

Part III – Category Theory (Ongoing course)

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0 Introduction

Lecture 1 Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

1 Definitions and Examples

1.1 Definition (Category). A **category** \mathcal{C} consists of

- (a) a collection \mathcal{C} of **objects** A, B, C, \dots
- (b) a collection $\text{mor } \mathcal{C}$ of **morphisms** f, g, h, \dots
- (c) two operations dom, cod assigning to each $f \in \text{mor } \mathcal{C}$ a pair of objects, its **domain** and **codomain**. We write $A \xrightarrow{f} B$ to mean ‘ f is a morphism and $\text{dom } f = A$ and $\text{cod } f = B$ ’.
- (d) an operation assigning to each $A \in \text{ob } \mathcal{C}$ a morphism $A \xrightarrow{1_A} A$, called its **identity**.
- (e) a partial binary operation **composition** $(f, g) \mapsto fg$ on morphisms, such that fg is defined iff $\text{dom } f = \text{cod } g$ and $\text{dom}(fg) = \text{dom } g$, $\text{cod}(fg) = \text{cod } f$ if fg is defined.

satisfying

- (f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$
- (g) $(fg)h = f(gh)$ whenever fg and gh are defined

1.2 Remark.

- (a) This definition is independent of a model of set theory. If we’re given a particular model of set theory, we call the **category** \mathcal{C} **small** if $\text{ob } \mathcal{C}$ and $\text{mor } \mathcal{C}$ are sets.
- (b) Some texts say fg means ‘ f followed by g ’, i.e. fg defined $\iff \text{cod } f = \text{dom } g$.
- (c) Note that a morphism f is an **identity** iff $fg = g$ and $hf = h$ whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

1.3 Examples.

- (a) The **category** **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms $A \longrightarrow B$ are pairs (f, B) where f is a set-theoretic function.)
- (b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, **Mod** $_R$ the category of R -modules.

- (c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.
- (d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given \mathcal{C} , we call an equivalence relation \simeq on $\text{mor } \mathcal{C}$ a **congruence** if $f \simeq g \implies \text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category \mathcal{C}/\simeq with the same objects as \mathcal{C} , but congruence classes as morphisms.
- (e) Given \mathcal{C} , the **opposite category** \mathcal{C}^{op} has the same objects and morphisms as \mathcal{C} , but dom and cod are interchanged, and fg in \mathcal{C}^{op} is gf in \mathcal{C} . This leads to the **Duality principle**: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.
- (f) A **small** category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (f is an **isomorphism** if $\exists g$ such that fg and gf are identities).
- (g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space X , the fundamental groupoid $\pi(X)$ has all points of X as objects and morphisms $x \rightarrow y$ are homotopy classes $\text{rel } \{0, 1\}$ of paths $u : [0, 1] \rightarrow X$ with $u(0) = x$, $u(1) = y$. (If you know how to prove that the fundamental group is a group, you can prove that $\pi(X)$ is a groupoid.)
- (h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category \mathcal{C} in which, for any pair (A, B) there is at most 1 morphism $A \rightarrow B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
- (i) The category **Rel** has the same objects as **Set**, but morphisms $A \rightarrow B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define

$$S \circ R = \{ (a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R \wedge (b, c) \in S) \}.$$

The identity $1_A : A \rightarrow A$ is $\{ (a, a) \mid a \in A \}$.

Similarly, the category **Part** of sets and partial functions (i.e. relations such that $(a, b) \in R, (a, b') \in R \implies b = b'$).

- (j) Let K be a field. The category **Mat** $_K$ has natural numbers as objects, and morphisms $n \rightarrow p$ are $(p \times n)$ matrices with entries from K . Composition is matrix multiplication.

1.4 Definition (Functor). Let \mathcal{C}, \mathcal{D} be **categories**. A **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of

- (a) a mapping $A \mapsto FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$
- (b) a mapping $f \mapsto Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$

such that $\text{dom}(Ff) = F(\text{dom } f)$, $\text{cod}(Ff) = F(\text{cod } f)$, $1_{FA} = F(1_A)$ and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Lecture 2 **1.3 Examples** (*Continued*).

- (k) We write **Cat** for the category whose objects are all **small categories**, and whose morphisms are **functors** between them.

1.5 Examples.

- (a) We have **forgetful functors** $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}, \mathbf{Rng} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set}, \mathbf{Rng} \rightarrow \mathbf{AbGp}$ (forgetting \times), $\mathbf{Rng} \rightarrow \mathbf{Mon}$ (forgetting $+$).
- (b) Given a set A , the free group FA has the property: given any group G and any function $A \xrightarrow{f} UG$, there's a unique homomorphism $FA \xrightarrow{f} G$ extending f . F is a functor $\mathbf{Set} \rightarrow \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$.

Functoriality follows from uniqueness: given $B \xrightarrow{g} C$, $F(gf)$ and $(Fg)(Ff)$ are both homoms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$. Call this the **free functor**.

- (c) Given a set A , we write $\mathcal{P}A$ for the set of all subsets of A . We can make \mathcal{P} into a functor $\mathbf{Set} \rightarrow \mathbf{Set}$: given $A \xrightarrow{f} B$, we define $\mathcal{P}f(A') = \{f(a) \mid a \in A'\}$ for $A' \subseteq A$. But we also have a functor $\mathcal{P}^* : \mathbf{Set} \rightarrow \mathbf{Set}^{\text{op}}$ defined on objects by \mathcal{P} , but $\mathcal{P}^*f(B') = \{a \in A \mid f(a) \in B'\}$ for $B' \subseteq B$.

By a **contravariant** functor $\mathcal{C} \rightarrow \mathcal{D}$, we mean a **functor** $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ (or $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$). (A **covariant** functor is one that doesn't reverse arrows).

- (d) Let K be a field. We have a functor $* : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{\text{op}}$ defined by $V^* = \{\text{linear maps } V \rightarrow K\}$ and if $V \xrightarrow{f} W$, $f^*(\theta : W \rightarrow K) = \theta f$.
- (e) We have a functor $op : \mathbf{Cat} \rightarrow \mathbf{Cat}$ which is the 'identity' on morphisms. (Note that this is **covariant**).
- (f) A functor between monoids is a monoid homomorphism.
- (g) A functor between posets is an order-preserving map.
- (h) Let G be a group. A functor $F : G \rightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A , i.e. a permutation representation of G (where we use $*$ to refer to the unique object of the group). Similarly a functor $G \rightarrow \mathbf{Mod}_K$ is a K -linear representation of G .
- (i) The construction of a fundamental group $\pi_1(X, x)$ of a space X with basepoint x is a functor $\mathbf{Top}_* \rightarrow \mathbf{Gp}$ where \mathbf{Top}_* is the set of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \rightarrow \mathbf{Gpd}$ where \mathbf{Gpd} is the category of groupoids and functors between them.

1.6 Definition (Natural transformation). Let \mathcal{C}, \mathcal{D} be **categories** and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two **functors**. A **natural transformation** $\alpha : F \rightarrow G$ consists of an assignment $A \mapsto \alpha_A$

from $\text{ob } \mathcal{C}$ to $\text{mor } \mathcal{D}$, such that $\text{dom } \alpha_A = FA$ and $\text{cod } \alpha_A = GA$ for all A , and for all $A \xrightarrow{f} B$ in \mathcal{C} the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)\alpha_A$).

1.3 Examples (Continued).

- (1) Given categories \mathcal{C}, \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$, and whose morphisms are natural transformations.

1.7 Examples.

- (a) Let K be a field, V a vector space over K . There is a linear map $\alpha_V : V \rightarrow V^{**}$ given by

$$\alpha_V(v)(\theta) = \theta(v)$$

for $\theta \in V^*$. This is the V -component of a natural transformation

$$1_{\mathbf{Mod}_K} \rightarrow ** : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K.$$

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- (b) For any set A , we have a mapping $\sigma_A : A \rightarrow \mathcal{P}A$ sending a to $\{a\}$. If $f : A \rightarrow B$, then $\mathcal{P}f\{a\} = \{f(a)\}$, so σ is a natural transformation $1_{\mathbf{Set}} \rightarrow \mathcal{P}$.
- (c) Let $F : \mathbf{Set} \rightarrow \mathbf{Gp}$ be the free group functor (Examples 1.5(b)) and $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \rightarrow UFA$ form a natural transformation $1_{\mathbf{Set}} \rightarrow UF$.
- (d) Let G, H be groups and $f, g : G \rightarrow H$ two homomorphisms. A natural transformation $\alpha : f \rightarrow g$ corresponds to an element $h = \alpha_*$ of H such that $h.f(x) = g(x).h$ for all $x \in G$, or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
- (e) Let A, B be two G -sets regarded as functors $G \rightarrow \mathbf{Set}$. A natural transformation $A \rightarrow B$ is a function f satisfying $f(g.a) = g.f(a)$ for all $a \in A$, i.e. a G -equivariant map.

1.8 Lemma. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors, and $\alpha : F \rightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in \mathcal{D} .

Proof.

\Rightarrow trivial

\Leftarrow Suppose each α_A has an inverse β_A . Given $f : A \rightarrow B$ in \mathcal{C} , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

commutes.

But

$$\begin{aligned}(Ff)\beta_A &= \beta_B\alpha_B(Ff)\beta_A \\ &= \beta_B(Gf)\alpha_A\beta_A \\ &= \beta_B(Gf).\end{aligned}$$

□

1.9 Definition (Equivalent category). Let \mathcal{C}, \mathcal{D} be categories. By an **equivalence** between \mathcal{C} and \mathcal{D} , we mean a pair of **functors** $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$ and $\beta : FG \rightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ if \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of **categories** is a **categorical property** if whenever \mathcal{C} has P and $\mathcal{C} \simeq \mathcal{D}$, then \mathcal{D} has P .

For instance, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

1.10 Examples.

- (a) The category **Part** is equivalent to the category **Set**_{*} of pointed sets (and basepoint-preserving functions). We define $F : \mathbf{Set}_* \rightarrow \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$ and if $f : (A, a) \rightarrow (B, b)$

$$Ff(x) = \begin{cases} f(x) & \text{if } f(x) \neq b \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $G : \mathbf{Part} \rightarrow \mathbf{Set}_*$ by $G(A) = A^+ = A \cup \{A\}$ and if $f : A \rightarrow B$ is a partial function, we define $Gf : A^+ \rightarrow B^+$ by

$$Gf = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

The composite FG is the identity on **Part**, but GF is not the identity, however there's an isomorphism

$$(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism $\mathbf{Set}_* \rightarrow \mathbf{Part}$ since **Part** has a 1-element isomorphism class $\{\emptyset\}$ and **Set**_{*} doesn't.

- (b) The category **FdMod** _{K} of finite-dimensional vector spaces over K is equivalent to **FdMod** _{K} ^{op}: the functors in both directions are $(-)^*$ and both isomorphisms are the natural transformations of [Examples 1.7\(a\)](#).
- (c) **FdMod** _{K} is also equivalent to **Mat** _{K} : We define $F : \mathbf{Mat}_K \rightarrow \mathbf{FdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A with respect to the standard bases of K^n and K^p .

To define $G : \mathbf{FdMod}_K \rightarrow \mathbf{Mat}_K$, choose a basis for each finite dimensional vector space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ as the matrix representing f with respect to the chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n ; $FG \neq 1$, but the chosen basis gives isomorphisms $FG(V) = K^{\dim V} \rightarrow V$ for each V , which form a natural isomorphism.

Lecture 4 **1.11 Definition** (Faithful, full, essentially surjective). Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a **functor**.

- (a) We say F is **faithful** if, given $f, f' \in \text{mor } \mathcal{C}$ with $\text{dom } f = \text{dom } f'$, $\text{cod } f = \text{cod } f'$ and $Ff = Ff'$ then $f = f'$.
- (b) We say F is **full** if, given $FA \xrightarrow{g} FB$ in \mathcal{D} , there exists $A \xrightarrow{f} B$ in \mathcal{C} with $Ff = g$.
- (c) We say F is **essentially surjective** if, for every $B \in \text{ob } \mathcal{D}$, there exists $A \in \text{ob } \mathcal{C}$ and an isomorphism $FA \rightarrow B$ in \mathcal{D} .

We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is **full** if the inclusion $\mathcal{C}' \rightarrow \mathcal{C}$ is a full functor.

Example. **Gp** is a **full subcategory** of **Mon**, but **Mon** is not a full subcategory of the category **Sgp** of semigroups.

1.12 Lemma. Assuming the axiom of choice, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is part of an **equivalence** $\mathcal{C} \simeq \mathcal{D}$ iff it is **full**, **faithful** and **essentially surjective**.

Proof.

\Rightarrow Given G, α, β as in **Definition 1.9**, for each $B \in \text{ob } \mathcal{D}$, β_B is an isomorphism $FGB \rightarrow B$, so F is essentially surjective.

Given $A \xrightarrow{f} B$ in \mathcal{C} , we can recover f from Ff as the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B.$$

Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$.

Given $FA \xrightarrow{g} FB$, define f to be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$$

Then $GFf = \alpha_B f \alpha_A^{-1} = Gg$, and G is faithful for the same reason as F , so $Ff = g$.

\Leftarrow For each $B \in \text{ob } \mathcal{D}$, choose $GB \in \text{ob } \mathcal{C}$ and an isomorphism $\beta_B : FGB \rightarrow B$ in \mathcal{D} . Given

$$B \xrightarrow{g} B'$$

define $Gg : GB \rightarrow GB'$ to be the unique morphism whose image under F is

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

Uniqueness implies functoriality: given

$$B' \xrightarrow{g'} B''$$

then note $(Gg')(Gg)$ and $G(g'g)$ have the same image under F , so they're equal.

By construction, β is a natural transformation $FG \rightarrow 1_{\mathcal{D}}$.

Given $A \in \text{ob } \mathcal{C}$, define $\alpha_A : A \rightarrow GFA$ to be the unique morphism whose image under F is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA$$

α_A is an isomorphism, since β_{FA} also has a unique pre-image under F .

Also α is a natural transformation, since any naturality square for α is mapped by F to a commutative square, and F is faithful. \square

1.13 Definition (Skeleton). By a **skeleton** of a category \mathcal{C} , we mean a **full subcategory** \mathcal{C}_0 containing one object from each isomorphism class. We say \mathcal{C} is **skeletal** if it's a skeleton of itself.

Example. \mathbf{Mat}_K is **skeletal**, and the image of $F : \mathbf{Mat}_K \rightarrow \mathbf{FdMod}_K$ of [Examples 1.10\(c\)](#) is a **skeleton** of \mathbf{FdMod}_K .

Warning. Almost any assertion about **skeletons** is equivalent to the axiom of choice. See question 2 on example sheet 1.

1.14 Definition (Monomorphism, epimorphism). Let $A \xrightarrow{f} B$ be a **morphism** in \mathcal{C}

- (a) We say f is a **monomorphism** (or f is **monic**) if, given any pair $C \xrightarrow[g]{f} A$, $fg = fh$ implies $g = h$
- (b) We say f is an **epimorphism** (or **epic**) if it's a monomorphism in C^{op} i.e. if $gf = hf$ implies $g = h$.

We denote **monomorphisms** by $A \rightarrowtail B$ and **epimorphisms** by $A \twoheadrightarrow B$

Any isomorphism is **monic** and **epic**: more generally if f has a left inverse (i.e. $\exists g$ such that gf is an identity) then it's monic. We call such monomorphisms **split**.

We say C is a **balanced category** if any morphism which is both **monic** and **epic** is an isomorphism.

1.15 Examples.

- (a) In **Set**, **mono** \iff injective (\implies easy; for \longleftarrow take $C = 1 = \{*\}$) and **epi** \iff surjective (\implies easy; for \longleftarrow use two morphisms $B \rightarrow 2 = \{0, 1\}$). So **Set** is **balanced**.
- (b) In **Gp** **mono** \iff injective (for \longleftarrow use homoms $\mathbb{Z} \rightarrow A$) and **epi** \iff surjective (\longleftarrow uses free products with amalgamation). So **Gp** is balanced.
- (c) In **Rng**, **mono** \iff injective (proof much as for **Gp**) but the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \xrightarrow[g]{f} R$ agree on all integers, they agree everywhere. So **Rng** isn't balanced.
- (d) In **Top**, **mono** \iff injective and **epi** \iff surjective (proofs as in **Set**). But **Top** isn't balanced since a continuous bijection needn't have a continuous inverse.

2 The Yoneda Lemma

Lecture 5 **2.1 Definition** (Locally small). We say a **category** \mathcal{C} is **locally small** if, for any two objects A, B , the morphisms $A \rightarrow B$ in \mathcal{C} form a set $\mathcal{C}(A, B)$.

If we fix A and let B vary, the assignment $B \mapsto \mathcal{C}(A, B)$ becomes a **functor** $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathcal{C}(A, f)$ is the mapping $g \mapsto fg$. Similarly, $A \mapsto \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.

2.2 Lemma (Yoneda Lemma). Let \mathcal{C} be a **locally small category**, $A \in \text{ob } \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathbf{Set}$ a **functor**.

- (i) Then natural transformations $\mathcal{C}(A, -) \rightarrow F$ are in bijection with elements of FA .
- (ii) Moreover, this bijection is natural in both A and F .

