Part III – Introduction to Discrete Analysis (Ongoing course, rough)

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Michaelmas 2018

Contents

1 The Discrete Fourier transform

Let N be some fixed positive integer. Write ω for $e^{\frac{2\pi i}{N}}$, and $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$.

Definition (Discrete Fourier transform)**.** Let $f : \mathbb{Z}_N \to \mathbb{C}$. Given $r \in \mathbb{Z}_N$, define $\hat{f}(r)$ to be

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x)\omega^{-rx}.$$

Notation. From now on, we shall use notation $\mathbb{E}_{x \in \mathbb{Z}_N}$ for $\frac{1}{N}\sum_{x \in \mathbb{Z}_N}$, where the subscript is omitted when it is clear from context.

Notice we can write

$$\hat{f}(r) = \mathop{\mathbb{E}}_x f(x) e^{-\frac{2\pi i r x}{N}},$$

highlighting the similarity with the usual Fourier transform.

If we write ω_r for the function $x \mapsto \omega^{rx}$, and $\langle f, g\rangle$ for $\mathbb{E}_x f(x)\overline{g(x)}$, then $\hat{f}(r) = \langle f, \omega_r\rangle$. Let us write $\|f\|_p$ for $(\mathbb{E}_x |f(x)|^p)^{\frac{1}{p}}$ and call the resulting space $L_p(\mathbb{Z}_N)$.

Important convention. We use *averages* for the 'original functions' in 'physical space' and *sums* for their Fourier transforms in 'frequency space'.

Lemma 1.1 (Parseval's identity)**.** If $f, g : \mathbb{Z}_N \to \mathbb{C}$, then $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$.

Proof.

$$\begin{aligned}
\langle \hat{f}, \hat{g}\rangle &= \sum_r \hat{f}(r)\overline{\hat{g}(r)} \\
&= \sum_r (\mathop{\mathbb{E}}_x f(x)\omega^{-rx})\overline{(\mathop{\mathbb{E}}_y g(y)\omega^{-ry})} \\
&= \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_y f(x)\overline{g(y)} \sum_r \omega^{-r(x-y)} \\
&= \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_y f(x)\overline{g(y)} \Delta_{xy} \\
&= \mathop{\mathbb{E}}_x f(x) \mathop{\mathbb{E}}_y \overline{g(y)} \Delta_{xy} \\
&= \mathop{\mathbb{E}}_x f(x)\overline{g(x)} = \langle f, g\rangle
\end{aligned}$$

where

$$\Delta_{xy} = \begin{cases} N & x = y \\ 0 & x \neq y. \end{cases}$$

□

Definition (Convolution)**.** The convolution $\widehat{f * g}(x)$ is defined to be

$$\mathop{\mathbb{E}}_{y+z=x} f(y)g(z) = \mathop{\mathbb{E}}_y f(y)g(x-y).$$

Lemma 1.2 (Convolution identity)**.**

$$\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r).$$

Proof.

$$\begin{aligned}
\widehat{f * g}(r) &= \mathop{\mathbb{E}}_x f * g(x)\omega^{-rx} \\
&= \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_{y+z=x} f(y)g(z)\omega^{-rx} \\
&= \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_{y+z=x} f(y)g(z)\omega^{-ry}\omega^{-rz} \\
&= \mathop{\mathbb{E}}_y f(y)\omega^{-ry} \mathop{\mathbb{E}}_z g(z)\omega^{-rz} = \hat{f}(r)\hat{g}(r).
\end{aligned}$$

□

Lemma 1.3 (Inversion formula)**.**

$$f(x) = \sum_r \hat{f}(r)\omega^{rx}$$

Proof.

$$\begin{aligned}
\sum_r \hat{f}(r)\omega^{rx} &= \sum_r \mathop{\mathbb{E}}_y f(y)\omega^{r(x-y)} \\
&= \mathop{\mathbb{E}}_y f(y) \sum_r \omega^{r(x-y)} \\
&= \mathop{\mathbb{E}}_y f(y)\Delta_{xy} = f(x).
\end{aligned}$$

□

Further observations:

- If f is real-valued, then $\hat{f}(-r) = \mathbb{E}_x f(x)\omega^{rx} = \overline{\mathbb{E}_x f(x)\omega^{-rx}} = \overline{\hat{f}(r)}$.
- If $A \subset \mathbb{Z}_n$, write A (instead of $\mathbb{1}_A$ or χ_A) for the characteristic function of A. Then $\hat{A}(0) = \mathbb{E}_x A(x) = \frac{|A|}{N}$, the density of A.
- Also, $\|\hat{A}\|_2^2 = \langle \hat{A}, \hat{A} \rangle = \langle A, A \rangle = \mathbb{E}_x A(x)^2 = \mathbb{E}_x A(x) = \frac{A}{N}$.

Let $f : \mathbb{Z}_N \to \mathbb{C}$. Given $\mu \in \mathbb{Z}_N$ with $(\mu, N) = 1$, define $f_\mu(x)$ to be $f(\mu^{-1}x)$. Then

$$\begin{aligned}
\hat{f}_\mu(r) &= \mathop{\mathbb{E}}_x f_\mu(x)\omega^{-rx} \\
&= \mathop{\mathbb{E}}_x f(x/\mu)\omega^{-rx} \\
&= \mathop{\mathbb{E}}_x f(x)\omega^{-r\mu x} \\
&= \hat{f}(\mu r).
\end{aligned}$$

1.1 Roth's Theorem

Theorem 1.4. For every $\delta > 0$, there exists N such that if $A \subseteq \{1, \ldots, N\}$ is a set of size at least δN then A must contain an arithmetic progression of length 3.

This is the $k = 3$ case of Szemerédi's theorem.

Basic strategy: show that if A has density $\geq \delta$ and no arithmetic progression of length 3, then there is a long arithmetic progression $P \subseteq \{1, \ldots, N\}$ such that

$$|A \cap P| \geq (\delta + c(\delta))|P|.$$

In particular, we have that $|P| \to \infty$ as $N \to \infty$.

The proof we give will produce a bound $\delta \geq \frac{C}{\log \log N}$, but this is not the best known. If the bound was reduced to $\frac{1}{\log N}$, this produces a combinatorial proof of the fact that there are arbitrarily long arithmetic progressions in the primes. The best known bound is $\frac{(\log \log N)^4}{\log N}$ by Thomas Bloom. In the other direction, we know $e^{-\sqrt{\log N}}$ does not work.

Lemma 1.5. Let $A, B, C \subset \mathbb{Z}_N$ have densities α, β, γ, for N odd. If $\max_{r \neq 0} |\hat{A}(r)| \leq \frac{\alpha(\beta\gamma)^{\frac{1}{2}}}{2}$ and $\frac{\alpha\beta\gamma}{2} > \frac{1}{N}$ then there exists $x, d \in \mathbb{Z}_N$ with $d \neq 0$ such that $(x, x+d, x+2d) \in A \times B \times C$.

Proof.

$$\begin{aligned}
\mathop{\mathbb{E}}_{x,d} A(x)B(x+d)C(x+2d) &= \mathop{\mathbb{E}}_{x+z=2y} A(x)B(y)C(z) \\
&= \mathop{\mathbb{E}}_{u}\Big(\mathop{\mathbb{E}}_{x+z=u} A(x)C(z)\Big) \mathop{\mathbb{E}}_{2y=u} B(y) \\
&= \mathop{\mathbb{E}}_{u}(A * C)(u)B_2(u) = \langle A * C, B_2 \rangle \\
&= \langle \widehat{A * C}, \hat{B}_2 \rangle \\
&= \langle \hat{A}\hat{C}, \hat{B}_2 \rangle \\
&= \sum_{r} \hat{A}(r)\hat{C}(r)\hat{B}(-2r) \\
&= \alpha\beta\gamma + \sum_{r \neq 0} \hat{A}(r)\hat{C}(r)\hat{B}(-2r).
\end{aligned}$$

We have a lower bound on the left term, so focus on the right.

$$\begin{aligned}
\left| \sum_{r \neq 0} \hat{A}(r)\hat{B}(-2r)\hat{C}(r) \right| &\leq \frac{\alpha(\beta\gamma)^{\frac{1}{2}}}{2} \sum_{r \neq 0} |\hat{B}(-2r)\hat{C}(r)| \\
&\leq \frac{\alpha(\beta\gamma)^{\frac{1}{2}}}{2} \left(\sum_{r} |\hat{B}(-2r)|^2 \right)^{\frac{1}{2}} \left(\sum_{r} |\hat{C}(r)|^2 \right)^{\frac{1}{2}} \\
&= \frac{\alpha(\beta\gamma)^{\frac{1}{2}}}{2} \|\hat{B}\|_2 \|\hat{C}\|_2 = \frac{\alpha(\beta\gamma)^{\frac{1}{2}}}{2} \|B\|_2 \|C\|_2 \\
&= \frac{\alpha\beta\gamma}{2}.
\end{aligned}$$

The contribution to $\mathbb{E}_{x,d} A(x)B(x+d)C(x+2d)$ from $d=0$ is at most $\frac{1}{N}$, so if $\frac{\alpha\beta\gamma}{2} > \frac{1}{N}$, we are done. $\square$

Now let A be a subset of $\{1,\dots,N\}$ of density $\geq \delta$ and let $B = C = A \cap (\frac{N}{3}, \frac{2N}{3}]$. If B has density $< \frac{\delta}{5}$, then either $A \cap [1, \frac{N}{3}]$ or $A \cap [\frac{2N}{3}, N]$ has density at least $\frac{2\delta}{5}$. So in that case we find an AP P of length about $\frac{N}{3}$ such that $\frac{|A\cap P|}{|P|} \geq \frac{6\delta}{5}$.

Otherwise, we find that if $\max_{r\neq 0} |\hat{A}(r)| \leq \frac{\delta^2}{10}$ and $\frac{\delta^3}{50} > \frac{1}{N}$ then $A \times B \times C$ contains a 3AP $\implies$ A contains a 3AP. So if A does not contain a 3AP, then either we find P of length about $\frac{N}{3}$ with $\frac{|A\cap P|}{|P|} \geq \frac{6\delta}{5}$ or $\exists r \neq 0$ such that $|\hat{A}(r)| \geq \frac{\delta^2}{10}$.

Definition. If X is a finite set and $f : X \to \mathbb{C}$, $Y \subseteq X$, write $\operatorname{osc}(f|_Y)$ to mean $\max_{y_1,y_2\in Y} |f(y_1) - f(y_2)|$.

Lemma 1.6. Let $r \in \hat{\mathbb{Z}}_n$ and let $\epsilon > 0$. Then there is a partition of $\{1,2,\dots,N\}$ into arithmetic progressions P_i of length at least $c(\epsilon)\sqrt{N}$ such that $\operatorname{osc}(\omega_r|_{P_i}) \leq \epsilon$ for each i.

