

# Part III – Category Theory (Ongoing course)

Based on lectures by Professor P. T. Johnstone

Notes taken by Bhavik Mehta

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## Contents

<b>0</b>	<b>Introduction</b>	<b>2</b>
<b>1</b>	<b>Definitions and Examples</b>	<b>2</b>
<b>2</b>	<b>The Yoneda Lemma</b>	<b>9</b>
<b>3</b>	<b>Adjunctions</b>	<b>14</b>
<b>4</b>	<b>Limits</b>	<b>20</b>
<b>5</b>	<b>Monads</b>	<b>31</b>
	<b>Index</b>	<b>39</b>

## 0 Introduction

*Lecture 1* Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

## 1 Definitions and Examples

**1.1 Definition** (Category). A **category**  $\mathcal{C}$  consists of

- (a) a collection  $\mathcal{C}$  of **objects**  $A, B, C, \dots$
- (b) a collection  $\text{mor } \mathcal{C}$  of **morphisms**  $f, g, h, \dots$
- (c) two operations  $\text{dom}, \text{cod}$  assigning to each  $f \in \text{mor } \mathcal{C}$  a pair of objects, its **domain** and **codomain**. We write  $A \xrightarrow{f} B$  to mean ‘ $f$  is a morphism and  $\text{dom } f = A$  and  $\text{cod } f = B$ ’.
- (d) an operation assigning to each  $A \in \text{ob } \mathcal{C}$  a morphism  $A \xrightarrow{1_A} A$ , called its **identity**.
- (e) a partial binary operation **composition**  $(f, g) \mapsto fg$  on morphisms, such that  $fg$  is defined iff  $\text{dom } f = \text{cod } g$  and  $\text{dom}(fg) = \text{dom } g$ ,  $\text{cod}(fg) = \text{cod } f$  if  $fg$  is defined.

satisfying

- (f)  $f1_A = f = 1_B f$  for any  $A \xrightarrow{f} B$
- (g)  $(fg)h = f(gh)$  whenever  $fg$  and  $gh$  are defined

**1.2 Remark.**

- (a) This definition is independent of a model of set theory. If we’re given a particular model of set theory, we call the **category**  $\mathcal{C}$  **small** if  $\text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{C}$  are sets.
- (b) Some texts say  $fg$  means ‘ $f$  followed by  $g$ ’, i.e.  $fg$  defined  $\iff \text{cod } f = \text{dom } g$ .
- (c) Note that a morphism  $f$  is an **identity** iff  $fg = g$  and  $hf = h$  whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

**1.3 Examples.**

- (a) The **category** **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms  $A \longrightarrow B$  are pairs  $(f, B)$  where  $f$  is a set-theoretic function.)
- (b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, **Mod** $_R$  the category of  $R$ -modules.

- (c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.
- (d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given  $\mathcal{C}$ , we call an equivalence relation  $\simeq$  on  $\text{mor } \mathcal{C}$  a **congruence** if  $f \simeq g \implies \text{dom } f = \text{dom } g$  and  $\text{cod } f = \text{cod } g$ , and  $f \simeq g \implies fh \simeq gh$  and  $kf \simeq kg$  whenever the composites are defined. Then we have a category  $\mathcal{C}/\simeq$  with the same objects as  $\mathcal{C}$ , but congruence classes as morphisms.
- (e) Given  $\mathcal{C}$ , the **opposite category**  $\mathcal{C}^{\text{op}}$  has the same objects and morphisms as  $\mathcal{C}$ , but  $\text{dom}$  and  $\text{cod}$  are interchanged, and  $fg$  in  $\mathcal{C}^{\text{op}}$  is  $gf$  in  $\mathcal{C}$ . This leads to the **Duality principle** if  $P$  is a true statement about categories, so is the statement  $P^*$  obtained from  $P$  by reversing all arrows.
- (f) A **small** category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism ( $f$  is an **isomorphism** if  $\exists g$  such that  $fg$  and  $gf$  are identities).
- (g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space  $X$ , the fundamental groupoid  $\pi(X)$  has all points of  $X$  as objects and morphisms  $x \longrightarrow y$  are homotopy classes  $\text{rel } \{0, 1\}$  of paths  $u : [0, 1] \longrightarrow X$  with  $u(0) = x$ ,  $u(1) = y$ . (If you know how to prove that the fundamental group is a group, you can prove that  $\pi(X)$  is a groupoid.)
- (h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category  $\mathcal{C}$  in which, for any pair  $(A, B)$  there is at most 1 morphism  $A \longrightarrow B$ . A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
- (i) The category **Rel** has the same objects as **Set**, but morphisms  $A \longrightarrow B$  are arbitrary relations  $R \subseteq A \times B$ . Given  $R$  and  $S \subseteq B \times C$ , we define

$$S \circ R = \{ (a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R \wedge (b, c) \in S) \}.$$

The identity  $1_A : A \longrightarrow A$  is  $\{ (a, a) \mid a \in A \}$ .

Similarly, the category **Part** of sets and partial functions (i.e. relations such that  $(a, b) \in R, (a, b') \in R \implies b = b'$ ).

- (j) Let  $K$  be a field. The category **Mat** $_K$  has natural numbers as objects, and morphisms  $n \longrightarrow p$  are  $(p \times n)$  matrices with entries from  $K$ . Composition is matrix multiplication.

**1.4 Definition (Functor).** Let  $\mathcal{C}, \mathcal{D}$  be **categories**. A **functor**  $F : \mathcal{C} \longrightarrow \mathcal{D}$  consists of

- (a) a mapping  $A \longmapsto FA$  from  $\text{ob } \mathcal{C}$  to  $\text{ob } \mathcal{D}$
- (b) a mapping  $f \longmapsto Ff$  from  $\text{mor } \mathcal{C}$  to  $\text{mor } \mathcal{D}$

such that  $\text{dom}(Ff) = F(\text{dom } f)$ ,  $\text{cod}(Ff) = F(\text{cod } f)$ ,  $1_{FA} = F(1_A)$  and  $(Ff)(Fg) = F(fg)$  whenever  $fg$  is defined.

Lecture 2 **1.3 Examples** (*Continued*).

- (k) We write **Cat** for the category whose objects are all **small categories**, and whose morphisms are **functors** between them.

**1.5 Examples.**

- (a) We have **forgetful functors**  $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}, \mathbf{Rng} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set}, \mathbf{Rng} \rightarrow \mathbf{AbGp}$  (forgetting  $\times$ ),  $\mathbf{Rng} \rightarrow \mathbf{Mon}$  (forgetting  $+$ ).
- (b) Given a set  $A$ , the free group  $FA$  has the property: given any group  $G$  and any function  $A \xrightarrow{f} UG$ , there's a unique homomorphism  $FA \xrightarrow{f} G$  extending  $f$ .  $F$  is a functor  $\mathbf{Set} \rightarrow \mathbf{Gp}$ : given  $A \xrightarrow{f} B$ , we define  $Ff$  to be the unique homomorphism extending  $A \xrightarrow{f} B \hookrightarrow UFB$ .

Functoriality follows from uniqueness: given  $B \xrightarrow{g} C$ ,  $F(gf)$  and  $(Fg)(Ff)$  are both homoms extending  $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$ . Call this the **free functor**.

- (c) Given a set  $A$ , we write  $\mathcal{P}A$  for the set of all subsets of  $A$ . We can make  $\mathcal{P}$  into a functor  $\mathbf{Set} \rightarrow \mathbf{Set}$ : given  $A \xrightarrow{f} B$ , we define  $\mathcal{P}f(A') = \{f(a) \mid a \in A'\}$  for  $A' \subseteq A$ . But we also have a functor  $\mathcal{P}^* : \mathbf{Set} \rightarrow \mathbf{Set}^{\text{op}}$  defined on objects by  $\mathcal{P}$ , but  $\mathcal{P}^*f(B') = \{a \in A \mid f(a) \in B'\}$  for  $B' \subseteq B$ .

By a **contravariant** functor  $\mathcal{C} \rightarrow \mathcal{D}$ , we mean a **functor**  $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$  (or  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ ). (A **covariant** functor is one that doesn't reverse arrows).

- (d) Let  $K$  be a field. We have a functor  $* : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{\text{op}}$  defined by  $V^* = \{\text{linear maps } V \rightarrow K\}$  and if  $V \xrightarrow{f} W$ ,  $f^*(\theta : W \rightarrow K) = \theta f$ .
- (e) We have a functor  $op : \mathbf{Cat} \rightarrow \mathbf{Cat}$  which is the 'identity' on morphisms. (Note that this is **covariant**).
- (f) A functor between monoids is a monoid homomorphism.
- (g) A functor between posets is an order-preserving map.
- (h) Let  $G$  be a group. A functor  $F : G \rightarrow \mathbf{Set}$  consists of a set  $A = F*$  together with an action of  $G$  on  $A$ , i.e. a permutation representation of  $G$  (where we use  $*$  to refer to the unique object of the group). Similarly a functor  $G \rightarrow \mathbf{Mod}_K$  is a  $K$ -linear representation of  $G$ .
- (i) The construction of a fundamental group  $\pi_1(X, x)$  of a space  $X$  with basepoint  $x$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Gp}$  where  $\mathbf{Top}_*$  is the set of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor  $\mathbf{Top} \rightarrow \mathbf{Gpd}$  where  $\mathbf{Gpd}$  is the category of groupoids and functors between them.

**1.6 Definition** (Natural transformation). Let  $\mathcal{C}, \mathcal{D}$  be **categories** and  $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$  two **functors**. A **natural transformation**  $\alpha : F \rightarrow G$  consists of an assignment  $A \mapsto \alpha_A$

from  $\text{ob } \mathcal{C}$  to  $\text{mor } \mathcal{D}$ , such that  $\text{dom } \alpha_A = FA$  and  $\text{cod } \alpha_A = GA$  for all  $A$ , and for all  $A \xrightarrow{f} B$  in  $\mathcal{C}$  the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e.  $\alpha_B(Ff) = (Gf)\alpha_A$ ).

### 1.3 Examples (Continued).

- (1) Given [categories](#)  $\mathcal{C}, \mathcal{D}$ , we write  $[\mathcal{C}, \mathcal{D}]$  for the category whose objects are [functors](#)  $\mathcal{C} \rightarrow \mathcal{D}$ , and whose morphisms are [natural transformations](#).

### 1.7 Examples.

- (a) Let  $K$  be a field,  $V$  a vector space over  $K$ . There is a linear map  $\alpha_V : V \rightarrow V^{**}$  given by

$$\alpha_V(v)(\theta) = \theta(v)$$

for  $\theta \in V^*$ . This is the  $V$ -component of a [natural transformation](#)

$$1_{\mathbf{Mod}_K} \rightarrow ** : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K.$$

*Lecture 3*

- (b) For any set  $A$ , we have a mapping  $\sigma_A : A \rightarrow \mathcal{P}A$  sending  $a$  to  $\{a\}$ . If  $f : A \rightarrow B$ , then  $\mathcal{P}f\{a\} = \{f(a)\}$ , so  $\sigma$  is a natural transformation  $1_{\mathbf{Set}} \rightarrow \mathcal{P}$ .
- (c) Let  $F : \mathbf{Set} \rightarrow \mathbf{Gp}$  be the [free group functor](#) ([Examples 1.5\(b\)](#)) and  $U : \mathbf{Gp} \rightarrow \mathbf{Set}$  the forgetful functor. The inclusions  $A \rightarrow UFA$  form a natural transformation  $1_{\mathbf{Set}} \rightarrow UF$ .
- (d) Let  $G, H$  be groups and  $f, g : G \rightarrow H$  two homomorphisms. A natural transformation  $\alpha : f \rightarrow g$  corresponds to an element  $h = \alpha_*$  of  $H$  such that  $h.f(x) = g(x).h$  for all  $x \in G$ , or equivalently  $f(x) = h^{-1}g(x)h$ , i.e.  $f$  and  $g$  are conjugate group homomorphisms.
- (e) Let  $A, B$  be two  $G$ -sets regarded as functors  $G \rightarrow \mathbf{Set}$ . A natural transformation  $A \rightarrow B$  is a function  $f$  satisfying  $f(g.a) = g.f(a)$  for all  $a \in A$ , i.e. a  $G$ -equivariant map.

**1.8 Lemma.** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two [functors](#), and  $\alpha : F \rightarrow G$  a [natural transformation](#). Then  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}]$  iff each  $\alpha_A$  is an isomorphism in  $\mathcal{D}$ .

*Proof.*

$\Rightarrow$  trivial

$\Leftarrow$  Suppose each  $\alpha_A$  has an inverse  $\beta_A$ . Given  $f : A \rightarrow B$  in  $\mathcal{C}$ , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

commutes.

But

$$\begin{aligned}(Ff)\beta_A &= \beta_B\alpha_B(Ff)\beta_A \\ &= \beta_B(Gf)\alpha_A\beta_A \\ &= \beta_B(Gf).\end{aligned}$$

□

**1.9 Definition** (Equivalent category). Let  $\mathcal{C}, \mathcal{D}$  be categories. By an **equivalence** between  $\mathcal{C}$  and  $\mathcal{D}$ , we mean a pair of **functors**  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  together with natural isomorphisms  $\alpha : 1_{\mathcal{C}} \rightarrow GF$  and  $\beta : FG \rightarrow 1_{\mathcal{D}}$ . We write  $\mathcal{C} \simeq \mathcal{D}$  if  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent.

