

Part III – Model Theory (Ongoing course, rough)

Based on lectures by Dr. S. Barbina

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0 Introduction

Lecture 1 Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: ‘how powerful is our description of these objects to pin them down’? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

1 Languages and structures

Definition 1.1 (Language). A **language** L consists of

- (i) a set \mathcal{F} of function symbols, and for each $f \in \mathcal{F}$ a positive integer m_f the **arity** of f .
- (ii) a set \mathcal{R} of relation symbols, and for each $R \in \mathcal{R}$, a positive integer m_R .
- (iii) a set \mathcal{C} of constant symbols.

Note: each of \mathcal{F} , \mathcal{R} and \mathcal{C} can be empty.

Example. Take $L = \{\{\cdot, {}^{-1}\}, \{1\}\}$, for \cdot a binary function and ${}^{-1}$ an unary function, 1 a constant. This is the **language** of groups, call it L_{gp} . Also, $L_{\text{lo}} = \{<\}$ a single binary relation, for linear orders.

Definition 1.2 (L -structure). Given a **language** L , say, an L -**structure** consists of

- (i) a set M , the **domain**
- (ii) for each $f \in \mathcal{F}$, a function $f^{\mathcal{M}} : M^{m_f} \rightarrow M$.
- (iii) for each $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$.
- (iv) for each $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$.

$f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ are the **interpretations** of f, R, c respectively.

Remark 1.3. We often fail to distinguish between the **symbols** in L and their **interpretations** in a **structure**, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$.

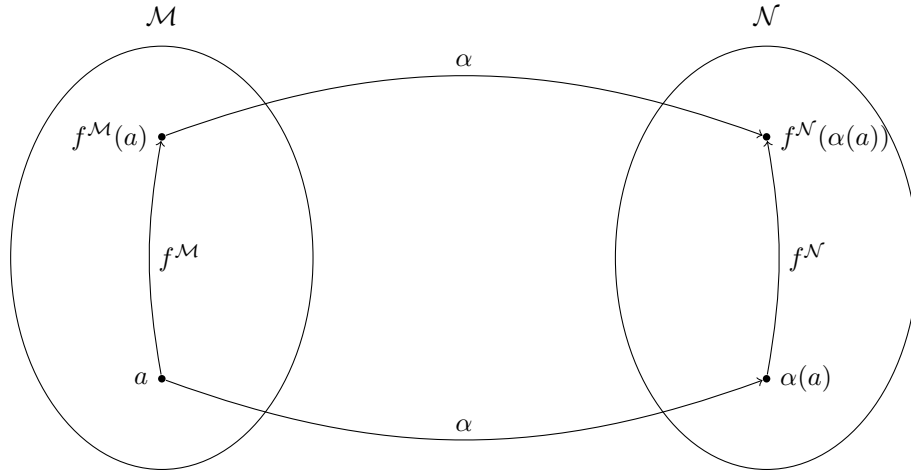
Example 1.4.

- (a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, {}^{-1}\}, 1 \rangle$ is an L_{gp} -**structure**.
- (b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is an L_{gp} -**structure**.
- (c) $\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo} -**structure**.

Definition 1.5 (Embedding). Let L be a **language**, let \mathcal{M}, \mathcal{N} be L -**structures**. An **embedding** of \mathcal{M} into \mathcal{N} is a one-to-one mapping $\alpha : M \rightarrow N$ such that

- (i) for all $f \in \mathcal{F}$, and $a_1, \dots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \dots, a_{m_f})) = f^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_{m_f}))$$



(ii) for all $R \in \mathcal{R}$, and $a_1, \dots, a_{n_R} \in M$

$$(a_1, \dots, a_{n_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathcal{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of \mathcal{M} into \mathcal{N} is a surjective embedding (onto).

Exercise 1.6. Let G_1, G_2 be groups, regarded as L_{gp} -structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an **isomorphism** $\alpha : G_1 \rightarrow G_2$ in the sense of [Definition 1.5](#).

2 Review: Terms, formulae and their interpretations

In addition to the [symbols](#) of L , we also have

- (i) infinitely many variables $\{x_i\}_{i \in I}$
- (ii) logical connectives \wedge, \neg (also expresses \vee, \implies, \iff)
- (iii) quantifier \exists (also expresses \forall)
- (iv) (,)
- (v) equality symbol =

Definition 2.1 (L -terms). L -terms are defined recursively as follows:

- any variable x_i is a term
- any constant symbol is a term
- for any $f \in \mathcal{F}$, $f(t_1, \dots, t_{m_f})$ for any terms t_1, \dots, t_{m_f} is a term
- nothing else is a term

Notation: we write $t(x_1, \dots, x_m)$ to mean that the variables appearing in t are among x_1, \dots, x_m .

Lecture 2 **Example.** Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1 \rangle$. Then $\cdot((x_1, x_2), x_3)$ is a [term](#), usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a [term](#), written $(1 \cdot x)^{-1}$

Definition 2.2. If \mathcal{M} is an L -structure, to each L -term $t(x_1, \dots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \rightarrow M$ defined as follows:

- (i) If $t = x_i$, $t^{\mathcal{M}}[a_1, \dots, a_k] = a_i$
- (ii) If $t = c$, $t^{\mathcal{M}}[a_1, \dots, a_k] = c^{\mathcal{M}}$.
- (iii) If $t = f(t(x_1, \dots, x_k), \dots, t_{m_f}(x_1, \dots, x_k))$, then

$$t^{\mathcal{M}}(a_1, \dots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_k), \dots, t_{m_f}^{\mathcal{M}}(a_1, \dots, a_k)).$$

Notice in L_{gp} , the term $x_2 \cdot x_3$ can be described as $t_1(x_1, x_2, x_3)$ or $t_2(x_1, x_2, x_3, x_4)$, or infinitely many other ways. In these cases, t_1 is [assigned](#) to $t_1^{\mathcal{M}} : M^3 \rightarrow M$, with $(a_1, a_2, a_3) \mapsto (a_2, a_3)$, and t_2 is assigned to $t_2^{\mathcal{M}} : M^4 \rightarrow M$, with $(a_1, a_2, a_3, a_4) \mapsto a_2 \cdot a_3$.

Fact 2.3. Let \mathcal{M}, \mathcal{N} be L -structures, and let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an [embedding](#). For any L -term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$ we have

$$\alpha(t^{\mathcal{M}}(a_1, \dots, a_k)) = t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k))$$

Proof. By induction on the complexity of t . Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_k)$. Then

- (i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, and $t^{\mathcal{N}}(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds.

- (ii) if $t = c$ a constant, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, and $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$, as required.
- (iii) if $t = f(t_1(\bar{x}), \dots, t_{m_f}(\bar{x}))$, then

$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{m_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \dots, \alpha(t_{m_f}^{\mathcal{M}}(\bar{a})))$$

since α is an [embedding](#). $t_1(\bar{x}), \dots, t_{m_f}(\bar{x})$ have lower complexity than t , so inductive hypothesis applies. \square

Exercise 2.4. Conclude the proof of [Fact 2.3](#).

Definition 2.5 (Atomic formula). The set of **atomic formulas** of L is defined as follows

- (i) if t_1, t_2 are L -terms, then $t_1 = t_2$ is an atomic formula
- (ii) if R is a relation symbol and t_1, \dots, t_{m_R} are terms, then $R(t_1, \dots, t_{m_R})$ is an atomic formula
- (iii) nothing else is an atomic formula.

Definition 2.6 (Formula). The set of L -**formulas** is defined as follows

- (i) any [atomic formula](#) is an L -formula
- (ii) if ϕ is an L -formula, then so is $\neg\phi$
- (iii) if ϕ and ψ are L -formulas, then so is $\phi \wedge \psi$
- (iv) if ϕ is an L -formula, for any $i \geq 1$, $\exists x_i \phi$ is an L -formula
- (v) nothing else is an L -formula

Example. In L_{gp} , $x_1 \cdot x_1 = x_2$ and $x_1 \cdot x_2 = 1$ are [atomic formulas](#), and $\exists x_1 (x_1 \cdot x_2) = 1$ is an L_{gp} -formula.

A variable occurs freely in a formula if it does not occur within the scope of a quantifier \exists (the variable is **free**). Otherwise the variable is **bound**. For instance, in $\exists x_1 (x_1 \cdot x_2) = 1$, x_1 is bound and x_2 is free.

Important convention: no variable occurs both [freely](#) and as a bound variable in the same formula.

A **sentence** is a [formula](#) with no [free](#) variables. $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp} -sentence. Notation: $\phi(x_1, \dots, x_k)$ means that the free variables in ϕ are among x_1, \dots, x_k .

Definition 2.7 (\models). Let $\phi(x_1, \dots, x_k)$ be an L -formula, let \mathcal{M} be an L -structure, and let $\bar{a} = (a_1, \dots, a_k)$ be elements of M . We define $\mathcal{M} \models \phi(\bar{a})$ recursively as follows.