Proof. Let $t = \lfloor\sqrt{N}\rfloor$. Of the numbers $1, \omega^r, \omega^{2r}, \dots, \omega^{tr}$ there must be two that differ by at most $\frac{2\pi}{t}$. If $|\omega^{ar} - \omega^{br}| \leq \frac{2\pi}{t}$ with $a < b$, then $|1 - \omega^{dr}| \leq \frac{2\pi}{t}$ where $d = b - a$. Now, by the triangle inequality, if $u < v$, then

$$|\omega^{urd} - \omega^{vrd}| \leq |\omega^{urd} - \omega^{(u+1)rd}| + |\omega^{urd} - \omega^{(u+1)rd}| + \cdots + |\omega^{urd} - \omega^{(u+1)rd}| \leq \frac{2\pi}{t}(v-u).$$

So if P is a progression with common difference d and length l, then $\operatorname{osc}(\omega_r|_P) \leq \frac{2\pi l}{t}$. So divide up $\{1,\dots,N\}$ into residue classes mod d and partition each residue class into parts of length between $\frac{\epsilon t}{4\pi}$ and $\frac{\epsilon t}{2\pi}$ (possible, since $d \leq t \leq \sqrt{N}$). We are done, with $c(\epsilon) = \frac{\epsilon}{16}$. $\square$

Now let us use the information that $r \neq 0$ and $|\hat{A}(r)| \geq \frac{\delta^2}{10}$. Define the balanced function f of A by $f(x) = A(x) - \frac{|A|}{N}$ for each x.

Note that $\hat{f}(0) = 0$ and $\hat{f}(r) = \hat{A}(r)$ for all $r \neq 0$. Now let $P_1, \dots, P_m$ be given by Lemma 1.6 with $\epsilon = \frac{\delta^2}{20}$. Then

$$\begin{aligned}\frac{\delta^2}{10} \leq \frac{1}{N}\left|\sum_x f(x)\omega^{-rx}\right| &\leq \frac{1}{N}\sum_{i=1}^m \left|\sum_{x\in P_i} f(x)\omega^{-rx}\right| \\ &\leq \frac{1}{N}\sum_{i=1}^m \left[\left|\sum_{x\in P_i} f(x)\omega^{-rx_i}\right| + \left|\sum_{x\in P_i} f(x)(\omega^{-rx} - \omega^{-rx_i})\right|\right]\end{aligned}$$

where $x_i \in P_i$ is arbitrary

$$\leq \frac{1}{N}\sum_{i=1}^m \left|\sum_{x\in P_i} f(x)\right| + \frac{\delta^2}{20}$$

So

$$\sum_{i=1}^N \left|\sum_{x\in P_i} f(x)\right| \geq \frac{\delta^2 N}{20}.$$

Also,

$$\sum_{i=1}^{m} \sum_{x \in P_i} f(x) = 0.$$

So

$$\sum_{i=1} \left(\left| \sum_{x \in P_i} f(x) \right| + \sum_{x \in P_i} f(x) \right) \geq \frac{\delta^2}{20} \sum_{i=1}^{m} |P_i|$$

Therefore, $\exists i$ such that

$$\left| \sum_{x \in P_i} f(x) \right| + \sum_{x \in P_i} f(x) \geq \frac{\delta^2}{20} |P_i|$$

$$\implies \sum_{x \in P_i} f(x) \geq \frac{\delta}{40} |P_i|$$

$$\implies |A \cap P_i| \geq \left(\delta + \frac{\delta^2}{40} \right) |P_i|$$

So now, either

1. A contains a $3AP$
2. N is even
3. $\exists P \subset \{1, \ldots, N\}$, $|P| \geq \frac{N}{3}$ such that $|A \cap P| \geq \frac{6\delta}{5} |P|$
4. $\exists P \subset \{1, \ldots, N\}$, $|P| \geq \frac{\delta^2}{320} \sqrt{N}$ such that $|A \cap P| \geq \left(\delta + \frac{\delta^2}{40} \right) |P|$

If 2 holds, write $N = N_1 + N_2$ with N_1, N_2 odd, $N_1 + N_2 \approx \frac{N}{2}$. Then A has density at least δ in one of $\{1, \ldots, N_1\}$ or $\{N_1 + 1, \ldots, N_1 + N_2\}$.

If 4 holds (note $3 \Rightarrow 4$) then we pass to P and start again. After $\frac{40}{\delta}$ iterations, the density at least doubles. So the total number of iterations we can have is $\leq \frac{40}{\delta} + \frac{40}{2\delta} + \frac{40}{4\delta} + \ldots \leq \frac{80}{\delta}$.

If $\frac{\delta^2}{320} \sqrt{N} \geq N^{\frac{1}{3}}$ at each iteration, and $\frac{\delta^3}{25} > N^{-1}$ (which follows from the first condition) then after $\frac{80}{\delta}$ iterations we have $N \geq N^{\left(\frac{1}{3}\right)^{\frac{80}{\delta}}}$. So the argument works provided

$$\begin{aligned} N^{\left(\frac{1}{3}\right)^{\frac{80}{\delta}}} \geq \left(\frac{320}{\delta^2} \right)^6 &\impliedby \left(\frac{1}{3} \right)^{\frac{80}{\delta}} \log N \geq 6 \left(\log 320 + 2 \log \frac{1}{\delta} \right) \\ &\impliedby -\frac{80}{\delta} \log 3 + \log \log N \geq \log 6 + \log \left(\log 320 + 2 \log \frac{1}{\delta} \right) \\ &\impliedby \log \log N \geq \frac{160}{\delta} \impliedby \delta \geq \frac{160}{\log \log N}. \end{aligned}$$

1.2 Bogolyubov's method

Let $K \subset \hat{\mathbb{Z}}_N$ and let $\delta > 0$.

Definition (Bohr set)**.** The **Bohr set** $B(K, \delta)$ has two definitions.

1. $B(K, \delta) = \{ x \in \mathbb{Z}_N \mid rx \in [-\delta N, \delta N] \ \forall r \in K \}$ (arc-length definition)
2. $B(K, \delta) = \{ x \in \mathbb{Z}_N \mid |1 - \omega^{rx}| < \delta \ \forall r \in K \}$ (chord-length definition)

Definition. Let G be an abelian group and let A, B be subsets of G. Then

$$\begin{aligned} A + B &= \{ a + b \mid a \in A, b \in B \} \\ A - B &= \{ a - b \mid a \in A, b \in B \} \\ rA &= \{ a_1 + \cdots + a_r \mid a_1, \ldots, a_r \in A \} \end{aligned}$$

Lemma 1.7 (Bogolyubov's Lemma)**.** Let $A \subset \mathbb{Z}_N$ be a set of density α. Then $2A - 2A$ contains a Bohr set (arc) with $|K| \le \alpha^{-2}$.

Proof. Observe that $x \in 2A - 2A$ iff $A * A * (-A) * (-A)(x) \neq 0$. But

$$\begin{aligned} A * A * (-A) * (-A)(x) &= \sum_r \widehat{A * A * (-A) * (-A)}(r) \omega^{rx} \\ &= \sum_r |\hat{A}(r)|^4 \omega^{rx}. \end{aligned}$$

Let $K = \{ r \mid |\hat{A}(r)| \ge \alpha^{\frac{3}{2}} \}$. Then $\alpha = \|\hat{A}\|_2^2 = \sum_r |\hat{A}(r)|^2 \ge \alpha^3 |K|$. So $|K| \le \alpha^{-2}$.

Now suppose that $x \in B(K, \frac{1}{4})$. Then

$$\sum_r |\hat{A}(r)|^4 \omega^{rx} = \alpha^4 + \sum_{r \in K \setminus \{0\}} |\hat{A}(r)|^4 \omega^{rx} + \sum_{r \notin K} |\hat{A}(r)|^4 \omega^{rx}.$$

The real part of the second term is non-negative, since $rx \in \left[-\frac{N}{4}, \frac{N}{4}\right]$ when $r \in K$. Also

$$\left| \sum_{r \notin K} |\hat{A}(r)|^4 \omega^{rx} \right| \le \sum_{r \notin K} |\hat{A}(r)|^4 < \alpha^3 \sum_{r \notin K} |\hat{A}(r)|^2 \le \alpha^4.$$

It follows that $\mathrm{Re}\left(\sum_r |\hat{A}(r)|^4 \omega^{rx} \right) > 0$, so $x \in 2A - 2A$. $\square$

Lemma 1.8. Let $K \subset \mathbb{Z}_N$ and let $\delta > 0$. Then

(i) $B(K, \delta)$ (arc) has density at least $\delta^{|K|}$

(ii) $B(K, \delta)$ contains a mod-N arithmetic progression of length $\ge \delta N^{\frac{1}{|K|}}$

Proof.

(i) Let $K = \{r_1, \ldots, r_k\}$. Consider the N k-tuples $(r_1 x, r_2 x, \ldots, r_k x) \in \mathbb{Z}_N^k$. If we intersect this set of k-tuples with a random 'box' $[t_1, t_1 + \delta N] \times \cdots \times [t_k, t_k + \delta N]$ then the expected number of the k-tuples in the box is $\delta^k N$ (since each one has a probability δ^k). But if $(r_1 x, \ldots, r_k x)$ and $(r_1 y, \ldots, r_k y)$ belong to this box, then $x - y \in B(K, \delta)$.

(ii) If we take $\eta > N^{\frac{1}{2}}$, then by (i) we get that $|B(K,\eta) > 1$, so $\exists x \in B(K,\eta)$ such that $x \neq 0$. But then $dx \in B(K, d\eta)$ for every d. So if $d\eta \leq \delta$ then $dx \in B(K,\delta)$. That gives us an AP of length at least $\frac{\delta}{\eta}$. So we get one of length at least $\delta N^{\frac{1}{k}}$. $\square$

Definition (Freiman homomorphism)**.** Let A, B be subsets of Abelian groups and let $\varphi : A \to B$. Then φ is a **Freiman homomorphism of order** k if

$$a_1 + \cdots + a_k = a_{k+1} + \ldots + a_{2k} \implies \varphi(a_1) + \cdots + \varphi(a_k) = \varphi(a_{k+1}) + \cdots + \varphi(a_{2k}).$$

If $k = 2$, we call this a **Freiman homomorphism**. In that case, the condition is equivalent to $a - b = c - d \implies \varphi(a) - \varphi(b) = \varphi(c) - \varphi(d)$.

If φ has an inverse which is also a Freiman homomorphism of order k, then φ is a **Freiman isomorphism of order** k.

Lemma 1.9. Assume $0 \notin K$ and N prime. If $\delta < \frac{1}{4}$, then $B(K,\delta)$ (arc) is Freiman isomorphic to the intersection in $\mathbb{R}^K$ of $[-\delta N, \delta N]^{|K|}$ with some lattice Λ.

Proof. Let $K = \{r_1, \ldots, r_k\}$ and let

$$\Lambda = N\mathbb{Z}^k + \{\, (r_1 x, \ldots, r_k x) \mid x \in \mathbb{Z} \,\}\,.$$

Write $\mathbf{r}$ for $(r_1, \ldots, r_k)$. Claim that $B(K,\delta) \cong \Lambda \cap [-\delta N, \delta N]^k$.

Define a map $\varphi : B(K,\delta) \to \Lambda \cap [-\delta N, \delta N]^k$ by sending x to $(\langle r_1 x\rangle, \ldots, \langle r_k x\rangle)$ where $\langle u\rangle$ means the least-modulus residue of u mod N. If $x + y = z + w$, then $\mathbf{r}x + \mathbf{r}y = \mathbf{r}z + \mathbf{r}w$ in $\mathbb{Z}_N^k$. But for each i, $\langle r_i x\rangle + \langle r_i y\rangle - \langle r_i z\rangle - \langle r_i w\rangle \in [-4\delta N, 4\delta N]$. Since $\delta < \frac{1}{4}$, that implies that $\langle r_i x\rangle + \langle r_i y\rangle - \langle r_i z\rangle - \langle r_i w\rangle = 0$. So $\langle \mathbf{r}x\rangle + \langle \mathbf{r}y\rangle = \langle \mathbf{r}z\rangle + \langle \mathbf{r}w\rangle$.

