Part III – Topics in Ergodic Theory (Ongoing course, rough)

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Contents

Ergodic theory is all about measure preserving systems.

Definition (Measure preserving system)**.** A **measure preserving system** $(X, \mathcal{B}, \mu, T)$ with X a set, $\mathcal{B}$ a σ-algebra, μ a probability measure ($\mu(A) \geq 0\ \forall A \in \mathcal{B}$ and $\mu(X) = 1$) and T is a measure preserving transformation. Recall a measure preserving transformation $T : X \to X$ is a measurable function such that $\mu(T^{-1}(A)) = \mu(A)\ \forall A \in \mathcal{B}$.

If Y is a random element of X with distribution μ, then $T(Y)$ also has distribution μ.

Example. For example, consider a circle rotation. We have $X = \mathbb{R}/\mathbb{Z}$, $\mathcal{B}$ is the Borel sets, μ the Lebesgue measure, and $T = R_\alpha$, with $x \mapsto x + \alpha$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ is a parameter.

We also have the 'times 2 map', with the same $X, \mathcal{B}, \mu$ and $T = T_2$, $x \mapsto 2 \cdot x$.

Proof that T_2 is measure preserving. First check for intervals: Let $I = (a, b)$, then $\mu(I) = b - a$. Also, $\mu(T_2^{-1} I) = \mu\left((\frac{a}{2}, \frac{b}{2}) \cup (\frac{a}{2} + \frac{1}{2}, \frac{b}{2} + \frac{1}{2})\right) = \frac{b}{2} - \frac{a}{2} + \frac{b}{2} - \frac{a}{2} = b - a$, as required.

Now, let $U \subset \mathbb{R}/\mathbb{Z}$ be open. Then $U = I_1 \sqcup I_2 \sqcup \cdots$ is a disjoint union of intervals:

$$\begin{aligned} \mu(T^{-1} U) &= \mu\left(\bigcup T^{-1} I_j\right) \\ &= \sum \mu(T^{-1} I_j) \\ &= \sum \mu(I_j) \\ &= \mu(U). \end{aligned}$$

Let $K \subset \mathbb{R}/\mathbb{Z}$ be a compact set.

$$\mu(T^{-1} K) = 1 - \mu((T^{-1} K)^c) = 1 - \mu(T^{-1} K^c) = 1 - \mu(K^c) = \mu(K).$$

Now let $A \in \mathcal{B}$ be arbitrary. Let $\epsilon > 0$. $\exists U$ open and $\exists K$ compact such that $K \subset A \subset U$ and $\mu(U \setminus K) < \epsilon$.

$$\mu(K) = \mu(T^{-1} K) \leq \mu(T^{-1} A) \leq \mu(T^{-1} U) = \mu(U).$$

We also have $\mu(K) \leq \mu(A) \leq \mu(U)$. Since $\mu(U) - \mu(K) < \epsilon$, $|\mu(A) - \mu(T^{-1} A)| < \epsilon$. ϵ was arbitrary, so $\mu(A) = \mu(T^{-1} A)$. $\square$

The two examples generalise to the Haar measure on a topological group and to endomorphisms respectively.

In ergodic theory, we study the long term behaviour of orbits.

Definition (Orbit)**.** The orbit of $x \in X$ is the sequence

$$x, Tx, T^2 x, \ldots$$

Some questions we might ask are:

- Let $A \in \mathcal{B}$ and $x \in A$. Does the orbit of x visit A infinitely often? (Recurrence)
- What is the proportion of times n such that $T^n x \in A$?
- What is $\mu(\{\, x \in A \mid T^n x \in A \,\})$ if n is large? (Mixing property)

Example. Let $A = [0, \frac{1}{4}) \subset \mathbb{R}/\mathbb{Z}$. Then ${T_2}^n x \in A \iff$ the $n+1$th and $n+2$th 'binary digits' of x are 0.

For some $x = 0.x_1x_2x_3\dots{}_2$, $x \in A$ corresponds to x_1, x_2 both being 0 and the doubling map sends x to $T_2x = x_2x_3\dots{}_2$, giving the required property above.

For example, $x = \frac{1}{6} = 0.00101010\dots{}_2$ starts in A but never comes back to A. Also, we have $\mu(\{\, x \in A \mid T_2^n x \,\}) = \frac{1}{16}$ if $n \geq 2$.