- (i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.
- (ii) if ϕ is $R(t_1, \dots, t_{m_k})$ then $\mathcal{M} \models \phi(\bar{a})$ iff

$$(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{m_k}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}.$$

- (iii) if ϕ is $\psi \wedge \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \chi(\bar{a})$.

- (iv) if $\phi = \neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$. (this is well-defined since $\psi(\bar{a})$ is shorter than $\phi(\bar{a})$)
- (v) if ϕ is $\exists x_j : \chi(x_1, \dots, x_k, x_j)$ (where $x_j \neq x_i$ for $i = 1, \dots, k$). Then $\mathcal{M} \models \phi(\bar{a})$ iff there is $b \in \mathcal{M}$ such that $\mathcal{M} \models \chi(a_1, \dots, a_k, b)$.

Example. For $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1 \rangle$, if $\phi(x_1) = \exists x_2 (x_2 \cdot x_2) = x_1$ then $\mathcal{R} \models \phi(1)$ but $\mathcal{R} \not\models \phi(-1)$.

Notation 2.8 (Useful abbreviations). We write

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$
- $\phi \rightarrow \psi$ for $\neg\phi \vee \psi$
- $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
- $\forall x_i \phi$ for $\neg\exists x_i (\neg\phi)$

Proposition 2.9. Let \mathcal{M}, \mathcal{N} be *L-structures*, let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an *embedding*. Let $\phi(\bar{x})$ be *atomic* and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a})).$$

Question: If ϕ is an *L-formula*, not necessarily *atomic*, does [Proposition 2.9](#) hold?

Lecture 3 Proof of [Proposition 2.9](#). Cases:

- (i) $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. (Exercise: complete this case, using [Fact 2.3](#))
- (ii) $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \dots, t_{m_R}(\bar{x}))$. Then $\mathcal{M} \models R(t_1(\bar{a}), \dots, t_{m_R}(\bar{a}))$ if and only if... (Exercise: complete this case)

□

Exercise 2.10. Show that [Proposition 2.9](#) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

Example 2.11. Do *embeddings* preserve *all formulas*? No. Take $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ an *L_{lo}-structure*. Then $\alpha : \mathbb{Z} \rightarrow \mathbb{Q}$ (inclusion) is an embedding, but

$$\begin{aligned} \phi(x_1, x_2) &= \exists x_3 (x_1 < x_3 \wedge x_3 < x_2). \\ \mathcal{Q} &\models \phi(1, 2) \text{ but } \mathcal{Z} \not\models \phi(1, 2). \end{aligned}$$

Fact 2.12. Let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an *isomorphism*. Then if $\phi(\bar{x})$ is an *L-formula* and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \phi(\alpha(\bar{a})).$$

Proof. Exercise.

□

3 Theories and elementarity

Throughout, L is a [language](#), \mathcal{M}, \mathcal{N} are [L-structures](#).

Definition 3.1 (*L-theory*). An *L-theory* T is a set of [L-sentences](#). \mathcal{M} is a **model** of T if $\mathcal{M} \models \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \models T$. The class of all the models of T is written $\text{Mod}(T)$. The theory of \mathcal{M} is the set

$$\text{Th}(\mathcal{M}) = \{ \sigma \mid \sigma \text{ is an } L\text{-structure and } \mathcal{M} \models \sigma \}.$$

Example 3.2. Let T_{gp} be the set of [L_{gp}-sentences](#)

- (i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$
- (ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$
- (iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly for a group G , $G \models T_{\text{gp}}$. For a specific G , clearly $\text{Th}(G)$ is larger than T_{gp} !

Definition 3.3 (Elementarily equivalent). Say \mathcal{M} and \mathcal{N} are **elementarily equivalent** if $\text{Th}(\mathcal{M}) = \text{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$.

Clearly if $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$ but if \mathcal{M} and \mathcal{N} are not [isomorphic](#), establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as [L_{lo}-structures](#).

Definition 3.4 (Elementary substructure).

- (i) an [embedding](#) $\beta : \mathcal{M} \rightarrow \mathcal{N}$ is **elementary** if for all [formulas](#) $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a})).$$

- (ii) if $M \subseteq N$ and $\text{id} : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding, then \mathcal{M} is said to be a **substructure** of \mathcal{N} , written $\mathcal{M} \subseteq \mathcal{N}$.
- (iii) if $M \subseteq N$ and $\text{id} : \mathcal{M} \rightarrow \mathcal{N}$ is an elementary embedding, then \mathcal{M} is said to be an **elementary substructure** of \mathcal{N} , written $\mathcal{M} \preceq \mathcal{N}$.

Example 3.5. Consider $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an [L_{lo}-structure](#), where $<$ is the usual order, and $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$ in the same way. Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo} -structures.

Is $\mathcal{M} \equiv \mathcal{N}$? Yes: they are isomorphic!

Is $\mathcal{M} \subseteq \mathcal{N}$? Yes (the ordering $<$ coincides on \mathcal{M} and \mathcal{N} .)

But $\mathcal{M} \not\preceq \mathcal{N}$, since if $\phi(x) = \exists y (x < y)$, then

$$\mathcal{N} \models \phi(1) \quad \text{and} \quad \mathcal{M} \not\models \phi(1).$$

Definition 3.6 (Parameter). Let \mathcal{M} be an [L-structure](#), $A \subseteq M$, then define

$$L(A) := L \cup \{ c_a \mid a \in A \}$$

for c_a each constant symbols. An [interpretation](#) of \mathcal{M} as an L -structure extends to an interpretation of \mathcal{M} as an $L(A)$ -structure in the obvious way ($c_a^{\mathcal{M}} = a$). The elements of A are called **parameters**. If \mathcal{M}, \mathcal{N} are L -structures and $A \subseteq M \cap N$, then we write $\mathcal{M} \equiv_A \mathcal{N}$ when \mathcal{M}, \mathcal{N} satisfy exactly the same $L(A)$ -sentences.

Lecture 4 **Exercise 3.7.** $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of \mathcal{M}).

Lemma 3.8 (Tarski-Vaught test). Let \mathcal{N} be an L -structure, let $A \subseteq N$. The following are equivalent:

- (i) A is the domain of a structure \mathcal{M} such that $\mathcal{M} \preceq \mathcal{N}$.
- (ii) for every $L(A)$ -formula $\phi(x)$ with one free variable, if $\mathcal{N} \models \exists x \phi(x)$, then $\mathcal{N} \models \phi(b)$ for some $b \in A$.

Proof.

(i) \Rightarrow (ii) Suppose $\mathcal{N} \models \phi(x)$. Then by elementarity, $\mathcal{M} \models \exists x \phi(x)$, and so $\mathcal{M} \models \exists x \phi(x)$ for $b \in \mathcal{M}$, so again by elementarity $\mathcal{N} \models \phi(b)$.

(ii) \Rightarrow (i) First we prove that A is the domain $\mathcal{M} \subseteq \mathcal{N}$. By exercise 4 on sheet 1, it is enough to check:

- (a) for each constant c , $c^{\mathcal{N}} \in A$.
- (b) for each function symbol f , $f^{\mathcal{N}}(\bar{a}) \in A$ (for all $\bar{a} \in A^{m_f}$).

For (a), use property (ii) with $\exists x (x = c)$. For (b) use property (ii) with $\exists x (f(\bar{a}) = x)$.

So we now have $\mathcal{M} \subseteq \mathcal{N}$, and the domain of \mathcal{M} is A . Let $\chi(\bar{x})$ be an L -formula. We show that for $\bar{a} \in A^{|\bar{x}|}$,

$$\mathcal{M} \models \chi(\bar{a}) \iff \mathcal{N} \models \chi(\bar{a}). \quad (*)$$

By induction on the complexity of $\chi(\bar{x})$:

- if $\chi(\bar{x})$ is atomic $(*)$ follows from $\mathcal{M} \subseteq \mathcal{N}$ (\mathcal{M} is a substructure).
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$: straightforward induction.
- if $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L -formula, suppose that $\mathcal{M} \models \chi(\bar{a})$. Then $\mathcal{M} \models \exists y \psi(\bar{a}, y)$, hence $\mathcal{M} \models \psi(\bar{a}, b)$ for some $b \in A = \text{dom } \mathcal{M}$. But then $\mathcal{N} \models \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \models \chi(\bar{a})$.
Now let $\mathcal{N} \models \chi(\bar{a})$, i.e. $\mathcal{N} \models \exists y \psi(\bar{a}, y)$. By property (ii), $\mathcal{N} \models \psi(\bar{a}, b)$ for some $b \in A = \text{dom}(\mathcal{M})$. By inductive hypothesis, $\mathcal{M} \models \psi(\bar{a}, b)$ and so $\mathcal{M} \models \chi(\bar{a})$. \square

Remark 3.9. Assume the set of variables is countably infinite. Then

- the cardinality of the set of L -formulas is $|L| + \omega$. (We abuse notation and write ω for the ordinal and cardinal, and define the cardinality of L as the number of symbols in it: $|L_{\text{gp}}| = 3$, $|L_{\text{lo}}| = 1$).
- if A is a set of parameters in some structure, the cardinality of the set of $L(A)$ -formulas is $|A| + |L| + \omega$.