We say a property  $P$  of **categories** is a **categorical property** if whenever  $\mathcal{C}$  has  $P$  and  $\mathcal{C} \simeq \mathcal{D}$ , then  $\mathcal{D}$  has  $P$ .

For instance, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

### 1.10 Examples.

- (a) The category **Part** is equivalent to the category **Set**<sub>\*</sub> of pointed sets (and basepoint-preserving functions). We define  $F : \mathbf{Set}_* \rightarrow \mathbf{Part}$  by  $F(A, a) = A \setminus \{a\}$  and if  $f : (A, a) \rightarrow (B, b)$

$$Ff(x) = \begin{cases} f(x) & \text{if } f(x) \neq b \\ \text{undefined} & \text{otherwise} \end{cases}$$

and  $G : \mathbf{Part} \rightarrow \mathbf{Set}_*$  by  $G(A) = A^+ = A \cup \{A\}$  and if  $f : A \rightarrow B$  is a partial function, we define  $Gf : A^+ \rightarrow B^+$  by

$$Gf = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

The composite  $FG$  is the identity on **Part**, but  $GF$  is not the identity, however there's an isomorphism

$$(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending  $a$  to  $A \setminus \{a\}$  and everything else to itself and this is natural.

Note that there can be no isomorphism  $\mathbf{Set}_* \rightarrow \mathbf{Part}$  since **Part** has a 1-element isomorphism class  $\{\emptyset\}$  and **Set**<sub>\*</sub> doesn't.

- (b) The category **FdMod** <sub>$K$</sub>  of finite-dimensional vector spaces over  $K$  is equivalent to **FdMod**<sup>op</sup> <sub>$K$</sub> : the functors in both directions are  $(-)^*$  and both isomorphisms are the natural transformations of [Examples 1.7\(a\)](#).
- (c) **FdMod** <sub>$K$</sub>  is also equivalent to **Mat** <sub>$K$</sub> : We define  $F : \mathbf{Mat}_K \rightarrow \mathbf{FdMod}_K$  by  $F(n) = K^n$ , and  $F(A)$  is the linear map represented by  $A$  with respect to the standard bases of  $K^n$  and  $K^p$ .

To define  $G : \mathbf{FdMod}_K \rightarrow \mathbf{Mat}_K$ , choose a basis for each finite dimensional vector space, and define  $G(V) = \dim V$ ,  $G(V \xrightarrow{f} W)$  as the matrix representing  $f$  with respect to the chosen bases.  $GF$  is the identity, provided we choose the standard bases for the spaces  $K^n$ ;  $FG \neq 1$ , but the chosen basis gives isomorphisms  $FG(V) = K^{\dim V} \rightarrow V$  for each  $V$ , which form a natural isomorphism.

Lecture 4 **1.11 Definition** (Faithful, full, essentially surjective). Let  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  be a **functor**.

- (a) We say  $F$  is **faithful** if, given  $f, f' \in \text{mor } \mathcal{C}$  with  $\text{dom } f = \text{dom } f'$ ,  $\text{cod } f = \text{cod } f'$  and  $Ff = Ff'$  then  $f = f'$ .
- (b) We say  $F$  is **full** if, given  $FA \xrightarrow{g} FB$  in  $\mathcal{D}$ , there exists  $A \xrightarrow{f} B$  in  $\mathcal{C}$  with  $Ff = g$ .
- (c) We say  $F$  is **essentially surjective** if, for every  $B \in \text{ob } \mathcal{D}$ , there exists  $A \in \text{ob } \mathcal{C}$  and an isomorphism  $FA \rightarrow B$  in  $\mathcal{D}$ .

We say a subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is **full** if the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  is a full functor.

**Example.** **Gp** is a **full subcategory** of **Mon**, but **Mon** is not a full subcategory of the category **Sgp** of semigroups.

**1.12 Lemma.** Assuming the axiom of choice, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is part of an **equivalence**  $\mathcal{C} \simeq \mathcal{D}$  iff it is **full**, **faithful** and **essentially surjective**.

*Proof.*

$\Rightarrow$  Given  $G, \alpha, \beta$  as in **Definition 1.9**, for each  $B \in \text{ob } \mathcal{D}$ ,  $\beta_B$  is an isomorphism  $FGB \rightarrow B$ , so  $F$  is essentially surjective.

Given  $A \xrightarrow{f} B$  in  $\mathcal{C}$ , we can recover  $f$  from  $Ff$  as the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B.$$

Hence if  $A \xrightarrow{f'} B$  satisfies  $Ff = Ff'$ , then  $f = f'$ .

Given  $FA \xrightarrow{g} FB$ , define  $f$  to be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$$

Then  $GFf = \alpha_B f \alpha_A^{-1} = Gg$ , and  $G$  is faithful for the same reason as  $F$ , so  $Ff = g$ .

$\Leftarrow$  For each  $B \in \text{ob } \mathcal{D}$ , choose  $GB \in \text{ob } \mathcal{C}$  and an isomorphism  $\beta_B : FGB \rightarrow B$  in  $\mathcal{D}$ . Given

$$B \xrightarrow{g} B'$$

define  $Gg : GB \rightarrow GB'$  to be the unique morphism whose image under  $F$  is

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

Uniqueness implies functoriality: given

$$B' \xrightarrow{g'} B''$$

then note  $(Gg')(Gg)$  and  $G(g'g)$  have the same image under  $F$ , so they're equal.

By construction,  $\beta$  is a natural transformation  $FG \rightarrow 1_{\mathcal{D}}$ .

Given  $A \in \text{ob } \mathcal{C}$ , define  $\alpha_A : A \rightarrow GFA$  to be the unique morphism whose image under  $F$  is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA$$

$\alpha_A$  is an isomorphism, since  $\beta_{FA}$  also has a unique pre-image under  $F$ .

Also  $\alpha$  is a natural transformation, since any naturality square for  $\alpha$  is mapped by  $F$  to a commutative square, and  $F$  is faithful.  $\square$

**1.13 Definition** (Skeleton). By a **skeleton** of a **category**  $\mathcal{C}$ , we mean a **full subcategory**  $\mathcal{C}_0$  containing one object from each isomorphism class. We say  $\mathcal{C}$  is **skeletal** if it's a skeleton of itself.

**Example.**  $\mathbf{Mat}_K$  is **skeletal**, and the image of  $F : \mathbf{Mat}_K \rightarrow \mathbf{FdMod}_K$  of **Examples 1.10(c)** is a **skeleton** of  $\mathbf{FdMod}_K$ .

**Warning.** Almost any assertion about **skeletons** is equivalent to the axiom of choice. See question 2 on example sheet 1.

**1.14 Definition** (Monomorphism, epimorphism). Let  $A \xrightarrow{f} B$  be a **morphism** in  $\mathcal{C}$

- (a) We say  $f$  is a **monomorphism** (or  $f$  is **monic**) if, given any pair  $C \xrightarrow[g]{f} A$ ,  $fg = fh$  implies  $g = h$
- (b) We say  $f$  is an **epimorphism** (or **epic**) if it's a monomorphism in  $C^{op}$  i.e. if  $gf = hf$  implies  $g = h$ .

We denote **monomorphisms** by  $A \xrightarrow{f} B$  and **epimorphisms** by  $A \xrightarrow{f} B$

Any isomorphism is **monic** and **epic**: more generally if  $f$  has a left inverse (i.e.  $\exists g$  such that  $gf$  is an identity) then it's monic. We call such monomorphisms **split**.

We say  $C$  is a **balanced category** if any morphism which is both **monic** and **epic** is an isomorphism.

**1.15 Examples.**

- (a) In **Set**, **mono**  $\iff$  injective ( $\implies$  easy; for  $\longleftarrow$  take  $C = 1 = \{*\}$ ) and **epi**  $\iff$  surjective ( $\implies$  easy; for  $\longleftarrow$  use two morphisms  $B \rightarrow 2 = \{0, 1\}$ ). So **Set** is **balanced**.
- (b) In **Gp** **mono**  $\iff$  injective (for  $\longleftarrow$  use homoms  $\mathbb{Z} \rightarrow A$ ) and **epi**  $\iff$  surjective ( $\longleftarrow$  uses free products with amalgamation). So **Gp** is balanced.
- (c) In **Rng**, **mono**  $\iff$  injective (proof much as for **Gp**) but the inclusion  $\mathbb{Z} \rightarrow \mathbb{Q}$  is an epimorphism, since if  $\mathbb{Q} \xrightarrow[g]{f} R$  agree on all integers, they agree everywhere. So **Rng** isn't balanced.
- (d) In **Top**, **mono**  $\iff$  injective and **epi**  $\iff$  surjective (proofs as in **Set**). But **Top** isn't balanced since a continuous bijection needn't have a continuous inverse.



## 2 The Yoneda Lemma

*Lecture 5* **2.1 Definition** (Locally small). We say a **category**  $\mathcal{C}$  is **locally small** if, for any two objects  $A, B$ , the morphisms  $A \rightarrow B$  in  $\mathcal{C}$  form a set  $\mathcal{C}(A, B)$ .

If we fix  $A$  and let  $B$  vary, the assignment  $B \mapsto \mathcal{C}(A, B)$  becomes a **functor**  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ : given  $B \xrightarrow{f} C$ ,  $\mathcal{C}(A, f)$  is the mapping  $g \mapsto fg$ . Similarly,  $A \mapsto \mathcal{C}(A, B)$  defines a functor  $\mathcal{C}(-, B) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ .

**2.2 Lemma** (Yoneda Lemma). Let  $\mathcal{C}$  be a **locally small category**,  $A \in \text{ob } \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathbf{Set}$  a **functor**.

- (i) Then natural transformations  $\mathcal{C}(A, -) \rightarrow F$  are in bijection with elements of  $FA$ .
- (ii) Moreover, this bijection is natural in both  $A$  and  $F$ .

*Proof of Yoneda Lemma(i).* Given  $\alpha : \mathcal{C}(A, -) \rightarrow F$ , we define

$$\Phi(\alpha) = \alpha_A(1_A) \in FA.$$

Given  $x \in FA$ , we define  $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$  by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB.$$

$\Psi(x)$  is natural: given  $g : B \rightarrow C$ , we have

$$\begin{aligned} \Psi(x)_C \mathcal{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x) \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x). \end{aligned}$$

$$\begin{array}{ccc} \mathcal{C}(A, B) & \xrightarrow{\mathcal{C}(A, g)} & \mathcal{C}(A, C) \\ \Psi(x)_B \downarrow & & \downarrow \Psi(x)_C \\ FB & \xrightarrow{Fg} & FC \end{array}$$

We also verify  $\Psi$  and  $\Phi$  are inverse:

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x.$$

Given  $\alpha$ ,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

so  $\Psi\Phi(\alpha) = \alpha$ . □

**2.3 Corollary.** The assignment  $A \mapsto \mathcal{C}(A, -)$  defines a **full** and **faithful functor**  $\mathcal{C}^{\text{op}} \rightarrow [\mathcal{C}, \mathbf{Set}]$ .

*Proof.* Put  $F = \mathcal{C}(B, -)$  in [Lemma 2.2\(i\)](#): we get a bijection between  $\mathcal{C}(B, A)$  and morphisms  $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$  in  $[\mathcal{C}, \mathbf{Set}]$ . We need to verify this is functorial: but it sends  $f : B \rightarrow A$  to the natural transformation  $g \mapsto gf$ . So functoriality follows from associativity.  $\square$

We call this **functor** (or the functor  $\mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ ) sending  $A$  to  $\mathcal{C}(-, A)$  the **Yoneda embedding** of  $\mathcal{C}$ , and denote it by  $Y$ .

*Proof of Yoneda Lemma(ii).* Suppose for the moment that  $\mathcal{C}$  is **small**, so that  $[\mathcal{C}, \mathbf{Set}]$  is **locally small**. Then we have two functors  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$ : One sends  $(A, F)$  to  $FA$ , and the other is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{\text{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-, -)} \mathbf{Set}$$

[Yoneda Lemma\(ii\)](#) says that these are naturally isomorphic.

We can translate this into an elementary statement, making sense even when  $\mathcal{C}$  isn't small, given  $A \xrightarrow{f} B$  and  $F \xrightarrow{\alpha} G$ , the two ways of producing an element of  $GB$  from a natural transformation  $\beta : \mathcal{C}(A, -) \rightarrow F$  give the same result, namely

$$\alpha_B(Ff)\beta_A(1_A) = (GF)\alpha_A\beta_A(1_A)$$

which is equal to  $\alpha_B\beta_B(f)$ .  $\square$

**2.4 Definition.** We say a **functor**  $F : \mathcal{C} \rightarrow \mathbf{Set}$  is **representable** if it's **isomorphic** to  $\mathcal{C}(A, -)$  for some  $A$ . By **representation** of  $F$ , we mean a pair  $(A, x)$  where  $x \in FA$  is such that  $\Psi(x)$  is an **isomorphism**. We also call  $x$  a **universal element** of  $F$ .

**2.5 Corollary.** If  $(A, x)$  and  $(B, y)$  are both **representations** of  $F$ , then there's a unique **isomorphism**  $f : A \rightarrow B$  such that  $(Ff)(x) = y$ .

