Part II – Graph Theory

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0 Introduction

0.1 Preliminary

0.2 Informal definitions

0.3 Where do such structures arise?

Theorem (Schur's Theorem)**.** Let n be a positive integer. Then if p is a sufficiently large positive integer, whenever $\{1, 2, \ldots, p\}$ is partitioned into n parts, we can solve $a + b = c$ with a, b, c all in some part.

1 Ramsey Theory

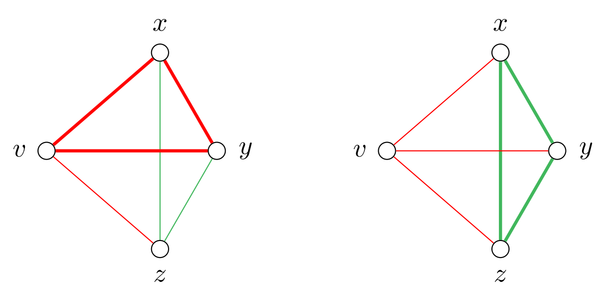
Theorem (Schur's Theorem reformulated)**.** Let k be a positive integer. Then there is a positive integer n such that if the set $[n] = \{1, 2, \ldots, n\}$ is coloured with k colours, we can find a, b, c with $a + b = c$ and a, b, c the same colour.

Proof. ($k = 2$ of Schur's Theorem, improved) Suppose $[5] = \{1, 2, 3, 4, 5\}$ are coloured red and green. Then some three of these are the same colour, and without loss of generality $i < j < k$ are red. If $j - i$ is red we are done, since $i + (j - i) = j$. Similarly if $k - i$ or $k - j$ is red, we are done. If not, all of $j - i$, $k - i$, $k - j$ are green, but then $k - i = (j - i) + (k - j)$, and we are done. □

Proof. (Schur's theorem, $k = 3$) Suppose $[16]$ are coloured red, green and blue. By the pigeonhole principle, some six numbers are the same colour, without loss of generality $x_1 < x_2 < \cdots < x_6$ are red. If $x_j - x_i$ is red for any $i < j$ then we are done: $x_i + (x_j - x_i) = x_j$. So assume all $x_j - x_i$ are blue or green. Consider the five numbers $x_2 - x_1$, $x_3 - x_1$, $x_4 - x_1$, $x_5 - x_1$, $x_6 - x_1$. By the pigeonhole principle, some three of these are the same colour: say $x_i - x_1$, $x_j - x_1$, $x_k - x_1$ are green, for $i < j < k$.

If $x_j - x_i$ is green, we are done: $(x_i - x_1) + (x_j - x_i) = x_j - x_1$, similarly if $x_k - x_i$ or $x_k - x_j$ is green. Otherwise, all of $x_j - x_i$, $x_k - x_i$, $x_k - x_j$ are blue, and we have $(x_j - x_i) + (x_k - x_j) = x_k - x_i$, so we are done. □

Proof. Pick $v \in K_6$. v is in five edges, so some three are the same colour, without loss of generality call them vx, vy, vz and say they are red. If any of xy, xz, yz is red, we have a red triangle with v. If not, xyz is a green triangle.



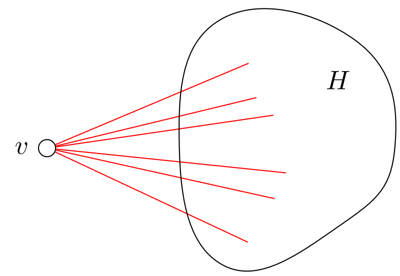
□

Proposition 1. Let k be a positive integer. Then there is a positive integer n such that whenever the edges of K_n are coloured with k colours we can find a monochromatic triangle.

Proof. Induction on k. For $k = 1$, $n = 3$ works, so consider $k > 1$.

By the induction hypothesis, there exists m such that if K_m is $(k-1)$-coloured, then there is a monochromatic triangle. Let $n = k(m - 1) + 2$.

Now k-colour the edges of K_n. Pick vertex v. The number of edges containing v is $n - 1 = k(m - 1) + 1$. So some m of them are the same colour, without loss of generality red. Let H be a K_m joined to v by red edges. If H contains a red edge, it makes a red triangle with v. If not, H is a $(k - 1)$-coloured K_m so by definition of m, it contains a monochromatic triangle.



∎

Theorem 2 (Ramsey's Theorem)**.** $R(s,t)$ exists for all $s,t \geq 2$. Moreover, if $s,t > 2$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Proof. Induction on $s+t$.

For $s = 2$, we have $R(2,t) = t$: If all edges of a K_t are green, then we have a green K_t, otherwise there is a red edge, which is exactly a K_2. Similarly $R(s,2) = s$.

In the case $s,t > 2$, let $a = R(s-1,t)$ and $b = R(s,t-1)$ which exist by the induction hypothesis. Let $n = a+b$. Suppose K_n has edges coloured red/green. Pick $v \in K_n$, which is in $a+b-1$ edges, so either v is in a red edges, or it is in b green edges.

- If v is in a red edges, let H be the K_a joined to v by red edges. Now $a = R(s-1,t)$, so either H has a red K_{s-1}, making a red K_s with v, or H has a green K_t, so done.
- If v is in b green edges, and we can make the same argument with colours reversed. ∎

Corollary 3. For all $s,t \geq 2$, $R(s,t) \leq 2^{s+t}$, so $R(s) \leq 4^s$.

Proof. Induction on $s+t$. Base cases $s = 2$, $R(2,t) = t \leq 2^{2+t}$ and similarly for $t = 2$, $R(s,2) = s \leq 2^{s+2}$. For $s,t \geq 2$

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq 2^{s-1+t} + 2^{s+1-t} \\ &= 2^{s+t}. \end{aligned}$$ ∎

Theorem 4 (Multicolour Ramsey Theorem)**.** Let $k \geq 1$ and $s \geq 2$. Then there exists some n such that whenever the edges of K_n are coloured with k colours, we can find a monochromatic K_s.

Proof. Induction on k. For the base case $k = 1$, we can take $n = s$.

For $k > 1$, by the induction hypothesis we can find m such that if K_m is $(k-1)$-coloured, then there is a monochromatic K_s. Let $n = R(s,m)$ and colour K_n with k colours, including red but not green. Re-colour by turning all non-red edges green. By definition of n, we have either

- A red K_s, so done
- A green K_m. Then in the original colouring, this K_m was $(k-1)$-coloured, so by definition of m, it contains a monochromatic K_s. ∎

Theorem 5 (Infinite Ramsey Theorem)**.** Let $k \geq 1$. Whenever the edges of K_∞ are k-coloured, we have a monochromatic K_∞ subgraph.

Proof. Take $v_1 \in K_\infty$. The vertex v_1 is in infinitely many edges, so infinitely many edges from v_1 are the same colour. Let A_1 be an infinite set of vertices of K_∞ such that for all $u \in A_1$, $v_1 u$ has colour c_1. Now pick $v_2 \in A_1$. Similarly, we can find an infinite $A_2 \subset A_1$ such that all edges $v_2 u$ for $(u \in A_2)$ have colour c_2. Keep going. We get infinite sequences $v_1, v_2, v_3, \dots$ of vertices, $c_1, c_2, c_3, \dots$ of colours and $A_1 \supset A_2 \supset A_3 \supset \dots$ such that

- for $i \geq 2$, $v_i \in A_{i-1}$
- for $i \geq 1$, for all $u \in A_i$, $v_i u$ is an edge of colour c_i.

In particular, if $i < j$ then $v_i v_j$ has colour c_i. Now, infinitely many of the c_i are the same. Say $i_1 < i_2 < i_3 < \dots$ such that $c_{i_1} = c_{i_2} = c_{i_3} = \dots$. Consider $v_{i_1}, v_{i_2}, v_{i_3}, \dots$. Any edge between two of these vertices has colour c_{i_1}. So we have a monochromatic K_∞. □

Corollary 6. Any bounded sequence has a convergent subsequence.

