Part IV – Connections between Model Theory and Combinatorics (Ongoing course)

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Contents

1 Introduction to Stability

1.1 History

Shelah was interested in $I_T(\kappa)$, the number of models of T of size κ up to isomorphism. Morley showed:

Theorem. Let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.

Example.

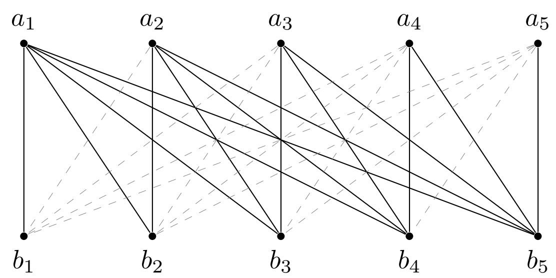
- theory of vector spaces over a fixed field
- algebraically closed fields

1.2 The order property

Definition. Let T be a theory with $\mathcal{M} \vDash T$ and $k \geq 1$ an integer. Then a formula $\varphi(x, y)$ is said to have the k-**order property** if there are sequences $(a_i)_{i=1}^k$, $(b_i)_{i=1}^k$ such that $\mathcal{M} \vDash \varphi(a_i, b_j)$ iff $i \leq j$. A formula $\varphi(x, y)$ is said to be k-**stable** if it does not have the k-order property.

Example.

- Take the theory of graphs, $G = \langle V, E\rangle$. The formula $E(x, y)$ is k-stable if G does not contain a **halfgraph** of height k as an *induced bipartite subgraph* (i.e. we don't care about the edges between the a_i, similarly for the b_j).



halfgraph of height 5

- Theory of abelian groups, $\langle G, +, -, 0, A\rangle$ where A is a unary predicate associated with a subset of G, then the formula $\varphi(x, y) =$ '$x+y \in A$' $= A(+(x, y))$ is k-stable if G does not contain $(a_i)_{i=1}^k, (b_j)_{j=1}^k$ such that $a_i + b_j \in A$ iff $i \leq j$. In this case, we say the subset A is k-stable.

Lemma. Let G be an abelian group. If $H \leq G$, then H is 2-stable.

Proof. Suppose a_1, a_2, b_1, b_2 such that $a_i + b_j \in H$ for $1 \leq i \leq j \leq 2$. Then

$$\underbrace{(a_1 + b_1)}_{\in H} - \underbrace{(a_1 + b_2)}_{\in H} + \underbrace{(a_2 + b_2)}_{\in H} = a_2 + b_1$$

so $a_2 + b_1 \in H$. $\square$

Lemma. Let G be an abelian group, $H \leq G$ and U a union of k cosets of H. Then U is $(k + 1)$-stable.

Proof. Suppose we had $a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k+1} \in G$ witnessing the $(k+1)$-order property. Then by pigeonhole $\exists 1 \leq i < j \leq k+1$ such that $a_i + b_i, a_i + b_j$ lie in the same coset of H, whence $b_i - b_j \in H$ so

$$a_j + b_i = (a_j + b_j) + \underbrace{(b_i - b_j)}_{\in H} \in U.$$

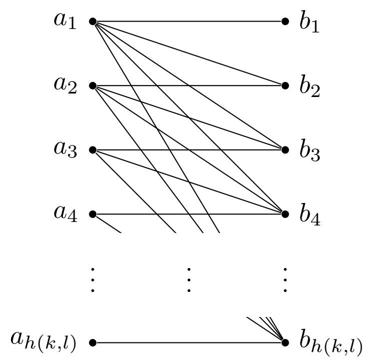
$\square$

Exercise. Let $A \subseteq G$ be a Sidon set, i.e. it contains no non-trivial solutions to $x + y = z + w$. Show that A is 3-stable. Are all 3-stable sets Sidon?

Exercise. Show that if $A \subseteq G$ is k-stable, then so is $A + g$ for any $g \in G$. Moreover, A^c is $(k+1)$-stable.

Lemma. Suppose $A_0, A_1 \subseteq G$ are l-stable and k-stable, respectively. Then $A_0 \cup A_1$ is $h(k,l)$-stable, where $h(k,l) = (k+l)2^{k+l}$.

Proof. Suppose not. Then $\exists a_1, \ldots, a_{h(k,l)}, b_1, \ldots, b_{h(k,l)}$ such that $a_i + b_j \in A_0 \cup A_1$ iff $i \leq j$.



Since $a_1 + b_j \in A_0 \cup A_1$ for every $1 \leq j \leq h(k,l)$, there must exist $i_1 \in \{0,1\}$ and $D_1 = \{j \mid a_1 + b_j \in A_{i_1}\}$ with $|D_1| \leq h(k,l)/2$. Label D_1 as $j_1 < j_2 < \cdots < j_{|D_1|}$ and define new sequences

$$\begin{aligned} a'_1, \ldots, a'_{|D_1|} &= a_1, a_{j_2}, \ldots, a_{j_{|D_1|}} \\ b'_1, \ldots, b'_{|D_1|} &= b_{j_1}, b_{j_2}, \ldots, b_{j_{|D_1|}}. \end{aligned}$$

By pigeonhole, $\exists i_2 \in \{0,1\}$ and $D_2 = \{j \mid a'_2 + b'_j \in A_{i_2}\}$. Label D_2 as $s_1 < s_2 < \cdots < s_{|D_2|}$ and define new sequences

$$\begin{aligned} a^2_1, \ldots, a^2_{|D_2|} &= a'_1, a'_2, a'_{s_3}, \ldots, a'_{s_{|D_2|}} \\ b^2_1, \ldots, b^2_{|D_2|} &= b'_{s_1}, b'_{s_2}, b'_{s_3}, \ldots, b'_{s_{|D_2|}} \end{aligned}$$

After $k+1$ steps, we will have sequences

$$\begin{aligned} a^{k+l}_1, \ldots, a^{k+l}_t \\ b^{k+l}_1, \ldots, b^{k+l}_t \end{aligned}$$

with $t \geq \frac{h(k,l)}{2^{k+l}} = k + l$ such that for every $1 \leq j < s \leq t$, $a^{k+1}_s + b^{k+l}_j \notin A_0 \cup A_1$ and for every $1 \leq s \leq j \leq t$, $a^{k+l}_s + b^{k+l}_j \in A_{i_s}$.

By pigeonhole again, either $|\{s \mid i_s = 0\}| \geq l$ or $|\{s \mid i_s = 1\}| \geq k$ contradicting the fact that A_0 (A_1) was l (k)-stable. $\square$

The typical model theoretic way of working with this is

Definition. A formula $\varphi(x, y)$ is said to have the **order property** (OP) if there are sequences $(a_i)_{i<\omega}$, $(b_j)_{i<\omega}$ such that $\vDash \varphi(a_i, b_j)$ iff $i < j$. A formula is **stable** if it does not have the order property.

Exercise. Show that any Boolean combination of stable formulas is stable.

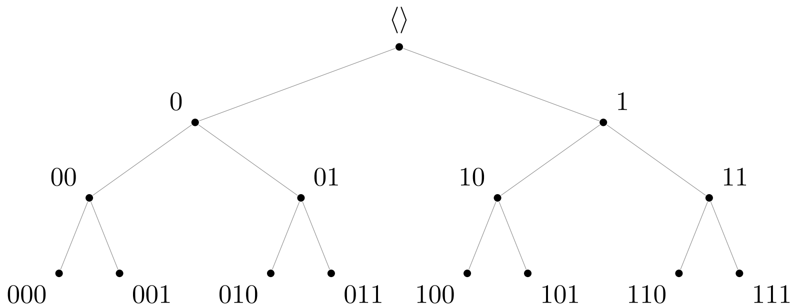
Definition. A theory has the **order property** if some formula in some model of the theory has the order property. A theory is **stable** if it does not have the order property.

1.3 Characterisation in terms of trees

A tree in the set theoretic sense is simply a partial order $(P, \lhd)$ such that $\forall p \in P$, $\{q \in P \mid q \lhd p\}$ is a well-order.

Notation.

$$2^{<n} = \bigcup_{i<n} \{0,1\}^i$$
$$\{0,1\}^0 = \langle\rangle, \text{ the empty string}$$
$$2^i = \{0,1\}^i$$



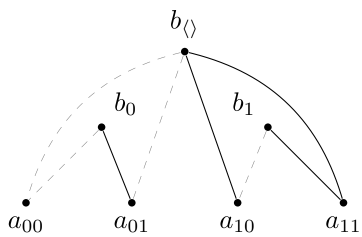
The set $2^{<n}$ has a natural tree structure: $\rho \trianglelefteq \eta$ iff $\rho = \langle\rangle$ or ρ is an initial segment of η.

If $\eta = \langle \eta_1, \ldots, \eta_i \rangle$, $j \in \{0,1\}$ then

$$\eta \wedge j := \langle \eta_1, \ldots, \eta_i, j \rangle.$$

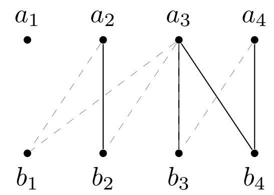
Definition. Given a graph $\Gamma = \langle V, E \rangle$, the tree bound $d(\Gamma)$ is the least integer d such that there do not exist sequences $(a_\eta)_{\eta \in 2^d}, (b_\rho)_{\rho \in 2^{<d}}$ of elements of V with the property that for each $\eta \in 2^d, \rho \in 2^{<d}$, if $\rho \lhd \eta$, then $a_\eta b_\rho \in E$ iff $\rho \wedge 1 \trianglelefteq \eta$.

Example. A graph has tree bound 2 if it does not contain the following:



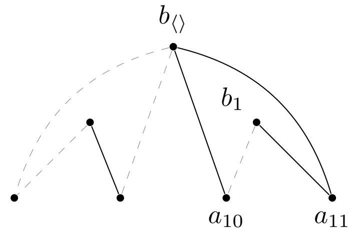
Theorem (Shelah 1978, Hodges 1996, Alon et al 2018)**.** For each k, $\exists d = d(k)$ such that if Γ is a k-stable graph, then $d(\Gamma) \leq d$. (We will get $d(k) = 2^k + 1$, Hodges gives $2^{k+2} - 2$).

Conversely, if Γ contains the 2^k-order property, then it contains a tree of height k. $k = 2$:



More generally, a formula φ admits a tree of height d if $\exists (a_\eta)_{\eta\in 2^d}, (b_\rho)_{\rho\in 2^{<d}} \in M$ such that if $\rho \lhd \eta$, then $\vDash \varphi(a_\eta, b_\rho) \leftrightarrow \rho \wedge 1 \unlhd \eta$.

Want to show: If G admits a tree of height 2^k+1, then G has the k-order property.



Unfortunately, the $k = 2$ case doesn't immediately generalise.

Exercise (Ramsey lemma)**.** Suppose p, q are positive integers and T is a tree of height $p + q - 1$ whose internal nodes are coloured red and blue. Then there is a subtree of height p all of whose internal nodes are red or a subtree of height q all of whose internal nodes are blue.

