Part III – Algebraic Topology (Ongoing course, rough)

Based on lectures by Professor I. Smith

Notes taken by Bhavik Mehta

Michaelmas 2018

Contents

0 Introduction

Algebraic topology concerns the connectivity properties of topological spaces. Recall a space X is **connected** if we cannot write $X = U \cup V$ where U, V are non-empty, open and disjoint.

Example. $\mathbb{R}$ is connected (with its Euclidean topology), $\mathbb{R} \setminus \{0\}$ is not connected.

Corollary (Intermediate value theorem)**.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0, f(y) < 0$, then there is some z lying between x, y such that $f(z) = 0$.

Proof. If $f(z) \neq 0$ for all z, then $\mathbb{R} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ is disconnected. $\qquad\square$

For nice spaces, connected $\iff$ path-connected. Recall: a space X is path-connected if $\forall x, y \in X, \exists \gamma : [0, 1] \to X$ continuous such that $\gamma(0) = x, \gamma(1) = y$. Informally, any two maps of a point to X can be continuously deformed into one another.
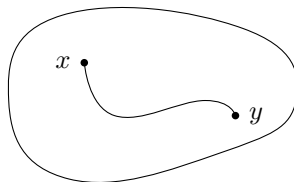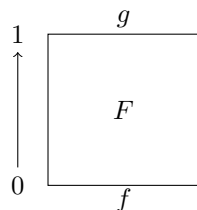

Definition (Homotopy)**.** If X, Y are topological spaces and $f, g : X \to Y$ are (continuous) maps, then f is **homotopic** to g if

$$\exists F : X \times [0, 1] \to Y$$

continuous such that

$$F|_{X \times \{0\}} = f, F|_{X \times \{1\}} = g.$$

Write $f \simeq g$ or $f \simeq_F g$ and schematically

Definition (Simply connected)**.** A path-connected space X is **simply connected** if every two continuous maps $S^1 \to X$ are homotopic. Here $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$ is the n dimensional sphere, S^1 is the circle $\subseteq \mathbb{C}$.

Example. $\mathbb{R}^2$ is simply connected but $\mathbb{R}^2 \setminus \{0\}$ is not. In fact, continuous maps $S^1 \xrightarrow{\gamma} \mathbb{R}^2 \setminus \{0\}$ have a degree $\deg(\gamma) \in \mathbb{Z}$, invariant under homotopy. (If γ was differentiable, we could set $\deg(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \in \mathbb{Z}$.) If $\gamma_n : S^1 \to \mathbb{R}^2 \setminus \{0\}$ has $t \mapsto e^{2\pi i n t}$ then $\deg(\gamma_n) = n$.

Corollary (Fundamental theorem of algebra)**.** Every nonconstant complex polynomial has a root.

Proof. Let $f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ be a complex polynomial and suppose $f(z) \neq 0 \ \forall z \in \mathbb{C}$. Let $\gamma_R(t) = f(Re^{2\pi it})$ so $\gamma_R : S^1 \to \mathbb{R}^2 \setminus \{0\}$. Now γ_0 is a constant map, so $\deg(\gamma_0) = 0$. By homotopy invariance of degree, $\deg(\gamma_R) = 0 \ \forall R$.

If $R \gg \sum_i |a_i|$, we can consider $f_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_n)$ for $0 \leq s \leq 1$, and on the circle $Re^{2\pi it}$, f_s also takes values in $\mathbb{R}^2 \setminus \{0\}$. If $\gamma_{R,s}(t) = f_s(Re^{2\pi it})$, then $\gamma_{R,1} = \gamma_R$ but $\gamma_{R,0} : z \mapsto z^n$, which has degree n. Now $\deg(\gamma_0) = \deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0})$, so $n = 0$ and f is constant. $\qquad\square$

Fact. Any two maps $S^n \to \mathbb{R}^{n+1}$ are homotopic but maps $S^n \xrightarrow{f} \mathbb{R}^{n+1} \setminus \{0\}$ have a degree $\deg(f) \in \mathbb{Z}$, invariant up to homotopy. Moreover, the degree of the constant map is 0 and the degree of inclusion is 1.

Corollary (Brouwer's fixed point theorem)**.** If $B^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$, any continuous map $f : B^n \to B^n$ has a fixed point.

Proof. Suppose f has no fixed point. Let $\gamma_R : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be the map $v \mapsto R_v - f(R_v)$ for $0 \leq R \leq 1$. So γ_0 is a constant, so has degree 0. Hence $\deg(\gamma_1) = 0$.

Let $\gamma_{1,s}(v) := v - sf(v)$, for $0 \leq s \leq 1$ and $v \in S^{n-1}$. Note $\gamma_{1,s}$ has image in $\mathbb{R}^n \setminus \{0\}$: if $s = 1$, this is because $v \neq f(v) \ \forall v$ and if $s < 1$ then $|v| > |sf(v)|$. Therefore $\gamma_1 = \gamma_{1,1}$ has the same degree as $\gamma_{1,0}$ which is the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$, a contradiction. $\qquad\square$

Definition (Homotopy equivalence)**.** We say spaces X, Y are **homotopy equivalent** if $\exists$ maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$.

Example.

- Trivial case: if X, Y are homeomorphic, i.e. $X \cong Y$, then clearly $X \simeq Y$.

- $\mathbb{R}^n \simeq \{0\}$, the single point. A space homotopy equivalent to a point is sometimes called contractible.

- $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. If $i : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ inclusion, and $p : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is projection $v \mapsto \frac{v}{\|v\|}$, then $p \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ p \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy

$$F : \mathbb{R}^n \setminus \{0\} \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}$$
$$(v,t) \longmapsto tv + (1-t)\frac{v}{\|v\|}$$

Algebraic topology is the study of the set of spaces up to homotopy equivalence via the set of groups up to isomorphism.

The first naive attempt would be homotopy groups: Loops (continuous maps $S^1 \to X$) with a common base-point can be concatenated and this induces a group structure on the set of homotopy classes of maps $(S^1, *) \to (X, x_0)$. This refers to continuous maps $S^1 \to X$ taking $* \mapsto x_0$ and a based homotopy $F : f \simeq g$ of two such is one such that $F|_{S \times \{t\}}$ sends $*$ to $x_0 \ \forall t$.

Again, there is a group structure on the set of based homotopy classes of maps $(S^n, *) \to (X, x_0)$ calld $\pi_n(X, x_0)$, the n-th homotopy group of X.

Fact. $\{\pi_n(S^2, x)\}_{n \geq 1}$ is not known. Indeed, there is no simply connected manifold of dimension > 0 for which all π_n are known.

Instead, we will focus on homology theory, more precisely singular (co)homology. We will obtain invariants of spaces in a two-step process:

(a) Associate to X a chain complex (or cochain complex).

(b) take (co)homology of that complex.

This will be rather computable for simple spaces. In this course, we will mostly focus on studying manifolds.

Definition (Chain complex, cochain complex). A **chain complex** (C_*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

(indexed by $\mathbb{N}$ or $\mathbb{Z}$) with the key property that $\forall n \; d_{n-1} \circ d_n = 0$ Then $\text{image}(d_{n+1}) \subseteq \ker(d_n)$ and the n-th homology group $H_n(C_*, d)$ of the chain complex is the quotient

$$H_n(C_*, d) := \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})}.$$

A **cochain complex** (C^*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} \equiv 0 \; \forall n$ The n-th cohomology group $H^n(C^*, d)$ is the quotient

$$H^n(C^*, d) := \frac{\ker(d^n)}{\operatorname{im}(d^{n-1})}.$$