Proof. Any bounded monotone sequence converges, so it is enough to show that any real sequence $(x_n)_{n\geq 1}$ has a monotone subsequence. Let G be a K_∞ with vertex set $\{1, 2, 3, \dots\}$. Colour ij, with $i < j$ blue if $x_i < x_j$ and yellow if not.

By infinite Ramsey, there is a monochromatic subgraph $H \cong K_\infty$. Let the vertices of H be $n_1 < n_2 < n_3 < \dots$. If H is blue, then $(x_{n_j})_{j\geq 1}$ is decreasing, whereas if H is yellow then $(x_{n_j})_{j\geq 1}$ is increasing. □

1.1 Basic Terminology

2 Extremal Graph Theory

2.1 Forbidden Subgraph Problem

Theorem 7. A graph is bipartite iff it contains no odd cycles.

Proof. May return to this later, an exercise for now. $\square$

Theorem 8 (Mantel's Theorem)**.** Let $n \geq 3$. Suppose $|G| = n$, $e(g) \geq \lfloor \frac{n^2}{4} \rfloor$ and $\triangle \not\subset G$. Then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Proof. Induction on n. Start with the $n = 3$ case: Consider $K_{2,1}$, it satisfies $|G| = 3$, $e(G) \geq 2$, $\triangle \subset G$, as required.

$n > 3$: Let $|G| = n$, $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$, $\triangle \not\subset G$. First, remove edges from G if necessary to get H with $|H| = n$, $e(H) = \lfloor \frac{n^2}{4} \rfloor$. Clearly $\triangle \not\subset H$. Let $v \in H$ with $d(v) = \delta(H)$ and let $K = H - v$ (i.e. H with vertex v and all edges including v removed). Now, $|K| = n - 1$, $\triangle \not\subset K$ and $e(K) = \lfloor \frac{n^2}{4} \rfloor - \delta(H)$.

Suppose n is even. Then $\delta(H) \leq$ average degree of $H = \frac{2e(H)}{H} = \frac{n^2/2}{n} = \frac{n}{2}$. Hence

$$e(K) \geq \frac{n^2}{4} - \frac{n}{2} = \frac{n^2 - 2n}{4} = \frac{n^2 - 2n + 1}{4} - \frac{1}{4} = \frac{(n-1)^2}{4} - \frac{1}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

Similarly if n odd, also get $e(K) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.

Hence by the induction hypothesis, $K \cong K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$. Also, $d(v) = e(H) - e(K)$. If n even, $d(v) = \frac{n^2}{4} - \frac{n^2-2n}{4} = \frac{n}{2}$. H is formed by adding a vertex v to $K \cong K_{\frac{n}{2}, \frac{n-2}{2}}$ and joining v to $\frac{n}{2}$ vertices of K, without creating a triangle.

If K has bipartition (X, Y), v cannot be joined both to a vertex in X and a vertex in Y. So v must be joined to all vertices in the larger of X, Y. Thus $H \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and similarly if n is odd. We recover G by adding edges to H without making a $\triangle$. But any new edge creates a $\triangle$, so $G \cong H$. $\square$

Theorem 9 (Turán's Theorem)**.** Let $r \geq 2$ and $|G| = n \geq r + 1$. If $e(G) \geq t_r(n)$ and $K_{r+1} \not\subset G$ then $G \cong T_r(n)$.

Proof. Induction on n. Take first $n = r + 1$. $T_r(r+1)$ has one class of 2 vertices, rest with 1 vertex each. So $T_r(r+1)$ is K_{r+1} with 1 edge removed. So $G \cong T_r(r+1)$.

Next consider $n > r + 1$. First delete edges from G to form subgraph H with $|H| = n$, $e(H) = t_r(n)$ and $K_{r+1} \not\subset H$. Let $v \in H$ have minimal degree and let $K = H - v$. We know $|H| = |T_r(n)|$ and $e(H) = e(T_r(n))$ so H and $T_r(n)$ have same average degree. But degrees in $T_r(n)$ are as equal as possible by property 4 earlier.

So $\delta(H) \leq \delta(T_r(n))$. Thus $|K| = n - 1$, $K_{r+1} \not\subset K$ and

$$e(K) = e(H) - \delta(H) \geq e(H) - \delta(T_r(n)) = t_r(n) - \delta(T_r(n)) = t_r(n-1)$$

by 5. So by induction hypothesis, $K \cong T_r(n-1)$. And $d(v) = e(H) - e(K) = t_r(n) - t_r(n-1)$. To recover H, we must add a vertex and $t_r(n) - t_r(n-1)$ edges to K without creating a K_{r+1}. So by 6, $H \cong T_r(n)$. To recover G, add edges to H without creating a K_{r+1}. But by property 1 we can't add any edges. So, $G \cong T_r(n)$. $\square$

Corollary 10. Let $r \geq 2$. As $n \to \infty$, $\mathrm{ex}(n; K_{r+1}) \sim (1 - \frac{1}{r})\binom{r}{2}$.

Theorem 11. Let $t \geq 2$. Then $\mathrm{ex}(n; K_{t,t}) = \mathcal{O}\left(n^{2-\frac{1}{t}}\right)$.

Proof. Let $|G| = n$, $e(G) = m = \mathrm{ex}(n; K_{t,t})$ and $K_{t,t} \not\subset G$. How many t-fans are there in G? Each $v \in G$ is in $\binom{d(v)}{t}$ t-fans. This total number is $\sum_{v \in G} \binom{d(v)}{t}$.

Given $W \subset V(G)$ with $|W| = t$, W is in at most $t-1$ t-fans (as $K_{t,t} \not\subset G$). So the total number of t-fans is $\leq \binom{n}{t}(t-1)$. Hence

$$\begin{aligned}\binom{n}{t}(t-1) &\geq \sum_{v \in G} \binom{d(v)}{t} \\ &\geq n \binom{\frac{1}{n}\sum_{v\in G} d(v)}{t} \qquad \text{by Jensen} \\ &= n\binom{\frac{2m}{n}}{t}\end{aligned}$$

So $\frac{n^t}{t!}(t-1) \geq \frac{n}{t!}(\frac{2m}{n} - t)^t$, taking the highest factor from the LHS and lowest factor from the RHS so

$$n^t(t-1) \geq n\left(\frac{2m}{n} - t\right)^t.$$

If n is sufficiently large (as we may assume) then $m \geq nt$ and so $\frac{m}{n} \geq t$ and so $\frac{2m}{n} - t \geq \frac{m}{n}$. Thus

$$\begin{aligned} n^t(t-1) &\geq n\left(\frac{m}{n}\right)^t \\ m^t &\leq n^{2t-1}(t-1) \\ m &\leq (t-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} = \mathcal{O}(n^{2-\frac{1}{t}}). \end{aligned}$$

□

Theorem 12. Let $t \geq 2$. Then $z(n,t) = \mathcal{O}(n^{2-\frac{1}{t}})$.

Proof. Let G be bipartite with classes X, Y with $|X| = |Y| = n$, $e(G) = m = z(n,t)$ and $K_{t,t} \not\subset G$. Count number of t-fans with vertex in X and set in Y. Similar to the proof of Theorem 11,

$$\binom{n}{t}(t-1) \geq \sum_{v \in X} \binom{d(v)}{t}$$

Now $\sum_{v \in X} d(v) = m$, so a similar calculation to Theorem 11 gives $m = \mathcal{O}(n^{2-\frac{1}{t}})$. □

Proposition 13. Let H be a graph with at least one edge, and for $n \geq |H|$, let $x_n = \frac{\mathrm{ex}(n;H)}{\binom{n}{2}}$. Then (x_n) converges.

