

Part III – Algebraic Topology (Ongoing course, rough)

Based on lectures by Professor I. Smith

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Contents

0	Introduction	2
0.1	Singular (co)chains	4

0 Introduction

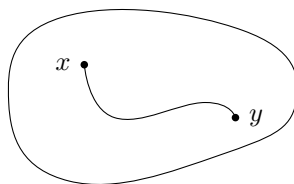
Algebraic topology concerns the connectivity properties of topological spaces. Recall a space X is **connected** if we cannot write $X = U \cup V$ where U, V are non-empty, open and disjoint.

Example. \mathbb{R} is connected (with its Euclidean topology), $\mathbb{R} \setminus \{0\}$ is not connected.

Corollary (Intermediate value theorem). If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x) > 0, f(y) < 0$, then there is some z lying between x, y such that $f(z) = 0$.

Proof. If $f(z) \neq 0$ for all z , then $\mathbb{R} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ is disconnected. \square

For nice spaces, connected \iff path-connected. Recall: a space X is path-connected if $\forall x, y \in X, \exists \gamma : [0, 1] \rightarrow X$ continuous such that $\gamma(0) = x, \gamma(1) = y$. Informally, any two maps of a point to X can be continuously deformed into one another.



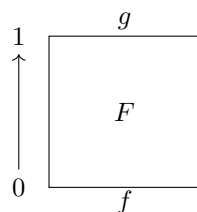
Definition (Homotopy). If X, Y are topological spaces and $f, g : X \rightarrow Y$ are (continuous) maps, then f is **homotopic** to g if

$$\exists F : X \times [0, 1] \rightarrow Y$$

continuous such that

$$F|_{X \times \{0\}} = f, F|_{X \times \{1\}} = g.$$

Write $f \simeq g$ or $f \simeq_F g$ and schematically



Definition (Simply connected). A path-connected space X is **simply connected** if every two continuous maps $S^1 \rightarrow X$ are homotopic. Here $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is the n dimensional sphere, S^1 is the circle $\subseteq \mathbb{C}$.

Example. \mathbb{R}^2 is simply connected but $\mathbb{R}^2 \setminus \{0\}$ is not. In fact, continuous maps $S^1 \xrightarrow{\gamma} \mathbb{R}^2 \setminus \{0\}$ have a degree $\deg(\gamma) \in \mathbb{Z}$, invariant under homotopy. (If γ was differentiable, we could set $\deg(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} \in \mathbb{Z}$.) If $\gamma_n : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ has $t \mapsto e^{2\pi i n t}$ then $\deg(\gamma_n) = n$.

Corollary (Fundamental theorem of algebra). Every nonconstant complex polynomial has a root.

Proof. Let $f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$ be a complex polynomial and suppose $f(z) \neq 0 \forall z \in \mathbb{C}$. Let $\gamma_R(t) = f(Re^{2\pi i t})$ so $\gamma_R : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$. Now γ_0 is a constant map, so $\deg(\gamma_0) = 0$. By homotopy invariance of degree, $\deg(\gamma_R) = 0 \forall R$.

If $R \gg \sum_i |a_i|$, we can consider $f_s(z) = z^n + s(a_1 z^{n-1} + \dots + a_n)$ for $0 \leq s \leq 1$, and on the circle $Re^{2\pi i t}$, f_s also takes values in $\mathbb{R}^2 \setminus \{0\}$. If $\gamma_{R,s}(t) = f_s(Re^{2\pi i t})$, then $\gamma_{R,1} = \gamma_R$ but $\gamma_{R,0} : z \mapsto z^n$, which has degree n . Now $\deg(\gamma_0) = \deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0})$, so $n = 0$ and f is constant. \square

Fact. Any two maps $S^n \rightarrow \mathbb{R}^{n+1}$ are homotopic but maps $S^n \xrightarrow{f} \mathbb{R}^{n+1} \setminus \{0\}$ have a degree $\deg(f) \in \mathbb{Z}$, invariant up to homotopy. Moreover, the degree of the constant map is 0 and the degree of inclusion is 1.

Corollary (Brouwer's fixed point theorem). If $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, any continuous map $f : B^n \rightarrow B^n$ has a fixed point.

Proof. Suppose f has no fixed point. Let $\gamma_R : S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ be the map $v \mapsto Rv - f(Rv)$ for $0 \leq R \leq 1$. So γ_0 is a constant, so has degree 0. Hence $\deg(\gamma_1) = 0$.