Definition 3.10 (Chain). Let λ be an ordinal. Then a **chain of length** λ of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$. A **chain of L -structures** is a sequence $\langle \mathcal{M}_i : i < \lambda \rangle$ such that $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i \leq j < \lambda$.

The **union** of this chain is the L -structure \mathcal{M} is defined as follows:

- the domain of \mathcal{M} is $\bigcup_{i < \lambda} M_i$
- $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$ (c is a constant).

- if f is a function symbol, $\bar{a} \in M^{m_f}$, $f^{\mathcal{M}}\bar{a} = f^{\mathcal{M}_i}\bar{a}$ where i is such that $\bar{a} \in M_i^{m_f}$.
- if R is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i < \lambda} R^{\mathcal{M}_i}$

Theorem 3.11 (Downward Löwenheim-Skolem). Let \mathcal{N} be an L -structure, and $|N| \geq |L| + \omega$. Let $A \subseteq N$. Then for any cardinal λ such that $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preceq \mathcal{N}$ such that

- (i) $A \subseteq M$
- (ii) $|M| = \lambda$.

(It helps to think about the case $|L| \leq \omega$, $|A| = \omega$ and $|N|$ is uncountable).

For instance, think of $(\mathbb{C}, +, \cdot, -, {}^{-1}, 0, 1)$ as a field. Then $\mathbb{Q} \subseteq \mathbb{C}$: it is a subset and a substructure. In particular, the property of being algebraically closed is in the theory of \mathbb{C} . Thus Theorem 3.11 gives a algebraically closed field, which is countable and contains \mathbb{Q} - a possibility is the algebraic closure of \mathbb{Q} .

Proof. We inductively build a chain $\langle A_i : i < \omega \rangle$, with $A_i \subseteq N$, such that $|A_i| = \lambda$. (Our goal is to define $M = \bigcup_{i < \omega} A_i$).

Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$. At stage $i + 1$, assume that A_i has been built, with $|A_i| = \lambda$. Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$ -formulas such that $\mathcal{N} \models \phi_k(x)$ (observe there are no more than λ , since $|L(A)| = |L| + |A| + \omega \leq \lambda$). Let a_k be such that $\mathcal{N} \models \phi_k(a_k)$ and let $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$. Then $|A_{i+1}| = \lambda$.

Now let $M = \bigcup_{i < \omega} A_i$. We use the Tarski-Vaught test to show that M is the domain of a structure $\mathcal{M} \preceq \mathcal{N}$, and $|M| = \lambda$:

Let $\mathcal{N} \models \exists x \psi(x, \bar{a})$, where \bar{a} is a tuple in M . Then \bar{a} is a *finite* tuple, so there is an i such that \bar{a} is in A_i . Then A_{i+1} , by construction, contains b such that $\mathcal{N} \models \psi(b, \bar{a})$. But $A_{i+1} \subseteq M$, so $b \in M$. \square

4 Two relational structures

4.1 Dense linear orders

Lecture 5 **Definition 4.1** (Dense linear orders). A **linear order** is an $L_{lo} = \{<\}$ -**structure** such that

- (i) $\forall x \neg(x < x)$
- (ii) $\forall xyz ((x < y \wedge y < z) \rightarrow x < z)$
- (iii) $\forall xy ((x < y) \wedge (y < x) \vee (x = y))$.

A linear order is **dense** if it also satisfies

- (iv) $\exists xy (x < y)$
- (v) $\forall xy (x < y \rightarrow \exists z (x < z < y))$ (density).

A linear order has no endpoints if

- (vi) $\forall x (\exists y (x < y) \wedge \exists z (z < x))$

T_{dlo} is the **theory** that includes axioms (i) to (vi), T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark: (iv) and (v) imply that if $\mathcal{M} \models T_{dlo}$ then $|\mathcal{M}| \geq \omega$.

Definition 4.2 ((Finite) Partial embedding). If $\mathcal{M}, \mathcal{N} \models T_{lo}$, then an injective map $p : A \subseteq M \rightarrow N$ is called a **partial embedding** if for all $a, b \in A$,

$$\mathcal{M} \models a < b \implies \mathcal{N} \models p(a) < p(b).$$

If $|\text{dom}(p)| < \omega$, then p is a **finite partial embedding**.

Lemma 4.3 (Extension lemma for dense linear orders). Suppose $\mathcal{M} \models T_{lo}$, $\mathcal{N} \models T_{dlo}$, let $p : A \subseteq M \rightarrow N$ be a **finite partial embedding**. Then if $c \in M$, there is a finite partial embedding \hat{p} such that $p \subseteq \hat{p}$ and $c \in \text{dom}(\hat{p})$.

Proof. Split into three cases:

1. $c > a$ for all $a \in \text{dom}(p)$. Then choose $d \in \mathcal{N}$ so that $d > b$ for all $b \in \text{img}(p)$.
2. $a_i < c < a_{i+1}$ for some $a_i, a_{i+1} \in \text{dom}(p)$. Then $\mathcal{N} \models p(a_i) < p(a_{i+1})$, so by density, $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$.
3. $c < a$ for all $a \in \text{dom}(p)$. Similar to case 1. □

Theorem 4.4. Let $\mathcal{M}, \mathcal{N} \models T_{dlo}$ such that $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \rightarrow N$ be a **finite partial embedding**. Then there is $\pi : \mathcal{M} \rightarrow \mathcal{N}$, an **isomorphism** such that $p \subseteq \pi$.

Proof. Enumerate M, N : say $M = \langle a_i : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ sequences of elements. We define inductively a chain of **finite partial embeddings** $\langle p_i : i < \omega \rangle$ (idea: $\pi = \bigcup_{i < \omega} p_i$).

Let $p_0 = p$. At stage $i + 1$, p_i is given. We want to include a_i in $\text{dom}(p_{i+1})$, and b_i in $\text{img}(p_{i+1})$.

Forward step: By **Lemma 4.3**, extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \text{dom}(p_{i+\frac{1}{2}})$. Backward step: By **Lemma 4.3** applied to $p_{i+\frac{1}{2}}^{-1}$ to include $b_i \in \text{dom}(p_{i+\frac{1}{2}}^{-1})$ (i.e. in the range of $p_{i+\frac{1}{2}}$). Then p_{i+1} extends p_i as required.

Let $\pi = \bigcup_{i < \omega} p_i$. Then (check) π is an **isomorphism** (i.e. order-preserving bijection). □

Definition 4.5 (Consistent, complete, \vdash). An L -theory T is **consistent** if there is \mathcal{M} such that $\mathcal{M} \models T$. If T is a **theory** in L and ϕ is an L -sentence, then we write $T \vdash \phi$ if for all \mathcal{M} such that $\mathcal{M} \models T$, we also have $\mathcal{M} \models \phi$. An L -theory T is **complete** if for all L -sentences ϕ , either $T \vdash \phi$ or $T \vdash \neg\phi$.

Is T_{dlo} complete?

Lecture 6 **Definition 4.6** (ω -categorical). A **theory** T in a **countable language** with a countably infinite **model** is called **ω -categorical** if any two countable models of T are **isomorphic**.

Corollary 4.7 (of Theorem 4.4). T_{dlo} is **ω -categorical**.

Proof. Say $\mathcal{M}, \mathcal{N} \models T_{\text{dlo}}$, and $|\mathcal{M}| = |\mathcal{N}| = \omega$. Then \emptyset (the empty map) is a **finite partial embedding**. By Theorem 4.4, $\mathcal{M} \simeq \mathcal{N}$. (Can also use any $\{\langle a, b \rangle\}$ where $a \in \mathcal{M}, b \in \mathcal{N}$ as initial finite partial embedding). \square

Theorem 4.8. If T is an **ω -categorical theory** in a countable language, and T has no finite models then T is **complete**.

Proof. Let $\mathcal{M} \models T$ and φ be an L -sentence.

If $\mathcal{M} \models \varphi$, suppose $\mathcal{N} \models T$. Then by Downward Löwenheim-Skolem, there are $\mathcal{M}' \preceq \mathcal{M}$, $\mathcal{N}' \preceq \mathcal{N}$ such that $|\mathcal{M}'| = |\mathcal{N}'| = \omega$. By ω -categoricity, $\mathcal{M}' \simeq \mathcal{N}'$, so in particular $\mathcal{M}' \equiv \mathcal{N}'$ and so $\mathcal{N}' \models \varphi$.

If $\mathcal{M} \models \neg\varphi$, similar. \square

Corollary 4.9. T_{dlo} is **complete**.

Definition 4.10 ((Partial) elementary map). If \mathcal{M}, \mathcal{N} are L -structures, a map f such that $\text{dom } f \subseteq M$ and $\text{img } f \subseteq N$ is called a **(partial) elementary map** if for all L -formulae $\phi(\bar{x})$ and $\bar{a} \in (\text{dom } f)^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(f(\bar{a})).$$

Remark 4.11. A map f is **elementary** iff every finite restriction of f is elementary.