Proof. Induction on k, where the induction statement is that the result is true, with one class a subset of leaves and the other class a subset of internal nodes. Assume $I(k)$, and we want $I(k+1)$. Given a leaf y, colour an internal node red if it is connected to y by an edge in G, and blue otherwise. Use the Ramsey lemma on our tree, which has height $2(2^k + 1) - 1$, giving two cases

- Case 1: there is a leaf y such that we get a red subtree T' of height $2^k + 1$, say its root is x (include the leaves too). Let T'' be the subtree of T' rooted at the left child of x. Note that T'' has height 2^k

  Let X', Y' be the set of leaves, nodes of T'', respectively Note that no element of Y' connects to η in G. By the inductive hypothesis, we find $X_0 \subseteq X'$, $Y_0 \subseteq Y'$ that give a half graph of height k.

  Observe $y \in Y_0$. Let $X = X_0 \cup \{x\}$, $Y = Y_0 \cup \{y\}$. y is connected to everything in X_0 but x is connected to nothing in Y_0, giving the required halfgraph.

- Case 2: Suppose no leaf y produces a red subtree of height $2^k + 1$. Say x is the root of T, and say T' is the subtree rooted at the right child of x, and consider only the leaves of T'.

  Pick a leaf of T'. This, by assumption, induces a blue subtree T'' in T' of height 2^k in T'.

  By the inductive hypothesis, there are X_0, Y_0 which give a halfgraph of height k using $X = \{x\} \cup X_0$, $Y = \{y\} \cup Y_0$ (attached to the front). $\square$

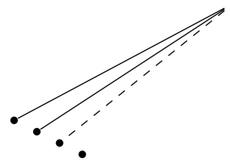
Exercise. Show that the theory of the random graph is unstable.

1.4 Characterisation of stability in terms of types

Definition. Let $\mathscr{M} \models T$, $A \subseteq M$ a set of parameters, $\varphi(x, y)$ a formula. Then a (partial) φ-type over A is a collection of formulas of the form $\varphi(x, a)$, $\neg\varphi(x, a)$ for some $a \in A$.

Definition. A complete φ-type over A is a maximal consistent partial type over A (i.e. $\forall a \in A$, either $\varphi(x, a)$ or $\neg\varphi(x, a)$ is in the type). Let $S\varphi(A)$ denote the space of complete φ-types over A.

Example. $G = \langle V, E\rangle$, $A \subseteq V$, $\varphi(x, y) = E(x, y)$. Suppose $A = \{a_1, a_2, a_3, a_4\}$ Then a possible type is $p(x) = \{E(x, a_1), E(x, a_2), \neg E(x, a_3)\}$, and the type defines the set of vertices which connect to a_1 and a_2 but not to a_3.



$p(x)$ is not complete, but if, say, $\neg E(x, a_4)$ were added, then it is a complete E-type over A.

Definition. Let $b \in M$, $A \subseteq M$. Then the type of b over A is the collecitno of all formulas with parameters in A that are satisfied by b:

$$\mathrm{tp}_\varphi(b/A) \coloneqq \{\varphi(x, a) \mid a \in A, \models \varphi(b, a)\}.$$

We have

$$S\varphi(A) \supseteq \{\mathrm{tp}_\varphi(b/A) \mid b \in M\}.$$

Exercise.

(i) Prove the Erdős-Makkai theorem: Let A be an infinite set and let $\mathcal{F} \subseteq \mathcal{P}(A)$ such that $|\mathcal{F}| > |A|$. Then there are sequences $(a_i)_{i<\omega}$, $a_i \in A$, $(F_j)_{j<\omega}$, $F_j \in \mathcal{F}$ such that either

$$\begin{aligned} \text{either } & a_i \in F_j \leftrightarrow j < i \ \forall i, j \in \omega \\ \text{or } & a_i \in F_j \leftrightarrow i < j \ \forall i, j \in \omega \end{aligned}$$

(ii) Deduce that if $|S\varphi(A)| > |A|$, then $\varphi(x, y)$ is unstable.

Theorem. Let $G = (V, E)$ be an infinite graph. Suppose $\exists$ countable $A \subseteq V$ such that

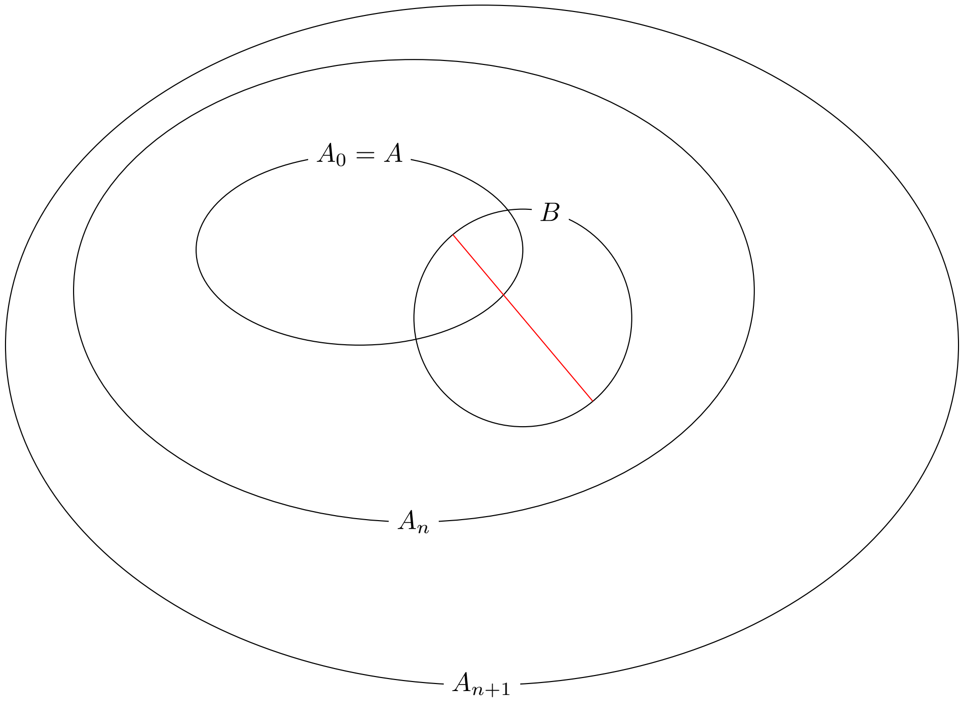
$$|\{\{a \in A \mid E(a, b)\} \mid b \in V\}| > \aleph_0.$$

Then G contains an infinite halfgraph.

Proof. Pick an uncountable sequence $(c_i)_{i<\omega_1}$, of distinct elements of V, each inducing a different partition of A. By induction on $n < \omega$, we define an increasing sequence of countable sets $A_n \subseteq V$ as follows:

- $A_0 \coloneqq A$
- having constructed A_n for some $n \geq 0$, we choose $A_{n+1} \supseteq A_n$ such that $\forall$ finite $B \subseteq A_n$, every partition of B which is induced by a vertex in V is already induced by a vertex in A_{n+1}.

Remark: A_{n+1} is countable, since there are countably many B, and we only need to add in one vertex each.



Claim: $\exists i < \omega_1$ such that $\forall n < \omega$, $\forall$ finite $B \subseteq A_n$, we can find two elements $v = v_{n,B}$ and $w = w_{n,B}$ in $A_{n+1} \setminus \{c_i\}$ such that v and w induce the same partition on B but $E(c_i, v)$ and $\neg E(c_i, w)$.

For now, assume this claim, and construct the halfgraph. Fix $c_* = c_i$ for $i < \omega$ in the claim. We will construct three vertex classes, and use a Ramsey argument to give the halfgraph. Construct sequences $(a_n)_{n<\omega}, (b_n)_{n<\omega}, (c_n)_{n<\omega}$ with $a_n, b_n, c_n \in A_{2n+2}$. Having completed step $n-1$, let

$$B_n = \bigcup_{m<n} \{a_m, b_m, c_m\}.$$

Note that $B_n \subseteq A_{2(n-1)+2} = A_{2n}$. By choice of c_*, $\exists a_n, b_n \in A_{2n+1} \setminus \{c_*\}$ such that $E(c_*, a_n), \neg E(c_*, b_n)$ and a_n and b_n induce the same partition on B_n.

To complete step n, choose $c_n \in A_{2n+2}$ such that it induces the same partition of $B_n \cup \{a_n, b_n\}$ as c^*. (Note c^* and c_n induce the same partition on B_n, but this doesn't have to be the same partition that a_n and b_n induce on B_n).

Observe:

- if $m > n$, then a_m and b_m relate to c_n in the same way: $E(a_m, c_n) \leftrightarrow E(b_m, c_n)$.
- if $m \leq n$, c_* and c_n relate to a_m and b_m in the same way, and $\forall m$, $E(a_m, c^*)$ and $\neg E(b_m, c^*)$. So $E(a_m, c_n)$ and $\neg E(b_m, c_n)$ for all $m \leq n$.

If $E(a_m, c_n)$ on an infinite subsequence, $E(b_m, c_n) \leftrightarrow n < m$. If not, $E(a_m, c_n) \leftrightarrow m \leq n$.

Finally, it remains to prove the claim. Suppose the conclusion fails. Then $\forall i < \omega_1$, $\exists n < \omega$, $\exists$ finite $B \subseteq A_n$ such that whether or not c_i connects to $v \in A_{n+1} \setminus \{c_i\}$ is entirely determined by the partition of B induced by v.

Replacing $(c_i)_{i<\omega_1}$ by a subsequence, may assume that n is constant and B is constant. Fix n, B. By construction, since B is finite, $\exists$ finite $C \subseteq A_{n+1}$ such that every partition of B is already induced by an element of C.

(1) By passing to a subsequence, may assume that all $(c_i)_{i<\omega_1}$ induce the same partition on C.

(2) Any two c_is induce distinct partitions on A, so $\exists a_* \in A$ such that $E(c_i, a_*)$ but $\neg E(c_j, a_*)$.

(3) By choice of C, there is $a_{**} \in C$ such that a_* and a_{**} induce the same partition on B.

(4) But B is such that whether or not $v \in A_{n+1} \setminus \{c_i\}$ is connected to c_i is entirely determined by the partition it induces on B.

□

2 Applications of stability

2.1 Stable Ramsey/Erdős-Hajnal

Definition. Let $A \subseteq 2^{<n}$ be closed under initial segments (CUIS) and let G be a graph on n vertices. We say G **is a type tree** on A if there is an indexing $V = \{a_\eta : \eta \in A\}$ such that $\forall \eta \in A$, the following holds.

(1) If $\eta \wedge 0$ is in A, then $\neg E(a_\eta, a_{\eta\wedge 0})$

(2) If $\eta \wedge 1$ is in A, then $E(a_\eta, a_{\eta\wedge 1})$

(3) If $\sigma, \tau \in A$ and $\eta \lhd \sigma \lhd \tau$, then

$$E(a_\eta, a_\sigma) \leftrightarrow E(a_\eta, a_\tau)$$

A type tree of A has height h if $A \subseteq 2^{<h}$ but $A \not\subseteq 2^{<h-1}$

Lemma. Every graph on n vertices is a type tree on A for some $A \subseteq 2^{<n}$ (CUIS).

Proof. Let $a_{\langle\rangle}$ be an arbitrary element of V. Let $A_0 = \{a_{\langle\rangle}\}$, $X_{\langle\rangle} = V$. Set $X_1 = N_G(a_{\langle\rangle})$, $X_0 = V \setminus (N_G(a_{\langle\rangle}) \cup \{a_{\langle\rangle}\})$ Observe X_0 and X_1 partition $V \setminus A_0$.