Proof. Let $n > |H|$. Let $|G| = n$, $e(G) = \mathrm{ex}(n; H) = x_n\binom{n}{2}$ and $H \not\subset G$. For any $v \in G$, $|G - v| = n-1$ and $H \not\subset G - v$, so

$$e(G-v) \leq \mathrm{ex}(n-1; H) = x_{n-1}\binom{n-1}{2}.$$

Each edge $xy \in E(G)$ is in $G - v$ for all $v \neq x, y$. Hence

$$(n-2)x_n\binom{n}{2} = (n-2)e(G) = \sum_{v \in G} e(G-v) \leq n x_{n-1}\binom{n-1}{2}.$$

So $x_n \leq x_{n-1}$, so (x_n) is decreasing and bounded below by zero. □

Theorem 14 (Erdős-Stone Theorem)**.** Let $r, t \geq 1$ be integers, and let $\epsilon > 0$ be real. Then $\exists n_0$ such that $\forall n \geq n_0$,

$$|G| = n,\ e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} \implies K_{r+1}(t) \subset G.$$

Proof. Coming up in Section 2.1, non-examinable. □

Corollary 15. Let H be a graph with at least one edge. Then

$$\mathrm{ex}(H) = 1 - \frac{1}{\chi(H) - 1}.$$

Proof. Let $r = \chi(H) - 1$. Then H is $(r+1)$-partite so we can find t such that $H \subset K_{r+1}(t)$ (for instance $t = |H|$ suffices). Let $\epsilon > 0$, and take n_0 as in Theorem 14. Then for all $n \geq n_0$,

$$\begin{aligned} |G| = n,\ e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} &\implies K_{r+1}(t) \subset G \\ &\implies H \subset G. \end{aligned}$$

So, for all $n \geq n_0$, $\mathrm{ex}(n; H) < (1 + \frac{1}{r} + \epsilon)\binom{n}{2}$. Hence

$$\mathrm{ex}(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(n; H)}{\binom{n}{2}} \leq 1 - \frac{1}{r} + \epsilon.$$

But $\epsilon > 0$ was arbitrary, so $\mathrm{ex}(H) \leq 1 - \frac{1}{r}$.

On the other hand, for all n, $H \not\subset T_r(n)$ as H is not r-partite. So $\mathrm{ex}(n, H) \geq t_r(n)$, and

$$\frac{t_r(n)}{\binom{n}{2}} \to 1 - \frac{1}{r} \quad \text{so} \quad \mathrm{ex}(H) \geq 1 - \frac{1}{r}.$$ □

Corollary 16. For any infinite graph G,

$$\mathrm{ud}(G) \in \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}.$$

Proof. It is enough to show that for $r = 1, 2, 3, \dots$,

$$\mathrm{ud}(G) > 1 - \frac{1}{r} \implies \mathrm{ud}(G) \geq 1 - \frac{1}{r+1}.$$

Suppose $\mathrm{ud}(G) > 1 - \frac{1}{r}$. Pick α such that $\mathrm{ud}(G) > \alpha > 1 - \frac{1}{r}$ and fix n. For sufficiently large N, G has subgraphs of order N and density $\geq \alpha > 1 - \frac{1}{r} = \mathrm{ex}(T_{r+1}(n))$. So $T_{r+1}(n) \subset G$, but $D(T_{r+1}(n)) \to 1 - \frac{1}{r+1}$ as $n \to \infty$. We can do this for every n, hence $\mathrm{ud}(G) \geq 1 - \frac{1}{r+1}$. □

Proof. Induction on r. Fix T such that

$$T > \left(\frac{2}{r\epsilon}\right)^t (t-1)$$

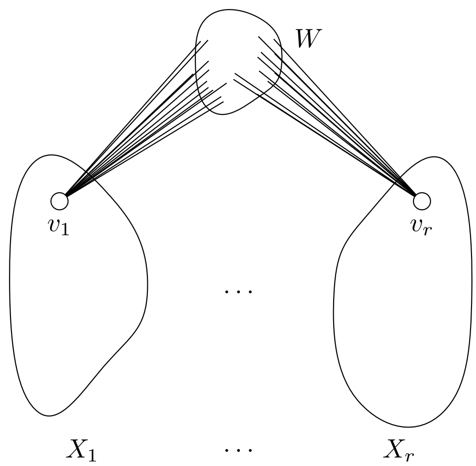
Then choose n_0 such that for all $n \geq n_0$,

$$|G| = n, \quad \delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n \implies K_r(T) \subset G.$$

(How? For $r = 1$, $K_1(T) \subset G \iff |G| \geq T$. For $r > 1$, $1 - \frac{1}{r} > 1 - \frac{1}{r-1}$ so it follows by inductive hypothesis.)

Suppose the result is not true. Then we can find arbitrarily large n and graphs G with $|G| = n$ and $\delta(G) \geq (1 - \frac{1}{r} + \epsilon)n$, $K_{r+1}(t) \not\subset G$. Pick such an n, G with $n \geq n_0$ and also $n \geq \frac{2t}{r\epsilon}$.

Then we can find $K_r(T) \subset G$, say with vertex classes $X_1, \dots, X_r$.



Let

$$A = \{\, (W, v_1, \dots, v_r) \mid W \subset V(G), |W| = t, \forall i\ v_i \in X_i, \forall i \forall w \in W : v_i \sim w \,\}$$

What can we say about $|A|$? First, given $v_1 \in X, \dots, v_r \in X_r$, we can check from the minimum degree condition that

$$|\Gamma(v_1) \cap \ldots \cap \Gamma(v_r)| \geq r\epsilon n$$

So there are at least $\binom{r\epsilon n}{t}$ choices for W. Hence $|A| \geq T^r \binom{r\epsilon n}{t}$.

On the other hand given the set W, as $K_{r+1}(t) \not\subset G$, we know that there is some X_i containing at most $t-1$ vertices joined to all of W. Hence $|A| \leq \binom{n}{t}(t-1)T^{r-1}$, thus

$$T^r \binom{r\epsilon n}{t} \leq \binom{n}{t}(t-1)T^{r-1}.$$

Now,

$$\text{RHS} \leq \frac{n^t}{t!}(t-1)T^{r-1}$$

while

$$\text{LHS} \geq T^r \frac{1}{t!}(r\epsilon n - t)^t \geq T^r \frac{1}{t!}\left(\frac{r\epsilon n}{2}\right)^t.$$

Combining, we get

$$T^r \left(\frac{r\epsilon n}{2}\right)^t \leq n^t (t-1) T^{r-1}$$

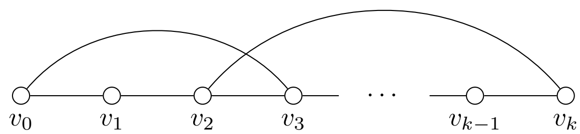
hence $T \leq (\frac{2}{r\epsilon})^t (t-1)$, a contradiction. $\square$

2.2 Hamiltonian graphs

Theorem 17 (Dirac's Theorem)**.** Let $|G| = n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.

Proof. First, observe G is connected. Indeed, if $x \neq y$, $x \nsim y$, then $|\Gamma(x) \cup \Gamma(y)| \leq n - 2$, but $|\Gamma(x)| + |\Gamma(y)| \geq \frac{n}{2} + \frac{n}{2} = n$ so $\Gamma(x) \cap \Gamma(y) \neq \varnothing$.

Let $v_0, v_1, \ldots, v_k$ be a path of maximal length in G, say length $k \leq n-1$. By maximality, $\Gamma(v_0) \subset \{v_1, \ldots, v_k\}$. Similarly, $\Gamma(v_k) \subset \{v_0, \ldots, v_{k-1}\}$.