Proof of Yoneda Lemma(i). Given $\alpha : \mathcal{C}(A, -) \rightarrow F$, we define

$$\Phi(\alpha) = \alpha_A(1_A) \in FA.$$

Given $x \in FA$, we define $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB.$$

$\Psi(x)$ is natural: given $g : B \rightarrow C$, we have

$$\begin{aligned} \Psi(x)_C \mathcal{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x) \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x). \end{aligned}$$

$$\begin{array}{ccc} \mathcal{C}(A, B) & \xrightarrow{\mathcal{C}(A, g)} & \mathcal{C}(A, C) \\ \Psi(x)_B \downarrow & & \downarrow \Psi(x)_C \\ FB & \xrightarrow{Fg} & FC \end{array}$$

We also verify Ψ and Φ are inverse:

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x.$$

Given α ,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

so $\Psi\Phi(\alpha) = \alpha$. □

2.3 Corollary. The assignment $A \mapsto \mathcal{C}(A, -)$ defines a **full** and **faithful functor** $\mathcal{C}^{\text{op}} \rightarrow [\mathcal{C}, \mathbf{Set}]$.

Proof. Put $F = \mathcal{C}(B, -)$ in [Lemma 2.2\(i\)](#): we get a bijection between $\mathcal{C}(B, A)$ and morphisms $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$ in $[\mathcal{C}, \mathbf{Set}]$. We need to verify this is functorial: but it sends $f : B \rightarrow A$ to the natural transformation $g \mapsto gf$. So functoriality follows from associativity. \square

We call this **functor** (or the functor $\mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$) sending A to $\mathcal{C}(-, A)$ the **Yoneda embedding** of \mathcal{C} , and denote it by Y .

Proof of Yoneda Lemma(ii). Suppose for the moment that \mathcal{C} is **small**, so that $[\mathcal{C}, \mathbf{Set}]$ is **locally small**. Then we have two functors $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$: One sends (A, F) to FA , and the other is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{\text{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-, -)} \mathbf{Set}$$

[Yoneda Lemma\(ii\)](#) says that these are naturally isomorphic.

We can translate this into an elementary statement, making sense even when \mathcal{C} isn't small, given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathcal{C}(A, -) \rightarrow F$ give the same result, namely

$$\alpha_B(Ff)\beta_A(1_A) = (GF)\alpha_A\beta_A(1_A)$$

which is equal to $\alpha_B\beta_B(f)$. \square

2.4 Definition. We say a **functor** $F : \mathcal{C} \rightarrow \mathbf{Set}$ is **representable** if it's **isomorphic** to $\mathcal{C}(A, -)$ for some A . By **representation** of F , we mean a pair (A, x) where $x \in FA$ is such that $\Psi(x)$ is an **isomorphism**. We also call x a **universal element** of F .

2.5 Corollary. If (A, x) and (B, y) are both **representations** of F , then there's a unique **isomorphism** $f : A \rightarrow B$ such that $(Ff)(x) = y$.

Proof. Consider the composite

$$\mathcal{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathcal{C}(A, -)$$

By [Corollary 2.3](#), this is of the form $Y(f)$ for a unique **isomorphism** $f : A \rightarrow B$ and the diagram

$$\begin{array}{ccc} \mathcal{C}(B, -) & \xrightarrow{Y(f)} & \mathcal{C}(A, -) \\ & \searrow \Psi(y) & \swarrow \Psi(x) \\ & F & \end{array}$$

commutes iff $(Ff)x = y$. \square

2.6 Examples.

- (a) The **forgetful functor** $\mathbf{Gp} \rightarrow \mathbf{Set}$ is **representable** by $(\mathbb{Z}, 1)$. Similarly, the forgetful functor $\mathbf{Rng} \rightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}[x], x)$ and the forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ is representable by $(\{\ast\}, \ast)$.

- (b) The functor $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ (see Examples 1.5(c)) is representable by $(\{0, 1\}, \{1\})$: this is the bijection between subsets and characteristic functions.
- (c) Let G be a group. The unique (up to isomorphism) representable functor $G(*, -) : G \rightarrow \mathbf{Set}$ is the *Cayley representation* of G , i.e. the set UG with G acting by left multiplication.
- (d) Let A, B be two objects of a locally small category \mathcal{C} . We have a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ sending C to $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$. A representation of this, if it exists, is called a (categorical) **product** of A and B , and denoted

$$(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B)).$$

This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$ there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.

Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top** they are 'just' cartesian products, in posets they are binary meets.

Dually we have the notion of **coproduct** $(A + B, (A \xrightarrow{\nu_1} A + B, B \xrightarrow{\nu_2} A + B))$. These also exist in many categories of interest.

Lecture 6

- (e) The dual-vector-space functor $\mathbf{Mod}_K^{\text{op}} \rightarrow \mathbf{Mod}_K$, when composed with the forgetful functor $\mathbf{Mod}_K \rightarrow \mathbf{Set}$, is representable by $(K, 1_K)$.
- (f) Let $A \rightrightarrows B$ be morphisms in a locally small category \mathcal{C} . We have a functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ defined by

$$F(C) = \{ h \in \mathcal{C}(C, A) \mid fh = gh \}.$$

A representation of F , if it exists, is called an **equalizer** of (f, g) . It consists of an objects E and a morphism $E \xrightarrow{e} A$ such $fe = ge$, and every h with $fh = gh$ factors uniquely through e . In **Set**, we can take $E = \{ x \in A \mid f(x) = g(x) \}$ and $e =$ inclusion. Similar constructions work in **Gp**, **Rng**, **Top**, ...

Dually, we have the notion of **coequalizer**.

2.7 Remark. If e occurs as an **equalizer**, then it's a **monomorphism**, since any h factors through it in at most one way. We say a monomorphism is **regular** if it occurs as an equalizer.

Split monomorphisms are **regular** (c.f. question 6i on sheet 1). Note that regular mono + **epi** \implies **iso**: if the equalizer e of (f, g) is epic, then $f = g$, so $e \cong 1_{\text{cod } e}$.

2.8 Definition (Separating, detecting families). Let \mathcal{C} be a category, and \mathcal{G} a class of objects of \mathcal{C} .

- (a) We say \mathcal{G} is a **separating family** for \mathcal{C} if, given $A \rightrightarrows B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathcal{G}$, then $f = g$ (i.e. the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are collectively faithful).

- (b) We say \mathcal{G} is a **detecting family** for \mathcal{C} if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f , then f is an isomorphism.

If $\mathcal{G} = \{G\}$, we call G a **separator/detector**.

2.9 Lemma.

- (i) If \mathcal{C} is a **balanced** category, then any **separating** family is **detecting**.
- (ii) If \mathcal{C} has **equalizers**, then any detecting family is separating.

Proof.

- (i) Suppose \mathcal{G} is **separating** and $A \xrightarrow{f} B$ satisfies the condition of **Definition 2.8(b)**. If $B \xrightarrow[g]{g} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is **epic**.
Similarly if $D \xrightarrow[k]{l} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorisations of fky through f . So $k = l$, i.e. f is **monic**.
- (ii) Suppose \mathcal{G} is **detecting** and $A \xrightarrow[g]{f} B$ satisfies the condition of 2.8(a). Then the **equalizer** $E \xrightarrow{e} A$ is an **isomorphism**, so $f = g$.

□

2.10 Examples.

- (a) In $[\mathcal{C}, \mathbf{Set}]$ the family

$$\{\mathcal{C}(A, -) \mid A \in \text{ob } \mathcal{C}\}$$

is both **separating** and **detecting** (this is just a restatement of **Yoneda Lemma**.)

- (b) In **Set**, $1 = \{*\}$ is both a separator and a detector since it represents the identity **functor** $\mathbf{Set} \longrightarrow \mathbf{Set}$.

Similarly, \mathbb{Z} is both in **Gp**, since it represents the **forgetful functor** $\mathbf{Gp} \longrightarrow \mathbf{Set}$.

And $2 = \{0, 1\}$ is a coseparator and a codetector in **Set**, since it represents $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Set}$.

- (c) In **Top**, $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \longrightarrow \mathbf{Set}$, but not a detector. In fact, **Top** has no detecting *set* of objects:

For any infinite cardinal κ , let X be a discrete space of cardinality κ and let Y be the same set with ‘co- $< \kappa$ ’ topology, i.e. $F \subseteq Y$ closed $\iff F = Y$ or $\text{card } F < \kappa$. The identity $X \longrightarrow Y$ is continuous, but not a homeomorphism.

So if $\{G_i \mid i \in I\}$ is any set of spaces, taking $\kappa > \text{card } G_i$ for all i yields an example to show that the set is not detecting.

(d) Let \mathcal{C} be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous maps. JHC Whitehead proved that if $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \rightarrow \pi_n(Y)$ for all n , then it's an isomorphism in \mathcal{C} . This says that $\{S^n \mid n \geq 1\}$ is a detecting set for \mathcal{C} .

But PJ Freyd showed there is no **faithful** functor $\mathcal{C} \rightarrow \mathbf{Set}$, so no separating *set*: if $\{G_i \mid i \in I\}$ were separating, then

$$x \mapsto \prod_{i \in I} \mathcal{C}(G_i, X)$$

would be faithful.

Note that any **functor** of the form $\mathcal{C}(A, -)$ preserves **monomorphisms**, but they don't normally preserve **epimorphisms**.

2.11 Definition (Projective). We say an object P is **projective** if, given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{e} & B \end{array}$$

there exists $P \xrightarrow{g} A$ with $eg = f$. (If \mathcal{C} is **locally small**, this says $\mathcal{C}(P, -)$ preserves **epimorphisms**).

Dually, an **injective** object of \mathcal{C} is a projective object of \mathcal{C}^{op} . Given a class \mathcal{E} of epimorphisms, we say P is \mathcal{E} -projective if it satisfies the condition for all $e \in \mathcal{E}$.

2.12 Lemma. **Representable functors** are (pointwise) **projective** in $[\mathcal{C}, \mathbf{Set}]$.

Proof. Take

$$\begin{array}{ccc} & \mathcal{C}(A, -) & \\ & \downarrow \beta & \\ F & \xrightarrow{a} & G \end{array}$$

where α is pointwise surjective. By **Yoneda Lemma**, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathcal{C}(A, -) \rightarrow F$ corresponds to x then naturality of the **Yoneda bijection** yields $\alpha\gamma = \beta$. \square

3 Adjunctions

Lecture 7 **3.1 Definition.** Let \mathcal{C} and \mathcal{D} be two **categories** and $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ two **functors**. By an **adjunction** between F and G we mean a bijection between **morphisms** $FA \xrightarrow{\hat{f}} B$ in \mathcal{D} and morphisms $A \xrightarrow{f} GB$ in \mathcal{C} which is **natural** in A and B , i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, we have $h\hat{f}(Fg) = \widehat{(Gh)fg} : FA' \rightarrow B'$.

We say F is **left adjoint** to G and write $F \dashv G$.

3.2 Examples.

- (a) The **free functor** $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$ is **left adjoint** to the **forgetful functor** $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, since any function $f : A \rightarrow UB$ extends uniquely to a homomorphism $\hat{f} : FA \rightarrow B$. Naturality in B is easy, naturality in A follows from the definition of F as a functor.
- (b) The **forgetful functor** $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology and a right adjoint I which equips a set A with the indiscrete topology $\{\emptyset, A\}$.
- (c) The functor $\mathbf{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$ has a left adjoint D sending A to the **discrete** category with $\mathbf{ob}(DA) = A$ and only identity morphisms. It also has a right adjoint I sending A to the (**indiscrete**) category with $\mathbf{ob}(IA) = A$ and one morphism $x \rightarrow y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category \mathcal{C} to its set of *connected components*, i.e. the quotient of $\mathbf{ob} \mathcal{C}$ by the smallest equivalent relation identifying $\mathbf{dom} f$ with $\mathbf{cod} f$ for all $f \in \mathbf{mor} \mathcal{C}$.
- (d) Let \mathcal{M} be the **monoid** $\{1, e\}$ with $e^2 = e$. An object of $[\mathcal{M}, \mathbf{Set}]$ is a pair (A, e) where $e : A \rightarrow A$ satisfies $e^2 = e$.

We have a functor $G : [\mathcal{M}, \mathbf{Set}] \rightarrow \mathbf{Set}$ sending (A, e) to

$$\{x \in A \mid e(x) = x\} = \{e(x) \mid x \in A\}$$

and a functor $F : \mathbf{Set} \rightarrow [\mathcal{M}, \mathbf{Set}]$ sending A to $(A, 1_A)$.

Claim $F \dashv G \dashv F$: given $f : (A, 1_A) \rightarrow (B, e)$: it must take values in $G(B, e)$, and any $g : (B, e) \rightarrow (A, 1_A)$ is determined by its values on the image of e .

- (e) Let $\mathbf{1}$ be the **discrete** category with one object $*$. For any \mathcal{C} , there's a unique functor $\mathcal{C} \rightarrow \mathbf{1}$: a **left adjoint** for this picks out an **initial object** of \mathcal{C} , i.e. an object I such that there exists a unique $I \rightarrow A$ for each $A \in \mathbf{ob} \mathcal{C}$. Dually, a right adjoint for $\mathcal{C} \rightarrow \mathbf{1}$ corresponds to a **terminal object** of \mathcal{C} .
- (f) Let $A \xrightarrow{f} B$ be a morphism in \mathbf{Set} . We can regard $\mathcal{P}A$ and $\mathcal{P}B$ as posets, and we have functors

$$\mathcal{P}A \xrightleftharpoons[\mathcal{P}^*f]{\mathcal{P}f} \mathcal{P}B$$

Claim $(\mathcal{P}f \dashv \mathcal{P}^*f)$: we have $\mathcal{P}f(A') \subseteq B' \iff f(x) \in B'$ for all $x \in A' \iff A' \subseteq \mathcal{P}^*f(B')$.

- (g) Suppose given sets A, B and a relation $R \subseteq A \times B$. We define mappings $(-)^l, (-)^r$ between $\mathcal{P}A$ and $\mathcal{P}B$ by

$$\begin{aligned} S^r &= \{ y \in B \mid (\forall x \in S)((x, y) \in R) \} \quad \text{for } S \subseteq A \\ T^l &= \{ x \in A \mid (\forall y \in T)((x, y) \in R) \} \quad \text{for } T \subseteq B. \end{aligned}$$

These mappings are order-reversing (i.e. [contravariant functors](#)) and

$$T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l.$$

We say $(-)^r$ and $(-)^l$ are **adjoint on the right**.

- (h) The functor $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ is self-[adjoint on the right](#), since functions $A \rightarrow \mathcal{P}B$ correspond bijectively to subsets of $A \times B$ and hence to functions $B \rightarrow \mathcal{P}A$.

Definition (Comma category). $(A \downarrow G)$ is the **comma category** with objects pairs (B, f) with $A \xrightarrow{f} GB$, and morphisms $(B, f) \rightarrow (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow f' \\ GB & \xrightarrow{Gg} & GB' \end{array}$$

commutes.

3.3 Theorem. Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a [functor](#). Then specifying a [left adjoint](#) for G is equivalent to specifying an [initial object](#) of $(A \downarrow G)$ for each $A \in \text{ob } \mathcal{C}$.

Proof. Suppose we are given $F \dashv G$. Consider the morphism $\eta_A : A \rightarrow GFA$ corresponding to $FA \xrightarrow{1} FA$. Then (FA, η_A) is an object of $(A \downarrow G)$. Moreover, given $g : FA \rightarrow B$ and $f : A \rightarrow GB$, the diagram

$$\begin{array}{ccc} & A & \\ \eta_A \swarrow & & \searrow f \\ GFA & \xrightarrow{Gg} & GB \end{array}$$

commutes iff

$$\begin{array}{ccc} & FA & \\ 1_{FA} \swarrow & & \searrow \hat{f} \\ FA & \xrightarrow{g} & B \end{array}$$

commutes, i.e. $g = \hat{f}$. So (FA, η_A) is [initial](#) in $(A \downarrow G)$.

Conversely, suppose we are given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $A \xrightarrow{f} A'$, we define $Ff : FA \rightarrow FA'$ to be the unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute. Functoriality follows from uniqueness: given $f' : A' \rightarrow A''$, both $F(f'f)$ and $(Ff')(Ff)$ are morphisms $(FA, \eta_A) \rightarrow (FA'', \eta_{A''}f'f)$ in $(A \downarrow G)$.

To show $F \dashv G$: given $A \xrightarrow{f} GB$, we define $\hat{f} : FA \rightarrow B$ to be the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$. This is a bijection with inverse

$$(FA \xrightarrow{f} B) \mapsto (A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB)$$

The latter mapping is natural in B since G is a functor, and in A since, by construction, η is a natural transformation $1_{\mathcal{C}} \rightarrow GF$. \square

3.4 Corollary. If F and F' are both [left adjoint](#) to $G : \mathcal{D} \rightarrow \mathcal{C}$, then they are [naturally isomorphic](#).

Proof. For any A , (FA, η_A) and $(F'A, \eta'_A)$ are both [initial](#) in $(A \downarrow G)$, so there's a unique [isomorphism](#) $\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A)$. In any naturality square for α , the two ways round are both morphisms in $(A \downarrow G)$ where the domain is initial, so they are equal. \square

3.5 Lemma. Given

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{H} \\ \xleftarrow{K} \end{array} \mathcal{E}$$

with $(F \dashv G)$ and $(H \dashv K)$ we have $(HF \dashv GK)$.

Proof. We have bijections between morphisms $A \rightarrow GKC$, morphisms $FA \rightarrow KC$ and morphisms $HFA \rightarrow C$, which are both natural in A and C . \square

3.6 Corollary. Given a commutative square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{F} \end{array}$$

of [categories](#) and [functors](#), if the functors all have left [adjoints](#), then the diagram of left adjoints commutes up to [natural isomorphism](#).