Suppose we've constructed $A_0, A_1, \ldots, A_m$ for $m \geq 0$, and that for each $\eta \in A_m$, we have a partition of X_η with the following properties:

1. $\{X_{\eta\wedge i} : \eta \in A_m, i = 0, 1\}$ partition $V \setminus \bigcup_{i=0}^{m} \{a_\eta : \eta \in A_i\}$
2. $\forall \eta \in A_m$, $X_{\eta\wedge 1} \subseteq N_G(a_\eta)$, $X_{\eta\wedge 0} \subseteq V \setminus (\{a_\eta\} \cup N_G(a_\eta))$.

Now for each $\eta \in A_m$ and $i \in \{0, 1\}$ let $a_{\eta\wedge i}$ be an arbitrary element of $X_{\eta\wedge i}$ be an arbitrary element of $X_{\eta\wedge i}$. If $X_{\eta\wedge i} \neq \varnothing$. Let A_{m+1} be the set of all these elements.

For each $\sigma \in A_{m+1}$, $i \in \{0, 1\}$,

$$X_{\sigma\wedge 1} = N(a_0) \cap X_0$$
$$X_{\sigma\wedge 0} = (V \setminus (\{a_0\} \cup N_G(a_d))) \cap X_\sigma$$

Check that the set $A = \bigcup_{i=1} A_i$ satisfies the properties of a type tree. □

Definition. Say $G = (V, E)$ contains a type tree of height h if $\exists V' \subseteq V$ such that the induced graph on V' is a type tree of height h on A for some $A \subseteq 2^{<h}$ (CUIS).

The tree height of G, denoted by $h(G)$ is the largest h such that G contains a type tree of height h.

Definition. Say $G = (V, E)$ contains a full binary type tree of height t if $\exists V' \subseteq V$ such that the induced graph on V' is a type tree on the set $2^{<t}$. The tree rank of G, $t(G)$ is the largest full binary type tree of height t.

Observe that $d(G) \leq k$ then $t(G) \leq k$. Observe also that if G has tree rank t, then G contains an independent set of size t.

Lemma. Let $h \geq 1$ and let $G = (V, E)$ be a finite graph of tree height h. Then G contains a clique or an independent set of size $\geq \frac{h}{2}$.

Proof. Since G has tree height h, $\exists V' \subseteq V$ such that $V' = \{ a_\eta \mid \eta \in A \}$ for some $A \subset 2^{<h}$ containing a branch B of length h. Let a_τ be the final element of B. By assumption, $|B| = h$. If $|N(a, \tau) \cap B| \geq \frac{h}{2}$, then let $I = N(a_\tau) \cap B$, otherwise let

$$I = (V \setminus N(a_\tau)) \cap B.$$

If $x, y \in I$, then by definition of I,

$$E(a_\tau, x) \leftrightarrow E(a_\tau, y).$$

But by (3), in the definition of a type tree, $E(a_\tau, x) \leftrightarrow E(x, y)$. □

Theorem (Malliaris-Shelah 2014, Malliaris-Terry 2016)**.** Let G be a graph on n vertices. Then

$$h(G) \geq \frac{1}{2} \left(\frac{n}{t(G)} \right)^{\frac{1}{t(G)+1}}$$

Corollary. If G is a graph of tree rank t, then it contains a clqiue or independent set of size $\geq \frac{1}{4} \left(\frac{n}{t}\right)^{\frac{1}{t+1}}$.

Exercise. Show that the generic $\triangle$-free graph has bounded tree rank but is unstable.

(Reference: Chernikov and Starchenko 2015)

Proof of theorem. Let $G = (V, E)$ be a graph of order n, and suppose it is a type tree on $A \subseteq 2^{\leq n}$. Let h be the height of a tallest branch $h \leq h(G)$. Let $t(a_\eta)$ denote the largest k such that there is a full binary type tree of height k below a_η. Let $t = \max\{t(a_\eta) | \eta \in A\}$. Note that $t \leq t(G)$. For all $0 \leq s \leq t$, $0 \leq l < h$, define

$$Z_l^s = \{ a_\eta \in V \mid |\eta| = l, t(a_\eta) = s \}.$$

Let $N_l^s = |Z_l^s|$, then

$$n = \sum_{s=0}^{t} \sum_{l=0}^{h} N_l^s.$$

Exercise. Show that for all $0 \leq s < t$, $0 \leq l < h$.

$$N_{l+1}^s \leq N_l^s + 2N_l^{s+1}$$

Deduce by induction that $N_{l+1}^{t-s} \leq 2^s(l+1)^s$ for $0 \leq s \leq t$, $0 \leq l < h$.

Now for all $0 \leq l < h$,

$$\sum_{s=0}^{t} N_{l+1}^s \leq \sum_{s=0}^{t}(2(l+1))^s \leq t(2n)^t$$

and

$$\begin{aligned} n = \sum_{l=0}^{h}\sum_{s=0}^{t} N_l^s &= \sum_{s=0}^{t} N_0^s + \sum_{l=0}^{h-1}\sum_{s=0}^{t} N_{l+1}^s \\ &\leq 1 + \sum_{t(2h)^t} \leq t(2h)^{t+1}. \end{aligned}$$

□

Malliaris and Terry deduced a result about the structure of prime graphs. A set of vertices X is called a module if every vertex $V \setminus X$ is connected to all of X. A graph is prime if it contains no nontrivial modules.

Chudnovsky et al 2015 showed Every sufficiently large prime graph contains one of the following as an induced subgraph.

- 1-subdivision of $K_{1,n}$
- line graph of $K_{2,n}$
- thin spider on n legs
- halfgraph of height n
- split halfgraph

2.2 Stable regularity

Theorem (Szemerédi 1975)**.** For any $\epsilon > 0$ $r \in \mathbb{N}$, $\exists M = M(\epsilon)$ such that the vertex set of any sufficiently large graph can be partitioned into $r \leq s \leq M$ sets $V_1, V_2, \ldots, V_s$ such that

- $||V_i| - |V_j|| \leq 1$ $\forall i, j \in \{1, \ldots, s\}$
- all but at most ϵs^2 pairs (V_i, V_j) are ϵ-regular in the sense that

  $$\forall X \subseteq V_i, Y \subseteq V_j, |X| \geq \epsilon|V_i|, |Y| \geq \epsilon|V_j|,$$

  then

  $$|d(X,Y) - d(V_i, V_j)| < \epsilon$$

  where $d(X,Y) = \frac{|E(X,Y)|}{|X||Y|}$.

Cannot rule out the existence of irregular pairs: Consider a halfgraph.

Theorem (Malliaris-Shelah 2014)**.** For every $\epsilon > 0$, every $k \in \mathbb{N}$, $\exists M = M(\epsilon, k)$ such that the vertex set of any sufficiently large k-stable graph G can be partitioned into sets $V_1, V_2, \ldots, V_s$ with $s \leq M(\epsilon, k)$ such that

- $||V_i| - |V_j|| \leq 1$ $\forall i, j \in [s]$
- $\forall i \neq j$, either $d(V_i, V_j) < \epsilon$ or $d(V_i, V_j) > 1 - \epsilon$.

Moreover, $M = \mathcal{O}(\epsilon^{-\mathcal{O}_k(1)})$.

We will only prove a key proposition, and omit the details of ensuring $||V_i|-|V_j|| \leq 1$.

Definition. Let $G = (V, E)$ be a finite graph. Let $\epsilon > 0$.

(i) Let $A \subseteq V$. A is ϵ-good if $\forall g \in V$, either $|N(g) \cap A| < \epsilon|A|$ or $|\neg N(g) \cap A| < \epsilon|A|$. In the first case, we write $t(g, A) = 0$, and $t(g, A) = 1$ for the second case.

(ii) Let A be ϵ-excellent, if it is ϵ-good, and $\forall\epsilon$-good $B \subseteq V$,

$$\begin{aligned} \text{either } & |\{a \in A : t(a, B) = 0\}| < \epsilon|A| \\ \text{or } & |\{a \in A : t(a, B) = 1\}| < \epsilon|A| \end{aligned}$$

Proposition. Suppose G contains no tree of height d, and $0 < \epsilon < \frac{1}{2^d}$. Then for any $A \subseteq V$ such that $|A| \geq \epsilon^{-d}$, we can find an ϵ-excellent $A' \subseteq A$ of size $|A'| \geq \epsilon^{d-1}|A|$.

Proof. Suppose not. We shall build sequences $(A_\eta)_{\eta \in 2^{\leq d}}$, $(B_\rho)_{\rho \in 2^{<d}}$ and choose representatives to form a tree of height d. Let $A_{\langle\rangle} \coloneqq A$. May assume A is not ϵ-excellent. Then there is an ϵ-good B such that $\{a \in A : t(a, B) = 0\} \geq \epsilon|A|$ and $\{a \in A : t(a, B) = 1\} \geq \epsilon|A|$. Let

$$\begin{aligned} B_{\langle\rangle} &\coloneqq B, \\ A_0 &\coloneqq \{a \in A : t(a, B) = 0\}, \\ A_1 &\coloneqq \{a \in A : t(a, B) = 1\} \end{aligned}$$

Having constructed $(A_\eta)_{\eta \in 2^{\leq n}}$, $(B_\rho)_{\rho \in 2^{<n}}$ such that $\forall \eta \in 2^{n-1}$,

- $A_{\eta \wedge i} \subseteq A_\eta$ for $i = 0, 1$
- $|A_{\eta \wedge i}| \geq \epsilon|A_\eta|$
- B_η is ϵ-good
- $a \in A_{\eta \wedge 0} \Rightarrow t(a, B_\eta) = 0$
- $a \in A_{\eta \wedge 1} \Rightarrow t(a, B_\eta) = 1$

We may assume A_η is not ϵ-excellent for any $\eta \in 2^n$.

So for each $\eta \in 2^n$, there is an ϵ-good $B_\eta \subseteq V$ such that

$$\begin{aligned} |\{a \in A_\eta : t(a, B_\eta) = 0\}| \geq \epsilon|A_\eta| \\ |\{a \in A_\eta : t(a, B_\eta) = 1\}| \geq \epsilon|A_\eta| \end{aligned}$$

and set these sets to be $A_{\eta \wedge 0}$, $A_{\eta \wedge 1}$ respectively.

For each $\eta \in 2^d$, pick an arbitrary $a_\eta \in A_\eta$, and for each $\rho \in 2^{<d}$, pick $b_\rho \in B_\rho$ such that $\forall \eta \in 2^d$ with $\rho \lhd \eta$, $t(a_\eta, b_\rho) = t(a_\eta, B_\rho)$. For fixed η and $\rho \lhd \eta$,

$$|\{b \in B_\rho : t(a_\eta, b_\rho) \neq t(a_\eta, B_\rho)\}| < \epsilon|B_\rho|$$

So,

$$\begin{aligned} |\{b \in B_\rho : t(a_\eta, b_\rho) = t(a_\eta, B_\rho) \ \forall \eta \in 2^d\}| &\geq |B_\rho| - \epsilon|B_\rho|2^d \\ &= (1 - \epsilon 2^d)|B_\rho| > 0 \end{aligned}$$

since $0 < \epsilon < \frac{1}{2^d}$. □

Exercise. $\forall \epsilon, \delta, \theta \geq 0$, $\exists N = N(\epsilon, \delta, \theta)$ such that if $|A| = n \geq N$, is ϵ-excellent then for any $n \geq m \geq \log\log n$, a randomly chosen subset of of size m will be θ-almost surely $(\epsilon + \delta)$-excellent.