*Proof.* Consider the composite

$$\mathcal{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathcal{C}(A, -)$$

By [Corollary 2.3](#), this is of the form  $Y(f)$  for a unique **isomorphism**  $f : A \rightarrow B$  and the diagram

$$\begin{array}{ccc} \mathcal{C}(B, -) & \xrightarrow{Y(f)} & \mathcal{C}(A, -) \\ & \searrow \Psi(y) & \swarrow \Psi(x) \\ & F & \end{array}$$

commutes iff  $(Ff)x = y$ .  $\square$

## 2.6 Examples.

- (a) The **forgetful functor**  $\mathbf{Gp} \rightarrow \mathbf{Set}$  is **representable** by  $(\mathbb{Z}, 1)$ . Similarly, the forgetful functor  $\mathbf{Rng} \rightarrow \mathbf{Set}$  is representable by  $(\mathbb{Z}[x], x)$  and the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  is representable by  $(\{\ast\}, \ast)$ .

- (b) The functor  $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$  (see Examples 1.5(c)) is representable by  $(\{0, 1\}, \{1\})$ : this is the bijection between subsets and characteristic functions.
- (c) Let  $G$  be a group. The unique (up to isomorphism) representable functor  $G(*, -) : G \rightarrow \mathbf{Set}$  is the *Cayley representation* of  $G$ , i.e. the set  $UG$  with  $G$  acting by left multiplication.
- (d) Let  $A, B$  be two objects of a locally small category  $\mathcal{C}$ . We have a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  sending  $C$  to  $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$ . A representation of this, if it exists, is called a (categorical) **product** of  $A$  and  $B$ , and denoted

$$(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B)).$$

This pair has the property that, for any pair  $(C \xrightarrow{f} A, C \xrightarrow{g} B)$  there's a unique  $C \xrightarrow{h} A \times B$  with  $\pi_1 h = f$  and  $\pi_2 h = g$ .

Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top** they are 'just' Cartesian products, in posets they are binary meets.

Dually we have the notion of **coproduct**  $(A + B, (A \xrightarrow{\nu_1} A + B, B \xrightarrow{\nu_2} A + B))$ . These also exist in many categories of interest.

Lecture 6

- (e) The dual-vector-space functor  $\mathbf{Mod}_K^{\text{op}} \rightarrow \mathbf{Mod}_K$ , when composed with the forgetful functor  $\mathbf{Mod}_K \rightarrow \mathbf{Set}$ , is representable by  $(K, 1_K)$ .
- (f) Let  $A \rightrightarrows B$  be morphisms in a locally small category  $\mathcal{C}$ . We have a functor  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  defined by

$$F(C) = \{ h \in \mathcal{C}(C, A) \mid fh = gh \}.$$

A representation of  $F$ , if it exists, is called an **equalizer** of  $(f, g)$ . It consists of an objects  $E$  and a morphism  $E \xrightarrow{e} A$  such  $fe = ge$ , and every  $h$  with  $fh = gh$  factors uniquely through  $e$ . In **Set**, we can take  $E = \{ x \in A \mid f(x) = g(x) \}$  and  $e =$  inclusion. Similar constructions work in **Gp**, **Rng**, **Top**, ...

Dually, we have the notion of **coequalizer**.

**2.7 Remark.** If  $e$  occurs as an **equalizer**, then it's a **monomorphism**, since any  $h$  factors through it in at most one way. We say a monomorphism is **regular** if it occurs as an equalizer.

**Split** monomorphisms are **regular** (c.f. question 6i on sheet 1). Note that regular mono + **epi**  $\implies$  **iso**: if the equalizer  $e$  of  $(f, g)$  is epic, then  $f = g$ , so  $e \cong 1_{\text{cod } e}$ .

**2.8 Definition** (Separating, detecting families). Let  $\mathcal{C}$  be a category, and  $\mathcal{G}$  a class of objects of  $\mathcal{C}$ .

- (a) We say  $\mathcal{G}$  is a **separating family** for  $\mathcal{C}$  if, given  $A \rightrightarrows B$  such that  $fh = gh$  for all  $G \xrightarrow{h} A$  with  $G \in \mathcal{G}$ , then  $f = g$  (i.e. the functors  $\mathcal{C}(G, -)$ ,  $G \in \mathcal{G}$  are collectively faithful).

- (b) We say  $\mathcal{G}$  is a **detecting family** for  $\mathcal{C}$  if, given  $A \xrightarrow{f} B$  such that every  $G \xrightarrow{h} B$  with  $G \in \mathcal{G}$  factors uniquely through  $f$ , then  $f$  is an isomorphism.

If  $\mathcal{G} = \{G\}$ , we call  $G$  a **separator/detector**.

## 2.9 Lemma.

- (i) If  $\mathcal{C}$  is a **balanced** category, then any **separating** family is **detecting**.
- (ii) If  $\mathcal{C}$  has **equalizers**, then any detecting family is separating.

*Proof.*

- (i) Suppose  $\mathcal{G}$  is **separating** and  $A \xrightarrow{f} B$  satisfies the condition of **Definition 2.8(b)**. If  $B \xrightarrow[g]{g} C$  satisfy  $gf = hf$ , then  $gx = hx$  for every  $G \xrightarrow{x} B$ , so  $g = h$ , i.e.  $f$  is **epic**.  
Similarly if  $D \xrightarrow[k]{l} A$  satisfy  $fk = fl$ , then  $ky = ly$  for any  $G \xrightarrow{y} D$ , since both are factorisations of  $fky$  through  $f$ . So  $k = l$ , i.e.  $f$  is **monic**.
- (ii) Suppose  $\mathcal{G}$  is **detecting** and  $A \xrightarrow[g]{f} B$  satisfies the condition of 2.8(a). Then the **equalizer**  $E \xrightarrow{e} A$  is an **isomorphism**, so  $f = g$ .

□

## 2.10 Examples.

- (a) In  $[\mathcal{C}, \mathbf{Set}]$  the family

$$\{\mathcal{C}(A, -) \mid A \in \text{ob } \mathcal{C}\}$$

is both **separating** and **detecting** (this is just a restatement of **Yoneda Lemma**.)

- (b) In **Set**,  $1 = \{*\}$  is both a separator and a detector since it represents the identity **functor**  $\mathbf{Set} \longrightarrow \mathbf{Set}$ .

Similarly,  $\mathbb{Z}$  is both in **Gp**, since it represents the **forgetful functor**  $\mathbf{Gp} \longrightarrow \mathbf{Set}$ .

And  $2 = \{0, 1\}$  is a coseparator and a codetector in **Set**, since it represents  $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Set}$ .

- (c) In **Top**,  $1 = \{*\}$  is a separator since it represents the forgetful functor  $\mathbf{Top} \longrightarrow \mathbf{Set}$ , but not a detector. In fact, **Top** has no detecting *set* of objects:

For any infinite cardinal  $\kappa$ , let  $X$  be a discrete space of cardinality  $\kappa$  and let  $Y$  be the same set with ‘co- $< \kappa$ ’ topology, i.e.  $F \subseteq Y$  closed  $\iff F = Y$  or  $\text{card } F < \kappa$ . The identity  $X \longrightarrow Y$  is continuous, but not a homeomorphism.

So if  $\{G_i \mid i \in I\}$  is any set of spaces, taking  $\kappa > \text{card } G_i$  for all  $i$  yields an example to show that the set is not detecting.

(d) Let  $\mathcal{C}$  be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous maps. JHC Whitehead proved that if  $X \xrightarrow{f} Y$  in this category induces isomorphisms  $\pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then it's an isomorphism in  $\mathcal{C}$ . This says that  $\{S^n \mid n \geq 1\}$  is a detecting set for  $\mathcal{C}$ .

But PJ Freyd showed there is no **faithful** functor  $\mathcal{C} \rightarrow \mathbf{Set}$ , so no separating *set*: if  $\{G_i \mid i \in I\}$  were separating, then

$$x \mapsto \prod_{i \in I} \mathcal{C}(G_i, X)$$

would be faithful.

Note that any **functor** of the form  $\mathcal{C}(A, -)$  preserves **monomorphisms**, but they don't normally preserve **epimorphisms**.

**2.11 Definition** (Projective). We say an object  $P$  is **projective** if, given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{e} & B \end{array}$$

there exists  $P \xrightarrow{g} A$  with  $eg = f$ . (If  $\mathcal{C}$  is **locally small**, this says  $\mathcal{C}(P, -)$  preserves **epimorphisms**).

Dually, an **injective** object of  $\mathcal{C}$  is a projective object of  $\mathcal{C}^{\text{op}}$ . Given a class  $\mathcal{E}$  of epimorphisms, we say  $P$  is  $\mathcal{E}$ -projective if it satisfies the condition for all  $e \in \mathcal{E}$ .

**2.12 Lemma.** **Representable functors** are (pointwise) **projective** in  $[\mathcal{C}, \mathbf{Set}]$ .

*Proof.* Take

$$\begin{array}{ccc} & \mathcal{C}(A, -) & \\ & \downarrow \beta & \\ F & \xrightarrow{a} & G \end{array}$$

where  $\alpha$  is pointwise surjective. By **Yoneda Lemma**,  $\beta$  corresponds to some  $y \in GA$ , and we can find  $x \in FA$  with  $\alpha_A(x) = y$ . Now if  $\gamma : \mathcal{C}(A, -) \rightarrow F$  corresponds to  $x$  then naturality of the **Yoneda bijection** yields  $\alpha\gamma = \beta$ .  $\square$

### 3 Adjunctions

*Lecture 7* **3.1 Definition.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two **categories** and  $\mathcal{C} \xrightarrow{F} \mathcal{D}$ ,  $\mathcal{D} \xrightarrow{G} \mathcal{C}$  two **functors**. By an **adjunction** between  $F$  and  $G$  we mean a bijection between **morphisms**  $FA \xrightarrow{\hat{f}} B$  in  $\mathcal{D}$  and morphisms  $A \xrightarrow{f} GB$  in  $\mathcal{C}$  which is **natural** in  $A$  and  $B$ , i.e. given  $A' \xrightarrow{g} A$  and  $B \xrightarrow{h} B'$ , we have  $h\hat{f}(Fg) = \widehat{(Gh)fg} : FA' \rightarrow B'$ .

We say  $F$  is **left adjoint** to  $G$  and write  $F \dashv G$ .

#### 3.2 Examples.

- (a) The **free functor**  $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$  is **left adjoint** to the **forgetful functor**  $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$ , since any function  $f : A \rightarrow UB$  extends uniquely to a homomorphism  $\hat{f} : FA \rightarrow B$ . Naturality in  $B$  is easy, naturality in  $A$  follows from the definition of  $F$  as a functor.
- (b) The **forgetful functor**  $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$  has a left adjoint  $D$  which equips any set with the discrete topology and a right adjoint  $I$  which equips a set  $A$  with the indiscrete topology  $\{\emptyset, A\}$ .
- (c) The functor  $\mathbf{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$  has a left adjoint  $D$  sending  $A$  to the **discrete** category with  $\mathbf{ob}(DA) = A$  and only identity morphisms. It also has a right adjoint  $I$  sending  $A$  to the (**indiscrete**) category with  $\mathbf{ob}(IA) = A$  and one morphism  $x \rightarrow y$  for each  $(x, y) \in A \times A$ . In this case  $D$  in turn has a left adjoint  $\pi_0$  sending a small category  $\mathcal{C}$  to its set of *connected components*, i.e. the quotient of  $\mathbf{ob} \mathcal{C}$  by the smallest equivalent relation identifying  $\mathbf{dom} f$  with  $\mathbf{cod} f$  for all  $f \in \mathbf{mor} \mathcal{C}$ .
- (d) Let  $\mathcal{M}$  be the **monoid**  $\{1, e\}$  with  $e^2 = e$ . An object of  $[\mathcal{M}, \mathbf{Set}]$  is a pair  $(A, e)$  where  $e : A \rightarrow A$  satisfies  $e^2 = e$ .

We have a functor  $G : [\mathcal{M}, \mathbf{Set}] \rightarrow \mathbf{Set}$  sending  $(A, e)$  to

$$\{x \in A \mid e(x) = x\} = \{e(x) \mid x \in A\}$$

and a functor  $F : \mathbf{Set} \rightarrow [\mathcal{M}, \mathbf{Set}]$  sending  $A$  to  $(A, 1_A)$ .

Claim  $F \dashv G \dashv F$ : given  $f : (A, 1_A) \rightarrow (B, e)$ : it must take values in  $G(B, e)$ , and any  $g : (B, e) \rightarrow (A, 1_A)$  is determined by its values on the image of  $e$ .

- (e) Let  $\mathbf{1}$  be the **discrete** category with one object  $*$ . For any  $\mathcal{C}$ , there's a unique functor  $\mathcal{C} \rightarrow \mathbf{1}$ : a **left adjoint** for this picks out an **initial object** of  $\mathcal{C}$ , i.e. an object  $I$  such that there exists a unique  $I \rightarrow A$  for each  $A \in \mathbf{ob} \mathcal{C}$ . Dually, a right adjoint for  $\mathcal{C} \rightarrow \mathbf{1}$  corresponds to a **terminal object** of  $\mathcal{C}$ .
- (f) Let  $A \xrightarrow{f} B$  be a morphism in  $\mathbf{Set}$ . We can regard  $\mathcal{P}A$  and  $\mathcal{P}B$  as posets, and we have functors

$$\mathcal{P}A \xrightleftharpoons[\mathcal{P}^*f]{\mathcal{P}f} \mathcal{P}B$$

Claim  $(\mathcal{P}f \dashv \mathcal{P}^*f)$ : we have  $\mathcal{P}f(A') \subseteq B' \iff f(x) \in B'$  for all  $x \in A' \iff A' \subseteq \mathcal{P}^*f(B')$ .