Proof. By [Lemma 3.5](#), both ways round the diagram of left adjoints are left adjoint to the composite $\mathcal{C} \rightarrow \mathcal{F}$, so by [Corollary 3.4](#) they are isomorphic. \square

Given an [adjunction](#) $(F \dashv G)$, the [natural transformation](#) $\eta : 1_{\mathcal{C}} \rightarrow GF$ emerging in the proof of [Theorem 3.3](#) is called the **unit** of the adjunction. Dually, we have a natural transformation $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ such that $\epsilon_B : FGB \rightarrow B$ corresponds to $GB \xrightarrow{1_{GB}} GB$ is called the **counit**.

3.7 Theorem. Given functors $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$ specifying an [adjunction](#) $(F \dashv G)$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \rightarrow GF$, $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ satisfying the commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \epsilon F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow 1_G & \downarrow G\epsilon \\ & & G \end{array}$$

called the **triangular identities**.

Proof. First suppose given $(F \dashv G)$. Define η and ϵ as in [Theorem 3.3](#) and its dual; now consider the composite

$$FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA.$$

Under the adjunction this corresponds to

$$A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$$

but this also corresponds to 1_{FA} , so $\epsilon_{FA} \cdot F\eta_A = 1_{FA}$. The other identity is [dual](#).

Conversely, suppose given η and ϵ satisfying the [triangular identities](#). Given $A \xrightarrow{f} GB$, let $\Phi(f)$ be the composite

$$FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B,$$

and given $FA \xrightarrow{g} B$, let $\Psi(g)$ be

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB.$$

Then Φ and Ψ are both natural; we need to show that $\Phi\Psi$ and $\Psi\Phi$ are identity mappings. But

$$\begin{aligned} \Psi\Phi \left(A \xrightarrow{f} GB \right) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f \end{aligned}$$

and dually $\Phi\Psi(g) = g$. □

3.8 Lemma. Suppose given

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

and [natural isomorphisms](#) $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. Then there are isomorphisms $\alpha' : 1_{\mathcal{C}} \rightarrow GF$, $\beta' : FG \rightarrow 1_{\mathcal{D}}$ which satisfy the [triangular identities](#), so $(F \dashv G)$ (and $(G \dashv F)$).

Proof. We define $\alpha' = \alpha$ and β' to be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}.$$

Note that $FG\beta = \beta FG$ since

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta FG & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$$

commutes by naturality of β and β is monic. Now $(\beta'_F)(F\alpha')$ is the composite

$$\begin{aligned} F &\xrightarrow{F\alpha} FGF \xrightarrow{(\beta FGF)^{-1}} FGF GF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\beta F} F \\ &= F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{FGF\alpha} FGF GF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\beta F} F \\ &= F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{\beta F} F = 1_F \end{aligned}$$

since $GF\alpha = \alpha_{GF}$. Similarly $(G\beta')(G'\alpha')$ is

$$\begin{aligned} G &\xrightarrow{\alpha G} GFG \xrightarrow{(GFG\beta)^{-1}} GFG FG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha GFG} GFG FG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{G\beta} G = 1_G. \end{aligned} \quad \square$$

3.9 Lemma. Suppose $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint F with counit $\epsilon : FG \rightarrow 1_{\mathcal{D}}$, then

- (i) G is faithful iff ϵ is pointwise epic.
- (ii) G is full and faithful iff ϵ is an isomorphism.

Proof.

- (i) Given $B \xrightarrow{g} B'$, Gg corresponds under the adjunction to the composite

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'.$$

Hence the mapping $g \mapsto Gg$ is injective on morphisms with domain B (and specified codomain) iff $g \mapsto g\epsilon_B$ is injective, i.e. iff ϵ_B is epic.

- (ii) Similarly, G is full and faithful iff $g \mapsto g\epsilon_B$ is bijective. If $\alpha : B \rightarrow FGB$ is such that $\alpha\epsilon_B = 1_{FGB}$, then $\epsilon_B\alpha\epsilon_B = \epsilon_B$, whence $\epsilon_B\alpha = 1_B$. So ϵ_B is an isomorphism for all B . \square

Lecture 9 **3.10 Definition** (Reflection). By a **reflection**, we mean an adjunction in which the right adjoint is full and faithful (equivalently, the counit is an isomorphism). We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is **reflective** if the inclusion $\mathcal{C}' \rightarrow \mathcal{C}$ has a left adjoint.

3.11 Examples.

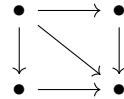
- (a) The category **AbGp** of abelian groups is reflective in **Gp**: the left adjoint sends a group G to its abelianization G/G' , where G' is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$, $x, y \in G$. (The unit of the adjunction is the quotient map $G \mapsto G/G'$.)
- (b) Given an abelian group A , let A_t denote the torsion subgroup, i.e. the subgroup of elements of finite order. The assignment $A \mapsto A/A_t$ gives a left adjoint to the inclusion **tfAbGp** \rightarrow **AbGp** where **tfAbGp** is the full subcategory of torsion-free abelian groups. And $A \mapsto A_t$ is right adjoint to the inclusion **tfAbGp** \rightarrow **AbGp** so this subcategory is coreflective.

- (c) Let $\mathbf{KHaus} \subseteq \mathbf{Top}$ be the full subcategory of compact Hausdorff spaces. The inclusion $\mathbf{KHaus} \rightarrow \mathbf{Top}$ has a left adjoint β , the Stone-Ćech compactification.
- (d) Let X be a topological space. We say $A \subseteq X$ is sequentially closed if $x_n \rightarrow x_\infty$ and $x_n \in A$ for all n implies $x_\infty \in A$. We say X is sequential if all sequentially closed sets are closed. Given a non-sequential space X , let X_s be the same set with topology given by the sequentially open sets in X ; the identity $X_s \rightarrow X$ is continuous, and defines the counit of an adjunction between the inclusion $\mathbf{Seq} \rightarrow \mathbf{Top}$ and the functor $X \mapsto X_s$.
- (e) If X is a topological space, the poset CX of closed subsets of X is reflective in PX , with reflector given by closure, and the poset OX of open subsets is coreflective, with coreflector given by interior.

4 Limits

4.1 Definition.

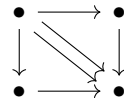
- (a) Let J be a **category** (almost always **small**, often finite). By a **diagram of shape J** in \mathcal{C} we mean a **functor** $D : J \rightarrow \mathcal{C}$. The objects $D(j)$, $j \in \text{ob } J$ are called **vertices** of the diagram, and the morphisms $D(\alpha)$, $\alpha \in \text{mor } J$ are called **edges** of D . For example, if J is the category



with 4 objects and 5 non-identity morphisms, a diagram of shape J is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

If J is

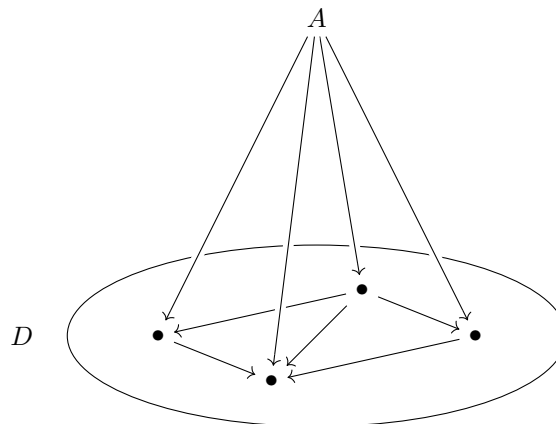


a diagram of shape J is a not-necessarily commutative square.

- (b) Given $D : J \rightarrow \mathcal{C}$, a **cone** over D consists of an object A of \mathcal{C} (the **apex** of the cone) together with morphisms $A \xrightarrow{\lambda_j} D(j)$ for each $j \in \text{ob } J$, such that

$$\begin{array}{ccc} & A & \\ \lambda_j \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array}$$

commutes for all $j \xrightarrow{\alpha} j'$ in $\text{mor } J$ (the λ_j are called the **legs** of the cone).



Given cones $(A, (\lambda_j)_{j \in \text{ob } J})$ and $(B, (\mu_j)_{j \in \text{ob } J})$, a morphism of cones between them is a morphism $A \xrightarrow{f} B$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \lambda_j \searrow & & \swarrow \mu_j \\ & D(j) & \end{array}$$

commutes for all j .

We write $\mathbf{Cone}(D)$ for the category of cones over D .

- (c) A **limit** for D is a **terminal object** of $\mathbf{Cone}(D)$, if this exists. Dually, we have the notion of **cone under a diagram**, and of **colimit** (**initial** cone under D).

Alternatively if \mathcal{C} is **locally small** and J is small, we have a **functor** $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ sending A to the set of **cones** with apex A . A **limit** for D is a **representation** of this functor.

If ΔA denotes the constant diagram of shape J with all vertices A and all edges 1_A , then a **cone** over D with **apex** A is the same thing as a **natural transformation** $\Delta A \rightarrow D$. Δ is a functor $\mathcal{C} \rightarrow [J, \mathcal{C}]$, and $\mathbf{Cone}(D)$ is the category $(\Delta \downarrow D)$ (a **comma category**, reversed). So to say that every **diagram of shape** J in \mathcal{C} has a **limit** is equivalent to saying that Δ has a **right adjoint**. (We say \mathcal{C} **has limits** of shape J). Dually, \mathcal{C} **has colimits** of shape J iff $\Delta : \mathcal{C} \rightarrow [J, \mathcal{C}]$ has a left adjoint.

4.2 Examples.

- (a) Suppose $J = \emptyset$. There's a unique **diagram of shape** J in \mathcal{C} ; a **cone** over it is just an object, and a morphism of cones is a morphism of \mathcal{C} . So a **limit** for this empty diagram is a **terminal object** of \mathcal{C} . (Dually, a **colimit** for it is an initial object).

Lecture 10

- (b) Let J be the category



A **diagram of shape** J is a pair of objects A, B ; a **cone** over it is a **span**

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ A & & B \end{array}$$

and a **limit** for it is a **product**

$$\begin{array}{ccc} & A \times B & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ A & & B \end{array}$$

Dually, a colimit for it is a **coproduct**

$$\begin{array}{ccc} A & & B \\ \nu_1 \searrow & & \swarrow \nu_2 \\ & A + B & \end{array}$$

- (c) More generally, if J is a [small](#) discrete category, a diagram of shape J is a J -indexed family $\{A_j \mid j \in J\}$, and a limit for it is a product

$$\left\{ \prod_{j \in J} A_j \xrightarrow{\pi_j} A_j \mid j \in J \right\}.$$

Dually, a colimit for it is a coproduct

$$\left\{ A_j \xrightarrow{\nu_j} \sum_{j \in J} A_j \mid j \in J \right\},$$

also written $\coprod_{j \in J} A_j$.

- (d) Let J be the category

$$\bullet \rightrightarrows \bullet$$

A diagram of shape J is a parallel pair

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

a cone over this is

$$\begin{array}{ccc} & C & \\ h \swarrow & & \searrow k \\ A & & B \end{array}$$

satisfying $fh = k = gh$, or equivalently a morphism $C \xrightarrow{h} A$ satisfying $fh = gh$. A (co)limit for the diagram is a [\(co\)equalizer](#).

- (e) Let J be the category

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

A diagram of shape J is a [cospan](#)

$$\begin{array}{ccc} & A & \\ & \downarrow f & \\ B & \xrightarrow{g} & C \end{array}$$

a cone over it is

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ \downarrow q & \searrow r & \\ B & & C \end{array}$$

satisfying $fp = r = gq$, or equivalently a span (p, q) completing the diagram to a commutative square. A limit for the diagram is called a **pullback** of (f, g) . In **Set**, the [apex](#) of the pullback is the ‘fibre product’

$$A \times_C B = \{ (x, y) \in A \times B \mid f(x) = g(y) \}$$

Dually, colimits of shape J^{op} are called **pushouts**, given

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

we ‘push g along f ’ to get the right hand side of the colimit square.

(f) Let J be the poset of natural numbers. A diagram of shape J is a **direct system**

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

A colimit for this is called a **direct limit**: it consists of A_∞ equipped with morphisms $A_n \xrightarrow{g_n} A_\infty$ satisfying $g_n = g_{n+1}f_n$ for all n , and universal among such. Dually, we have **inverse system** and **inverse limit**.

4.3 Theorem.

- (i) Suppose \mathcal{C} has **equalizers** and all finite (resp. **small**) **products**. Then \mathcal{C} has all finite (resp. small) **limits**.
- (ii) Suppose \mathcal{C} has **pullbacks** and a **terminal object**. Then \mathcal{C} has all finite limits.

Proof.

- (i) Suppose given $D : J \rightarrow \mathcal{C}$. Form the **products**

$$P = \prod_{j \in \text{ob } J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha).$$

We have morphisms $P \xrightleftharpoons[g]{f} Q$ defined by $\pi_\alpha f = \pi_{\text{cod } \alpha}$, $\pi_\alpha g = D(\alpha)\pi_{\text{dom } \alpha}$ for all α .

Let $E \xrightarrow{e} P$ be an **equalizer** of (f, g) . The composites $\lambda_j = \pi_j e : E \rightarrow D(j)$ form a **cone** over D : given $\alpha : j \rightarrow j'$ in J ,

$$D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e = \lambda_{j'}.$$

Given any cone $(A, \{\mu_j \mid j \in \text{ob } J\})$ over D , there’s a unique $\mu : A \rightarrow P$ with $\pi_j \mu = \mu_j$ for each j , and

$$\pi_\alpha f \mu = \mu_{\text{cod } \alpha} = D(\alpha)\mu_{\text{dom } \alpha} = \pi_\alpha g \mu$$

for all α , and hence $f\mu = g\mu$, so $\exists! \nu : A \rightarrow E$ with $e\nu = \mu$. So $(E, \{\lambda_j \mid j \in \text{ob } J\})$ is a **limit cone**.

- (ii) It’s enough to construct finite products and equalizers. But if 1 is the **terminal object**, then a pullback for

$$\begin{array}{ccc} & A & \\ & \downarrow & \\ B & \longrightarrow & 1 \end{array}$$

has the universal property of a product $A \times B$, and we can form $\prod_{i=1}^n A_i$ inductively as $A_1 \times (A_2 \times (A_3 \times \cdots (A_{n-1} \times A_n) \cdots))$.

Now, to form the equalizer of $A \xrightleftharpoons[f]{g} B$, consider the **cospan**

$$\begin{array}{ccc} & A & \\ & \downarrow (1_A, f) & \\ A & \xrightarrow{(1_A, g)} & A \times B. \end{array}$$

A cone over this consists of

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \\ A & & \end{array}$$

satisfying $(1_A, f)h = (1_A, g)k$ or equivalently, $1_A h = 1_A k$ and $fh = gk$, or equivalently a morphism $P \xrightarrow{h} A$ satisfying $fh = gh$. So a pullback for $(1_A, f)$ and $(1_A, g)$ is an equalizer of (f, g) . \square

Definition (Complete). We say a **category** \mathcal{C} is **complete** if it has all **small limits**. (Dually, **cocomplete** = all small colimits).

Set is **complete** and cocomplete: **products** are cartesian products, **coproducts** are disjoint unions. Similarly, **Gp**, **AbGp**, **Rng**, **Mod_R**, ... are all complete and cocomplete. **Top** is also complete and cocomplete.

4.4 Definition. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a **functor**.

- (a) We say F **preserves limits** of **shape** J if, given $D : J \rightarrow \mathcal{C}$ and a **limit cone** $(L, \{\lambda_j \mid j \in \text{ob } J\})$ in \mathcal{C} , $(FL, \{F\lambda_j \mid j \in \text{ob } J\})$ is a limit for FD .
- (b) We say F **reflects limits** of shape J if, given $D : J \rightarrow \mathcal{C}$ and a cone $(L, (\lambda_j)_j)$ such that $(FL, (F\lambda_j)_j)$ is a limit for FD , then $(L, (\lambda_j)_j)$ is a limit for D .
- (c) We say F **creates limits** of shape J if, given $D : J \rightarrow \mathcal{C}$ and a limit $(M, (\mu_j)_j)$ for FD , there exists a cone $(L, (\lambda_j)_j)$ over D whose image under F is isomorphic to the limit cone, and any such that cone is a limit in \mathcal{C} .

Lecture 11 **4.5 Remark.**

- (a) If \mathcal{C} has **limits of shape** J , $F : \mathcal{C} \rightarrow \mathcal{D}$ **preserves** them and F reflects **isomorphisms**, then F **reflects** limits of shape J .
- (b) F reflects limits of shape **1** $\iff F$ reflects isomorphisms.
- (c) If \mathcal{D} has limits of shape J and $F : \mathcal{C} \rightarrow \mathcal{D}$ **creates them**, then F both preserves and reflects them.
- (d) In any of the statements of **Theorem 4.3**, we may replace both instances of ‘ \mathcal{C} has’ by either ‘ \mathcal{C} has and $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves’ or ‘ \mathcal{D} has and $F : \mathcal{C} \rightarrow \mathcal{D}$ creates’.

4.6 Examples.

- (a) $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ (forgetful) creates all small limits: Given a family $\{G_i \mid i \in I\}$ of groups, there's a unique group structure on $\prod_{i \in I} UG_i$ making the projections π_i into homomorphisms and this makes it into a product in \mathbf{Gp} . Similarly for equalizers.
But U doesn't preserve coproducts: $U(G * H) \not\cong UG \sqcup UH$.
- (b) $U : \mathbf{Top} \rightarrow \mathbf{Set}$ (forgetful) preserves all small limits and colimits, but doesn't reflect them: if L is a limit for $D : J \rightarrow \mathbf{Top}$, and L is not discrete, there's another cone with apex L_d mapping to the limit in \mathbf{Set} .
- (c) The inclusion functor $I : \mathbf{AbGp} \rightarrow \mathbf{Gp}$ reflects coproducts but doesn't preserve them. The direct sum $A \oplus B$ (coproduct in \mathbf{AbGp}) is not normally isomorphic to the free product $A * B$; $A * B$ is not abelian unless either A or B is $\{e\}$, but if $A \cong \{e\}$ then $A * B \cong A \oplus B \cong B$.

