Part III – Model Theory (Ongoing course, rough)

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0 Introduction

Lecture 1 Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: 'how powerful is our description of these objects to pin them down'? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

1 Languages and structures

Definition 1.1 (Language)**.** A **language** L consists of

(i) a set $\mathscr{F}$ of function symbols, and for each $f \in \mathscr{F}$ a positive integer m_f the **arity** of f.

(ii) a set $\mathscr{R}$ of relation symbols, and for each $R \in \mathscr{R}$, a positive integer m_R.

(iii) a set $\mathscr{C}$ of constant symbols.

Note: each of $\mathscr{F}, \mathscr{R}$ and $\mathscr{C}$ can be empty.

Example. Take $L = \{\{\cdot, ^{-1}\}, \{1\}\}$, for $\cdot$ a binary function and $^{-1}$ an unary function, 1 a constant. This is the language of groups, call it L_{gp}. Also, $L_{\text{lo}} = \{<\}$ a single binary relation, for linear orders.

Definition 1.2 (L-structure)**.** Given a language L, say, an **L-structure** consists of

(i) a set M, the **domain**

(ii) for each $f \in \mathscr{F}$, a function $f^{\mathcal{M}} : M^{m_f} \to M$.

(iii) for each $R \in \mathscr{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$.

(iv) for each $c \in \mathscr{C}$, an element $c^{\mathcal{M}} \in M$.

f^M, R^M, c^M are the **interpretations** of f, R, c respectively.

Remark 1.3. We often fail to distinguish between the symbols in L and their interpretations in a structure, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathscr{F}, \mathscr{R}, \mathscr{C} \rangle$.

Example 1.4.

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1 \rangle$ is an L_{gp}-structure.

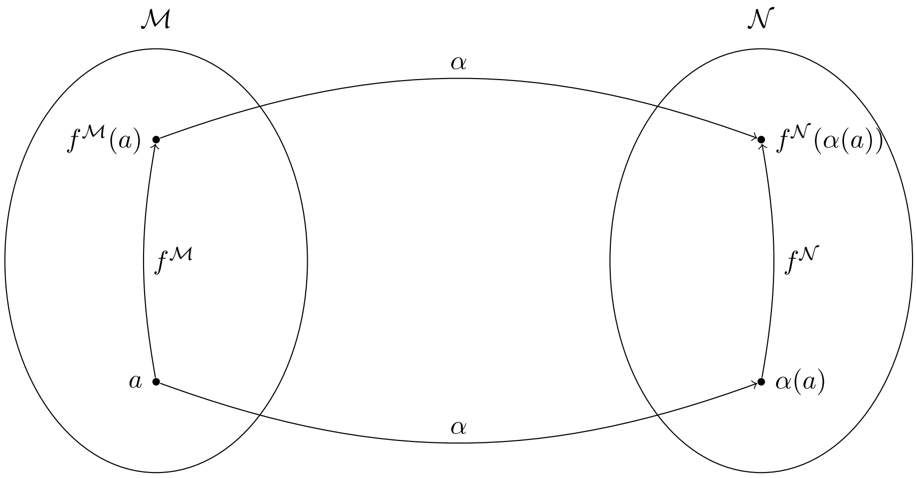
(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp}-structure.

(c) $\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo}-structure.

Definition 1.5 (Embedding)**.** Let L be a language, let $\mathcal{M}, \mathcal{N}$ be L-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is a one-to-one mapping $\alpha : M \to N$ such that

(i) for all $f \in \mathscr{F}$, and $a_1, \dots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \dots, a_{m_f})) = f^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_{m_f}))$$



(ii) for all $R \in \mathscr{R}$, and $a_1, \ldots, a_{m_R} \in M$

$$(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \ldots, \alpha(a_{m_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathscr{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of $\mathcal{M}$ into $\mathcal{N}$ is a surjective embedding (onto), written $\mathcal{M} \simeq \mathcal{N}$.

Exercise 1.6. Let G_1, G_2 be groups, regarded as L_{gp}-structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an isomorphism $\alpha : G_1 \to G_2$ in the sense of Definition 1.5.

2 Review: Terms, formulae and their interpretations

In addition to the symbols of L, we also have

(i) infinitely many variables $\{x_i\}_{i\in I}$

(ii) logical connectives $\wedge, \neg$ (also expresses $\vee$, $\implies$, $\iff$)

(iii) quantifier $\exists$ (also expresses $\forall$)

(iv) (,)

(v) equality symbol $=$

Definition 2.1 (L-terms)**.** **L-terms** are defined recursively as follows:

- any variable x_i is a term
- any constant symbol is a term
- for any $f \in \mathscr{F}$, $f(t_1, \dots, t_{m_f})$ for any terms $t_1, \dots, t_{m_f}$ is a term
- nothing else is a term

Notation: we write $t(x_1, \dots, x_m)$ to mean that the variables appearing in t are among $x_1, \dots, x_m$.

Lecture 2 **Example.** Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1\rangle$. Then $\cdot(\cdot(x_1, x_2), x_3)$ is a term, usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a term, written $(1 \cdot x)^{-1}$

Definition 2.2. If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, \dots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \to M$ defined as follows:

(i) If $t = x_i$, $t^{\mathcal{M}}[a_1, \dots, a_k] = a_i$

(ii) If $t = c$, $t^{\mathcal{M}}[a_1, \dots, a_k] = c^{\mathcal{M}}$.

(iii) If $t = f(t_1(x_1, \dots, x_k), \dots, t_{m_f}(x_1, \dots, x_k))$, then

$$t^{\mathcal{M}}(a_1, \dots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_k), \dots, t_{m_f}^{\mathcal{M}}(a_1, \dots, a_k)).$$

Notice in L_{gp}, the term $x_2 \cdot x_3$ can be described as $t_1(x_1, x_2, x_3)$ or $t_2(x_1, x_2, x_3, x_4)$, or infinitely many other ways. In these cases, t_1 is assigned to $t_1^{\mathcal{M}} : M^3 \to M$, with $(a_1, a_2, a_3) \mapsto (a_2, a_3)$, and t_2 is assigned to $t_2^{\mathcal{M}} : M^4 \to M$, with $(a_1, a_2, a_3, a_4) \mapsto a_2 \cdot a_3$.

Fact 2.3. Let $\mathcal{M}, \mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$ we have

$$\alpha(t^{\mathcal{M}}(a_1, \dots, a_k)) = t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k))$$

Proof. By induction on the complexity of t. Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_k)$. Then

(i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, and $t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds.

(ii) if $t = c$ a constant, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, and $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$, as required.

(iii) if $t = f(t_1(\bar{x}), \ldots, t_{m_f}(\bar{x}))$, then

$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \ldots, \alpha(t_{m_f}^{\mathcal{M}}(\bar{a})))$$

since α is an embedding. $t_1(\bar{x}), \ldots, t_{m_f}(\bar{x})$ have lower complexity than t, so inductive hypothesis applies. $\square$

Exercise 2.4. Conclude the proof of Fact 2.3.

Definition 2.5 (Atomic formula)**.** The set of **atomic formulas** of L is defined as follows

(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula

(ii) if R is a relation symbol and $t_1, \ldots, t_{m_R}$ are terms, then $R(t_1, \ldots, t_{m_R})$ is an atomic formula

(iii) nothing else is an atomic formula.

Definition 2.6 (Formula)**.** The set of **L-formulas** is defined as follows

(i) any atomic formula is an L-formula

(ii) if ϕ is an L-formula, then so is $\neg\phi$

(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$

(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i\ \phi$ is an L-formula

(v) nothing else is an L-formula

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$ and $x_1 \cdot x_2 = 1$ are atomic formulas, and $\exists x_1\ (x_1 \cdot x_2) = 1$ is an L_{gp}-formula.

A variable occurs freely in a formula if it does not occur within the scope of a quantifier $\exists$ (the variable is **free**). Otherwise the variable is **bound**. For instance, in $\exists x_1\ (x_1 \cdot x_2) = 1$, x_1 is bound and x_2 is free.

Important convention: no variable occurs both freely and as a bound variable in the same formula.

A **sentence** is a formula with no free variables.

$$\exists x_1 \exists x_2\ (x_1 \cdot x_2 = 1)$$

is an L_{gp}-sentence. Notation: $\phi(x_1, \ldots, x_k)$ means that the free variables in ϕ are among $x_1, \ldots, x_k$.

Definition 2.7 ($\models$)**.** Let $\phi(x_1, \ldots, x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = (a_1, \ldots, a_k)$ be elements of M. We define $\mathcal{M} \models \phi(\bar{a})$ recursively as follows.

(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

(ii) if ϕ is $R(t_1, \ldots, t_{m_k})$ then $\mathcal{M} \models \phi(\bar{a})$ iff

$$(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_k}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}.$$

(iii) if ϕ is $\psi \wedge \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \chi(\bar{a})$.

(iv) if $\phi = \neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$. (this is well-defined since $\psi(\bar{a})$ is shorter than $\phi(\bar{a})$)

(v) if ϕ is $\exists x_j\ \chi(x_1, \ldots, x_k, x_j)$ (where $x_j \neq x_i$ for $i = 1, \ldots, k$). Then $\mathcal{M} \vDash \phi(\bar{a})$ iff there is $b \in \mathcal{M}$ such that $\mathcal{M} \vDash \chi(a_1, \ldots, a_k, b)$.

Example. For $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1\rangle$, if $\phi(x_1) = \exists x_2\ (x_2 \cdot x_2) = x_1$ then $\mathcal{R} \vDash \phi(1)$ but $\mathcal{R} \nvDash \phi(-1)$.

Notation 2.8 (Useful abbreviations)**.** We write

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$
- $\phi \to \psi$ for $\neg\phi \vee \psi$
- $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$
- $\forall x_i\ \phi$ for $\neg\exists x_i\ (\neg\phi)$

Proposition 2.9. Let $\mathcal{M}, \mathcal{N}$ be L-structures, let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. Let $\phi(\bar{x})$ be atomic and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a})).$$

Question: If ϕ is an L-formula, not necessarily atomic, does Proposition 2.9 hold?

Lecture 3 *Proof of Proposition 2.9.* Cases:

(i) $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. (Exercise: complete this case, using Fact 2.3)

(ii) $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \ldots, t_{m_R}(\bar{x}))$. Then $\mathcal{M} \vDash R(t_1(\bar{a}), \ldots, t_{m_R}(\bar{a}))$ if and only if... (Exercise: complete this case)

□

Exercise 2.10. Show that Proposition 2.9 holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

Example 2.11. Do embeddings preserve *all* formulas? No. Take $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ an L_{lo}-structure. Then $\alpha : \mathbb{Z} \to \mathbb{Q}$ (inclusion) is an embedding, but

$$\phi(x_1, x_2) = \exists x_3\, (x_1 < x_3 \wedge x_3 < x_2).$$
$$\mathcal{Q} \vDash \phi(1, 2) \text{ but } \mathcal{Z} \nvDash \phi(1, 2).$$

Fact 2.12. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{M} \vDash \phi(\alpha(\bar{a})).$$

Proof. Exercise. □

3 Theories and elementarity

Throughout, L is a language, $\mathcal{M}, \mathcal{N}$ are L-structures.

Definition 3.1 (L-theory)**.** An **L-theory** T is a set of L-sentences. $\mathcal{M}$ is a **model** of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$. The class of all the models of T is written $\mathrm{Mod}(T)$. The theory of $\mathcal{M}$ is the set

$$\mathrm{Th}(\mathcal{M}) = \{\, \sigma \mid \sigma \text{ is an } L\text{-sentence and } \mathcal{M} \vDash \sigma \,\}.$$

Example 3.2. Let T_{gp} be the set of L_{gp}-sentences

(i) $\forall x_1 x_2 x_3 \, (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$

(ii) $\forall x_1 \, (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1 \, (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly for a group G, $G \vDash T_{\mathrm{gp}}$. For a specific G, clearly $\mathrm{Th}(G)$ is larger than T_{gp}!

Definition 3.3 (Elementarily equivalent)**.** Say $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$.

Clearly if $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$ but if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as L_{lo}-structures.

Definition 3.4 (Elementary substructure)**.**

(i) an embedding $\beta : \mathcal{M} \to \mathcal{N}$ is **elementary** if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a})).$$

(ii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is said to be a **substructure** of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$.

