Part III – Model Theory (Ongoing course, rough)

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0 Introduction

Lecture 1 Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: 'how powerful is our description of these objects to pin them down'? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

1 Languages and structures

Definition 1.1 (Language)**.** A **language** L consists of

(i) a set $\mathscr{F}$ of function symbols, and for each $f \in \mathscr{F}$ a positive integer m_f the **arity** of f.

(ii) a set $\mathscr{R}$ of relation symbols, and for each $R \in \mathscr{R}$, a positive integer m_R.

(iii) a set $\mathscr{C}$ of constant symbols.

Note: each of $\mathscr{F}, \mathscr{R}$ and $\mathscr{C}$ can be empty.

Example. Take $L = \{\{\cdot, ^{-1}\}, \{1\}\}$, for $\cdot$ a binary function and $^{-1}$ an unary function, 1 a constant. This is the language of groups, call it L_{gp}. Also, $L_{lo} = \{<\}$ a single binary relation, for linear orders.

Definition 1.2 (L-structure)**.** Given a language L, say, an **L-structure** consists of

(i) a set M, the **domain**

(ii) for each $f \in \mathscr{F}$, a function $f^{\mathcal{M}} : M^{m_f} \to M$.

(iii) for each $R \in \mathscr{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$.

(iv) for each $c \in \mathscr{C}$, an element $c^{\mathcal{M}} \in M$.

f^M, R^M, c^M are the **interpretations** of f, R, c respectively.

Remark 1.3. We often fail to distinguish between the symbols in L and their interpretations in a structure, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathscr{F}, \mathscr{R}, \mathscr{C} \rangle$.

Example 1.4.

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1 \rangle$ is an L_{gp}-structure.

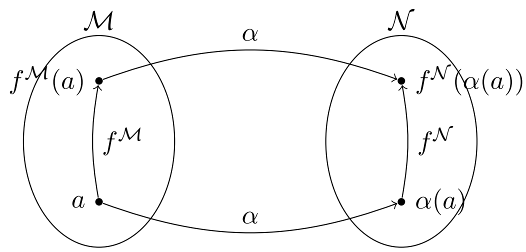
(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp}-structure.

(c) $\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo}-structure.

Definition 1.5 (Embedding)**.** Let L be a language, let $\mathcal{M}, \mathcal{N}$ be L-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is a one-to-one mapping $\alpha : M \to N$ such that

(i) for all $f \in \mathscr{F}$, and $a_1, \ldots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\alpha(a_1), \ldots, \alpha(a_{n_f}))$$



(ii) for all $R \in \mathscr{R}$, and $a_1, \ldots, a_{n_R} \in M$

$$(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \ldots, \alpha(a_{n_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathscr{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of $\mathcal{M}$ into $\mathcal{N}$ is a surjective embedding (onto).

Exercise 1.6. Let G_1, G_2 be groups, regarded as L_{gp}-structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an isomorphism $\alpha : G_1 \to G_2$ in the sense of Definition 1.5.

2 Review: Terms, formulae and their interpretations

In addition to the symbols of L, we also have

(i) infinitely many variables $\{x_i\}_{i\in I}$

(ii) logical connectives $\wedge, \neg$ (also expresses $\vee$, $\implies$, $\iff$)

(iii) quantifier $\exists$ (also expresses $\forall$)

(iv) (,)

(v) equality symbol =

Definition 2.1 (L-terms)**.** ***L*-terms** are defined recursively as follows:

– any variable x_i is a term

– any constant symbol is a term

– for any $f \in \mathscr{F}$, $f(t_1, \dots, t_{m_f})$ for any terms $t_1, \dots, t_{m_f}$ is a term

– nothing else is a term

Notation: we write $t(x_1, \dots, x_m)$ to mean that the variables appearing in t are among $x_1, \dots, x_m$.

Lecture 2 **Example.** Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1\rangle$. Then $\cdot(\cdot(x_1, x_2), x_3)$ is a term, usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a term, written $(1 \cdot x)^{-1}$

Definition 2.2. If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, \dots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \to M$ defined as follows:

(i) If $t = x_i$, $t^{\mathcal{M}}[a_1, \dots, a_k] = a_i$

(ii) If $t = c$, $t^{\mathcal{M}}[a_1, \dots, a_k] = c^{\mathcal{M}}$.