4.7 Lemma. If \mathcal{D} has limits of shape J , then so does the functor category $[\mathcal{C}, \mathcal{D}]$ for any \mathcal{C} , and the forgetful functor $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$ creates them.

Proof. Suppose we are given a diagram of shape J in $[\mathcal{C}, \mathcal{D}]$; think of it as a functor $D : J \times \mathcal{C} \rightarrow \mathcal{D}$. For each $A \in \text{ob } \mathcal{C}$, let $(LA, \{\lambda_{j,A} \mid j \in \text{ob } J\})$ be a limit cone for the diagram $D(-, A) : J \rightarrow \mathcal{D}$. Given $A \xrightarrow{f} B$ in \mathcal{C} , the composites

$$A \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, since the squares

$$\begin{array}{ccc} D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \\ \downarrow D(\alpha, A) & & \downarrow D(\alpha, B) \\ D(j', A) & \xrightarrow{D(j',f)} & D(j', B) \end{array}$$

commute. So there's a unique $Lf : LA \rightarrow LB$ making

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_{j,A}} & D(j, A) \\ \downarrow Lf & & \downarrow D(j,f) \\ LB & \xrightarrow{\lambda_{j,B}} & D(j, B) \end{array}$$

commute for all j .

Uniqueness implies functoriality: given $g : B \rightarrow C$, $L(gf)$ and $(Lg)(Lf)$ are factorizations of the same cone through the limit LC . And this is the unique functor structure on $(A \mapsto LA)$ making the $\lambda_{j,-}$ into natural transformations.

The cone $(L, \{\lambda_{j,-} \mid j \in \text{ob } J\})$ is a limit: suppose given another cone $(M, \{\mu_{j,-} \mid j \in \text{ob } J\})$, then for each A , $(MA, \{\mu_{j,A} \mid j \in \text{ob } J\})$ is a cone over $D(-, A)$, so induces a unique $\alpha_A : MA \rightarrow LA$. Naturality of α follows from uniqueness of factorisations through a limit. So $(M, (\mu_j))$ factors uniquely through $(L, (\lambda_j))$. \square

4.8 Remark. In any category, a morphism $A \xrightarrow{f} B$ is **monic** iff

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a **pullback**. Hence any **functor** which preserves pullbacks preserves monomorphisms. In particular, if \mathcal{D} has pullbacks, then monomorphisms in $[\mathcal{C}, \mathcal{D}]$ are just pointwise monomorphisms.

4.9 Theorem. Suppose $G : \mathcal{D} \rightarrow \mathcal{C}$ has a **left adjoint** F . Then G **preserves** all **limits** which exist in \mathcal{D} .

Proof 1. Suppose \mathcal{C} and \mathcal{D} both have **limits of shape** J . We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

and all the **functors** in it have **right adjoints** (in particular $[J, F] \dashv [J, G]$). So by **Corollary 3.6**, the diagram of right adjoints

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ \lim_J \uparrow & & \lim_J \uparrow \\ [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \end{array}$$

commutes up to **isomorphism**, i.e. G **preserves limits** of shape J . □

Proof 2. Suppose given $D : J \rightarrow \mathcal{D}$ and a **limit cone** $(L, \{L \xrightarrow{\lambda_j} D(j) \mid j \in \text{ob } J\})$. Given a cone $(A, \{A \xrightarrow{\alpha_j} GD(j) \mid j \in \text{ob } J\})$ over GD the morphisms $FA \xrightarrow{\hat{\alpha}_j} D(j)$ form a cone over D , so they induce a unique $FA \xrightarrow{\hat{\beta}} L$ such that $\lambda_j \hat{\beta} = \hat{\alpha}_j$ for all j . Then $A \xrightarrow{\beta} GL$ is the unique morphism satisfying $(G\lambda_j)\beta = \alpha_j$ for all j . So $(GL, \{G\lambda_j \mid j \in \text{ob } J\})$ is a limit cone in \mathcal{C} . □

Lecture 12

The ‘Primeval’ Adjoint Functor Theorem says that the converse of **Theorem 4.9** is true: if \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ **preserves all** limits, then G has a **left adjoint**.

4.10 Lemma. Suppose \mathcal{D} **has** and $G : \mathcal{D} \rightarrow \mathcal{C}$ **preserves limits** of shape J . Then for any $A \in \text{ob } \mathcal{C}$, the **arrow category** $(A \downarrow G)$ has **limits** of shape J , and the **forgetful functor** $U : (A \downarrow G) \rightarrow \mathcal{D}$ **creates them**.

Proof. Suppose given $D : J \rightarrow (A \downarrow G)$, write $D(j)$ as $(UD(j), f_j)$. Let $(L, (\lambda_j : L \rightarrow UD(j))_{j \in \text{ob } J})$ be a **limit** for UD ; then $(GL, (G\lambda_j)_{j \in \text{ob } J})$ is a limit for GUD .

Since the edges of UD are morphisms in $(A \downarrow G)$, the f_j form a **cone** over GUD . So there’s a unique $h : A \rightarrow GL$ such that $(G\lambda_j)h = f_j$ for all j , i.e. there’s a unique h such that the λ_j are all morphisms $(L, h) \rightarrow (UD(j), f_j)$ in $(A \downarrow G)$.

If $((C, k)(\mu_j)_{j \in \text{ob } J})$ is any cone over D , then $(C, (\mu_j)_{j \in \text{ob } J})$ is a cone over UD , so there's a unique $l : C \rightarrow L$ with $\lambda_j l = \mu_j$ for all j . We need to show $(Gl)k = h$: but $(G\lambda_j)(Gl)k = (G\mu_j)k = f_j = (G\lambda_j)h$ for all j so $(Gl)k = h$ by uniqueness of factorizations through limits. \square

4.11 Lemma. A category \mathcal{C} has an **initial object** iff $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$, regarded as a **diagram** of shape \mathcal{C} in \mathcal{C} , has a **limit**.

Proof. Suppose \mathcal{C} has an **initial object** I . Then the unique morphisms $\{I \rightarrow A \mid A \in \text{ob } \mathcal{C}\}$ form a **cone** over $1_{\mathcal{C}}$; and given any cone $\{C \xrightarrow{\lambda_A} A \mid A \in \text{ob } \mathcal{C}\}$, then for any A the triangle

$$\begin{array}{ccc} C & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow \\ & & A \end{array}$$

commutes, so λ_I is the unique factorization of $\{\lambda_A \mid A \in \text{ob } \mathcal{C}\}$ through the cone

$$\{I \rightarrow A \mid A \in \text{ob } \mathcal{C}\}.$$

Conversely, suppose $(I, \{\lambda_A : I \rightarrow A \mid A \in \text{ob } \mathcal{C}\})$ is a **limit**. Then, for any $I \xrightarrow{f} A$, the diagram

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow f \\ & & A \end{array}$$

commutes. In particular, putting $f = \lambda_A$, we see that λ_I is a factorization of the limit cone through itself, so $\lambda_I = 1_I$. Hence every $f : I \rightarrow A$ satisfies $f = \lambda_A$. \square

The **primeval Adjoint Functor Theorem** now follows immediately from **Lemma 4.10**, **Lemma 4.11** and **Theorem 3.3**. However, it only applies to **functors** between preorders (cf question 6 on example sheet 2).

4.12 Theorem (General Adjoint Functor Theorem). Suppose that \mathcal{D} is **locally small** and **complete**. Then $G : \mathcal{D} \rightarrow \mathcal{C}$ has a **left adjoint** if and only if G satisfies the **solution set condition**: the solution set condition says that G preserves all small limits and, for each $A \in \text{ob } \mathcal{C}$ there exists a set of morphisms $\{A \xrightarrow{f_i} GB_i \mid i \in I\}$ such that every $A \xrightarrow{h} GC$ factors as $A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$ for some i and some $g : B_i \rightarrow C$.

Proof. (\Rightarrow) . If $F \dashv G$, G preserves limits by **Theorem 4.9**, and $\{A \xrightarrow{\eta_A} GFA\}$ is a singleton **solution set**, by **Theorem 3.3**.

(\Leftarrow) . By **Lemma 4.10**, $(A \downarrow G)$ is **complete**, and it inherits **local smallness** from \mathcal{D} . So we need to show: if $\mathcal{A} := (A \downarrow G)$ is complete and locally small, and has a weakly initial set of objects $\{B_i \mid i \in I\}$, then \mathcal{A} has an initial object. (An object is **weakly initial** if it has a (not necessarily unique) morphism to any other object.)

First form $P = \prod_{i \in I} B_i$, then P is **weakly initial**. Now form the limit of

$$P \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \vdots \\ \xrightarrow{\quad} \end{array} P \quad (*)$$

where edges are all the endomorphisms of P : denote the limit $I \xrightarrow{i} P$. I is also **weakly initial** in \mathcal{A} : suppose given $I \xrightarrow[f]{f} C$. Form the equalizer $E \xrightarrow{e} I$ of (f, g) , then there exists $P \xrightarrow{h} E$ since P is weakly initial. $ieh : P \rightarrow P$, and 1_P are edges of the diagram $(*)$ so $i = iehi$. But i is monic, so $ehi = 1_I$, so e is split epic, so $f = g$. Hence I is initial. \square

4.13 Examples.

- (a) Consider the **forgetful functor** $U : \mathbf{Gp} \rightarrow \mathbf{Set}$. By **Examples 4.6(a)**, U creates all small limits, so \mathbf{Gp} has them and U preserves them. \mathbf{Gp} is **locally small**, given a set A , any $f : A \rightarrow UG$ factors as $A \rightarrow UG' \rightarrow UG$, where G' is the subgroup generated by $\{f(x) \mid x \in A\}$ and $\text{card } G' \leq \max\{\aleph_0, \text{card } A\}$.

Let B be a set of this cardinality; consider all subsets $B' \subseteq B$, all group structures on B' , and all mappings $A \rightarrow B'$. These give us a **solution set** at A .

- (b) Consider the category \mathbf{CLat} of complete lattices (posets with all meets and joins). Again $U : \mathbf{CLat} \rightarrow \mathbf{Set}$ creates all small limits. But A. W. Hales (1964) showed that, for any cardinal κ , there exist complete lattices of $\text{card} \geq \kappa$ generated by three elements, so the **solution set condition** fails at $A = \{x, y, z\}$, and U doesn't have a left adjoint.

Lecture 13 **4.14 Definition** (Subobject). By a **subobject** of an object A of \mathcal{C} , we mean a **monomorphism** $A' \rightarrowtail A$. Dually, we have quotient objects. The subobjects of A are preordered by $A'' \leq A'$ if there exists a factorization

$$\begin{array}{ccc} A'' & \xrightarrow{\quad} & A' \\ & \searrow & \swarrow \\ & A. & \end{array}$$

We say \mathcal{C} is **well-powered** if each $A \in \text{ob } \mathcal{C}$ has a set of subobjects $\{A_i \rightarrowtail A \mid i \in I\}$ such that every subobject of A is isomorphic to some A_i (e.g. in \mathbf{Set} we can take the inclusions $\{A' \hookrightarrow A \mid A' \in \mathcal{P}A\}$).

If C^{op} is well-powered, we say \mathcal{C} is **well-copowered** (not cowell-powered).

4.15 Lemma. Suppose given a **pullback** square

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

with f **monic**. Then k is monic.

Proof. Suppose $D \xrightarrow[x]{x} P$ satisfy $kx = ky$. Then $fhx = gkx = gky = fhy$, but f is **monic** so $hx = hy$. So x and y are factorizations of the same **cone** through the **limit** cone (h, k) . \square

4.16 Theorem (Special adjoint functor theorem). Suppose \mathcal{C} and \mathcal{D} are both **locally small**, and that \mathcal{D} is **complete** and **well-powered** and has a **coseparating** set. Then a **functor** $G : \mathcal{D} \rightarrow \mathcal{C}$ has a **left adjoint** iff it **preserves all small limits**.

Proof. (\Rightarrow) by **Theorem 4.9**.

(\Leftarrow) . For any $A \in \text{ob } \mathcal{C}$, $(A \downarrow G)$ is **complete** by **Lemma 4.10**, **locally small**, and **well-powered** since the **subobjects** of (B, f) in $(A \downarrow G)$ are just those **subobjects** $B' \rightarrowtail B$ in \mathcal{D} for which f factors through $GB' \rightarrowtail GB$.

Also, if $\{S_i \mid i \in I\}$ is a **coseparating** set for \mathcal{D} , then the set

$$\{(s_i, f) \mid i \in I, f \in \mathcal{C}(A, GS_i)\}$$

is coseparating in $(A \downarrow G)$: given $(B, f) \xrightarrow[h]{g} (B', f')$ in $(A \downarrow G)$ with $g \neq h$, there exists $k : B' \rightarrow S_i$ for some i with $kg \neq kh$, and then k is also a morphism $(B', f') \rightarrow (S_i, (Gk)f')$ in $(A \downarrow G)$.

So we need to show that if \mathcal{A} is complete, locally small and well powered, and has a coseparating set $\{S_i \mid i \in I\}$ then \mathcal{A} has an initial object. Form the product $P = \prod_{i \in I} S_i$. Now consider the diagram

$$\begin{array}{ccc} & & P_i \\ & \nearrow & \downarrow \\ & P_j & \\ P' & \longrightarrow & P \end{array}$$

whose edges are a representative set of subobjects of P , and form its **limit**

$$\begin{array}{ccc} I & \longrightarrow & P_i \\ & \searrow & \downarrow \\ & P_j & \\ I & \longrightarrow & P \\ \downarrow & & \downarrow \\ P' & \longrightarrow & P \end{array}$$

By the argument of **Lemma 4.15**, the legs of this cone are all **monic**; in particular $I \rightarrowtail P$ is monic, and it's a **least** subobject of P . Hence I has no proper subobjects, so, given $I \xrightarrow[f]{f} A$, their **equalizer** is an **isomorphism** and hence $f = g$.

Now let A be any object of \mathcal{A} ; form the **product**

$$Q = \prod_{\substack{i \in I \\ f \in \mathcal{A}(A, S_i)}} S_i.$$

There's an obvious $h : A \rightarrow Q$ defined by $\pi_{i,f}h = f$; and h is monic, since the S_i are a coseparating set. We also have a morphism $k : P \rightarrow Q$ defined by $\pi_{i,f}k = \pi_i$.

Now form the [pullback](#)

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow h \\ P & \xrightarrow{k} & Q \end{array}$$

by [Lemma 4.15](#), $B \rightarrow P$ is monic, so B is a subobject of P . Hence there exists

$$\begin{array}{ccc} I & \longrightarrow & B \\ & \searrow & \swarrow \\ & P & \end{array}$$

and hence a morphism $I \rightarrow B \rightarrow A$. □

4.17 Examples. Consider the inclusion $\mathbf{KHaus} \xrightarrow{I} \mathbf{Top}$, where \mathbf{KHaus} is the [full subcategory](#) of compact Hausdorff spaces. \mathbf{KHaus} [has](#) and I [preserves small products](#) (by Tychonoff's Theorem) and [equalizers](#) (since equalizers of pairs $X \xrightarrow[f]{g} Y$ with Y Hausdorff are closed subspaces). Both categories are [locally small](#) and \mathbf{KHaus} is [well-powered](#) (subobjects of X are all isomorphic to closed subspaces). The closed interval $[0, 1]$ is a [coseparator](#) in \mathbf{KHaus} by Urysohn's Lemma. So by [Theorem 4.16](#), I has a [left adjoint](#) β .

4.18 Remark.

- (a) Čech's construction of β : given X form $P = \prod_{f: X \rightarrow [0,1]} [0,1]$ and define $h : X \rightarrow P$ by $\pi_f h = f$. Define βX to be the closure of the image of h .
Čech's proof that this works is essentially the same as [Theorem 4.16](#).
- (b) We could have used [General Adjoint Functor Theorem](#) to construct β : we get a solution set at X by considering all continuous $X \xrightarrow{f} Y$ with Y compact Hausdorff, and $\text{im } f$ dense in Y and such Y have cardinality $\leq 2^{2^{\text{card } X}}$.

5 Monads

Lecture 14 Suppose given

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

with $(F \dashv G)$. How much of this structure can we describe without mentioning \mathcal{D} ?

We have the **functor** $T = GF : \mathcal{C} \rightarrow \mathcal{C}$, and the **unit** $\eta : 1_{\mathcal{C}} \rightarrow T = GF$ and the **natural transformation**

$$\mu = G\epsilon_F : TT = GF GF \rightarrow GF = T.$$

These satisfy the commutative diagrams

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta_T} & T \\ & \searrow (1) & \downarrow \mu & \swarrow (2) & \\ & 1_T & T & 1_T & \end{array}$$

by **triangular identities** and

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \downarrow \mu_T & (3) & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

by naturality of ϵ .

5.1 Definition (Monad). A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a **category** \mathcal{C} consists of a **functor** $T : \mathcal{C} \rightarrow \mathcal{C}$ and **natural transformation** $\eta : 1_{\mathcal{C}} \rightarrow T$, $\mu : TT \rightarrow T$ satisfying equations (1)-(3). η and μ are called the **unit** and **multiplication** of \mathbb{T} .