Proof.

\Leftarrow Suppose f is not **elementary**. Then there are $\varphi(\bar{x})$ and $\bar{a} \in (\text{dom } f)^{|\bar{x}|}$ such that

$$\mathcal{M} \models \varphi(\bar{a}) \not\iff \mathcal{N} \models \varphi(f(\bar{a})).$$

Then $f|_{\bar{a}}$ is a finite restriction of f that is not elementary.

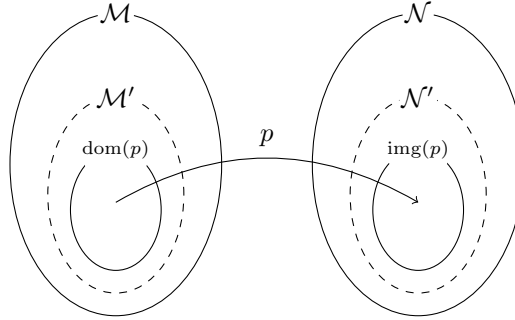
\Rightarrow Clear. \square

Proposition 4.12. Let $\mathcal{M}, \mathcal{N} \models T_{\text{dlo}}$ and let $p : A \subseteq M \rightarrow N$ be a **partial embedding**. Then p is **elementary**.

Proof. By Remark 4.11, it suffices to consider p finite. By Downward Löwenheim-Skolem, we choose $\mathcal{M}', \mathcal{N}'$ such that

- (i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.
- (ii) $\mathcal{M}' \preceq \mathcal{M}, \mathcal{N}' \preceq \mathcal{N}$

$$(iii) \text{ dom}(p) \subseteq \mathcal{M}', \text{img}(p) \subseteq \mathcal{N}'$$



Now p is a [finite partial embedding](#) between countable models, so p extends to an [isomorphism](#) $\pi : \mathcal{M}' \rightarrow \mathcal{N}'$ by [Theorem 4.4](#). In particular, π is an [elementary map](#) between \mathcal{M} and \mathcal{N} . \square

Corollary 4.13. $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$.

Proof. Use [Proposition 4.12](#) with $\text{id} : \mathbb{Q} \rightarrow \mathbb{R}$. \square

4.2 Random graph

Definition 4.14 (Random graph). Let $L_{\text{gph}} = \{R\}$, a binary relation symbol. An L_{gph} -[structure](#) is a **graph** if

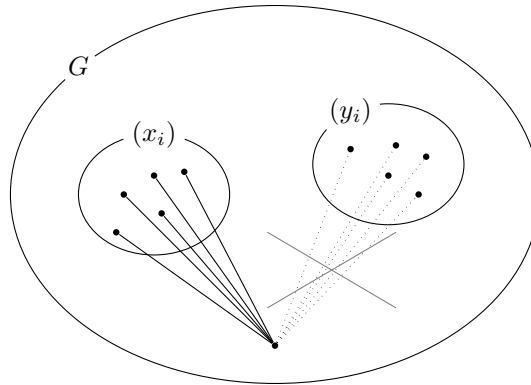
- (i) $\forall x \neg R(x, x)$
- (ii) $\forall xy (R(x, y) \leftrightarrow R(y, x))$

An L_{gph} -[structure](#) is a **random graph** if it is a graph such that, for all $n \in \omega$, axiom (r_n) holds:

$$\forall x_0 \dots x_n, y_0 \dots y_n \left(\bigwedge_{i,j=0}^n x_i \neq y_j \rightarrow \exists z \left(\bigwedge_{i=0}^n (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

- (iii) $\exists xy (x \neq y)$.

Axiom (r_n) effectively says that for disjoint subsets (x_i) and (y_i) each of size n , there is a (different) node z connected to each x_i and none of the y_i .



Remark. A [random graph](#) is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

Fact 4.15. There is a [random graph](#).

Proof. Let the domain be ω , let $i, j \in \omega$ such that $i < j$. Write j as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff 2^i appears in the sum. \square

Exercise. Prove that ω with this definition of R is a [random graph](#).

Definition 4.16 (Graph theories, partial embedding). T_{gph} consists of the axioms (i),(ii) above, and $T_{\text{rg}} = T_{\text{gph}} \cup \{(iii), (r_n) : n \in \omega\}$. If $\mathcal{M}, \mathcal{N} \models T_{\text{gph}}$, a **partial embedding** is an injective map $p : A \subseteq M$ to N such that

$$\mathcal{M} \models R(a, b) \iff \mathcal{N} \models R(p(a), p(b))$$

for all a, b in the domain. Just as before, if $|\text{dom}(p)| < \omega$ then p is called a **finite partial embedding**.

Lemma 4.17 (Extension lemma for random graphs). Let $\mathcal{M} \models T_{\text{gph}}, \mathcal{N} \models T_{\text{rg}}$, let $p : A \subseteq M \rightarrow N$ be a [finite partial embedding](#), and let $c \in M$. Then there is a partial embedding $\hat{p} : \hat{A} \subseteq M \rightarrow N$ such that, $c \in \text{dom}(\hat{p})$, and $p \subseteq \hat{p}$.

Lecture 7 Proof. Take $c \in M, c \notin \text{dom}(p)$.

diagram coming soon

Find $d \in N$ such that $N \models R(d, p(a)) \iff M \models R(c, a)$. \square

Theorem 4.18. Let $\mathcal{M}, \mathcal{N} \models T_{\text{rg}}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $p : A \subset M \rightarrow N$ a [finite partial embedding](#). Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends p .

Proof. Same as proof of [Theorem 4.4](#), but with [Lemma 4.17](#) instead of [Lemma 4.3](#). \square

Corollary 4.19. T_{rg} is ω -categorical and complete. Moreover, every [finite partial embedding](#) between [models](#) of T_{rg} is an [elementary map](#).

Remark 4.20. The unique (up to isomorphism) countable model of T_{rg} is *the* countable random graph, or the **Rado graph**. It is universal with respect to finite and countable graphs (i.e. it embeds them all). It is **ultrahomogeneous** i.e. every [isomorphism](#) between finite [substructures](#) extends to an automorphism of the whole graph.

5 Compactness

Definition 5.1. Take an L -theory T .

- (i) T is **finitely satisfiable** if every finite subset of **sentences** in T has a **model**.
- (ii) T is **maximal** if for all L -sentences σ , either $\sigma \in T$ or $\neg\sigma \in T$.
- (iii) T has the **witness property** if for all $\phi(x)$ (L -formula with one **free** variable) there is a constant $c \in \mathcal{C}$ such that

$$(\exists x \phi(x)) \rightarrow \phi(c) \in T.$$

Lemma 5.2. If T is **maximal** and **finitely satisfiable** and φ is an L -sentence, and $\Delta \subseteq^{\text{finite}} T$ with $\Delta \models \varphi$, then $\varphi \in T$.

Proof. If $\varphi \in T$ then $\neg\varphi \in T$ (by maximality). But then $\Delta \cup \{\neg\varphi\}$ is a finite subset of T which does not have a model. \square

Lemma 5.3. Let T be a **maximal**, **finitely satisfiable** theory with the **witness property**. Then T has a **model**. Moreover, if λ is a cardinal and $|\mathcal{C}| \leq \lambda$, then T has a model of size at most λ .

Proof. Let $c, d \in \mathcal{C}$, define $c \sim d$ iff $c = d \in T$.

Claim: \sim is an equivalence relation. **Proof:** For transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$, so $c = e \in T$ (by [Lemma 5.2](#)), and so $c \sim e$. Reflexivity follows from **maximality**, and symmetry is immediate. \blacksquare

We denote $[c] \in \mathcal{C} / \sim$ by c^* . Now, define a **structure** \mathcal{M} whose domain is $\mathcal{C} / \sim = M$. Clearly, $|M| \leq \lambda$ if $|\mathcal{C}| \leq \lambda$. We must define **interpretations** in \mathcal{M} for symbols of L .

- If $c \in \mathcal{C}$, then $c^{\mathcal{M}} = c^*$.
- If $R \in \mathcal{R}$, define

$$R^{\mathcal{M}} := \{ (c_1^*, \dots, c_{n_R}^*) \mid R(c_1, \dots, c_{n_R}) \in T \}.$$

Claim: $R^{\mathcal{M}}$ is well defined. **Proof:** Suppose $\bar{c}, \bar{d} \in \mathcal{C}^{n_R}$ and suppose $c_i \sim d_i$. That is, $c_i = d_i \in T$ for $i = 1, \dots, n_R$ so by [Lemma 5.2](#)

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T. \quad \blacksquare$$

- If $f \in \mathcal{F}$, and $\bar{c} \in \mathcal{C}^{n_f}$, then $f\bar{c} = d \in T$ for some $d \in \mathcal{C}$. (This is because $\exists x (f(\bar{c}) = x) \in T$ by **maximality**, then apply **witness property**.)