- (g) Suppose given sets  $A, B$  and a relation  $R \subseteq A \times B$ . We define mappings  $(-)^l, (-)^r$  between  $\mathcal{P}A$  and  $\mathcal{P}B$  by

$$\begin{aligned} S^r &= \{ y \in B \mid (\forall x \in S)((x, y) \in R) \} \quad \text{for } S \subseteq A \\ T^l &= \{ x \in A \mid (\forall y \in T)((x, y) \in R) \} \quad \text{for } T \subseteq B. \end{aligned}$$

These mappings are order-reversing (i.e. [contravariant functors](#)) and

$$T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l.$$

We say  $(-)^r$  and  $(-)^l$  are **adjoint on the right**.

- (h) The functor  $\mathcal{P}^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$  is self-[adjoint on the right](#), since functions  $A \rightarrow \mathcal{P}B$  correspond bijectively to subsets of  $A \times B$  and hence to functions  $B \rightarrow \mathcal{P}A$ .

**Definition** (Comma category).  $(A \downarrow G)$  is the **comma category** with objects pairs  $(B, f)$  with  $A \xrightarrow{f} GB$ , and morphisms  $(B, f) \rightarrow (B', f')$  are morphisms  $B \xrightarrow{g} B'$  such that

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow f' \\ GB & \xrightarrow{Gg} & GB' \end{array}$$

commutes.

**3.3 Theorem.** Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a [functor](#). Then specifying a [left adjoint](#) for  $G$  is equivalent to specifying an [initial object](#) of  $(A \downarrow G)$  for each  $A \in \text{ob } \mathcal{C}$ .

*Proof.* Suppose we are given  $F \dashv G$ . Consider the morphism  $\eta_A : A \rightarrow GFA$  corresponding to  $FA \xrightarrow{1} FA$ . Then  $(FA, \eta_A)$  is an object of  $(A \downarrow G)$ . Moreover, given  $g : FA \rightarrow B$  and  $f : A \rightarrow GB$ , the diagram

$$\begin{array}{ccc} & A & \\ \eta_A \swarrow & & \searrow f \\ GFA & \xrightarrow{Gg} & GB \end{array}$$

commutes iff

$$\begin{array}{ccc} & FA & \\ 1_{FA} \swarrow & & \searrow \hat{f} \\ FA & \xrightarrow{g} & B \end{array}$$

commutes, i.e.  $g = \hat{f}$ . So  $(FA, \eta_A)$  is [initial](#) in  $(A \downarrow G)$ .

Conversely, suppose we are given an initial object  $(FA, \eta_A)$  for each  $(A \downarrow G)$ . Given  $A \xrightarrow{f} A'$ , we define  $Ff : FA \rightarrow FA'$  to be the unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute. Functoriality follows from uniqueness: given  $f' : A' \rightarrow A''$ , both  $F(f'f)$  and  $(Ff')(Ff)$  are morphisms  $(FA, \eta_A) \rightarrow (FA'', \eta_{A''}f'f)$  in  $(A \downarrow G)$ .

To show  $F \dashv G$ : given  $A \xrightarrow{f} GB$ , we define  $\hat{f} : FA \rightarrow B$  to be the unique morphism  $(FA, \eta_A) \rightarrow (B, f)$  in  $(A \downarrow G)$ . This is a bijection with inverse

$$(FA \xrightarrow{f} B) \mapsto (A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB)$$

The latter mapping is natural in  $B$  since  $G$  is a functor, and in  $A$  since, by construction,  $\eta$  is a natural transformation  $1_{\mathcal{C}} \rightarrow GF$ .  $\square$

**3.4 Corollary.** If  $F$  and  $F'$  are both [left adjoint](#) to  $G : \mathcal{D} \rightarrow \mathcal{C}$ , then they are [naturally isomorphic](#).

*Proof.* For any  $A$ ,  $(FA, \eta_A)$  and  $(F'A, \eta'_A)$  are both [initial](#) in  $(A \downarrow G)$ , so there's a unique [isomorphism](#)  $\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A)$ . In any naturality square for  $\alpha$ , the two ways round are both morphisms in  $(A \downarrow G)$  where the domain is initial, so they are equal.  $\square$

**3.5 Lemma.** Given

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{H} \\ \xleftarrow{K} \end{array} \mathcal{E}$$

with  $(F \dashv G)$  and  $(H \dashv K)$  we have  $(HF \dashv GK)$ .

*Proof.* We have bijections between morphisms  $A \rightarrow GKC$ , morphisms  $FA \rightarrow KC$  and morphisms  $HFA \rightarrow C$ , which are both natural in  $A$  and  $C$ .  $\square$

**3.6 Corollary.** Given a commutative square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{F} \end{array}$$

of [categories](#) and [functors](#), if the functors all have left [adjoints](#), then the diagram of left adjoints commutes up to [natural isomorphism](#).

*Proof.* By [Lemma 3.5](#), both ways round the diagram of left adjoints are left adjoint to the composite  $\mathcal{C} \rightarrow \mathcal{F}$ , so by [Corollary 3.4](#) they are isomorphic.  $\square$

Given an [adjunction](#)  $(F \dashv G)$ , the [natural transformation](#)  $\eta : 1_{\mathcal{C}} \rightarrow GF$  emerging in the proof of [Theorem 3.3](#) is called the **unit** of the adjunction. Dually, we have a natural transformation  $\epsilon : FG \rightarrow 1_{\mathcal{D}}$  such that  $\epsilon_B : FGB \rightarrow B$  corresponds to  $GB \xrightarrow{1_{GB}} GB$  is called the **counit**.

**3.7 Theorem.** Given functors  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$  specifying an [adjunction](#)  $(F \dashv G)$  is equivalent to specifying natural transformations  $\eta : 1_{\mathcal{C}} \rightarrow GF$ ,  $\epsilon : FG \rightarrow 1_{\mathcal{D}}$  satisfying the commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \epsilon F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow 1_G & \downarrow G\epsilon \\ & & G \end{array}$$



called the **triangular identities**.

*Proof.* First suppose given  $(F \dashv G)$ . Define  $\eta$  and  $\epsilon$  as in [Theorem 3.3](#) and its dual; now consider the composite

$$FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA.$$

Under the adjunction this corresponds to

$$A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$$

but this also corresponds to  $1_{FA}$ , so  $\epsilon_{FA} \cdot F\eta_A = 1_{FA}$ . The other identity is [dual](#).

Conversely, suppose given  $\eta$  and  $\epsilon$  satisfying the [triangular identities](#). Given  $A \xrightarrow{f} GB$ , let  $\Phi(f)$  be the composite

$$FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B,$$

and given  $FA \xrightarrow{g} B$ , let  $\Psi(g)$  be

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB.$$

Then  $\Phi$  and  $\Psi$  are both natural; we need to show that  $\Phi\Psi$  and  $\Psi\Phi$  are identity mappings. But

$$\begin{aligned} \Psi\Phi \left( A \xrightarrow{f} GB \right) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f \end{aligned}$$

and dually  $\Phi\Psi(g) = g$ . □

**3.8 Lemma.** Suppose given

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

and [natural isomorphisms](#)  $\alpha : 1_{\mathcal{C}} \rightarrow GF$ ,  $\beta : FG \rightarrow 1_{\mathcal{D}}$ . Then there are isomorphisms  $\alpha' : 1_{\mathcal{C}} \rightarrow GF$ ,  $\beta' : FG \rightarrow 1_{\mathcal{D}}$  which satisfy the [triangular identities](#), so  $(F \dashv G)$  (and  $(G \dashv F)$ ).

*Proof.* We define  $\alpha' = \alpha$  and  $\beta'$  to be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}.$$

Note that  $FG\beta = \beta FG$  since

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta FG & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$$

commutes by naturality of  $\beta$  and  $\beta$  is monic. Now  $(\beta'_F)(F\alpha')$  is the composite

$$\begin{aligned} & F \xrightarrow{F\alpha} FGF \xrightarrow{(\beta FGF)^{-1}} FGF GF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\beta F} F \\ &= F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{FGF\alpha} FGF GF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\beta F} F \\ &= F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{\beta F} F = 1_F \end{aligned}$$

since  $GF\alpha = \alpha_{GF}$ . Similarly  $(G\beta')(G'\alpha')$  is

$$\begin{aligned} & G \xrightarrow{\alpha G} GFG \xrightarrow{(GFG\beta)^{-1}} GFG FG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha GFG} GFG FG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{G\beta} G = 1_G. \end{aligned} \quad \square$$

**3.9 Lemma.** Suppose  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a left adjoint  $F$  with counit  $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ , then

- (i)  $G$  is faithful iff  $\epsilon$  is pointwise epic.
- (ii)  $G$  is full and faithful iff  $\epsilon$  is an isomorphism.

*Proof.*

- (i) Given  $B \xrightarrow{g} B'$ ,  $Gg$  corresponds under the adjunction to the composite

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'.$$

Hence the mapping  $g \mapsto Gg$  is injective on morphisms with domain  $B$  (and specified codomain) iff  $g \mapsto g\epsilon_B$  is injective, i.e. iff  $\epsilon_B$  is epic.

- (ii) Similarly,  $G$  is full and faithful iff  $g \mapsto g\epsilon_B$  is bijective. If  $\alpha : B \rightarrow FGB$  is such that  $\alpha\epsilon_B = 1_{FGB}$ , then  $\epsilon_B\alpha\epsilon_B = \epsilon_B$ , whence  $\epsilon_B\alpha = 1_B$ . So  $\epsilon_B$  is an isomorphism for all  $B$ .  $\square$

*Lecture 9* **3.10 Definition** (Reflection). By a **reflection**, we mean an adjunction in which the right adjoint is full and faithful (equivalently, the counit is an isomorphism). We say a subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is **reflective** if the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  has a left adjoint.

### 3.11 Examples.

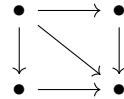
- (a) The category **AbGp** of abelian groups is reflective in **Gp**: the left adjoint sends a group  $G$  to its abelianization  $G/G'$ , where  $G'$  is the subgroup generated by all commutators  $[x, y] = xyx^{-1}y^{-1}$ ,  $x, y \in G$ . (The unit of the adjunction is the quotient map  $G \mapsto G/G'$ .)
- (b) Given an abelian group  $A$ , let  $A_t$  denote the torsion subgroup, i.e. the subgroup of elements of finite order. The assignment  $A \mapsto A/A_t$  gives a left adjoint to the inclusion **tfAbGp**  $\rightarrow$  **AbGp** where **tfAbGp** is the full subcategory of torsion-free abelian groups. And  $A \mapsto A_t$  is right adjoint to the inclusion **tfAbGp**  $\rightarrow$  **AbGp** so this subcategory is coreflective.

- (c) Let  $\mathbf{KHaus} \subseteq \mathbf{Top}$  be the full subcategory of compact Hausdorff spaces. The inclusion  $\mathbf{KHaus} \longrightarrow \mathbf{Top}$  has a left adjoint  $\beta$ , the Stone-Ćech compactification.
- (d) Let  $X$  be a topological space. We say  $A \subseteq X$  is sequentially closed if  $x_n \longrightarrow x_\infty$  and  $x_n \in A$  for all  $n$  implies  $x_\infty \in A$ . We say  $X$  is sequential if all sequentially closed sets are closed. Given a non-sequential space  $X$ , let  $X_s$  be the same set with topology given by the sequentially open sets in  $X$ ; the identity  $X_s \longrightarrow X$  is continuous, and defines the counit of an adjunction between the inclusion  $\mathbf{Seq} \longrightarrow \mathbf{Top}$  and the functor  $X \longmapsto X_s$ .
- (e) If  $X$  is a topological space, the poset  $CX$  of closed subsets of  $X$  is reflective in  $PX$ , with reflector given by closure, and the poset  $OX$  of open subsets is coreflective, with coreflector given by interior.

## 4 Limits

### 4.1 Definition.

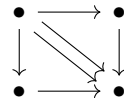
- (a) Let  $J$  be a **category** (almost always **small**, often finite). By a **diagram of shape  $J$**  in  $\mathcal{C}$  we mean a **functor**  $D : J \rightarrow \mathcal{C}$ . The objects  $D(j)$ ,  $j \in \text{ob } J$  are called **vertices** of the diagram, and the morphisms  $D(\alpha)$ ,  $\alpha \in \text{mor } J$  are called **edges** of  $D$ . For example, if  $J$  is the category



with 4 objects and 5 non-identity morphisms, a diagram of shape  $J$  is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

If  $J$  is

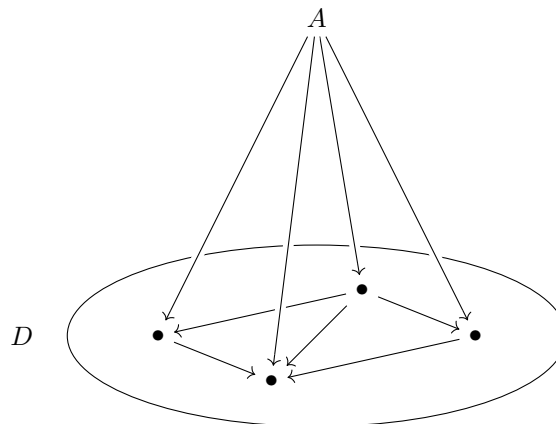


a diagram of shape  $J$  is a not-necessarily commutative square.