5.2 Examples.

- (a) Any **adjunction** $(F \dashv G)$ induces both a **monad** $(GF, \eta, G\epsilon_F)$ on \mathcal{C} and a **comonad** $(FG, \epsilon, F\eta_G)$ on \mathcal{D} .
- (b) Let M be a **monoid**. The functor $(M \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$ has a monad structure with **unit** given by $\eta_A(a) = (1_M, a)$ and **multiplication** $\mu_A(m, m', a) = (mm', a)$. The monad identities follow from the monoid ones.
- (c) Let \mathcal{C} be any **category** with finite **products**, $A \in \text{ob } \mathcal{C}$. The **functor** $(A \times -) : \mathcal{C} \rightarrow \mathcal{C}$ has a comonad structure with counit $\epsilon_B : A \times B \rightarrow B$ given by π_2 and comultiplication $\delta_B : A \times B \rightarrow A \times A \times B$ given by (π_1, π_1, π_2) .

Does every **monad** arise from an **adjunction**?

In **Examples 5.2(b)** we have the **category** $[M, \mathbf{Set}]$. Its **forgetful functor** to \mathbf{Set} has a left **adjoint**, sending A to $M \times A$ with M acting by multiplication on the left factor. This adjunction gives rise to the **monad** of **Examples 5.2(b)**.

5.3 Definition (Eilenberg-Moore category). Let \mathbb{T} be a [monad](#) on \mathcal{C} . A \mathbb{T} -**algebra** is a pair (A, α) with $A \in \text{ob } \mathcal{C}$ and $TA \xrightarrow{\alpha} A$ satisfying the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array} \quad (4)$$

$$\begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow \mu_A & (5) & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A. \end{array}$$

A **homomorphism** $f : (A, \alpha) \rightarrow (B, \beta)$ is a morphism $A \xrightarrow{f} B$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & (6) & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes. The category of \mathbb{T} -algebras is denoted $\mathcal{C}^{\mathbb{T}}$, and called the **Eilenberg-Moore category**.

5.4 Lemma. The [forgetful functor](#) $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ has a [left adjoint](#) $F^{\mathbb{T}}$ and the adjunction induces \mathbb{T} .

Proof. We define $F^{\mathbb{T}}A = (TA, \mu_A)$ (an [algebra](#) by (2) and (3)) and $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$ (a [homomorphism](#) by [naturality](#) of μ).

Clearly $G^{\mathbb{T}}F^{\mathbb{T}} = T$, the [unit](#) of the [adjunction](#) is η . We define the [counit](#)

$$\epsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \rightarrow (A, \alpha)$$

(a homomorphism by (5)) ϵ is natural by (6); for the triangular identities, $\epsilon_{FA}(F\eta_A) = 1_{FA}$ is (1), $G\epsilon_{(A, \epsilon)}\eta_A = 1_A$ is (4).

The [monad](#) induced by $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ has [functor](#) T and unit η , and $G^{\mathbb{T}}\epsilon_{F^{\mathbb{T}}A} = \mu_A$ by definition of $F^{\mathbb{T}}A$. \square

Kleisli took a ‘minimalist’ approach: if

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

induces \mathbb{T} , then so does

$$\mathcal{C} \xrightleftharpoons[G|_{\mathcal{D}'}]{F} \mathcal{D}'$$

where \mathcal{D}' is the [full subcategory](#) of \mathcal{D} on objects FA .

So in trying to construct \mathcal{D} , we may assume F is surjective (or indeed bijective) on objects. But then morphisms $FA \rightarrow FB$ correspond bijectively to morphisms $A \rightarrow GFB = TB$ in \mathcal{C} .

5.5 Definition (Kleisli category). Given a **monad** \mathbb{T} on \mathcal{C} , the **Kleisli category** $\mathcal{C}_{\mathbb{T}}$ has $\text{ob } \mathcal{C}_{\mathbb{T}} = \text{ob } \mathcal{C}$, and morphisms $A \xrightarrow{\quad} B$ are morphisms $A \rightarrow TB$ in \mathcal{C} . The composite $A \xrightarrow{f} B \xrightarrow{g} C$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$$

and the identity $A \xrightarrow{\quad} A$ is $A \xrightarrow{\eta_A} TA$.

To verify associativity, suppose given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$. Then

$$\begin{array}{ccccccc} A & \xrightarrow{f} & TB & \xrightarrow{Tg} & TTC & \xrightarrow{TTh} & TTTD \xrightarrow{T\mu_D} TTD \\ & & & & \downarrow \mu_C & & \downarrow \mu_{TD} \quad (3) \\ & & & & TC & \xrightarrow{Th} & TTD \xrightarrow{\mu_D} TD \end{array}$$

commutes by naturality: the upper way round is $(hg)f$ and the lower is $h(gf)$.

The unit laws for the **category** similarly follow from

$$\begin{array}{ccc} A & \xrightarrow{f} & TB \\ & \searrow 1_{TB} & \downarrow \mu_B \\ & & TB \end{array} \quad (1)$$

and

$$\begin{array}{ccc} A & \xrightarrow{f} & TB \\ \downarrow \eta_A & & \downarrow \eta_{TB} \\ TA & \xrightarrow{Tf} & TTB \xrightarrow{\mu_B} TB \end{array} \quad (2)$$

Lecture 15 **5.6 Lemma.** There exists an **adjunction**

$$\mathcal{C} \xrightleftharpoons[G_{\mathbb{T}}]{F_{\mathbb{T}}} \mathcal{C}_{\mathbb{T}}$$

inducing the **monad** \mathbb{T} .

Proof. We define $F_{\mathbb{T}}A = A$,

$$F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB.$$

$F_{\mathbb{T}}$ preserves identities by definition; for composites, consider $A \xrightarrow{f} B \xrightarrow{g} C$. We get

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\eta_B} & TB \\ & & \downarrow g & & \downarrow Tg \\ & & C & \xrightarrow{\eta_C} & TC \end{array} \quad \begin{array}{ccc} & & \downarrow T\eta_C \\ & & TTC \\ & \searrow 1_{TC} & \downarrow \mu_C \\ & & TC \end{array} \quad (1)$$

We define $G_{\mathbb{T}}A = TA$,

$$G_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB.$$

$G_{\mathbb{T}}$ preserves identities by (1); for composites, consider $A \xrightarrow{f} B \xrightarrow{g} C$. We get

$$\begin{array}{ccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{TTg} & TTTC & \xrightarrow{T\mu_C} & TTC \\ & & \downarrow \mu_B & & \downarrow \mu_{TC} & (3) & \downarrow \mu_C \\ & & TB & \xrightarrow{Tg} & TTC & \xrightarrow{\mu_C} & TC \end{array}$$

We have

$$\begin{aligned} G_{\mathbb{T}}F_{\mathbb{T}}A &= TA \\ G_{\mathbb{T}}F_{\mathbb{T}}f &= \mu_B(T\eta_B)Tf = Tf \end{aligned}$$

so we take $\eta : 1_{\mathbb{C}} \rightarrow T$ as the **unit** of $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$. The counit $TA \xrightarrow{\epsilon_A} A$ is 1_{TA} . To verify **naturality**, consider the square

$$\begin{array}{ccc} TA & \xrightarrow{F_{\mathbb{T}}G_{\mathbb{T}}f} & TB \\ \downarrow \epsilon_A & & \downarrow \epsilon_B \\ A & \xrightarrow{f} & B \end{array}$$

This expands to

$$\begin{array}{ccccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{\mu_B} & TB & \xrightarrow{\eta_{TB}} & TTB \\ & & & & \searrow 1_{TB} & (2) & \downarrow \mu_B \\ & & & & & & TB \end{array}$$

so ϵ is natural.

$$G_{\mathbb{T}}(TA \xrightarrow{\epsilon_A} A) = \mu_A, \text{ so } G_{\mathbb{T}}(\epsilon_A)\eta_{G_{\mathbb{T}}A} = \mu_A \cdot \eta_{TA} = 1_{TA}$$

and $(\epsilon_{F_{\mathbb{T}}A})(F_{\mathbb{T}}\eta_A)$ is

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & TA & \xrightarrow{\eta_{TA}} & TTA \\ & & \searrow 1_{TA} & (1) & \downarrow \mu_A \\ & & & & TA \end{array}$$

which is $1_{F_{\mathbb{T}}A}$. Also $G_{\mathbb{T}}(\epsilon_{F_{\mathbb{T}}A}) = \mu_A$, so $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ induces \mathbb{T} . □

5.7 Theorem. Given a **monad** \mathbb{T} on \mathcal{C} , let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are the **adjunctions** $\left(\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}\right)$ inducing \mathbb{T} , and whose morphisms

$$\left(\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}\right) \rightarrow \left(\mathcal{C} \xrightleftharpoons[F']{F'} \mathcal{D}'\right)$$

are functors $H : \mathcal{D} \rightarrow \mathcal{D}'$ satisfying $HF = F'$ and $G'H = G$. Then the **Kleisli adjunction** is an **initial** object of $\mathbf{Adj}(\mathbb{T})$, and the **Eilenberg-Moore** adjunction is **terminal**.

Proof. Let $\left(\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}\right)$ be an object of $\mathbf{Adj}(\mathbb{T})$. We define $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ (the **Eilenberg-Moore comparison functor**) by $KB = (GB, G\epsilon_B)$ where ϵ is the counit of $(F \dashv G)$; note this is an algebra by one of the **triangular identities** for $(F \dashv G)$ and **naturality** of ϵ and $K(B \xrightarrow{g} B') = Gg$ (a homomorphism by naturality of ϵ). Clearly $G^{\mathbb{T}}K = G$ and

$$\begin{aligned} KFA &= (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A, \\ KF(A \xrightarrow{f} A') &= Tf = F^{\mathbb{T}}f. \end{aligned}$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Suppose $K' : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ is another such; then since $G^{\mathbb{T}}K' = G$, we know $K'B = (GB, \beta_B)$ where β is a natural transformation $GFG \rightarrow G$. Also, since $K'F = F^{\mathbb{T}}$, we have $\beta_{FA} = \mu_A = G\epsilon_{FA}$.

Now, given any $B \in \text{ob } \mathcal{D}$, consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GFG\epsilon_B} & GFGB \\ \downarrow G\epsilon_{FGB} \quad \downarrow \beta_{FGB} & & \downarrow G\epsilon_B \quad \downarrow \beta_B \\ GFGB & \xrightarrow{G\epsilon_B} & GB \end{array}$$

Both squares commute so $G\epsilon_B$ and β_B have the same composite with $GFG\epsilon_B$. But this is **split epic**, with splitting $GF\eta_{GB}$, so $\beta = G\epsilon$. Hence $K' = K$.

We now define the **Kleisli comparison functor** $L : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ by $LA = FA$,

$$L(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB.$$

L preserves identities by one of the **triangular identities** for $(F \dashv G)$; given $A \xrightarrow{f} B \xrightarrow{g} C$, we have

$$\begin{array}{ccccccc} FA & \xrightarrow{Ff} & FGFB & \xrightarrow{FGFg} & FGFGFC & \xrightarrow{FG\epsilon_{FC}} & FGFC \\ & & \downarrow \epsilon_{FB} & & \downarrow \epsilon_{FGFC} & & \downarrow \epsilon_{FC} \\ & & FB & \xrightarrow{Fg} & FGFC & \xrightarrow{\epsilon_{FC}} & FC \end{array}$$

$$GLA = TA = G_{\mathbb{T}}A,$$

$$GL(A \xrightarrow{f} B) = (G\epsilon_{FB})(GFf) = \mu_B(Tf).$$

$$LF_{\mathbb{T}}A = FA,$$

$$LF_{\mathbb{T}}(A \xrightarrow{f} B) = (\epsilon_{FB})(F\eta_B)(Ff) = Ff = G_{\mathbb{T}}f.$$

Note that L is full and faithful; its effect on morphisms (with given domain and codomain) is that of transposition across $(F \dashv G)$. Suppose $L' : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ is a morphism of $\mathbf{Adj}(\mathbb{T})$. We must have $L'A = FA$, and L' maps the counit $TA \rightarrow A$ to the counit $FGFA \xrightarrow{\epsilon_{FA}} FA$. For any $A \xrightarrow{f} B$, we have $f = 1_{TA}(F_{\mathbb{T}}f)$ so $L'(f) = \epsilon_{FA} \cdot (Ff) = Lf$. \square

Lecture 16 If \mathcal{C} has **coproducts**, then so does $\mathcal{C}_{\mathbb{T}}$, since $F_{\mathbb{T}}$ preserves them. But in general, it has few other **limits** or colimits. In contrast, we have

5.8 Theorem.

- (i) The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ creates all limits which exist in \mathcal{C} .
- (ii) If \mathcal{C} has colimits of shape J , then $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ creates them $\iff T$ preserves them.

Proof.

- (i) Suppose given $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$; write $D(j) = (GD(j), \delta_j)$ and suppose

$$(L, \{\mu_j : L \rightarrow GD(j) \mid j \in \text{ob } J\})$$

is a limit cone for GD . Then the composites

$$TL \xrightarrow{T\mu_j} TGD(j) \xrightarrow{\delta_j} GD(j)$$

form a cone over GD , since the edges of GD are homomorphisms, so they induce a unique $\lambda : TL \rightarrow L$ such that $\mu_j \lambda = \delta_j(T\mu)$ for all j . The fact that λ is a \mathbb{T} -algebra structure on L follows from the fact that the δ_j are algebra structures and uniqueness of factorisations through limits.

So $((L, \lambda), \{\mu_j \mid j \in \text{ob } J\})$ is the unique lifting of the limit cone over GD to a cone over D ; and it's a limit, since given a cone over D with apex (A, α) , we get a unique factorisation $A \xrightarrow{f} L$ in \mathcal{C} and F is an algebra homomorphism by uniqueness of factorisations through L .

- (ii) (\Rightarrow) $F : \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{T}}$ preserves colimits since it's a left adjoint, so $T = GF$ preserves colimits of shape J .

(\Leftarrow) Suppose given $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$ as in (i), and a colimit cone

$$\{GD(j) \xrightarrow{\mu_j} L \mid j \in \text{ob } J\}$$

in \mathcal{C} . Then

$$\{TGD(j) \xrightarrow{T\mu_j} TL \mid j \in \text{ob } J\}$$

is also a colimit cone, so the composites

$$TGD(j) \xrightarrow{\delta_j} GD(j) \xrightarrow{\mu_j} L$$

induce a unique $\lambda : TL \rightarrow L$. The rest of the argument is like (i). □

5.9 Definition (Monadic adjunction). Given an adjunction $\left(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}\right)$, with $(F \dashv G)$ we say the adjunction (or the functor G) is **monadic** if the comparison functor $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ is part of an equivalence of categories.

(Note that, since the Kleisli comparison $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ is full and faithful, it's part of an equivalence if and only if it (equivalently, F) is essentially surjective on objects.)

Remark. Given any [adjunction](#) $(F \dashv G)$, for each object B of \mathcal{D} we have a diagram

$$FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB \xrightarrow{\epsilon_B} B$$

with equal composites. The ‘primeval monadicity theorem’ asserts that $\mathcal{C}^{\mathbb{T}}$ is characterised in $\mathbf{Adj}(\mathbb{T})$ by the fact that these diagrams are all [coequalizers](#).

5.10 Definition (Reflexive coequalizers).

- (a) We say a parallel pair $A \xrightleftharpoons[g]{f} B$ is **reflexive** if there exists $B \xrightarrow{r} A$ such that $fr = gr = 1_B$. (Note that $FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB$ is reflexive, with $r = F\eta_{GB}$). We say \mathcal{C} has **reflexive coequalizers** if it has [coequalizers](#) of all reflexive pairs (equivalently, colimits of shape J where $J = \begin{array}{ccc} & \bullet & \\ \curvearrowright & \rightleftarrows & \bullet \\ \curvearrowleft & & \end{array}$)

- (b) By a **split coequalizer** diagram, we mean a diagram

$$\begin{array}{ccccc} A & \xrightleftharpoons[g]{f} & B & \xrightleftharpoons[s]{h} & C \\ & \curvearrowright & & \curvearrowleft & \\ & t & & s & \end{array}$$

satisfying

$$hf = hg \quad hs = 1_C \quad gt = 1_B \quad ft = sh.$$

These equations imply that h is a coequalizer of (f, g) ; if $B \xrightarrow{x} D$ satisfies $xf = xg$ then $x = xgt = xft = xsh$, so x factors through h , and the factorisation is unique since h is [split epic](#). Note that split coequalizers are preserved by all [functors](#).

- (c) Given a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, a parallel pair $A \xrightleftharpoons[g]{f} B$ is called **G -split** if there exists a split coequalizer diagram

$$\begin{array}{ccccc} GA & \xrightleftharpoons[Gg]{Gf} & GB & \xrightleftharpoons[s]{h} & C \\ & \curvearrowright & & \curvearrowleft & \\ & t & & s & \end{array}$$

in \mathcal{C} .

Note that $FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB$ is **G -split**, since

$$\begin{array}{ccccc} GFGFGB & \xrightleftharpoons[G\epsilon_{FGB}]{GFG\epsilon_B} & GFGB & \xrightarrow{G\epsilon_B} & GB \\ & \curvearrowright & & \curvearrowleft & \\ & \eta_{GFGB} & & \eta_{GB} & \end{array}$$

is a [split coequalizer](#).