(iii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is said to be an **elementary substructure** of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example 3.5. Consider $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an L_{lo}-structure, where $<$ is the usual order, and $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$ in the same way. Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo}-structures.

Is $\mathcal{M} \equiv \mathcal{N}$? Yes: they are isomorphic!

Is $\mathcal{M} \subseteq \mathcal{N}$? Yes (the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$.)

But $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if $\phi(x) = \exists y \; (x < y)$, then

$$\mathcal{N} \vDash \phi(1) \quad \text{and} \quad \mathcal{M} \nvDash \phi(1).$$

Definition 3.6 (Parameter)**.** Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, then define

$$L(A) \coloneqq L \cup \{\, c_a \mid a \in A \,\}$$

for c_a each constant symbols. An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way ($c_a^{\mathcal{M}} = a$). The elements of A are called **parameters**. If $\mathcal{M}, \mathcal{N}$ are L-structures and $A \subseteq M \cap N$, then we write $\mathcal{M} \equiv_A \mathcal{N}$ when $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$-sentences.

Lecture 4 **Exercise 3.7.** $\mathcal{M} \preccurlyeq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of $\mathcal{M}$).

Lemma 3.8 (Tarski-Vaught test)**.** Let $\mathcal{N}$ be an L-structure, let $A \subseteq N$. The following are equivalent:

(i) A is the domain of a structure $\mathcal{M}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$.

(ii) for every $L(A)$-formula $\phi(x)$ with one free variable, if $\mathcal{N} \vDash \exists x\ \phi(x)$, then $\mathcal{N} \vDash \phi(b)$ for some $b \in A$.

Proof.

(i) $\Rightarrow$ (ii) Suppose $\mathcal{N} \vDash \phi(x)$. Then by elementarity, $\mathcal{M} \vDash \exists x\ \phi(x)$, and so $\mathcal{M} \vDash \exists x\ \phi(x)$ for $b \in \mathcal{M}$, so again by elementarity $\mathcal{N} \vDash \phi(b)$.

(ii) $\Rightarrow$ (i) First we prove that A is the domain $\mathcal{M} \subseteq \mathcal{N}$. By exercise 4 on sheet 1, it is enough to check:

(a) for each constant c, $c^{\mathcal{N}} \in A$.

(b) for each function symbol f, $f^{\mathcal{N}}(\bar{a}) \in A$ (for all $\bar{a} \in A^{m_f}$).

For (a), use property (ii) with $\exists x\ (x = c)$. For (b) use property (ii) with $\exists x\ (f(\bar{a}) = x)$.

So we now have $\mathcal{M} \subseteq \mathcal{N}$, and the domain of $\mathcal{M}$ is A. Let $\chi(\bar{x})$ be an L-formula. We show that for $\bar{a} \in A^{|\bar{x}|}$,

$$\mathcal{M} \vDash \chi(\bar{a}) \iff \mathcal{N} \vDash \chi(\bar{a}). \tag{$*$}$$

By induction on the complexity of $\chi(\bar{x})$:

- if $\chi(\bar{x})$ is atomic ($*$) follows from $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M}$ is a substructure).
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$: straightforward induction.
- if $\chi(\bar{x}) = \exists y\ \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L-formula, suppose that $\mathcal{M} \vDash \chi(\bar{a})$. Then $\mathcal{M} \vDash \exists y\ \psi(\bar{a}, y)$, hence $\mathcal{M} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom} \mathcal{M}$. But then $\mathcal{N} \vDash \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \vDash \chi(\bar{a})$.

  Now let $\mathcal{N} \vDash \chi(\bar{a})$, i.e. $\mathcal{N} \vDash \exists y\ \psi(\bar{a}, y)$. By property (ii), $\mathcal{N} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom}(\mathcal{M})$. By inductive hypothesis, $\mathcal{M} \vDash \psi(\bar{a}, b)$ and so $\mathcal{M} \vDash \chi(\bar{a})$. □

Remark 3.9. Assume the set of variables is countably infinite. Then

- the cardinality of the set of L-formulas is $|L| + \omega$. (We abuse notation and write ω for the ordinal and cardinal, and define the cardinality of L as the number of symbols in it: $|L_{\mathrm{gp}}| = 3$, $|L_{\mathrm{lo}}| = 1$).

- if A is a set of parameters in some structure, the cardinality of the set of $L(A)$-formulas is $|A| + |L| + \omega$.

Definition 3.10 (Chain)**.** Let λ be an ordinal. Then **a chain of length** λ of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$. A **chain of L-structures** is a sequence $\langle \mathcal{M}_i : i < \lambda \rangle$ such that $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i \leq j < \lambda$.

The **union** of this chain is the L-structure $\mathcal{M}$ is defined as follows:

- the domain of $\mathcal{M}$ is $\bigcup_{i<\lambda} M_i$

- $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$ (c is a constant).

- if f is a function symbol, $\bar{a} \in M^{m_f}$, $f^{\mathcal{M}}\bar{a} = f^{\mathcal{M}_i}\bar{a}$ where i is such that $\bar{a} \in M_i^{m_f}$.

– if R is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i<\lambda} R^{\mathcal{M}_i}$

Theorem 3.11 (Downward Löwenheim-Skolem)**.** Let $\mathcal{N}$ be an L-structure, and $|N| \geq |L| + \omega$. Let $A \subseteq N$. Then for any cardinal λ such that $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preccurlyeq \mathcal{N}$ such that

(i) $A \subseteq M$

(ii) $|\mathcal{M}| = \lambda$.

(It helps to think about the case $|L| \leq \omega$, $|A| = \omega$ and $|N|$ is uncountable).

For instance, think of $(\mathbb{C}, +, \cdot, -, ^{-1}, 0, 1)$ as a field. Then $\mathbb{Q} \subseteq \mathbb{C}$: it is a subset and a substructure. In particular, the property of being algebraically closed is in the theory of $\mathbb{C}$. Thus Theorem 3.11 gives a algebraically closed field, which is countable and contains $\mathbb{Q}$ - a possibility is the algebraic closure of $\mathbb{Q}$.

Proof. We inductively build a chain $\langle A_i : i < \omega \rangle$, with $A_i \subseteq N$, such that $|A_i| = \lambda$. (Our goal is to define $M = \bigcup_{i<\omega} A_i$).

Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$. At stage $i+1$, assume that A_i has been built, with $|A_i| = \lambda$. Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$-formulas such that $\mathcal{N} \vDash \exists x\ \phi_k(x)$ (observe there are no more than λ, since $|L(A)| = |L| + |A| + \omega \leq \lambda$). Let a_k be such that $\mathcal{N} \vDash \phi_k(a_k)$ and let $A_{i+1} = A_i \cup \{ a_k : k < \lambda \}$. Then $|A_{i+1}| = \lambda$.

Now let $M = \bigcup_{i<\omega} A_i$. We use the Tarski-Vaught test to show that M is the domain of a structure $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|M| = \lambda$:

Let $\mathcal{N} \vDash \exists x\ \psi(x, \bar{a})$, where $\bar{a}$ is a tuple in M. Then $\bar{a}$ is a *finite* tuple, so there is an i such that $\bar{a}$ is in A_i. Then A_{i+1}, by construction, contains b such that $\mathcal{N} \vDash \phi(b, \bar{a})$. But $A_{i+1} \subseteq M$, so $b \in M$. □

4 Two relational structures

4.1 Dense linear orders

Lecture 5 **Definition 4.1** (Dense linear orders)**.** A **linear order** is an $L_{\text{lo}} = \{<\}$-structure such that

(i) $\forall x \neg(x < x)$

(ii) $\forall xyz\ ((x < y \land y < z) \to x < z)$

(iii) $\forall xy\ ((x < y) \land (y < x) \lor (x = y))$.

A linear order is **dense** if it also satisfies

(iv) $\exists xy\ (x < y)$

(v) $\forall xy\ (x < y \to \exists z\ (x < z < y))$ (density).

A linear order has no endpoints if

(vi) $\forall x\ (\exists y\ (x < y) \land \exists z\ (z < x))$

T_{dlo} is the theory that includes axioms (i) to (vi), T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark: (iv) and (v) imply that if $\mathcal{M} \models T_{\text{dlo}}$ then $|\mathcal{M}| \geq \omega$.

Definition 4.2 ((Finite) Partial embedding)**.** If $\mathcal{M}, \mathcal{N} \models T_{\text{lo}}$, then an injective map $p : A \subseteq M \to N$ is called a **partial embedding** if for all $a, b \in A$,

$$\mathcal{M} \models a < b \iff \mathcal{N} \models p(a) < p(b).$$

If $|\operatorname{dom}(p)| < \omega$, then p is a **finite partial embedding**.

Lemma 4.3 (Extension lemma for dense linear orders)**.** Suppose $\mathcal{M} \models T_{\text{lo}}$, $\mathcal{N} \models T_{\text{dlo}}$, let $p : A \subseteq M \to N$ be a finite partial embedding. Then if $c \in M$, there is a finite partial embedding $\hat{p}$ such that $p \subseteq \hat{p}$ and $c \in \operatorname{dom}(\hat{p})$.

Proof. Split into three cases:

1. $a < c$ for all $a \in \operatorname{dom}(p)$. Then choose $d \in \mathcal{N}$ so that $b < d$ for all $b \in \operatorname{img}(p)$.
2. $a_i < c < a_{i+1}$ for some $a_i, a_{i+1} \in \operatorname{dom}(p)$. Then $\mathcal{N} \models p(a_i) < p(a_{i+1})$, so by density, $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$.
3. $c < a$ for all $a \in \operatorname{dom} p$. Similar to case 1. □

Theorem 4.4. Let $\mathcal{M}, \mathcal{N} \models T_{\text{dlo}}$ such that $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \to N$ be a finite partial embedding. Then there is $\pi : \mathcal{M} \to \mathcal{N}$, an isomorphism such that $p \subseteq \pi$.

Proof. Enumerate M, N: say $M = \langle a_i : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ sequences of elements. We define inductively a chain of finite partial embeddings $\langle p_i : i < \omega \rangle$ (idea: $\pi = \bigcup_{i<\omega} p_i$).

Let $p_0 = p$. At stage $i + 1$, p_i is given. We want to include a_i in $\operatorname{dom}(p_{i+1})$, and b_i in $\operatorname{img}(p_{i+1})$.

Forward step: By Lemma 4.3, extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \operatorname{dom}(p_{i+\frac{1}{2}})$. Backward step: By Lemma 4.3 applied to $p_{i+\frac{1}{2}}^{-1}$ to include $b_i \in \operatorname{dom}(p_{i+\frac{1}{2}}^{-1})$ (i.e. in the range of p_{i+1}). Then p_{i+1} extends p_i as required.

Let $\pi = \bigcup_{i<\omega} p_i$. Then (check) π is an isomorphism (i.e. order-preserving bijection). □

Definition 4.5 (Consistent, complete, $\vdash$)**.** An L-theory T is **consistent** if there is $\mathcal{M}$ such that $\mathcal{M} \vDash T$. If T is a theory in L and ϕ is an L-sentence, then we write $T \vdash \phi$ if for all $\mathcal{M}$ such that $\mathcal{M} \vDash T$, we also have $\mathcal{M} \vDash \phi$. An L-theory T is **complete** if for all L-sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$.

Is T_{dlo} complete?

Lecture 6 **Definition 4.6** (ω-categorical)**.** A theory T in a countable language with a countably infinite model is called ω-categorical if any two countable models of T are isomorphic.

Corollary 4.7 (of Theorem 4.4)**.** T_{dlo} is ω-categorical.

Proof. Say $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$, and $|\mathcal{M}| = |\mathcal{N}| = \omega$. Then $\varnothing$ (the empty map) is a finite partial embedding. By Theorem 4.4, $\mathcal{M} \simeq \mathcal{N}$. (Can also use any $\{\langle a, b\rangle\}$ where $a \in \mathcal{M}, b \in \mathcal{N}$ as initial finite partial embedding). $\square$

Theorem 4.8. If T is an ω-categorical theory in a countable language, and T has no finite models then T is complete.

Proof. Let $\mathcal{M} \vDash T$ and φ be an L-sentence.

If $\mathcal{M} \vDash \varphi$, suppose $\mathcal{N} \vDash T$. Then by Downward Löwenheim-Skolem, there are $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$ such that $|\mathcal{M}'| = |\mathcal{N}'| = \omega$. By ω-categoricity, $\mathcal{M}' \simeq \mathcal{N}'$, so in particular $\mathcal{M}' \equiv \mathcal{N}'$ and so $\mathcal{N}' \vDash \varphi$.