That already implies that φ is an injection. If $\mathbf{r}x + \mathbf{a}N \in [-\delta N, \delta N]^k$ then $r_i x \in [-\delta N, \delta N]$ mod N for each i, so $x \in B(K,\delta)$ and $\varphi(x) = \mathbf{r}x + \mathbf{a}N$. So φ is a surjection.

If $\mathbf{r}x + \mathbf{a}N + \mathbf{r}y + \mathbf{b}N = \mathbf{r}z + \mathbf{c}N + \mathbf{r}w + \mathbf{d}N$, then $r_1(x+y) = r_1(z+w)$ mod N, so $x + y = z + w$ mod N. So the inverse of φ is also a Freiman homomorphism. $\square$

Lemma 1.10. Let Λ be a lattice and let C be a symmetric convex body, both in $\mathbb{R}^k$. Then $|\Lambda \cap C| \leq 5^k |\Lambda \cap \frac{C}{2}|$.

Proof. Let $x_1, \ldots, x_m$ be a maximal subset of $\Lambda \cap C$ such that for all $i \neq j$, $x_j \notin x_i + \frac{C}{2}$. Then by maximality, the sets $x_i + \frac{C}{2}$ cover all of $\Lambda \cap C$. Also, the sets $x_i + \frac{C}{4}$ are disjoint subsets of $\mathbb{R}^k$, and they are all contained in $C + \frac{C}{4} = \frac{5}{4}C$. So

$$m \leq \frac{\mathrm{vol}(\frac{5}{4}C)}{\mathrm{vol}(\frac{1}{4}C)} = 5^k. \qquad \square$$

Corollary 1.11. If N is prime, $0 \notin K$, $|K| = k$, $\delta < \frac{1}{4}$, then $|B(K,\delta)| \leq 5^k |B(K, \frac{\delta}{2})|$.

2 Sumsets and their structure

The idea is to show that for $A \subset \mathbb{Z}$, if $|A+A| \le K|A|$ then $|rA - sA| \le K^{r+s}|A|$.

Lemma 2.1 (Petridis)**.** Let A_0, B be finite subsets of an abelian group such that $|A_0 + B| \le K_0|A_0|$. Then there exists a non-empty subset $A \subset A_0$ and $K \le K_0$ such that $|A+B+C| \le K|A+C|$ for every finite subset C of the group.

Proof. Let A minimise the ratio $\frac{|A+B|}{|A|}$ and let the minimal ratio be K. Claim: this works. We prove this by induction on C.

If $C = \varnothing$, then the result holds. Now assume it for C and let $x \notin C$. Then

$$A + (C \cup \{x\}) = (A+C) \cup (A + \{x\}) = (A+C) \cup [(A+x) \setminus (A'+x)]$$

where $A' = \{\, a \in A \mid a + x \in A + C \,\}$. This is a disjoint union, so

$$|A + (C \cup \{x\})| = |A+C| + |A| - |A'|.$$

Similarly,

$$A + B + (C \cup \{x\}) = (A+B+C) \cup ((A+B+x) \setminus (A'+B+x))$$

(since if $a + x \in A + C$ then $a + B + x \subset A + B + C$)

$$\begin{aligned} \implies |A+B+(C\cup\{x\})| &\le |A+B+C| + |A+B| - |A'+B| \\ &\le K|A+C| + K|A| - K|A'| \end{aligned}$$

by induction and minimality property of A. $\square$

Corollary 2.2. If A, B are finite subsets of an Abelian group and $|A+B| \le K|A|$, then there exists $A' \subseteq A$, $A' \neq \varnothing$ such that $|A' + rB| \le K^r|A'|$ for every positive integer r.

Proof. Choose A' as we chose A in the proof of Lemma 2.1. Then

$$|A' + rB| = |A' + B + (r-1)B| \le K|A' + (r-1)B|$$

and $|A' + B| \le K|A'|$ so we are done by induction. $\square$

Corollary 2.3. If $|A+A| \le K|A|$ or $|A-A| \le K|A|$, then $|rA| \le K^r|A|$.

Proof. Set $B = A$ or $-A$ in Corollary 2.2 $\square$

Lemma 2.4 (Rusza inequality)**.** Let A, B, C be finite subsets of an abelian group. Then $|A||B-C| \le |A-B||A-C|$.

Proof. Define a map $\phi : A \times (B - C) \to (A-B) \times (A-C)$ as follows. Given (a, x) with $a \in A, x \in B - C$, choose, somehow, $b(x) \in B$ and $c(x) \in C$ such that $b(x) - c(x) = x$ and set $\phi(a,x) = (a - b(x), a - c(x))$.

Note that $(a - c(x)) - (a - b(x)) = b(x) - c(x) = x$. Having worked out x, we know $b(x)$ and $a = a - b(x) + b(x)$, so a is determined too. So ϕ is an injection. $\square$

Why is it called a triangle inequality? We can write it as

$$\frac{|B-C|}{|B|^{\frac{1}{2}}|C|^{\frac{1}{2}}} \leq \frac{|A-B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} + \frac{|A-C|}{|A|^{\frac{1}{2}}|C|^{\frac{1}{2}}}$$

so if we define the Rusza distance $d(A,B)$ to be

$$\frac{|A-B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}},$$

then the inequality says $d(B,C) \leq d(A,B)d(A,C)$.

Corollary 2.5. If $|A-B| \leq K|A|$, then $|rB - sB| \leq K^{r+s}|A|$ for all r, s.

Proof. Pick A' as before. Then by Corollary 2.2 with B replaced by $-B$, $|A' - rB| \leq K'|A'|$ and $|A' - sB| \leq K^s|A'|$. Therefore, by Rusza inequality,

$$|A'||rB - sB| \leq K^{r+s}|A'|^2 \implies |rB - sB| \leq K^{r+s}|A|. \qquad \square$$

Corollary 2.6 (Plünnecke's theorem)**.** If $|A+A| \leq K|A|$ or $|A-A| \leq K|A|$, then $|rA-sA| \leq K^{r+s}|A|$.

Proof. Apply Corollary 2.5 with $B = -A$ or $B = A$. $\square$

Lemma 2.7 (Ruzsa's embedding theorem)**.** Let $A \subseteq \mathbb{Z}$ be finite and suppose that $|kA - kA| \leq C|A|$. Then there exists a prime $p \leq 4C|A|$ and a subset $A' \subseteq A$ of size at least $|A|/k$ such that A' is Freiman isomorphic of order k to a subset of $\mathbb{Z}_p$.

Proof. Consider the following composition of maps

$$\mathbb{Z} \xrightarrow{\text{reduce mod } q} \mathbb{Z}_q \xrightarrow[\text{non-zero } r]{\times \text{ by some}} \mathbb{Z}_q \xrightarrow[\text{residue}]{\text{least non-negative}} \mathbb{Z} \xrightarrow{\text{reduce mod } p} \mathbb{Z}_p$$

where q is a prime bigger than $\operatorname{diam} A$ and p is a prime $\in (2C|A|, 4C|A|]$.

Let ϕ be the composition. The first, second and fourth parts are group homomorphisms, and thus Freiman homomorphisms of all orders. Also, the third map is a Freiman homomorphism of order k if you restrict to a subinterval of $[0, q-1]$ of length $\leq \frac{q}{k}$. To see this, write $\langle u \rangle$ for the least non-negative residue. Then if I has length $\leq \frac{q}{k}$ (and therefore $< \frac{q}{k}$) and $u_1, \ldots, u_{2k} \in I$, then if $u_1 + \cdots + u_k - u_{k+1} - \cdots - u_{2k} = 0$, then

$$\langle u_1 \rangle + \cdots + \langle u_k \rangle - \langle u_{k+1} \rangle - \cdots - \langle u_{2k} \rangle \equiv 0 \pmod q$$

and also has modulus less than q. So it is zero.

By the pigeonhole principe, for any r we can find I of length $\leq \frac{q}{k}$ such that

$$A' = \{\, a \in A \mid ra \in I \,\}$$

has size at least $|A|/k$.

Remains to prove that ϕ is an isomorphism to its image. That is, we must show that if

$$a_1 + \cdots + a_k - a_{k+1} - \cdots a_{2k} \equiv 0 \quad (a_i \in A)$$

then

$$\langle ra_1\rangle + \cdots + \langle ra_k\rangle - \langle ra_{k+1}\rangle - \langle ra_{2k}\rangle \neq 0 \pmod p$$

But if the a_i are chosen such that the ra_i all belong to the same interval of length $\frac{q}{k}$,

$$|\langle ra_1\rangle + \cdots \langle ra_k\rangle - \langle ra_{k+1}\rangle - \cdots - \langle ra_{2k}\rangle| < q$$

and

$$\langle ra_1\rangle + \cdots + \langle ra_k\rangle - \langle ra_{k+1}\rangle - \cdots - \langle ra_{2k}\rangle \equiv r(a_1 + \cdots + a_k - a_{k+1} - \cdots - a_{2k}) \bmod q$$

So all that can go wrong is if $r(a_1+\cdots+a_k-a_{k+1}-\cdots-a_{2k})$ is xp for some $x \neq 0$ with $|x| < \frac{q}{p}$. The number of values to avoid is at most $\frac{2q}{p}$, so for each $a_1 + \cdots + a_k - a_{k+1} - \ldots - a_{2k}$ the probability of going wrong if r is chosen randomly is at most $\frac{2}{p}$. So since $|kA - kA| \leq C|A|$, the probability of going wrong is at most $\frac{2}{p}C|A|$. Since $p > 2C|A|$, there exists r such that we get a Freiman isomorphism of order k. □

2.1 Freiman's theorem (a version of)

We shall now start with a set $A \subseteq \mathbb{Z}$ with $|A + A| \leq C|A|$ and put together several of the previous results to say a lot about the structure of A.

By Plünnecke's theorem, $|8A - 8A| \leq C^{16}|A|$. By Ruzsa's embedding lemma, A has a subset A' of size at least $\frac{A}{8}$ that is 8-isomorphic to a subset $A'' \subset \mathbb{Z}_p$ with $p \leq 4C^{16}|A|$. The density of A'' in $\mathbb{Z}_p$ is $\alpha \geq \frac{1}{32C^{16}}$. By Bogolyubov's lemma, $2A'' - 2A''$ contains a Bohr set $B(K, \frac{1}{4})$ with $|K| \leq \alpha^{-2}$, which is 2-isomorphic to a set B' that is the intersection of a symmetric convex body with a lattice of dimension at most α^{-2}.

On the example sheet, prove that if A is 8-isomorphic to B, then $2A-2A$ is 2-isomorphic to $2B - 2B$. Thus $2A'' - 2A'' \overset{2}{\cong} 2A' - 2A'$, and therefore $2A' - 2A'$ has a subset B that is isomorphic to B'.

Now let $X \subset A$ be maximal such that the sets $x + B$ with $x \in X$ are disjoint. Then $A \subset X + B - B$, by maximality. Also, $|X||B| = |X + B| \leq |3A - 2A| \leq C^5|A| \implies |X| \leq C^5\frac{|A|}{|B|}$. But by basic facts about Bohr sets, $|B| \geq 4^{-\alpha^{-2}}|A|$, so $|X| \leq 4^{\alpha^{-2}}C^5$. So A is the union of at most $4^{1024C^{32}}C^5$ translates of $B - B$. If $B = \Lambda \cap K_0$ then $B - B \subset \Lambda \cap 2K_0$ and also $|B - B| \leq 5^{\alpha^{-2}}|B|$.