2.3 Stabilisers

Hrushovski 'stable group theory and approximate groups'. Let G be a non-standard finite group definable in a sufficiently saturated structure M.

Given a complete type $p(x) \in S(M)$ over a small model M. Let

$$\operatorname{st}(p) = \{g \in G : gp \cap p \text{ is 'wide'}\}$$

Identify p with its set of realizations, and think of 'wide' as meaning positive measure.

Theorem (Hrushovski's Stabiliser Theorem)**.** Let $\operatorname{Stab}(p) \coloneqq \langle \operatorname{st}(p) \rangle$. Then

- $\operatorname{Stab}(p) \lhd G$ of bounded index
- $\operatorname{Stab}(p) = (pp^{-1})^2 = \operatorname{st}(p)^2$
- $pp^{-1}p$ is a coset of $\operatorname{Stab}(p)$
- $\operatorname{Stab}(p) \setminus \operatorname{st}(p)$ is contained in a union of non-wide definable sets.

In fact, $\operatorname{Stab}(p) = G_M^{00}$ is the connected component of G, i.e. the smallest type-definable subgroup of G of bounded index.

This has applications to approximate groups, product-free subsets of groups and arithmetic regularity

Theorem (Sanders 2012, special case)**.** Let $A \subseteq \mathbb{F}_p^n$ be a set of density $\alpha > 0$, i.e. $\frac{|A|}{|\mathbb{F}_p^n|} = \alpha$. Then there is a subspace $H \leq \mathbb{F}_p^n$ of codimension $\mathcal{O}(\log^4 \alpha^{-1})$ such that $H \subseteq 2A - 2A$.

Theorem (Sanders 2009)**.** Let $m \in \mathbb{N}$. Let G be a group, $A \subseteq G$ a finite non-empty subset with $|A^2| \leq K|A|$. Then there is a symmetric neighbour S of the identity such that A^2A^{-2} and $|S| \geq e^{-K^{\mathcal{O}(m)}}|A|$.

In particular, this used $S = \operatorname{Sym}_{1-\epsilon}(A'A)$ for some $A' \subseteq A$, where

$$\begin{aligned}\operatorname{Sym}_\eta(A) &= \{x \in G \mid 1_A * 1_{A'}(x) \geq \eta\alpha\} \\ &= \{x \in G \mid |A \cap xA| \geq \eta|A|\},\end{aligned}$$

which has a similar shape to $\operatorname{st}(A)$. (Recall $1_A * 1_{A'}(x) = \mathbb{E}_y 1_A(y) 1_{A^{-1}}(xy^{-1})$).

A key step in the proof of the stabiliser thoerem is to show that the relation R, defined by

$$uRv = \text{'}up \cap vp \text{ is wide'}$$

is a stable formula.

3 Beyond stability: NIP

3.1 The independence property

Definition. Let $k \geq 1$ be an integer. A formula $\varphi(x, y)$ is said to have the k-independence property (k-IP) if there are sequences $(a_i)_{i=1}^k$, $(b_\sigma)_{\sigma \in 2^k}$ such that

$$\models \varphi(a_i, b_\sigma) \leftrightarrow \sigma(i) = 1.$$

If $\varphi(x, y)$ does not have the k-independence property, then it is said to be k-NIP. A formula $\varphi(x, y)$ is said to have the independence property (IP) if there are sequences $(a_i)_{i=1}^\omega$, $(b_\sigma)_{\sigma \in 2^\omega}$ such that

$$\models \varphi(a_i, b_\sigma) \leftrightarrow \sigma(i) = 1$$

(and NIP as above).

Example. Let $T = \text{DLO}$ (dense linear orders). For instance, $(\mathbb{Q}, <)$ is a model. The formula $\varphi(x, y) =$ '$x < y$' has the 1-IP but not 2-IP. Suppose a_1, a_2 with $a_1 < a_2$. But there is not b_{01} such that $\neg\varphi(a_1, b_{01}) \wedge \varphi(a_2, b_{01})$.

Exercise.

(a) Let T be the theory of arithmetic. Show that $\varphi(x, y) =$ 'x divides y' has IP.

(b) Let T be the theory of the random graph. Show that $\varphi(x, y) = xEy$ has IP.

Exercise. Any Boolean combination of NIP formulas is NIP.

Lemma. If $\varphi(x, y)$ has k-IP, then it has k-OP.

Proof. Suppose $(a_i)_{i=1}^k$, $(b_\sigma)_{\sigma \in 2^k}$ that witness k-IP. For each $i = 1, 2, \ldots, k$, let $\sigma_i = (1, \ldots, 1, 0, \ldots, 0)$. Note $\sigma_i(j) = 1 \leftrightarrow i \geq j$. Then $\varphi(a_i, b_{\sigma_j}) = 1 \leftrightarrow \sigma_j(i) = 1 \leftrightarrow i \leq j$. $\square$

Definition. A theory is NIP if all of its formulas are.

Example. Any stable theory is NIP.

Exercise. Show that any o-minimal theory is NIP.

Definition. Let $\mathcal{F}$ be a family of subsets of X. A set E is shattered by $\mathcal{F}$ if any subset of E can be obtained as $E \cap F$ for some $F \in \mathcal{F}$.

The VC-dimension of $\mathcal{F}$, denoted by $\dim_{VC}(\mathcal{F})$, is the size of a largest set shattered by $\mathcal{F}$.

Example. If $X = \mathbb{R}^2$ and $\mathcal{F}$ is the set of half-planes, then $\dim_{VC}(\mathcal{F}) = 3$.

If $\varphi(x, y)$ is k-NIP, then

$$\begin{aligned} \mathcal{F}_\varphi &= \{\varphi(M, b) \mid b \in M\} \\ &= \{\mathrm{tp}_\varphi(b/M) \mid b \in M\} \end{aligned}$$

has $\dim_{VC}(\mathcal{F}_\varphi) \leq k$

3.2 Counting types

Exercise. Prove the Sauer-Shelah Lemma: Let X be a set of size n, and let $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

$$|\mathcal{F}| > \sum_{j=0}^{k-1} \binom{n}{j}.$$

Then $\mathcal{F}$ shatters a set of size k (which implies $\dim_{\mathrm{VC}}(\mathcal{F}) \geq k$).

Lecture 1 On second thoughts, let us prove this using the polynomial method

Lemma (Sauer-Shelah(-Perles))**.** Let X be a set of size n, and let $\mathcal{F} \subseteq \mathcal{P}(X)$ with $\dim_{\mathrm{VC}}(\mathcal{F}) = k$. Then

$$|\mathcal{F}| \leq \sum_{j=0}^{k} \binom{n}{j}$$

Proof. Fro each $F \in \mathcal{F}$, define a polynomial

$$p_F(x_1, \dots, x_n) = \prod_{i \in F} x_i \prod_{i \notin F} (1 - x_i).$$

Let $W_{\mathcal{F}} = \langle \{p_F : F \in \mathcal{F}\} \rangle$, a subspace of functions $f : V_{\mathcal{F}} \to \mathbb{R}$, where $V_{\mathcal{F}} = \{v_F : F \in \mathcal{F}\}$. Note $\dim(W_{\mathcal{F}}) = |\mathcal{F}|$.

Claim: $W_{\mathcal{F}}$ is contained in the linear span of all monomials of the form $x_{i_1} x_{i_2} \cdots x_{i_d}$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ with $d \leq k$. The lemma follows since there are precisely $\sum_{j=0}^{k} \binom{n}{j}$ such multilinear monomials of degree $\leq k$.

Proof of claim: Need to show that any multilinear monomial of degree $k+1$ can be written as a linear combination of multilinear monomials of degree $\leq k$. Suppose $x_{i_1} x_{i_2} \cdots x_{i_{k+1}}$, then since $\dim_K(\mathcal{F}) = k$, $\exists I' \subseteq I = \{i_1, \dots, i_{k+1}\}$ which is not shattered by $\mathcal{F}$. Define $q(x_1, \dots, x_n) = \prod_{i \in I'} x_i \prod_{i \in I \setminus I'} (x_i - 1)$. q is the zero polynomial, and can be written as $x_{i_1} x_{i_2} \cdots x_{i_{k+1}} + r(x_1, \dots, x_n)$, where r is a linear combination of multilinear monomials of degree $\leq k$. $\square$

Corollary. Let X be an infinite set, $\mathcal{F} \subseteq \mathcal{P}(X)$. For each $k \in \mathbb{N}$, define the **shatter function**

$$\pi_{\mathcal{F}}(k) = \max\{|\{F \cap Y : F \in \mathcal{F}\}| : Y \subseteq X,\ |Y| \leq k\}.$$

Then either $\pi_{\mathcal{F}}(k) = 2^k \quad \forall k$ or $\exists r$ such that $\pi_{\mathcal{F}}(k) \leq k^r \quad \forall k$.

Proof. If $\pi_{\mathcal{F}}(l) < 2^l$ for some l, then $\forall Y \subseteq X$, $|Y| = l$ such that $\{F \cap Y : F \in \mathcal{F}\} \neq \mathcal{P}(Y)$. Then

$$\pi_{\mathcal{F}}(k) \leq \sum_{j=0}^{l} \binom{k}{j} \leq (\frac{ek}{l})^l,$$

for if not,

$$\pi_{\mathcal{F}}(k) > \sum_{j=0}^{l} \binom{k}{l}$$

implies that there is $Z \subseteq X$ with $|Z| = k$,

$$|\{F \cap Z : F \in \mathcal{F}\}| > \sum_{j=0}^{l} \binom{k}{j}$$

which by Sauer-Shelah implies that

$$\mathcal{F}' = \{F \cap Z : F \in \mathcal{F}\} \subseteq \mathcal{P}(Z)$$

shatters some $W \subseteq Z \subseteq X$ so in particular $\mathcal{F}$ does. $\square$

Exercise. Given a set X, $\mathcal{F} \subseteq \mathcal{P}(X)$, define $X^* = \{F \in \mathcal{F}\}$, and $\mathcal{F}^* = \{\{F \in \mathcal{F} : F \ni x\} : x \in X\}$. Show that $\dim_K(\mathcal{F}^*) < 2^{\dim_K(\mathcal{F})+1}$.

- If $|A| = n$, i.e. A finite, then if $|S\varphi(A)|$ grows faster than a polynomial in n, then you have k-IP (which implies k-OP).
- If $|A| = \kappa$ for some infinite κ,

  φ is stable $\leftrightarrow |S\varphi(A)| \leq \kappa$.

  Can prove

  - If φ has IP, then $\forall \kappa, \exists A$ with $|A| = \kappa$ such that $|S\varphi(A)| = 2^\kappa$.
  - If φ is NIP, then $\forall \kappa, \forall A$ with $|A| = \kappa$, $|S\varphi(A)| \leq \mathrm{ded}(\kappa)$.

3.3 NIP regularity

(Fox-Pach-Suk 2017)

Theorem (Lovasz-Szegedy 2010)**.** For every $\epsilon \in (0, \frac{1}{4})$, $d \in \mathbb{N}$ $\exists M = M(\epsilon, d)$ such that the following holds. Let $G = (V, E)$ be a graph such that $\dim_{\mathrm{VC}}(E^*) \leq d$.