- (b) Given  $D : J \rightarrow \mathcal{C}$ , a **cone** over  $D$  consists of an object  $A$  of  $\mathcal{C}$  (the **apex** of the cone) together with morphisms  $A \xrightarrow{\lambda_j} D(j)$  for each  $j \in \text{ob } J$ , such that

$$\begin{array}{ccc} & A & \\ \lambda_j \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array}$$

commutes for all  $j \xrightarrow{\alpha} j'$  in  $\text{mor } J$  (the  $\lambda_j$  are called the **legs** of the cone).



Given cones  $(A, (\lambda_j)_{j \in \text{ob } J})$  and  $(B, (\mu_j)_{j \in \text{ob } J})$ , a morphism of cones between them is a morphism  $A \xrightarrow{f} B$  such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \lambda_j \searrow & & \swarrow \mu_j \\ & D(j) & \end{array}$$

commutes for all  $j$ .

We write  $\mathbf{Cone}(D)$  for the category of cones over  $D$ .

- (c) A **limit** for  $D$  is a **terminal object** of  $\mathbf{Cone}(D)$ , if this exists. Dually, we have the notion of **cone under a diagram**, and of **colimit** (**initial** cone under  $D$ ).

Alternatively if  $\mathcal{C}$  is **locally small** and  $J$  is small, we have a **functor**  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  sending  $A$  to the set of **cones** with apex  $A$ . A **limit** for  $D$  is a **representation** of this functor.

If  $\Delta A$  denotes the constant diagram of shape  $J$  with all vertices  $A$  and all edges  $1_A$ , then a **cone** over  $D$  with **apex**  $A$  is the same thing as a **natural transformation**  $\Delta A \rightarrow D$ .  $\Delta$  is a functor  $\mathcal{C} \rightarrow [J, \mathcal{C}]$ , and  $\mathbf{Cone}(D)$  is the category  $(\Delta \downarrow D)$  (a **comma category**, reversed). So to say that every **diagram of shape**  $J$  in  $\mathcal{C}$  has a **limit** is equivalent to saying that  $\Delta$  has a **right adjoint**. (We say  $\mathcal{C}$  **has limits** of shape  $J$ ). Dually,  $\mathcal{C}$  **has colimits** of shape  $J$  iff  $\Delta : \mathcal{C} \rightarrow [J, \mathcal{C}]$  has a left adjoint.

## 4.2 Examples.

- (a) Suppose  $J = \emptyset$ . There's a unique **diagram of shape**  $J$  in  $\mathcal{C}$ ; a **cone** over it is just an object, and a morphism of cones is a morphism of  $\mathcal{C}$ . So a **limit** for this empty diagram is a **terminal object** of  $\mathcal{C}$ . (Dually, a **colimit** for it is an initial object).

Lecture 10

- (b) Let  $J$  be the category



A **diagram of shape**  $J$  is a pair of objects  $A, B$ ; a **cone** over it is a **span**

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ A & & B \end{array}$$

and a **limit** for it is a **product**

$$\begin{array}{ccc} & A \times B & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ A & & B \end{array}$$

Dually, a colimit for it is a **coproduct**

$$\begin{array}{ccc} A & & B \\ \nu_1 \searrow & & \swarrow \nu_2 \\ & A + B & \end{array}$$

- (c) More generally, if  $J$  is a [small](#) discrete category, a diagram of shape  $J$  is a  $J$ -indexed family  $\{A_j \mid j \in J\}$ , and a limit for it is a product

$$\left\{ \prod_{j \in J} A_j \xrightarrow{\pi_j} A_j \mid j \in J \right\}.$$

Dually, a colimit for it is a coproduct

$$\left\{ A_j \xrightarrow{\nu_j} \sum_{j \in J} A_j \mid j \in J \right\},$$

also written  $\coprod_{j \in J} A_j$ .

- (d) Let  $J$  be the category

$$\bullet \rightrightarrows \bullet$$

A diagram of shape  $J$  is a parallel pair

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

a cone over this is

$$\begin{array}{ccc} & C & \\ h \swarrow & & \searrow k \\ A & & B \end{array}$$

satisfying  $fh = k = gh$ , or equivalently a morphism  $C \xrightarrow{h} A$  satisfying  $fh = gh$ . A (co)limit for the diagram is a [\(co\)equalizer](#).

- (e) Let  $J$  be the category

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

A diagram of shape  $J$  is a [cospan](#)

$$\begin{array}{ccc} & A & \\ & \downarrow f & \\ B & \xrightarrow{g} & C \end{array}$$

a cone over it is

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ \downarrow q & \searrow r & \\ B & & C \end{array}$$

satisfying  $fp = r = gq$ , or equivalently a span  $(p, q)$  completing the diagram to a commutative square. A limit for the diagram is called a **pullback** of  $(f, g)$ . In **Set**, the [apex](#) of the pullback is the ‘fibre product’

$$A \times_C B = \{ (x, y) \in A \times B \mid f(x) = g(y) \}$$

Dually, colimits of shape  $J^{\text{op}}$  are called **pushouts**, given

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

we ‘push  $g$  along  $f$ ’ to get the right hand side of the colimit square.

(f) Let  $J$  be the poset of natural numbers. A diagram of shape  $J$  is a **direct system**

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

A colimit for this is called a **direct limit**: it consists of  $A_\infty$  equipped with morphisms  $A_n \xrightarrow{g_n} A_\infty$  satisfying  $g_n = g_{n+1}f_n$  for all  $n$ , and universal among such. Dually, we have **inverse system** and **inverse limit**.

#### 4.3 Theorem.

- (i) Suppose  $\mathcal{C}$  has **equalizers** and all finite (resp. **small**) **products**. Then  $\mathcal{C}$  has all finite (resp. small) **limits**.
- (ii) Suppose  $\mathcal{C}$  has **pullbacks** and a **terminal object**. Then  $\mathcal{C}$  has all finite limits.

*Proof.*

- (i) Suppose given  $D : J \rightarrow \mathcal{C}$ . Form the **products**

$$P = \prod_{j \in \text{ob } J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha).$$

We have morphisms  $P \xrightleftharpoons[g]{f} Q$  defined by  $\pi_\alpha f = \pi_{\text{cod } \alpha}$ ,  $\pi_\alpha g = D(\alpha)\pi_{\text{dom } \alpha}$  for all  $\alpha$ .

Let  $E \xrightarrow{e} P$  be an **equalizer** of  $(f, g)$ . The composites  $\lambda_j = \pi_j e : E \rightarrow D(j)$  form a **cone** over  $D$ : given  $\alpha : j \rightarrow j'$  in  $J$ ,

$$D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e = \lambda_{j'}.$$

Given any cone  $(A, \{\mu_j \mid j \in \text{ob } J\})$  over  $D$ , there’s a unique  $\mu : A \rightarrow P$  with  $\pi_j \mu = \mu_j$  for each  $j$ , and

$$\pi_\alpha f \mu = \mu_{\text{cod } \alpha} = D(\alpha)\mu_{\text{dom } \alpha} = \pi_\alpha g \mu$$

for all  $\alpha$ , and hence  $f\mu = g\mu$ , so  $\exists! \nu : A \rightarrow E$  with  $e\nu = \mu$ . So  $(E, \{\lambda_j \mid j \in \text{ob } J\})$  is a **limit cone**.

- (ii) It’s enough to construct finite products and equalizers. But if  $1$  is the **terminal object**, then a pullback for

$$\begin{array}{ccc} & A & \\ & \downarrow & \\ B & \longrightarrow & 1 \end{array}$$

has the universal property of a product  $A \times B$ , and we can form  $\prod_{i=1}^n A_i$  inductively as  $A_1 \times (A_2 \times (A_3 \times \cdots (A_{n-1} \times A_n) \cdots))$ .

Now, to form the equalizer of  $A \xrightleftharpoons[g]{f} B$ , consider the **cospan**

$$\begin{array}{ccc} & A & \\ & \downarrow (1_A, f) & \\ A & \xrightarrow{(1_A, g)} & A \times B. \end{array}$$

A cone over this consists of

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \\ A & & \end{array}$$

satisfying  $(1_A, f)h = (1_A, g)k$  or equivalently,  $1_A h = 1_A k$  and  $fh = gk$ , or equivalently a morphism  $P \xrightarrow{h} A$  satisfying  $fh = gh$ . So a pullback for  $(1_A, f)$  and  $(1_A, g)$  is an equalizer of  $(f, g)$ .  $\square$

**Definition (Complete).** We say a **category**  $\mathcal{C}$  is **complete** if it has all **small limits**. (Dually, **cocomplete** = all small colimits).

Set is **complete** and **cocomplete**: **products** are Cartesian products, **coproducts** are disjoint unions. Similarly, **Gp**, **AbGp**, **Rng**, **Mod<sub>R</sub>**, ... are all complete and cocomplete. **Top** is also complete and cocomplete.

**4.4 Definition.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a **functor**.

- (a) We say  $F$  **preserves limits** of **shape**  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a **limit cone**  $(L, \{\lambda_j \mid j \in \text{ob } J\})$  in  $\mathcal{C}$ ,  $(FL, \{F\lambda_j \mid j \in \text{ob } J\})$  is a limit for  $FD$ .
- (b) We say  $F$  **reflects limits** of shape  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a cone  $(L, (\lambda_j)_j)$  such that  $(FL, (F\lambda_j)_j)$  is a limit for  $FD$ , then  $(L, (\lambda_j)_j)$  is a limit for  $D$ .
- (c) We say  $F$  **creates limits** of shape  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a limit  $(M, (\mu_j)_j)$  for  $FD$ , there exists a cone  $(L, (\lambda_j)_j)$  over  $D$  whose image under  $F$  is isomorphic to the limit cone, and any such that cone is a limit in  $\mathcal{C}$ .

*Lecture 11* **4.5 Remark.**

- (a) If  $\mathcal{C}$  has **limits of shape**  $J$ ,  $F : \mathcal{C} \rightarrow \mathcal{D}$  **preserves** them and  $F$  and  $F$  reflects **isomorphisms**, then  $F$  **reflects** limits of shape  $J$ .
- (b)  $F$  reflects limits of shape **1**  $\iff$   $F$  reflects isomorphisms.
- (c) If  $\mathcal{D}$  has limits of shape  $J$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$  **creates them**, then  $F$  both preserves and reflects them.
- (d) In any of the statements of **Theorem 4.3**, we may replace both instances of ' $\mathcal{C}$  has' by either ' $\mathcal{C}$  has and  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves' or ' $\mathcal{D}$  has and  $F : \mathcal{C} \rightarrow \mathcal{D}$  creates'.



#### 4.6 Examples.

- (a)  $U : \mathbf{Gp} \rightarrow \mathbf{Set}$  (forgetful) creates all small limits: Given a family  $\{G_i \mid i \in I\}$  of groups, there's a unique group structure on  $\prod_{i \in I} UG_i$  making the projections  $\pi_i$  into homomorphisms and this makes it into a product in  $\mathbf{Gp}$ . Similarly for equalizers.  
But  $U$  doesn't preserve coproducts:  $U(G * H) \not\cong UG \sqcup UH$ .
- (b)  $U : \mathbf{Top} \rightarrow \mathbf{Set}$  (forgetful) preserves all small limits and colimits, but doesn't reflect them: if  $L$  is a limit for  $D : J \rightarrow \mathbf{Top}$ , and  $L$  is not discrete, there's another cone with apex  $L_d$  mapping to the limit in  $\mathbf{Set}$ .
- (c) The inclusion functor  $I : \mathbf{AbGp} \rightarrow \mathbf{Gp}$  reflects coproducts but doesn't preserve them. The direct sum  $A \oplus B$  (coproduct in  $\mathbf{AbGp}$ ) is not normally isomorphic to the free product  $A * B$ ;  $A * B$  is not abelian unless either  $A$  or  $B$  is  $\{e\}$ , but if  $A \cong \{e\}$  then  $A * B \cong A \oplus B \cong B$ .

**4.7 Lemma.** If  $\mathcal{D}$  has limits of shape  $J$ , then so does the functor category  $[\mathcal{C}, \mathcal{D}]$  for any  $\mathcal{C}$ , and the forgetful functor  $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$  creates them.

*Proof.* Suppose we are given a diagram of shape  $J$  in  $[\mathcal{C}, \mathcal{D}]$ ; think of it as a functor  $D : J \times \mathcal{C} \rightarrow \mathcal{D}$ . For each  $A \in \text{ob } \mathcal{C}$ , let  $(LA, \{\lambda_{j,A} \mid j \in \text{ob } J\})$  be a limit cone for the diagram  $D(-, A) : J \rightarrow \mathcal{D}$ . Given  $A \xrightarrow{f} B$  in  $\mathcal{C}$ , the composites

$$A \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over  $D(-, B)$ , since the squares

$$\begin{array}{ccc} D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \\ \downarrow D(\alpha, A) & & \downarrow D(\alpha, B) \\ D(j', A) & \xrightarrow{D(j',f)} & D(j', B) \end{array}$$

commute. So there's a unique  $Lf : LA \rightarrow LB$  making

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_{j,A}} & D(j, A) \\ \downarrow Lf & & \downarrow D(j,f) \\ LB & \xrightarrow{\lambda_{j,B}} & D(j, B) \end{array}$$

commute for all  $j$ .

Uniqueness implies functoriality: given  $g : B \rightarrow C$ ,  $L(gf)$  and  $(Lg)(Lf)$  are factorizations of the same cone through the limit  $LC$ . And this is the unique functor structure on  $(A \mapsto LA)$  making the  $\lambda_{j,-}$  into natural transformations.

