpha \mapsto F(\alpha, n)$. Then $R_n = \{F_n(\alpha) \mid \aleph_n \le a < \aleph_{n+1}\} \neq \aleph_\omega$. So, pick b_n such that $b_n \notin R_n$.

Claim that $b : n \mapsto b_n$, $b \in \aleph_\omega^{\aleph_0}$. Suppose it is: find $\alpha < \aleph_\omega$ such that $b = F(\alpha)$. Find n such that $\aleph_n \le \alpha < \aleph_{n+1}$.

$$b_n = b(n) \notin R_n.$$

so $b \neq F(\alpha, n)$. □

How many nice names does $\mathrm{Fn}(\aleph_\omega^M \times \mathbb{N}, \mathbb{N})$ have? $\aleph_\omega^{\aleph_0}$ many.

What if we in addition assume $2^{\aleph_\omega} = \aleph_{\omega+1}$ (true in L).

$$\aleph_\omega < \aleph_\omega^{\aleph_0} \le (2^{\aleph_\omega})^{\aleph_0} = 2^{\aleph_\omega \cdot \aleph_0} = 2^{\aleph_\omega} = \aleph_{\omega+1}$$

so $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.

Corollary. If $M \vDash 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_\omega} = \aleph_{\omega+1}$ and G is $\mathrm{Fn}(\aleph_\omega^M \times \mathbb{N}, \mathbb{N})$-generic over M, then

$$M[G] \vDash 2^{\aleph_0} = \aleph_{\omega+1}.$$

In general, if $\kappa > \aleph_0$ is regular and $M \vDash \mathsf{GCH}$ then $\kappa^{\aleph_0} = \kappa$ and thus forcing with $\mathrm{Fn}(\kappa \times \mathbb{N}, \mathbb{N})$ gives a model of $2^{\aleph_0} = \kappa$.

2. We've seen that $\mathsf{V{=}L} \Rightarrow \mathsf{CH}$. What about the converse? We've seen that forcing with any splitting poset forces $\mathsf{V} \neq \mathsf{L}$. Consider $\mathrm{Fn}(\mathbb{N}, 2)$. This has $\aleph_0^{\aleph_0}$ many nice names.

   So forcing with $\mathbb{P}$ over $M \vDash \mathsf{CH}$, we preserve CH. So, taking everything together, we get that $M[G] \vDash \mathsf{CH} + \mathsf{V{\neq}L}$.

3. Remember the excursion where we looked at

$$\mathrm{Fn}(\aleph_0, \aleph_1).$$

   This does not preserve cardinals: it adds a surjection $\aleph_0 \to \aleph_1$. ($p_\alpha := \{(0, \alpha)\}$ is an antichain of size $\aleph_1$)

Excursion Assume $M \vDash 2^{\aleph_0} = \aleph_2$, G is $\mathbb{P}$-generic over M. We try to prove $M[G] \vDash \mathsf{CH}$. For this, we need:

(a) $\aleph_2$ stays a cardinal

(b) $|\mathbb{R}^{M[G]}| = |\mathbb{R}^M|$

(a) Show that $\mathrm{Fn}(\aleph_0, \aleph_1)$ has the $\aleph_2$-cc; thus by Sheet 4, it preserves cardinals $\geq \aleph_2$. This gives us (a).

(b) Count nice names: how many nice names are there? $2^{\aleph_1} = \aleph_1^{\aleph_1}$ many. If $M \vDash 2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} = \aleph_3$, this does not help, since we'd only get

$$M[G] \vDash 2^{\aleph_0} \leq \aleph_3^M$$

If we knew that $M \vDash 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, then we'd be done.

4. What about 2^κ in general for $\kappa > \aleph_0$? Take $\mathrm{Fn}(\kappa \times \aleph_1, 2)$, the canonical poset for adding κ many subsets of $\aleph_1$. This has the countable chain condition; analysis of nice names (assuming that $M \vDash \mathsf{GCH}$) still gives $M[G] \vDash 2^{\aleph_1} = \kappa$.

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