Let $\gamma_{1,s}(v) := v - sf(v)$, for $0 \leq s \leq 1$ and $v \in S^{n-1}$. Note $\gamma_{1,s}$ has image in $\mathbb{R}^n \setminus \{0\}$: if $s = 1$, this is because $v \neq f(v) \forall v$ and if $s < 1$ then $|v| > |sf(v)|$. Therefore $\gamma_1 = \gamma_{1,1}$ has the same degree as $\gamma_{1,0}$ which is the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$, a contradiction. \square

Definition (Homotopy equivalence). We say spaces X, Y are **homotopy equivalent** if \exists maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. We write $X \simeq Y$.

Example.

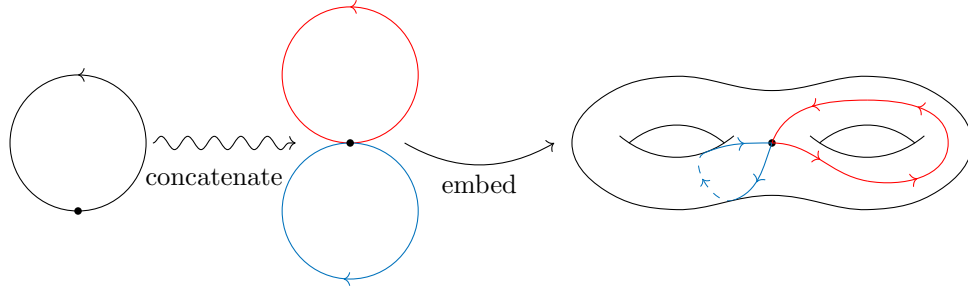
- Trivial case: if X, Y are homeomorphic, i.e. $X \cong Y$, then clearly $X \simeq Y$.
- $\mathbb{R}^n \simeq \{0\}$, the single point. A space homotopy equivalent to a point is sometimes called contractible.
- $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. If $i : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ inclusion, and $p : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ is projection $v \mapsto \frac{v}{\|v\|}$, then $p \circ i = \text{id}_{S^{n-1}}$ and $i \circ p \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy

$$F : \mathbb{R}^n \setminus \{0\} \times [0, 1] \longrightarrow \mathbb{R}^n \setminus \{0\}$$

$$(v, t) \mapsto tv + (1 - t) \frac{v}{\|v\|}$$

Algebraic topology is the study of the set of spaces up to homotopy equivalence via the set of groups up to isomorphism.

The first naive attempt would be homotopy groups: Loops (continuous maps $S^1 \rightarrow X$) with a common base-point can be concatenated and this induces a group structure on the set of homotopy classes of maps $(S^1, *) \rightarrow (X, x_0)$. Recall this refers to continuous maps $S^1 \rightarrow X$ taking $*$ to x_0 and a based homotopy $F : f \simeq g$ of two such is one such that $F|_{S \times \{t\}}$ sends $*$ to $x_0 \forall t$.



Again, there is a group structure on the set of based homotopy classes of maps $(S^n, *) \rightarrow (X, x_0)$ called $\pi_n(X, x_0)$, the n -th homotopy group of X .

Fact. $\{\pi_n(S^2, x)\}_{n \geq 1}$ is not known. Indeed, there is no simply connected manifold of dimension > 0 for which all π_n are known.

Instead, we will focus on homology theory, more precisely singular (co)homology. We will obtain invariants of spaces in a two-step process:

- (a) Associate to X a chain complex (or cochain complex).
- (b) take (co)homology of that complex.

This will be rather computable for simple spaces. In this course, we will mostly focus on studying manifolds.

Definition (Chain complex, cochain complex). A **chain complex** (C_*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

(indexed by \mathbb{N} or \mathbb{Z}) with the key property that $\forall n \ d_{n-1} \circ d_n = 0$. Then $\text{image}(d_{n+1}) \subseteq \ker(d_n)$ and the n -th homology group $H_n(C_*, d)$ of the chain complex is the quotient

$$H_n(C_*, d) := \frac{\ker(d_n)}{\text{im}(d_{n+1})}.$$

A **cochain complex** (C^*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} \equiv 0 \ \forall n$. The n -th cohomology group $H^n(C^*, d)$ is the quotient