Then V has an equitable partition $V_1, \ldots, V_k$ with $\frac{8}{\epsilon} \leq k \leq M$ such that all but an ϵ-fraction of pairs (V_i, V_j) are ϵ-homogeneous, meaning that

$$d(V_i, V_j) < \epsilon \text{ or } \geq 1 - \epsilon.$$

Moreover, M can be taken to be $\mathcal{O}_d(\epsilon^{-(2d+1)})$.

Definition. Let $\mathcal{F}$ be a family of sets. We say $F, F' \in \mathcal{F}$ are δ-separated if $|F \triangle F'| \geq \delta$. We say $\mathcal{F}$ is δ-separated if every pair of distinct elements is δ-separated.

Lemma (Haussler Packing)**.** For all $d \in \mathbb{N}$, $\exists c = c(d)$ such that the following holds. Let $\delta \in \mathbb{R}$, X be a set of size n, $\mathcal{F} \subseteq \mathcal{P}(X)$ be a δ-separated family with $\dim_{\mathrm{VC}}(\mathcal{F}) \leq d$. Then $\mathcal{F} \leq c(\frac{n}{\delta})^d$.

Let $UD(\mathcal{F})$ be the unit distance graph of $\mathcal{F}$, i.e. $V(UD(\mathcal{F})) = \mathcal{F}$, $E(UD(\mathcal{F})) = \{\{F, F'\} : F, F' \in \mathcal{F}, |F \triangle F'| = 1\}$.

Lemma. Let X be a set of size n, $\mathcal{F} \subseteq \mathcal{P}(X)$ with $\dim_{\mathrm{VC}}(\mathcal{F}) = d$. Then

$$|E(UD(\mathcal{F}))| \leq d|\mathcal{F}|.$$

Lecture 2 *Proof.* By induction on d and n. $d = 1, n = 1$ easy. Fix $x \in X$, and let $\mathcal{F}_1 = \{F \cap (X \setminus \{x\}) : F \in \mathcal{F}\}$

$$\mathcal{F}_1 = \{F \cap (X \setminus \{x\}) : F \in \mathcal{F}\}$$
$$\mathcal{F}_2 = \{F \in \mathcal{F} : x \notin F, F \cup \{x\} \in \mathcal{F}\}$$

Now $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2|$, and $\dim_{VC}(\mathcal{F}_1) \leq d$, $\dim_{VC}(\mathcal{F}_2) \leq d - 1$ for if $A \cap X \setminus \{x\}$ is shattered by $\mathcal{F}_2$, then $A \cup \{x\}$ will be shattered by $\mathcal{F}$.

Let $E_1 = E(UD(\mathcal{F}_1))$ and $E_2 = E(UD(\mathcal{F}_2))$ then by the inductive hypothesis,

$$|E_1| \leq d|\mathcal{F}_1| \quad \text{and} \quad |E_2| \leq (d-1)|\mathcal{F}_2|.$$

Claim:

$$|E| \leq |\mathcal{F}_2| + |E_1| + |E_2|.$$

Then $|E| \leq |\mathcal{F}_2| + d|\mathcal{F}_1| + (d-1)|\mathcal{F}_2| \leq d|\mathcal{F}|$ as desired.

Now prove the claim: Consider an edge $e = (F, F')$ with $F, F' \in \mathcal{F}$ such that $F \triangle F' = \{y_e\}$. There are at most $|\mathcal{F}_2|$ many edges with $y_e = x$. If $y_e \neq x$, consider $e' = \{F \setminus \{x\}, F' \setminus \{x\}\}$. Clearly $e' \in E_1$, but two distinct edges $e_1, e_2 \in E$ could give rise to the same e' in this. This happens if and only if $e_1 = \{F, F'\}$ and $e_2 = \{F \setminus \{x\}, F' \setminus \{x\}\}$ so if and only if e' is also in E_2. □

Proof of the packing lemma. Choose a random subset $A \subseteq X$ of size $s = \lceil \frac{4dn}{\delta} \rceil$. Let $\mathcal{Q} = F \mid A = \{F \cap A : F \in \mathcal{F}\}$. For each $Q \in \mathcal{Q}$, let the weight of Q be

$$w(Q) = \{F \in \mathcal{F} : F \cap A = Q\}.$$

Note that $|\mathcal{F}| = \sum_{Q \in \mathcal{Q}} \omega(Q)$.

Let E be the set of edges of $UD(Q)$, and define the weight of $e = \{Q, Q'\}$ to be

$$w(e) = \min\{w(Q), w(Q')\}.$$

Let $W = \sum_{e \in E} w(e)$.

1. Observe that for all $A \subseteq X$, $W \leq 2d \sum_{Q \in \mathcal{Q}} w(Q) = 2d|\mathcal{F}|$.

   By the lemma, there is a vertex $Q \in \mathcal{Q}$ of degree $\leq 2d$ in $UD(\mathcal{Q})$. Removing Q, the total edge weight reduces by $\leq 2d\omega(Q)$ since Q has at most $2d$ neighbours Q', and $\min\{\omega(Q), \omega(Q')\} \leq \omega(Q)$.

2. We bound $\mathbb{E}W$ from below by considering a 2-step process: First pick an $(s-1)$-element set $A' \subseteq X$, then a random $a \in X \setminus A'$. (Then $A = A' \cup \{A\}$ is indistinguishable from a random A). Consider $A = A' \cup \{a\}$ and the family Q from before.

   Let $E_1 \subseteq E$ be the edges of $UD(Q)$ for which the symmetric difference is a, and let $W_1 = \sum_{e \in E_1} w(e)$. By symmetry, $\mathbb{E}W = s\mathbb{E}W_1$.

   But $\mathbb{E}W_1 = \mathbb{E}\mathbb{E}(W_1 \mid A)$. So now consider A fixed. Divide $\mathbb{F}$ into equivalence classes $\mathcal{F}_1, \ldots, \mathcal{F}_t$ according to their intersection with A'. Note that $t \leq \pi_{\mathcal{F}}(s-1) \leq s^d \leq (\frac{5dn}{\delta})^\delta = c'(d)(\frac{n}{d})^\delta$.

   $\mathcal{F}_i$ gives rise to an edge in E_1 of weight $\min\{b_1, b_2\} \geq \frac{b_1 b_2}{b}$. Note further that if $F_1, F_2 \in \mathcal{F}_i$ differ in $\geq \delta$ elements, then the probability that they differ in a is $\geq \frac{\delta}{n-(s-1)} \geq \frac{\delta}{n}$. Hence the expected contribution of each pair $F_1, F_2 \in \mathcal{F}_i$ to $b_1 b_2$ is $\geq \frac{\delta}{n}$ and so $\mathbb{E}b_1 b_2 \geq b(b-1)\frac{\delta}{n}$ and thus the expected contribution of each $\mathcal{F}_i$ is $\geq (b-1)\frac{\delta}{n}$. Hence

$$\mathbb{E}W_1 \geq \frac{\delta}{n} \sum_{i=1}^{t} |\mathcal{F}_i - 1 = \frac{\delta}{n}(|\mathcal{F}| - t).$$

Now,

$$\begin{aligned}2d|\mathcal{F}| \geq s \cdot \mathbb{E}W_1 &\geq s\frac{\delta}{n}(|\mathcal{F}| - t)\\ &\geq \frac{4dn}{\delta}\frac{\delta}{n}(\mathcal{F} - t)\\ &= 4d(|\mathcal{F}| - t).\end{aligned}$$

Thus

$$\begin{aligned}2d|\mathcal{F}| &\leq 4dt\\ |F| &\leq 2 \cdot c'(d)\left(\frac{n}{\delta}\right)^d.\end{aligned}$$

□

Lecture 3 Recall

Theorem (Lovasz-Szegedy, Alon-Fischer-Newman, Fox-Pach-Suk)**.** For every $\epsilon \in (0, \frac{1}{4})$, $d \in \mathbb{N}$ $\exists M = M(\epsilon, d)$ such that the following holds.

Let $G = (V, E)$ be a graph of VC-dimension d (i.e. $\dim_{\mathrm{VC}}(\{N_G(v) : v \in V\}) = d$), then V has an equitable partition $V = V_1 \cup \cdots \cup V_k$ with $\frac{8}{\epsilon} \leq k \leq M$, such that all but an ϵ-proportion of pairs (V_i, V_j) is ϵ-homogeneous, i.e. $d(V_i, V_j) < \epsilon$ or $d(V_i, V_j) > 1 - \epsilon$. Moreover, $M(\epsilon, d) = \mathcal{O}_d(\epsilon^{-(2d+1)})$.

Proof. Let $\delta = \frac{\epsilon^2 n}{12}$ and greedily construct a maximal $S \subseteq V$ such that $\{N_G(s) : s \in S\}$ is δ-separated. By Haussler's packing lemma,

$$|S| \leq c(\frac{n}{\epsilon^2 n/12})^d = c(\frac{12}{\epsilon^2})^d,$$

where c depends only on d.

Let $S = \{s_1, \ldots, s_{|S|}\}$. Define a partition $V = U_1 \cup \cdots \cup U_{|S|}$ of V such that $v \in U_i$ if i is the lesat index such that $|N(v) \triangle N(s_i)| < \delta$. Since S is maximal, such an i always exists.

Note if $u, v \in U_i$, then by the triangle inequality, $|N(u) \triangle N(v)| < 2\delta$.

Let $K = \lfloor \frac{16|S|}{\epsilon} \rfloor$. Partition each U_i into pieces of size $\frac{n}{K}$, possibly one additional part of size $< \frac{n}{K}$. Collect the remainders for all i, and partition the union into sets of size $\frac{n}{K}$ or $\frac{n}{K} - 1$. This gives $V = V_1 \cup \cdots \cup V_k$.

The fraction of pairs $\{V_i, V_j\}$ such that at least one of V_i, V_j is not fully contained in some $U_l \leq \frac{\epsilon}{8}$. (Since it is $\frac{|s|K}{\binom{K}{2}}$). Let X be the set of pairs $\{V_i, V_j\}$ which are fully contained in some U_l, U_m repsectively, but which are not ϵ-homogeneous.

Let T be the set of pairs of pairs of vertices $(e, e') \in V^{(2)} \times V^{(2)}$

- e and e' have a vertex in common,
- $e \in E$, $e' \notin E$,
- $|e \cap V_i| = |e' \cap V_i| = |e \cap V_j| = |e' \cap V_j| = 1$ whenever $\{V_i, V_j\} \in X$.

Observe that if $(e, e') \in T$, and $e \cap V_j = \{x\}$, $e' \cap V_j = \{y\}$, $x \neq y$ then $|N(x) \triangle N(y)| < 2\delta$. It follows that $|T| \leq K(\frac{n}{K})^2 \cdot 2\delta$.