2.2 The Balog-Szemerédi-Gowers theorem

Definition (Additive quadruple)**.** Let A be a subset of an Abelian group. An additive quadruple in A is a quadruple $(a, b, c, d) \in A^4$ such that $a + b = c + d$.

(Equivalently, it's a quadruple such that $a - b = c - d$.)

If $|A| = n$, then the number of additve quadruples in A is at most n^3. We shall show that if A^4 contains at least cn^3 additive quadruples, then A has a subset A' of size at least $c'n$ with $|A' - A'| \leq C|A|$ where c' and C depend (nicely) on c only.

Lecture 7 **Lemma 2.8.** Let $A_1, \dots, A_m$ be subsets of $[n]$ of average density at least δ. Then $\forall \eta > 0$, we can find a set $B \subset [m]$ of size at least $\frac{\delta m}{\sqrt{2}}$ such that the proportion of pairs $(i, j) \in B^2$ with $|A_i \cap A_j| \ge \frac{\eta \delta^2 n}{2}$ is at least $1 - \eta$.

Proof. Choose $y \in [n]$ uniformly at random, and set $B = \{\, i : y \in A_i \,\}$. The probability that we pick both i and j is $\frac{|A_i \cap A_j|}{n}$, which we can write as $\langle A_i, A_j \rangle$. So the expected number of pairs we pick is

$$\sum_{i,j} \langle A_i, A_j \rangle = \left\| \sum_i A_i \right\|_2^2 \ge \left\| \sum_i A_i \right\|_1^2 = (\delta m)^2.$$

The probability that we pick i, j if $|A_i \cap A_j| \le \frac{\eta \delta^2 n}{2}$ is at most $\frac{\eta \delta^2}{2}$. Call such a pair (i, j) bad. Then

$$\mathbb{E}|B|^2 - \eta^{-1} \mathbb{E}(\#\text{bad pairs picked}) \ge \delta^2 m^2 - \eta^{-1} \frac{\eta \delta^2}{2} m^2 = \frac{\delta^2 m^2}{2}.$$

Therefore, there exists B such that $|B| \ge \frac{\delta m}{\sqrt{2}}$ and the proportion of pairs in B^2 that are bad is at most η. □

Corollary 2.9. Let G be a bipartite graph with finite vertex sets X, Y and density at least δ. Then there is a subset $B \subset X$ of size at least $\frac{\delta |X|}{\sqrt{2}}$ such that for at least $(1 - \eta)|B|^2$ pairs $(x_1, x_2) \in B^2$ there are at least $\frac{\eta \delta^2}{2} |Y|$ paths of length 2 from x_1 to x_2.

Corollary 2.10. Let G be a bipartite graph as above. Then there is a subset $B' \subset X$ of size at least $\frac{\delta |X|}{2\sqrt{2}}$ such that for every $x_1, x_2 \in B'$, there are at least $\frac{\delta^5}{2048\sqrt{2}} |X||Y|^2$ paths of length 4 from x_1 to x_2.

Proof. Choose B as in Corollary 2.9, with $\eta = \frac{1}{16}$. Define a graph Γ with vertex set B, joining x_1 to x_2 if there are at least $\frac{\delta^2}{32} |Y|$ P_2s from x_1 to x_2 (in G). By Corollary 2.9, the average degree in Γ is at least $\frac{15}{16}|B|$. Therefore, there are at least $\frac{|B|}{2}$ vertices in B of degree at least $\frac{7}{8}|B|$. Let B' be the set of all such vertices. If $x_1, x_2 \in B'$, then there are at least $\frac{3}{4}|B|$ vertices in B joined to both x_1, x_2 in Γ. Therefore, there are at least

$$\frac{\delta^4}{1024} |Y|^2 \cdot \frac{3}{4} \cdot \frac{\delta |X|}{\sqrt{2}}$$

P_4s from x_1 to x_2. □

Note: setting $\eta = \frac{1}{8}$ would give a bound of $\frac{\delta^4}{256} \cdot \frac{1}{2} \cdot \frac{\delta}{\sqrt{2}} |Y|^2 |X|$.

Lemma 2.11 (BSG Lemma)**.** Let A be a subset of size n of an abelian group. Suppose that there are at least cn^3 additive quadruples in A. Then A has a subset A'' of size at least $c'n$ with $|A' - A'| \le C|A'|$, where c' and C depend on c only.

Proof. For each d, let $f(d)$ be the number of ways of writing $d = a_1 - d_2$ with $a_1 - a_2 \in A$. Then $\sum_d f(d)^2 \geq cn^3$. Call d popular if $f(d) \geq \frac{cn}{2}$.

$$\begin{aligned} cn^3 \leq \sum_d f(d)^2 &= \sum_{d \text{ popular}} f(d)^2 + \sum_{d \text{ unpopular}} \\ &\leq (\#\text{popular } d) \times n^2 + \frac{cn}{2} \cdot n^2 \end{aligned}$$

since $\sum_d f(d) = n^2$.

Therefore, the number of popular d is at least $\frac{cn}{2}$. Now define a graph with vertex set A, by joining a_1 to a_2 if $a_1 - a_2$ (or $a_2 - a_1$) is popular. Each popular difference contributes at least $\frac{cn}{2}$ edges so average degree of the graph is at least $\frac{c^2}{4}n$.

By duplicating the vertex set, create a corresponding bipartite graph G. By Corollary 2.10 with $\delta = \frac{c^2}{4}$ we can find a subset $B \subset A$ of size at least $\frac{c^2}{8\sqrt{2}}|A|$ such that for any $a_1, a_2 \in B$ there are at least $\frac{\delta^5}{2048\sqrt{2}}|A|^3$ $P4$s from a_1 to a_2. Each such $P4$ gives us at least $\left(\frac{c}{2}|A|\right)^4$ ways of writing $a_2 - a_1$ as $b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8$ with all $b_1 \in A$. So

$$|B - B| \cdot \frac{\delta^5}{2048\sqrt{2}}|A|^3 \cdot \left(\frac{c}{2}|A|\right)^4 \leq |A|^8.$$

So $|B \cdot B| \leq C'|A| \leq C|B|$. $\square$

3 Quasirandom graphs

3.1 The box norm

Let X and Y be finite sets and let $f : X \times Y \to \mathbb{C}$.

Definition. We define the **box norm** $\|f\|_\square$ of f by the formula

$$\|f\|_\square^4 = \mathop{\mathbb{E}}_{x_1,y_1,x_2,y_2} f(x_1,y_1)\overline{f(x_1,y_2)f(x_2,y_1)}f(x_2,y_2).$$

If $f_1, f_2, f_3, f_4 : X \times Y \to \mathbb{C}$, then their **box inner product** $[f_1, f_2, f_3, f_4]$ is

$$\mathop{\mathbb{E}}_{x_1,y_1,x_2,y_2} f_1(x_1,y_1)\overline{f_2(x_1,y_2)f_3(x_2,y_1)}f_4(x_2,y_2)$$

We use the notation $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$.

Lemma 3.1 (Box Cauchy-Schwarz)**.** For any four functions $f_{00}, f_{01}, f_{10}, f_{11}$, we have

$$\left|\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}\right| \le \|f_{00}\|_\square \|f_{01}\|_\square \|f_{10}\|_\square \|f_{11}\|_\square$$

Proof.

$$\begin{aligned}
&\left| \mathop{\mathbb{E}}_{x_0,y_0,x_1,y_1} f_{00}(x_0,y_0)\overline{f_{01}(x_0,y_1)f_{10}(x_1,y_0)}f_{11}(x_1,y_1) \right| \\
&= \left| \mathop{\mathbb{E}}_{x_0,x_1} \left(\mathop{\mathbb{E}}_{y_0} f_{00}(x_0,y_0)\overline{f_{10}(x_1,y_0)} \right) \left(\mathop{\mathbb{E}}_{y_1} f_{01}(x_0,y_1)\overline{f_{11}(x_1,y_1)} \right) \right| \\
&\le \left(\mathop{\mathbb{E}}_{x_0,x_1} \left| \mathop{\mathbb{E}}_{y_0} f_{00}(x_0,y_0)\overline{f_{10}(x_1,y_0)} \right|^2 \right)^{\frac{1}{2}} \left(\mathop{\mathbb{E}}_{x_0,x_1} \left| \mathop{\mathbb{E}}_{y_1} f_{01}(x_0,y_1)\overline{f_{11}(x_1,y_1)} \right|^2 \right)^{\frac{1}{2}} \\
&= \begin{bmatrix} f_{00} & f_{00} \\ f_{10} & f_{10} \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} f_{01} & f_{01} \\ f_{11} & f_{11} \end{bmatrix}^{\frac{1}{2}}.
\end{aligned}$$

By symmetry (interchanging the roles of x and y) we also have

$$\left|\begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix}\right| \le \begin{bmatrix} f_{00} & f_{01} \\ f_{00} & f_{01} \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} f_{10} & f_{11} \\ f_{10} & f_{11} \end{bmatrix}^{\frac{1}{2}}.$$

Combining these two inequalities in the obvious way gives the result. $\square$

Corollary 3.2. $\|\cdot\|_\square$ is a norm.

Proof. The only property that is not straightforward is the triangle inequality. Let $f_0, f_1 : X \times Y \to \mathbb{C}$. Then

$$\begin{aligned}
\|f_0 + f_1\|_\square^4 &= [f_0 + f_1, f_0 + f_1, f_0 + f_1, f_0 + f_1] \\
&= \sum_{\epsilon \in \{0,1\}^4} [f_{\epsilon_1}, f_{\epsilon_2}, f_{\epsilon_3}, f_{\epsilon_4}] \\
&\leq \sum_{\epsilon \in \{0,1\}^4} \|f_{\epsilon_1}\|_\square \|f_{\epsilon_2}\|_\square \|f_{\epsilon_3}\|_\square \|f_{\epsilon_4}\|_\square \\
&= (\|f_0\|_\square + \|f_1\|_\square)^4 .
\end{aligned}$$

□

Remark. Suppose that $f(x, y) = g(x)$ for every x, y. Then

$$\|f\|_\square^4 = \mathop{\mathbb{E}}_{x_1,x_2} g(x_1)\overline{g(x_1)g(x_2)}g(x_2) = \|g\|_2^4.$$

So $\|f\|_\square = \|g\|_2$.

Corollary 3.3 (Box-norm inequality)**.** If $f : X \times Y \to \mathbb{C}$, $u : X \to \mathbb{C}$, $v : Y \to \mathbb{C}$, then

$$|\mathop{\mathbb{E}}_{x,y} f(x, y)u(x)v(y)| \leq \|f\|_\square \|u\|_2 \|v\|_2.$$

Proof. Apply the Box Cauchy-Schwarz to $f_1 = f$, $f_2(x, y) = u(x)$, $f_3(x, y) = v(y)$, $f_4(x, y) \equiv 1$, and use the remark above. □

Lemma 3.4. Let $f : X \times Y \to \mathbb{C}$, then $\|f\|_\square \geq |\mathbb{E}_{x,y} f(x, y)|$.