Example (Markov shift)**.** Let $P_1, P_2, \dots, P_n$ be a probability vector. Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be the 'matrix of transition probabilities'. Assume

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} P_1 & P_2 & \dots & P_n \end{pmatrix} A = \begin{pmatrix} P_1 & P_2 & \dots & P_n \end{pmatrix}$$

Take $X = \{1, \dots, n\}^{\mathbb{Z}}$, $\mathcal{B}$ the Borel σ-algebra generated by the product topology of the discrete topology on $\{1, \dots, n\}$, $T = \sigma$ the shift map: $(\sigma x)_m = x_{m+1}$. Finally, set the measure

$$\mu(\{\, x \in X \mid x_m = i_0, x_{m+1} = i_1, \dots, x_{m+n} = i_n \,\}) = P_{i_0} a_{i_0 i_1} \cdots a_{i_{n-1} i_n}.$$

Theorem (Szemerédi)**.** Let $S \subset \mathbb{Z}$ of positive upper Banach density. That is,

$$\bar{d}(S) \coloneqq \limsup_{N,M : M-N \to \infty} \frac{1}{M-N} \left| S \cap [N, M-1] \right|$$

and $\bar{d}(S) > 0$. Then S contains arbitrarily long arithmetic progressions. That is, $\forall l, \exists a \in \mathbb{Z}, d \in \mathbb{Z}_{>0}$,

$$a, a+d, \dots, a+(l-1)d \in S.$$

Theorem (Furstenberg, multiple recurrence)**.** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Let $A \in \mathcal{B}$ such that $\mu(A) > 0$. Let $l \in \mathbb{Z}_{>0}$. Then

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n} A \cap \dots \cap T^{-(l-1)n} A) > 0.$$

Let

- $X = \{0, 1\}^{\mathbb{Z}}$
- $\mathcal{B} =$ Borel σ-algebra
- $\sigma =$ the shift map $\mathbf{x} \mapsto (x_{n+1})_n$

Let $\mathbf{x}^S \in X$ be defined by

$$\mathbf{x}_n^S = \begin{cases} 1 & n \in S \\ 0 & n \notin S. \end{cases}$$

Also let $A \in \beta$ be given by $A = \{\, x \in X \mid x_0 = 1 \,\}$. Observe then that

$$\mathbf{x}_n^S = 1 \iff n \in S \iff \sigma^n \mathbf{x}^S \in A \iff (\sigma^n \mathbf{x}^S)_0 = 1.$$

Let $\{M_m\}$ and $\{N_m\}$ be sequences s.t. $M_n - N_m \to \infty$ and

$$\bar{d}(S) = \lim_{m\to\infty} \frac{1}{M_m - N_m} |S \cap [N_m, M_m - 1]|$$

Let

$$\mu_m = \frac{1}{M_m - N_m} \sum_{n=N_m}^{M_m - 1} \delta_{\sigma^n \mathbf{x}^S}$$

where δ_x is a measure on X defined as

$$\delta_x(B) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

Let μ be the weak limit of *a* subsequence of μ_m. Note how the μ could be different dependent on subsequence choice.

Definition (Weak limit)**.** Let X be a compact metric space. Let μ_m be a sequence of Borel measures on X, and let μ be another Borel measure. Then μ_m converges weakly to μ if for any $f \in C(X)$, we have

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

Theorem. (Banach-Alaoglu, or Helly) Let X be a compact metric space. Then $\mathcal{M}(X)$, the set of Borel probability measures on X, endowed with the topology of weak convergence, is compact and metrizable. That is, there is a weakly convergent subsequence in any sequence of Borel probability measures.

Lemma. $(X, \mathcal{B}, \mu, \sigma)$ as defined above is a measure preserving system.

Proof sketch. Let $B \in \mathcal{B}$. Then

$$\begin{aligned} \mu_m(B) &= \frac{1}{M_m - N_m} \left| \{ n \in [N_m, M_m - 1] \mid \sigma^n \mathbf{x}^S \in \mathcal{B} \} \right| \\ \mu_m(\sigma^{-1} B) &= \frac{1}{M_m - N_m} \left| \{ n \in [N_m, M_m - 1] \mid \sigma^n \mathbf{x}^S \in \sigma^{-1}\mathcal{B} \} \right| \\ &= \frac{1}{M_m - N_m} \left| \{ n \in [N_m + 1, M_m] \mid \sigma^n \mathbf{x}^S \in \mathcal{B} \} \right| \end{aligned}$$

So the difference is such that

$$\left| \mu_m(B) - \mu_m(\sigma^{-1} B) \right| \leq \frac{1}{M_m - N_m} \to 0$$

It can be shown that we can pass to the limit on m and conclude that $\mu(B) = \mu(\theta^{-1} B)$. $\square$

Remark. If B is a cylinder set, i.e. $\exists L \in \mathbb{Z}_{>0}$ and $\tilde{B} \subseteq \{0,1\}^{2L+1}$ such that

$$B = \{ x \in X \mid (x_{-L}, \ldots, x_L) \in \tilde{B} \},$$

then B is both closed and open. Therefore χ_B, the characteristic function of B is continuous. Hence $\lim_{n\to\infty} \mu_m(B) = \mu(B)$, since $\mu_m(B) = \int \chi_B \, d\mu_m$ and $\mu(B) = \int \chi_B \, d\mu$.