If we have some situation like in the diagram, we get a cycle. To be more precise, if

$$A = \{\, i \in [k] \mid v_0 \sim v_i \,\} \quad \text{and} \quad B = \{\, i \in [k] \mid v_k \sim v_{i-1} \,\}$$

then $A \cap B \neq \varnothing \implies$ we have a cycle. But $|A \cup B| \leq k < n$ while $|A| + |B| \geq \frac{n}{2} + \frac{n}{2} = n$. Hence $\exists i \in A \cap B$ so we have a cycle $C = v_0 v_1 \ldots v_{i-1} v_k v_{k-1} \ldots v_i v_0$ of length $k+1$.

If $k = n - 1$, we have a Hamiltonian cycle as required. If $k < n - 1$, relabel the cycle $C = u_0 u_1 \ldots u_k u_0$. By connectedness, we have some $u_i \in C$ and $w \notin C$ with $w \sim u_i$. Then $w u_i u_{i+1} \ldots u_k u_0 \ldots u_{i-1}$ is a path of length $k+1$, contradicting maximality. $\square$

Proposition 18. Let G be a connected graph. Then

$$G \text{ Eulerian if and only if } \forall v \in G,\ d(v) \text{ is even.}$$

Proof. $(\Rightarrow)$ is obvious: an Eulerian circuit must go in and out of a given vertex the same number of times.

$(\Leftarrow)$: use induction on $e(G)$. For $e(G) = 0$ it is clearly true.

Consider $e(G) > 0$. Let $v_0 v_1 \ldots v_k = C$ be a circuit in G of maximal length. If C uses all edges of G then we are done. If not, delete all edges used in C from G to form H. In H, every vertex still has even degree. Let H_1 be a component of H with at least one edge.

By induction hypothesis, H_1 has an Euler circuit D. Certainly C, D meet at some vertex v. Join them at v to produce a longer circuit in G, a contradiction. (Walk along C until we get to v, then walk all round D starting/ending at v, then walk along the rest of C). $\square$

3 Graph Colouring

3.1 Planar Graphs

Theorem 19 (Kuratowski's Theorem)**.** Let G be a graph. Then G planar iff G contains no subdivision of K_5 or $K_{3,3}$.

Proposition 20. Every tree of order at least 2 has a leaf.

Proof. Let T be a tree $|T| \geq 2$ and let $v_0 v_1 \dots v_k$ be a path of maximum length in T. Now $v_k \sim v_{k-1}$, but v_k has no other neighbours in the path (as T acyclic) and v_k has no neighbours outside path (by maximality). Hence v_k is a leaf. □

Proposition 21. Let T be a tree, $|T| = n \geq 1$. Then $e(T) = n - 1$.

Proof. Use induction on n. For $n = 1$, $e(T) = 0$ as required.

For $n > 1$, let v be a leaf. Then $T - v$ is a tree with $|T - v| = n - 1$ so by the induction hypothesis, $e(T - v) = n - 2$, so $e(T) = n - 1$. □

Proposition 22. Every tree is planar.

Proof. Let T be a tree, $|T| = n$ and use induction on n. For $n = 0, 1$ we are done immediately.

For $n > 1$, let $v \in T$ be a leaf. By the induction hypothesis, $T - v$ can be drawn. Let $u \in T$ be the neighbour of v. Take a small circle around u in the drawing; so the circle contains only the centre and some radii in the drawing. Hence, we can easily add v and uv to the drawing. □

Theorem 23 (Euler's Formula)**.** Take G connected and planar. Take $|G| = n \geq 1$, $e(G) = m$ with l faces. Then $n - m + l = 2$.

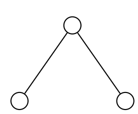
Proof. Induction on m. If G is a tree: $m = n - 1$, $l = 1$, and we are done.

Otherwise, G has a cycle. Pick an edge e in the cycle and consider $G - e$. Then $|G - e| = n$, $e(G - e) = m - 1$. Moreover, in our drawing, removing e combines two faces so we have drawn $G - e$ with $l - 1$ faces. So, by the induction hypothesis $n - (m-1) + (l-1) = 2$, so $n - m + l = 2$. □

Corollary 24. Let G be planar, $|G| = n \geq 3$. Then $e(G) \leq 3n - 6$.

Proof. Let $e(G) = m$. Draw G with l faces. Without loss of generality, G is connected (if not, add edges to make it so).

We have a special case where

$G =$ 

with $n = 3$, $m = 2$ and $3n - 6 = 3 \geq 2$.

Otherwise, we know $n - m + l = 2$ and each face has at least 3 edges in its boundary, and each edge is in the boundary of at most 2 faces. So, $l \leq \frac{2m}{3}$. Thus, $n - m + \frac{2}{3}m \geq 2$ so $\frac{1}{3}m \geq n - 2$ so $m \geq 3n - 6$. □

Proposition 25 (Six colour theorem)**.** Any planar graph is 6-colourable.

Proof. Let G be planar, $|G| = n$. Induction on n. For $n \leq 6$, give every vertex a different colour, as required.

For $n > 6$, by Corollary 24, $e(G) \leq 3n - 6$. Hence

$$\delta(G) \leq \frac{2e(G)}{|G|} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6.$$

So, $\delta(G) \leq 5$.

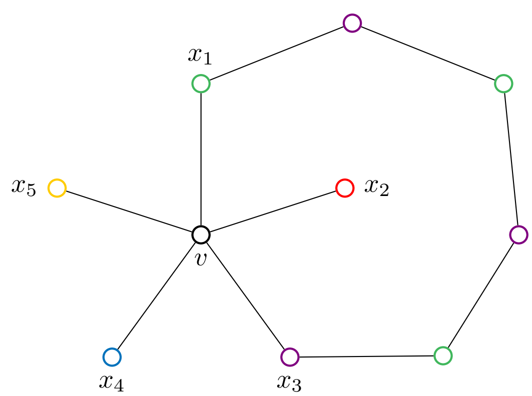
Pick $v \in G$ with $d(v) \leq 5$. By inductive hypothesis, $G - v$ can be 6-coloured. Some colour is missing from $\Gamma(v)$, use this colour to colour v. □

Theorem 26 (Five colour theorem)**.** Any planar graph is 5-colourable.

Proof. Let G be planar, $|G| = n$. Use induction on n. The base case $n \leq 5$ is trivial, so take $n > 5$. As in Proposition 25, we can find $v \in G$ with $\delta(v) \leq 5$ and 5-colour $G - v$. If there is a colour missing from $\Gamma(v)$, use that colour at v. Otherwise, draw G; WLOG v has neighbours $x_1, \ldots, x_5$ in clockwise order around v with colours $1, 2, 3, 4, 5$ respectively.

Call a path in $G - v$ an ij**-path** if all its vertices have colour i or j. Given $x \in G - v$, the ij**-component** of x consists of those vertices reachable from x along ij-paths.

If x_1, x_3 are in different 13 components, then swap the colours of $1, 3$ on the 13 component of x_1 and give v colour 1.



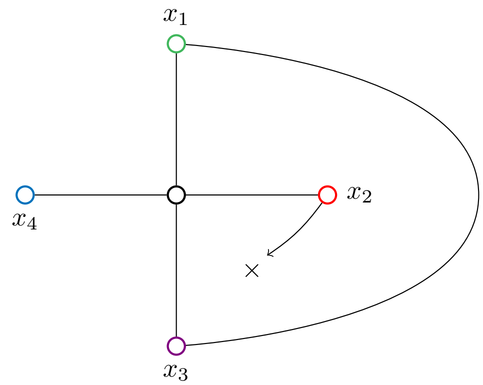
If not, then x_2, x_4 are in different 24 components as we see in the diagram, so swap colours $2, 4$ on the 24 component of x_2 and give v colour 2. □

Theorem 27 (Four colour theorem)**.** Any planar graph is 4-colourable.