Proof.

$$\begin{aligned}
\|f\|_\square^4 &= \mathop{\mathbb{E}}_{x_1,x_2} \left| \mathop{\mathbb{E}}_{y} f(x_1, y)\overline{f(x_2, y)} \right|^2 \\
&\geq |\mathop{\mathbb{E}}_{x_1,x_2} \mathop{\mathbb{E}}_{y} f(x_1, y)\overline{f(x_2, y)}|^2 \\
&= \left| \mathop{\mathbb{E}}_{y} |\mathop{\mathbb{E}}_{x} f(x, y)|^2 \right|^2 \\
&\geq |\mathop{\mathbb{E}}_{y} \mathop{\mathbb{E}}_{x} f(x, y)|^4.
\end{aligned}$$

□

Lemma 3.5. Let $F : X \times Y \to \mathbb{R}$ be such that $\mathbb{E}_{x,y} F(x, y) = \delta > 0$ and $\|F\|_\square^4 \leq \delta^4(1+c)$. Let $f(x, y) = F(x, y) - \delta$. Then $\|f\|_\square \leq \dots$.

Proof. For each x, let $g(x) = \mathbb{E}_y f(x, y)$ and let $h(x, y) = f(x, y) - g(x)$. Then

$$\mathop{\mathbb{E}}_{x} g(x) = \mathop{\mathbb{E}}_{x,y} f(x, y) = \mathop{\mathbb{E}}_{x,y} F(x, y) - \delta = 0,$$

and for every x,

$$\mathop{\mathbb{E}}_{y} h(x, y) = g(x) - g(x) = 0.$$

Now,

$$\begin{aligned}\|F\|_\square^4 &= \mathop{\mathbb{E}}_{x_1,x_2}\left|\mathop{\mathbb{E}}_{y}(\delta+g(x_1)+h(x_1,y))(\bar{\delta}+\overline{g(x_2)}+\overline{h(x_2,y)})\right|^2\\ &= \mathop{\mathbb{E}}_{x_1,x_2}\left|\mathop{\mathbb{E}}_{y}(\delta+g(x_1))(\bar{\delta}+\overline{g(x_2)})+\mathop{\mathbb{E}}_{y}h(x,y)\overline{h(x_2,y)}\right|^2\end{aligned}$$

when we expand the modulus squared, we get three terms as follows. The first is

$$\begin{aligned}&\mathop{\mathbb{E}}_{x_1,x_2}(\delta+g(x_1))(\overline{\delta+g(x_1)})(\overline{\delta+g(x_2)})(\delta+g(x_2))\\ &=\delta^4+|\delta|^2\,\mathbb{E}\,|g(x_1)|^2+|\delta|^2\,\mathbb{E}\,|g(x_2)|^2+\mathbb{E}\,|g(x_1)|^2|g(x_2)|^2\\ &=(|\delta|^2+\|g\|_2^2)^2\end{aligned}$$

The second is

$$\mathop{\mathbb{E}}_{y}2\operatorname{Re}\mathop{\mathbb{E}}_{x_1,x_2}(\delta+g(x_1))(\overline{\delta+g(x_2)})\overline{h(x_1,y)}h(x_2,y)=\mathop{\mathbb{E}}_{y}2\operatorname{Re}\left|\mathop{\mathbb{E}}_{x}(\delta+g(x))\overline{h(x,y)}\right|^2\geq 0.$$

The third is $\|h\|_\square^4$.

So $\|F\|_\square^4\geq(|\delta|^2+\|g\|_2^2)^2+\|h\|_\square^4$. Therefore, if $\|F\|_\square^4\leq|\delta|^4(1+c)$, we have $\|h\|_\square^4\leq c|\delta|^4$ and

$$|\delta|^2+\|g\|_2^2\leq|\delta|^2(1+c)^{\frac{1}{2}}\leq|\delta|^2(1+\frac{c}{2})$$

so

$$\|g\|^2\leq\sqrt{\frac{c}{2}}|\delta|.$$

Therefore, $\|F-\delta\|_\square\leq|\delta|(c^{\frac{1}{4}}+(\frac{c}{2})^{\frac{1}{2}})$. □

Lemma 3.6. Let G be a bipartite graph of density δ with finite vertex sets X,Y. Then the following statements are 'equivalent' in the sense that if one holds for sufficiently small c_i, then the others hold for small c_j.

(i) $\|G\|_\square^4\leq\delta^4(1+c_1)$.

(ii) For each vertex $x\in X$, let $\delta(x)=\mathbb{E}_y G(x,y)$ = density of neighbourhood of x in Y, similarly for $y\in Y$. Let $\delta(x_1,x_2)=\mathbb{E}_y G(x_1,y)G(x_2,y)$ = the density of neighbourhood intersection. Similarly for $y_1.y_2\in Y$. Then

$$\mathbb{E}(\delta(x_1,x_2)-\delta^2)^2\leq c_2\delta^4.$$

(iii) For every $A\subset X$ of density α and $B\subset Y$ of density β,

$$\left|\mathop{\mathbb{E}}_{x,y}G(x,y)A(x)B(y)-\delta\alpha\beta\right|\leq c_3\delta.$$

Proof. (i) $\iff$ (ii). Observe first that $\|G\|_\square^4 = \mathbb{E}_{x_1,x_2} \delta(x_1,x_2)^2$. Also,

$$\mathbb{E}(\delta(x_1,x_2)-\delta^2)^2 = \mathop{\mathbb{E}}_{x_1,x_2} \delta(x_1,x_2)^2 - 2\delta^2 \mathop{\mathbb{E}}_{x_1,x_2} \delta(x_1,x_2) + \delta^4.$$

$$\mathbb{E}\,\delta(x_1,x_2) = \mathop{\mathbb{E}}_{y} \mathop{\mathbb{E}}_{x_1,x_2} G(x_1,y)G(x_2,y) = \mathop{\mathbb{E}}_{y} \delta(y)^2 \geq (\mathop{\mathbb{E}}_{y} \delta(y))^2 = \delta^2.$$

So if $\|G\|_\square^4 \leq \delta^4(1+c)$, then $\mathbb{E}(\delta(x_1,x_2)-\delta^2)^2 \leq \delta^4(1+c) - 2\delta^4 + \delta^4 = \delta^4 c$.

If $\mathbb{E}(\delta(x_1,x_2)-\delta^2)^2 \leq c\delta^4$, then $\mathbb{E}(\delta(x_1,x_2)-\delta^2) \leq c^{\frac{1}{2}}\delta^2$ so $\mathbb{E}\,\delta(x_1,x_2) \leq (1+c^{\frac{1}{2}})\delta^2$. Then

$$\|G\|_\square^4 = \mathop{\mathbb{E}}_{x_1,x_2} \delta(x_1,x_2)^2 \leq c\delta^4 + 2\delta^2(1+c^{\frac{1}{2}})\delta^2 - \delta^4 = \delta^4(1+c+2c^{\frac{1}{2}})$$

(i) $\implies$ (iii). Suppose (i). Then

$$|\mathop{\mathbb{E}}_{x,y} G(x,y)A(x)B(y) - \delta\alpha\beta| = |\mathop{\mathbb{E}}_{x,y} (G-\delta)(x,y)A(x)B(y)|.$$

But by Lemma 3.5, $\|G-\delta\|_\square \leq \delta(c_1^{\frac{1}{4}} + (\frac{c_1}{2})^{\frac{1}{2}})$ so by the box norm inequality, this is at most

$$\delta\left(c_1^{\frac{1}{4}} + \left(\frac{c_1}{2}\right)^{\frac{1}{2}}\right)\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \leq \delta\left(c_1^{\frac{1}{4}} + \left(\frac{c_1}{2}\right)^{\frac{1}{2}}\right).$$

(iii) $\Rightarrow$ (i). If G is regular (on both sides) then the proof is very short. If (i) is false, then

$$\mathop{\mathbb{E}}_{x_2,y_2} \mathop{\mathbb{E}}_{x_1,y_1} G(x_1,y_1)G(x_1,y_2)G(x_2,y_1)G(x_2,y_2) > \delta^4(1+c_1).$$

Since $\mathbb{P}[G(x_1,x_2)=1] = \delta$, it follows that there exist x_2, y_2 such that

$$\mathop{\mathbb{E}}_{x_1,y_1} G(x_1,y_1)G(x_1,y_2)G(x_2,y_1) > \delta^3(1+c_1).$$

Let $A = \{\, x \mid G(x,y_2) = 1 \,\}$, $B = \{\, y \mid G(x_2,y) = 1 \,\}$. Then A and B have density δ by regularity and

$$|\mathop{\mathbb{E}}_{x,y} G(x,y)A(x)B(y) - \delta^3| > \delta^3 c_1$$

Now consider the non-regular case. Assume (i) is false, so

$$\mathop{\mathbb{E}}_{x_2,y_2} \mathop{\mathbb{E}}_{x_1,y_1} G(x_1,y_1)G(x_1,y_2)G(x_2,y_1)G(x_2,y_2) > \delta^4 + \delta^4 c_1.$$

Now, either

$$\mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}} G(x_1,y_1)G(x_1,y_2)G(x_2,y_1)G(x_2,y_2) \geq \delta \mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}} G(x_1,y_2)G(x_2,y_1)G(x_2,y_2) + \frac{c}{2}\delta^4$$

or

$$\mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}} G(x_1,y_2)G(x_2,y_1)G(x_2,y_2) \geq \delta^3 + \frac{c}{2}\delta^3 = \delta \mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}} G(x_1,y_2)G(x_2,y_1) + \frac{c}{2}\delta^3.$$

In the first case, we can rewrite the inequality as

$$\mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}}(G(x,y_1)-\delta)G(x_1,y_2)G(x_2,y_1)G(x_2,y_2)\geq\frac{c\delta^4}{2}.$$

Since $G(x_2,y_2)=1$ with probability δ, it follows that there exist x_2,y_2 such that

$$\mathop{\mathbb{E}}_{x_1,y_1}(G(x_1,y_1)-\delta)G(x_1,y_2)G(x_2,y_1)\geq\frac{c\delta^3}{2}$$

Then, as before, take $A=N(y_2)$, $B=N(x_2)$ and we get

$$\left|\mathop{\mathbb{E}}_{x,y}G(x,y)A(x)B(y)-\delta\frac{|A|}{|X|}\frac{|B|}{|Y|}\right|\geq\frac{c\delta^3}{2}.$$

In the second case, we can rewrite the inequality as

$$\mathop{\mathbb{E}}_{\substack{x_1,x_2\\y_1,y_2}}G(x_1,y_2)G(x_2,y_1)\left(G(x_2,y_2)-\delta\right)\geq\frac{c\delta^3}{2}.$$

It follows that there exist x_1,y_1 such that

$$\mathop{\mathbb{E}}_{x_2,y_2}G(x_2,y_2)G(x_1,y_2)G(x_2,y_1)-\delta\mathop{\mathbb{E}}_{x_2,y_2}G(x_1,y_2)G(x_2,y_1)\geq\frac{c\delta^3}{2}.$$

This time set $A=N(y_1),B=N(x_1)$. □

Lemma 3.7 (Counting lemma)**.** Let G be a k-partite graph with vertex sets $X_1,\dots,X_k$. Write $G(X_i,X_j)$ for the induced bipartite subgraph with vertex sets (X_i,X_j). Suppose that the density of $G(X_i,X_j)$ is α_{ij} and that writing G_{ij} for the restriction of G to $X_i\times X_j$, $\|G_{ij}-\alpha_{ij}\|_\square\leq c$. Let H be a graph with vertex set $[k]$ and let $x_i\in X_i$ be chosen independently at random. Then

$$\left|\mathop{\mathbb{E}}_{x_1,\dots,x_k}\prod_{i,j\in E(H)}G_{ij}(x_i,x_j)-\prod_{ij\in E(H)}\alpha_{ij}\right|\leq 2^{|E(H)|}c.$$

Proof. Write $G_{ij}=f_{ij}+\alpha_{ij}$. Then $\|f_{ij}\|_\square\leq c$ for all i,j. Then

$$\mathbb{E}\prod_{ij\in E(H)}G_{ij}(x_i,x_j)=\mathbb{E}\prod_{ij\in E(H)}(\alpha_{ij}+f_{ij}(x_i,x_j)).$$

The main term is $\prod_{ij\in E(H)}\alpha_{ij}$ (when we expand the product into $2^{|E(H)|}$ terms). Every other term can be written in the form

$$\mathop{\mathbb{E}}_{x_i,x_j}f_{ij}(x_i,x_j)u(x_i)v(x_j)$$

with $\|u\|_\infty,\|v\|_\infty\leq 1$, if we fix all variables other than x_i and x_j. Therefore, by the box-norm inequality, it has size at most c, and this remains true after averaging over the other variables. □

3.2 Szemerédi's Regularity Lemma

Let X be a finite set and let $\mathcal{P}$ be a partition $X = X_1 \cap \cdots \cap X_k$. Given a function $f : X \to \mathbb{R}$, define $\mathbb{E}[f|\mathcal{P}]$ by

$$\mathbb{E}[f \mid \mathcal{P}](x) = \mathbb{E}[f(y) \mid y \text{ is in the same cell as } x]$$

Let us temporarily write $\mathcal{P}f$ for $\mathbb{E}[f \mid P]$.