If $\mathcal{M} \vDash \neg\varphi$, similar. $\square$

Corollary 4.9. T_{dlo} is complete.

Definition 4.10 ((Partial) elementary map)**.** If $\mathcal{M}, \mathcal{N}$ are L-structures, a map f such that $\operatorname{dom} f \subseteq M$ and $\operatorname{img} f \subseteq N$ is called a **(partial) elementary map** if for all L-formulae $\phi(\bar{x})$ and $\bar{a} \in (\operatorname{dom} f)^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(f(\bar{a})).$$

Remark 4.11. A map f is elementary iff every finite restriction of f is elementary.

Proof.

$\Leftarrow$ Suppose f is not elementary. Then there are $\varphi(\bar{x})$ and $\bar{a} \in (\operatorname{dom} f)^{|\bar{x}|}$ such that

$$\mathcal{M} \vDash \phi(\bar{a}) \not\iff \mathcal{N} \vDash \phi(f(\bar{a})).$$

Then $f|_{\bar{a}}$ is a finite restriction of f that is not elementary.

$\Rightarrow$ Clear. $\square$

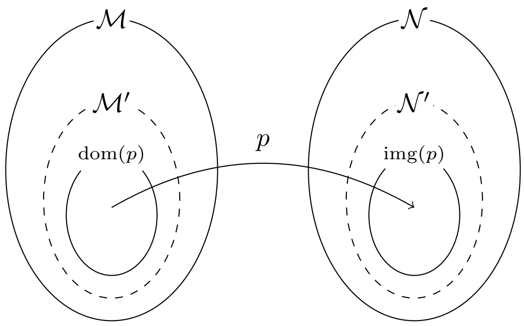
Proposition 4.12. Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$ and let $p : A \subseteq M \to N$ be a partial embedding. Then p is elementary.

Proof. By Remark 4.11, it suffices to consider p finite. By Downward Löwenheim-Skolem, we choose $\mathcal{M}', \mathcal{N}'$ such that

(i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.

(ii) $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$

(iii) $\operatorname{dom}(p) \subseteq \mathcal{M}', \operatorname{img}(p) \subseteq \mathcal{N}'$



Now p is a finite partial embedding between countable models, so p extends to an isomorphism $\pi : \mathcal{M}' \to \mathcal{N}'$ by Theorem 4.4. In particular, π is an elementary map between $\mathcal{M}$ and $\mathcal{N}$. $\square$

Corollary 4.13. $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

Proof. Use Proposition 4.12 with $\mathrm{id} : \mathbb{Q} \to \mathbb{R}$. $\square$

4.2 Random graph

Definition 4.14 (Random graph)**.** Let $L_{\mathrm{gph}} = \{R\}$, a binary relation symbol. An L_{gph}-structure is a **graph** if

(i) $\forall x \, \neg R(x, x)$

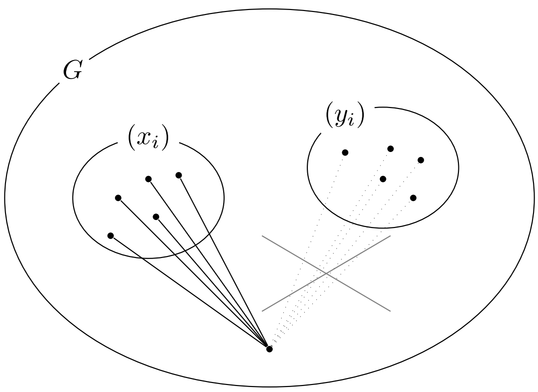
(ii) $\forall xy \, (R(x, y) \leftrightarrow R(y, x))$

An L_{gph}-structure is a **random graph** if it is a graph such that, for all $n \in \omega$, axiom (r_n) holds:

$$\forall x_0 \dots x_n, y_0 \dots y_n \left(\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left(\bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii) $\exists xy \, (x \neq y)$.

Axiom (r_n) effectively says that for disjoint subsets (x_i) and (y_i) each of size n, there is a (different) node z connected to each x_i and none of the y_i.



Remark. A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

Fact 4.15. There is a random graph.

Proof. Let the domain be ω, let $i, j \in \omega$ such that $i < j$. Write j as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff 2^i appears in the sum. $\square$

Exercise. Prove that ω with this definition of R is a random graph.

Definition 4.16 (Graph theories, partial embedding)**.** T_{gph} consists of the axioms (i),(ii) above, and $T_{\text{rg}} = T_{\text{gph}} \cup \{(\text{iii}), (r_n) : n \in \omega\}$. If $\mathcal{M}, \mathcal{N} \models T_{\text{gph}}$, a **partial embedding** is an injective map $p : A \subseteq M$ to N such that

$$\mathcal{M} \models R(a, b) \iff \mathcal{N} \models R(p(a), p(b))$$

for all a, b in the domain. Just as before, if $|\operatorname{dom}(p)| < \omega$ then p is called a **finite partial embedding**.

Lemma 4.17 (Extension lemma for random graphs)**.** Let $\mathcal{M} \models T_{\text{gph}}$, $\mathcal{N} \models T_{\text{rg}}$, let $p : A \subseteq M \to N$ be a finite partial embedding, and let $c \in M$. Then there is a partial embedding $\hat{p} : \hat{A} \subseteq M \to N$ such that, $c \in \operatorname{dom}(\hat{p})$, and $p \subseteq \hat{p}$.

Lecture 7 *Proof.* Take $c \in M$, $c \notin \operatorname{dom}(p)$.

diagram coming soon

Find $d \in N$ such that $N \models R(d, p(a)) \iff M \models R(c, a)$. $\square$

Theorem 4.18. Let $\mathcal{M}, \mathcal{N} \models T_{\text{rg}}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $p : A \subset M \to N$ a finite partial embedding. Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends p.

Proof. Same as proof of Theorem 4.4, but with Lemma 4.17 instead of Lemma 4.3. $\square$

Corollary 4.19. T_{rg} is ω-categorical and complete. Moreover, every finite partial embedding between models of T_{rg} is an elementary map.

Remark 4.20. The unique (up to isomorphism) countable model of T_{rg} is *the* countable random graph, or the **Rado graph**. It is universal with respect to finite and countable graphs (i.e. it embeds them all). It is **ultrahomogeneous** i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.

5 Compactness

Definition 5.1. Take an L-theory T.

(i) T is **finitely satisfiable** if every finite subset of sentences in T has a model.

(ii) T is **maximal** if for all L-sentences σ, either $\sigma \in T$ or $\neg\sigma \in T$.

(iii) T has the **witness property** if for all $\phi(x)$ (L-formula with one free variable) there is a constant $c \in \mathscr{C}$ such that

$$(\exists x\ \phi(x)) \to \phi(c) \in T.$$

Lemma 5.2. If T is maximal and finitely satisfiable and φ is an L-sentence, and $\Delta \overset{\text{finite}}{\subseteq} T$ with $\Delta \vdash \varphi$, then $\varphi \in T$.

Proof. If $\varphi \notin T$ then $\neg\varphi \in T$ (by maximality). But then $\Delta \cup \{\neg\varphi\}$ is a finite subset of T which does not have a model. □

Lemma 5.3. Let T be a maximal, finitely satisfiable theory with the witness property. Then T has a model. Moreover, if λ is a cardinal and $|\mathscr{C}| \leq \lambda$, then T has a model of size at most λ.

Proof. Let $c, d \in \mathscr{C}$, define $c \sim d$ iff $c = d \in T$.

Claim: $\sim$ is an equivalence relation. **Proof:** For transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$, so $c = e \in T$ (by Lemma 5.2), and so $c \sim e$. Reflexivity follows from maximality, and symmetry is immediate. ■

We denote $[c] \in \mathscr{C}/\sim$ by c^*. Now, define a structure $\mathcal{M}$ whose domain is $\mathscr{C}/\sim\, = M$. Clearly, $|M| \leq \lambda$ if $|\mathscr{C}| \leq \lambda$. We must define interpretations in $\mathcal{M}$ for symbols of L.

- If $c \in \mathscr{C}$, then $c^{\mathcal{M}} = c^*$.
- If $R \in \mathscr{R}$, define

$$R^{\mathcal{M}} := \{\, (c_1^*, \dots, c_{n_R}^*) \mid R(c_1, \dots, c_n) \in T \,\}.$$

Claim: $R^{\mathcal{M}}$ is well defined. **Proof:** Suppose $\bar{c}, \bar{d} \in \mathscr{C}^{n_R}$ and suppose $c_i \sim d_i$. That is, $c_i = d_i \in T$ for $i = 1, \dots, n_R$ so by Lemma 5.2

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T. \qquad ■$$

- If $f \in \mathscr{F}$, and $\bar{c} \in \mathscr{C}^{n_R}$, then $f\bar{c} = d \in T$ for some $d \in \mathscr{C}$. (This is because $\exists x\ (f(\bar{c}) = x) \in T$ so apply witness property.)

Then define $f^{\mathcal{M}}(\bar{c}^*) = d^*$. Exercise: Check $f^{\mathcal{M}}(\bar{c}^*)$ is well-defined!

Claim: if $t(x_1, \dots, x_n)$ is an L-term and $c_1, \dots, c_n, d \in \mathscr{C}$, then

$$t(c_1, \dots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*.$$

Proof:

($\Rightarrow$) by induction on the complexity of t.

($\Leftarrow$) Assume $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. Then

$$t(c_1, \dots, c_n) = e \in T$$

for some constant e by witness property and Lemma 5.2. Use ($\Rightarrow$) to get that $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$. But then $d^* = e^*$, i.e. $d = e \in T$. Then $t(c_1, \dots, c_n) = d \in T$. ■

Claim: For all L-formulas $\varphi(\bar{x})$, and $\bar{c} \in \mathscr{C}^{|\bar{x}|}$,

$$\mathcal{M} \vDash \varphi(\bar{c}) \iff \varphi(\bar{c}) \in T.$$

Proof: By induction on $\varphi(\bar{x})$. (Exercise: Fill in the details). ■ This shows $\mathcal{M} \vDash T$. □

Lecture 8 **Lemma 5.4.** Let T be a finitely satisfiable L-theory. Then there are $L^* \supseteq L$ and a finitely satisfiable L^*-theory $T^* \supseteq T$ such that

(i) $|L^*| = |L| + \omega$.

(ii) any L^*-theory extending T^* has the witness property.

Proof. We define $\langle L_i : i < \omega \rangle$ a chain of languages containing L and such that $|L_i| = |L| + \omega$, and $\langle T_i : i < \omega \rangle$ of finitely satisfiable theories such that $\forall i, T_i$ is an L_i-theory and $T_i \supseteq T$.

Set $L_0 = L$ and $T_0 = T$. At stage $i+1$, L_i and T_i are given. List all L_i-formulas $\varphi(x)$ (one free variable) and let

$$L_{i+1} = L_i \cup \{ c_\varphi \mid \varphi(x) \text{ an } L_i \text{ formula} \}.$$

For all $\varphi(x)$, an L_i formula in one free variable, let Φ_φ be the L_{i+1}-sentence

$$\exists x\, \varphi(x) \to \varphi(c_\varphi).$$

Then let

$$T_{i+1} = T_i \cup \{ \Phi_\varphi \mid \varphi(x) \text{ is an } L_i \text{ formula} \}.$$

Claim: T_{i+1} is finitely satisfiable.
Proof: Let $\Delta \subseteq T_{i+1}$ be finite. Then

$$\Delta = \Delta_0 \cup \{\Phi_{\varphi_1}, \dots, \Phi_{\varphi_n}\}$$

where $\Delta_0 \subseteq T_i$. Let $\mathcal{M} \vDash \Delta_0$ ($\mathcal{M}$ is an L_i structure; it exists because T_i is finitely satisfiable).

We define an L_{i+1}-structure $\mathcal{M}'$ with domain M. Define the interpretation of new constants as follows: if $\mathcal{M} \vDash \exists x\ \varphi(x)$, then let a be such that $\mathcal{M} \vDash \varphi(a)$, and set $c_\varphi^{\mathcal{M}'} := a$. Otherwise, $c_\varphi^{\mathcal{M}'}$ is arbitrary. Then $\mathcal{M}' \vDash \Delta$. ■

Let

$$L^* = \bigcup_{i<\omega} L_i, \qquad T^* = \bigcup_{i<\omega} T_i.$$

By construction, any extension of T^* has the witness property (check this!) and T^* is finitely satisfiable. (If $\Delta \overset{\text{finite}}{\subseteq} T^*$ then $\Delta \subseteq T_i$ for some i). □

Lemma 5.5. If T is finitely satisfiable, there exists a maximal finitely satisfiable $T' \supseteq T$.