Then define $f^{\mathcal{M}}(\bar{c}^*) = d^*$. Exercise: Check $f^{\mathcal{M}}(\bar{c}^*)$ is well-defined!

Claim: if $t(x_1, \dots, x_n)$ is an L -term and $c_1, \dots, c_n, d \in \mathcal{C}$, then

$$t(c_1, \dots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*.$$

Proof:

(\Rightarrow) by induction on the complexity of t .

(\Leftarrow) Assume $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. Then

$$t(c_1, \dots, c_n) = e \in T$$

for some constant e by [witness property](#) and [Lemma 5.2](#). Use (\Rightarrow) to get that $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$. But then $d^* = e^*$, i.e. $d = e \in T$. Then $t(c_1, \dots, c_n) = d \in T$. ■

Claim: For all L -formulas $\varphi(\bar{x})$, and $\bar{c} \in \mathcal{C}^{|\bar{x}|}$,

$$\mathcal{M} \models \varphi(\bar{c}) \iff \varphi(\bar{c}) \in T.$$

Proof: By induction on $\varphi(\bar{x})$. (Exercise: Fill in the details). ■ This shows $\mathcal{M} \models T$. □

Lecture 8 **Lemma 5.4.** Let T be a [finitely satisfiable \$L\$ -theory](#). Then there are $L^* \supseteq L$ and a finitely satisfiable $T^* \supseteq T$ such that

$$(i) \quad |L^*| = |L| + \omega.$$

(ii) any L^* -theory extending T^* has the [witness property](#).

Proof. We define $\langle L_i : i < \omega \rangle$ a [chain of languages](#) containing L and such that $|L_i| = |L| + \omega$, and $\langle T_i : i < \omega \rangle$ of [finitely satisfiable theories](#) such that $\forall i, T_i$ is an L_i -theory and $T_i \supseteq T$.

Set $L_0 = L$ and $T_0 = T$. At stage $i + 1$, L_i and T_i are given. List all [\$L_i\$ -formulas](#) $\varphi(x)$ (one [free variable](#)) and let

$$L_{i+1} = L_i \cup \{ c_\varphi \mid \varphi(x) \text{ an } L_i \text{ formula} \}.$$

For all $\varphi(x)$, an L_i formula in one free variable, let Φ_φ be the L_{i+1} -sentence

$$\exists x \varphi(x) \rightarrow \varphi(c_\varphi).$$

Then let

$$T_{i+1} = T_i \cup \{ \Phi_\varphi \mid \varphi(x) \text{ is an } L_i \text{ formula} \}.$$

Claim: T_{i+1} is [finitely satisfiable](#).

Proof: Let $\Delta \subseteq T_{i+1}$ be finite. Then

$$\Delta = \Delta_0 \cup \{ \Phi_{\varphi_1}, \dots, \Phi_{\varphi_n} \}$$

where $\Delta_0 \subseteq T_i$. Let $\mathcal{M} \models \Delta_0$ (\mathcal{M} is an [\$L_i\$ structure](#); it exists because T_i is [finitely satisfiable](#)).

We define an L_{i+1} -structure \mathcal{M}' with domain M . Define the [interpretation](#) of new constants as follows: if $\mathcal{M} \models \exists x \varphi(x)$, then let a be such that $\mathcal{M} \models \varphi(a)$, and set $c_\varphi^{\mathcal{M}'} := a$. Otherwise, $c_\varphi^{\mathcal{M}'}$ is arbitrary. Then $\mathcal{M}' \models \Delta$. ■

Let

$$L^* = \bigcup_{i < \omega} L_i, \quad T^* = \bigcup_{i < \omega} T_i.$$

By construction, any extension of T^* has the [witness property](#) (check this!) and T^* is finitely satisfiable. (If $\Delta \subseteq T^*$ then $\Delta \subseteq T_i$ for some i). □

Lemma 5.5. If T is [finitely satisfiable](#), there exists a [maximal](#) finitely satisfiable $T' \supseteq T$.

Proof. Let

$$I := \{ S \mid S \text{ is a finitely satisfiable } L\text{-theory such that } T \subseteq S \}.$$

I is partially ordered by inclusion, and non-empty.

If $\langle C_i : i < \lambda \rangle$ is a chain in I , then $\bigcup_{i < \lambda} C_i$ is an upper bound for the chain - it is finitely satisfiable. Then by Zorn's lemma, I has a maximal element (with respect to \subseteq).

Claim: the maximal element T' of I is the required extension of T (check that for all L -sentences σ , $\sigma \in T'$ or $\neg\sigma \in T'$). \square

Theorem 5.6 (Compactness). If T is a finitely satisfiable L -theory and $\lambda \geq |L| + \omega$, then there is $\mathcal{M} \models T$ such that $|\mathcal{M}| \leq \lambda$.

Proof sketch. Extend T to T^* , an L^* -theory that is finitely satisfiable and such that any $S \supseteq T^*$ has the witness property (by Lemma 5.4).

By Lemma 5.5, there is $T' \supseteq T^*$, which is maximal and finitely satisfiable. Then T' has the witness property. Then by Lemma 5.3 there is $\mathcal{M} \models T'$ with $|\mathcal{M}| \leq \lambda$, and $\mathcal{M} \models T$. \square

Definition 5.7 (Type). Let L be a language.

- An L -type $p(\bar{x})$ is a set of L -formulas whose free variables are in \bar{x} (and $\bar{x} = \langle x_i : i < \lambda \rangle$).
- An L -type is **satisfiable** if there is an L -structure \mathcal{M} and an assignment $\bar{a} \in \mathcal{M}^{|\bar{x}|}$ to \bar{x} such that $\mathcal{M} \models \varphi(\bar{a})$ for all $\varphi(\bar{x}) \in p(\bar{x})$ (we also say $p(\bar{x})$ **consistent**, and that \bar{a} **realizes** $p(\bar{x})$ in \mathcal{M}). We write $\mathcal{M} \models p(\bar{a})$ or $\mathcal{M}, \bar{a} \models p(\bar{x})$. We also say that $p(\bar{x})$ is **satisfied** in \mathcal{M} .
- A type $p(\bar{x})$ is **finitely satisfiable** if every finite subset of $p(\bar{x})$ is satisfiable (we may say $p(\bar{x})$ is **finitely consistent**).

Remark. An L -type may be finitely satisfiable in \mathcal{M} (i.e. every finite subset is satisfiable in \mathcal{M}) but not satisfiable in \mathcal{M} .

Example. Take $\mathcal{M} = (\mathbb{N}, <)$. Let $\phi_n(x)$ say 'there are at least n elements less than x '.

$$p(x) := \{ \phi_n(x) \mid n < \omega \}$$

Is $p(x)$ finitely satisfiable in \mathcal{M} ? Yes. But $p(x)$ is not satisfiable in \mathcal{M} .

Theorem 5.8 (Compactness theorem for types). Every finitely satisfiable L -type $p(\bar{x})$ is satisfiable.

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$, let $\langle c_i : i < \lambda \rangle$ be new constants (not in L). Expand L to $L' = L \cup \{c_i : i < \lambda\}$. Then $p(\bar{c})$ is a finitely satisfiable L' -theory and Theorem 5.6 applied to $p(\bar{c})$ gives an L' -structure \mathcal{M}' such that $\mathcal{M}' \models p(\bar{c})$. But \mathcal{M}' reduces to an L structure \mathcal{M} , so $\mathcal{M}, \bar{c}^{\mathcal{M}'} \models p(\bar{x})$. \square

Lecture 9 **Lemma 5.9.** Let \mathcal{M} be a structure, let $\bar{a} = \langle a_i : i < \lambda \rangle$ an enumeration of \mathcal{M} . Let

$$q(\bar{x}) = \{ \varphi(\bar{x}) \mid \mathcal{M} \models \varphi(\bar{a}) \},$$

where $|\bar{x}| < \lambda$. Then $q(\bar{x})$ is satisfiable in \mathcal{N} iff there is $\beta : \mathcal{M} \rightarrow \mathcal{N}$ that is an elementary embedding.

Proof.

(\Rightarrow) If $q(\bar{x})$ is **satisfiable** in \mathcal{N} , there is $\bar{b} \in N^{|\bar{x}|}$ such that

$$\mathcal{N} \models \varphi(\bar{b}) \quad \forall \varphi(\bar{x}) \in q(\bar{x}).$$

Then $\beta : a_i \mapsto b_i$ for $i < \lambda$ is an **elementary embedding**. (β preserves, for example, **atomic formulas** of the form $f(a_{i_1}, \dots, a_{i_n}) = a_{i_{n+1}}$). More generally, for any $\varphi(\bar{x})$ an **L -formula**,

$$\mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{N} \models \varphi(\bar{b})$$

but $\beta(\bar{a}) = \bar{b}$ so we have **elementarity**.