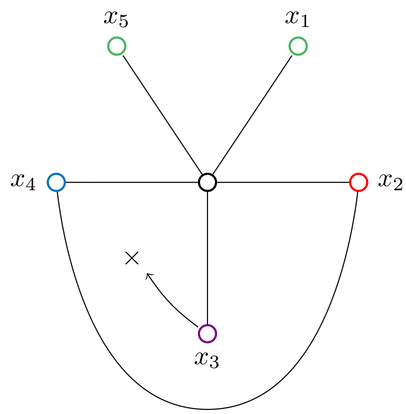
Proof. Let G be planar, $|G| = n$ and use induction on n. For $n = 4$, we are immediately done.

For $n > 4$, pick $v \in G$ with $d(v) \leq 5$, then draw G and 4-colour $G - v$. If some colour is missing on $\Gamma(v)$ then we are done. If not, there are three cases.

Case 1: $d(v) = 4$. There cannot be both a 13-path from x_1 to x_3 and a 24 path from x_2 to x_4, so done as in the proof of Theorem 26.

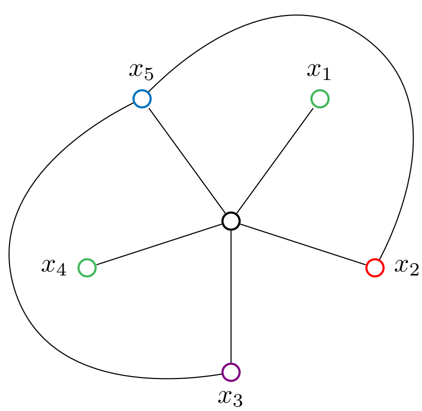


Case 2: $d(v) = 5$, v has neighbours $x_1, \ldots, x_5$ clockwise with colours 1, 2, 3, 4, 1.



Then there is either no 24 path from x_2 to x_4 or no 13 path from x_3 to x_1, so again we are done.

Case 3: $d(v) = 5$, colours 1, 2, 3, 1, 4.



If there is no 24 path from x_2 to x_5, we are done. If there is no 34 path from x_3 to x_5, we are done.

Otherwise, swap colours $1, 3$ on the 13 component of x_1 and swap colours $1, 2$ on the 12 component of x_4. Then, use colour 1 at v, but this is false. $\square$

3.2 General Graphs

Theorem 28 (Brooks' theorem)**.** Let G be a connected graph that is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Induction on $|G|$. Write $\Delta = \Delta(G)$. We cannot have $\Delta = 0, 1$ as $G \ncong K_1, K_2$. If $\Delta = 2$ then G is a path or an even cycle, so $\chi(G) = 2$. So assume $\Delta \geq 3$. Pick $v \in G$ and let H be a component of $G - v$.

Either $\Delta(H) < \Delta$, in which case by greedy algorithm bound (**??**) we have

$$\chi(H) \leq \Delta(H) + 1 \leq \Delta.$$

Or $\Delta(H) = \Delta$. Then H is connected and not an odd cycle (as $\Delta \geq 3$). Moreover, $\exists u \in H$ with $u \sim v$ in G. In G, $d(u) \leq \Delta$ so in H, $d(u) \leq \Delta - 1$. So H is not regular, so not complete. Hence by induction hypothesis, $\chi(H) \leq \Delta$.

Do this for each component of $G - v$ to obtain a Δ-colouring c of $G - v$. If there is a colour missing from $\Gamma(v)$, then use that colour at v, as required. So assume that is not the case.

So have $\Gamma(v) = \{x_1, \ldots, x_\Delta\}$ with $\forall i, c(x_i) = i$. We can also assume:

(i) if $i \neq j$ then there is an ij-path P_{ij} from x_i to x_j,

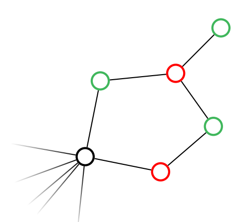
(ii) if $i \neq j$ then P_{ij} is entire ij-component containing x_i, x_j, and

(iii) if i, j, k distinct then P_{ij}, P_{ik} meet only at x_i.

(Why? If any of these fail then it is easy to check that the colouring c can be modified to change the colour of x_i and allowing us to use colour i at v).

(i) is as in previous proofs.

(ii) if P_{ij} is not the entire ij-component, then at some point in the path we get a point (wlog colour j) which 'branches off'.



Then this vertex has 3 green neighbours, so there are at most $\Delta - 3$ colours used in the rest of its neighbours, so there is a colour not in the colours of its neighbours$\cup\{i, j\}$. We can swap it to that, breaking the chain P_{ij}.

(iii) if P_{ij}, P_{ik} meet at some point which must have colour i, then that point has 2 j neighbours and 2 k neighbours, so as before there is a colour other than i that none of its neighbours have, which we can swap it to, breaking P_{ij} and P_{ik}.

As $G \not\cong K_{\Delta+1}$, there are some i, j with $i \neq j$, $x_i \nsim x_j$. As $\Delta \geq 3$, pick $k \in [\Delta] \setminus \{i, j\}$. Let u be the neighbour of x_i of colour j. Swap colours i, k on the ik-component of x_i (i.e. on P_{ik}). This gives a new colouring c', with $c'(x_i) = k$, $c'(x_k) = i$.

Also, if $w \in P_{ij}$ with $w \neq x_i$ then $c'(w) = c(w)$ so $c'(x_j) = c'(u) = j$. c' must satisfy conditions (i),(ii),(iii) as before. Then by (i) there is a kj-path from x_i to x_j , P'_{kj} . By (ii), $u \in P'_{kj}$.

By (i), there is a ji-path P'_{ji} from x_j to x_k. By (ii), $u \in P'_{ji}$. But now P'_{kj} and P'_{ji} meet at u, contradicting (iii). $\square$

3.3 Graphs on surfaces

Theorem 29 (Heawood's Theorem)**.** Let S be a closed boundaryless surface of Euler characteristic $E \leq 1$. Then

$$\chi(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

Proof. Let $\chi = \chi(S)$. Let G be a minimal χ-colourable graph that can be drawn on S - i.e. $\chi(G) = \chi$ but

$$H \subset G, H \neq G \implies \chi(G) \leq \chi - 1.$$

Clearly G is connected and $|G| \geq \chi$.

Let $|G| = n \geq \chi$, $e(G) = m$ and draw G on S with l faces. Then by Euler-Poincaré, $n - m + l \geq E$. As before, $l \leq \frac{2}{3}m$ so $n - \frac{1}{3}m \geq E$ so $m \leq 3n - 3E$. Hence

$$\delta(G) \leq \frac{2m}{n} \leq \frac{6n - 6E}{n} = 6 - \frac{6E}{n}. \qquad (*)$$

On the other hand, if $v \in G$ then $G - v$ is $(\chi - 1)$-colourable and this colouring does not extend to a $(\chi - 1)$-colouring of G so $d(v) \geq \chi - 1$. Hence $\delta(G) \geq \chi - 1$.

Combining this with $(*)$: If $E \leq 0$: $\chi - 1 \leq \delta(G) \leq 6 - \frac{6E}{n} \leq 6 - \frac{6E}{\chi}$ as $n \geq \chi$. Hence $\chi^2 - 7\chi + 6E \leq 0$ and so $\chi \leq \frac{7+\sqrt{49+24E}}{2}$.

If $E = 1$, $\delta(G) \leq 6 - \frac{6}{n} < 6$ so $\delta(G) \leq 5$. Hence $\chi - 1 = 5$ so $\chi \leq 6 = \frac{7+\sqrt{49+24}}{2}$. $\square$

3.4 Edge Colouring

Theorem 30 (Vizing's theorem)**.** Let G be a graph. Then

$$\chi'(G) \leq \Delta(G) + 1$$

Proof. Induction on $e(G)$. The $e(G) = 0$ case is immediate.

