Part III – Category Theory (Ongoing course, rough)

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0 Introduction

Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

1 Definitions and Examples

Definition 1.1 (Category)**.** A **category** $\mathscr{C}$ consists of

(a) a collection $\mathscr{C}$ of **objects** $A, B, C, \ldots$

(b) a collection $\operatorname{mor} \mathscr{C}$ of **morphism** $f, g, h, \ldots$

(c) two operations $\operatorname{dom}, \operatorname{cod}$ assigning to each $f \in \operatorname{mor} \mathscr{C}$ a pair of objects, its **domain** and **codomain**. We write $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'.

(d) an operation assigning to each $A \in \operatorname{ob} \mathscr{C}$ a morphism $A \xrightarrow{1_A} A$, called its **identity**.

(e) a partial binary operation **composition** $(f, g) \mapsto fg$ on morphisms, such that fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod} f$ if fg is defined.

satisfying

(f) $f 1_A = f = 1_B f$ for any $A \xrightarrow{f} B$

(g) $(fg)h = f(gh)$ whenever fg and gh are defined

Remark 1.2.

(a) This definition is independent of a model of set theory. If we're given a particular model of set theory, we call the category $\mathscr{C}$ **small** if $\operatorname{ob} \mathscr{C}$ and $\operatorname{mor} \mathscr{C}$ are sets.

(b) Some texts say fg means 'f followed by g', i.e. fg defined $\iff$ $\operatorname{cod} f = \operatorname{dom} g$.

(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

Examples 1.3.

(a) The category **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms $A \to B$ are pairs (f, B) where f is a set-theoretic function.)

(b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, $\mathbf{Mod}_R$ the category of R-modules.

(c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.

(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given $\mathscr{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathscr{C}$ a **congruence** if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathscr{C}/\simeq$ with the same objects as $\mathscr{C}$, but congruence classes as morphisms.

(e) Given $\mathscr{C}$, the **opposite category** $\mathscr{C}^{op}$ has the same objects and morphisms as $\mathscr{C}$, but dom and cod are interchanged, and fg in $\mathscr{C}^{op}$ is gf in $\mathscr{C}$. This leads to the **Duality principle** if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

(f) A small category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (i.e. for all f, $\exists g$ such that fg and gf are identities).

(g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space X, the fundamental groupoid $\pi(X)$ has all points of X as objects and morphisms $x \to y$ are homotopy classes rel $\{0, 1\}$ of paths $u : [0, 1] \to X$ with $u(0) = x$, $u(1) = y$. (If you know how to prove that the fundamental group is a group, you can prove that $\pi(X)$ is a groupoid.)

(h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category $\mathscr{C}$ in which, for any pair (A, B) there is at most 1 morphism $A \to B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

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