(\Leftarrow) If $\beta : \mathcal{M} \rightarrow \mathcal{N}$ is elementary, then $\beta(\bar{a})$ satisfies $q(\bar{x})$ in \mathcal{N} . \square

This lemma is sometimes also called the Diagram Lemma, and stated as: Suppose $\text{Th}(\mathcal{M}_M)$ is a theory in $L(M)$. Then if $\mathcal{N} \models \text{Th}(\mathcal{M}_M)$, then \mathcal{M} **embeds elementarily** in \mathcal{N} .

Remark 5.10. We can consider types in $L(A)$, where $A \subseteq M$. In particular, we can have $M = A$.

Types of this kind are said to have **parameters in A** (or to be over A). If $p(\bar{x})$ is a type over M , then there is \bar{a} , an enumeration of M , and a type $p'(\bar{x}, \bar{z})$ in L where the \bar{z} are new constants, $|\bar{z}| = |\bar{a}|$, and $p(\bar{x}) = p'(\bar{x}, \bar{a})$.

Theorem 5.11. If \mathcal{M} is a **structure**, and $p(\bar{x})$ is a **type** in $L(M)$ that is **finitely satisfiable** in \mathcal{M} , then $p(\bar{x})$ is **satisfiable** in some \mathcal{N} such that $\mathcal{M} \preceq \mathcal{N}$.

Example. Take $\mathcal{M} = (\mathbb{Q}, <)$, and let $\langle a_i : i < \omega \rangle$ a sequence in \mathbb{Q} that converges to $\sqrt{2}$ from below, and let $\langle b_i : i < \omega \rangle \subseteq \mathbb{Q}$ tend to $\sqrt{2}$ from above. Set $\phi_n(x) := a_n < x < b_n$. Then let $p(x) = \{ \phi_n(x) \mid n < \omega \}$. Then $p(x)$ is an $L(\mathbb{Q})$ -**type** which is **finitely satisfiable** in \mathbb{Q} . But $p(x)$ is not **satisfiable** in \mathcal{M} . It is, however, satisfiable in $(\mathbb{R}, <) \succ (\mathbb{Q}, <)$.

Proof of Theorem 5.11. Let $\langle a_i : i < \lambda \rangle$ enumerate \mathcal{M} , let

$$q(\bar{z}) := \{ \varphi(\bar{z}) \mid \mathcal{M} \models \varphi(\bar{a}) \}$$

where $|\bar{z}| = \lambda$ and the z_i are new variables (so not among the \bar{x}). Write $p(\bar{x})$ as $p'(\bar{x}, \bar{a})$ for some $p'(\bar{x}, \bar{z})$ (an **L -type**).

Claim: $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is **finitely satisfiable** in \mathcal{M} .

Proof: $p'(\bar{x}, \bar{a})$ is finitely satisfiable by hypothesis and $q(\bar{z})$ is **realized** by \bar{a} .

Then, by **Compactness theorem for types**, $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is satisfiable. That is, there is \mathcal{N} and $\bar{b} \in \mathcal{N}^{|\bar{z}|}$ and $\bar{c} \in \mathcal{N}^{|\bar{x}|}$ such that

$$\mathcal{N} \models p'(\bar{c}, \bar{b}) \cup q(\bar{b}).$$

In particular, $\mathcal{N} \models q(\bar{b})$, then by **Lemma 5.9**, $\beta : a_i \mapsto b_i$ is an **elementary embedding**. \square

Theorem 5.12 (Upward Löwenheim-Skolem). Let \mathcal{M} be such that $|\mathcal{M}| \geq \omega$. Then for any $\lambda \geq |\mathcal{M}| + |L|$, there is \mathcal{N} such that $\mathcal{M} \preceq \mathcal{N}$, and $|\mathcal{N}| = \lambda$.

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$ a tuple of distinct variables. Let

$$p(\bar{x}) = \{ x_i \neq x_j \mid i < j < \lambda \}.$$

Then $p(\bar{x})$ is **finitely consistent** in \mathcal{M} . By **Theorem 5.11**, $p(\bar{x})$ is **realized** in some $\mathcal{M} \preceq \mathcal{N}$, and $|\mathcal{N}| \geq \lambda$. By **Downward Löwenheim-Skolem**, we may assume $|\mathcal{N}| = \lambda$. \square

6 Saturation

Definition 6.1 (Saturated). Let λ be an infinite cardinal, let $|\mathcal{M}| \geq \omega$. Then \mathcal{M} is λ -saturated if \mathcal{M} realizes every type $p(x)$ with one free variable such that

- (i) $p(x)$ has parameters in $A \subseteq M$ and $|A| < \lambda$.
- (ii) $p(x)$ is finitely consistent in \mathcal{M} .

\mathcal{M} is **saturated** if it is $|\mathcal{M}|$ -saturated.

Can \mathcal{M} be λ -saturated if $\lambda > |\mathcal{M}|$? If so, \mathcal{M} would satisfy finitely satisfiable types in $L(M)$. For example,

$$p(x) = \{x \neq a_i \mid i < |\mathcal{M}|\}$$

where $\langle a_i : i < |\mathcal{M}| \rangle$ enumerates \mathcal{M} . $p(x)$ is finitely satisfiable, but not satisfied in \mathcal{M} .

Lecture 10 **Definition 6.2** (Type of tuple). Let \mathcal{M} be an L -structure, $A \subseteq M$, \bar{b} a tuple in M (possibly infinite). The **type of \bar{b} over A** is the following $L(A)$ -type:

$$\text{tp}_{\mathcal{M}}(\bar{b}/A) := \{ \varphi(\bar{x}) \in L(A) \mid \mathcal{M} \models \varphi(\bar{b}) \}.$$

The subscript \mathcal{M} is often omitted if clear from context.

Remark 6.3.

- (i) $\text{tp}_{\mathcal{M}}(\bar{b}/A)$ is **complete**, i.e. for every $L(A)$ formula $\phi(\bar{x})$, either $\phi(\bar{x}) \in \text{tp}(\bar{b}/A)$ or $\neg\phi(\bar{x}) \in \text{tp}(\bar{b}/A)$.
- (ii) If $\mathcal{M} \preceq \mathcal{N}$, then for $A \subseteq M$, \bar{b} a tuple:

$$\text{tp}_{\mathcal{M}}(\bar{b}/A) = \text{tp}_{\mathcal{N}}(\bar{b}/A).$$

Fact 6.4.

- (i) If $f : A \subseteq \mathcal{M} \rightarrow \mathcal{N}$ is a (partial) elementary map, then in particular f preserves L -sentences, so $\mathcal{M} \equiv \mathcal{N}$.
- (ii) If $\mathcal{M} \equiv \mathcal{N}$, then \emptyset , the empty map, is an elementary map, as it preserves sentences.
- (iii) If $f : A \subseteq \mathcal{M} \rightarrow \mathcal{N}$ is elementary, and \bar{a} is an enumeration of $A = \text{dom}(f)$, then

$$\text{tp}(\bar{a}/\emptyset) = \text{tp}(f(\bar{a})/\emptyset).$$

More generally, if $f : \mathcal{M} \rightarrow \mathcal{N}$ is (partial) elementary and there is $A \subseteq M \cap N$ such that $A \subseteq \text{dom } f$, $f|_A = \text{id}$, then for every \bar{b} , a tuple in $\text{dom}(f)$,

$$\text{tp}_{\mathcal{M}}(\bar{b}/A) = \text{tp}_{\mathcal{N}}(f(\bar{b})/A).$$

- (iv) Let \bar{a} enumerate $A \subseteq M$, $A = \text{dom}(f)$ where $f : \mathcal{M} \rightarrow \mathcal{N}$ is elementary. Let $p(\bar{x}, \bar{a})$ be a type in $L(A)$ that is finitely satisfiable in \mathcal{M} . Then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in \mathcal{N} :

Let

$$\{\varphi_1(\bar{x}, \bar{a}), \dots, \varphi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a}).$$

By finite satisfiability of $p(\bar{x}, \bar{a})$,

$$\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{a}).$$

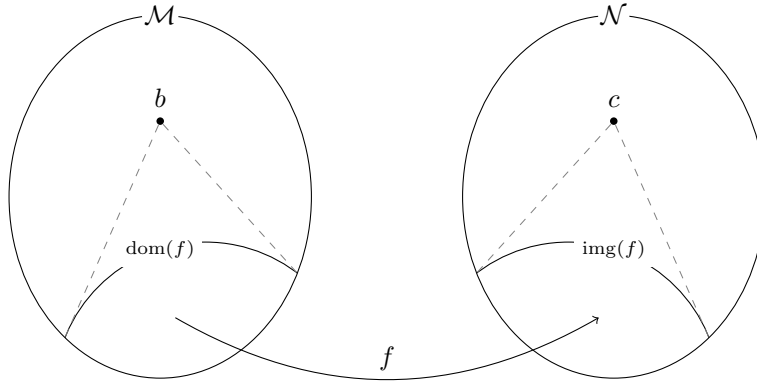
Then $\mathcal{N} \models \exists x \bigwedge_{i=1}^m \varphi_i(\bar{x}, f(\bar{a}))$ by elementarity of f . (Does $p(\bar{x}, \bar{a})$ satisfiable in \mathcal{M} imply $p(\bar{x}, f(\bar{a}))$ satisfiable in \mathcal{N} ? No.)