5.11 Lemma. Suppose given an **adjunction** $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ where $F \dashv G$, inducing a **monad** \mathbb{T} on \mathcal{C} . Then $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ has a left adjoint provided, for every \mathbb{T} -algebra (A, α) , the pair $FGFA \xrightleftharpoons[\epsilon_{FA}]{F\alpha} FA$ has a **coequalizer** in \mathcal{D} .

Proof. We define $L : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{D}$ by taking $FA \rightarrow L(A, \alpha)$ to be a **coequalizer** for $(F\alpha, \epsilon_{FA})$. Note that this is a functor $\mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{D}$.

Recall that K is defined by $KB = (GB, G\epsilon_B)$. For any B , morphisms $LA \rightarrow B$ correspond bijectively to morphisms $FA \xrightarrow{f} B$ satisfying $f(F\alpha) = f(\epsilon_{FA})$. These correspond to morphisms $A \xrightarrow{\check{f}} GB$ satisfying

$$\check{f}\alpha = Gf = G(\epsilon_B(F\check{f})) = (G\epsilon_B)(T\check{f})$$

i.e. to **algebra homomorphisms** $(A, \alpha) \rightarrow KB$. And these bijections are natural in (A, α) and in B . \square

Lecture 17 **5.12 Theorem** (Precise Monadicity Theorem). $G : \mathcal{D} \rightarrow \mathcal{C}$ is **monadic** iff G has a **left adjoint** and **creates coequalizers** of **G -split** pairs.

5.13 Theorem (Refined/Reflexive Monadicity Theorem). Suppose \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ preserves **reflexive coequalizers** and that G reflects isomorphisms and has a **left adjoint**. Then G is **monadic**.

Proof of Theorem 5.12 \Rightarrow . It is sufficient to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ **creates coequalizers** of $G^{\mathbb{T}}$ **split-pairs**. But this follows from the argument of **Theorem 5.8(ii)**, since if $(A, \alpha) \xrightleftharpoons[g]{f} (B, \beta)$ is a $G^{\mathbb{T}}$ -split pair, the coequalizer of $A \xrightleftharpoons[g]{f} B$ is preserved by T and TT . \square

Proof of Theorem 5.12 \Leftarrow and Theorem 5.13. Let \mathbb{T} be the **monad** induced by $(F \dashv G)$. For any \mathbb{T} -algebra (A, α) , the pair $FGFA \xrightleftharpoons[\epsilon_{FA}]{F\alpha} FA$ is both **reflexive** and **G -split**, so has a **coequalizer** in \mathcal{D} , and hence by **Lemma 5.11**, $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ has a **left adjoint** L .

The **unit** of $(L \dashv K)$ at an **algebra** (A, α) : the coequalizer defining $L(A, \alpha)$ is mapped by K to the diagram

$$\begin{array}{ccc} F^{\mathbb{T}}TA \xrightleftharpoons[\mu_A]{F^{\mathbb{T}}A} F^{\mathbb{T}}A & \xrightarrow{\quad} & KL(A, \alpha) \\ & \searrow \alpha & \nearrow \iota_{A, \alpha} \\ & (A, \alpha) & \end{array}$$

and $\iota_{A, \alpha}$ is the factorisation of this through the $(G^{\mathbb{T}}$ -split) coequalizer α . But either set of hypotheses implies that G preserves the coequalizer defining $L(A, \alpha)$ so $\iota_{(A, \alpha)}$ is an **isomorphism**.

For the counit $\zeta_B : LKB \rightarrow B$, we have a coequalizer

$$\begin{array}{ccc} FGFG B & \xrightarrow[\epsilon_{FGB}]{FG\epsilon_B} & FGB \longrightarrow LKB \\ & & \searrow \epsilon_B \quad \downarrow \zeta_B \\ & & B \end{array}$$

Again, either set of hypotheses implies that ϵ_B is a coequalizer of $(FG\epsilon_B, \epsilon_{FGB})$ so ζ_B is an isomorphism. \square

5.14 Examples.

- (a) The **forgetful functors** $\mathbf{Gp} \rightarrow \mathbf{Set}$, $\mathbf{Rng} \rightarrow \mathbf{Set}$, $\mathbf{Mod}_R \rightarrow \mathbf{Set}, \dots$ all satisfy the hypotheses of [Theorem 5.13](#); for the **reflexive coequalizers**, use question 3 on example sheet 4 which shows that if

$$A \xrightarrow[g]{f} B \xrightarrow{h} C$$

is a reflexive coequalizer diagram in \mathbf{Set} , then so is

$$A^n \xrightarrow[g^n]{f^n} B^n \xrightarrow{h^n} C^n.$$

- (b) Any **reflection** is **monadic**: this follows from q2 on example sheet 3, but also can be proved using [Theorem 5.12](#). Let \mathcal{D} be a **reflective (full)** subcategory of \mathcal{C} and suppose a pair $A \xrightarrow[g]{f} B$ in \mathcal{D} fits into a **split coequalizer** diagram

$$\begin{array}{ccccc} A & \xrightarrow[g]{f} & B & \xrightarrow{h} & C \\ & \searrow t & \swarrow s & & \\ & & & & \end{array}$$

in \mathcal{C} . Then t and $ft = sh$ belong to \mathcal{D} since \mathcal{D} is full, and hence s is in \mathcal{D} since it's an equalizer of $(1_B, sh)$ and \mathcal{D} is closed under limits in \mathcal{C} . Hence also $h \in \text{mor } \mathcal{D}$.

- (c) Consider the composite adjunction

$$\mathbf{Set} \xrightleftharpoons[G]{F} \mathbf{AbGp} \xrightleftharpoons[I]{L} \mathbf{tfAbGp}.$$

The two factors are monadic by (a) and (b) respectively, but the composite isn't, since the monad it induces on \mathbf{Set} is isomorphic to that induced by $(F \dashv U)$.

- (d) Consider the forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$. This is **faithful** and has both **left and right adjoints** (so preserves all **coequalizers**), but the monad induced on \mathbf{Set} is $(1, 1, 1)$ and the category of algebras is \mathbf{Set} .

- (e) Consider the composite adjunction

$$\mathbf{Set} \xrightleftharpoons[U]{D} \mathbf{Top} \xrightleftharpoons[I]{\beta} \mathbf{KHaus}.$$

We'll show that this satisfies the hypotheses of [Theorem 5.12](#). Let

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow[g]{t} \end{array} Y \begin{array}{c} \xrightarrow{h} \\ \xleftarrow[s]{s} \end{array} Z$$

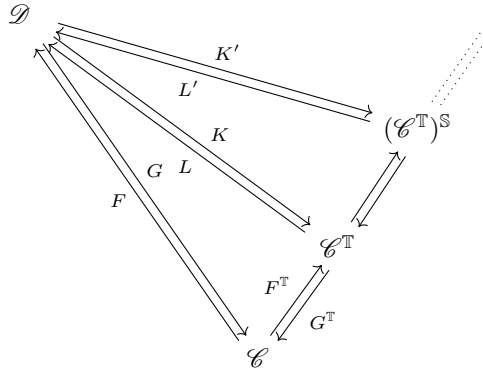
be a split coequalizer in **Set**, where X, Y have compact Hausdorff topologies and f, g are continuous. Note that the quotient topology on $Z \cong Y/R$ is compact, so it's the only possible candidate for a compact Hausdorff topology making h continuous.

We use the lemma from general topology: if Y is compact Hausdorff, then a quotient Y/R is Hausdorff $\iff R \subseteq Y \times Y$ is closed. We note

$$\begin{aligned} R &= \{ (y, y') \mid h(y) = h(y') \} = \{ (y, y') \mid sh(y) = sh(y') \} \\ &= \{ (y, y') \mid ft(y) = ft(y') \}. \end{aligned}$$

So if we define $S \subseteq X \times X = \{ (x, x') \mid f(x) = f(x') \}$ then $R \subseteq (g \times g)(S)$, but the reverse inclusion also holds. But $S \longrightarrow X \times X \xrightarrow[f \pi_2]{f \pi_1} Y$ is an equalizer, Y is Hausdorff, so S is closed in $X \times X$ and hence compact. So $R = (g \times g)(S)$ is compact and hence closed in $Y \times Y$.

Lecture 18 **5.15 Definition** (Monadic tower). Let $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ be an [adjunction](#), and suppose \mathcal{D} has [reflexive coequalizers](#). The **monadic tower** of $F \dashv G$ is the diagram



where \mathbb{T} is the [monad](#) induced by $(F \dashv G)$, K is as in [Theorem 5.7](#), L as in [Lemma 5.11](#), \mathbb{S} is the monad induced by $(L \dashv K)$, and so on.

We say $(F \dashv G)$ has **monadic length** n if we reach an [equivalence](#) after n steps.

For example, the [adjunction](#) of [Examples 5.14\(c\)](#) has [monadic length](#) 2, and the adjunction of [Examples 5.14\(d\)](#) has monadic length ∞ .

5.16 Theorem. Suppose given an [adjunction](#) $\mathcal{C} \xrightleftharpoons[L]{R} \mathcal{D}$ and [monads](#) \mathbb{T}, \mathbb{S} on \mathcal{C}, \mathcal{D} respectively, and a [functor](#) $\bar{R} : \mathcal{D}^{\mathbb{S}} \rightarrow \mathcal{C}^{\mathbb{T}}$ such that

$$\begin{array}{ccc} \mathcal{D}^{\mathbb{S}} & \xrightarrow{\bar{R}} & \mathcal{C}^{\mathbb{T}} \\ \downarrow G^{\mathbb{S}} & & \downarrow G^{\mathbb{T}} \\ \mathcal{D} & \xrightarrow{R} & \mathcal{C} \end{array}$$

commutes up to isomorphism.

Suppose also that $\mathcal{D}^{\mathbb{S}}$ has [reflexive coequalizers](#). Then \bar{R} has a [left adjoint](#) \bar{L} .

Proof. Note that if \bar{L} exists, we must have $\bar{L}F^{\mathbb{T}} \cong F^{\mathbb{S}}L$, by [Corollary 3.6](#). So we'd expect $\bar{L}(A, \alpha)$ to be a [coequalizer](#) of two morphisms

$$F^{\mathbb{S}}LTA \xrightarrow[\text{?}]{F^{\mathbb{S}}L\alpha} F^{\mathbb{S}}LA$$

To construct the second morphism, note first that we can assume wlog $G^{\mathbb{T}}\bar{R} = RG^{\mathbb{S}}$, by transporting [T-algebra](#) structures along the isomorphism $G^{\mathbb{T}}R(B, \beta) \rightarrow RB$. We obtain $\theta : TR \rightarrow RS$ by

$$\begin{aligned} R &\xrightarrow{R_L} RS = RG^{\mathbb{S}}F^{\mathbb{S}} = G^{\mathbb{T}}\bar{R}F^{\mathbb{S}} \\ F^{\mathbb{T}}R &\longrightarrow \bar{R}F^{\mathbb{S}} \\ TR = G^{\mathbb{T}}F^{\mathbb{T}}R &\xrightarrow{\theta} G^{\mathbb{T}}\bar{R}F^{\mathbb{S}} = RG^{\mathbb{S}}F^{\mathbb{S}} = RS \end{aligned}$$

Convert it into $\varphi : LT \rightarrow SL$ by

$$LT \xrightarrow{LT\gamma} LTRL \xrightarrow{L\theta_L} LRS L \xrightarrow{\delta_{SL}} SL$$

where γ and δ are the unit and counit of $(L \dashv R)$. Transposing across $(F^{\mathbb{S}} \dashv G^{\mathbb{S}})$, we get $F^{\mathbb{S}}LT \xrightarrow{\bar{\varphi}} F^{\mathbb{S}}L$. The pair $(F^{\mathbb{S}}L\alpha, \bar{\varphi}_A)$ is [reflexive](#), with common splitting $F^{\mathbb{S}}L\eta$.

It can be verified that the coequalizer of this pair has the universal property we require for $\bar{L}(A, \alpha)$. \square

6 Cartesian Closed Categories

6.1 Definition (Exponentiable object). Let \mathcal{C} be a [category](#) with [finite products](#). We say $A \in \mathbf{ob} \mathcal{C}$ is **exponentiable** if the [functor](#) $(-) \times A : \mathcal{C} \rightarrow \mathcal{C}$ has [right adjoint](#) $(-)^A$. If every object of \mathcal{C} is exponentiable, we say \mathcal{C} is **cartesian closed**.

6.2 Examples.

- (a) **Set** is [cartesian closed](#), with $B^A = \mathbf{Set}(A, B)$. A function $f : C \times A \rightarrow B$ corresponds to $\bar{f} : C \rightarrow B^A$.
- (b) **Cat** is cartesian closed, with $\mathcal{D}^{\mathcal{C}} = [\mathcal{C}, \mathcal{D}]$.
- (c) In **Top**, if an [exponential](#) Y^X exists, its points must be the continuous maps $X \rightarrow Y$. The compact-open topology on $\mathbf{Top}(X, Y)$ has the universal property of an [exponential](#) iff X is locally compact.

Note that [finite products](#) of exponentiable objects are exponentiable: since

$$(-) \times (A \times B) \cong (- \times A) \times B,$$

we have $(-)^{A \times B} \cong ((-)^B)^A$. However, even if X, Y are locally compact, Y^X needn't be, so the exponentiable objects don't form a cartesian closed [full subcategory](#).

- (d) A cartesian closed poset is called a **Heyting semilattice**: it's a poset with finite meets and a binary operation \Rightarrow satisfying $a \leq (b \Rightarrow c)$ iff $a \wedge b \leq c$. For example, a complete poset is a Heyting semilattice iff it satisfies the infinite distributive law

$$a \wedge \bigvee \{b_i \mid i \in I\} = \bigvee \{a \wedge b_i \mid i \in I\}.$$

For any topological space X , the lattice $\mathcal{O}(X)$ of open subsets satisfies this condition, since \wedge and \bigvee coincide with \cap and \bigcup .

Recall from example sheets that, if $B \in \mathbf{ob} \mathcal{C}$, we define the **slice category** \mathcal{C}/B to have objects which are morphisms

$$\begin{array}{c} A \\ \downarrow \\ B \end{array}$$

in \mathcal{C} and morphisms are commutative triangles

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A' \\ & \searrow & \swarrow \\ & B & \end{array}$$

The [forgetful functor](#) $\mathcal{C}/B \rightarrow \mathcal{C}$ will be denoted Σ_B . If \mathcal{C} has [finite products](#), Σ_B has a

right adjoint B^* which sends A to $\begin{pmatrix} A \times B \\ \downarrow \pi_2 \\ B \end{pmatrix}$, since morphisms

$$\begin{array}{ccc} C & \xrightarrow{(f,g)} & A \times B \\ & \searrow g \quad \swarrow \pi_2 & \\ & B & \end{array}$$

correspond to morphisms $C = \Sigma_B g \xrightarrow{f} A$.

6.3 Lemma. If \mathcal{C} has all [finite limits](#), then an object B is [exponentiable](#) iff $B^* : \mathcal{C} \rightarrow \mathcal{C}/B$ has a [right adjoint](#) Π_B .

Proof.

\Leftarrow The composite $\Sigma_B B^*$ is equal to $(-) \times B$, so we take $(-)^B$ to be $\Pi_B B^*$.

\Rightarrow If B is [exponentiable](#), for any $A \xrightarrow{f} B$ we define $\Pi_B(f)$ to be the [pullback](#)

$$\begin{array}{ccc} \Pi_B(f) & \longrightarrow & A^B \\ \downarrow & & \downarrow f^B \\ 1 & \xrightarrow{\pi_2} & B^B. \end{array}$$

The morphisms $C \rightarrow \Pi_B(f)$ correspond to morphisms $C \rightarrow A^B$ making

$$\begin{array}{ccc} C & \longrightarrow & A^B \\ \downarrow & & \downarrow f^B \\ 1 & \xrightarrow{\pi_2} & B^B \end{array}$$

i.e. to homomorphisms $C \times B \rightarrow A$ making

$$\begin{array}{ccc} C \times B & \longrightarrow & A \\ & \searrow \pi_2 & \downarrow f \\ & & B \end{array}$$

commute. □

Lecture 19 **6.4 Lemma.** Suppose \mathcal{C} has [finite limits](#). If A is [exponentiable](#) in \mathcal{C} , then $B^* A$ is [exponentiable](#) in \mathcal{C}/B for any B . Moreover, B^* [preserves](#) exponentials.

Proof. Given an object $C \downarrow_f B$, form the [pullback](#)

$$\begin{array}{ccc} P & \longrightarrow & C^A \\ f^{(B^* A)} \downarrow & & \downarrow f^A \\ B & \xrightarrow{\pi_1} & B^A. \end{array}$$

Then, for any $D \downarrow_g B$, morphisms $g \rightarrow f^{B^* A}$ in \mathcal{C}/B correspond to morphisms $D \xrightarrow{\bar{h}} C^A$

making

$$\begin{array}{ccc} D & \xrightarrow{\bar{h}} & C^A \\ g \downarrow & & \downarrow f^A \\ B & \xrightarrow{\bar{\pi}_1} & B^A \end{array}$$

commute, and hence to morphisms $D \times A \xrightarrow{h} C$ making

$$\begin{array}{ccc} D \times A & \xrightarrow{h} & C \\ & \searrow g\pi_1 & \downarrow f \\ & & B \end{array}$$

commute. But

$$\begin{array}{ccc} D \times A & \xrightarrow{g \times 1_A} & B \times A \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ D & \xrightarrow{g} & B \end{array}$$

is a pullback in \mathcal{C} , i.e. a product in \mathcal{C}/B .