Proof. Let

$$I := \{ S \mid S \text{ is a finitely satisfiable } L\text{-theory such that } T \subseteq S \}.$$

I is partially ordered by inclusion, and non-empty.

If $\langle C_i : i < \lambda \rangle$ is a chain in I, then $\bigcup_{i<\lambda} C_i$ is an upper bound for the chain - it is finitely satisfiable. Then by Zorn's lemma, I has a maximal element (with respect to $\subseteq$).

Claim: the maximal element T' of I is the required extension of T (check that for all L-sentences σ, $\sigma \in T'$ or $\neg\sigma \in T'$). □

Theorem 5.6 (Compactness)**.** If T is a finitely satisfiable L-theory and $\lambda \geq |L| + \omega$, then there is $\mathcal{M} \vDash T$ such that $|\mathcal{M}| \leq \lambda$.

Proof sketch. Extend T to T^*, an L^*-theory that is finitely satisfiable and such that any $S \supseteq T^*$ has the witness property (by Lemma 5.4).

By Lemma 5.5, there is $T' \supseteq T^*$, which is maximal and finitely satisfiable. Then T' has the witness property. Then by Lemma 5.3 there is $\mathcal{M} \vDash T'$ with $|\mathcal{M}| \leq \lambda$, and $\mathcal{M} \vDash T$. ☐

Definition 5.7 (Type)**.** Let L be a language.

- An L-**type** $p(\bar{x})$ is a set of L-formulas whose free variables are in $\bar{x}$ (and $\bar{x} = \langle x_i : i < \lambda \rangle$).
- An L-type is **satisfiable** if there is an L-structure $\mathcal{M}$ and an assignment $\bar{a} \in \mathcal{M}^{|\bar{x}|}$ to $\bar{x}$ such that $\mathcal{M} \vDash \varphi(\bar{a})$ for all $\varphi(\bar{x}) \in p(\bar{x})$ (we also say $p(\bar{x})$ **consistent**, and that $\bar{a}$ **realizes** $p(\bar{x})$ in $\mathcal{M}$). We write $\mathcal{M} \vDash p(\bar{a})$ or $\mathcal{M}, \bar{a} \vDash p(\bar{x})$. We also say that $p(\bar{x})$ is **satisfied** in $\mathcal{M}$.
- A type $p(\bar{x})$ is **finitely satisfiable** if every finite subset of $p(x)$ is satisfiable (we may say $p(\bar{x})$ is **finitely consistent**).

Remark. An L-type may be finitely satisfiable in $\mathcal{M}$ (i.e. every finite subset is satisfiable in $\mathcal{M}$) but not satisfiable in $\mathcal{M}$.

Example. Take $\mathcal{M} = (\mathbb{N}, <)$. Let $\phi_n(x)$ say 'there are at least n elements less than x'.

$$p(x) \coloneqq \{ \phi_n(x) \mid n < \omega \}$$

Is $p(x)$ finitely satisfiable in $\mathcal{M}$? Yes. But $p(x)$ is not satisfiable in $\mathcal{M}$.

Theorem 5.8 (Compactness theorem for types)**.** Every finitely satisfiable L-type $p(\bar{x})$ is satisfiable.

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$, let $\langle c_i : i < \lambda \rangle$ be new constants (not in L). Expand L to $L' = L \cup \{c_i : i < \lambda\}$. Then $p(\bar{c})$ is a finitely satisfiable L'-theory and Theorem 5.6 applied to $p(\bar{c})$ gives an L'-structure $\mathcal{M}'$ such that $\mathcal{M}' \vDash p(\bar{c})$. But $\mathcal{M}'$ reduces to an L structure $\mathcal{M}$, so $\mathcal{M}, \bar{c}^{\mathcal{M}'} \vDash p(\bar{x})$. ☐

Lecture 9 **Lemma 5.9.** Let $\mathcal{M}$ be a structure, let $\bar{a} = \langle a_i : i < \lambda \rangle$ an enumeration of $\mathcal{M}$. Let

$$q(\bar{x}) = \{ \varphi(\bar{x}) \mid \mathcal{M} \vDash \varphi(\bar{a}) \},$$

where $|\bar{x}| < \lambda$. Then $q(\bar{x})$ is satisfiable in $\mathcal{N}$ iff there is $\beta : \mathcal{M} \to \mathcal{N}$ that is an elementary embedding.

Proof.

($\Rightarrow$) If $q(\bar{x})$ is satisfiable in $\mathcal{N}$, there is $\bar{b} \in N^{|\bar{x}|}$ such that

$$\mathcal{N} \vDash \varphi(\bar{b}) \quad \forall \varphi(\bar{x}) \in q(\bar{x}).$$

Then $\beta : a_i \mapsto b_i$ for $i < \lambda$ is an elementary embedding. (β preserves, for example, atomic formulas of the form $f(a_{i_1}, \ldots, a_{i_n}) = a_{i_{n+1}}$). More generally, for any $\varphi(\bar{x})$ an L-formula,

$$\mathcal{M} \vDash \varphi(\bar{a}) \iff \mathcal{N} \vDash \varphi(\bar{b})$$

but $\beta(\bar{a}) = \bar{b}$ so we have elementarity.

($\Leftarrow$) If $\beta : \mathcal{M} \to \mathcal{N}$ is elementary, then $\beta(\bar{a})$ satisfies $q(\bar{x})$ in $\mathcal{N}$. □

This lemma is sometimes also called the Diagram Lemma, and stated as: Suppose $\mathrm{Th}(\mathcal{M}_M)$ is a theory in $L(M)$. Then if $\mathcal{N} \vDash \mathrm{Th}(\mathcal{M}_M)$, then $\mathcal{M}$ embeds elementarily in $\mathcal{N}$.

Remark 5.10. We can consider types in $L(A)$, where $A \subseteq M$. In particular, we can have $M = A$.

Types of this kind are said to have **parameters in** A (or to be over A). If $p(\bar{x})$ is a type over M, then there is $\bar{a}$, an enumeration of M, and a type $p'(\bar{x}, \bar{z})$ in L where the $\bar{z}$ are new constants, $|\bar{z}| = |\bar{a}|$, and $p(\bar{x}) = p'(\bar{x}, \bar{a})$.

Theorem 5.11. If $\mathcal{M}$ is a structure, and $p(\bar{x})$ is a type in $L(M)$ that is finitely satisfiable in $\mathcal{M}$, then $p(\bar{x})$ is satisfiable in some $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example. Take $\mathcal{M} = (\mathbb{Q}, <)$, and let $\langle a_i : i < \omega \rangle$ a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ from below, and let $\langle b_i : i < \omega \rangle \subseteq \mathbb{Q}$ tend to $\sqrt{2}$ from above. Set $\phi_n(x) \coloneqq a_n < x < b_n$. Then let $p(x) = \{\, \phi_n(x) \mid n < \omega \,\}$. Then $p(x)$ is an $L(\mathbb{Q})$-type which is finitely satisfiable in $\mathbb{Q}$. But $p(x)$ is not satisfiable in $\mathcal{M}$. It is, however, satisfiable in $(\mathbb{R}, <) \succcurlyeq (\mathbb{Q}, <)$.

Proof of Theorem 5.11. Let $\langle a_i : i < \lambda \rangle$ enumerate $\mathcal{M}$, let

$$q(\bar{z}) \coloneqq \{\, \varphi(\bar{z}) \mid \mathcal{M} \vDash \varphi(\bar{a}) \,\}$$

where $|\bar{z}| = \lambda$ and the z_i are new variables (so not among the $\bar{x}$). Write $p(\bar{x})$ as $p'(\bar{x}, \bar{a})$ for some $p'(\bar{x}, \bar{z})$ (an L-type).

Claim: $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is finitely satisfiable in $\mathcal{M}$.

Proof: $p'(\bar{x}, \bar{a})$ is finitely satisfiable by hypothesis and $q(\bar{z})$ is realized by $\bar{a}$.

Then, by Compactness theorem for types, $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is satisfiable. That is, there is $\mathcal{N}$ and $\bar{b} \in \mathcal{N}^{|\bar{z}|}$ and $\bar{c} \in \mathcal{N}^{|\bar{x}|}$ such that

$$\mathcal{N} \vDash p'(\bar{c}, \bar{b}) \cup q(\bar{b}).$$

In particular, $\mathcal{N} \vDash q(\bar{b})$, then by Lemma 5.9, $\beta : a_i \mapsto b_i$ is an elementary embedding. □

Theorem 5.12 (Upward Löwenheim-Skolem)**.** Let $\mathcal{M}$ be such that $|\mathcal{M}| \geq \omega$. Then for any $\lambda \geq |\mathcal{M}| + |L|$, there is $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|\mathcal{N}| = \lambda$.

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$ a tuple of distinct variables. Let

$$p(\bar{x}) = \{\, x_i \neq x_j \mid i < j < \lambda \,\}.$$

Then $p(\bar{x})$ is finitely consistent in $\mathcal{M}$. By Theorem 5.11, $p(\bar{x})$ is realized in some $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|\mathcal{N}| \geq \lambda$. By Downward Löwenheim-Skolem, we may assume $|\mathcal{N}| = \lambda$. □

6 Saturation

Definition 6.1 (Saturated)**.** Let λ be an infinite cardinal, let $|\mathcal{M}| \geq \omega$. Then $\mathcal{M}$ is **λ-saturated** if $\mathcal{M}$ realizes every type $p(x)$ with one free variable such that

(i) $p(x)$ has parameters in $A \subseteq M$ and $|A| < \lambda$.

(ii) $p(x)$ is finitely consistent in $\mathcal{M}$.

$\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated.

Can $\mathcal{M}$ be λ-saturated if $\lambda > |\mathcal{M}|$? If so, $\mathcal{M}$ would satisfy finitely satisfiable types in $L(M)$. For example,

$$p(x) = \{\, x \neq a_i \mid i < |\mathcal{M}| \,\}$$

where $\langle a_i : i < |\mathcal{M}| \rangle$ enumerates $\mathcal{M}$. $p(x)$ is finitely satisfiable, but not satisfied in $\mathcal{M}$.

Lecture 10 **Definition 6.2** (Type of tuple)**.** Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, $\bar{b}$ a tuple in M (possibly infinite). The **type of $\bar{b}$ over** A is the following $L(A)$-type:

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) := \{\, \varphi(\bar{x}) \in L(A) \mid \mathcal{M} \vDash \varphi(\bar{b}) \,\}\,.$$

The subscript $\mathcal{M}$ is often omitted if clear from context.

Remark 6.3.

(i) $\mathrm{tp}_{\mathcal{M}}(\bar{b}/A)$ is complete, i.e. for every $L(A)$ formula $\phi(\bar{x})$, either $\phi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$ or $\neg\phi(x) \in \mathrm{tp}(\bar{b}/A)$.

(ii) If $\mathcal{M} \preccurlyeq \mathcal{N}$, then for $A \subseteq M$, $\bar{b}$ a tuple:

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A).$$

Fact 6.4.

(i) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is a (partial) elementary map, then in particular f preserves L-sentences, so $\mathcal{M} \equiv \mathcal{N}$.

(ii) If $\mathcal{M} \equiv \mathcal{N}$, then $\varnothing$, the empty map, is an elementary map, as it preserves sentences.