Theorem 6.5. Let \mathcal{N} be such that $|\mathcal{N}| \geq \lambda \geq |L| + \omega$. The following are equivalent:

- (i) \mathcal{N} is λ -saturated.
- (ii) if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$ and $f : \mathcal{M} \rightarrow \mathcal{N}$ partial elementary map such that $|f| < \lambda$, then there is a partial elementary $\hat{f} \supseteq f$ and such that $b \in \text{dom}(\hat{f})$.
- (iii) If $p(\bar{z})$ is an $L(A)$ -type where $|\bar{z}| \leq \lambda$ and $|A| < \lambda$ and $p(\bar{z})$ is finitely satisfiable in \mathcal{N} , then $p(\bar{z})$ is satisfiable in \mathcal{N} .

Proof. (i) \Rightarrow (ii). Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be as in (ii), let $b \in M$. Let \bar{a} be an enumeration of $\text{dom}(f)$, so $|\bar{a}| < \lambda$. Let

$$p(x/\bar{a}) := \text{tp}_{\mathcal{M}}(b/\bar{a}).$$



Then $p(x/\bar{a})$ is finitely satisfiable in \mathcal{M} , hence $\text{tp}(x/f(\bar{a}))$ is finitely satisfiable in \mathcal{N} (by Fact 6.4(iv)). Since $|f(\bar{a})| < \lambda$ and \mathcal{N} is λ -saturated, $\text{tp}(x/f(\bar{a}))$ is realized in \mathcal{N} by some c . Then $f \cup \{(b, c)\}$ is the required extension of f :

$$\mathcal{M} \models \phi(b, \bar{a}) \iff \mathcal{N} \models \phi(c, f(\bar{a}))$$

Lecture 11

(ii) \Rightarrow (iii). Let $p(\bar{z})$ be as in (iii). There is \mathcal{M} such that $\mathcal{N} \preceq \mathcal{M}$ and $\mathcal{M} \models p(\bar{b})$. The identity map $\text{id}_A : \mathcal{M} \rightarrow \mathcal{N}$ is partial elementary. Idea: build $\langle f_i : i < |\bar{b}| \rangle$ of partial elementary maps extending id_A . Then $\bigcup_i f_i$ is partial elementary, and $\bar{b} \in \text{dom} \bigcup_{i < |\bar{a}|} f_i$.

Set $f_0 = \text{id}_A$, at stage $i + 1$ use (ii) to put b_i in $\text{dom}(f_{i+1})$. At limit stages, $\mu < \lambda$, let $f_\mu = \bigcup_{i < \mu} f_i$.

(iii) \Rightarrow (i) is trivial. □

Corollary 6.6. If \mathcal{M} and \mathcal{N} are saturated and $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}|$ then any elementary $f : \mathcal{M} \rightarrow \mathcal{N}$ extends to an isomorphism (in particular $\mathcal{M} \simeq \mathcal{N}$).

Proof. Use [Theorem 6.5\(ii\)](#) to extend $f : \mathcal{M} \rightarrow \mathcal{N}$ to an **isomorphism** by back-and-forth (take unions at limit stages). \square

Corollary 6.7. Models of T_{dlo} and T_{rg} are ω -saturated.

Proof. By [Theorem 6.5](#) and [Lemma 4.3](#) for T_{dlo} and [Lemma 4.17](#) for T_{rg} . \square

So $(\mathbb{Q}, <)$ is ω -saturated. Is $(\mathbb{R}, <)$ ω_1 saturated? No. It does not realize

$$p(x) := \{x > q \mid q \in \mathbb{Q}\}.$$

Definition 6.8 (Automorphism). An isomorphism $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ is called an **automorphism**. The automorphisms of \mathcal{N} form a group denoted by $\text{Aut}(\mathcal{N})$. If $A \subseteq N$, then

$$\text{Aut}(\mathcal{N}/A) := \{\alpha \in \text{Aut}(\mathcal{N}) \mid \alpha|_A = \text{id}\}.$$

Definition 6.9 (Universality, homogeneity).

- (i) An L -structure \mathcal{N} is λ -**universal** if for every $\mathcal{M} \equiv \mathcal{N}$ such that $|\mathcal{M}| \leq \lambda$ there is an **elementary embedding** $\beta : \mathcal{M} \rightarrow \mathcal{N}$. \mathcal{N} is **universal** if it is $|\mathcal{N}|$ -universal.
- (ii) \mathcal{N} is λ -**homogeneous** if every elementary map $f : \mathcal{N} \rightarrow \mathcal{N}$ such that $|f| < \lambda$ extends to an **isomorphism** of \mathcal{N} .

Theorem 6.10. Let \mathcal{N} be such that $|\mathcal{N}| \geq |L| + \omega$. The following are equivalent

- (i) \mathcal{N} is **saturated**
- (ii) \mathcal{N} is **universal** and **homogeneous**.

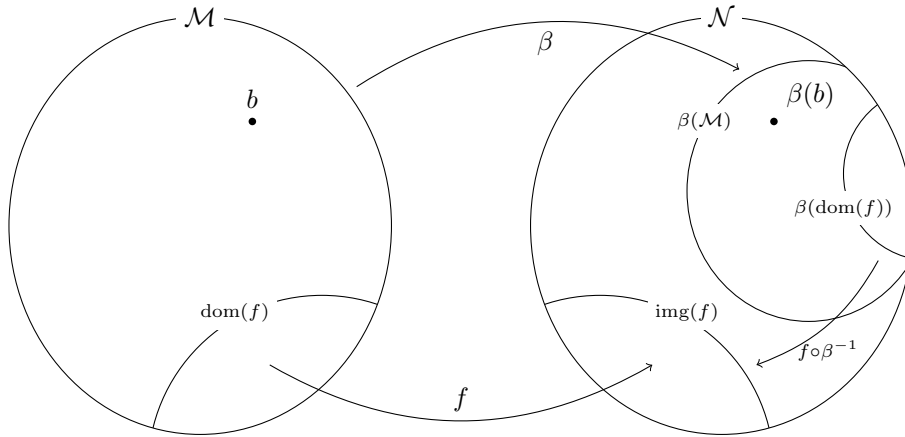
Proof. (i) \Rightarrow (ii). Assume \mathcal{N} is **saturated**, and $\mathcal{M} \equiv \mathcal{N}$ is such that $|\mathcal{M}| \leq |\mathcal{N}|$. Then let \bar{a} enumerate \mathcal{M} , let $p(\bar{x}) = \text{tp}(\bar{a}/\emptyset)$. Then $p(\bar{x})$ is **finitely satisfiable** in \mathcal{M} .

Claim: $p(\bar{x})$ is finitely satisfiable in \mathcal{N} . Indeed, let $\{\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x})$, and so $\mathcal{N} \models \exists \bar{x} \bigwedge_{i=1}^n \varphi_i(\bar{x})$ since $\mathcal{M} \equiv \mathcal{N}$.

Since $|\bar{x}| \leq |\mathcal{N}|$, \mathcal{N} realizes $p(\bar{x})$ by saturation ([Theorem 6.5](#)). **Homogeneity** follows from [Corollary 6.6](#).

(ii) \Rightarrow (i). We show that if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$, $f : \mathcal{M} \rightarrow \mathcal{N}$ **elementary** such that $|f| < |\mathcal{N}|$ then there is $\hat{f} \supseteq f$ elementary defined on b .

By working in $\mathcal{M}' \preccurlyeq \mathcal{M}$ such that $\text{dom}(f) \cup \{b\} \subseteq \mathcal{M}'$ if necessary (using [Theorem 3.11](#)), we may assume $|\mathcal{M}| \leq |\mathcal{N}|$. Since $\mathcal{M} \equiv \mathcal{N}$, by **universality** there is an **elementary embedding** $\beta : \mathcal{M} \rightarrow \mathcal{N}$. Then $\beta(\mathcal{M}) \preccurlyeq \mathcal{N}$.



Then the map $f \circ \beta^{-1} : \beta(\text{dom}(f)) \rightarrow \text{img}(f)$ is elementary. By [homogeneity](#), there is $\alpha \in \text{Aut}(\mathcal{N})$ such that $f \circ \beta^{-1} \subseteq \alpha$. Then $f \cup \{\langle b, \alpha(\beta(b)) \rangle\}$ is elementary (it is a restriction of $\alpha \circ \beta$). \square

Definition 6.11 (Orbit, defined set). Let \bar{a} be a tuple in \mathcal{N} and $A \subseteq N$. The **orbit** of \bar{a} over A is the set

$$O_{\mathcal{N}}(\bar{a}/A) = \{ \alpha(\bar{a}) \mid \alpha \in \text{Aut}(\mathcal{N}/A) \}.$$

If $\varphi(\bar{x})$ is an $L(A)$ -formula, then

$$\varphi(\mathcal{N}) := \{ \bar{a} \in N^{|\bar{x}|} \mid \mathcal{N} \models \varphi(\bar{a}) \}$$

is the **set defined by** $\varphi(\bar{x})$. A set is **definable** over A if it is defined by some $L(A)$ -formula. There are analogous notions of a type defining a set, and a set being type-definable.