Each pair $\{V_i, V_j\}$ which is not ϵ-homogeneous gives rise to at least

$$\epsilon(1 - \epsilon)\left(\frac{n}{K}\right)^3$$

of pairs $(e, e') \in T$, so $|T| \geq |X| \cdot \epsilon(1-\epsilon)(\frac{n}{K})^3$. Therefore,

$$|X| \leq \frac{|T|}{\epsilon(1-\epsilon)(\frac{n}{K})^3} \leq \frac{K(\frac{n}{K})^2 \cdot 2\delta}{\epsilon(1-\epsilon)(\frac{n}{K})^3}$$
$$\leq \frac{K2 \cdot \frac{\epsilon^2 n}{12}}{\epsilon(1-\epsilon)\frac{n}{K}} < \frac{2}{3}\epsilon\binom{K}{2}. \qquad \square$$

3.4 NIP Erdős-Hajnal

Theorem (Fox-Pach-Suk, 2017)**.** Let $d \in \mathbb{N}$. Let G be a graph on n vertices of VC-dimension at most d. Then G contains a clique or an independent set of size

$$\exp((\log n)^{1-o_d(1)})$$

Definition. The family $\mathcal{G}$ of **cographs** (or complement reducible graphs) is defined as follows:

- K_1 is in $\mathcal{G}$
- if $G, H \in \mathcal{G}$, then $G \sqcup H \in \mathcal{G}$
- if $G, H \in \mathcal{G}$, then $G \boxtimes H \in \mathcal{G}$

Exercise. Show that every cograph on n vertices contains a clique or independent set of size $\sqrt{n}$.

Lecture 4 **Theorem.** For each $d \in \mathbb{N}$, let

$$f_d(n) = \max\left\{m \;\middle|\; \begin{array}{c}\text{every graph } G \text{ on } n \text{ vertices of VC-dimension} \leq d\\ \text{contains an induced cograph of size } m\end{array}\right\}.$$

For $\delta \in (0, \frac{1}{2})$, $d \in \mathbb{N}$ $\exists c = c(d, \delta)$ such that

$$f_d(n) \geq \exp(c \log^{1-\delta} n) \quad \forall n.$$

Proof. By induction on n. Let $\epsilon = \frac{1}{32}\exp(-3c\log^{1-\delta} n)$ for some $c = c(d, \delta)$ to be determined later. Apply the ultrastrong regularity lemma to obtain a partition $V = V_1 \cup \cdots \cup V_K$ with $K = \epsilon^{-C(d)}$ parts, where $C(d) = \mathcal{O}(d)$ such that all but an ϵ-fraction of pairs (V_i, V_j) are ϵ-homogeneous.

Call a pair of vertices $\{u, v\}$ bad if at least one of the following holds:

1. u, v lie in the same part
2. $u \in V_i$, $v \in V_j$, $i \neq j$ and (V_i, V_j) is not ϵ-homogeneous
3. $u \in V_i$, $v \in V_j$, $i \neq j$ and $uv \in E(G)$ but $d(V_i, V_j) < \epsilon$
4. $u \in V_i$, $v \in V_j$, $i \neq j$ and $uv \notin E(G)$ but $d(V_i, V_j) > 1 - \epsilon$

Notice: Between homogeneous pairs V_i, V_j, can have $\leq \epsilon(\frac{n}{K})^2$ bad pairs. The number of bad pairs in G is at most

$$K \cdot \binom{\frac{n}{K}}{2} + \epsilon\binom{K}{2}\left(\frac{n}{K}\right)^2 + \binom{K}{2}\epsilon\left(\frac{n}{K}\right)^2 \leq 3\epsilon\binom{n}{2}.$$

By Turán, $\exists W \subseteq V$ of size $\geq (4\epsilon)^{-1}$ such that no pair of vertices in W is bad. The induced graph $G[W]$ has VC-dimension $\leq d$, so $G[W]$ contains a cograph of size

$t = f_d((4\epsilon)^{-1})$, induced on some subset $U_0 \subseteq W$ ($|U_0| = t$). Without loss of generality, denote the parts of the partition that correspond to U_0 by $V_1, V_2, \ldots, V_t$.

For each $u \in V_1$, let

$$\overline{d}(u) = |\{ uv : v \in V_i, i = 2, \ldots, t \text{ such that } uv \text{ is a bad pair } \}|$$

Claim: $\exists V_1' \subseteq V_1$, $|V_1'| \geq \frac{n}{2K}$ such that

$$\overline{d}(u) < 8t\epsilon \frac{n}{K} \quad \forall u \in V_1'.$$

Proof of claim: If not,

$$\frac{n}{2K} \cdot 8t\epsilon \frac{n}{K} \leq \sum_{u \in V_1'} \overline{d}(u) \leq \sum_{u \in V_1} \overline{d}(u) < \epsilon(t-1) \left(\frac{n}{K}\right)^2,$$

a contradiction.

By the inductive hypothesis, we can find $U_1 \subseteq V_1'$ such that the induced subgraph on U_1 is a cograph and $|U_1| = f_d\left(\frac{n}{2K}\right)$.

Split into cases. Case 1:

$$f_d(\frac{n}{2k}) 8t\epsilon \frac{n}{K} > \frac{n}{4tK},$$

then

$$\begin{aligned} f_d(n)^3 &> t^2 f_d\left(\frac{n}{2k}\right) \\ &> t^2 \frac{n}{4tK} / (8t\epsilon \frac{n}{K}) = \frac{1}{32\epsilon} = \exp(c(\log n)^{1-\delta}). \end{aligned}$$

Case 2:

$$f(\frac{n}{2k}) 8t\epsilon \frac{n}{K} \leq \frac{n}{4tK}$$

This means that when we delete all vertices $v \in V_2 \cup V_3 \cup \cdots \cup V_t$ which are in a bad pair with a vertex in U, we will have deleted at most $\frac{n}{4tK}$ vertices from $V_2 \cup V_3 \cup \cdots \cup V_t$.

Repeat the process on $V_2, V_3, \ldots, V_t$. At step i, find $U_i \subseteq V_i$ which induces a cograph of size

$$f\left(\frac{n}{2K} - i\frac{n}{4tK}\right) \geq f\left(\frac{n}{4K}\right)$$

and if $f(\frac{n}{4K}) \cdot 8t\epsilon \frac{n}{K} > \frac{n}{4tK}$, then we're done. Otherwise find a cograph $G[U_i]$ such that there are $\leq \frac{n}{4tK}$ vertices $\bigcup_{j>i} V_j$ that form a bad pair with vertices in U_i.

At the end, have $U_1, \ldots, U_t$ inducing a cograph of size $t \cdot f(\frac{n}{4k}) = f((4\epsilon)^{-1}) f(\frac{n}{4K})$. Calculations finish the argument. □

3.5 NIP groups

Lecture 5 Assume throughout the theory is complete, and models are saturated.

Definition. A group G is **stable/NIP** if it is definable in a stable/NIP theory, i.e. the underlying set G is a definable subset of $\mathcal{M}^n$ for some $n < \omega$, ad the group operation $\cdot : G(\mathcal{M}^n) \to G(\mathcal{M}^n)$ is a definable function.

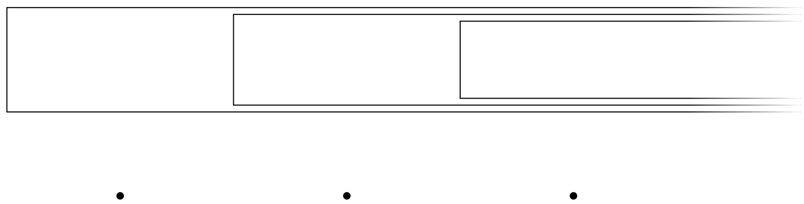
Example. Affine algebraic groups over algebraically closed fields.

Lemma (Chain lemma)**.** Let G be a stable group, and let $\varphi(x, y)$ be a formula. Then $\exists n \in \mathbb{N}$ such that every chain

$$H_1 \gneq H_2 \gneq \cdots$$

of subgroups of H which are uniformly definable by φ (in the sense that $H_i = \varphi(\mathcal{M}, a_i)$ for some parameter a_i) has length at most n.

Proof. Find $b_j \in H_j \setminus H_{j+1}$. Then $\vDash \varphi(b_j, a_i) \leftrightarrow i \leq j$, since $H_i = \varphi(\mathcal{M}, a_i)$.



$b_1 \in H_1 \setminus H_2$ $\quad$ $b_2 \in H_2 \setminus H_3$ $\quad$ $b_3 \in H_3 \setminus H_4$

□

Note we didn't use that they were subgroups, only nested subsets.

Theorem. Let G be an NIP group, and let $\varphi(x, y)$ be a formula. Then $\exists N \in \mathbb{N}$ such that whenever J is finite and $(H_j)_{j \in J}$ is a family of subgroups of G that is uniformly definable by φ (i.e. $\forall j \in J,\ H_j = \varphi(\mathcal{M}, a_j)$) then $\exists I \subseteq J$, $|I| \leq N$ such that

$$\bigcap_{j \in J} H_j = \bigcap_{i \in I} H_i.$$

Proof. Suppose not. Then $\forall N \in \mathbb{N}$, there are $(H_j)_{j=1}^{N}$ with $H_j = \varphi(\mathcal{M}, a_j)$ such that

$$\bigcap_{j=1}^{N} H_j \neq \bigcap_{j=1, j \neq i}^{N} H_j \quad \forall i = 1, 2, \ldots, N.$$

For each $i = 1, 2, \ldots, N$, let $b_i \in \bigcap_{j=1, j\neq i}^{N} H_j \setminus \bigcap_{j=1}^{N} H_j$. For $I \subseteq \{1, 2, \ldots, N\}$, let $b_I = \prod_{i \in I} b_i$. Observe that

$$\begin{aligned}
&b_I \in H_i = \varphi(\mathcal{M}, a_i) \leftrightarrow i \notin I \\
\iff &\vDash \varphi(b_I, a_i) \leftrightarrow i \notin I \\
\iff &\vDash \varphi^{\mathrm{opp}}(a_i, b_I) \leftrightarrow i \notin I \\
\iff &\vDash \neg\varphi^{\mathrm{opp}}(a_i, b_I) \leftrightarrow i \in I.
\end{aligned}$$

□

Corollary (Baldwin-Saxl)**.** Let G be a stable group, and let $\varphi(x, y)$ be a formula. Then $\exists k \in \mathbb{N}$ such that any descending chain of intersections of φ-definable subgroups has length at most k.

Proof.

$$H_1 \gneq H_1 \cap H_2 \gneq H_1 \cap H_2 \cap H_3 \gneq \cdots$$

Every element of the chain is a finite intersection of φ-definable subgroups, so by the theorem (and stable $\Rightarrow$ NIP), every element of the chain is in fact an intersection of

$N(\varphi)$-many subgroups. This means that every element of the chain is itself uniformly definable, by the formula

$$\varphi(x, \bar{y}) = \bigwedge_{i \leq N(\varphi)} \varphi(x, y_i).$$

Now by the Chain Lemma, the chain can have length at most $n(\psi) = n(N(\varphi))$. $\square$

We can deduce that in a stable group, all centralisers are definable. Let $A \subseteq G$ (not necessarily definable)

$$C_G(A) = \{g \in G : ga = ag \ \forall a \in A\}$$

is definable; in fact $\exists A_0 \subseteq A$, A_0 finite such that

$$C_G(A) = C_G(A_0).$$

Exercise. Let G be a stable group, $N \leq G$ a nilpotent subgroup. Show that $\exists$ definable nilpotent $N' \leq G$ such that $N' \supseteq N$.

Corollary. Let G be a stable group, and let $\varphi(x, y)$ be a formula. Then

$$G^0_\varphi := \bigcap \{H \leq G : H = \varphi(\mathcal{M}, a) \text{ for some } a, \ [G : H] < \infty\}$$

is a definable subgroup of G of finite index.