Approximating any Borel set by such cylinder sets would help complete the proof, but we in fact can get this result on spaces where χ is not continuous on nice set of sets. So we leave full proof till a more general theorem.

Proposition. Let $S \subseteq \mathbb{Z}$, let $\mathbf{x}^S, A, (X, \mathcal{B}, \mu, \sigma)$ as defined above. Let $l \in \mathbb{Z}_{>0}$. Suppose that $\exists n \in \mathbb{Z}_{>0}$ such that

$$\mu\left(A \cap \sigma^{-n}(A) \cap \cdots \cap \sigma^{-n(l-1)}(A)\right) > 0.$$

Then S contains an arithmetic progression of length l.

Proof. Without loss of generality, we can assume $\mu = \lim \mu_m$ - if not, pass to a subsequence. Let $B = A \cap \sigma^{-n} A \cap \cdots \cap \sigma^{-n(l-1)}(A)$. Observe that B is a cylinder set. Then by the earlier remark, $\mu(B) = \lim \mu_m(B)$, hence $\exists m$ such that $\mu_m(B) > 0$.

By definition of μ_m, $\exists k \in [N_m, M_m - 1]$ such that $\sigma^k \mathbf{x}^S \in B$. Hence

$$\sigma^k \mathbf{x}^S \in A, \sigma^k \mathbf{x}^S \in \sigma^{-n}(A), \ldots, \sigma^k \mathbf{x}^S \in \sigma^{-n(l-1)}(A).$$

Thus, $k, k+n, \ldots, k+n(l-1) \in S$. $\square$

Returning to the overall proof, we note A is a cylinder set. Then $\mu_m(A) \to \mu(A)$, i.e.

$$\mu(A) = \underbrace{\lim_{m\to\infty} \frac{1}{M_m - N_m} |\{ n \in [N_m, M_m - 1] : n \in S \}|}_{\bar{d}(S)} > 0$$

where the inequality comes from satisfying the conditions of Furstenberg.

Lemma. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then $\exists u \in \mathbb{Z}_{>0}$ such that $\mu(A \cap T^{-n} A) > 0$.

Proof. Suppose $\mu(A \cap T^{-n} A) = 0$ for all $n > 0$. Then $\mu(T^{-k} A \cap T^{-n} A) = \mu(A \cap T^{-(n-k)} A) = 0$ for all $n > k \geq 0$.

Then the set $A, T^{-1}A, \ldots$ are 'almost pairwise disjoint'. Then

$$\begin{aligned}
\mu(A \cup T^{-1} A \cup \cdots \cup T^{-n} A) &= \mu(A) \\
&\quad + \mu(T^{-1}A) - \underbrace{\mu(T^{-1} A \cap A)}_{=0} \\
&\quad + \mu(T^{-2}A) - \underbrace{\mu(T^{-2} A \cap (A \cup T^{-1} A))}_{=0} \\
&\quad + \cdots \\
&\quad + \mu(T^{-n}A) - \underbrace{\mu(T^{-n} A \cap (A \cup T^{-1} A \cup \cdots \cup T^{-(n-1)} A))}_{=0} \\
&= (n+1)\mu(A),
\end{aligned}$$

a contradiction if $n + 1 > \mu(A)^{-1}$. $\square$

Theorem (Poincaré recurrence)**.** Let $(X, \mathcal{B}, \mu, T)$ be a MPS. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then a.e. $x \in A$ returns to A infinitely often. That is:

$$\mu(A \setminus \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n} A) = 0.$$

Remark. $x \in T^{-n}A \iff T^n x \in A$. $\bigcup_{n=N}^{\infty} T^{-n}A$ are the points that visit A at least once after time N.

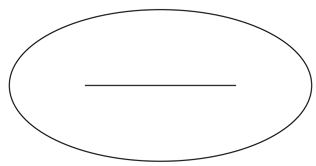
Proof. Let A_0 be the set of points in A that never return to A. We first show $\mu(A_0) = 0$. Note that $\mu(A_0 \cap T^{-n}A_0) \le \mu(A_0 \cap T^{-n}A) = \mu(\varnothing) = 0\ \forall n > 0$. By the lemma, $\mu(A_0) = 0$. Note that if $x \in (A \setminus \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} T^{-n}A)$, then there is a maximal $m \in \mathbb{Z}_{\ge 0}$ such that $T^m x \in A$. This means that

$$A \setminus \bigcap \bigcup T^{-n}A \subset \bigcup_{m=0}^{\infty} T^{-m}A$$

where the right hand side has measure 0. $\square$

This effectively answers one of the questions we asked earlier.