For the second assertion, note that if $\begin{array}{c} C \\ \downarrow f \\ B \end{array}$ is of the form $\begin{array}{c} B \times E \\ \downarrow \pi_1 \\ B \end{array}$, then the pullback

defining f^{B^*A} becomes

$$\begin{array}{ccc} B \times E^A & \xrightarrow{\bar{\pi}_1 \times 1} & B^A \times E^A \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ B & \xrightarrow{\bar{\pi}_1} & B^A \end{array}$$

so $f^{B^*A} \cong B^*(E^A)$. □

Remark. \mathcal{C}/B is isomorphic to the category of [coalgebras](#) for the [monad](#) structure on $(-) \times B$ ([Examples 5.2\(a\)](#)); so the first part of [Lemma 6.4](#) could have been proved using [Theorem 5.16](#).

6.5 Definition (Locally cartesian closed). We say \mathcal{C} is **locally cartesian closed** if it has all finite [limits](#) and each \mathcal{C}/B is [cartesian closed](#). (Note that this includes the fact that $\mathcal{C} \cong \mathcal{C}/1$ is cartesian closed).

6.6 Examples.

- (a) **Set** is [locally cartesian closed](#), since $\mathbf{Set}/B \cong \mathbf{Set}^B$ for any B .
- (b) For any [small](#) category \mathcal{C} , $[\mathcal{C}, \mathbf{Set}]$ is [cartesian closed](#): by Yoneda,

$$\begin{aligned} G^F(A) &\cong [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), G^F) \\ &\cong [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -) \times F, G), \end{aligned}$$

so we take the right hand side as a definition of $G^F(A)$ and define G^F on morphisms $A \xrightarrow{f} B$ by composition with $\mathcal{C}(f, -) \times 1_F$. Note that the class of functors H for which we have

$$[\mathcal{C}, \mathbf{Set}](H, G^F) \cong [\mathcal{C}, \mathbf{Set}](H \times F, G)$$

is closed under **colimits**; but every **functor** $\mathcal{C} \rightarrow \mathbf{Set}$ is a colimit of **representables**.

In fact $[\mathcal{C}, \mathbf{Set}]$ is locally cartesian closed, since all its slice categories $[\mathcal{C}, \mathbf{Set}]/F$ are of the same form (see question 6 on example sheet 4).

- (c) Any **Heyting semilattice** H is locally cartesian closed, since $H/B \cong \downarrow(b)$, the poset of elements $\leq b$, and $b^* = (-) \wedge b$ is surjective.
- (d) **Cat** is cartesian closed and not locally cartesian closed, since not all strong **epis** are regular (cf question 6 on example sheet 3).

Note that, given $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ in \mathcal{C}/B , the iterated slice $(\mathcal{C}/B)/f$ is isomorphic to \mathcal{C}/A , and this identifies

$$f^* : \mathcal{C}/B \rightarrow (\mathcal{C}/B)/f$$

with the operation of pulling back morphisms along f . So by **Lemma 6.3**, \mathcal{C} is **locally cartesian closed** iff it has **finite limits** and $f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A$ has a **right adjoint** Π_f for every $A \xrightarrow{f} B$ in \mathcal{C} .

6.7 Theorem. Suppose \mathcal{C} is **locally cartesian closed** and has **reflexive coequalizers**. Then every morphism $A \xrightarrow{f} B$ factors as

$$\begin{array}{ccc} & I & \\ q \nearrow & & \nwarrow m \\ A & \xrightarrow{f} & B \end{array}$$

where q is **regular epic** and m is **monic**.

Proof. First form the **pullback**

$$\begin{array}{ccc} R & \xrightarrow{a} & A \\ \downarrow b & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

and then form the **coequalizer**

$$R \xrightarrow[b]{a} A \xrightarrow{q} I.$$

Since $fa = fb$, f factors as $A \xrightarrow{q} I \xrightarrow{m} B$.

Suppose given $D \xrightarrow[h]{g} I$ with $mg = mh$. Form the pullback

$$\begin{array}{ccc} E & \xrightarrow{n} & D \\ (k,l) \downarrow & & \downarrow (g,h) \\ A \times A & \xrightarrow{q \times q} & I \times I. \end{array}$$

Since $q \times q$ factors as $(q \times 1_I)(1_A \times q)$ and both factors are pullbacks of q , it's an **epimorphism**, and hence so is n . Now $fk = m q k = m g n = m h n = m q l = f l$, so $\exists E \xrightarrow{p} R$ with $ap = k$, $bp = l$. Now $qk = qap = qbp = ql$, i.e. $gn = hn$. But n is epic, so $g = h$. Hence m is monic. \square

Note that this implies any strong epimorphism $A \xrightarrow{f} B$ is **regular**, since the **monic** part of its image factorization is an **isomorphism**. In particular, regular epimorphisms are stable under composition.

6.8 Definition (Cartesian closed functor). If \mathcal{C} and \mathcal{D} are **cartesian closed categories** and $F : \mathcal{C} \rightarrow \mathcal{D}$ **preserves finite products**, then for each pair of objects A, B of \mathcal{C} we get a natural morphism

$$\theta : F(B^A) \rightarrow FB^{FA},$$

namely the transpose of

$$F(B^A) \times FA \cong F(B^A \times A) \xrightarrow{F(\text{ev})} FB$$

where ev is the **counit** of $((-) \times A \dashv (-)^A)$. We say F is a **cartesian closed functor** if θ is an **isomorphism** for every pair (A, B) .

Note that the second part of **Lemma 6.4** says that if \mathcal{C} is **locally cartesian closed** then $f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A$ is a **cartesian closed functor** for any $A \xrightarrow{f} B$.

6.9 Theorem. Let \mathcal{C} and \mathcal{D} be **cartesian closed categories** and $F : \mathcal{C} \rightarrow \mathcal{D}$ a **functor** having a **left adjoint** L . Then F is **cartesian closed** iff the canonical morphism

$$L(B \times FA) \xrightarrow{(L\pi_1, L\pi_2)} LB \times LFA \xrightarrow{1_{LB} \times \epsilon_A} LB \times A$$

is an **isomorphism** for all $A \in \text{ob } \mathcal{D}$, $B \in \text{ob } \mathcal{C}$.

This condition is called **Frobenius reciprocity**. Note that if \mathcal{C} is **locally cartesian closed**, then $f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A$ has a **left adjoint** Σ_f given by composition with f and it's easy to verify that $\Sigma_f(g \times f^*h) \cong \Sigma_f g \times h$.

Proof.

Lecture 20 (\Rightarrow) Given an inverse for $\theta : F(C^A) \rightarrow FC^{FA}$, we define $\varphi_{A,B}^{-1}$ to be the composite

$$\begin{array}{ccccc} LB \times A & \xrightarrow{L\lambda \times 1} & L((B \times FA)^{FA}) & \xrightarrow{L(\eta^{FA}) \times 1} & L(FL(B \times FA)^{FA}) \times A \\ & & & & \downarrow L\theta^{-1} \times 1 \\ L(B \times FA) & \xleftarrow{\text{ev}} & L(B \times FA)^A \times A & \xleftarrow{\epsilon \times 1} & LF(L(B \times FA)^A) \times A \end{array}$$

(\Leftarrow) Given an inverse for φ , we define θ^{-1} to be

$$\begin{array}{ccccc} F((L(FC^{FA}) \times A)^A) & \xleftarrow{F\lambda} & FL(FC^{FA}) & \xleftarrow{\eta} & FC^{FA} \\ \downarrow F(\varphi^{-1A}) & & & & \\ F((L(FC^{FA} \times FA))^A) & \xrightarrow{F((L(\text{ev}))^A)} & F((LFC)^A) & \xrightarrow{F(\epsilon^A)} & F(C^A). \end{array}$$

We omit checking that these work. \square

6.10 Corollary. Suppose \mathcal{C} and \mathcal{D} are cartesian closed, and $F : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint L which preserves finite products. Then F is cartesian closed $\iff F$ is full and faithful.

Proof.

(\Rightarrow) L preserves **1**, so if we substitute $B = 1$ in the definition of φ , we get

$$LFA \xrightarrow{\epsilon_A} A.$$

But ϵ an isomorphism $\iff F$ is full and faithful by Lemma 3.9.

(\Leftarrow) If L preserves binary products and ϵ is isomorphic, then both factors in the definition of φ are isomorphisms. \square

6.11 Definition (Exponential ideal). Let \mathcal{C} be a cartesian closed category. By an **exponential ideal** in \mathcal{C} , we mean a class of objects (or a full subcategory) \mathcal{E} such that $B \in \mathcal{E} \implies B^A \in \mathcal{E}$ for all $A \in \text{ob } \mathcal{C}$.

6.12 Examples.

- (a) We say A is **subterminal** if $A \rightarrow 1$ is monic. In any cartesian closed category \mathcal{C} , the class $\text{Sub}_{\mathcal{C}}(1)$ of subterminal objects is an exponential ideal, since

$$\begin{aligned} A \text{ subterminal} &\iff \exists \leq 1 \quad B \rightarrow A, \text{ for any } B \\ &\implies \exists \leq 1 \quad C \times B \rightarrow A, \text{ for any } B \text{ and } C, \\ &\iff \exists \leq 1 \quad C \rightarrow A^B \\ &\iff A^B \text{ subterminal} \end{aligned}$$

More generally, if \mathcal{C} is locally cartesian closed, then $\text{Sub}_{\mathcal{C}}(A)$ is an exponential ideal in \mathcal{C}/A , for any A . If \mathcal{C} also satisfies the hypotheses of Theorem 6.7, then $\text{Sub}_{\mathcal{C}}(A)$ is reflective in \mathcal{C}/A .

- (b) Let X be a topological space. By a **presheaf** on X , we mean a functor

$$\mathcal{O}(X)^{\text{op}} \xrightarrow{F} \mathbf{Set},$$

where $\mathcal{O}(X)$ is the poset of open subsets of X . So F has sets $F(U)$ for each open U , and restriction maps $x \mapsto x|_V : F(U) \rightarrow F(V)$ whenever $V \subseteq U$. We say F is a **sheaf** if, whenever $U = \bigcup_{i \in I} U_i$ and we're given $x_i \in F(U_i)$ for each i such that

$$x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$$

for all (i, j) , there exists a unique $x \in F(U)$ such that $x|_{U_i} = x_i$ for all i .

We write $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ for the full subcategory of sheaves. We'll show $\mathbf{Sh}(X)$ is an exponential ideal:

Given presheaves F, G , $G^F(U)$ is the set of natural transformations

$$F \times \mathcal{O}(X)(-, U) \longrightarrow G$$

or equivalently the set of natural transformations $F|_U \longrightarrow G|_U$, where

$$F|_U : \mathcal{O}(U)^{\text{op}} \longrightarrow \mathbf{Set}$$

is the presheaf obtained by restricting F to open sets $\subseteq U$.

Now suppose G is a sheaf, suppose $U = \bigcup_{i \in I} U_i$ and suppose given $\alpha_i : F|_{U_i} \longrightarrow G|_{U_i}$ for each, such that

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

for all i, j . Given $x \in F(V)$ where $V \subseteq U$, write $V_i = V \cap U_i$, then $V = \bigcup_{i \in I} V_i$.

The elements $x|_{V_i}$, $i \in I$, satisfy the compatibility condition, and hence so do the elements

$$(\alpha_i)_{V_i}(x|_{V_i}) \in G(V_i).$$

So there's a unique $y \in G(V)$ such that $y|_{V_i} = (\alpha_i)_{V_i}(x|_{V_i})$ for all i , and we define this to be $\alpha_V(x)$. This defines a natural transformation $\alpha : F|_U \longrightarrow G|_U$, and it's the unique transformation where restriction to U_i is α_i , for each i .

Note that, since $\mathbf{Sh}(X)$ is closed under finite products (and in fact all limits) in $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$, it is itself cartesian closed.

6.13 Lemma. Suppose \mathcal{C} is cartesian closed, $\mathcal{D} \subseteq \mathcal{C}$ is a (full) reflective subcategory, with reflector $L : \mathcal{C} \longrightarrow \mathcal{D}$. Then \mathcal{D} is an exponential ideal $\iff L$ preserves binary products.

Proof.

(\Rightarrow) Suppose $A, B \in \text{ob } \mathcal{C}$, $C \in \text{ob } \mathcal{D}$. Then we have bijections

$$\begin{aligned} A \times B &\longrightarrow C \\ A &\longrightarrow C^B \\ LA &\longrightarrow C^B \\ LA \times B &\longrightarrow C \\ B &\longrightarrow C^{LA} \\ LB &\longrightarrow C^{LA} \\ LA \times LB &\longrightarrow C \end{aligned}$$

so $LA \times LB$ has the universal property of $L(A \times B)$.

(\Leftarrow) Suppose $B \in \text{ob } \mathcal{D}$, $A, C \in \text{ob } \mathcal{C}$. We have bijections

$$\begin{aligned}
 C &\longrightarrow B^A \\
 C \times A &\longrightarrow B \\
 LC \times LA &\cong L(C \times A) \longrightarrow B \\
 L(LC \times A) &\longrightarrow B \\
 LC \times A &\longrightarrow B \\
 LC &\longrightarrow B^A
 \end{aligned}$$

So every $C \longrightarrow B^A$ factors through $C \longrightarrow LC$, hence $B^A \in \text{ob } \mathcal{D}$. □

7 Toposes

Lecture 21 Grothendieck (around 1963) introduced toposes as [categories](#) of ‘generalized [sheaves](#)’. J. Giraud gave a characterization of such categories by (set-theoretic) categorical properties.

F. W. Lawvere and M. Tierney (1969-70) investigated the elementary categorical properties of these categories and came up with the elementary definition. In fact a Grothendieck topos is exactly a Lawvere-Tierney topos which is [\(co\)complete](#) and [locally small](#), and has a [separating set of objects](#).

7.1 Definition (Subobject classifier, topos).

- (a) Let \mathcal{E} be a [category](#) with finite [limits](#). A **subobject classifier** for \mathcal{E} is a [monomorphism](#) $\Omega' \xrightarrow{\top} \Omega$ such that for every monomorphism $A' \xrightarrow{m} A$ in \mathcal{E} , there is a unique $\chi_m : A \rightarrow \Omega$ for which there is a [pullback](#) square

$$\begin{array}{ccc} A' & \longrightarrow & \Omega' \\ \downarrow m & & \downarrow \top \\ A & \xrightarrow{\chi_m} & \Omega \end{array}$$

Note that, for any A , there's a unique $A \rightarrow \Omega$ which factors through $\Omega' \xrightarrow{\top} \Omega$, so the [domain](#) of \top is actually a [terminal object](#).

If \mathcal{E} is [well-powered](#), we have a [functor](#)

$$\text{Sub}_{\mathcal{E}}(-) : \mathcal{E}^{\text{op}} \rightarrow \mathbf{Set}$$

sending A to the set of (isomorphism classes of) [subobjects](#) of A and acting on morphisms by pullback, and a subobject classifier is [representation](#) of this functor.

- (b) \mathcal{A} is a category which has finite limits, is [cartesian closed](#) and has a subobject classifier.
(c) If \mathcal{E} and \mathcal{F} are toposes, a **logical functor** $F : \mathcal{E} \rightarrow \mathcal{F}$ is one which preserves finite limits, [exponentials](#) and the subobject classifier.

7.2 Examples.

- (a) **Set** is a [topos](#), with $\Omega = \{0, 1\}$ and $\top = 1 : 1 \rightarrow \{0, 1\}$. Here

$$\chi_m(a) = \begin{cases} 1 & \text{if } a \in A' \\ 0 & \text{if } a \notin A'. \end{cases}$$

So also is the category \mathbf{Set}_f of finite sets, or the category $\mathbf{Set}_{<\kappa}$ of sets of cardinality $< \kappa$, where κ is an infinite cardinal such that $\lambda < \kappa \implies 2^\lambda < \kappa$.

- (b) For any [small](#) category \mathcal{C} , $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is a topos: we've seen that it's [cartesian closed](#), and Ω is determined by [Yoneda Lemma](#):

$$\Omega(A) \cong [\mathcal{C}^{\text{op}}, \mathbf{Set}](\mathcal{C}(-, A), \Omega) \cong \{\text{subfunctors of } \mathcal{C}(-, A)\}$$

So we define $\Omega(A)$ to be the set of **sieves** on A , i.e. sets of morphisms with $\text{cod} = A$, such that $f \in R \implies fg \in R$ for any g .

Given $B \xrightarrow{f} A$ and a sieve R on A , we define f^*R to be the set of g with codomain B such that $fg \in R$. This makes Ω into a **functor** $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, as follows. $\top : 1 \rightarrow \Omega$ is defined by $\top_A(*) = \{\text{all morphisms with codomain } A\}$. Given a subfunctor $F' \xrightarrow{m} F$, we define $\chi_m : F \rightarrow \Omega$ by

$$(\chi_m)_A(x) = \{f : B \rightarrow A \mid Ff(x) \in F'(B)\}.$$

This is the unique **natural transformation** making

$$\begin{array}{ccc} F' & \longrightarrow & 1 \\ \downarrow m & & \downarrow \top \\ F & \longrightarrow & \Omega \end{array}$$

a **pullback**.

- (c) For any space X , $\mathbf{Sh}(X)$ is a topos. It's **cartesian closed** by [Examples 6.12\(ii\)](#); for the subobject classifier, we take

$$\Omega(U) = \{V \in \mathcal{O}(X) \mid V \subseteq U\},$$

$\Omega(U' \rightarrow U)$ is the map $V \mapsto V \cap U'$ and Ω is a **sheaf** since if we have $U = \bigcup_{i \in I} U_i$, and $V_i \subseteq U_i$ such that $V_i \cap U_j = V_j \cap U_i$ for each i, j , then $V = \bigcup_{i \in I} V_i$ is the unique open subset of U with $V \cap U_i = V_i$ for each i .