(iii) If $t = f(t(x_1, \dots, x_k), \dots, t_{m_f}(x_1, \dots, x_k))$,

$$t^{\mathcal{M}}(a_1, \dots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_k), \dots, t_{m_f}^{\mathcal{M}}(a_1, \dots, a_k))$$

Notice in L_{gp}, the term $x_2 \cdot x_3$ can be described as $t_1(x_1, x_2, x_3)$ or $t_2(x_1, x_2, x_3, x_4)$, or infinitely many other ways. Then t_1 is assigned to $t_1^{\mathcal{M}} : M^3 \to M$, with $(a_1, a_2, a_3) \mapsto (a_2, a_3)$, and t_2 is assigned to $t_2^{\mathcal{M}} : M^4 \to M$, with $(a_1, a_2, a_3, a_4) \mapsto a_2 \cdot a_3$.

Fact 2.3. Let $\mathcal{M}, \mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$ we have

$$\alpha(t^{\mathcal{M}}(a_1, \dots, a_k)) = t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k))$$

Proof. By induction on the complexity of t. Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_k)$. Then

(i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, and $t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds.

(ii) if $t = c$ a constant, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, and $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$, as required.

(iii) if $t = f(t_1(\bar{x}), \ldots, t_{m_f}(\bar{x}))$, then

$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \ldots, \alpha(t_{m_f}^{\mathcal{M}}(\bar{a})))$$

since α is an embedding. $t_1(\bar{x}), \ldots, t_{m_f}(\bar{x})$ have lower complexity than t, so inductive hypothesis applies.

□

Example 2.4. Exercise: conclude the proof of Fact 2.3.

Definition 2.5 (Atomic formula)**.** The set of **atomic formulas** of L is defined as follows

(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula

(ii) if R is a relation symbol and $t_1, \ldots, t_{m_R}$ are terms, then $R(t_1, \ldots, t_{m_R})$ is an atomic formula

(iii) nothing else is an atomic formula.

Definition 2.6 (Formula)**.** The set of L**-formulas** is defined as follows

(i) any atomic formula is an L-formula

(ii) if ϕ is an L-formula, then so is $\neg\phi$

(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$

(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i \phi$ is an L-formula

(v) nothing else is an L-formula

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$ and $x_1 \cdot x_2 = 1$ are atomic formulas, and $\exists x_1 (x_1 \cdot x_2) = 1$ is an L_{gp}-formula.

A variable occurs freely in a formula if it does not occur within the scope of a quantifier $\exists$ (the variable is **free**). Otherwise the variable is **bound**. For instance, in $\exists x_1 (x_1 \cdot x_2) = 1$, x_1 is bound and x_2 is free.

Important convention: no variable occurs both freely and as a bound variable in the same formula.

A **sentence** is a formula with no free variables. $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp}-sentence.

Notation: $\phi(x_1, \ldots, x_k)$ means that the free variables in ϕ are among $x_1, \ldots, x_k$.

Definition 2.7 ($\vDash$)**.** Let $\phi(x_1, \ldots, x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = (a_1, \ldots, a_k)$ be elements of $\mathcal{M}$. We define $\mathcal{M} \vDash \phi(\bar{a})$ as follows.

(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

(ii) if ϕ is $R(t_1, \ldots, t_{m_k})$ then $\mathcal{M} \vDash \phi(\bar{a})$ iff

$$(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_k}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}.$$

(iii) if ϕ is $\psi \wedge \chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$.

(iv) if $\phi = \neg\psi$ then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$. (this is well-defined since $\psi(\bar{a})$ is shorter than $\phi(\bar{a})$)

(v) if ϕ is $\exists x_j : \chi(x_1, \dots, x_k, x_j)$ (where $x_j \neq x_i$ for $i = 1, \dots, k$). Then $\mathcal{M} \vDash \phi(\bar{a})$ iff there is $b \in \mathcal{M}$ such that $\mathcal{M} \vDash \chi(a_1, \dots, a_k, b)$.