(i) Note that $\mathbb{E}[\mathbb{E}[f \mid \mathcal{P}]] = \mathbb{E}f$

(ii)
$$\langle f, \mathcal{P}f \rangle = \mathbb{E}_x f(x) \mathbb{E}[f \mid \mathcal{P}](x) = \mathbb{E}\mathbb{E}[f \mid \mathcal{P}]^2 = \langle \mathcal{P}f, \mathcal{P}f \rangle.$$

Therefore, $\langle (1 - \mathcal{P})f, \mathcal{P}f \rangle = 0$.

(iii) Also, $\mathcal{P}(\mathcal{P}f) = \mathbb{E}[\mathbb{E}[f \mid P] \mid P] = \mathbb{E}[f \mid \mathcal{P}] = \mathcal{P}f$. So $\mathcal{P}$ is an orthogonal projection.

(iv) It follows that $\mathbb{E}f^2 = \|f\|_2^2 = \|\mathcal{P}f\|_2^2 + \|(1 - \mathcal{P})f\|^2 \geq \|\mathcal{P}f\|_2^2 = \mathbb{E}\mathbb{E}[f \mid \mathcal{P}]^2$

(v) Let $\mathcal{Q}$ be a refinement of $\mathcal{P}$, then

$$\mathbb{E}[\mathbb{E}[f \mid \mathcal{Q}] \mid \mathcal{P}] = \mathbb{E}[f \mid \mathcal{P}]$$

It follows from (iv) applied to $\mathbb{E}[f \mid \mathcal{Q}]$ that

$$\mathbb{E}\mathbb{E}[f \mid \mathcal{Q}]^2 \geq \mathbb{E}\mathbb{E}[f \mid \mathcal{P}]^2.$$

Definition. Let G be a bipartite graph with (finite) vertex sets X, Y. Let $A \subset X$, $B \subset Y$. Then the density is

$$\mathop{\mathbb{E}}_{\substack{x \in A \\ y \in B}} AG(x, y).$$

We shall say that (A, B) is ϵ-regular if $\forall A' \subset A,\ B' \subset B$,

$$\left| \mathop{\mathbb{E}}_{\substack{x \in A \\ y \in B}} G(x, y) A'(x) B'(y) - d(A, B) \mathop{\mathbb{E}}_{\substack{x \in A \\ y \in B}} A'(x) B'(y) \right| \leq \epsilon.$$

Note that this is one of the quasirandomness conditions on the subgraph of G induced by A' and B'.

Theorem 3.8 (Szemerédi's Regularity Lemma)**.** Let G be a bipartite graph with vertex sets X, Y and let $\epsilon > 0$. Then there exist partitions $X = X_1 \cup \cdots \cup X_r$, $Y = Y_1 \cup \cdots \cup Y_s$ with $r, s \leq k(\epsilon)$ such that if you choose $(x, y) \in X \times Y$ at random then the probability that it belongs to a ϵ-regular pair (X_i, Y_j) is at least $1 - \epsilon$.

Definition. Given partitions $X_1 \cup \cdots \cup X_r$ of X and $Y_1 \cup \cdots \cup Y_s$ of Y, let $\mathcal{P}$ be the partition of $X \times Y$ into the sets $X_i \times Y_j$. The **mean square density** of G with respect to $X_1 \cup \cdots \cup X_r$, $Y_1 \cup \cdots \cup Y_s$ is $\mathbb{E}(\mathbb{E}[G \mid \mathcal{P}])^2$.

That is, it is the expectation of $d(X_i, Y_j)^2$ where (X_i, Y_j) is the pair containing a random edge (x, y).

Lemma 3.9. Let $A \subset X, B \subset Y$ and suppose that (A, B) is not ϵ-regular. Then there are partitions $A = A_0 \cup A_1$, $B = B_0 \cup B_1$ such that the mean-square density of $G|_{A\times B}$ is at least $d(A,B)^2 + \epsilon^2$.

Proof. By hypothesis, we can find $A_0 \subset A$, $B_0 \subset B$ such that

$$\left| \mathbb{E}_{\substack{x\in A\\ y\in B}} (G(x,y) - d(A,B))A_0(x)B_0(y) \right| \geq \epsilon.$$

Set $A_1 = A \setminus A_0$, $B_1 = B \setminus B_0$. Then

$$\begin{aligned}\mathbb{E}(\mathbb{E}[G \mid \mathcal{P}]^2) &= (\mathbb{E}\mathbb{E}[G \mid \mathcal{P}])^2 + \mathrm{Var}(\mathbb{E}[G \mid \mathcal{P}]) \\ &= d(A,B)^2 + \mathbb{E}(\mathbb{E}[G \mid \mathcal{P}] - d(A,B))^2\end{aligned}$$

But

$$\begin{aligned}\mathbb{E}(\mathbb{E}[G \mid \mathcal{P}] - d(A,B))^2 &\geq \mathbb{P}[x \in A_0, y \in B_0]\left(d(A_0,B_0) - d(A,B)\right)^2 \\ &= \frac{|A_0||B_0|}{|A||B|}\frac{1}{|A_0||B_0|}\left(\sum_{\substack{x\in A_0\\ y\in B_0}} (G(x,y)-d(A,B))^2\right) \\ &\geq \left(\sum_{\substack{x\in A\\ y\in B}} (G(x,y) - d(A,B))^2\right) \geq \epsilon^2.\end{aligned}$$

□

Lemma 3.10. Suppose that the partitions $X_1 \cup \cdots \cup X_r$, $Y_1 \cup \cdots \cup Y_s$ do not satisfy the conclusion of Theorem 3.8. Suppose also that $r, s \leq m$. Then there are refinements of the partitions into at most $m \cdot 2^m$ sets each, such that the mean square density goes up by at least ϵ^3.

Proof. For each pair (X_i, Y_j) that is not ϵ-regular, find a partition $X_i = X_{ij}^0 \cup X_{ij}^1$, $Y_j = Y_{ji}^0 \cup Y_{ji}^1$ with respect to the mean-square identity of $G|_{X_i\times Y_j}$ is at least $d(X_i,Y_j)^2 + \epsilon^2$. For each i take a common refinement of the partitions $X_{ij}^0 \times X_{ij}^1$ into at most $m2^m$ sets. Similarly for Y_j.

Pick a random edge x, y and suppose that it belongs to (X_i, Y_j). If (X_i, Y_j) is not ϵ-regular (which happens with probability $\geq \epsilon$), then the mean square density of the cell of the refined partition is at least $d(X_i,Y_j)^2 + \epsilon^2$, because inside (X_i, Y_j) the refined partition refines the partitions $X_{ij}^0 \cup X_{ij}^1$, $Y_{ji}^0 \cup Y_{ji}^1$. Otherwise, it is at least $d(X_i,Y_j)^2$, so the result follows. □

Proof of Szemerédi's Regularity Lemma. Start with the trivial partition of X and Y. If those don't work, apply Lemma 3.10 repeatedly. Since mean-square density is ≤ 1, there can be at most ϵ^{-3} iterations before partitions are found that work. □

This gives an upper bound on k obtained by iterating the function $m \mapsto m2^m$ ϵ^{-3} times. Since $m2^m \leq 4^m$, this gives a bound of

$$4^{4^{4^{\cdot^{\cdot^{\cdot^{4}}}}}}$$

where the tower has height ϵ^{-3}.

Theorem 3.11 (The triangle-removal lemma)**.** For every $\epsilon > 0$ there exists $\delta > 0$ such that if G is any graph with n vertices and at most δn^3 triangles then there is a subgraph $H \subset G$ such that $G \setminus H$ contains at most ϵn^2 edges and H is triangle-free.

Before we prove this, a quick remark. The **graphs** version (as opposed to bipartite graphs version) of Szemerédi's regularity lemma is essentially the same but with just one partition of $V(G)$. To prove it, form a bipartite graph with two copies of $V(G)$ and run the previous argument, but at each iteration do a further refinement to keep the two partitions 'the same'.

Proof. Apply the regularity lemma with parameter θ. It gives us a partition of the vertex set into sets $V_1, \dots, V_k$. Throw away

- all edges with a vertex in some V_i of density $\frac{\epsilon}{6k}$
- all edges joining V_i to V_j if $d(V_i, V_j) < \frac{\epsilon}{3}$.
- all edges joining V_i to V_j if (V_i, V_j) is not a θ-regular pair.

As long as $\theta \le \frac{\epsilon}{3}$, the total density of removed edges is at most

$$2k \cdot \frac{\epsilon}{6k} + \frac{\epsilon}{3} + \theta \le \epsilon.$$

Now suppose that after these deletions, there is still a triangle. Let its vertices belong to V_i, V_j, V_k (not necessarily distinct). Then the pairs (V_i, V_j), (V_j, V_k), (V_i, V_k) are all θ-regular of density $\delta \ge \frac{\epsilon}{3}$. So

$$\begin{aligned}
&|\mathop{\mathbb{E}}_{\substack{x \in V_i \\ y \in V_j \\ z \in V_k}} G(x,y)G(y,z)G(x,z) - \delta^3| \\
&\qquad \le |\mathop{\mathbb{E}}_{x,y,z} (G(x,y) - \delta)G(y,z)G(x,z)| + \delta |\mathop{\mathbb{E}}_{x,y,z} (G(y,z) - \delta)G(x,z)| \\
&\qquad \le \theta + \delta\theta \le 2\theta
\end{aligned}$$

So the number of triangles in G is at least $|V_i||V_j||V_k| \cdot ((\frac{\epsilon}{3})^3 - 2\theta)$ which is at least

$$\left(\frac{\epsilon}{6K}\right)^2 \frac{\epsilon^3}{54} n^3$$

if $\theta \le \frac{\epsilon}{108}$.