The cone  $(L, \{\lambda_{j,-} \mid j \in \text{ob } J\})$  is a limit: suppose given another cone  $(M, \{\mu_{j,-} \mid j \in \text{ob } J\})$ , then for each  $A$ ,  $(MA, \{\mu_{j,A} \mid j \in \text{ob } J\})$  is a cone over  $D(-, A)$ , so induces a unique  $\alpha_A : MA \rightarrow LA$ . Naturality of  $\alpha$  follows from uniqueness of factorisations through a limit. So  $(M, (\mu_j))$  factors uniquely through  $(L, (\lambda_j))$ .  $\square$

**4.8 Remark.** In any **category**, a morphism  $A \xrightarrow{f} B$  is **monic** iff

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a **pullback**. Hence any **functor** which preserves pullbacks preserves monomorphisms. In particular, if  $\mathcal{D}$  has pullbacks, then monomorphisms in  $[\mathcal{C}, \mathcal{D}]$  are just pointwise monomorphisms.

**4.9 Theorem.** Suppose  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a **left adjoint**  $F$ . Then  $G$  **preserves** all **limits** which exist in  $\mathcal{D}$ .

*Proof 1.* Suppose  $\mathcal{C}$  and  $\mathcal{D}$  both have **limits of shape**  $J$ . We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

and all the **functors** in it have **right adjoints** (in particular  $[J, F] \dashv [J, G]$ ). So by **Corollary 3.6**, the diagram of right adjoints

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ \lim_J \uparrow & & \lim_J \uparrow \\ [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \end{array}$$

commutes up to **isomorphism**, i.e.  $G$  **preserves limits** of shape  $J$ . □

*Proof 2.* Suppose given  $D : J \rightarrow \mathcal{D}$  and a **limit cone**  $(L, \{L \xrightarrow{\lambda_j} D(j) \mid j \in \text{ob } J\})$ . Given a cone  $(A, \{A \xrightarrow{\alpha_j} GD(j) \mid j \in \text{ob } J\})$  over  $GD$  the morphisms  $FA \xrightarrow{\hat{\alpha}_j} D(j)$  form a cone over  $D$ , so they induce a unique  $FA \xrightarrow{\hat{\beta}} L$  such that  $\lambda_j \hat{\beta} = \hat{\alpha}_j$  for all  $j$ . Then  $A \xrightarrow{\beta} GL$  is the unique morphism satisfying  $(G\lambda_j)\beta = \alpha_j$  for all  $j$ . So  $(GL, \{G\lambda_j \mid j \in \text{ob } J\})$  is a limit cone in  $\mathcal{C}$ . □

*Lecture 12* The ‘Primeval’ Adjoint Functor Theorem says that the converse of **Theorem 4.9** is true: if  $\mathcal{D}$  has and  $G : \mathcal{D} \rightarrow \mathcal{C}$  **preserves all** limits, then  $G$  has a **left adjoint**.

**4.10 Lemma.** Suppose  $\mathcal{D}$  **has** and  $G : \mathcal{D} \rightarrow \mathcal{C}$  **preserves limits** of shape  $J$ . Then for any  $A \in \text{ob } \mathcal{C}$ , the **arrow category**  $(A \downarrow G)$  has **limits** of shape  $J$ , and the **forgetful functor**  $U : (A \downarrow G) \rightarrow \mathcal{D}$  **creates them**.

*Proof.* Suppose given  $D : J \rightarrow (A \downarrow G)$ , write  $D(j)$  as  $(UD(j), f_j)$ . Let  $(L, (\lambda_j : L \rightarrow UD(j))_{j \in \text{ob } J})$  be a **limit** for  $UD$ ; then  $(GL, (G\lambda_j)_{j \in \text{ob } J})$  is a limit for  $GUD$ .

Since the edges of  $UD$  are morphisms in  $(A \downarrow G)$ , the  $f_j$  form a **cone** over  $GUD$ . So there’s a unique  $h : A \rightarrow GL$  such that  $(G\lambda_j)h = f_j$  for all  $j$ , i.e. there’s a unique  $h$  such that the  $\lambda_j$  are all morphisms  $(L, h) \rightarrow (UD(j), f_j)$  in  $(A \downarrow G)$ .

If  $((C, k)(\mu_j)_{j \in \text{ob } J})$  is any cone over  $D$ , then  $(C, (\mu_j)_{j \in \text{ob } J})$  is a cone over  $UD$ , so there's a unique  $l : C \rightarrow L$  with  $\lambda_j l = \mu_j$  for all  $j$ . We need to show  $(Gl)k = h$ : but  $(G\lambda_j)(Gl)k = (G\mu_j)k = f_j = (G\lambda_j)h$  for all  $j$  so  $(Gl)k = h$  by uniqueness of factorizations through limits.  $\square$

**4.11 Lemma.** A category  $\mathcal{C}$  has an **initial object** iff  $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ , regarded as a **diagram** of shape  $\mathcal{C}$  in  $\mathcal{C}$ , has a **limit**.

*Proof.* Suppose  $\mathcal{C}$  has an **initial object**  $I$ . Then the unique morphisms  $\{I \rightarrow A \mid A \in \text{ob } \mathcal{C}\}$  form a **cone** over  $1_{\mathcal{C}}$ ; and given any cone  $\{C \xrightarrow{\lambda_A} A \mid A \in \text{ob } \mathcal{C}\}$ , then for any  $A$  the triangle

$$\begin{array}{ccc} C & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow \\ & & A \end{array}$$

commutes, so  $\lambda_I$  is the unique factorization of  $\{\lambda_A \mid A \in \text{ob } \mathcal{C}\}$  through the cone

$$\{I \rightarrow A \mid A \in \text{ob } \mathcal{C}\}.$$

Conversely, suppose  $(I, \{\lambda_A : I \rightarrow A \mid A \in \text{ob } \mathcal{C}\})$  is a **limit**. Then, for any  $I \xrightarrow{f} A$ , the diagram

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow f \\ & & A \end{array}$$

commutes. In particular, putting  $f = \lambda_A$ , we see that  $\lambda_I$  is a factorization of the limit cone through itself, so  $\lambda_I = 1_I$ . Hence every  $f : I \rightarrow A$  satisfies  $f = \lambda_A$ .  $\square$

The **primeval Adjoint Functor Theorem** now follows immediately from **Lemma 4.10**, **Lemma 4.11** and **Theorem 3.3**. However, it only applies to **functors** between preorders (cf question 6 on example sheet 2).

**4.12 Theorem** (General Adjoint Functor Theorem). Suppose that  $\mathcal{D}$  is **locally small** and **complete**. Then  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a **left adjoint** if and only if  $G$  satisfies the **solution set condition**: the solution set condition says that  $G$  preserves all small limits and, for each  $A \in \text{ob } \mathcal{C}$  there exists a set of morphisms  $\{A \xrightarrow{f_i} GB_i \mid i \in I\}$  such that every  $A \xrightarrow{h} GC$  factors as  $A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$  for some  $i$  and some  $g : B_i \rightarrow C$ .

*Proof.*  $(\Rightarrow)$ . If  $F \dashv G$ ,  $G$  preserves limits by **Theorem 4.9**, and  $\{A \xrightarrow{\eta_A} GFA\}$  is a singleton **solution set**, by **Theorem 3.3**.

$(\Leftarrow)$ . By **Lemma 4.10**,  $(A \downarrow G)$  is **complete**, and it inherits **local smallness** from  $\mathcal{D}$ . So we need to show: if  $\mathcal{A} := (A \downarrow G)$  is complete and locally small, and has a weakly initial set of objects  $\{B_i \mid i \in I\}$ , then  $\mathcal{A}$  has an initial object. (An object is **weakly initial** if it has a (not necessarily unique) morphism to any other object.)

First form  $P = \prod_{i \in I} B_i$ , then  $P$  is **weakly initial**. Now form the limit of

$$P \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \vdots \\ \rightrightarrows \end{array} P \quad (*)$$

where edges are all the endomorphisms of  $P$ : denote the limit  $I \xrightarrow{i} P$ .  $I$  is also **weakly initial** in  $\mathcal{A}$ : suppose given  $I \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C$ . Form the equalizer  $E \xrightarrow{e} I$  of  $(f, g)$ , then there exists  $P \xrightarrow{h} E$  since  $P$  is weakly initial.  $ieh : P \rightarrow P$ , and  $1_P$  are edges of the diagram  $(*)$  so  $i = iehi$ . But  $i$  is monic, so  $ehi = 1_I$ , so  $e$  is split epic, so  $f = g$ . Hence  $I$  is initial.  $\square$

#### 4.13 Examples.

- (a) Consider the **forgetful functor**  $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ . By **Examples 4.6(a)**,  $U$  creates all small limits, so  $\mathbf{Gp}$  has them and  $U$  preserves them.  $\mathbf{Gp}$  is **locally small**, given a set  $A$ , any  $f : A \rightarrow UG$  factors as  $A \rightarrow UG' \rightarrow UG$ , where  $G'$  is the subgroup generated by  $\{f(x) \mid x \in A\}$  and  $\text{card } G' \leq \max\{\aleph_0, \text{card } A\}$ .

Let  $B$  be a set of this cardinality; consider all subsets  $B' \subseteq B$ , all group structures on  $B'$ , and all mappings  $A \rightarrow B'$ . These give us a **solution set** at  $A$ .

- (b) Consider the category  $\mathbf{CLat}$  of complete lattices (posets with all meets and joins). Again  $U : \mathbf{CLat} \rightarrow \mathbf{Set}$  creates all small limits. But A. W. Hales (1964) showed that, for any cardinal  $\kappa$ , there exist complete lattices of  $\text{card} \geq \kappa$  generated by three elements, so the **solution set condition** fails at  $A = \{x, y, z\}$ , and  $U$  doesn't have a left adjoint.

*Lecture 13* **4.14 Definition** (Subobject). By a **subobject** of an object  $A$  of  $\mathcal{C}$ , we mean a **monomorphism**  $A' \rightarrowtail A$ . Dually, we have quotient objects. The subobjects of  $A$  are preordered by  $A'' \leq A'$  if there exists a factorization

$$\begin{array}{ccc} A'' & \xrightarrow{\quad} & A' \\ & \searrow & \swarrow \\ & A. & \end{array}$$

We say  $\mathcal{C}$  is **well-powered** if each  $A \in \text{ob } \mathcal{C}$  has a set of subobjects  $\{A_i \rightarrowtail A \mid i \in I\}$  such that every subobject of  $A$  is isomorphic to some  $A_i$  (e.g. in  $\mathbf{Set}$  we can take the inclusions  $\{A' \hookrightarrow A \mid A' \in \mathcal{P}A\}$ ).

If  $C^{op}$  is well-powered, we say  $\mathcal{C}$  is **well-copowered** (not cowell-powered).

**4.15 Lemma.** Suppose given a **pullback** square

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

with  $f$  **monic**. Then  $k$  is monic.

*Proof.* Suppose  $D \xrightarrow[x]{x} P$  satisfy  $kx = ky$ . Then  $fhx = gkx = gky = fhy$ , but  $f$  is **monic** so  $hx = hy$ . So  $x$  and  $y$  are factorizations of the same **cone** through the **limit** cone  $(h, k)$ .  $\square$

**4.16 Theorem** (Special adjoint functor theorem). Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are both **locally small**, and that  $\mathcal{D}$  is **complete** and **well-powered** and has a **coseparating** set. Then a **functor**  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a **left adjoint** iff it **preserves all small limits**.

*Proof.*  $(\Rightarrow)$  by **Theorem 4.9**.

$(\Leftarrow)$ . For any  $A \in \text{ob } \mathcal{C}$ ,  $(A \downarrow G)$  is **complete** by **Lemma 4.10**, **locally small**, and **well-powered** since the **subobjects** of  $(B, f)$  in  $(A \downarrow G)$  are just those **subobjects**  $B' \rightarrowtail B$  in  $\mathcal{D}$  for which  $f$  factors through  $GB' \rightarrowtail GB$ .

Also, if  $\{S_i \mid i \in I\}$  is a **coseparating** set for  $\mathcal{D}$ , then the set

$$\{(s_i, f) \mid i \in I, f \in \mathcal{C}(A, GS_i)\}$$

is coseparating in  $(A \downarrow G)$ : given  $(B, f) \xrightarrow[h]{g} (B', f')$  in  $(A \downarrow G)$  with  $g \neq h$ , there exists  $k : B' \rightarrow S_i$  for some  $i$  with  $kg \neq kh$ , and then  $k$  is also a morphism  $(B', f') \rightarrow (S_i, (Gk)f')$  in  $(A \downarrow G)$ .

So we need to show that if  $\mathcal{A}$  is complete, locally small and well powered, and has a coseparating set  $\{S_i \mid i \in I\}$  then  $\mathcal{A}$  has an initial object. Form the product  $P = \prod_{i \in I} S_i$ . Now consider the diagram

$$\begin{array}{ccc} & & P_i \\ & \nearrow & \downarrow \\ & P_j & \\ P' & \longrightarrow & P \end{array}$$

whose edges are a representative set of subobjects of  $P$ , and form its **limit**

$$\begin{array}{ccc} I & \longrightarrow & P_i \\ & \searrow & \downarrow \\ & P_j & \\ I & \longrightarrow & P \\ \downarrow & & \downarrow \\ P' & \longrightarrow & P \end{array}$$

By the argument of **Lemma 4.15**, the legs of this cone are all **monic**; in particular  $I \rightarrowtail P$  is monic, and it's a **least** subobject of  $P$ . Hence  $I$  has no proper subobjects, so, given  $I \xrightarrow[f]{f} A$ , their **equalizer** is an **isomorphism** and hence  $f = g$ .