(iii) If $f : A \subseteq \mathcal{M} \to \mathcal{N}$ is elementary, and $\bar{a}$ is an enumeration of $A = \mathrm{dom}(f)$, then

$$\mathrm{tp}(\bar{a}/\varnothing) = \mathrm{tp}(f(\bar{a})/\varnothing).$$

More generally, if $f : \mathcal{M} \to \mathcal{N}$ is (partial) elementary and there is $A \subseteq M \cap N$ such that $A \subseteq \mathrm{dom}\, f$, $f|_A = \mathrm{id}$, then for every $\bar{b}$, a tuple in $\mathrm{dom}(f)$,

$$\mathrm{tp}_{\mathcal{M}}(\bar{b}/A) = \mathrm{tp}_{\mathcal{N}}(f(\bar{b})/A).$$

(iv) Let $\bar{a}$ enumerate $A \subseteq M$, $A = \mathrm{dom}(f)$ where $f : \mathcal{M} \to \mathcal{N}$ is elementary. Let $p(\bar{x}, \bar{a})$ be a type in $L(A)$ that is finitely satisfiable in $\mathcal{M}$. Then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in $\mathcal{N}$:

Let

$$\{\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a}).$$

By finite satisfiability of $p(\bar{x}, \bar{a})$,

$$\mathcal{M} \vDash \exists \bar{x} \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{a}).$$

Then

$$\mathcal{N} \vDash \exists x \bigwedge_{i=1}^{m} \varphi_i(\bar{x}, f(\bar{a}))$$

by elementarity of f. (Does $p(\bar{x}, \bar{a})$ satisfiable in $\mathcal{M}$ imply $p(\bar{x}, f(\bar{a}))$ satisfiable in $\mathcal{N}$? No.)

Theorem 6.5. Let $\mathcal{N}$ be such that $|\mathcal{N}| \geq \lambda \geq |L| + \omega$. The following are equivalent:

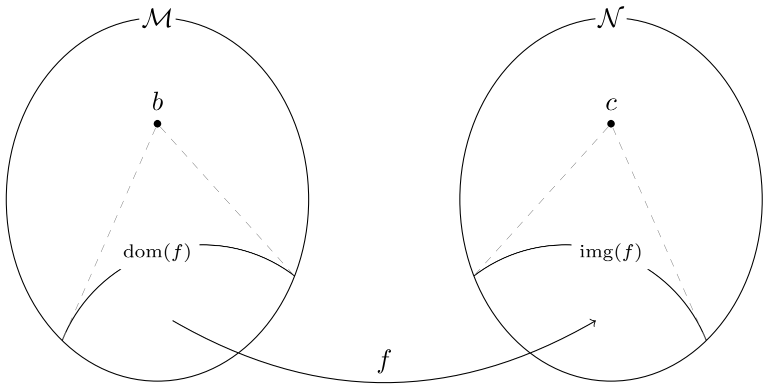
(i) $\mathcal{N}$ is λ-saturated.

(ii) if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$ and $f : \mathcal{M} \to \mathcal{N}$ partial elementary map such that $|f| < \lambda$, then there is a partial elementary $\hat{f} \supseteq f$ and such that $b \in \operatorname{dom}(\hat{f})$.

(iii) If $p(\bar{z})$ is an $L(A)$-type where $|\bar{z}| \leq \lambda$ and $|A| < \lambda$ and $p(\bar{z})$ is finitely satisfiable in $\mathcal{N}$, then $p(\bar{z})$ is satisfiable in $\mathcal{N}$.

Proof. (i) $\Rightarrow$ (ii). Let $f : \mathcal{M} \to \mathcal{N}$ be as in (ii), let $b \in M$. Let $\bar{a}$ be an enumeration of $\operatorname{dom}(f)$, so $|\bar{a}| < \lambda$. Let

$$p(x/\bar{a}) := \operatorname{tp}_{\mathcal{M}}(b/\bar{a}).$$



Then $p(x/\bar{a})$ is finitely satisfiable in $\mathcal{M}$, hence $\operatorname{tp}(x/f(\bar{a}))$ is finitely satisfiable in $\mathcal{N}$ (by Fact 6.4(iv)). Since $|f(\bar{a})| < \lambda$ and $\mathcal{N}$ is λ-saturated, $\operatorname{tp}(x/f(\bar{a}))$ is realized in $\mathcal{N}$ by some c. Then $f \cup \{\langle b, c\rangle\}$ is the required extension of f:

$$\mathcal{M} \vDash \phi(b, \bar{a}) \iff \mathcal{N} \vDash \phi(c, f(\bar{a}))$$

Lecture 11 (ii) $\Rightarrow$ (iii). Let $p(\bar{z})$ be as in (iii). There is $\mathcal{M}$ such that $\mathcal{N} \preccurlyeq \mathcal{M}$ and $\mathcal{M} \vDash p(\bar{b})$. The identity map $\mathrm{id}_A : \mathcal{M} \to \mathcal{N}$ is partial elementary. Idea: build $\langle f_i : i < |\bar{b}|\rangle$ of partial elementary maps extending id_A. Then $\bigcup_i f_i$ is partial elementary, and $\bar{b} \in \operatorname{dom} \bigcup_{i<|\bar{a}|} f_i$.

Set $f_0 = \mathrm{id}_A$, at stage $i+1$ use (ii) to put b_i in $\operatorname{dom}(f_{i+1})$. At limit stages, $\mu < \lambda$, let $f_\mu = \bigcup_{i<\mu} f_i$.

(iii) $\Rightarrow$ (i) is trivial. □

Corollary 6.6. If $\mathcal{M}$ and $\mathcal{N}$ are saturated and $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}|$ then any elementary $f : \mathcal{M} \to \mathcal{N}$ extends to an isomorphism (in particular $\mathcal{M} \simeq \mathcal{N}$).

Proof. Use Theorem 6.5(ii) to extend $f : \mathcal{M} \to \mathcal{N}$ to an isomorphism by back-and-forth (take unions at limit stages). □

Corollary 6.7. Models of of T_{dlo} and T_{rg} are ω-saturated.

Proof. By Theorem 6.5 and Lemma 4.3 for T_{dlo} and Lemma 4.17 for T_{rg}. □

So $(\mathbb{Q}, <)$ is ω-saturated. Is $(\mathbb{R}, <)$ ω_1 saturated? No. It does not realize

$$p(x) \coloneqq \{\, x > q \mid q \in \mathbb{Q} \,\}\,.$$

Definition 6.8 (Automorphism)**.** An isomorphism $\alpha : \mathcal{N} \to \mathcal{N}$ is called an **automorphism**. The automorphisms of $\mathcal{N}$ form a group denoted by $\mathrm{Aut}(\mathcal{N})$. If $A \subseteq N$, then

$$\mathrm{Aut}(\mathcal{N}/A) \coloneqq \{\, \alpha \in \mathrm{Aut}(\mathcal{M}) \mid \alpha|_A = \mathrm{id} \,\}\,.$$

Definition 6.9 (Universality, homogeneity)**.**

(i) An L-structure $\mathcal{N}$ is **λ-universal** if for every $\mathcal{M} \equiv \mathcal{N}$ such that $|\mathcal{M}| \leq \lambda$ there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$. $\mathcal{N}$ is **universal** if it is $|\mathcal{N}|$-universal.

(ii) $\mathcal{N}$ is **λ-homogeneous** if every elementary map $f : \mathcal{N} \to \mathcal{N}$ such that $|f| < \lambda$ extends to an isomorphism of $\mathcal{N}$.

Theorem 6.10. Let $\mathcal{N}$ be such that $|\mathcal{N}| \geq |L| + \omega$. The following are equivalent

(i) $\mathcal{N}$ is saturated

(ii) $\mathcal{N}$ is universal and homogeneous.

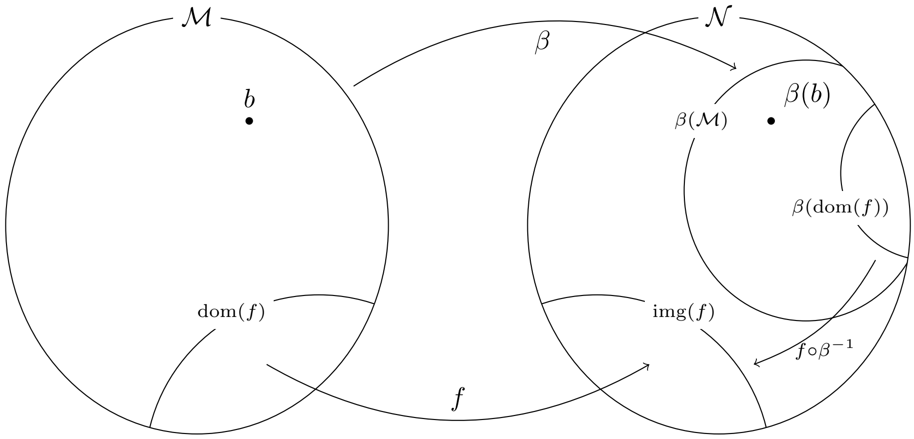
Proof. (i) $\Rightarrow$ (ii). Assume $\mathcal{N}$ is saturated, and $\mathcal{M} \equiv \mathcal{N}$ is such that $|\mathcal{M}| \leq |\mathcal{N}|$. Then let $\bar{a}$ enumerate $\mathcal{M}$, let $p(\bar{x}) = \mathrm{tp}(\bar{a}/\varnothing)$. Then $p(\bar{x})$ is finitely satisfiable in $\mathcal{M}$.

Claim: $p(\bar{x})$ is finitely satisfiable in $\mathcal{N}$. Indeed, let $\{\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\mathcal{M} \vDash \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x})$, and so $\mathcal{N} \vDash \exists x \bigwedge \varphi_i(\bar{x})$ since $\mathcal{M} \equiv \mathcal{N}$.

Since $|\bar{x}| \leq |\mathcal{N}|$, $\mathcal{N}$ realizes $p(\bar{x})$ by saturation (Theorem 6.5). Homogeneity follows from Corollary 6.6.

(ii) $\Rightarrow$ (i). We show that if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$, $f : \mathcal{M} \to \mathcal{N}$ elementary such that $|f| < |\mathcal{N}|$ then there is $\hat{f} \supseteq f$ elementary defined on b.

By working in $\mathcal{M}' \preccurlyeq \mathcal{M}$ such that $\mathrm{dom}(f) \cup \{b\} \subseteq \mathcal{M}'$ if necessary (using Theorem 3.11), we may assume $|\mathcal{M}| \leq |\mathcal{N}|$. Since $\mathcal{M} \equiv \mathcal{N}$, by universality there is an elementary embedding $\beta : \mathcal{M} \to \mathcal{N}$. Then $\beta(\mathcal{M}) \preccurlyeq \mathcal{N}$.



Then the map $f \circ \beta^{-1} : \beta(\mathrm{dom}(f)) \to \mathrm{img}(f)$ is elementary. By homogeneity, there is $\alpha \in \mathrm{Aut}(\mathcal{N})$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b))\}$ is elementary (it is a restriction of $\alpha \circ \beta$). $\square$

Definition 6.11 (Orbit, defined set)**.** Let $\bar{a}$ be a tuple in $\mathcal{N}$ and $A \subseteq N$. The **orbit** of $\bar{a}$ over A is the set

$$O_{\mathcal{N}}(\bar{a}/A) = \{\, \alpha(\bar{a}) \mid \alpha \in \mathrm{Aut}(\mathcal{N}/A) \,\}\,.$$

If $\varphi(\bar{x})$ is an $L(A)$-formula, then

$$\varphi(\mathcal{N}) \coloneqq \{\, \bar{a} \in N^{|\bar{x}|} \mid \mathcal{N} \vDash \varphi(\bar{a}) \,\}$$

is the **set defined by** $\varphi(\bar{x})$. A set is **definable** over A if it is defined by some $L(A)$-formula. There are analogous notions of a type defining a set, and a set being type-definable.

Lecture 12 **Remark 6.12.** If $\bar{a}$, $\bar{b}$ are tuples in $\mathcal{N}$ of the same length, and $A \subseteq N$, then the following are equivalent.

(i) $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$

(ii) $\{\, a_i \mapsto b_i \mid i < |\bar{a}| \,\} \cup \mathrm{id}_A$ is an elementary map from $\mathcal{N}$ to $\mathcal{N}$

Proposition 6.13. Let $\mathcal{N}$ be λ-homogeneous, $A \subseteq N$, with $|A| < \lambda$ and let $\bar{a}$ a tuple in $\mathcal{N}$ such that $|\bar{a}| < \lambda$. Then

$$O_{\mathcal{N}}(\bar{a}/A) = p(\mathcal{N})$$

where $p(\bar{x}) = \mathrm{tp}_{\mathcal{N}}(\bar{a}/A)$.

Proof. If $\alpha(\bar{a}) = \bar{b}$, where $\alpha \in \mathrm{Aut}(\mathcal{N}/A)$, then $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$.