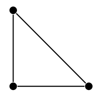
So taking $\delta = \left(\frac{\epsilon}{6K}\right)^2 \frac{\epsilon^3}{54}$ gives the result. □

The idea of this proof is fundamentally that having one triangle gives many triangles, once the deletions have been made.

Theorem 3.12 (The corners theorem)**.** For every $\delta_0 > 0$, there exists n such that every subset A of $[n]^2$ of density $\delta \ge \delta_0$ contains a triple

$$(x, y), (x + d, y), (x, y + d)$$

with $d \ne 0$.



Proof. Form a tripartite graph with vertex sets $X = Y = [n]$, $Z = [2n]$, and edges:

- Join $x \in X$ to $y \in Y$ if $(x, y) \in A$.
- Join $x \in X$ to $z \in Z$ if $(x, z - x) \in A$.
- Join $y \in Y$ to $z \in Z$ if $(z - y, y) \in A$.

If (x, y, z) is triangle, then, writing $d = z - x - y$, we have

$$(x, y) \in A \quad (x, z - x) = (x, y + d) \in A \quad (z - y, y) = (x + d, y) \in A.$$

So if A contains no corners, then the only triangles are 'degenerate' triangles where $x+y = z$, so $d = 0$. So the number of triangles is at most δn^2.

But the degenerate triangles are edge-disjoint, since any two of x, y, z determine the third if $x + y = z$. So the number of edges you have to remove to get one of all triangles is at least δn^2. So the number of triangles is exactly $\delta n^2 = o(n^3)$. For sufficiently large n, this contradicts the triangle removal lemma. $\square$

4 The Polynomial Method

On the example sheet, we proved Roth's theorem in $\mathbb{F}_3^n$, with a bound $\frac{C}{n}$. This was beaten massively by a bound c^n, and we present the proof now.

Definition (Slice rank)**.** Let X be a finite set and let $f : X^3 \to \mathbb{F}$ for some field $\mathbb{F}$. The **slice rank** of f is the smallest k such that there are functions $u_1, \ldots, u_k : X \to \mathbb{F}$, $v_1, \ldots, v_k : X^2 \to \mathbb{F}$, k_1, k_2 such that

$$f(x,y,z) = \sum_{i=1}^{k_1} u_i(x)v_i(y,z) + \sum_{i=k+1}^{k_2} u_i(y)v_i(x,z) + \sum_{i=k_2+1}^{k} u_i(z)v_i(x,y).$$

Lemma 4.1. Let X be a finite set, let $A \subset X$ and let $f : X^3 \to \mathbb{F}$ be a function such that for $x, y, z \in A$, $f(x,y,z) \neq 0$ iff $x = y = z \in A$. Then f has slice rank $|A|$.

Proof. We can write $f(x,y,z)$ as

$$\sum_{a \in \Lambda} \lambda_a \delta_a(x)\delta_a(y)\delta_a(z)$$

so the slice rank is at most $|A|$. Now suppose that

$$f(x,y,z) = \sum_{i=1}^{k} k_1 u_i(x)v_i(y,z) + \sum_{i=k_1+1}^{k_2} u_i(y)v_i(x,z) + \sum_{i=k_2+1}^{k} u_i(z)v_i(x,y).$$

Let V be the subspace of $\mathbb{F}^*$ that consists of functions orthogonal to all of $u_1, \ldots, u_{k_1}$. That is, $h \in V$ iff

$$\sum_x h(x)u_i(x) = 0$$

for all $i \leq k_1$. Then V has codimension at most k_1, it follows that V contains some function h that vanishes at most k_1 times.

To see this, form an $m \times n$ matrix whose rows are a basis for V (so $m \geq n - k_1$). Put the matrix into reduced row echelon form. Add up the rows to get the desired function. Now define $g : X^2 \to \mathbb{F}$ by

$$g(y,z) = \sum_x f(x,y,z)h(x).$$

Then $g(y,z) \neq 0$ iff $y = z \in A$ and $h(y) \neq 0$. This holds at least $|A| - k_1$ times, so g has rank $\geq |A| - k_1$.

But also, $g(y,z)$ has a formula of the formula of the form

$$\sum_{i=k_1+1}^{k_2} u_i(y)w_i(z) + \sum_{k_2+1}^{k} u_i(z)w_i(y)$$

for some functions w_i (if $k_1 < i \leq k_2$ then $w_i(z) = \sum_x u_i(x,z)h(x)$). Therefore, g has rank at most $k - k_1$. Therefore, $|A| \leq k$. $\square$

Let $F(n)$ be the number of $\{0,1,2\}$ sequences of length n that at up to at most $\frac{2n}{3}$.

Lemma 4.2. Let A be a subset of $\mathbb{F}_3^n$ such that if $x, y, z \in A$ and $x + y + z = 0$ then $x = y = z$. Then $|A| \le 3F(n)$.

Proof. The function
$$\sum_{a \in A} \delta_a(x)\delta_a(y)\delta_a(z)$$
on A^3 can be expressed by the formula
$$f(x, y, z) = \sum_{i=1}^{n} (1 - (x_i + y_i + z_i)^2).$$
This is a polynomial in $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n$ with each x_i, y_i, z_i occurring with degree $0, 1$ or 2. Also, it has total degree $2n$. It is therefore a linear combination of monomials, and in each one, either x variables or y variables or z variables occur with total degree at most $\frac{2n}{3}$. Partition the monomials into three sets accordingly.

It follows that we can write
$$\sum_{a \in A} \delta_a(x)\delta_a(y)\delta_a(z)$$
as
$$\sum_{i=1}^{m_1} u_i(x)v_i(y, z) + \sum_{i=m_1+1} m_2 u_i(y)v_i(x, z) + \sum_{i=m_2+1}^{m} u_i(z)v_i(x, y)$$
where the u_i are monomials of total degree at most $\frac{2n}{3}$ and the v_i are polynomials (in $2n$ variables). The number of monomials in $x_1, \dots, x_n$ such that each x_i occurs with degree $0, 1$ or 2 and the total degree is $\le \frac{2n}{3}$ is $F(n)$ so $|A| \le 3F(n)$ by Lemma 4.1. □

Lemma 4.3.
$$F(n) \le e^{-\frac{n}{12}} \cdot 3^n.$$

Proof. Let x be a random sequence in $\{0, 1, 2\}^n$ and let $X_1, \dots, X_n$ be independent random variables uniformly distributed in $\{-1, 0, 1\}$. Then
$$\begin{aligned}
\mathbb{P}\left[\sum x_i \le \frac{2n}{3}\right] &= \mathbb{P}\left[\sum_{i=1}^{n} X_i \ge \frac{n}{3}\right] \\
&= \mathbb{P}\left[e^{\lambda \sum_{i=1}^{n} X_i} \ge e^{\lambda \frac{n}{3}}\right] \\
&\le e^{-\frac{\lambda n}{3}} \mathbb{E} e^{\lambda \sum_{i=1}^{n} X_i} \\
&= e^{-\frac{\lambda n}{3}} \prod_{i=1}^{n} (\mathbb{E} e^{\lambda x_i}) \\
&= e^{-\frac{\lambda n}{3}} \left(\frac{1 + e^{\lambda} + e^{-\lambda}}{3}\right)^n.
\end{aligned}$$

But

$$\begin{aligned}\frac{1+e^{\lambda}+e^{-\lambda}}{3} &= 1+\frac{2}{3}\frac{\lambda^2}{2!}+\frac{2}{3}\frac{\lambda^4}{4!}+\cdots \\ &= 1+\frac{\lambda^2}{3}+\left(\frac{\lambda^2}{3}\right)^2\frac{1}{2!}+\left(\frac{\lambda^2}{3}\right)^3\frac{1}{3!}+\cdots \\ &= e^{\frac{\lambda^2}{3}}\end{aligned}$$

So the probability is at most $e^{-\frac{\lambda n}{3}+\frac{\lambda^2 n}{3}}$. By choosing $\lambda = \frac{1}{2}$ gives an upper bound of $e^{-\frac{n}{12}}$. ☐

Theorem 4.4. Let $A \subset \mathbb{F}_3^n$ contain no nontrivial solutions to $x+y+z=0$. Then $|A| \leq e^{-\frac{n}{12}} \cdot 3^n$.

Our next target is the finite-fields Kakeya theorem of Zeev Dvir.

Lemma 4.5. Let $A \subset \mathbb{F}_p^N$. If $|A| < \binom{n+d}{n}$ then there exists a non-zero polynomial of degree at most d that vanishes everywhere on A.

Proof. A polynomial P of degree $\leq d$ has a formula

$$P(x) = \sum_{|E|\leq d} \lambda_E \prod_{i\in E} x_i$$

where E is a multiset. So for P to vanish on A, we need

$$\sum_{|E|\leq d} \lambda_E \prod_{i\in E} a_i$$

to be zero for every $a \in A$. This gives $|A|$ linear equations that the coefficients λ_E must solve. Given a subset $m_1 < m_2 < \cdots < m_n$ of $[n+d]$, we can define a monomial $x_1^{r_1}\cdots x_n^{r_n}$ of degree $\leq d$ by setting $r_1 = m_1 - 1, r_i = m_i - m_{i-1} - 1$ for $i > 1$, since $r_1 + \cdots + r_n = m_n - n \leq d$. Moreover, this is a bijection. So the number of monomials is $\binom{n+d}{n}$. So if $|A| < \binom{n+d}{n}$ we get a non-trivial choice for the λ_t. ☐

Lemma 4.6. Let P be a polynomial of degree $< p$ that vanishes everywhere on $\mathbb{F}_p^n$. Then P is the zero polynomial.

Proof. If $n = 1$, then P has p roots and degree $< p$, so it must be the zero polynomial. If $n > 1$, then we can write

$$P(x) = P_0(x_2,\ldots,x_n) + x_1 P_1(x_2,\ldots,x_n) + x_1^2 P_2(x_2,\ldots,x_n) + \cdots + x_1^{p-1} P_{p-1}(x_2,\ldots,x_n).$$

For every choice of $x_2,\ldots,x_n$, the RHS vanishes for all x_1 so is the zero polynomial and therefore all the P_i vanish everywhere. Hence by induction, the P_i are all the zero polynomial. ☐

Theorem. Let $A \subset \mathbb{F}_p^n$ be a set and suppose that for every $\mathbf{d} \neq 0$ there exists $\mathbf{a}$ such that the set

$$\{\, \mathbf{a} + \lambda \mathbf{d} \mid \lambda \in \mathbb{F}_p \,\} \subset A.$$

Then $|A| \geq \frac{p^n}{n!}$.

Proof. By Lemma 4.5, if $|A| < \binom{n+p-1}{n}$ then there is a non-zero polynomial Q of degree $< p$ that vanishes on A. Let $\deg Q = m$ and let Q_m be the homogeneous polynomial consisting of the degree m terms of Q.