Now let  $A$  be any object of  $\mathcal{A}$ ; form the **product**

$$Q = \prod_{\substack{i \in I \\ f \in \mathcal{A}(A, S_i)}} S_i.$$

There's an obvious  $h : A \rightarrow Q$  defined by  $\pi_{i,f}h = f$ ; and  $h$  is monic, since the  $S_i$  are a coseparating set. We also have a morphism  $k : P \rightarrow Q$  defined by  $\pi_{i,f}k = \pi_i$ .

Now form the [pullback](#)

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow h \\ P & \xrightarrow{k} & Q \end{array}$$

by [Lemma 4.15](#),  $B \rightarrow P$  is monic, so  $B$  is a subobject of  $P$ . Hence there exists

$$\begin{array}{ccc} I & \longrightarrow & B \\ & \searrow & \swarrow \\ & P & \end{array}$$

and hence a morphism  $I \rightarrow B \rightarrow A$ . □

**4.17 Examples.** Consider the inclusion  $\mathbf{KHaus} \xrightarrow{I} \mathbf{Top}$ , where  $\mathbf{KHaus}$  is the [full subcategory](#) of compact Hausdorff spaces.  $\mathbf{KHaus}$  [has](#) and  $I$  [preserves small products](#) (by Tychonoff's Theorem) and [equalizers](#) (since equalizers of pairs  $X \xrightarrow[f]{g} Y$  with  $Y$  Hausdorff are closed subspaces). Both categories are [locally small](#) and  $\mathbf{KHaus}$  is [well-powered](#) (subobjects of  $X$  are all isomorphic to closed subspaces). The closed interval  $[0, 1]$  is a [coseparator](#) in  $\mathbf{KHaus}$  by Urysohn's Lemma. So by [Theorem 4.16](#),  $I$  has a [left adjoint](#)  $\beta$ .

**4.18 Remark.**

- (a) Čech's construction of  $\beta$ : given  $X$  form  $P = \prod_{f: X \rightarrow [0,1]} [0,1]$  and define  $h : X \rightarrow P$  by  $\pi_f h = f$ . Define  $\beta X$  to be the closure of the image of  $h$ .  
Čech's proof that this works is essentially the same as [Theorem 4.16](#).
- (b) We could have used [General Adjoint Functor Theorem](#) to construct  $\beta$ : we get a solution set at  $X$  by considering all continuous  $X \xrightarrow{f} Y$  with  $Y$  compact Hausdorff, and  $\text{im } f$  dense in  $Y$  and such  $Y$  have cardinality  $\leq 2^{2^{\text{card } X}}$ .

## 5 Monads

Lecture 14 Suppose given

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

with  $(F \dashv G)$ . How much of this structure can we describe without mentioning  $\mathcal{D}$ ?

We have the **functor**  $T = GF : \mathcal{C} \rightarrow \mathcal{C}$ , and the **unit**  $\eta : 1_{\mathcal{C}} \rightarrow T = GF$  and the **natural transformation**

$$\mu = G\epsilon_F : TT = GFGF \rightarrow GF = T.$$

These satisfy the commutative diagrams

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta_T} & T \\ & \searrow 1_T & \downarrow \mu & \swarrow 1_T & \\ & & T & & \end{array} \quad \begin{array}{c} (1) \\ (2) \end{array}$$

by **triangular identities** and

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \downarrow \mu_T & (3) & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

by naturality of  $\epsilon$ .

**5.1 Definition** (Monad). A **monad**  $\mathbb{T} = (T, \eta, \mu)$  on a **category**  $\mathcal{C}$  consists of a **functor**  $T : \mathcal{C} \rightarrow \mathcal{C}$  and **natural transformation**  $\eta : 1_{\mathcal{C}} \rightarrow T$ ,  $\mu : TT \rightarrow T$  satisfying equations (1)-(3).  $\eta$  and  $\mu$  are called the **unit** and **multiplication** of  $\mathbb{T}$ .

### 5.2 Examples.

- (a) Any **adjunction**  $(F \dashv G)$  induces both a **monad**  $(GF, \eta, G\epsilon_F)$  on  $\mathcal{C}$  and a **comonad**  $(FG, \epsilon, F\eta_G)$  on  $\mathcal{D}$ .
- (b) Let  $M$  be a **monoid**. The functor  $(M \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$  has a monad structure with **unit** given by  $\eta_A(a) = (1_M, a)$  and **multiplication**  $\mu_A(m, m', a) = (mm', a)$ . The monad identities follow from the monoid ones.
- (c) Let  $\mathcal{C}$  be any **category** with finite **products**,  $A \in \text{ob } \mathcal{C}$ . The **functor**  $(A \times -) : \mathcal{C} \rightarrow \mathcal{C}$  has a comonad structure with counit  $\epsilon_B : A \times B \rightarrow B$  given by  $\pi_2$  and comultiplication  $\delta_B : A \times B \rightarrow A \times A \times B$  given by  $(\pi_1, \pi_1, \pi_2)$ .

Does every **monad** arise from an **adjunction**?

In **Examples 5.2(b)** we have the **category**  $[M, \mathbf{Set}]$ . Its **forgetful functor** to  $\mathbf{Set}$  has a left **adjoint**, sending  $A$  to  $M \times A$  with  $M$  acting by multiplication on the left factor. This adjunction gives rise to the **monad** of **Examples 5.2(b)**.

**5.3 Definition** (Eilenberg-Moore category). Let  $\mathbb{T}$  be a [monad](#) on  $\mathcal{C}$ . A  $\mathbb{T}$ -**algebra** is a pair  $(A, \alpha)$  with  $A \in \text{ob } \mathcal{C}$  and  $TA \xrightarrow{\alpha} A$  satisfying the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array} \quad (4)$$

$$\begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow \mu_A & (5) & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A. \end{array}$$

A **homomorphism**  $f : (A, \alpha) \rightarrow (B, \beta)$  is a morphism  $A \xrightarrow{f} B$  such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & (6) & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes. The category of  $\mathbb{T}$ -algebras is denoted  $\mathcal{C}^{\mathbb{T}}$ , and called the **Eilenberg-Moore category**.

**5.4 Lemma.** The [forgetful functor](#)  $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$  has a [left adjoint](#)  $F^{\mathbb{T}}$  and the adjunction induces  $\mathbb{T}$ .

*Proof.* We define  $F^{\mathbb{T}}A = (TA, \mu_A)$  (an [algebra](#) by (2) and (3)) and  $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$  (a [homomorphism](#) by [naturality](#) of  $\mu$ ).

Clearly  $G^{\mathbb{T}}F^{\mathbb{T}} = T$ , the [unit](#) of the [adjunction](#) is  $\eta$ . We define the [counit](#)

$$\epsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \rightarrow (A, \alpha)$$

(a homomorphism by (5))  $\epsilon$  is natural by (6); for the triangular identities,  $\epsilon_{FA}(F\eta_A) = 1_{FA}$  is (1),  $G\epsilon_{(A, \epsilon)}\eta_A = 1_A$  is (4).

The [monad](#) induced by  $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$  has [functor](#)  $T$  and unit  $\eta$ , and  $G^{\mathbb{T}}\epsilon_{F^{\mathbb{T}}A} = \mu_A$  by definition of  $F^{\mathbb{T}}A$ .  $\square$

Kleisli took a ‘minimalist’ approach: if

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

induces  $\mathbb{T}$ , then so does

$$\mathcal{C} \xrightleftharpoons[G|_{\mathcal{D}'}]{F} \mathcal{D}'$$

where  $\mathcal{D}'$  is the [full subcategory](#) of  $\mathcal{D}$  on objects  $FA$ .

So in trying to construct  $\mathcal{D}$ , we may assume  $F$  is surjective (or indeed bijective) on objects. But then morphisms  $FA \rightarrow FB$  correspond bijectively to morphisms  $A \rightarrow GFB = TB$  in  $\mathcal{C}$ .



**5.5 Definition** (Kleisli category). Given a **monad**  $\mathbb{T}$  on  $\mathcal{C}$ , the **Kleisli category**  $\mathcal{C}_{\mathbb{T}}$  has  $\text{ob } \mathcal{C}_{\mathbb{T}} = \text{ob } \mathcal{C}$ , and morphisms  $A \rightarrowtail B$  are morphisms  $A \rightarrow TB$  in  $\mathcal{C}$ . The composite  $A \xrightarrow{f} B \xrightarrow{g} C$  is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$$

and the identity  $A \rightarrowtail A$  is  $A \xrightarrow{\eta_A} TA$ .

To verify associativity, suppose given  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ . Then

$$\begin{array}{ccccccc} A & \xrightarrow{f} & TB & \xrightarrow{Tg} & TTC & \xrightarrow{TT h} & TTTD \xrightarrow{T\mu_D} TTD \\ & & \downarrow \mu_C & & \downarrow \mu_{TD} & (3) & \downarrow \mu_D \\ & & TC & \xrightarrow{Th} & TTD & \xrightarrow{\mu_D} & TD \end{array}$$

commutes by naturality: the upper way round is  $(hg)f$  and the lower is  $h(gf)$ .

The unit laws for the **category** similarly follow from

$$\begin{array}{ccc} A & \xrightarrow{f} & TB \xrightarrow{T\eta_B} TTB \\ & \searrow 1_{TB} & \downarrow \mu_B \\ & & TB \end{array} \quad (1)$$

(commutes by (1)) and

$$\begin{array}{ccc} A & \xrightarrow{f} & TB \\ \downarrow \eta_A & \eta_{TB} \downarrow & \searrow 1_{TB} \\ TA & \xrightarrow{Tf} & TTB \xrightarrow{\mu_B} TB \end{array} \quad (2)$$

which commutes by (2).

*Lecture 15* **5.6 Lemma.** There exists an **adjunction**

$$\mathcal{C} \xrightleftharpoons[G_{\mathbb{T}}]{F_{\mathbb{T}}} \mathcal{C}_{\mathbb{T}}$$

inducing the **monad**  $\mathbb{T}$ .

*Proof.* We define  $F_{\mathbb{T}}A = A$ ,

$$F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB.$$

$F_{\mathbb{T}}$  preserves identities by definition; for composites, consider  $A \xrightarrow{f} B \xrightarrow{g} C$ . We get

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\eta_B} & TB \\ & & \downarrow g & & \downarrow Tg \\ & & C & \xrightarrow{\eta_C} & TC \xrightarrow{T\eta_C} TTC \\ & & & \searrow 1_{TC} & \downarrow \mu_C \\ & & & & TC. \end{array} \quad (1)$$

We define  $G_{\mathbb{T}}A = TA$ ,

$$G_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB.$$

$G_{\mathbb{T}}$  preserves identities by (1); for composites, consider  $A \xrightarrow{f} B \xrightarrow{g} C$ . We get

$$\begin{array}{ccccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{TTg} & TTTC & \xrightarrow{T\mu_C} & TTC \\ & & \downarrow \mu_B & & \downarrow \mu_{TC} & (3) & \downarrow \mu_C \\ & & TB & \xrightarrow{Tg} & TTC & \xrightarrow{\mu_C} & TC \end{array}$$

We have

$$\begin{aligned} G_{\mathbb{T}}F_{\mathbb{T}}A &= TA \\ G_{\mathbb{T}}F_{\mathbb{T}}f &= \mu_B(T\eta_B)Tf = Tf \end{aligned}$$

so we take  $\eta : 1_{\mathbb{C}} \rightarrow T$  as the **unit** of  $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ . The counit  $TA \xrightarrow{\epsilon_A} A$  is  $1_{TA}$ . To verify **naturality**, consider the square

$$\begin{array}{ccc} TA & \xrightarrow{F_{\mathbb{T}}G_{\mathbb{T}}f} & TB \\ \downarrow \epsilon_A & & \downarrow \epsilon_B \\ A & \xrightarrow{f} & B \end{array}$$

This expands to

$$\begin{array}{ccccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{\mu_B} & TB & \xrightarrow{\eta_{TB}} & TTB \\ & & & & \searrow 1_{TB} & (2) & \downarrow \mu_B \\ & & & & & & TB \end{array}$$

so  $\epsilon$  is natural.

$$G_{\mathbb{T}}(TA \xrightarrow{\epsilon_A} A) = \mu_A, \text{ so } G_{\mathbb{T}}(\epsilon_A)\eta_{G_{\mathbb{T}}A} = \mu_A \cdot \eta_{TA} = 1_{TA}$$

and  $(\epsilon_{F_{\mathbb{T}}A})(F_{\mathbb{T}}\eta_A)$  is

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & TA & \xrightarrow{\eta_{TA}} & TTA \\ & & \searrow 1_{TA} & (1) & \downarrow \mu_A \\ & & & & TA \end{array}$$

which is  $1_{F_{\mathbb{T}}A}$ . Also  $G_{\mathbb{T}}(\epsilon_{F_{\mathbb{T}}A}) = \mu_A$ , so  $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$  induces  $\mathbb{T}$ . □

**5.7 Theorem.** Given a **monad**  $\mathbb{T}$  on  $\mathcal{C}$ , let  $\mathbf{Adj}(\mathbb{T})$  be the category whose objects are the **adjunctions**  $\left(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}\right)$  inducing  $\mathbb{T}$ , and whose morphisms

$$\left(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}\right) \rightarrow \left(\mathcal{C} \xrightleftharpoons[G']{F'} \mathcal{D}'\right)$$

are functors  $H : \mathcal{D} \rightarrow \mathcal{D}'$  satisfying  $HF = F'$  and  $G'H = G$ . Then the **Kleisli adjunction** is an **initial** object of  $\mathbf{Adj}(\mathbb{T})$ , and the **Eilenberg-Moore** adjunction is **terminal**.