For $e(G) > 0$, let $\Delta = \Delta(G)$. Pick an edge xy. By the induction hypothesis we can find an $(\Delta + 1)$-edge colouring of $G - xy_1$, call it φ.

As $\Delta(G) < \Delta + 1$, there some colour 'missing' at each vertex. Let c_1 be missing at y_1. If c_1 is also missing at x, then colour xy_1 with colour c, so done.

If not, let $y_2 \in \Gamma(x)$ with $\varphi(xy_2) = c_1$ and let c_2 be missing at y_2. Continue inductively $(*)$: Given distinct $y_1, \ldots, y_n \in \Gamma(x)$, distinct colours $c_1, \ldots, c_k$ such that c_i is missing at y_i

(for $1 \leq i \leq k$) and $\varphi(xy_{i+1}) = c_i$ (for $1 \leq i \leq k-1$). If c_k is missing at x, re-colour xy_i in colour c_i. Otherwise, let $y_{k+1} \in \Gamma(x)$ with $\varphi(xy_{k+1}) = c_k$. Let c_{k+1} be missing at y_k. If $c_{k+1} \notin \{c_1, \dots, c_k\}$, continue as at $(*)$.

Otherwise assume WLOG $c_{k+1} = c_1$. (If instead $c_{k+1} = c_j$ for some $j > 1$, then uncolour xy_j, recolour xy_i in colour c_i for $1 \leq i < j$ and relabel $y_j, y_{j+1}, \dots$ as $y_1, y_2, \dots$). Let c be a colour missing at x. Consider the cc_1 subgraph of G, i.e. only edges coloured c or c_1. This subgraph has maximum degree ≤ 2 so each component is a path or a cycle. Moreover, x, y_1, y_{k+1} have degree ≤ 1 in this subgraph so x_1, y_1, y_{k+1} are not all in the same component.

If x, y_1 in different cc_1 component, then swap c, c_1 on the component of y_1 and give xy_1 colour c. Otherwise x, y_{k+1} in different components. In this case, uncolour xy_{k+1} and recolour xy_i with colour c_i for $1 \leq i \leq k$. Then swap colours c, c_1 on the cc_1-component of y_{k+1} and give xy_{k+1} colour c. □

4 Connectivity

4.1 The Marriage Problem

Theorem 31 (Hall's Marriage Theorem)**.** Let G be a bipartite graph with bipartition (X, Y). Then G has a matching from X to Y iff G satisfies **Hall's condition**:

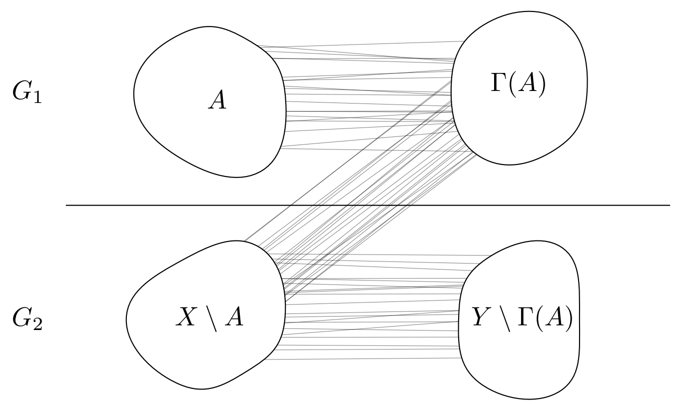
$$\forall A \subset X, |\Gamma(A)| \geq |A|$$

Proof. $(\Rightarrow)$ is obvious. For $(\Leftarrow)$, use induction on $|X|$. $|X| = 0, 1$ are immediate.

For $|X| \geq 2$, there are two cases. The easy case has $|\Gamma(A)| > |A|$ for all $A \subset X$ with $A \neq \varnothing, X$. Pick any $x \in X$. $|\Gamma(x)| \geq 1$ so pick $y \in \Gamma(x)$. Look at $G - \{x, y\}$; this graph satisfies Hall's condition, so has a matching from $X - \{x\}$ to $Y - \{y\}$. Add edge xy, and done.

The harder case has $|\Gamma(A)| = |A|$ for some $A \neq 0, X$. Let

$$\begin{aligned} G_1 &= G[A \cup \Gamma(A)] \\ G_2 &= G[(X \setminus A) \cup (Y \setminus \Gamma(A))]. \end{aligned}$$



First, G_1 obviously satisfies Hall's condition so $\exists$ a matching A to $\Gamma(A)$. What about G_2? Take $B \subset X \setminus A$. Then

$$|\Gamma_{G_2}(B)| = |\Gamma(A \cup B) \setminus \Gamma(A)| = |\Gamma(A \cup B)| - |\Gamma(A)| \geq |A \cup B| - |A| = |B|.$$

Hence G_2 also satisfies Hall and has a matching from $X \setminus A$ to $Y \setminus \Gamma(A)$. Combine the two matchings to get a matching from X to Y in G. □

Corollary 32 (Defect Hall)**.** Let G be a bipartite graph with bipartition (X, Y) and let $d \geq 1$. Then G contains $|X| - d$ independent edges if and only if $\forall A \subset X,\ |\Gamma(A)| \geq |A| - d$.

Proof. $(\Rightarrow)$ is immediate. $(\Leftarrow)$. In marriage terminology: Introduce d imaginary perfect men, suitable husbands for all the women. This satisfies Hall's condition so has matching from X to Y. In real life, at most d women unmarried. □

Corollary 33 (Polyandrous Hall)**.** Let G be a bipartite graph, bipartition (X, Y), $d \geq 2$. Then G contains a set of $d|X|$ edges, each vertex in X in precisely d of them, each vertex in Y in at most one $\iff \forall A \subset X, |\Gamma(A)| \geq d|A|$.

Proof. ($\Rightarrow$) is immediate. ($\Leftarrow$). In marriage terminology: clone each woman $d-1$ times so there are d copies of each. This satisfies Hall's condition so have matching from X to Y. Destroy the clones. $\square$

4.2 Connectivity

Theorem 34. Let G be a graph and $A, B \subset V(G)$. Let

$$k = \min\{\, |W| \mid W \text{ is an } AB\text{-separator} \,\}.$$

Then G contains k vertex-disjoint AB-paths.

Proof. Coming soon. $\square$

Corollary 35 (Menger's Theorem)**.** Let G be an incomplete k-connected graph and let $a, b \in V(G)$, $a \neq b$. Then G contains k independent ab-paths.

Proof. Suppose first $a \nsim b$. Let $A = \Gamma(a)$ and $B = \Gamma(b)$. We have G k-connected and $G - A, G - B$ disconnected, so $|A|, |B| \geq k$.

Hence, any AB-separator W has $|W| \geq k$, so by Theorem 34, there are k vertex-disjoint AB paths. Extend these to a,b.

If instead $a \sim b$: $G - ab$ is $(k-1)$-connected, so has $k-1$ independent ab paths by first part. ab is another, as required. $\square$

Proof of Theorem 34. Induction on $e(G)$. If $e(G) = 0$, the smallest AB-separator is $A \cap B$, and each vertex of $A \cap B$ gives an AB-path (of length zero), so done.

If $e(G) > 0$, pick $xy \in E(G)$. Let W be an AB-separator of minimum order in $G - xy$. If $|W| \geq k$ then by the induction hypothesis there are k vertex-disjoint AB-paths in $G - xy$ and so also in G.

So assume $|W| < k$. Then $W \cup \{x\}$ is an AB-separator in G. Hence $|W \cup \{x\}| \geq k$ so $|W| \geq k-1$ so $|W| = k-1$. Write $W = \{w_1, w_2, \ldots, w_{k-1}\}$. As $|W| < k$, $G - W$ contains an AB-path. This path must include the edge xy. Assume WLOG x comes before y in this path (if not, swap x and y).

