

Part III – Topics in Set Theory (Unfinished course)

Based on lectures by Professor B. Löwe

Notes taken by Bhavik Mehta

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0 Introduction

Lecture 1 The main ‘topic in set theory’ covered in this course will be one of the most important: solving the Continuum Problem. A priori, set theory does not seem intrinsically related to logic, but the continuum hypothesis showed that logic was a very important tool in set theory. In contrast to many other disciplines of mathematics, in set theory we typically try to prove things are *impossible*, rather than showing what is possible.

The second international congress of mathematicians in 1900 was in Paris, where Hilbert spoke. At that time, Hilbert was a ‘universal’ mathematician, and had worked in every major field of mathematics. He gave a list of problems for the century, the 23 Hilbert Problems. The first on this list was the Continuum Problem.

0.1 Continuum Hypothesis

Here is Hilbert’s formulation of the Continuum hypothesis (CH): Every set of infinitely many real numbers is either equinumerous with the set of natural numbers or the set of real numbers. More formally, we might write

$$\forall X \subseteq \mathbb{R}, (X \text{ is infinite} \Rightarrow X \sim \mathbb{N} \text{ or } X \sim \mathbb{R})$$

In more modern terms, we write this as the claim $2^{\aleph_0} = \aleph_1$. These two statements are equivalent (in ZFC).

Assume that $2^{\aleph_0} > \aleph_1$, in particular $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \rightarrow \mathbb{R}$. Consider $X := i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i|_{\aleph_1}$ is a bijection between \aleph_1 and X , so $X \sim \aleph_1$. But $\aleph_1 \approx \mathbb{N}$ and $\aleph_1 \approx \mathbb{R}$. Thus X refutes CH (in its earlier formulation). So: $2^{\aleph_0} \neq \aleph_1 \Rightarrow \neg\text{CH}$.

If $2^{\aleph_0} = \aleph_1$. Let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \rightarrow \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either \aleph_0 , i.e. $X \sim \mathbb{N}$ or \aleph_1 , i.e. $X \sim \mathbb{R}$. So, $2^{\aleph_0} = \aleph_1 \Rightarrow \text{CH}$.

0.2 History of CH

- 1938, Gödel: ZFC does not prove $\neg\text{CH}$.
- 1963, Cohen: ZFC does not prove CH.

Gödel’s proof used the technique of inner models; Cohen’s proof used forcing, sometimes referred to as outer models.

Gödel’s Completeness Theorem:

$$\text{Cons}(T) \iff \exists (M, E)(M, E) \models T$$

From this, we might guess that Gödel’s and Cohen’s proof will show there is a model of ZFC + CH, and a model of ZFC + $\neg\text{CH}$, but by the incompleteness phenomenon, we cannot prove there is a model of ZFC! So, we are not going to be able to prove $\text{Cons}(\text{ZFC}+\text{CH})$, but instead

$$\text{Cons}(\text{ZFC}) \rightarrow \text{Cons}(\text{ZFC}+\text{CH})$$

Or, equivalently,

$$\text{if } M \models \text{ZFC}, \text{ then there is } N \models \text{ZFC} + \text{CH}.$$

1 Model theory of set theory

Let's assume for a moment that

$$(M, \epsilon) \models ZFC.$$

We refer to the canonical objects in M by the usual symbols, e.g., $0, 1, 2, 3, 4, \dots, \omega, \omega+1, \dots$

What would an “inner model” be? Take $A \subseteq M$, and consider (A, ϵ) . This is a substructure of (M, ϵ) .

Note: the language of set theory has no function or constant symbols. But we write down

$$X = \emptyset, X = \{Y\}, X = \{Y, Z\}, X = \bigcup Z, X = \mathcal{P}(Z)$$

which appear to use function or constant symbols. These are technically not part of the language of set theory; they are abbreviations:

$X = \emptyset$	abbreviates	$\forall w (\neg w \in X)$
$X = \{Y\}$	abbreviates	$\forall w (w \in X \leftrightarrow w = Y)$
$X \subseteq Y$	abbreviates	$\forall w (w \in X \rightarrow w \in Y)$

and so on.

Definition. If φ is a formula in n free variables. We say

- (1) φ is **upwards absolute** between A and M if

$$\text{for all } a_1, \dots, a_n \in A, \quad (A, \epsilon) \models \varphi(a_1, \dots, a_n) \implies (M, \epsilon) \models \varphi(a_1, \dots, a_n)$$

- (2) φ is **downwards absolute** between A and M if

$$\text{for all } a_1, \dots, a_n \in A, \quad (M, \epsilon) \models \varphi(a_1, \dots, a_n) \implies (A, \epsilon) \models \varphi(a_1, \dots, a_n)$$

- (3) φ is **absolute** between A and M if it is upwards absolute and downwards absolute.

Definition. We say that a formula is Σ_1 if it is of the form

$$\exists x_1 \dots \exists x_n \varphi(x_1, \dots, x_n) \text{ where } \varphi \text{ is quantifier-free}$$

or Π_1 if it is of the form

$$\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n) \text{ where } \varphi \text{ is quantifier-free}$$

Remark.

- (a) If φ is quantifier-free, then φ is **absolute** between A and M .
- (b) If φ is Π_1 , then it's downward absolute
- (c) If φ is Σ_1 , then it's upward absolute

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