If $\mathrm{tp}_{\mathcal{N}}(\bar{a}/A) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/A)$, then $\{\, \langle a_i, b_i \rangle \mid i < |\bar{a}| \,\} \cup \mathrm{id}_A$ is elementary, and by homogeneity it extends to $\alpha \in \mathrm{Aut}(\mathcal{N})$, and in particular $\alpha \in \mathrm{Aut}(\mathcal{N}/A)$. $\square$

7 The Monster Model

Given a complete theory T with an infinite model, we work in a saturated structure $\mathcal{U}$ (sometimes denoted $\mathbb{M}$) that is a model of T, which is sufficiently large such that any other model of T we might be interested in is an elementary substructure of $\mathcal{U}$. ($\mathcal{U}$ is an expository device - see Tent/Ziegler for more details, also Marker).

Definition 7.1 (Terminology and conventions)**.** When working in $\mathcal{U}$, we say

- '$\varphi(\bar{x})$ **holds**' to mean that $\mathcal{U} \vDash \forall \bar{x}\ \varphi(\bar{x})$
- '$\varphi(\bar{x})$ is **consistent**' to mean $\mathcal{U} \vDash \exists \bar{x}\ \varphi(\bar{x})$
- 'the type $p(\bar{x})$ is **consistent/satisfiable**' to mean $\mathcal{U} \vDash \exists \bar{x}\ p(\bar{x})$
- A cardinality λ is **small** if $\lambda < |U|$ (usually denote $|U|$ by κ)
- a **model** is some $\mathcal{M} \preccurlyeq \mathcal{U}$ such that $|M|$ is small

Conventions:

- all tuples assumed to have small length, unless specified otherwise
- formulas have parameters in U
- types have parameters in small sets
- definable sets have the form $\varphi(U)$ for some $L(U)$-formula $\varphi(\bar{x})$
- type definable sets have the form $p(U)$ for some type $p(\bar{x}, A)$ where $|A| < \kappa$.
- Orbits and types of tuples are within $\mathcal{U}$, so $\mathrm{tp}(\bar{a}/A)$ means $\mathrm{tp}_{\mathcal{U}}(\bar{a}/A)$,

$$O(\bar{a}/A) = O_{\mathcal{U}}(\bar{a}/A)$$

- If $p(\bar{x})$, $q(\bar{x})$ are types, we write $p(\bar{x}) \to q(\bar{x})$ to mean $p(\mathcal{N}) \subseteq q(\mathcal{N})$ (think of $p(\bar{x})$ as an infinite conjunction of formulas)

Fact 7.2. Let $p(\bar{x})$ be a satisfiable $L(A)$-type, and $q(\bar{x})$ a satisfiable $L(B)$-type, such that

$$p(\bar{x}) \to \neg q(\bar{x})$$

(explicitly, $p(\bar{x})$ and $q(\bar{x})$ have no common realisations).

Then there are $\varphi_i(\bar{x}) \in p(\bar{x})$ and $\psi_i(\bar{x}) \in q(\bar{x})$ such that

$$\bigwedge_{i=1}^{n} \varphi_i(\bar{x}) \to \neg \left(\bigwedge_{i=1}^{m} \psi_i(\bar{x}) \right).$$

Proof. $p(\bar{x}) \cup q(\bar{x})$ is not realized in $\mathcal{U}$. By saturation of $\mathcal{U}$, $p(\bar{x}) \cup q(\bar{x})$ is not finitely satisfiable, hence there exist finite subsets $\{\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\{\psi_1(\bar{x}), \ldots, \psi_n(\bar{x})\} \subseteq q(\bar{x})$ such that their union is not satisfiable. Then

$$\bigwedge \varphi_i(\bar{x}) \to \neg \left(\bigwedge \psi_i(\bar{x}) \right). \qquad \square$$

Remark 7.3. Let $\varphi(\mathcal{U}, \bar{b})$ be such that $\varphi(\bar{x}, \bar{z})$ is an L-formula, $\bar{b} \in \mathcal{U}^{|\bar{z}|}$. If $\alpha \in \mathrm{Aut}(\mathcal{U})$, then

$$\begin{aligned}\alpha[\varphi(\mathcal{U}, \bar{b})] &= \{\, \alpha(\bar{a}) \mid \varphi(\bar{a}, \bar{b}), \bar{a} \in \mathcal{U}^{|\bar{x}|} \,\} \\ &= \{\, \alpha(\bar{a}) \mid \varphi(\alpha(\bar{a}), \alpha(\bar{b})), \bar{a} \in \mathcal{U}^{|\bar{x}|} \,\} \\ &= \varphi(\mathcal{U}, \alpha(\bar{b}))\end{aligned}$$

So $\mathrm{Aut}(\mathcal{U})$ acts on the definable sets in a natural way. (Similarly for the type-definable sets)

Definition 7.4 (Invariant)**.** A set $D \subseteq \mathcal{U}$ is **invariant** under $\mathrm{Aut}(\mathcal{U}/A)$ (**invariant over** A) if $\alpha(D) = D$ for every $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$.

Equivalently, for all $\bar{a} \in D$, $O(\bar{a}/A) \subseteq D$.

If $\bar{a} \in D$, $q(\bar{x}) = \mathrm{tp}(\bar{a}/A)$ and $\bar{b} \vDash q(\bar{x})$, then $\bar{b} \in D$. ($\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{a}/A)$, so there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ s.t. $\alpha(\bar{a}) = \bar{b}$ by homogeneity of $\mathcal{U}$). Hence we could also define invariance over A as

$$\forall \bar{a} \in D, \qquad \bar{b} \equiv_A \bar{a} \implies \bar{b} \in D.$$

Proposition 7.5. Let $\varphi(\bar{x})$ be an $L(U)$-formula, then the following are equivalent:

(i) $\varphi(\bar{x})$ is equivalent to some $L(A)$-formula $\psi(\bar{x})$

(ii) $\varphi(\mathcal{U})$ is invariant over A

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i): Let $\varphi(\bar{x}, \bar{z})$ be an L-formula such that $\varphi(\mathcal{U}, \bar{b})$ is invariant over A, for suitable $\bar{b} \in U^{|\bar{z}|}$.

Let $q(\bar{z})$ be the type $\mathrm{tp}(\bar{b}/A)$. If $\bar{c} \vDash q(\bar{z})$, then there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(\bar{b}) = \bar{c}$. Then

$$\begin{aligned}\varphi(\mathcal{U}, \bar{c}) &= \alpha(\varphi(\mathcal{U}, \bar{b})) && \text{by Remark 7.3} \\ &= \varphi(\mathcal{U}, \bar{b}) && \text{by invariance}\end{aligned}$$

Hence

$$q(\bar{z}) \to \forall \bar{x}\ (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b})).$$

By an argument similar to Fact 7.2, there is $\theta(\bar{z}) \in q(\bar{z})$ such that $\theta(\bar{z}) \to \forall \bar{x}\ (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b}))$. Then $\theta(\bar{z})$ is an $L(A)$-formula and $\exists z\ [\theta(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})]$ defines $\varphi(\mathcal{U}, \bar{b})$. ☐

Lecture 13 **Definition 7.6.** An injective map $p : A \subseteq \mathcal{M} \to \mathcal{N}$ is a **partial embedding** if for all tuples in $A = \mathrm{dom}(p)$, p satisfies conditions (i), (ii), (iii) in Definition 1.5.

Idea: a partial embedding preserves quantifier-free formulas.

Proposition 7.7. Let $\varphi(\bar{x})$ be an L-formula. The following are equivalent:

(i) there is $\psi(\bar{x})$, a quantifier-free L-formula such that

$$\mathcal{U} \vDash \forall x\ [\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})].$$

(ii) for all partial embeddings $p : \mathcal{U} \to \mathcal{U}$, for all $\bar{a}$ from $\mathrm{dom}(\bar{p})$,

$$\varphi(\bar{a}) \leftrightarrow \varphi(p(\bar{a}))$$

Proof. (i) ⇒ (ii): clear.

(ii) ⇒ (i). For $\bar{a} \in U$, set

$$\mathrm{qftp}(\bar{a}) := \{\, \psi(\bar{x}) \mid \psi(\bar{a}) \text{ and } \psi(\bar{x}) \text{ is quantifier free} \,\}.$$

Let

$$D = \{\, q(\bar{x}) \mid q(\bar{x}) = \mathrm{qftp}(\bar{a}) \text{ for some } \bar{a} \text{ such that } \varphi(\bar{a}) \,\}.$$

Claim: $\varphi(U) = \bigcup_{q(\bar{x}) \in D} q(U)$.

By (an argument similar to) Fact 7.2, there is $\theta_q(\bar{x})$ in $q(\bar{x})$ a finite conjunction of formulas such that $\theta_q(\bar{x}) \to \varphi(x)$. So we have

$$\varphi(\bar{x}) \leftrightarrow \bigvee_{q(\bar{x}) \in D} \{\theta_q(\bar{x})\}.$$

By Fact 7.2, there are $\psi_{q_1}(\bar{x}), \dots, \psi_{q_m}(\bar{x})$ such that

$$\varphi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \psi_{q_i}(\bar{x}).$$

So $\bigvee \psi_{q_i}(\bar{x})$ is the required quantifier-free formula. □

Definition 7.8. An L-theory T has **quantifier elimination** if for every L-formula $\varphi(\bar{x})$ there is $\psi(\bar{x})$ quantifier free such that

$$T \vdash \forall \bar{x}\, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Theorem 7.9. Let T be a complete theory with an infinite model. Then the following are equivalent:

(i) T has quantifier elimination

(ii) every $p : \mathcal{U} \to \mathcal{U}$ partial embedding is elementary

(iii) If $p : \mathcal{U} \to \mathcal{U}$ is partial embedding and $|\operatorname{dom} p| < |\mathcal{U}|$ and $b \in \mathcal{U}$, then there is a partial embedding $\hat{p} \supseteq p$ such that $b \in \operatorname{dom} \hat{p}$.

Proof. (i) ⇔ (ii). Follows from Proposition 7.7.

(ii) ⇒ (iii). If $p : \mathcal{U} \to \mathcal{U}$ is a partial embedding, then it is elementary. Let $b \in \mathcal{U}$. By homogeneity of $\mathcal{U}$, there is $\alpha \in \mathrm{Aut}(\mathcal{U})$ such that $p \subseteq \alpha$, and so $p \cup \{\langle b, \alpha(b) \rangle\}$ is the required extension of p.

(iii) ⇒ (ii). Let $p : \mathcal{U} \to \mathcal{U}$ be a partial embedding. Consider $p_0 \subseteq p$, p_0 finite or small. Use property (iii) and saturation to extend p_0 to $\alpha \in \mathrm{Aut}(U)$ by back and forth. □

Remark. There is a fourth condition equivalent to (i), (ii), (iii):

(iv) for every finite partial embedding $p : \mathcal{U} \to \mathcal{U}$ and $b \in \mathcal{U}$ there is $\hat{p} \supseteq p$, a partial embedding such that $b \in \mathrm{dom}(\hat{p})$.

Proof: Later, exercise.

This gives quantifier elimination for T_{rg} and T_{dlo}.

Remark. If T has quantifier elimination and $\mathcal{M} \vDash T$, any substructure of $\mathcal{M}$ is an elementary substructure (T is 'model-complete').

Definition 7.10. An element $a \in \mathcal{U}$ is **definable** over $A \subseteq U$ if there is an $L(A)$-formula $\varphi(x)$ such that $\varphi(U) = \{a\}$. (In particular, any element of A is definable over A; $x = a$ for $a \in A$).

An element $a \in \mathcal{U}$ is **algebraic** over $A \subseteq U$ if there is an $L(A)$-formula $\varphi(x)$ such that $|\varphi(U)| < \omega$ and $a \in \phi(\mathcal{U})$.

The **definable closure** of A is

$$\mathrm{dcl}(A) = \{\, a \in \mathcal{U} \mid a \text{ definable over } A \,\}$$

and the **algebraic closure** of A is

$$\mathrm{acl}(A) = \{\, a \in \mathcal{U} \mid a \text{ algebraic over } A \,\}\,.$$

Proposition 7.11. For $a \in \mathcal{U}$ and $A \subseteq \mathcal{U}$, the following are equivalent

(i) $a \in \mathrm{dcl}(A)$

(ii) $O(a/A) = \{a\}$.