For each $\mathbf{d} \neq 0$ we know that Q vanishes on some line $\{\mathbf{a} + \lambda\mathbf{d}\}$. But Q restricted to $\{\mathbf{a} + \lambda\mathbf{d}\}$ is a polynomial in λ of degree $< p$, so it is the zero polynomial. The degree-m coefficient of this restriction is $Q_m(\mathbf{d})$ (since 'to get λ^m we must choose λd_i from each bracket'). So $Q_m(\mathbf{d}) = 0$ for every $\mathbf{d} = 0$. Trivially $Q_m(\mathbf{0}) = 0$ too. So Q_m is the zero polynomial. Therefore

$$|A| \geq \binom{n+p-1}{n} = \frac{(n+p-1)(n+p-2)\cdots p}{n!} \geq \frac{p^n}{n!}. \qquad \square$$

5 A taste of higher-order Fourier analysis

Let $k \geq 1$. Given finite sets $X_1, \ldots, X_k$ and a function $f : X_1 \times \cdots \times X_k \to \mathbb{C}$, we define the k-dimensional box-norm $\|f\|_\square$ of f by the formula

$$\|f\|_\square^{2^k} = \mathop{\mathbb{E}}_{\substack{x_1^0,\ldots,x_k^0 \\ x_1^1,\ldots,x_k^1}} \prod_{\epsilon\in\{0,1\}^k} C^{|\epsilon|} f(x_1^{\epsilon_1}, \ldots, x_k^{\epsilon_k})$$

where $|\epsilon| = \epsilon_1 + \cdots + \epsilon_k$ and C is complex conjugation. It can be shown that this is a norm.

5.1 The U^k norms

Let G be an Abelian group and let $f : G \to \mathbb{C}$. Then we define $\|f\|_{U^2}$ by the formula

$$\begin{aligned} \|f\|_{U^2}^4 &= \mathop{\mathbb{E}}_{x,a,b} f(x)\overline{f(x+a)f(x+b)}f(x+a+b) \\ &= \mathop{\mathbb{E}}_{x+y=z+w} f(x)f(y)\overline{f(z)f(w)} \\ &= \langle f * f, f * f\rangle = \|f \times f\|_2^2 = \|\hat{f}^2\|^2 = \sum_\chi |\hat{f}(\chi)|^4 \end{aligned}$$

We also define $\|r\|_{U^3}$ by the formula

$$\|f\|_{U^3}^8 = \mathop{\mathbb{E}}_{x,a,b,c} f(x)\overline{f(x-a)f(x-b)}f(x-a-b)\overline{f(c)}f(x-a-c)f(x-b-c)\overline{f(x-a-b-c)}.$$

If $F(x,y,z) = f(x+y+z)$, then $\|f\|_{U^3} = \|F\|_\square$. (Exercise). In fact, if r, s, t are coprime to the order of G, then the same is true of $F(x,y,z) = f(rx+sy+tz)$. The 3d box norm satisfies the box-norm inequality (from sheet 3)

$$\left| \mathop{\mathbb{E}}_{x,y,z} f(x,y,z)u(x,y)v(x,z)w(y,z) \right| \leq \|f\|_\square \|u\|_\infty \|v\|_\infty \|w\|_\infty.$$

Lemma 5.1. Let G be a finite Abelian group with order not divisible by 2 or 3, and let $f_1, f_2, f_3, f_4 : G \to \mathbb{C}$. Then

$$\left| \mathop{\mathbb{E}}_{x,d} f_1(x)f_2(x+d)f_3(x+2d)f_4(x+3d) \right| \leq \|f_1\|_{U^3} \|f_2\|_\infty \|f_3\|_\infty \|f_4\|_\infty.$$

Proof.

$$\begin{aligned}\text{LHS} &= |\mathop{\mathbb{E}}_{x,y,z} f_1(-3x-2y-z)f_2(-2x-y)f_3(-x+z)f_4(y+2z)| \\ &\leq \|f_1\|_{U^3}\|f_2\|_\infty\|f_3\|_\infty\|f_4\|_\infty\end{aligned}$$

by the previous remarks. □

Corollary 5.2. Let A, B be subsets of a finite Abelian group G with order coprime to 2 and 3. Let A have density α, B have density β. Suppose that $f = A - \alpha$ has the property that $\|f\|_{U^3} \leq c$. Then $|\mathbb{E}_{x,d} A(x)A(x+d)B(x+2d)B(x+3d) - \alpha^2\beta^2| \leq 2c$.

Proof. Let $B = \beta + g$. Then

$$\mathop{\mathbb{E}}_{x,d} A(x)A(x+d)B(x+2d)B(x+3d) = \mathop{\mathbb{E}}_{x,d}(\alpha + f(x))(\alpha + f(x+d))(\beta + g(x+2d))(\beta + g(x+3d)).$$

This splits into 16 terms, one of which is $\alpha^2\beta^2$. Of the remaining terms, we have a contribution of

$$\mathop{\mathbb{E}}_{x,d} f(x)A(x+d)B(x+2d)B(x+3d)$$

from the terms where we pick $f(x)$ from the first bracket. By lemma 1, this is at most c. If we choose α from the first bracket and $f(x+d)$ from the second, we get a contribution $\alpha\,\mathbb{E}_{x,d} f(x+d)B(x+2d)B(x+3d) = \alpha\,\mathbb{E}_{x,d} f(x)B(x+d)B(x+2d)$, which also has magnitude at most c. All other contributions are zero. □

If $\|f\|_{U^3}$ is small, then A contains roughly the right number of 4APs. What happens if $\|f\|_{U^3}$ is large?

Definition. Let G be a finite abelian group, $f : G \to \mathbb{C}$, $a \in G$. Write $\partial_a f$ for the function $\partial_a f(x) = f(x)\overline{f(x-a)}$.

This is in some sense a multiplicative discrete derivative: if $f(x) = \omega^{\phi(x)}$ then $\partial_a f(x) = \omega^{\phi(x)-\phi(x-a)}$.

Observe that

$$\begin{aligned}\|f\|_{U^3}^8 &= \mathop{\mathbb{E}}_{x,a,b,c} f(x)\overline{f(x-a)}\cdots\overline{f(x-a-b-c)} \\ &= \mathop{\mathbb{E}}_{x,a,b,c} \partial_a f(x)\overline{\partial_a f(x-b)\partial_a f(x-c)}\partial_a(f-b-c) \\ &= \mathop{\mathbb{E}}_{a} \|\partial_a f\|_{U^2}^4 \\ &= \mathop{\mathbb{E}}_{a}\sum_r |\widehat{\partial_a f}(r)|^4 \leq \mathop{\mathbb{E}}_{a}\max_r \left|\widehat{\partial_a f}(r)\right|^2 \sum_r |\widehat{\partial_a f}(r)|^2.\end{aligned}$$

So if $\|f\|_{U^3}^8 \geq c$, $\mathbb{E}_a \max_r \left|\widehat{\partial_a f}(r)\right|^2 \geq c$.

$$\implies \exists B \text{ such that } |B| \geq \frac{c}{2}|G| \text{ and } \max_r \left|\widehat{\partial_a f}(r)\right|^2 \geq \frac{c}{2} \text{ for every } a \in B.$$

$$\implies \exists B, \varphi : B \to \hat{G} \text{ such that } \mathop{\mathbb{E}}_{a} B(a)\left|\widehat{\partial_a f}(\phi(a))\right|^2 \geq \frac{c^2}{4}.$$

So

$$
\begin{aligned}
\frac{c^2}{4} &\le \mathop{\mathbb{E}}_a B(a) \left|\widehat{\partial_a f}(\phi(a))\right|^2 \\
&= \mathop{\mathbb{E}}_a B(a) \mathop{\mathbb{E}}_{x,y} f(x)\overline{f(x-a)f(y)}f(y-a)\omega^{-\phi(a)\cdot(x-y)} \\
&= \mathop{\mathbb{E}}_{b,x,z} f(x)\overline{f(z)f(x-b)}f(z-b)\omega^{-\phi(x-z)\cdot b)}B(x-z) \\
&\le \mathop{\mathbb{E}}_b \|F_b\|_\square \qquad \text{where } F_b(x,y) = \omega^{-b\cdot\phi(x-z)}B(x-z) \\
&\le (\mathop{\mathbb{E}}_b \|F_b\|_\square^4)^{\frac{1}{4}} \\
&= \left(\mathop{\mathbb{E}}_b \mathop{\mathbb{E}}_{\substack{x_1,x_2 \\ z_1,z_2}} B(x_1-z_1)B(x_1-z_2)B(x_2-z_1)B(x_2-z_2)\omega^{-b\cdot(\phi(x_1-z_1)-\phi(x_1-z_2)-\phi(x_2-z_1)+\phi(x_2-z_2))}\right)^{\frac{1}{4}} \\
&= \left(\mathop{\mathbb{E}}_{t+u=v+w} B(t)B(v)B(w)B(u) \underbrace{\mathop{\mathbb{E}}_b \omega^{-b\cdot(\phi(t)+\phi(u)-\phi(v)+\phi(w))}}_{=\mathbb{1}_{\phi(t)+\phi(u)=\phi(v)+\phi(w)}}\right)^{\frac{1}{4}} \\
&= \mathbb{P}_{t+u=v+w}\left[t,u,v,w \in B \text{ and } \phi(t)+\phi(u)=\phi(v)+\phi(w)\right].
\end{aligned}
$$

Equivalently, if $\Gamma = \{\, (x,\phi(x)) : x \in B \,\}$ is the graph of ϕ, then Γ contains at least $\frac{c^8}{256}|\Gamma|^3$ additive quadruples. So by BSG, Γ has a large subset Γ' such that $|\Gamma' - \Gamma'| \le C|\Gamma'|$. Then by the proof of Freiman's theorem, $2\Gamma' - 2G'$ contains a subset isomorphic to a low dimensional Bohr set B. But low dimensional Bohr sets contain longish arithmetic APs. So $2\Gamma' - 2\Gamma'$ contains a fairly long AP P.

Let $X \subset \Gamma'$ be maximal such that sets $x + P$, $x \in X$ are disjoint. Then $|X||P| = |X + P| \le |3\Gamma' - 2\Gamma'| \le C^5|\Gamma'|$. Also, $\Gamma' \subset X + P - P$ by maximality. So Γ' is contained in the union of $\le \frac{C^5|\Gamma'|}{|P|}$ translates of P. So one of these translates contains at least $\frac{|P|}{C^5}$ points of Γ'.

Now let's oversimplify and assume that $B = \mathbb{Z}_N$ and $\phi(x) = 2\lambda x$ for every x. Then

$$c \le \mathop{\mathbb{E}}_a |\widehat{\partial_a f}(2\lambda a)|^2 = \mathop{\mathbb{E}}_{a,b} f(x)\overline{f(x-a)f(x-b)}f(x-a-b)\omega^{-2\lambda ab}$$

But $2ab = x^2 - (x-a)^2 - (x-b)^2 + (x-a-b)^2$, so the right hand side is

$$= \mathop{\mathbb{E}}_{a,x,b} g(x)\overline{g(x-a)g(x-b)}g(x-a-b)$$

where $g(x) = f(x)\omega^{\lambda x^2}$.

$$= \|g\|_{U^2}^4 = \sum_r |\hat{g}(r)|^4 \le \max_r |\hat{g}(r)|^2$$

So $\exists r$ such that $c^{\frac{1}{2}} \le |\hat{g}(r)| = \left|\mathbb{E}_x f(x)\omega^{\lambda x^2 - rx}\right|$.

Claim: for all r, we can find $d \le m$ such that ω^{rd^2} is close to 1.

Divide the circle into arcs of length $\leq \epsilon$. By van der Waerden's theorem we can find $x-d, x, x+d$ such that $\frac{(x-d)^2}{2}, \frac{x^2}{2}, \frac{(x+d)^2}{2}$ all lie in the same interval. But then $d^2 = \frac{(x-d)^2}{2} - 2 \cdot \frac{x^2}{2} + \frac{(x+d)^2}{2}$ is small, so $\omega^{d^2} \approx 1$.