Let $X = W \cup \{x\}$ and $Y = W \cup \{y\}$. Let U be an AX-separator in $G - xy$. Then U is an AB-separator in G, so $|U| \geq k$. So by the induction hypothesis, we have k vertex-disjoint AX-paths in $G - xy$, say $P_0, P_1, \ldots, P_{k-1}$ ending at $x, w_1, \ldots, w_{k-1}$ respectively.

Similarly, there are k vertex-disjoint YB paths in $G - xy$, say $Q_0, Q_1, \ldots, Q_{k-1}$ starting at $y, w_1, w_2, \ldots, w_{k-1}$ respectively.

Given path $P = u_0 u_1 \cdots u_l$ and $Q = v_0 v_1 \cdots v_m$ meeting only at $u_l = v_0$, write $P \vee Q$ for the path $u_0 \cdots u_l v_1 \cdots v_m$. Then $P_0 \vee xy \vee Q_)$ and $P_i \vee Q_i$ (for $1 \leq i \leq k-1$) are k vertex-disjoint AB-paths in G. $\square$

4.3 Edge connectivity

Corollary 36 (Edge Menger)**.** Let G be l-edge connected and $a, b \in V(G)$ be distinct. Then G has l edge-disjoint ab-paths.

Proof. The **line graph** of $G = (V, E)$ is the graph $L(G) = (E, F)$ where

$$F = \{\, ee' \mid e, e' \in E,\ e, e' \text{ share precisely one vertex} \,\}.$$

Then $L(G)$ is l-connected. Add extra vertices a', b' to $L(G)$ with

- a' joined to each $e \in E$ with $a \in e$ and
- b' joined to each $e \in E$ with $b \in e$.

By Corollary 35, $L(G)$ has l independent $a'b'$-paths. This yields l edge-disjoint ab-paths in G. □

5 Probabilistic Techniques

5.1 The Probabilistic Method

Theorem 37 (Erdős)**.**

$$R(s) = \Omega(\sqrt{2}^s)$$

Proof. Fix n, s. Colour each edge of K_n red/green at random, independently, each colour equally likely. Given $H \subset K_n$ with $H \cong K_s$,

$$\mathbb{P}(H \text{ monochromatic}) = 2\mathbb{P}(H \text{ red}) = 2 \times \left(\frac{1}{2}\right)^{\binom{s}{2}}.$$

So

$$\begin{aligned}
\mathbb{P}(K_n \text{ has a monochromatic } K_5) &\leq \binom{n}{s} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{s}{2}} \\
&\leq \frac{n^s}{s!} \cdot 2 \left(\frac{1}{2}\right)^{\binom{s}{2}} \\
&\leq n^s \cdot 2^{-\frac{s(s-1)}{2}} \\
&= \left(\frac{n}{\sqrt{2}^{s-1}}\right)^s < 1 \quad \text{if } n < \sqrt{2}^{s-1}.
\end{aligned}$$

So if $n < \sqrt{2}^{s-1}$ then there is *some* colouring with no monochromatic K_5. So $R(s) \geq \sqrt{2}^{s-1}$. □

5.2 Modifying a Random Graph

Theorem 38. If $t \geq 2$ then $z(n,t) = \Omega(n^{2-\frac{2}{t+1}})$.

Proof. Strategy: Given n, p, let G be a random bipartite graph with vertex classes X, Y with $|X| = |Y| = n$, where for each $x \in X$, $y \in Y$ we have $xy \in E(G)$ with probability p, independently. Let $A = e(G)$ and B be the number of copies of $K_{t,t}$ in G (so A,B are random variables).

Aim: Given n, try to choose p such that $\mathbb{E}(A-B)$ is large, specifically $\mathbb{E}(A-B) = \Omega(n^{2-\frac{2}{t+1}})$. Then we can find a specific graph G with $A - B = \Omega(n^{2-\frac{2}{t+1}})$. Remove an edge from each $K_{t,t}$ in G to form a graph H with no $K_{t,t}$ and $e(H) = \Omega(n^{2-\frac{2}{t+1}})$, as required.

Now, $\mathbb{E}(A) = n^2 p$ and $\mathbb{E}(B) = \binom{n}{t}^2 p^{t^2} \leq \frac{1}{(t!)^2} n^{2t} p^{t^2}$. So $\mathbb{E}(A-B) \geq n^2 p - \frac{1}{t!^2} n^{2t} p^{t^2}$.

We want $n^2 p = n^{2t} p^{t^2}$ i.e. $p = n^{\frac{2-2t}{t^2-1}} = n^{-\frac{2}{t+1}}$. So, take $p = n^{-\frac{2}{t+1}}$. Then

$$\mathbb{E}(A-B) \geq \left(1 - \frac{1}{(t!)^2}\right) n^{2-\frac{2}{t+1}}.$$ □

Theorem 39. Let $g \geq 3$, $k \geq 2$. Then there is a graph G with no cycles of length $\leq g$ and $\chi(G) \geq k$.

Proof. Strategy: Fix n and p. Let G be a random graph with n vertices, each possible edge present independently with probability p. Let X be the number of short cycles in G. Recall that $\chi(G) \geq \frac{|G|}{\alpha(G)}$ where $\alpha(G)$ is the independence number of G. For cycles, by 'short', we mean of length $\leq g$.

Aim: Pick n and p such that

1. $\mathbb{P}(X > \frac{n}{2}) < \frac{1}{2}$ and
2. $\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) > \frac{1}{2}$.

Then $\mathbb{P}(X > \frac{n}{2}$ or $\alpha(G) \geq \frac{n}{2k}) < 1$ so can pick a specific G such that $X \leq \frac{n}{2}$ and $\alpha(G) < \frac{n}{2k}$. Remove a vertex from each short cycle to get H with $|H| \geq \frac{n}{2}$ and $\alpha(H) < \frac{n}{2k}$ so $\chi(H) > \frac{n/2}{n/(2k)} = k$, as required.

1. For $3 \leq i \leq g$, let X_i be the number of cycles of length i appearing as subgraphs of G. Then
$$\mathbb{E}(X_i) = \binom{n}{i}\frac{i!}{2i}p^i \leq (np)^i.$$
Now $X = \sum_{i=3}^{g} X_i$ so
$$\mathbb{E}(X) \leq \sum_{i=3}^{g}(np)^i < g(np)^g \text{ as long as } np \geq 1. \tag{$*$}$$
By Markov,
$$\mathbb{P}\left(X > \frac{n}{2}\right) \leq \frac{\mathbb{E}(X)}{n/2} < 2gn^{g-1}p^g \leq \frac{1}{2}$$
if $p \leq (\frac{1}{4g})^{\frac{1}{g}} n^{\frac{1}{g}-1}$. Take $p = (\frac{1}{4g})^{\frac{1}{g}} n^{\frac{1}{g}-1}$. Then $np = \left(\frac{1}{4g}\right)^{\frac{1}{g}} n^{\frac{1}{g}} \geq 1$ if n sufficiently large, satisfying the condition of ($*$).

2.
$$\begin{aligned}\mathbb{P}\left(\alpha(G) \geq \frac{n}{2k}\right) &\leq \binom{n}{\frac{n}{2k}}(1-p)^{\binom{n/2k}{2}} \\ &\leq n^{\frac{n}{2k}} e^{-p\frac{n^2}{16k^2}} \\ &= \exp\left\{\frac{2}{2k}\log n - \frac{n^2}{16k^2}\left(\frac{1}{4g}\right)^{\frac{1}{g}} n^{\frac{1}{g}-1}\right\} \to 0 \text{ as } n \to \infty.\end{aligned}$$
So if n is sufficiently large, $\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) < \frac{1}{2}$. □

5.3 The Structure of Random Graphs

Proposition 40. $p = \frac{1}{n}$ is a sharp threshold for $G \in \mathcal{G}(n,p)$ to contain a $\triangle$, in the sense that:

- if $p = o(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has no $\triangle$, whereas
- if $p = \omega(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has a $\triangle$.