*Proof.* Let  $\left(\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}\right)$  be an object of  $\mathbf{Adj}(\mathbb{T})$ . We define  $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  (the **Eilenberg-Moore comparison functor**) by  $KB = (GB, G\epsilon_B)$  where  $\epsilon$  is the counit of  $(F \dashv G)$ ; note this is an algebra by one of the **triangular identities** for  $(F \dashv G)$  and **naturality** of  $\epsilon$  and  $K(B \xrightarrow{g} B') = Gg$  (a homomorphism by naturality of  $\epsilon$ ). Clearly  $G^{\mathbb{T}}K = G$  and

$$\begin{aligned} KFA &= (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A, \\ KF(A \xrightarrow{f} A') &= Tf = F^{\mathbb{T}}f. \end{aligned}$$

So  $K$  is a morphism of  $\mathbf{Adj}(\mathbb{T})$ .

Suppose  $K' : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  is another such; then since  $G^{\mathbb{T}}K' = G$ , we know  $K'B = (GB, \beta_B)$  where  $\beta$  is a natural transformation  $GFG \rightarrow G$ . Also, since  $K'F = F^{\mathbb{T}}$ , we have  $\beta_{FA} = \mu_A = G\epsilon_{FA}$ .

Now, given any  $B \in \text{ob } \mathcal{D}$ , consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GFG\epsilon_B} & GFGB \\ \downarrow G\epsilon_{FGB} \quad \downarrow \beta_{FGB} & & \downarrow G\epsilon_B \quad \downarrow \beta_B \\ GFGB & \xrightarrow{G\epsilon_B} & GB \end{array}$$

Both squares commute so  $G\epsilon_B$  and  $\beta_B$  have the same composite with  $GFG\epsilon_B$ . But this is **split epic**, with splitting  $GF\eta_{GB}$ , so  $\beta = G\epsilon$ . Hence  $K' = K$ .

We now define the **Kleisli comparison functor**  $L : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$  by  $LA = FA$ ,

$$L(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB.$$

$L$  preserves identities by one of the **triangular identities** for  $(F \dashv G)$ ; given  $A \xrightarrow{f} B \xrightarrow{g} C$ , we have

$$\begin{array}{ccccccc} FA & \xrightarrow{Ff} & FGFB & \xrightarrow{FGFg} & FGFGFC & \xrightarrow{FG\epsilon_{FC}} & FGFC \\ & & \downarrow \epsilon_{FB} & & \downarrow \epsilon_{FGFC} & & \downarrow \epsilon_{FC} \\ & & FB & \xrightarrow{Fg} & FGFC & \xrightarrow{\epsilon_{FC}} & FC \end{array}$$

$$GLA = TA = G_{\mathbb{T}}A,$$

$$GL(A \xrightarrow{f} B) = (G\epsilon_{FB})(GFf) = \mu_B(Tf).$$

$$LF_{\mathbb{T}}A = FA,$$

$$LF_{\mathbb{T}}(A \xrightarrow{f} B) = (\epsilon_{FB})(F\eta_B)(Ff) = Ff = G_{\mathbb{T}}f.$$

Note that  $L$  is full and faithful; its effect on morphisms (with given domain and codomain) is that of transposition across  $(F \dashv G)$ . Suppose  $L' : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$  is a morphism of  $\mathbf{Adj}(\mathbb{T})$ . We must have  $L'A = FA$ , and  $L'$  maps the counit  $TA \rightarrow A$  to the counit  $FGFA \xrightarrow{\epsilon_{FA}} FA$ . For any  $A \xrightarrow{f} B$ , we have  $f = 1_{TA}(F_{\mathbb{T}}f)$  so  $L'(f) = \epsilon_{FA} \cdot (Ff) = Lf$ .  $\square$

*Lecture 16* If  $\mathcal{C}$  has **coproducts**, then so does  $\mathcal{C}_{\mathbb{T}}$ , since  $F_{\mathbb{T}}$  preserves them. But in general, it has few other **limits** or colimits. In contrast, we have

### 5.8 Theorem.

- (i) The forgetful functor  $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$  creates all limits which exist in  $\mathcal{C}$ .
- (ii) If  $\mathcal{C}$  has colimits of shape  $J$ , then  $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$  creates them  $\iff T$  preserves them.

*Proof.*

- (i) Suppose given  $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$ ; write  $D(j) = (GD(j), \delta_j)$  and suppose

$$(L, \{\mu_j : L \rightarrow GD(j) \mid j \in \text{ob } J\})$$

is a limit cone for  $GD$ . Then the composites

$$TL \xrightarrow{T\mu_j} TGD(j) \xrightarrow{\delta_j} GD(j)$$

form a cone over  $GD$ , since the edges of  $GD$  are homomorphisms, so they induce a unique  $\lambda : TL \rightarrow L$  such that  $\mu_j \lambda = \delta_j(T\mu)$  for all  $j$ . The fact that  $\lambda$  is a  $\mathbb{T}$ -algebra structure on  $L$  follows from the fact that the  $\delta_j$  are algebra structures and uniqueness of factorisations through limits.

So  $((L, \lambda), \{\mu_j \mid j \in \text{ob } J\})$  is the unique lifting of the limit cone over  $GD$  to a cone over  $D$ ; and it's a limit, since given a cone over  $D$  with apex  $(A, \alpha)$ , we get a unique factorisation  $A \xrightarrow{f} L$  in  $\mathcal{C}$  and  $F$  is an algebra homomorphism by uniqueness of factorisations through  $L$ .

- (ii)  $(\Rightarrow)$   $F : \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{T}}$  preserves colimits since it's a left adjoint, so  $T = GF$  preserves colimits of shape  $J$ .

$(\Leftarrow)$  Suppose given  $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$  as in (i), and a colimit cone

$$\{GD(j) \xrightarrow{\mu_j} L \mid j \in \text{ob } J\}$$

in  $\mathcal{C}$ . Then

$$\{TGD(j) \xrightarrow{T\mu_j} TL \mid j \in \text{ob } J\}$$

is also a colimit cone, so the composites

$$TGD(j) \xrightarrow{\delta_j} GD(j) \xrightarrow{\mu_j} L$$

induce a unique  $\lambda : TL \rightarrow L$ . The rest of the argument is like (i). □

**5.9 Definition** (Monadic adjunction). Given an adjunction  $\left(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}\right)$ , with  $(F \dashv G)$  we say the adjunction (or the functor  $G$ ) is **monadic** if the comparison functor  $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  is part of an equivalence of categories.

(Note that, since the Kleisli comparison  $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$  is full and faithful, it's part of an equivalence if and only if it (equivalently,  $F$ ) is essentially surjective on objects.)

**Remark.** Given any [adjunction](#)  $(F \dashv G)$ , for each object  $B$  of  $\mathcal{D}$  we have a diagram

$$FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB \xrightarrow{\epsilon_B} B$$

with equal composites. The ‘primeval monadicity theorem’ asserts that  $\mathcal{C}^{\mathbb{T}}$  is characterised in  $\mathbf{Adj}(\mathbb{T})$  by the fact that these diagrams are all [coequalizers](#).

### 5.10 Definition.

- (a) We say a parallel pair  $A \xrightleftharpoons[g]{f} B$  is **reflexive** if there exists  $B \xrightarrow{r} A$  such that  $fr = gr = 1_B$ . (Note that  $FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB$  is reflexive, with  $r = F\eta_{GB}$ ). We say  $\mathcal{C}$  has **reflexive coequalizers** if it has [coequalizers](#) of all reflexive pairs (equivalently,

colimits of shape  $J$  where  $J = \begin{array}{ccc} & \bullet & \bullet \\ \curvearrowright & \rightleftarrows & \rightleftarrows \\ \curvearrowleft & & \end{array}$ )

- (b) By a **split coequalizer** diagram, we mean a diagram

$$A \xrightleftharpoons[t]{g} B \xrightleftharpoons[s]{h} C$$

satisfying

$$hf = hg \quad hs = 1_C \quad gt = 1_B \quad ft = sh.$$

These equations imply that  $h$  is a coequalizer of  $(f, g)$ ; if  $B \xrightarrow{x} D$  satisfies  $xf = xg$  then  $x = xgt = xft = xsh$ , so  $x$  factors through  $h$ , and the factorisation is unique since  $h$  is [split epic](#). Note that split coequalizers are preserved by all [functors](#).

- (c) Given a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$ , a parallel pair  $A \xrightleftharpoons[g]{f} B$  is called  **$G$ -split** if there exists a split coequalizer diagram

$$GA \xrightleftharpoons[t]{Gg} GB \xrightleftharpoons[s]{h} C$$

in  $\mathcal{C}$ .

Note that  $FGFGB \xrightleftharpoons[\epsilon_{FGB}]{FG\epsilon_B} FGB$  is  $G$ -split, since

$$GFGFGB \xrightleftharpoons[\eta_{GFGB}]{GFG\epsilon_B} GFGB \xrightarrow{\eta_{GB}} GB$$

is a split coequalizer.

**5.11 Lemma.** Suppose given an **adjunction**  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$  where  $F \dashv G$ , inducing a **monad**  $\mathbb{T}$  on  $\mathcal{C}$ . Then  $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  has a left adjoint provided, for every  $\mathbb{T}$ -algebra  $(A, \alpha)$ , the pair  $FGFA \xrightleftharpoons[\epsilon_{FA}]{F\alpha} FA$  has a **coequalizer** in  $\mathcal{D}$ .

*Proof.* We define  $L : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{D}$  by taking  $FA \rightarrow L(A, \alpha)$  to be a **coequalizer** for  $(F\alpha, \epsilon_{FA})$ . Note that this is a functor  $\mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{D}$ .

Recall that  $K$  is defined by  $KB = (GB, G\epsilon_B)$ . For any  $B$ , morphisms  $LA \rightarrow B$  correspond bijectively to morphisms  $FA \xrightarrow{f} B$  satisfying  $f(F\alpha) = f(\epsilon_{FA})$ . These correspond to morphisms  $A \xrightarrow{\check{f}} GB$  satisfying

$$\check{f}\alpha = Gf = G(\epsilon_B(F\check{f})) = (G\epsilon_B)(T\check{f})$$

i.e. to algebra homomorphisms  $(A, \alpha) \rightarrow KB$ . And these bijections are natural in  $(A, \alpha)$  and in  $B$ .  $\square$

# Index

- adjoint, [14](#)
- Adjoint Functor Theorem
  - Primeval, [26](#)
- adjunction, [14](#)
- algebra
  - homomorphism, [32](#)
- apex, [20](#)
  
- balanced, [8](#)
  
- categorical property, [6](#)
- category, [2](#)
  - axioms, [2](#)
  - balanced, [8](#)
  - discrete, [14](#)
  - equivalence, [6](#)
  - indiscrete, [14](#)
  - small, [2](#)
- codomain, [2](#)
- colimit, [21](#)
- comma category, [15](#)
- complete, [24](#)
- composition, [2](#)
- cone, [20](#)
- coproduct, [11](#)
- cospan, [21](#)
- counit, [16](#)
- creates limits, [24](#)
  
- detecting family, [12](#)
- diagram, [20](#)
- domain, [2](#)
- duality, [3](#)
  
- Eilenberg-Moore category, [32](#)
- epimorphism, [8](#)
- equivalent, [6](#)
  
- free, [4](#)
- functor, [3](#)
  - contravariant, [4](#)
  - essentially surjective, [7](#)
  - faithful, [7](#)
  - forgetful, [4](#)
  - free, [4](#)
  - full, [7](#)
  
- has limits, [21](#)
  
- identity, [2](#)
- initial object, [14](#)
- isomorphism, [3](#)
  
- Kleisli category, [33](#)
  
- limit, [21](#)
  - created, [24](#)
  - preserved, [24](#)
  - reflected, [24](#)
- locally small, [9](#)
  
- monad, [31](#)
- monoid, [3](#)
- monomorphism, [8](#)
  - regular, [11](#)
  - split, [8](#)
- morphism, [2](#)
  - epi-, [8](#)
  - iso-, [3](#)
  - mono-, [8](#)
  
- natural transformations, [4](#)
  
- object, [2](#)
- opposite, [3](#)
  
- preserves limits, [24](#)
- product, [11](#)
- projective, [13](#)
  
- reflection, [18](#)
- reflects limits, [24](#)
- reflexive, [37](#)
- regular, [11](#)
- representable, [10](#)
- representation, [10](#)
  
- separating family, [11](#)
- small, [2](#)
- small category, [2](#)
- solution set condition, [27](#)
- span, [21](#)
- split, [8](#)
- split coequalizer, [37](#)
- subobject, [28](#)

$\mathbb{T}$ -algebra, [32](#)

terminal object, [14](#)

unit, [16](#)

universal element, [10](#)

weakly initial, [27](#)

well-powered, [28](#)

Yoneda

embedding, [10](#)

lemma, [9](#)