If $F' \xrightarrow{m} F$ is a subsheaf, then for any $x \in F(U)$, the sieve $\{V \subseteq U \mid x|_V \in F'(V)\}$ has a greatest element since F' is a sheaf, so we define $\chi_m : F \rightarrow \Omega$ to send x to this element.

- (d) Let \mathcal{C} be a group G . The topos structure on $[G, \mathbf{Set}]$ is particularly simple: B^A is the set of all G -equivariant maps $A \times G \xrightarrow{f} B$, but such an f is determined by its values at elements of the form $(a, 1)$ since $f(a, g) = g \cdot f(g^{-1} \cdot a, 1)$, and this restriction can be any mapping $A \times \{1\} \rightarrow B$. So we can take B^A to be the set of functions $A \rightarrow B$, with G acting by $(g \cdot f)(a) = g(f(g^{-1} \cdot a))$, and $\Omega = \{0, 1\}$ with trivial G -action.

So the **forgetful functor** $[G, \mathbf{Set}] \rightarrow \mathbf{Set}$ is **logical**, as is the functor which equips a set A with trivial G -action. Moreover, even if G is infinite, $[G, \mathbf{Set}_f]$ is a topos, and the inclusion $[G, \mathbf{Set}_f] \rightarrow [G, \mathbf{Set}]$ is logical. Similarly, if \mathcal{G} is a large group, then $[\mathcal{G}, \mathbf{Set}]$ is a topos.

- (e) Let \mathcal{C} be a category such that every **slice** \mathcal{C}/A is equivalent to a finite category. Then $[\mathcal{C}^{\text{op}}, \mathbf{Set}_f]$ is a topos; similarly if \mathcal{C} is large but all \mathcal{C}/A are small then $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is a topos. In particular, $[\mathbf{On}^{\text{op}}, \mathbf{Set}]$ is a topos, but it's not **locally small**.

Lecture 22 7.3 Lemma. Suppose \mathcal{C} has **finite limits** and a **subobject classifier**. Then every **monomorphism** in \mathcal{C} is **regular**. In particular, \mathcal{C} is **balanced**.

Proof. The universal **monomorphism** $1 \xrightarrow{\top} \Omega$ is **split** and hence **regular**. But any **pullback** of a regular monomorphism is regular: if f is an **equalizer** of (g, h) then $k^*(f)$ is an equalizer of (gk, hk) . The second assertion follows since regular monomorphism together with epimorphism implies isomorphism. \square

Given an object A in a **topos** \mathcal{E} , we write PA for the **exponential** Ω^A , and $\exists_A \rightharpoonup PA \times A$ for the **subobject** corresponding to $PA \times A \xrightarrow{\text{ev}} \Omega$. This has the property that, for any B and any $R \xrightarrow{m} B \times A$ there is a unique $\ulcorner m \urcorner : B \rightarrow PA$ such that

$$\begin{array}{ccc} R & \xrightarrow{\quad} & \exists_A \\ \downarrow m & & \downarrow \\ B \times A & \xrightarrow{\ulcorner m \urcorner \times 1_A} & PA \times A \end{array}$$

is a **pullback**.

7.4 Definition (Power object, weak topos). By a **power object** for A in a **category** \mathcal{E} with **finite limits**, we mean an object PA equipped with $\exists_A \rightharpoonup PA \times A$ satisfying the above.

We say \mathcal{E} is a **weak topos** if every $A \in \text{ob } \mathcal{E}$ has a power-object. Similarly, we say $F : \mathcal{E} \rightarrow \mathcal{F}$ is **weakly logical** if $F(\exists_A) \rightharpoonup F(PA) \times FA$ is a power-object for FA , for every $A \in \text{ob } \mathcal{E}$.

7.5 Lemma. P is a **functor** $\mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$. Moreover, it is **self-adjoint on the right**.

Proof. Given $A \xrightarrow{f} B$, we define $PB \xrightarrow{Pf} PA$ to correspond to the **pullback**

$$\begin{array}{ccc} E_f & \xrightarrow{\quad} & \exists_B \\ \downarrow & & \downarrow \\ PB \times A & \xrightarrow{1 \times f} & PB \times B. \end{array}$$

For any $C \xrightarrow{\ulcorner m \urcorner} PB$, it's easy to see that $(Pf)\ulcorner m \urcorner$ corresponds to $(1_C \times f)^*(m)$; hence $f \mapsto Pf$ is **functorial**. For any A, B , we have a bijection between **subobjects** of $A \times B$ and of $B \times A$, given by composition with $(\pi_2, \pi_1) : A \times B \rightarrow B \times A$; this yields a (natural) bijection between morphisms $A \rightarrow PB$ and $B \rightarrow PA$. \square

We write $\{\}_A : A \rightarrow PA$ for the morphism corresponding to $A \xrightarrow{(1_A, 1_A)} A \times A$.

7.6 Lemma. Given $A \xrightarrow{f} B$, $\{\}_B f$ corresponds to $A \xrightarrow{(1_A, f)} A \times B$ and $(Pf)\{\}_B$ corresponds to $A \xrightarrow{(f, 1_A)} B \times A$.

Proof. The square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow (1, f) & & \downarrow (1, 1) \\ A \times B & \xrightarrow{f \times 1} & B \times B \end{array}$$

is a **pullback**. Similarly for the second assertion. \square

7.7 Corollary.

- (i) $\{\} : A \rightarrow PA$ is **monic**.

(ii) P is **faithful**.

Proof.

- (i) If $\{ \} f = \{ \} g$, then $(1_A, f)$ and $(1_A, g)$ are isomorphic as **subobjects** of $A \times B$, which forces $f = g$.
- (i) Similarly, if $Pf = Pg$ then $(Pf)\{ \} = (Pg)\{ \}$, so we again deduce $f = g$. \square

Given a **monomorphism** $A \xrightarrow{f} B$ in \mathcal{E} , we define $\exists f : PA \rightarrow PB$ to correspond to the composite

$$\exists_A \rightarrowtail PA \times A \xrightarrow{1 \times f} PA \times B$$

Then, for any $C \xrightarrow{\ulcorner m \urcorner} PA$, $(\exists f)^{\ulcorner m \urcorner}$ corresponds to

$$R \rightarrowtail C \times A \xrightarrow{1 \times f} C \times B.$$

So $f \mapsto \exists f$ is a functor $\mathbf{Mono}(\mathcal{E}) \rightarrow \mathcal{E}$.

7.8 Lemma (Beck-Chevalley condition). Suppose

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

is a **pullback** with f **monic**. Then the diagram

$$\begin{array}{ccc} PA & \xrightarrow{\exists f} & PC \\ \downarrow Ph & & \downarrow Pg \\ PD & \xrightarrow{\exists k} & PB \end{array}$$

commutes.

Proof. Consider the diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\quad} & \exists_A \\ \downarrow & & \downarrow \\ PA \times D & \xrightarrow{1 \times h} & PA \times A \\ \downarrow 1 \times k & & \downarrow 1 \times f \\ PA \times B & \xrightarrow{1 \times g} & PA \times C \end{array}$$

The lower square is a **pullback**, so the upper square is a pullback if and only if the composite is a pullback. \square

7.9 Theorem (Paré). The **functor** $P : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$ is **monadic**.

Proof. It has a **left adjoint** $P : \mathcal{E} \longrightarrow \mathcal{E}^{\text{op}}$ by [Lemma 7.5](#). It's **faithful** by [Corollary 7.7\(ii\)](#) and hence reflects **isomorphisms** by [Lemma 7.3](#). \mathcal{E}^{op} has **coequalizers**, since \mathcal{E} has equalizers. Suppose

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{r} \\ \xrightarrow{g} \end{array} B$$

is a **coreflexive pair** in \mathcal{E} ; then f and g are (split) **monic**, and the equalizer $E \xrightarrow{e} A$ makes

$$\begin{array}{ccc} E & \xrightarrow{e} & A \\ \downarrow e & & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

a **pullback square** since any cone over

$$\begin{array}{ccc} & & A \\ & & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

has both legs equal. So by [Lemma 7.8](#) we have $(Pf)(\exists g) = (\exists e)(Pe)$; but we also have $(Pg)(\exists g) = 1_{PA}$ since

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow 1 & & \downarrow g \\ A & \xrightarrow{g} & B \end{array}$$

is a pullback, and similarly $(Pe)(\exists e) = 1_{PE}$. So

$$\begin{array}{ccccc} PB & \xrightarrow{Pf} & PA & \xrightarrow{Pe} & PE \\ \swarrow \scriptstyle Pg & & \swarrow \scriptstyle \exists g & & \swarrow \scriptstyle \exists e \\ & & & & \end{array}$$

is a **split coequalizer**, and in particular a coequalizer. Hence by [Theorem 5.13](#), P is **monadic**. \square

7.10 Corollary.

- (i) A **weak topos** has finite **colimits**. Moreover, if it has any infinite **limits**, then it has the corresponding colimits.
- (ii) If a **weakly logical** functor has a **left adjoint**, then it has a right adjoint.

Proof.

- (i) P **creates all limits** which exist, by [Theorem 5.8](#).
- (ii) By definition, if F is **weakly logical** then

$$\begin{array}{ccc} \mathcal{E}^{\text{op}} & \xrightarrow{F} & \mathcal{F}^{\text{op}} \\ \downarrow P & & \downarrow P \\ \mathcal{E} & \xrightarrow{F} & \mathcal{F} \end{array}$$

commutes up to **isomorphism**. So this follows from [Theorem 5.16](#).

□

Lecture 23 **7.11 Lemma.** Let \mathcal{E} be a **category** with **finite limits**, suppose $A \in \text{ob } \mathcal{E}$ has a **power object** PA . Then for any B , $B^*(PA)$ is a power object for B^*A in \mathcal{E}/B .

Proof. Given $\begin{array}{c} C \\ \downarrow g \\ B \end{array}$, we have a **pullback square**

$$\begin{array}{ccc} C \times A & \xrightarrow{g \times 1} & B \times A \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ C & \xrightarrow{g} & B \end{array}$$

So $\Sigma_B(g \times B^*A) \cong C \times A$. Hence $\text{Sub}_{\mathcal{E}/B}(g \times B^*A) \cong \text{Sub}_{\mathcal{E}}(C \times A)$, but if $C \xrightarrow{h} PA$

corresponds to $\begin{array}{c} R \\ \downarrow \\ C \times A \end{array}$, then the upper square of the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad} & B \times \exists_A \\ \downarrow & & \downarrow \\ C \times A & \xrightarrow{(g,h) \times 1_A} & B \times PA \times A \\ & \searrow g\pi_1 & \swarrow \pi_1 \\ & B & \end{array}$$

is a pullback. So $\begin{array}{c} B \times PA \\ \downarrow \pi_1 \\ B \end{array}$, equipped with $B^*(\exists_A) \rightarrow B^*(PA \times A)$ is a power object for B^*A . □

7.12 Theorem. Suppose \mathcal{E} is a **weak topos**. Then for any $B \in \text{ob } \mathcal{E}$, \mathcal{E}/B is a weak topos and $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ is **weakly logical**.

Proof. The second assertion follows from **Lemma 7.11**. For the first, we need to construct a

power object for an arbitrary $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ in \mathcal{E}/B . Then the **pullback**

$$\begin{array}{ccc} \Sigma_B(g \times f) & \longrightarrow & f \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

is a **subobject** of $C \times A$, namely the **equalizer** of

$$C \times A \xrightarrow[g\pi_1]{f\pi_2} B.$$

Define $\wedge : PA \times PA \longrightarrow PA$ to correspond to the intersection of $\pi_{13}^*(\exists_A \rightharpoonup PA \times A)$ and $\pi_{23}^*(\exists_A \rightharpoonup PA \times A)$ and define $P_1A \rightharpoonup PA \times PA$ to be the equalizer of $PA \times PA \xrightarrow[\pi_1]{\wedge} PA$. Then, for any C ,

$$C \xrightarrow{(\ulcorner m \urcorner, \ulcorner n \urcorner)} PA \times PA$$

factors through P_1A iff $m \leq n$ in $\mathbf{Sub}_{\mathcal{E}}(C \times A)$. Now form the pullback

$$\begin{array}{ccc} Q & \xrightarrow{\quad} & P_1A \\ \downarrow (h,k) & & \downarrow \\ PA \times B & \xrightarrow{1 \times \{\}} PA \times PB & \xrightarrow{1 \times Pf} PA \times PA. \end{array}$$

Given any $\begin{array}{c} C \\ \downarrow g \\ B \end{array}$, morphisms $g \xrightarrow{l} k$ in \mathcal{E}/B correspond to morphisms $C \xrightarrow{hl} PA$ such that

the subobject named by hl is contained in that named by $(Pf)(\{\})g$. But the latter is indeed $\Sigma_B(g \times f) \rightharpoonup C \times A$. So k is a power-object for f in \mathcal{E}/B . \square

7.13 Corollary. A weak topos is locally cartesian closed (in particular, it's a topos).

Proof. For any $f : A \longrightarrow B$ in \mathcal{E} , we can identify $(\mathcal{E}/B)/f$ with \mathcal{E}/A , and $f^* : \mathcal{E}/B \longrightarrow \mathcal{E}/A$ with pullback along f . Hence all such functors are weakly logical.

But f^* has a left adjoint Σ_f , so by Corollary 7.10(ii) it has a right adjoint Π_f . Hence by Lemma 6.3, \mathcal{E}/B is cartesian closed for any B . \square

Remark. It can be shown that a weakly logical functor is cartesian closed (and hence logical).

7.14 Corollary.

- (i) Any epimorphism in a topos is regular.
- (ii) Any $A \xrightarrow{f} B$ in a topos factors uniquely up to isomorphism as

$$A \xrightarrow{q} \twoheadrightarrow I \rightharpoonup^m B.$$

Proof. For a topos \mathcal{E} , it is locally cartesian closed by Corollary 7.13 and has coequalizers by Corollary 7.10(i), so by Theorem 6.7 every f factors uniquely as regular epi + mono. If f itself is epic, then the monic part of this factorization is isomorphic by Lemma 7.3, so f is regular epic. \square

*Sheaves and local operators

Recall that $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ is a **full subcategory** closed under **limits**: in fact it's **reflective** and the reflector $L : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Sh}(X)$ **preserves finite limits**. This suggests considering reflective subcategories $\mathcal{D} \subseteq \mathcal{E}$ for which the reflector preserves finite limits (equivalently, **pullbacks**).

7.15 Lemma. Given such a **reflective subcategory**, and a **monomorphism** $A' \rightarrowtail A$ in \mathcal{E} , define $c(A') \rightarrowtail A$ by the **pullback** diagram

$$\begin{array}{ccc} c(A') & \longrightarrow & LA' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\eta_A} & LA. \end{array}$$

Then $A' \mapsto c(A')$ is a closure operation on $\text{Sub}_{\mathcal{E}}(A)$, and commutes with pullback along a fixed morphism of \mathcal{E} .

Proof. Since

$$\begin{array}{ccc} A' & \longrightarrow & LA' \\ \downarrow \eta_{A'} & & \downarrow \\ A & \xrightarrow{\eta_A} & LA. \end{array}$$

commutes, we have $A' \leq c(A)'$ and $A' \leq A''$ in $\text{Sub}(A)$ implies $LA' \leq LA''$ in $\text{Sub}(LA)$ and hence $c(A') \leq c(A'')$. Since $L\eta$ is iso,

$$\begin{array}{ccc} LA' & \longrightarrow & LLA' \\ \downarrow L\eta_{A'} & & \downarrow \\ LA & \xrightarrow{L\eta_A} & LLA. \end{array}$$

is a **pullback**, and since L **preserves** pullbacks we deduce $Lc(A') \cong LA'$ is $\text{Sub}(LA)$. Hence $c(c(A')) \cong c(A')$. For stability under pullback, suppose

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback. Then in the cube

$$\begin{array}{ccccc} c(A') & \longrightarrow & & \longrightarrow & LA' \\ & \searrow & & & \searrow \\ & & c(B') & \longrightarrow & LB' \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & LA & \xrightarrow{Lf} & LB \\ & \searrow & \downarrow & & \downarrow \\ & & B & \xrightarrow{\eta_B} & LB \end{array}$$

the front, back and right faces are pullbacks, whence the left face is too. \square

7.16 Definition (Local operator). Let \mathcal{E} be a [topos](#). By a **local operator** on \mathcal{E} , we mean a morphism $j : \Omega \longrightarrow \Omega$ satisfying the commutative diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\top} & \Omega \\ & \searrow \top & \downarrow j \\ & & \Omega \end{array} \quad \begin{array}{c} \longleftarrow \Omega \\ \nearrow j \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega_1 & \longrightarrow & \Omega_1 \\ \downarrow & & \downarrow \\ \Omega \times \Omega & \xrightarrow{j \times j} & \Omega \times \Omega \end{array}$$

where Ω_1 is the order-relation in Ω , defined as in [Theorem 7.12](#).

Given a closure operator on [subobjects](#) as in [Lemma 7.15](#), define $J \rightharpoonup \Omega$ to be the closure of $1 \xrightarrow{\top} \Omega$ and $j : \Omega \longrightarrow \Omega$ to be the classifying map of $J \rightharpoonup \Omega$. Then for any $A' \rightharpoonup A$ with classifying map $\chi_m : A \longrightarrow \Omega$, the composite $j\chi_m$ classifies $c(A') \rightharpoonup A$.

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