Part III – Model Theory (Ongoing course, rough)

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Contents

0 Introduction

Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: 'how powerful is our description of these objects to pin them down'? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

1 Languages and structures

Definition 1.1 (Language)**.** A **language** L consists of

(i) a set $\mathscr{F}$ of function symbols, and for each $f \in \mathscr{F}$ a positive integer m_f the **arity** of f.

(ii) a set $\mathscr{R}$ of relation symbols, and for each $R \in \mathscr{R}$, a positive integer m_R.

(iii) a set $\mathscr{C}$ of constant symbols.

Note: each of $\mathscr{F}, \mathscr{R}$ and $\mathscr{C}$ can be empty.

Example. Take $L = \{\{\cdot, ^{-1}\}, \{1\}\}$, for $\cdot$ a binary function and $^{-1}$ an unary function, 1 a constant. This is the language of groups, call it L_{gp}. Also, $L_{lo} = \{<\}$ a single binary relation, for linear orders.

Definition 1.2 (L-structure)**.** Given a language L, say, an ***L*-structure** consists of

(i) a set M, the **domain**

(ii) for each $f \in \mathscr{F}$, a function $f^{\mathcal{M}} : M^{m_f} \to M$.

(iii) for each $R \in \mathscr{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$.

(iv) for each $c \in \mathscr{C}$, an element $c^{\mathcal{M}} \in M$.

f^M, R^M, c^M are the **interpretations** of f, R, c respectively.

Remark 1.3. We often fail to distinguish between the symbols in L and their interpretations in a structure, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathscr{F}, \mathscr{R}, \mathscr{C} \rangle$.

Example 1.4.

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1\rangle$ is an L_{gp}-structure.

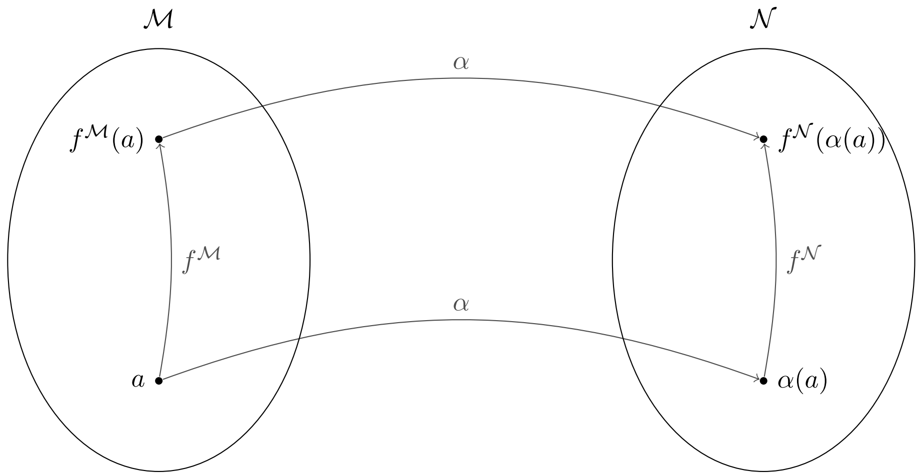
(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0\rangle$ is an L_{gp}-structure.

(c) $\mathcal{Q} = \langle \mathbb{Q}, <\rangle$ is an L_{lo}-structure.

Definition 1.5 (Embedding)**.** Let L be a language, let $\mathcal{M}, \mathcal{N}$ be L-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is a one-to-one mapping $\alpha : M \to N$ such that

(i) for all $f \in \mathscr{F}$, and $a_1, \ldots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\alpha(a_1), \ldots, \alpha(a_{n_f}))$$



(ii) for all $R \in \mathscr{R}$, and $a_1, \dots, a_{n_R} \in M$

$$(a_1, \dots, a_{n_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathscr{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of $\mathcal{M}$ into $\mathcal{N}$ is a surjective embedding (onto).

Exercise 1.6. Let G_1, G_2 be groups, regarded as L_{gp}-structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an isomorphism $\alpha : G_1 \to G_2$ in the sense of Definition 1.5

2 Review: Terms, formulae and their interpretations

In addition to the symbols of L, we also have

(i) infinitely many variables $\{x_i\}_{i \in I}$

(ii) logical connectives $\wedge, \neg$ (also expresses $\vee, \Rightarrow, \iff$)

(iii) quantifier $\exists$ (also expresses $\forall$)

(iv) (,)

Definition 2.1 (L-terms)**.** **L-terms** are defined recursively as follows:

– any variable x_i is a term

– any constant symbol is a term

– for any $f \in \mathscr{F}$, $f(t_1, \dots, t_{m_f})$ for any terms $t_1, \dots, t_{m_f}$ is a term

– nothing else is a term

Notation: we write $t(x_1, \dots, x_m)$ to mean that the variables appearing in t are among $x_1, \dots, x_m$.

Example. Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1\rangle$. Then $\cdot(\cdot(x_1, x_2), x_3)$ is a term, usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a term, written $(1 \cdot x)^{-1}$

Definition 2.2. If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, \ldots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \to M$ defined as follows:

(i) If $t = x_i$, $t^{\mathcal{M}}[a_1, \ldots, a_k] = a_i$

(ii) If $t = c$, $t^{\mathcal{M}}[a_1, \ldots, a_k] = c^{\mathcal{M}}$.

(iii) If $t = f(t(x_1, \ldots, x_k), \ldots, t_{m_f}(x_1, \ldots, x_k))$,

$$t^{\mathcal{M}}(a_1, \ldots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \ldots, a_k), \ldots, t_{m_f}^{\mathcal{M}}(a_1, \ldots, a_k))$$