Furthermore, $G^0 = \bigcap_{\varphi \in L} G^0_\varphi$ is a normal group of G of bounded index, where bounded means small with respect to the level of saturation.

G^0 is the connected component of G.

Proof. Follows from Baldwin-Saxl that $\forall \varphi, \exists N$ such that any finite intersection of subgroups of the form $\varphi(\mathcal{M}, b_1)$ of index k has index $\leq k^N$.

But this is also true of infinite intersections. Indeed, suppose $\exists H = \bigcap_{i \in I} \varphi(\mathcal{M}, b_i)$ of index $> k^N$. This means $\exists a_1, \ldots, a_{k^N+1} \in G$ which are pairwise H-inequivalent, i.e. $\forall 1 \leq s < t \leq k^N + 1$, $a_s^{-1} a_t \notin H$. Then,

$$\forall 1 \leq s < t \leq k^N + 1, \ \exists i_{s,t} \in I \ \neg\varphi(a_s^{-1} a_t, b_{i_{s,t}}).$$

Let $I_0 = \{i_{s,t} : 1 \leq s < t \leq k^N + 1\}$, by Baldwin-Saxl.

$$H_0 = \bigcap_{i \in I_0} \varphi(\mathcal{M}, b_i)$$

is a subgroup of index $\leq k^N$. But $a_1, \ldots, a_{k^N+1}$ are also pairwise H_0-inequivalent.

Therefore

$$G_{\varphi,k} := \bigcap \{H \leq G, H = \varphi(\mathcal{M}, b_i), [G : H] \leq k]\}$$

has index $\leq k^N$ (and is definable: exercise).

Since G is stable, $G_\varphi = \bigcap_k G_{\varphi,k}$ is definable, of finite index. $\square$

G/G_0 with the logic topology is a compact Hausdorff group. Take $G = (\mathbb{Z}, +)$. The finite index subgroups are $n\mathbb{Z}$. Then G^0 is the set divisible by all n, which is infinite in a suitable extension. $G/G_0 \cong \lim \mathbb{Z}/n\mathbb{Z}$ a profinite group.

$G = (\mathbb{R}, +)$, then $G = G_0$.

For NIP groups, it is no longer true that G_φ is definable and finite index. Instead, we consider G^{00}, and G/G^{00} is compact Hausdorff (hard).

4 Beyond stability: NSOP

4.1 Indiscernible sequences

Lecture 6 **Definition.** Given a linear order I, a set of parameters A and a set of formulas Δ, a sequence $(a_i)_{i \in I}$ is Δ-indiscernible over A if $\forall k$, any two increasing tuples $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_k \in I$, any $b \in A$, and any $\varphi(x_1, \ldots, x_k, y) \in \Delta$, we have

$$\varphi(a_{i_1}, a_{i_2}, \ldots, a_{i_k}, b) \leftrightarrow \varphi(a_{j_1}, a_{j_2}, \ldots, a_{j_k}, b).$$

An indiscernible (over A) sequence is an infinite sequence which is Δ-indiscernible (over A) for all Δ.

If A is not mentioned, we usually take $A = \varnothing$.

Example. (i) Any constant sequence (in any model of any theory) is indiscernible over any set.

(ii) DLO, consider $(\mathbb{Q}, <)$. Any increasing sequence is indiscernible.

(iii) Let T be theory of an equivalence relation with infinitely many infinite classes. Then there are two types of indiscernibles (a sequence all in one class, or one from each class)

(iv) Let $\mathcal{M} \vDash T_{rg}$. An infinite sequence (a_i) is indiscernible if and only if $\operatorname{tp}(a_i/A) = \operatorname{tp}(a_j/A)$ for all $i < j < \omega$ and the a_is form a clique or an independent set: $\operatorname{tp}(a_0 a_1) = \operatorname{tp}(a_1 a_2)$.

(v) Any basis in a vector space is indiscernible.

(vi) $(\mathbb{Z}, x \mapsto x+1)$. Then the type of $1, 2, 3, 4, \ldots$ is the same as the type of $2, 3, 4, \ldots$, but the type of $(1, 2)$ is not the same as the type of $(1, 4)$, so we do not have an indiscernible sequence.

Note in (iii) and (v), the order of the elements didn't matter. Such indiscernible sequences are called totally indiscernible.

Exercise. Show that a theory T is stable if and only if every indiscernible sequence is in fact totally indiscernible (i.e. every permutation is also indiscernible).

Definition. The Ehrenfeucht-Mostowski type or EM-type of $(a_i)_{i \in I}$ over A is defined to be the set of $L(A)$-formulas φ such that $\mathcal{M} \vDash \varphi(a_{i_1}, \ldots, a_{i_n})$, $\forall i_1 < i_2 < \cdots < i_n \in I$ for $n < \omega$. For $\bar{a} = (a_i)_{i<\omega}$, we write $EM(\bar{a}/A)$. If $\bar{b} = (b_j)_{j \in J}$ and I is any infinite linear order then using Ramsey and compactness, we can find $\bar{a} = (a_i)_{i \in I}$ indiscernible which realises $EM(\bar{b}/A)$.

Proposition. Let $\bar{b} = (b_j)_{j \in J}$ be an arbitrary sequence in $\mathcal{M}$. Then for any small set of parameters A and any small linear order I, we can find in $\mathcal{M}$ an A-indiscernible $\bar{a} = (a_i) i \in I$ based on $\bar{b}$, meaning that

$$\forall i_0 < i_1 < \cdots < i_n \in I \text{ any finite } \Delta \subseteq L(A),$$
$$\exists j_0 < j_1 < \cdots < j_n \in J \text{ such that } \forall \varphi(x_1, \ldots, x_n) \in \Delta, \varphi(a_{i_0}, \cdots, a_{i_n}) \leftrightarrow \varphi(b_{j_0}, \ldots, b_{j_n}).$$

Proof. Let $(c_i)_{i \in I}$ be a new set of constants and let $T' \supseteq T$ in the language $L' = L \cup \{c_i : i \in I\}$ which in addition contains the following axioms:

1. $\varphi(c_{i_0}, \ldots, c_{i_n})$ for all $i_0 < \cdots < i_n \in I$ and $\varphi \in EM(\bar{b}/A)$.
2. $\psi(c_{j_0}, \ldots, c_{j_n}) \leftrightarrow \psi(c_{i_0}, \ldots, c_{i_n})$ $\forall \psi, \forall i_0 < \cdots < i_n \in I$, $\forall j_0 < \cdots < j_n$

Need to show that T' is consistent. By compactness, it is enough to show that any finite $T_0 \subseteq T'$ is consistent.

Suppose T_0 only involves formulas from some finite set Δ of $L(A)$-formulas, in at most n free variables. Using dummy variables, if necessary, we may assume

$$\Delta = \{\psi_1(\bar{x}), \dots, \psi_k(\bar{x})\} \quad |\bar{x}| = n.$$

Consider the colouring of $\mathbb{N}^{(n)}$ by 2^k colours, given by,

$$\theta(i_1, \dots, i_n) = \mathrm{tp}_\Delta(b_{i_1}, \dots, b_{i_n}).$$

By Ramsey, there is an infinite subsueqnece $\bar{b}' = (b_j : j \in J')$ with $J' \subseteq J$ such that

$$\psi(b'_{i_1}, \dots, b'_{i_n}) \leftrightarrow \psi(b'_{j_1}, \dots, b'_{j_n})$$

for all $i_1 < \ldots < i_n, j_1 < \cdots < j_n \in J'$ for all $\psi \in \Delta$. □

4.2 Stable = NIP ∩ NSOP

Definition. A formula $\varphi(x, y)$ has the k-strict order property (k-SOP) if there is $(b_i)_{i=1}^k$ such that $\forall i < k$,

$$\varphi(\mathcal{M}, b_i) \subsetneq \varphi(\mathcal{M}, b_{i+1}).$$

(or $(\exists x\ (\varphi(x, b_j) \wedge \neg\varphi(x, b_i))) \leftrightarrow i < j$).

A formula has SOP if $\exists (b_i)_{i<\omega}$, with properties as before.

A theory is NSOP if no formula has SOP.

Example. DLO has SOP: $(\mathbb{Q}, <)$, with $\varphi(x, y) = x < y$.

Exercise. T_{rg} is NSOP.

Exercise. Show that if φ has k-SOP, then it has k-OP.

Theorem (Shelah)**.** Suppose T is unstable, i.e. it has the order property. Then at least one of the following holds

1. There is a formula $\varphi(x, y)$ which has the independence property.
2. There is a formula $\varphi(x, y)$ which has the strict order property.

Proof. Suppose we have $\varphi(x, y)$ which is unstable, and suppose T is NIP. Then, there is an indiscernible sequence $(a_i b_i)_{i\in\mathbb{Q}}$ witnessing the order property of φ_i, i.e. $\varphi(a_i, b_j) \leftrightarrow i < j$.

Fact (Problem 11, sheet 2)**.** NIP formulas have finite alternation: for any indiscernible (b_i) and any x, there are at most k increasing indices $i_0 < i_1 < \cdots < i_{k-1}$ such that

$$\varphi(x, b_i) \leftrightarrow \neg\varphi(x, b_{i+1}).$$

Using this fact: since T is NIP, there is $k = k(\varphi)$ such that

$$\{\varphi^{i \bmod 2}(x, b_i) \mid i \in \mathbb{N}, i < k\}$$

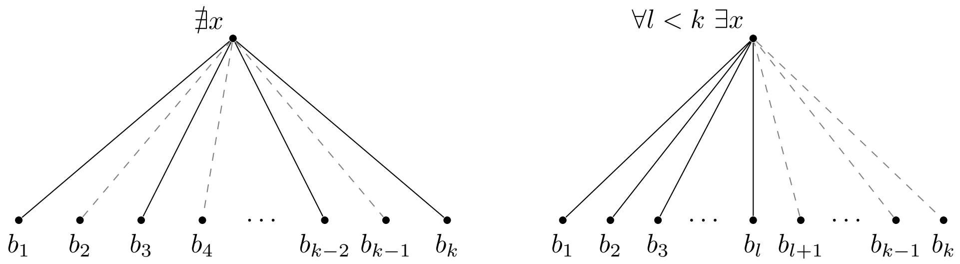
is inconsistent (writing $\varphi^0 = \neg\varphi$ and $\varphi^1 = \phi$).

On the other hand, because of instability, $\forall l < k$,

$$\{\neg\varphi(x, b_i) \mid i < l\} \cup \{\varphi(x, b_i) \mid i \geq l\}$$

is consistent.

Together, these facts say:



Our idea is to switch from the left picture to the right by replacing $\varphi(x, b_i) \wedge \neg\varphi(x, b_{i+1})$ by $\neg\varphi(x, b_i) \wedge \varphi(x, b_{i+1})$ one at a time. Along the way, we will switch from an inconsistent set of formulas to a consistent one.

Specifically, there is $\eta : [k] \to \{0, 1\}$ and $l < k$ such that

$$\{\varphi^{\eta(i)}(x, b_i) \mid i \neq l, l+1\} \cup \{\varphi(x, b_l, \neg\varphi(x, b_{l+1})\}$$

is consistent, but

$$\{\varphi^{\eta(i)}(x, b_i) \mid i \neq l, l+1\} \cup \{\neg\varphi(x, b_l), \varphi(x, b_{l+1})\}$$

is inconsistent.