Example. For $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1 \rangle$, $\phi(x_1) = \exists x_2 (x_2 \cdot x_2) = x_1$ then $\mathcal{R} \vDash \phi(1)$ but $\mathcal{R} \nvDash \phi(-1)$.

Notation 2.8 (Useful abbreviations)**.** We write

– $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$

– $\phi \to \psi$ for $\neg\phi \vee \psi$

– $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$

– $\forall x_i \ \phi$ for $\neg\exists x_i \ (\neg\phi)$

Proposition 2.9. Let $\mathcal{M}, \mathcal{N}$ be L-structures, let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. Let $\phi(\bar{x})$ be atomic and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a})).$$

Question: If ϕ is an L-formula, not necessarily atomic, does Proposition 2.9 hold?

Lecture 3

Proof of Proposition 2.9. Cases:

(i) $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. (Exercise: complete this case, using Fact 2.3)

(ii) $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \dots, t_{m_R}(\bar{x}))$. Then $\mathcal{M} \vDash R(t_1(\bar{a}), \dots, t_{m_R})$ if and only if... (Exercise: complete this case)

□

Exercise 2.10. Show that Proposition 2.9 holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

Example 2.11. Do embeddings preserve *all* formulas? No. Take $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ an L_{lo}-structure Then $\alpha : \mathbb{Z} \to \mathbb{Q}$ (inclusion) is an embedding, but

$$\phi(x_1, x_2) = \exists x_3 \, (x_1 < x_3 \wedge x_3 < x_2). \tag{1}$$

$$\mathcal{Q} \vDash \phi(1, 2) \text{ but } \mathcal{Z} \nvDash \phi(1, 2). \tag{2}$$

Fact 2.12. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{M} \vDash \phi(\alpha(\bar{a})).$$

Proof. Exercise. □

3 Theories and elementarity

Throughout, L is a language, $\mathcal{M}, \mathcal{N}$ are L-structures.

Definition 3.1 (L-theory)**.** An **L-theory** T is a set of L-sentences. $\mathcal{M}$ is a **model** of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$. The class of all the models of T is written $\mathrm{Mod}(T)$. The theory of $\mathcal{M}$ is the set

$$\mathrm{Th}(\mathcal{M}) = \{\, \sigma \mid \sigma \text{ is an } L\text{-structure and } \mathcal{M} \vDash \sigma \,\}.$$

Example 3.2. Let T_{gp} be the set of L_{gp}-sentences.

(i) $\forall x_1 x_2 x_3 \, (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$

(ii) $\forall x_1 \, (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1 \, (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly for a group G, $G \vDash T_{gp}$. For a specific G, clearly $\mathrm{Th}(G)$ is larger than T_{gp}!

Definition 3.3. —hyperpage
We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as L_{lo}-structures.

Definition 3.4.

(i) an embedding $\beta : \mathcal{M} \to \mathcal{N}$ is **elementary** if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a}))$$

(ii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is said to be a **substructure** of $\mathcal{N}$.

(iii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is said to be an **elementary substructure** of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example 3.5. Consider $\mathcal{M} = [0,1] \subseteq \mathbb{R}$, an L_{lo}-structure, where $<$ is the usual order, and $\mathcal{N} = [0,2] \subseteq \mathbb{R}$ in the same way. Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo}-structures. Is $\mathcal{M} \equiv \mathcal{N}$? Yes they are isomorphic!

Is $\mathcal{M} \subseteq \mathcal{N}$? Yes (the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$.) But $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if $\phi(x) = \exists y(x < y)$, then

$$\mathcal{N} \vDash \phi(1) \quad \text{and} \quad \mathcal{M} \nvDash \phi(1).$$

Definition 3.6. Let $\mathcal{M}$ an L-structure, $A \subseteq M$, then

$$L(A) \coloneqq L \cup \{\, c_a : a \in A \,\}$$

for c_a each constant symbols. An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way ($c_a^{\mathcal{M}} = a$). The elements of A are called **parameters**. If $\mathcal{M}, \mathcal{N}$ are L-structures and $A \subseteq M \cap N$, then $\mathcal{M} \equiv_A \mathcal{N}$ when $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$ sentences.

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