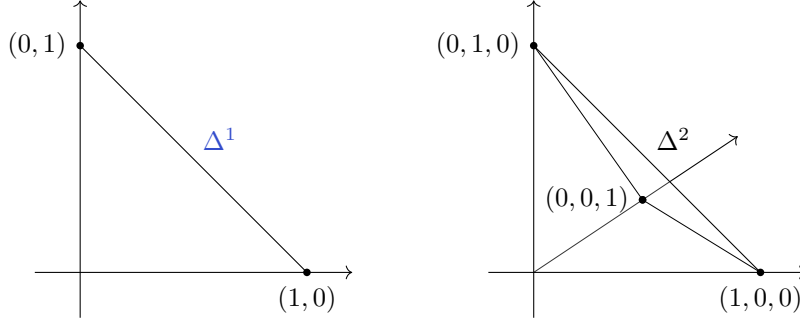
$$H^n(C^*, d) := \frac{\ker(d^n)}{\text{im}(d^{n-1})}.$$

0.1 Singular (co)chains

Definition (Simplex). A **simplex** in a topological space X is defined as follows.

- An n -simplex is the convex hull of $(n+1)$ ordered points v_0, \dots, v_n in \mathbb{R}^m such that $\{v_i - v_0 \mid 1 \leq i \leq n\}$ are linearly independent. Write this as $[v_0, \dots, v_n] = \sigma$.

The **standard n -simplex** is $\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \forall i \}$, e.g.



Note *any* n -simplex is canonically the image of Δ^n under a linear homeomorphism $\Delta^n \rightarrow \sigma$ given by $(t_i) \mapsto \sum t_i v_i \in \sigma$.

An n simplex in X is a continuous map $\sigma : \Delta^n \rightarrow X$, or from any n -simplex to X . Note any n -simplex has faces $\Delta_i^{n-1} \subseteq \Delta^n$ is defined by $\{t_i = 0\}$ and then this defines a corresponding face of any σ via the map $\Delta^n \rightarrow \sigma$. Write i th face of σ as $[v_0, \dots, \hat{v}_i, \dots, v_n] \subseteq [v_0, \dots, v_n]$ (so a hat over a vertex means omit it).

Note the edges of any simplex are canonically oriented via ' $v_i \rightarrow v_j$ if $i < j$ '.

Definition (Singular chain complex). If X is a space, the **singular chain complex** $C_*(X; \mathbb{Z})$ or just $C_*(X)$ is defined as follows:

$$\left\{ \sum_{i=1}^N h_i \sigma_i \mid N \in \mathbb{N}_{\geq 0}, h_i \in \mathbb{Z}, \sigma_i : \Delta^n \rightarrow X \text{ an } n\text{-simplex in } X \right\}$$

the free abelian group on n -simplices in X .

Definition (Boundary map). The boundary map $d : C_n(X) \rightarrow C_{n-1}(X)$ is defined by

$$d\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]}$$

on $\sigma = [v_0 \dots v_n]$ and extend to $C_n(X)$ by linearity.

Example. For the simplex $\sigma = [v_0 v_1 v_2]$, $d(\sigma) = [v_0 v_1] - [v_0 v_2] + [v_1 v_2]$.

Lemma. $d^2 = 0$, i.e. $d_{n-1} \circ d_n = 0 \forall n$.

Proof.

$$\begin{aligned} d \circ d(\sigma) &= d\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]}\right) \\ &= \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0 \dots \hat{v}_j \dots \hat{v}_i \dots v_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} \sigma|_{[v_0 \dots \hat{v}_i \dots \hat{v}_j \dots v_n]}. \end{aligned}$$

Exchanging i and j , these two terms exactly cancel. □

The resulting homology theory $H_*(X)$ or $H_*(X; \mathbb{Z})$ is called **singular homology**.

The \mathbb{Z} keeps track of the fact that $h_i \in \mathbb{Z}$, we could similarly define $C_*(X, G)$ and $H_*(X, G)$ for any abelian group G . Note $H_*(X, \mathbb{Z})$ is tautologically a homeomorphism invariant of X .

The idea is that d takes the boundary of a region covered by simplices.

Elements of $\ker(d : C_i(X) \rightarrow C_{i-1}(X))$ are called cycles or i -cycles. Elements in $\text{im}(d)$ are called boundaries.

