Part III – Topics in Ergodic Theory (Ongoing course, rough)

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Michaelmas 2018

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Ergodic theory is all about measure preserving systems.

Definition (Measure preserving system)**.** A **measure preserving system** $(X, \mathcal{B}, \mu, T)$ with X a set, $\mathcal{B}$ a σ-algebra, μ a probability measure ($\mu(A) \geq 0\ \forall A \in \mathcal{B}$ and $\mu(X) = 1$) and T is a measure preserving transformation. Recall a measure preserving transformation $T : X \to X$ is a measurable function such that $\mu(T^{-1}(A)) = \mu(A)\ \forall A \in \mathcal{B}$.

If Y is a random element of X with distribution μ, then $T(Y)$ also has distribution μ.

Example. For example, consider a circle rotation. We have $X = \mathbb{R}/\mathbb{Z}$, $\mathcal{B}$ is the Borel sets, μ the Lebesgue measure, and $T = R_\alpha$, with $x \mapsto x + \alpha$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ is a parameter.

We also have the 'times 2 map', with the same $X, \mathcal{B}, \mu$ and $T = T_2$, $x \mapsto 2 \cdot x$.

Proof that T_2 is measure preserving. First check for intervals: Let $I = (a, b)$, then $\mu(I) = b - a$. Also, $\mu(T_2^{-1} I) = \mu\left((\frac{a}{2}, \frac{b}{2}) \cup (\frac{a}{2} + \frac{1}{2}, \frac{b}{2} + \frac{1}{2})\right) = \frac{b}{2} - \frac{a}{2} + \frac{b}{2} - \frac{a}{2} = b - a$, as required.

Now, let $U \subset \mathbb{R}/\mathbb{Z}$ be open. Then $U = I_1 \sqcup I_2 \sqcup \cdots$ is a disjoint union of intervals:

$$\begin{aligned} \mu(T^{-1}U) &= \mu(\bigcup T^{-1} I_j) \\ &= \sum \mu(T^{-1} I_j) \\ &= \sum \mu(I_j) \\ &= \mu(U). \end{aligned}$$

Let $K \subset \mathbb{R}/\mathbb{Z}$ be a compact set.

$$\mu(T^{-1}K) = 1 - \mu((T^{-1}K)^c) = 1 - \mu(T^{-1}K^c) = 1 - \mu(K^c) = \mu(K).$$

Let $A \in \mathcal{B}$ be arbitrary. Let $\epsilon > 0$. $\exists U$ open and $\exists K$ compact such that $K \subset A \subset U$ and $\mu(U \setminus K) < \epsilon$.

$$\mu(K) = \mu(T^{-1}K) \leq \mu(T^{-1}A) \leq \mu(T^{-1}U) = \mu(U).$$

We also have $\mu(K) \leq \mu(A) \leq \mu(U)$. Since $\mu(U) - \mu(K) < \epsilon$, $|\mu(A) - \mu(T^{-1}A)| < \epsilon$. ϵ was arbitrary, so $\mu(A) = \mu(T^{-1}A)$. $\square$

The two examples generalise to the Haar measure on a topological group and to endomorphisms respectively.

In ergodic theory, we study the long term behaviour of orbits.

Definition (Orbit)**.** The orbit of $x \in X$ is the sequence

$$x, Tx, T^2x, \ldots$$

Some questions we might ask are:

- Let $A \in \mathcal{B}$ and $x \in A$. Does the orbit of x visit A infinitely often? (Recurrence)
- What is the proportion of times n such that $T^n x \in A$?
- What is $\mu(\{\, x \in A \mid T^n x \in A \,\})$ if n is large? (Mixing property)

Example. Let $A = [0, \frac{1}{4}) \subset \mathbb{R}/\mathbb{Z}$. Then $T_2^n x \in A \iff$ the $n+1$th and $n+2$th 'binary digits' of x are 0.

For some $x = 0.x_1x_2x_3\dots_2$, $x \in A$ corresponds to x_1, x_2 both being 0 and the doubling map sends x to $T_2 x = x_2x_3\dots_2$, giving the property above.

For example, $x = \frac{1}{6} = 0.00101010\dots_2$ starts in A but never comes back to A. Also, we have $\mu(\{\, x \in A \mid T_2^n x \,\}) = \frac{1}{16}$ if $n \geq 2$.

Example (Markov shift)**.** Let $P_1, P_2, \dots, P_n$ be a probability vector. Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be the 'matrix of transition probabilities'. Assume

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} P_1 & P_2 & \dots & P_n \end{pmatrix} A = \begin{pmatrix} P_1 & P_2 & \dots & P_n \end{pmatrix}$$

Take $X = \{1, \dots, n\}^{\mathbb{Z}}$, $\mathcal{B}$ the Borel σ-algebra generated by the product topology of the discrete topology on $\{1, \dots, n\}$, $T = \sigma$ the shift map: $(\sigma x)_m = x_{m+1}$. Finally, set the measure

$$\mu(\{\, x \in X \mid x_m = i_0, x_{m+1} = i_1, \dots, x_{m+n} = i_n \,\}) = P_{i_0} a_{i_0 i_1} \cdots a_{i_{n-1} i_n}.$$

Theorem (Szemerédi)**.** Let $S \subset \mathbb{Z}$ of positive upper Banach density. That is,

$$\bar{d}(S) \coloneqq \limsup_{N, M : M - N \to \infty} \frac{1}{M - N} \left| S \cap [N, M-1] \right|$$

and $\bar{d}(S) > 0$. Then S contains arbitrarily long arithmetic progressions. That is, $\forall l, \exists a \in \mathbb{Z}, d \in \mathbb{Z}_{>0}$,

$$a, a+d, \dots, a+(l-1)d \in S.$$

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Theorem (Furstenberg, multiple recurrence)**.** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Let $A \in \mathcal{B}$ such that $\mu(A) > 0$. Let $l \in \mathbb{Z}_{>0}$. Then

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n} A \cap \dots \cap T^{-(l-1)n} A) > 0.$$

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