# Part III – Ramsey Theory (Incomplete)

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# 0 Introduction

*Lecture 1* If you liked Graph Theory, you'll almost certainly like Ramsey Theory. If you didn't like Graph Theory, you probably won't like Ramsey Theory. Ramsey theory is an unusual part of maths, in that it's all about answering one question. The basic question is:

*Can we find some order in enough disorder?*

As usual in discrete mathematics, the key ideas of the course are in the proofs rather than in the definitions.

The course is structured into three sections.

**Chapter 1:** Monochromatic systems (abstract and concrete)

**Chapter 2:** Partition regular equations (concrete)

**Chapter 3:** Infinite Ramsey Theory (abstract)

There are not many prerequisites to this course, only basic concepts of topology (compact spaces).

No single book covers all of the course, but there are two books which cover the relevant content:

- Bollobás, *Combinatorics*, C.U.P., 1986 (for chapter 3). An excellent survey of the material.
- Graham, Rothschild, Spencer, *Ramsey Theory*, Wiley, 1990 (for chapters 1,2).

As well as lots of nice proofs in the area, there are many open problems we will come across.

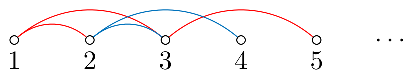
# 1 Monochromatic systems

## 1.1 Ramsey's Theorem

Write $\mathbb{N} = \{1, 2, 3, \dots\}$, and write $[n]$ for $\{1, \dots, n\}$. For any set $X$, write

$$X^{(r)} = \{A \subseteq X : |A| = r\}.$$

Suppose we have the natural numbers listed, and each pair of naturals is connected by an edge coloured either red or blue.




Formally, we have a 2-colouring $c$ of $\mathbb{N}^{(2)}$, i.e. $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Can we always find an infinite set $M$ that is **monochromatic**, i.e. $c$ is constant on $M^{(2)}$?

**Example.**

(i) Colour $ij$ red if $i+j$ even and blue if it is odd. Then we can find an $M$ that works, by using the evens.

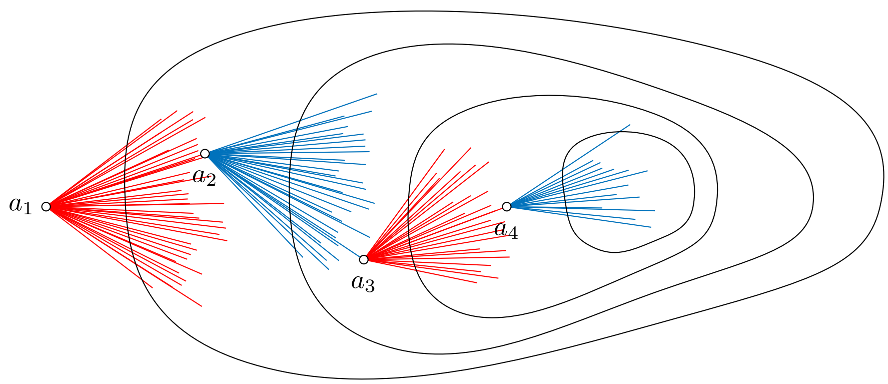
(ii) Colour $ij$ red if $\max\{n : 2^n \mid i+j\}$ even, and blue otherwise.
Again yes, we can use $M = \{4^0, 4^1, 4^2, \dots\}$ or $M = \{x : x \equiv 1 \pmod 4\}$.

(iii) Colour $ij$ red if $i+j$ has an even number of (distinct) prime factors, and blue if odd. Now the answer is less clear...

It turns out that the answer is always yes.

**Theorem 1.1** (Ramsey's Theorem)**.** Let $c$ be a 2-colouring of $\mathbb{N}^{(2)}$. Then $c$ has an infinite monochromatic set.

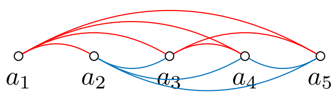
*Proof.*




Pick $a_1 \in \mathbb{N}$. There are infinitely many edges from $a$, so there is an infinite set $B_1 \subseteq \mathbb{N} - \{a_1\}$ such that all edges from $a_1$ to $B_1$ have the same colour, say $C_1$.

Pick $a_2 \in B_1$. There are infinitely maby edges from $a_2$ inside $B_1$, so there is an infinite set $B_2 \subset B_1 - \{a_2\}$ such that all edges from $a_2$ to $B_2$ have same colour, say $C_2$.

Continue inductively. We obtain distinct points $a_1, a_2, \dots$ and colours $C_1, C_2, \dots$ such that $a_i a_j$ (for $i < j$) has colour $C_i$.




We must have $C_{i_1} = C_{i_2} = C_{i_3} = \cdots$ for some $i_1 < i_2 < \cdots$ (as there are only two colours), so $\{a_{i_1}, a_{i_2}, \dots\}$ is monochromatic. $\square$

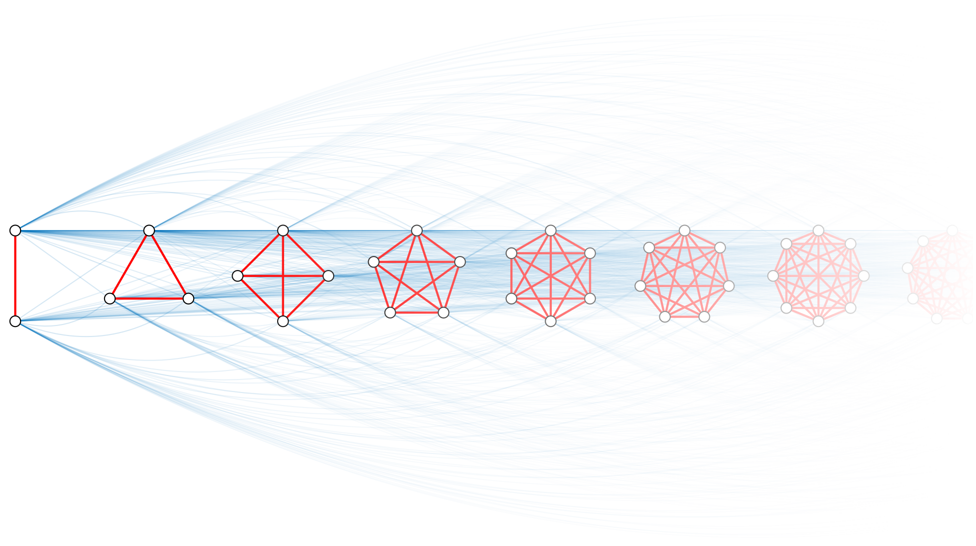
**Remark.**

(i) This is called a two-pass proof.

(ii) In example 3, no explicit example is known.

(iii) What about a $k$-colouring? (i.e. $c : \mathbb{N}^{(2)} \to [k]$). The same proof would show there is an infinite monochromatic set. Alternatively, we can deduce this from Ramsey's Theorem, by 'turquoise spectacles': view our colouring as a 2-colouring by colours '1' and '2 or 3 or ... or $k$' and apply Ramsey's Theorem and induction.

(iv) Asking for an infinite monochromatic set is much more than asking for arbitrarily large finite monochromatic sets, e.g. in



we have no infinite red set, but arbitrarily large finite red sets.

**Example.** Any sequence $x_1, x_2, \dots$ in $\mathbb{R}$ (or in any totally ordered set) has a monotone subsequence. Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving $ij$ (for $i < j$) colour up if $x_i < x_j$ and colour down if $x_i \geq x_j$ and apply Ramsey's Theorem.

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