Proof. Let $G \in \mathcal{G}(n,p)$ and let X be the number of $\triangle$s in G. Let $\mu = \mathbb{E}X$ and $\sigma^2 = \mathrm{Var}(X)$. Then $\mu = \binom{n}{3}p^3 \sim \frac{1}{6}(np)^3$. Also,

$$\sigma^2 = \binom{n}{3}(p^3 - p^6) + \binom{n}{3} \cdot 3 \cdot (n-3)(p^5 - p^6) \le n^3p^3 + n^4p^5.$$

Suppose first $p = \wr(\frac{1}{n})$, i.e. $np \to 0$. Then by Markov,

$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \ge 1) \ge 1 - \mu \to 1 \text{ as } n \to \infty.$$

Suppose instead $p = \omega(\frac{1}{n})$ so $np \to \infty$. Then with Chebyshev,

$$\frac{\sigma^2}{\mu^2} \le \frac{1}{\mu^2}(n^3p^3 + n^4p^5) \sim \frac{36}{n^6p^6}(n^3p^3 + n^4p^5) = \frac{36}{(np)^3} + \frac{36}{n \cdot np} \to 0 \text{ as } n \to \infty. \qquad \square$$

Theorem 41. There exists a function $d : \mathbb{N} \to \mathbb{N}$ such that a.e. $G \in \mathcal{G}(n,p)$ has $\omega(G) \in \{d-1, d, d+1\}$ (where $d = d(n)$).

Proof sketch. **(Currently missing).** $\square$

Corollary 42. Almost every $G \in \mathcal{G}(n,p)$ has

$$\chi(G) \ge (1 + o(1))\frac{n \log \frac{1}{q}}{2 \log n}$$

where $q = 1 - p$.

Proof. Let $G \in \mathcal{G}(n,p)$. Then $\overline{G} \in \mathcal{G}(n,q)$. So by Theorem 41, with probability tending to 1 as $n \to \infty$ we have

$$\begin{aligned} \omega(\overline{G}) &\sim \frac{2\log n}{\log \frac{1}{q}} \\ \implies \alpha(G) &\sim \frac{2 \log n}{\log \frac{1}{q}} \\ \implies \chi(G) &\ge \frac{|G|}{\alpha(G)} \sim \frac{n \log \frac{1}{q}}{2 \log n}. \end{aligned} \qquad \square$$

6 Algebraic Methods

6.1 The Chromatic Polynomial

Theorem 43 (Cut-fuse relation)**.** Let G be a graph, $e \in E(G)$, $k \geq 1$. Then $f_G(k) = f_{G-e}(k) - f_{G/e}(k)$.

Proof. Let $e = uv$. Let c be a k-colouring of $G - e$. If $c(u) \neq c(v)$ then c is a k-colouring of G and every k-colouring of G arises uniquely like this. If $c(u) = c(v)$ then c yields a k-colouring of G/e and every k-colouring of G/e arises uniquely like this.

$$f_{G-e}(k) = f_G(k) + f_{G/e}(k).$$ □

Corollary 44. Let G be a graph. Then f_G is a polynomial.

Proof. Induction on $e(G)$. For $e(G) = 0$, $f_G(k) = k^{|G|}$. For $e(G) > 0$, pick $e \in E(G)$. Then f_{G-e}, $f_{G/e}$ are polynomials by the induction hypothesis and hence $f_G = f_{G-e} - f_{G/e}$ is a polynomial. □

Corollary 45. If $|G| = n$, $e(G) = m$ then

$$f_G(X) = X^n - mX^{n-1} + \dots$$

Proof. Induction on $e(G)$. If $e(G) = 0$, then $f(X) = X^n$, as required. For $e(G) > 0$, pick $e \in E(G)$. Then

$$\begin{aligned} f_G(X) &= f_{G-e}(X) - f_{G/e}(X) = (X^r(m-1)X^{n-1} + \dots) - (X^{n-1} + \dots) \\ &= X^n - mX^{n-1} + \dots \end{aligned}$$ □

6.2 Eigenvalues

Theorem 46. Let G be a graph, $\Delta(G) = \Delta$, λ an eigenvalue of G. Then $|\lambda| \leq \Delta$. Moreover if G is connected then Δ is an eigenvalue $\iff$ G is Δ-regular; in this case Δ has multiplicity 1 and eigenvector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Proof. Let A be the adjacency matrix and x an eigenvector with eigenvalue λ. Let i be such that $|x_i|$ is maximal. Without loss of generality, $x_i = 1$ and $\forall j, |x_j| \leq 1$. Then

$$\begin{aligned} |\lambda| = |\lambda x_i| = |(Ax)_i| = \left| \sum_{j=1}^n A_{ij} x_j \right| &= \left| \sum_{j \in \Gamma(i)} x_j \right| \\ \leq \sum_{j \in \Gamma(i)} |x_j| \leq d(i) \leq \Delta. \end{aligned}$$

Assume now G is connected. ($\Leftarrow$). If G is Δ-regular then clearly $(1, \dots, 1)^T$ is an eigenvector with eigenvalue Δ.

($\Rightarrow$). Suppose Δ is an eigenvalue. Then taking $\lambda = \Delta$ in previous:

$$\Delta = (Ax)_i = \sum_{j \in \Gamma(i)} x_j.$$

Hence $d(i) = \Delta$ and $\forall j \in \Gamma(i), x_j = 1$. Repeat: $\forall j \in \Gamma(i)$ we have $d(j) = \Delta$ and $\forall k \in \Gamma(j)$, $x_k = 1$. Continuing, as G connected, $\forall k, d(k) = \Delta$ and $x_k = 1$. □

6.3 Strongly Regular Graphs

Theorem 47 (Rationality condition)**.** Let G be (k, a, b)-strongly regular. Then

$$\frac{1}{2}\left\{(n-1) \pm \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\} \in \mathbb{Z}.$$

Proof. Let A be the adjacency matrix of G. Let $|G| = n$. By Theorem 46, k is an eigenvalue of multiplicity 1 with eigenvector $(1, 1, \ldots, 1)^T = v$.

What about other eigenvalues? Let $\lambda \neq k$ be an eigenvalue with eigenvector x. Now

$$(A^2)_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i \sim j \\ b & \text{if } i \nsim j \end{cases}$$

Thus $A^2 = kI + aA + b(J - I - A)$ where J is the matrix with a 1 in every place. Applying this to x, and noting that $x \perp v$, giving $Jx = 0$, we get $\lambda^2 x = kx + a\lambda x - bx - b\lambda x$. Also $x \neq 0$, so $\lambda^2 + (b-a)\lambda + (b-k) = 0$.

So the remaining eigenvalues are

$$\lambda = \frac{(a-b) + \sqrt{(b-a)^2 - 4(b-k)}}{2} \text{ and } \mu = \frac{(a-b) - \sqrt{(b-a)^2 - 4(b-k)}}{2}$$

with multiplicities r, s say, respectively.

Now A is diagonalizable so

$$r + s + 1 = n \tag{1}$$

and $\operatorname{Tr} A = 0$ so

$$\lambda r + \mu s + k = 0 \tag{2}$$

Now take $\lambda \times (1) - (2)$: $(\lambda - \mu)s = \lambda(n-1) + k$.

We have $\lambda - \mu = \sqrt{(b-a)^2 - 4(b-k)}$ so

$$s = \frac{1}{2}\left\{(n-1) + \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\}$$

and so

$$r = \frac{1}{2}\left\{(n-1) - \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right\}. \qquad \square$$