Proof. $a \in \mathrm{dcl}(A)$ iff there is $\varphi(x) \in L(A)$ such that $\varphi(U) = \{a\}$. By Proposition 7.5 this is equivalent to invariance under $\mathrm{Aut}(U/A)$. □

Theorem 7.12. Let $A \subseteq \mathcal{U}$, $a \in \mathcal{U}$, the following are equivalent:

(i) $a \in \mathrm{acl}(A)$

(ii) $|O(a/A)| < \omega$

(iii) $a \in \mathcal{M}$ for any model $\mathcal{M}$ which contains A.

Lecture 14

Proof. (i) ⇒ (ii). If $a \in \mathrm{acl}(A)$, then there is an $L(A)$-formula $\varphi(x)$ such that $\varphi(a)$ holds and $|\varphi(U)| < \omega$. But $\varphi(U)$ is invariant over A, and so $O(a/A) \subseteq \varphi(U)$, and so $|\mathcal{O}(a/A)| < \omega$.

(ii) ⇒ (i). If $|O(a/A)| < \omega$, then $O(a/A)$ is definable by $\bigvee_{i=1}^{n}(x = a_i)$ where $O(a/A) = \{a_1, \ldots, a_n\}$. Also $O(a/A)$ is invariant over A, so by Proposition 7.5, there is an $L(A)$-formula $\varphi(x)$ that defines $O(a/A)$.

(i) ⇒ (iii). $a \in \mathrm{acl}(A)$, so there is $\varphi(x)$, an $L(A)$-formula such that there is $n \in \omega \setminus \{0\}$ with

$$\varphi(a) \wedge \exists^{\leq n} x\, \varphi(x).$$

Then by elementarity, $\varphi(a) \wedge \exists^{\leq n} x\, \varphi(x)$ holds in every $\mathcal{M} \supseteq A$, and the n realizations of $\varphi(x)$ in $\mathcal{U}$ must coincide with the realizations in $\mathcal{M}$. Therefore $a \in \mathcal{M}$.

(iii) ⇒ (i). Suppose $a \notin \mathrm{acl}(A)$, let $p(x) = \mathrm{tp}(a/A)$. Then for $\varphi(x) \in p(x)$, $|\varphi(\mathcal{U})| \geq \omega$. Then from sheet 2, $|p(\mathcal{U})| \geq \omega$. By an argument similar to the one in exercise 7 on sheet 2, $|p(\mathcal{U})| = |\mathcal{U}|$.

Let $\mathcal{M} \supseteq A$, then $p(\mathcal{U}) \setminus \mathcal{M} \neq \varnothing$. So there is $b \in p(\mathcal{U}) \setminus \mathcal{M}$. Since $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$, there is $\alpha \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\alpha(b) = a$.

But then $\alpha[\mathcal{M}]$ is a model that contains A, but $a \notin \alpha[\mathcal{M}]$ while $a = \alpha(b)$. □

Proposition 7.13. Let $a \in \mathcal{U}$, $A \subseteq \mathcal{U}$. Then:

(i) if $a \in \mathrm{acl}(A)$, then there is finite $A_0 \subseteq A$ such that $a \in \mathrm{acl}(A_0)$.

(ii) if $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.

(iii) $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$

(iv) $A \subseteq \mathrm{acl}(A)$.

(v) $\mathrm{acl}(A) = \bigcap_{A \subseteq \mathcal{M}} \mathcal{M}$ where $\mathcal{M}$ is a small elementary substructure of $\mathcal{U}$.

Proof.

(iv) $a \in A$ is definable over A, hence algebraic.

(iii) $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$ by monotonicity. For $\supseteq$, let $a \in \mathrm{acl}(\mathrm{acl}(A))$. By Theorem 7.12, $a \in \mathcal{M}$ for every $\mathcal{M} \supseteq \mathrm{acl}(A)$. But $\mathrm{acl}(A) \subseteq \mathcal{M} \iff A \subseteq \mathcal{M}$, so $a \in \mathcal{M}$ for every $\mathcal{M} \supseteq A$, i.e. $a \in \mathrm{acl}(A)$.

(v) follows from Theorem 7.12. □

Proposition 7.14. If $\beta \in \mathrm{Aut}(\mathcal{U})$, $A \subseteq \mathcal{U}$, then $\beta[\mathrm{acl}(A)] = \mathrm{acl}(\beta[A])$.

Proof. $\subseteq$: Let $a \in \mathrm{acl}(A)$, let $\varphi(x, \bar{z})$ be an L-formula such that $\varphi(a, \bar{b})$ holds for $\bar{b}$ in A and $|\varphi(U, \bar{b}) < \omega$. Then $\varphi(\beta(a), \beta(\bar{b}))$ holds, $|\varphi(U, \beta(\bar{b}))| < \omega$, and so $\beta(a)$ is algebraic over $\beta[\bar{b}]$.

The same proof with β^{-1} in place of β and $\beta[A]$ in place of A shows $\supseteq$. □

8 Strongly Minimal Theories

Definition 8.1 (Cofinite)**.** For $\mathcal{M}$ a structure, $A \subseteq M$ is **cofinite** if $M \setminus A$ is finite.

Remark 8.2. Finite and cofinite sets are definable in every structure.

In this chapter, we'll look at structures where these are the only definable sets.

Definition 8.3 (Minimality, strong minimality)**.** A structure $\mathcal{M}$ is **minimal** if all its definable subsets are finite or cofinite. $\mathcal{M}$ is **strongly minimal** if it is minimal and all its elementary extensions are minimal.

If T is a consistent theory without finite models, T is **strongly minimal** if for every formula $\varphi(x, \bar{z})$ there is $n \in \omega \setminus \{0\}$ such that

$$T \vdash \forall \bar{z}\, [\exists^{\leq n} x\, \varphi(x, \bar{z}) \vee \exists^{\leq n} x\, \neg\varphi(x, \bar{z})].$$

Example. Take $L = \{E\}$, a binary relation, let $\mathcal{M}$ be the L-structure where E is an equivalence relation with exactly one class of size n for all $n \in \omega$ and no infinite classes. Then can show $\mathcal{M}$ is minimal (can only say things like 'x is in the same class as a').

But, there is $\mathcal{N} \succcurlyeq \mathcal{M}$ where $\mathcal{N}$ has an infinite class. Then if the equivalence class of $a \in \mathcal{N}$ is infinite, the set defined by $E(x, a)$ is infinite/coinfinite, so $\mathcal{M}$ is not strongly minimal.

(Remark: strongly minimal theories have monster models). From now on: T is strongly minimal, complete, and has an infinite model.

Definition 8.4 (Independence)**.** Let $a \in \mathcal{U}$, $B \subseteq \mathcal{U}$. Then a is **independent** from B if $a \notin \mathrm{acl}(B)$. The set B is **independent** if for all $a \in B$, $a \notin \mathrm{acl}(B \setminus \{a\})$.

Example.

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- Vector spaces. Fix an infinite field K, and use $L = \{+, -, \mathbf{0}, \{\lambda\}_{\lambda \in K}\}$, where λ are unary functions (for scalar multiplication). The theory of vector spaces over K, T_{VSK} includes:
  - axioms in $\{+, -, \mathbf{0}\}$ for abelian group
  - axiom schemata for scalar multiplication:
    * $\forall xy\, [\lambda(x + y) = \lambda x + \lambda y]$ for each $\lambda \in K$, λx means $\lambda(x)$.
    * $\vdots$
    * $\forall x\, [1x = x]$ (since $1 \in K$).
    * $\exists x\, (x \neq \mathbf{0})$.

  Then it can be shown T_{VSK} is complete and has quantifier elimination.

  Atomic formulas express equality of linear combinations, any atomic formula in one variable and with parameters is equivalent to '$\lambda x = a$', so atomic formulas in one variable define singletons. Quantifier-free formulas in one variable and with parameters define sets that are either finite or cofinite.

  By quantifier elimination, T_{VSK} is strongly minimal. Also, $\mathrm{acl}(A) = \langle A \rangle$, the linear span, and a is independent from A if a is linearly independent from A, and A is independent if it is linearly independent.

- Fields. Take $L_{\text{ring}} = \{+, \cdot, -, 0, 1\}$. Then ACF is the theory that includes
  - axioms for abelian group in $\{+, -, 0\}$

– axioms for multiplicative monoids in $\{\cdot, 1\}$
– $\forall xyz\ [x \cdot (y+z) = x \cdot y + x \cdot z]$
– $\forall x\ [x = 0 \vee \exists y\ (x \cdot y) = 1]$
– $0 \neq 1$
– axioms for algebraic closure: for all n,

$$\forall x_0 \cdots x_n\ \exists y\ [x_n y^n + \cdots + x_1 y + x_0 = 0].$$

If

$$\chi_p \equiv \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0,$$

for p prime, then $ACF \cup \{\chi_p\} =: ACF_p$, which can be shown to be complete and have quantifier elimination. By adding $\{\neg \chi_n \mid n \in \omega\}$ to ACF, get ACF_0 (also complete with quantifier elimination).

Now, atomic formulas with parameters are polynomial equations. An atomic formula with one variable (and parameters in A) is equivalent to $p(x) = 0$, where $p(x)$ is a polynomial in the subfield generated by A. So such atomic formulas define finite sets, and quantifier free formulas define finite or cofinite sets, and so by quantifier elimination, ACF_p (ACF_0) is strongly minimal. If $a \in \mathcal{M} \vDash ACF_p$, $A \subseteq \mathcal{M}$, then $a \in \operatorname{acl}(A)$ if a is algebraic over the field generated by A.

Notation. We write $\operatorname{acl}(a, B)$ for $\operatorname{acl}(\{a\} \cup B)$ and $\operatorname{acl}(B \setminus a)$ for $\operatorname{acl}(B \setminus \{a\})$.

Theorem 8.5. Let $B \subseteq \mathcal{U}$, and $a, b \notin \operatorname{acl}(B)$. ($a, b \in \mathcal{U} \setminus \operatorname{acl}(B)$). Then

$$b \in \operatorname{acl}(a, B) \iff a \in \operatorname{acl}(b, B).$$

Proof. Let $a, b \in \operatorname{acl}(B)$. Assume $b \notin \operatorname{acl}(a, B)$ and $a \in \operatorname{acl}(b, B)$. Let $\varphi(x, y)$ be an L-formula such that for some n,

$$\varphi(a, b) \wedge \exists^{\leq n} x\ \varphi(x, b).$$

Since $b \notin \operatorname{acl}(a, B)$

$$\psi(a, y) := \varphi(a, y) \wedge \exists^{\leq n} x\ \varphi(x, y)$$

is such that $|\psi(a, \mathcal{U})| \geq \omega$. By question 7, example sheet 2, $|\psi(a, U)| = |\mathcal{U}|$. By strong minimality, $|\neg\psi(a, U)| < \omega$. By cardinality considerations, if $\mathcal{M} \supseteq B$, then $\mathcal{M}$ contains c such that $\psi(a, c)$. But then $a \in \operatorname{acl}(c, B)$, so $a \in \mathcal{M}$. Therefore a is in all models that contain B, so $a \in \operatorname{acl}(B)$ by Theorem 7.12, a contradiction. □

Definition 8.6 (Basis)**.** Let $B \subseteq C \subseteq \mathcal{U}$. Then B is a **basis** of C if

(i) B is independent,

(ii) $C \subseteq \operatorname{acl}(B)$ (or equivalently, $\operatorname{acl}(B) = \operatorname{acl}(C)$).

Lemma 8.7. If B is independent and $a \notin \operatorname{acl}(B)$, then $\{a\} \cup B$ is independent.

Proof. Let $a \notin \operatorname{acl}(B)$, and suppose (for contradiction) that $\{a\} \cup B$ is not independent. Then there is $b \in B$ such that $b \in \operatorname{acl}(a, B \setminus b)$. But $b \notin \operatorname{acl}(B \setminus b)$. Since $a \notin \operatorname{acl}(B \setminus b)$, by Theorem 8.5 we have

$$a \in \operatorname{acl}(b, B \setminus b) = \operatorname{acl}(B),$$

a contradiction. □

Corollary 8.8. If $B \subseteq C$, the following are equivalent:

(i) B is a basis of C

(ii) if $B \subseteq B' \subset C$ and B' is independent, then $B = B'$.

Proof. By Lemma 8.7. □

Theorem 8.9. Let $C \subseteq \mathcal{U}$, then

(i) every independent subset $B \subseteq C$ can be extended to a basis.

(ii) if A, B are bases of C, then $|A| = |B|$.