Lecture 12 **Remark 6.12.** If \bar{a}, \bar{b} are tuples in \mathcal{N} of the same length, and $A \subseteq N$, then the following are equivalent.

- (i) $\text{tp}_{\mathcal{N}}(\bar{a}/A) = \text{tp}_{\mathcal{N}}(\bar{b}/A)$
- (ii) $\{ a_i \mapsto b_i \mid i < |\bar{a}| \} \cup \text{id}_A$ is an [elementary map](#) from \mathcal{M} to \mathcal{N}

Proposition 6.13. Let \mathcal{N} be λ -homogeneous, $A \subseteq N$, with $|A| < \lambda$ and let \bar{a} a tuple in \mathcal{N} such that $|\bar{a}| < \lambda$. Then

$$O_{\mathcal{N}}(\bar{a}/A) = p(\mathcal{N})$$

where $p(\bar{x}) = \text{tp}_{\mathcal{N}}(\bar{b}/A)$.

Proof. If $\alpha(\bar{a}) = \bar{b}$, where $\alpha \in \text{Aut}(\mathcal{N}/A)$, then $\text{tp}_{\mathcal{N}}(\bar{a}/A) = \text{tp}_{\mathcal{N}}(\bar{b}/A)$.

If $\text{tp}_{\mathcal{N}}(\bar{a}/A) = \text{tp}_{\mathcal{N}}(\bar{b}/A)$, then $\{ \langle a_i, b_i \rangle \mid i < |\bar{a}| \} \cup \text{id}_A$ is [elementary](#), and by [homogeneity](#) it extends to $\alpha \in \text{Aut}(\mathcal{N})$, and in particular $\alpha \in \text{Aut}(\mathcal{N}/A)$. \square

7 The Monster Model

Given a **complete theory** T with an infinite **model**, we work in a **saturated structure** \mathcal{U} (sometimes denoted \mathbb{M}) that is a model of T , which is sufficiently large such that any other model of T we might be interested in is an **elementary substructure** of \mathcal{U} . (\mathcal{U} is an expository device - see Tent/Ziegler for more details, also Marker).

Definition 7.1 (Terminology and conventions). When working in \mathcal{U} , we say

- ‘ $\varphi(\bar{x})$ **holds**’ to mean that $\mathcal{U} \models \forall \bar{x} \varphi(\bar{x})$
- ‘ $\varphi(\bar{x})$ is **consistent**’ to mean $\mathcal{U} \models \exists \bar{x} \varphi(\bar{x})$
- ‘the type $p(\bar{x})$ is **consistent/satisfiable**’ to mean $\mathcal{U} \models \exists \bar{x} p(\bar{x})$
- A cardinality λ is **small** if $\lambda < |U|$ (usually denote $|U|$ by κ)
- a **model** is some $\mathcal{M} \preceq \mathcal{U}$ such that $|M|$ is small

Conventions:

- all tuples assumed to have small length, unless specified otherwise
- **formulas** have parameters in U
- **types** have parameters in small sets
- **definable sets** have the form $\varphi(U)$ for some $L(U)$ -formula $\varphi(\bar{x})$
- **type definable sets** have the form $p(U)$ for some type $p(\bar{x}, A)$ where $|A| < \kappa$.
- Orbits and types of tuples are within \mathcal{U} , so $\text{tp}(\bar{a}/A)$ means $\text{tp}_{\mathcal{U}}(\bar{a}/A)$,

$$O(\bar{a}/A) = O_{\mathcal{U}}(\bar{a}/A)$$

- If $p(\bar{x})$, $q(\bar{x})$ are **types**, we write $p(\bar{x}) \rightarrow q(\bar{x})$ to mean $p(\mathcal{N}) \subseteq q(\mathcal{N})$ (think of $p(\bar{x})$ as an infinite conjunction of formulas)

Fact 7.2. Let $p(\bar{x})$ be a **satisfiable** $L(A)$ -type, and $q(\bar{x})$ a satisfiable $L(B)$ -type, such that

$$p(\bar{x}) \rightarrow \neg q(\bar{x})$$

(explicitly, $p(\bar{x})$ and $q(\bar{x})$ have no common realisations).

Then there are $\varphi_i(\bar{x}) \in p(\bar{x})$ and $\psi_i(\bar{x}) \in q(\bar{x})$ such that

$$\bigwedge_{i=1}^n \varphi_i(\bar{x}) \rightarrow \neg \left(\bigwedge_{i=1}^m \psi_i(\bar{x}) \right).$$

Proof. $p(\bar{x}) \cup q(\bar{x})$ is not **realized** in \mathcal{U} . By **saturation** of \mathcal{U} , $p(\bar{x}) \cup q(\bar{x})$ is not **finitely satisfiable**, hence there exist finite subsets $\{\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$, $\{\psi_1(\bar{x}), \dots, \psi_m(\bar{x})\} \subseteq q(\bar{x})$ such that their union is not satisfiable. Then

$$\bigwedge \varphi_i(\bar{x}) \rightarrow \neg \left(\bigwedge \psi_i(\bar{x}) \right). \quad \square$$

Remark 7.3. Let $\varphi(\mathcal{U}, \bar{b})$ be such that $\varphi(\bar{x}, \bar{z})$ is an L -formula, $\bar{b} \in \mathcal{U}^{|\bar{z}|}$. If $\alpha \in \text{Aut}(\mathcal{U})$, then

$$\begin{aligned}\alpha[\varphi(\mathcal{U}, \bar{b})] &= \{ \alpha(\bar{a}) \mid \varphi(\bar{a}, \bar{b}), \bar{a} \in \mathcal{U}^{|\bar{x}|} \} \\ &= \{ \alpha(\bar{a}) \mid \varphi(\alpha(\bar{a}), \alpha(\bar{b})), \bar{a} \in \mathcal{U}^{|\bar{x}|} \} \\ &= \varphi(\mathcal{U}, \alpha(\bar{b}))\end{aligned}$$

So $\text{Aut}(\mathcal{U})$ acts on the definable sets in a natural way. (Similarly for the type-definable sets)

Definition 7.4 (Invariant). A set $D \subseteq U^\lambda$ is **invariant** under $\text{Aut}(\mathcal{U}/A)$ (**invariant over** A) if $\alpha(D) = D$ for every $\alpha \in \text{Aut}(\mathcal{U}/A)$.

Equivalently, for all $\bar{a} \in D$, $O(\bar{a}/A) \subseteq D$.

If $\bar{a} \in D$, $q(\bar{x}) = \text{tp}(\bar{a}/A)$ and $\bar{b} \models q(\bar{x})$, then $\bar{b} \in D$. ($\text{tp}(\bar{b}/A) = \text{tp}(\bar{a}/A)$, so there is $\alpha \in \text{Aut}(\mathcal{U}/A)$ s.t. $\alpha(\bar{a}) = \bar{b}$ by **homogeneity** of \mathcal{U}). Hence we could also define invariance over A as

$$\forall \bar{a} \in D, \quad \bar{b} \equiv_A \bar{a} \implies \bar{b} \in D.$$

Proposition 7.5. Let $\varphi(\bar{x})$ be an $L(U)$ -formula, then the following are equivalent:

- (i) $\varphi(\bar{x})$ is equivalent to some $L(A)$ -formula $\psi(\bar{x})$
- (ii) $\varphi(\mathcal{U})$ is **invariant** over A

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (i): Let $\varphi(\bar{x}, \bar{z})$ be an L -formula such that $\varphi(\mathcal{U}, \bar{b})$ is **invariant** over A , for suitable $\bar{b} \in U^{|\bar{z}|}$.

Let $q(\bar{z})$ be the **type** $\text{tp}(\bar{b}/A)$. If $\bar{c} \models q(\bar{z})$, then there is $\alpha \in \text{Aut}(\mathcal{U}/A)$ such that $\alpha(\bar{b}) = \bar{c}$. Then

$$\begin{aligned}\varphi(\mathcal{U}, \bar{c}) &= \alpha(\varphi(\mathcal{U}, \bar{b})) && \text{by Remark 7.3} \\ &= \varphi(\mathcal{U}, \bar{b}) && \text{by invariance}\end{aligned}$$

Hence

$$q(\bar{z}) \rightarrow \forall \bar{x} (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b}))$$

by an argument similar to **Fact 7.2**.

There is $\theta(\bar{z}) \in q(\bar{z})$ such that $\theta(\bar{z}) \rightarrow \forall \bar{x} (\varphi(\bar{x}, \bar{z}) \leftrightarrow \varphi(\bar{x}, \bar{b}))$. Then $\theta(\bar{z})$ is an $L(A)$ -formula and $\exists z [\theta(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})]$ defines $\varphi(\mathcal{U}, \bar{b})$. \square

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