The main issue that can occur is that X splits into parts, which are preserved under T:



Definition (Ergodic)**.** A measure preserving system is called **ergodic** if $A = T^{-1}A$ implies $\mu(A) = 0$ or 1 for all $A \in \mathcal{B}$.

If the MPS is not ergodic, and $A \in \mathcal{B}$ with $0 < \mu(A) < 1$ such that $T^{-1}A = A$, then we can restrict the MPS to A. That is, we consider the MPS: $(A, \mathcal{B}_A, \mu_A, T|_A)$ where $\mathcal{B}_A = \{ B \in \mathcal{B} \mid B \subseteq A \}$ and $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$ for all $B \in \mathcal{B}_A$.

Theorem. The following are equivalent for an measure preserving system $(X, \mathcal{B}, \mu, T)$.

(1) $(X, \mathcal{B}, \mu, T)$ is ergodic.

(2) For all $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\mu(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}A) = 1.$$

(3) $\mu(A \triangle T^{-1}A) = 0$ implies $\mu(A) = 0$ or 1 $\forall A \in \mathcal{B}$.

(4) For all bounded measurable functions $f : X \to \mathbb{R}$, $f = f \circ T$ a.e. implies f is constant a.e.

(5) For all bounded measurable functions $f : X \to \mathbb{C}$, $f = f \circ T$ a.e. implies f is constant a.e.

Proof. (1) $\Rightarrow$ (2). Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Let $B = \bigcap \bigcup T^{-n}A$. By Poincaré recurrence, $\mu(B) \ge \mu(A) > 0$. So if we show that $B = T^{-1}B$, then $\mu(B) = 1$ follows by ergodicity. $x \in B$ iff x visits A infinitely often $\iff$ Tx visits A infinitely often $\iff$ $T_x \in B$. So we have proved $B = T^{-1}B$.

(2) $\Rightarrow$ (3). Let $A \in \mathcal{B}$ such that $\mu(A \triangle T^{-1}A) = 0$. If $\mu(A) = 0$, there is nothing to prove. Suppose $\mu(A) > 0$. Let $B = \bigcap \bigcup T^{-n}A$. By (2), we know that $\mu(B) = 1$.

We show $\mu(B \setminus A) = 0$, which completes the proof. Let $x \in B \setminus A$, then there is a first time m such that $T^m x \in A$, and $m > 0$. Hence $x \in T^{-m}A \setminus T^{-(m-1)}A$. We proved: $B \setminus A \subseteq \bigcup_m \underbrace{T^{-m}A \setminus T^{-(m-1)}A}_{\text{measure 0 because } \mu(T^{-m}A \setminus T^{-(m-1)}A) = \mu(T^{-1}A \setminus A) = 0}$. So $\mu(B \setminus A) = 0$.

(3) $\implies$ (4). Let $f : X \to \mathbb{R}$ be a bounded measurable function such that $f = f \circ T$ almost everywhere. For all $t \in \mathbb{R}$, let $A_t = \{\, x \in X \mid f(x) \le t \,\}$. Then $\mu(A_t \triangle T^{-1}A_t) = 0$. By (3), we have $\mu(A_t) = 0$ or 1 for all t. If t is very small, then $\mu(A_t) = 0$. If t is very large, $\mu(A_t) = 1$. $t \mapsto \mu(A_t)$ is monotone, hence $\exists c \in \mathbb{R}$ such that $\mu(A_t) = 0$ for all $t < c$ and $\mu(A_t) = 1\ \forall t > c$. Then $f(x) = c$ a.e.

(4) $\Leftrightarrow$ (5) is left as an exercise. (4) $\Rightarrow$ (1). Let $A \in \mathcal{B}$ with $A = T^{-1}A$. Then $\chi(A) = \chi(A) \circ T$ everywhere so $\chi(A)$ is constant a.e. $\square$

Example. The circle rotation map $(\mathbb{R}/\mathbb{Z}, \mathcal{B}, \mu, R_\alpha)$ is ergodic iff α is irrational. Let $f : X \to \mathbb{R}$ be measurable. $f(x) = \sum_{n \in \mathbb{Z}} a_n \exp(2\pi i n x)$.

$$f \circ R_\alpha(x) = f(x + \alpha) = \sum_{n \in \mathbb{Z}} a_n \exp(2\pi i n (x + \alpha)) \tag{1}$$

$$= \sum_{n \in \mathbb{Z}} a_n \exp(2\pi i n \alpha) \exp(2\pi i n x) \tag{2}$$

So $f = f \circ R_\alpha \iff a_n = a_n = \exp(2\pi i n \alpha)\ \forall n$. If α is irrational, then $\exp(2\pi i n \alpha) \neq 1$ for all $n \neq 0$, then $a_n = 0$.

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