Let

$$\psi_1(x) = \bigwedge_{i \neq l, l+1} \varphi^{\eta(i)}(x, b_i).$$

By indiscernibility of (b_i), $\forall i < j \in \mathbb{Q} \cap [l, l+1]$, we have

(a) $\psi_1(x) \wedge \{\varphi(x, b_i), \neg\varphi(x, b_j)\}$ is inconsistent

(b) $\psi_1(x) \wedge \{\neg\varphi(x, b_i), \varphi(x, b_j)\}$ is consistent.

Claim: $\psi(x, y) = \psi_1(x) \wedge \varphi(x, y)$ has SOP. Proof of claim: Take $(b'_i) = (b_i \mid i \in \mathbb{Q} \cap [l, l+1])$. Then on b', we have $\exists x(\neg\psi(x, b_i) \wedge \psi(x, b_j)) \leftrightarrow i < j$. Indeed, if $i < j$, then

$$\begin{aligned} &\neg\psi(x, b_i) \wedge \psi(x, b_j) \\ &= \neg(\psi_1(x) \wedge \psi(x, b_i)) \wedge (\psi_1(x) \wedge \varphi(x, b_j)) \\ &= (\neg\psi_1(x) \vee \neg\psi(x, b_i)) \wedge (\psi_1(x) \wedge \varphi(x, b_j)) \end{aligned}$$

has a solution by b), while if $j \leq i$, it doesn't by a). $\square$

Note we are no longer in the 'local' formula setting: this is a statement about theories rather than properties of formulae.

4.3 Simplicity, forking and dividing

Throughout, take T countable, complete with infinite models.

Definition. Let $k \in \mathbb{N}$. A formula $\varphi(x, y)$ is said to have the k-**tree property** (k-TP) if $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- $\forall \eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta \wedge i})\}_{i<\omega}$ is k-inconsistent
- $\forall \sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|n})\}_{n<\omega}$ is consistent.

A theory is k-NTP if no formula has k-TP. A theory is **simple** if it is k-NTP for all $k \in \mathbb{N}$.

Informally, the k-tree property means that for each node every k-family of its children is inconsistent, and every initial segment of the tree is consistent.

Example.

- Stable theories are simple (exercise)
- DLO is not simple (exercise)
- The theory of the random graph is simple
- The theory of the generic K_n^r-free r uniform hypergraph is simple for $n > r > 2$, but not for $n = 3$, $r = 2$.

Definition. Let $k \in \mathbb{N}$, A a small set. A formula $\varphi(x, b)$ is said to k-**divide** over A if there is $(a_i)_{i<\omega}$ such that $\mathrm{tp}(a_i/A) = \mathrm{tp}(b/A)$ for all $i < \omega$, and $\{\varphi(x, a_i)\}_{i<\omega}$ is k-inconsistent.

A formula **divides** over A if it k-divides over A for every $k \in \mathbb{N}$.

Equivalently, by compactness and the Standard Lemma, $\varphi(x, b)$ k-divides over A if there is A-indiscernible $(a_i)_{i<\omega}$ starting with b such that $\{\varphi(x, a_i)\}_{i<\omega}$ is k-inconsistent.

Example.

1. In DLO, the formula $\varphi(x, a) =$ '$x < a$' does not divide over $\varnothing$: given any sequence $(b_i)_{i<\omega}$, the type

$$\{x < b_i : i < \omega\}$$

   is finitely satisfiable.

   For $a < b$, the formula $\varphi(x, a, b) =$ '$a < x < b$' does divide over $\varnothing$:

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

   $\mathrm{tp}(a_i b_i/\varnothing) = \mathrm{tp}(ab/\varnothing)$, but $\{\varphi(x, a_i, b_i)\}_{i<\omega}$ is 2-inconsistent.

2. In an arbitrary theory, the formula $\varphi(x, b) =$ '$x = b$' divides over A whenever $b \notin \mathrm{acl}(A)$ (exercise).

3. In the theory of the random graph, the formula $\varphi(x, a) =$ '$x = a$' divides over $\varnothing$. Take an indiscernible sequence starting with a, $\{\varphi(x, a_i)\}_{i<\omega}$ will be 2-inconsistent.

   On the other hand, $\varphi(x, a) =$ '$x \sim a$', '$x \nsim a$', '$x \neq a$' do not divide over $\varnothing$ (the only real way to be inconsistent in the random graph is to try to be equal to two different vertices at once). Given any indiscernible sequence $(a_i)_{i<\omega}$ with $a = a_0$, $\{\varphi(x, a_i)\}_{i<\omega}$ is k-inconsistent $\forall k \in \mathbb{N}$ by the random graph axioms, for example for any $I \subseteq \omega$ and $|I| = k$,

$$\exists x\ x \sim a_i\ \forall i \in I.$$

A formula forks over a set if it implies a disjunction of formulas, each of which divide over the set. This is often useful because it satisfies certain Boolean closure properties.

For simple theories, these two notions coincide.

Lecture 16

Proposition (Tent & Ziegler, 7.24)**.** A formula $\varphi(x, y)$ has k-TP if and only if $\exists (b_i)_{i<\omega}$ such that $\{\varphi(x, b_i)\}_{i<\omega}$ is consistent but each $\varphi(x, b_i)$ k-divides over $\{b_j\}_{j<i}$.

Corollary. T is simple if and only if $\forall\varphi(x,y)\ \forall k\in\mathbb{N}$, $\nexists(b_i)_{i<\omega}$ such that $\{\varphi(x,b_i)\}_{i<\omega}$ is consistent and $\varphi(x,b_i)$ k-divides over $\{b_j\}_{j<i}$.

Corollary. If the only formulas that divide are those with finitely many solutions, then T is simple.

Proof. Suppose that T is not simple. By the Corollary, there are $\varphi(x,y)$ $k\in\mathbb{N}$, $(b_i)_{i<\omega}$ such that $\{\varphi(x,b_i)\}_{i<\omega}$ is consistent and $\varphi(x,b_i)$ k-divides over $\{b_j\}_{j<i}$. This means that for each $i<\omega$, $\exists(b_i^n)_{n<\omega}$ indiscernible over $\{b_j\}_{j<i}$ such that $\{\varphi(x,b_i^n)\}_{n<\omega}$ are k-inconsistent and $(b_i^n)_{n<\omega}$ satisfies the same $L(\{b_j\}_{j<i})$ formulas as b_i. Consider

$$\varphi(\mathcal{M},b_0)\supseteq\varphi(\mathcal{M},b_0)\cap\varphi(\mathcal{M},b_1)\supseteq\varphi(\mathcal{M},b_0)\cap\varphi(\mathcal{M},b_1)\cap\varphi(\mathcal{M},b_2)\supseteq\cdots.$$

Claim: Containment

$$\bigwedge_{j<i}\varphi(\mathcal{M},b_j)\subsetneq\bigwedge_{j<i}\varphi(\mathcal{M},b_j)$$

is proper (so long as the left hand side is non-empty). Indeed, let $\psi(y)=\varphi(\mathcal{M},b_j)\wedge\varphi(\mathcal{M},y)$. Then $\psi(b_i^n)\subsetneq\bigwedge_{j<i}\varphi(\mathcal{M},b_j)$ for some $n<\omega$ for otherwise $\varphi(\mathcal{M},b_i^n)\supseteq\bigwedge_{j<i}\varphi(\mathcal{M},b_i)$ for all $n<\omega$ contradicting the k-inconsistency of the b_i^n (if $\bigwedge_{j<i}\varphi(\mathcal{M},b_j)\neq\varnothing$).

Claim leads to contradiction with consistency of $\{\varphi(x,b_i)\}_{i<\omega}$. □

Example.

1. In the theory of the random graph, no positive Boolean combination $xRy,\neg xRz$ divides. But the theory has QE, so the only formulas which do divide are of the form $\bigwedge\bigvee x=y_i$. Each of these only have finitely many solutions, so the random graph is simple.

   Can show that $\varphi(x,\bar b)$ divides over A if and only if $\varphi(\bar x,\bar b)\rhd x_j=b_i$ for some $x_j\in\bar x$ and $b_i\in\bar b$.

2. For $2\le k<m$, let $T_{k,m}$ be the theory of the random $K_m^{(k)}$-free k-uniform hypergraph. (The extension axiom is replaced with: for any A which does not contain a $K_{m-1}^{(k)}$ and any B, there is an x etc.) This has QE, and is $\aleph_0$-categorical, and is generic in the sense that it contains any $K_m^{(k)}$-free graph.

   (a) $T_{2,3}$ (and more generally, $T_{2,n}$, the Henson graphs) is not simple. $\varphi(x,a)=$‘$x=a$’ divides in the same way as in random graphs. On the other hand, $\varphi(x,a)=xRa$ does not divide. Indiscernible sequences must be independent sets, so there are no restrictions on types. But, the formula $\varphi(x,a,b)=xRa\wedge xRb$ does divide for $a\neq b$. Construct (a_i,b_i) with a_iRb_j iff $i\neq j$ (which is triangle-free, so can be found). Now $\{\varphi(x,a_i,b_i)\}_{i<\omega}$ is 2-inconsistent.

   (b) $T_{3,4}$ is simple (Hrushovski) Let $\Gamma\vDash T_{3,4}$ be the countable model, and denote by R the ternary edge relation. We look at formulas with infinitely many solutions, and show that none of these divide over the empty set.

   Let $\bar a$ be a finite tuple from Γ, consider $\varphi(x,\bar a)$ with infinitely many solutions. We will show that $\varphi(x,\bar a)$ does not k-divide over the empty set for any k.

   Let $(\bar a_i)_{i<\omega}$ be an indiscernible with $\bar a_0=\bar a$. Need to show that $\{\varphi(x,\bar a_i\}_{i<\omega}$ is consistent.

   Fix $N\in\mathbb{N}$. For each $i\le N$, let A_i be the set of elements of $\bar a_i$. Construct a 3-uniform hypergraph Δ as follows: the vertex set of Δ is $\bigcup_{i<N}A_i\cup\{c\}$ for some new point c (not in Γ). The edges on triples of Δ are as induced by Γ:

- if $\bar{b} \subseteq \Gamma$, then $\bar{b}$ is an edge in Δ if and only if it is an edge in Γ, i.e. $R(\bar{b})$ holds.
- if $c \in \bar{b}$, $\bar{b}$ is an edge in Δ if and only if $\bar{b}\ \{c\} \subseteq A_i$ for some i and $\varphi(x, \bar{a}) \vdash R(x, \bar{b} \setminus \{c\})$
(For instance, if $\varphi(x, \bar{a}_i))$ might say $R(xa_i^1 a_i^2)$ but $\neg R(xa_i^2 a_i^3)$. Then $\bar{b} = ca_i^1 a_i^2$ will be an edge in Δ, while $ca_i^2 a_i^3$ will not).

Claim: Δ is $K_4^{(3)}$-free. Once we have this claim, we have that Δ embeds into Γ, so there is a point in Γ that behaves like c, so $\{\varphi(x, \bar{a}_i)\}_{i \leq N}$ is consistent.

Proof of claim. Suppose we have a $K_4^{(3)}$. Then c is one of the vertices. Aim to say that $B \setminus \{c\} \subseteq A_i$ for some i.

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