Proof.

(i) If $\langle B_i : i < \lambda \rangle$ is a chain of independent sets containing B, then $\bigcup_{i<\lambda} B_i$ is independent (by Proposition 7.13(i)). By Zorn's lemma, there is a maximal independent subset of C that contains B. By Corollary 8.8, that maximal subset is a basis of C.

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(ii) Let $|B| \geq \omega$, assume (for contradiction) that $|A| < |B|$. Then $a \in A$ is also in acl(B). Let $D_a \subseteq B$ be finite such that $a \in \operatorname{acl}(D_a)$. Let $D = \bigcup_{a \in A} D_a$. Then $A \subseteq \operatorname{acl}(D)$ and $C \subseteq \operatorname{acl}(D)$, but $|D| < |B|$ contradicting the independence of B.

If A and B are finite, show that $|A| \leq |B|$ (and symmetrically) by using: if there is $a \in A \setminus B$, then there is $b \in B \setminus A$ such that $\{b\} \cup A \setminus \{a\}$ is independent. This holds because if $a \in A \setminus B$, then since $a \in \operatorname{acl}(B)$, we have that $B \nsubseteq \operatorname{acl}(A \setminus \{a\})$ (otherwise A is not independent). So let $b \in B \setminus \operatorname{acl}(A \setminus a)$. Then $\{b\} \cup (A \setminus a)$ is independent by Lemma 8.7.

Use finite induction argument to get $|A| \leq |B|$. □

Definition 8.10 (Dimension)**.** Let $C \subseteq \mathcal{U}$ be algebraically closed. Then the **dimension** of C is $\dim(C) = |A|$ where A is any basis of C.

Proposition 8.11. Let $f : \mathcal{U} \to \mathcal{U}$ be (partial) elementary. Let $b \notin \operatorname{acl}(\operatorname{dom}(f))$ and $c \notin \operatorname{acl}(\operatorname{img}(f))$. Then $f \cup \{\langle b, c\rangle\}$ is elementary.

Proof. Let $\bar{a}$ enumerate dom(f), let $\varphi(x, \bar{a})$ be a formula with parameters in $\bar{a}$. Claim: $\varphi(b, \bar{a})) \leftrightarrow \varphi(c, f(\bar{a}))$. Cases:

1. $|\varphi(\mathcal{U}, \bar{a})| < \omega$. Then $|\varphi(\mathcal{U}, f(\bar{a}))| < \omega$. Then $b \notin \varphi(\mathcal{U}, \bar{a})$ (because $b \notin \operatorname{acl}(\bar{a})$) and $c \notin \varphi(\mathcal{U}, f(\bar{a}))$. Then
$$\neg\varphi(b, \bar{a}) \wedge \neg\varphi(c, f(\bar{a})).$$

2. $|\varphi(U, \bar{a})| \geq \omega$. Then $|\neg\varphi(\mathcal{U}, \bar{a})| < \omega$, and so
$$\varphi(b, \bar{a}) \wedge \varphi(c, f(\bar{a})).$$
□

Corollary 8.12. Every bijection between independent subsets of $\mathcal{U}$ is elementary.

Proof. Pick $A, B \subseteq C$ independent and let $f : A \to B$ be any bijection. Let $\bar{a}$ enumerate A, write $f(a_i) = b_i$. Then $a_0 \notin \operatorname{acl}(\varnothing)$ and $b_0 \notin \operatorname{acl}(\varnothing)$ (otherwise A, B not independent). By Proposition 8.11, $\{\langle a_0, b_0\rangle\}$ is an elementary map.

At stage $i + 1$, $a_{i+1} \notin \operatorname{acl}(a_0, \ldots, a_i)$ so use the same argument. □

Remark 8.13. If $\mathcal{M} \subseteq \mathcal{U}$, then by Proposition 7.13, $\mathcal{M}$ is algebraically closed.

Theorem 8.14. Suppose that $\mathcal{M}, \mathcal{N} \subseteq \mathcal{U}$ are such that $\dim(M) = \dim(N)$, then $\mathcal{M} \simeq \mathcal{N}$.

Proof. Let A, B be bases of $\mathcal{M}, \mathcal{N}$ respectively. Then a bijection $f : A \to B$ is elementary (by Corollary 8.12). Then there is $\alpha \in \mathrm{Aut}(\mathcal{U})$ such that $f \subseteq \alpha$. Then by Proposition 7.14,

$$\alpha(\mathcal{M}) = \alpha(\mathrm{acl}(\mathcal{M})) = \mathrm{acl}(\alpha(A)) = \mathrm{acl}(B) = \mathcal{N}. \qquad \square$$

Corollary 8.15. Let T be strongly minimal, let $\lambda > |L|$. Then T is λ-categorical.

Proof. If $A \subseteq \mathcal{U}$, then $|\mathrm{acl}(A)| \leq |L(A)| + \omega$ (there are at most $|L(A)| + \omega$ formulas, each element m in $\mathrm{acl}(A)$ is one of finitely many solutions of one of those formulas). If $|\mathcal{M}| = \lambda$, then a basis of $\mathcal{M}$ must have cardinality λ. $\square$

In T_{VSK}, if K is infinite countable, the vector space can have finite dimension (ω-categoricity fails). If K is finite, the vector space must have dimension $\geq \omega$.

9 Bonus Lecture: Existence of saturated models

If $\mathcal{M}$ is saturated, then

- $\mathcal{M}$ is homogeneous.
- $\mathcal{M}$ is universal.

If $\mathcal{M}$ is λ-saturated, then:

- $\mathcal{M}$ is weakly λ-homogeneous, i.e. for all $f : \mathcal{M} \to \mathcal{M}$ (partial) elementary such that $|f| < \lambda$, for every $b \in \mathcal{M}$, then $\exists \hat{f} \supseteq f$ elementary and such that $b \in \operatorname{dom} f$.

Can prove: λ-homogeneous is equivalent to homogeneity when $|\mathcal{M}| = \lambda$.

Definition (Cofinality)**.** If α is a limit ordinal $\geq \omega$, $\operatorname{cof}(\alpha)$ (**cofinality** of α) is the least λ such that there is $f : \lambda \to \alpha$ such that $\operatorname{img}(f)$ is unbounded in α.

Example.

$$\operatorname{cof}(\omega) = \aleph_0 \qquad \operatorname{cof}(\omega_\omega) = \aleph_0.$$

Definition (Regular)**.** A cardinal κ is **regular** if $\operatorname{cof}(\kappa) = \kappa$.

Example. $\aleph_0$ is regular. Also, every successor cardinal is regular.

Are there any limit cardinals other than $\aleph_0$ that are regular?

Definition ($S_1^{\mathcal{M}}$)**.** If $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, then define

$$S_1^{\mathcal{M}}(A) := \{\, p(x) \mid p(x) \text{ is a complete type in a single variable with parameters in } A \,\}$$

Lemma. If $\mathcal{M}$ is such that $|\mathcal{M}| \geq |L| + \omega$, let $\kappa > \aleph_0$. Then there is $\mathcal{M}' \succcurlyeq \mathcal{M}$ such that for all $A \subseteq \mathcal{M}$ with $|A| < \kappa$, if $p(x) \in S_1^{\mathcal{M}}(A)$, then $p(x)$ is realized in $\mathcal{M}'$, $|\mathcal{M}'| \leq |\mathcal{M}|^\kappa$.

Proof. First, note

$$|\{\, A \subseteq \mathcal{M} \mid |A| \leq \kappa \,\}| \leq |\mathcal{M}|^\kappa$$
$$\left|S_1^{\mathcal{M}}(A)\right| \leq 2^\kappa.$$

Enumerate $S_1^{\mathcal{M}}(A)$ as $\langle p_\alpha : \alpha < |\mathcal{M}|^\kappa \rangle$. Build $\langle \mathcal{M}_\alpha : \alpha < |\mathcal{M}|^\kappa \rangle$ as follows:

- $\mathcal{M}_0 = \mathcal{M}$
- $\mathcal{M}_\alpha = \bigcup_{\beta < \alpha} \mathcal{M}_\beta$ when α is a limit.
- $\mathcal{M}_\alpha \preccurlyeq \mathcal{M}_{\alpha+1}$ such that $\mathcal{M}_{\alpha+1}$ realizes $p_\alpha(x)$ and $|\mathcal{M}_{\alpha+1}| = |\mathcal{M}_\alpha|$. Then $\bigcup_{\alpha < |\mathcal{M}|^\kappa} \mathcal{M}_\alpha$ realizes all types in $S_1^{\mathcal{M}}(A)$ and

$$\left| \bigcup_{\alpha < |\mathcal{M}|^\kappa} \mathcal{M}_\alpha \right| \leq |\mathcal{M}|^\kappa. \qquad \square$$

Theorem. Let $\kappa > \aleph_0$, let $\mathcal{M} \models T$. Then there is a κ^+-saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ such that $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$.

Proof. Build an elementary chain $\langle \mathcal{N} : \alpha < \kappa^+ \rangle$ such that

- $\mathcal{N}_0 = \mathcal{M}$

- take unions at limit stages
- Given $\mathcal{N}_\alpha$, find $\mathcal{N}_{\alpha+1} \succcurlyeq \mathcal{N}_\alpha$ such that all types in $S_1^{\mathcal{N}_\alpha}(A)$ with $|A| \leq \kappa$ are realized.

Moreover, $|\mathcal{N}_\alpha| \leq |\mathcal{M}|^\kappa$ (follows from previous result). Let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. Since $\kappa^+ \leq |\mathcal{M}|^\kappa$, $\mathcal{N}$ is the union of at most $|\mathcal{M}|^\kappa$ sets each of size at most $|\mathcal{M}|^\kappa$, hence $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$.

To see that $\mathcal{N}$ is κ^+ saturated, pick $A \subseteq \mathcal{N}$ such that $|A| \leq \kappa$. By the regularity of κ^+, there is α such that $A \subseteq \mathcal{N}_\alpha$, hence all types $/A$ with one free variable are realized in $\mathcal{N}$. □

Recap: For arbitrarily large κ, there is a κ^+ saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ with $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$. If κ, $|\mathcal{M}|$ are such that $|\mathcal{M}| \leq 2^\kappa$, then $|\mathcal{M}|^\kappa = 2^\kappa$ so you get a κ^+-saturated $\mathcal{N} \succcurlyeq \mathcal{M}$ such that $|\mathcal{N}| = 2^\kappa$. So GCH implies saturated models exist.

Alternatively, suppose there are arbitrarily large cardinals κ such that

$$\kappa^{<\kappa} = \bigcup \{ \kappa^\alpha \mid \alpha < \kappa \} = \kappa$$

(strongly inaccessible cardinals). Then the chain stabilises, giving the required structure.

Definition. Take T a complete theory in a countable language, $\kappa \geq \aleph_0$ a cardinal. Then T is κ-**stable** if for all $\mathcal{M} \vDash T$, $A \subseteq \mathcal{M}$, $|A| \leq \kappa$, $\forall n \leq \omega$, we have

$$|S_n^{\mathcal{M}}(A)| \leq \kappa$$

where $S_n^{\mathcal{M}}(A)$ is the set of complete types with n variables and parameters in A.

Theorem. Let κ be a regular cardinal, and T κ-stable. Then there is a $\mathcal{M} \vDash T$, $|\mathcal{M}| = \kappa$, $\mathcal{M}$ saturated.

Proof. We build an elementary chain $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ where $|\mathcal{M}_\alpha| < \kappa$ as follows:

- $\mathcal{M}_0 \vDash T$
- unions at limit stages
- given $\mathcal{M}_\alpha$, $|\mathcal{M}_\alpha| = \kappa \Rightarrow S_1^{\mathcal{M}_\alpha}(\mathcal{M}_\alpha) = \kappa$, there is $\mathcal{M}_{\alpha+1} \succcurlyeq \mathcal{M}_\alpha$ that realizes all types in $S_1^{\mathcal{M}_\alpha}(\mathcal{M}_\alpha)$ and $|\mathcal{M}_{\alpha+1}| = |\mathcal{M}_\alpha|$. Let $\bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$, then $|\bigcup \mathcal{M}_\alpha| = \kappa$ and $\bigcup \mathcal{M}_\alpha$ is κ-saturated by construction.

Now, $\mathcal{M}$ κ-saturated, κ-strongly homogeneous, $|\mathcal{M}| \gg \kappa$. □

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