Definition (Cochain complex). The singular **cochain** complex of a space X , $C^*(X, \mathbb{Z})$ or $C^*(X)$, has cochain groups $C^n(X) := \text{Hom}(C_n(X), \mathbb{Z})$ and coboundary map $d^* : C^n(X) \rightarrow C^{n+1}(X)$ by $(d^*\psi)(\sigma) := \psi(d\sigma)$. Here $\sigma \in C_{n+1}(X)$ and $d\sigma \in C_n(X)$, i.e. $d\sigma = d_{n+1}\sigma$.

Observe $d^*(d^*\psi)(\sigma) = d^*(\psi|_{d\sigma}) = \psi|_{dod(\sigma)}$ and $d \circ d(\sigma) = 0$, so $(d^*)^2 = 0$. So indeed $(C^*(X), d^*)$ is a cochain complex and the cohomology $H^*(X, \mathbb{Z})$ or $H^*(X)$ is called singular cohomology.

Note $H^*(X, \mathbb{Z}) \neq \text{Hom}_{\mathbb{Z}}(H_*(X, \mathbb{Z}), \mathbb{Z})$ in general.

Observe if $f : X \rightarrow Y$ is continuous and $\sigma : \Delta^n \rightarrow X$ is continuous, then I get $f \circ \sigma : \Delta^n \rightarrow Y$ is an n -simplex in Y , so I get

$$\begin{aligned} f_* : C_*(X) &\rightarrow C_*(Y) \\ f_* : C_n(X) &\rightarrow C_n(Y) \quad \forall n \end{aligned}$$

are group homomorphisms.

Key observation: $df_* = f_*d$ since $f \circ (\sigma|_{[v_0 \dots \hat{v}_i \dots v_n]}) = (f \circ \sigma)|_{[v_0 \dots \hat{v}_i \dots v_n]}$ i.e. a continuous maps $f : X \rightarrow Y$ induces a **chain map** of chain complexes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{d} & C_n(X) & \xrightarrow{d} & C_{n-1}(X) & \xrightarrow{d} & \dots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \dots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{d} & C_n(Y) & \xrightarrow{d} & C_{n-1}(Y) & \xrightarrow{d} & \dots \end{array}$$

and each square commutes.

Lemma. If C_* and D_* are chain complexes and $f_* : C_* \rightarrow D_*$ is a chain map. then f_* indexes homomorphisms $f_* : H_i(C_*) \rightarrow H_i(D_*)$ for every i .

Proof. Let $a \in H_i(C_*) = \frac{\ker(d_i : C_i \rightarrow C_{i-1})}{\text{im}(d_{i+1} : C_{i+1} \rightarrow C_i)}$, so a is represented by some i -cycle $\alpha \in C_i$, where $d\alpha = 0$.

Then $f_*(d\alpha) = 0 = d(f_*\alpha) \implies f_*(\alpha) \in D_i$ is a cycle in the D_* -chain complex, and hence defines an element in $H_i(D_*) = \frac{\ker(d : D_i \rightarrow D_{i-1})}{\text{im}(d : D_{i+1} \rightarrow D_i)}$. Call this element b and set $f_*(a) = b$. This is well-defined. If α' also represents $a \in H_i(C_*)$, then $\alpha - \alpha'$ is a boundary, i.e. $\alpha - \alpha' = d_{i+1}(\gamma)$ for some $\gamma \in C_{i+1}$. Then $f_*(\alpha) - f_*(\alpha') = f_*(d_{i+1}\gamma) = d_{i+1}f_*(\gamma)$ so $f_*(\alpha')$ and $f_*(\alpha)$ differ by a boundary, so define some element $b \in H_i(D_*)$. It is an easy exercise to check that this map $f_* : H_i(C_*) \rightarrow H_i(D_*)$ is indeed a homomorphism of groups. \square

The upshot of this is that if $f : X \rightarrow Y$ is a continuous map of spaces, it induces maps $f_* : H_i(X) \rightarrow H_i(Y)$ for all i .

Lemma. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_* \circ f_*$, $id_* = id$.

Proof. Exercise. □

In category-theoretic language, the association $X \mapsto H_*(X)$ is a functor from the category of topological spaces to the set of graded abelian groups. Observe $f : X \rightarrow Y$ induces $f_* : C_*(X) \rightarrow C_*(Y)$ and this has an adjoint $f^* : C^*(Y) \rightarrow C^*(X)$. Note this goes the other way. This again induces a map $H^*(Y) \rightarrow H^*(X)$.