Part II – Graph Theory

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0 Introduction

0.1 Preliminary

0.2 Informal definitions

0.3 Where do such structures arise?

Theorem (Schur's Theorem)**.** Let n be a positive integer. Then if p is a sufficiently large positive integer, whenever $\{1, 2, \ldots, p\}$ is partitioned into n parts, we can solve $a + b = c$ with a, b, c all in some part.

1 Ramsey Theory

Theorem (Schur's Theorem reformulated)**.** Let k be a positive integer. Then there is a positive integer n such that if the set $[n] = \{1, 2, \ldots, n\}$ is coloured with k colours, we can find a, b, c with $a + b = c$ and a, b, c the same colour.

Proposition 1. Let k be a positive integer. Then there is a positive integer n such that whenever the edges of K_n are coloured with k colours we can find a monochromatic triangle.

Theorem 2 (Ramsey's Theorem)**.** $R(s,t)$ exists for all $s, t \geq 2$. Moreover, if $s, t > 2$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Corollary 3. For all $s, t \geq 2$, $R(s,t) \leq 2^{s+t}$, so $R(s) \leq 4^s$.

Theorem 4 (Multicolour Ramsey Theorem)**.** Let $k \geq 1$ and $s \geq 2$. Then there exists some n such that whenever the edges of K_n are coloured with k colours, we can find a monochromatic K_s.

Theorem 5 (Infinite Ramsey Theorem)**.** Let $k \geq 1$. Whenever the edges of K_∞ are k-coloured, we have a monochromatic K_∞ subgraph.

Corollary 6. Any bounded sequence has a convergent subsequence.

1.1 Basic Terminology

2 Extremal Graph Theory

2.1 Forbidden Subgraph Problem

2.1.1 Triangles

Theorem 7. A graph is bipartite iff it contains no odd cycles.

Theorem 8 (Mantel's Theorem)**.** Let $n \geq 3$. Suppose $|G| = n$, $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$ and $\triangle \not\subset G$. Then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

2.1.2 Complete graphs

Theorem 9 (Turán's Theorem)**.** Let $r \geq 2$ and $|G| = n \geq r + 1$. If $e(G) \geq t_r(n)$ and $K_{r+1} \not\subset G$ then $G \cong T_r(n)$.

Corollary 10. Let $r \geq 2$. As $n \to \infty$, $\mathrm{ex}(n; K_{r+1}) \sim (1 - \frac{1}{r})\binom{n}{2}$.

2.1.3 Bipartite graphs

Theorem 11. Let $t \geq 2$. Then $\mathrm{ex}(n; K_{t,t}) = \mathcal{O}(n^{2-\frac{1}{t}})$.

Theorem 12. Let $t \geq 2$. Then $z(n, t) = \mathcal{O}(n^{2-\frac{1}{t}})$.

2.1.4 General graphs

Proposition 13. Let H be a graph with at least one edge, and for $n \geq |H|$, let $x_n = \frac{\mathrm{ex}(n;H)}{\binom{n}{2}}$. Then (x_n) converges.

Theorem 14 (Erdős-Stone Theorem)**.** Let $r, t \geq 1$ be integers, and let $\epsilon > 0$ be real. Then $\exists n_0$ such that $\forall n \geq n_0$,

$$|G| = n,\ e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} \implies K_{r+1}(t) \subset G.$$

Corollary 15. Let H be a graph with at least one edge. Then

$$\mathrm{ex}(H) = 1 - \frac{1}{\chi(H) - 1}.$$

Corollary 16. For any infinite graph G,

$$\mathrm{ud}(G) \in \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}.$$

2.1.5 Proof of Erdős-Stone (non-examinable)

2.2 Hamiltonian graphs

Theorem 17 (Dirac's Theorem)**.** Let $|G| = n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.

Proposition 18. Let G be a connected graph. Then

$$G \text{ Eulerian if and only if } \forall v \in G,\ d(v) \text{ is even.}$$

3 Graph Colouring

3.1 Planar Graphs

Theorem 19 (Kuratowski's Theorem)**.** Let G be a graph. Then G planar iff G contains no subdivision of K_5 or $K_{3,3}$.

Proposition 20. Every tree of order at least 2 has a leaf.

Proposition 21. Let T be a tree, $|T| = n \geq 1$. Then $e(T) = n - 1$.

Proposition 22. Every tree is planar.

Theorem 23 (Euler's Formula)**.** Take G connected and planar. Take $|G| = n \geq 1$, $e(G) = m$ with l faces. Then $n - m + l = 2$.

Corollary 24. Let G be planar, $|G| = n \geq 3$. Then $e(G) \leq 3n - 6$.

Proposition 25 (Six colour theorem)**.** Any planar graph is 6-colourable.

Theorem 26 (Five colour theorem)**.** Any planar graph is 5-colourable.

Theorem 27 (Four colour theorem)**.** Any planar graph is 4-colourable.

3.2 General Graphs

Theorem 28 (Brooks' theorem)**.** Let G be a connected graph that is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

3.3 Graphs on surfaces

Theorem 29 (Heawood's Theorem)**.** Let S be a closed boundaryless surface of Euler characteristic $E \leq 1$. Then

$$\chi(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

3.4 Edge Colouring

Theorem 30 (Vizing's theorem)**.** Let G be a graph. Then

$$\chi'(G) \leq \Delta(G) + 1$$

4 Connectivity

4.1 The Marriage Problem

Theorem 31 (Hall's Marriage Theorem)**.** Let G be a bipartite graph with bipartition (X, Y). Then G has a matching from X to Y iff G satisfies **Hall's condition**:

$$\forall A \subset X, |\Gamma(A)| \geq |A|$$

Corollary 32 (Defect Hall)**.** Let G be a bipartite graph with bipartition (X, Y) and let $d \geq 1$. Then G contains $|X| - d$ independent edges if and only if $\forall A \subset X,\ |\Gamma(A)| \geq |A| - d$.

Corollary 33 (Polyandrous Hall)**.** Let G be a bipartite graph, bipartition (X, Y), $d \geq 2$. Then G contains a set of $d|X|$ edges, each vertex in X in precisely d of them, each vertex in Y in at most one $\iff \forall A \subset X, |\Gamma(A)| \geq d|A|$.

4.2 Connectivity

Theorem 34. Let G be a graph and $A, B \subset V(G)$. Let

$$k = \min \{\, |W| \mid W \text{ is an } AB\text{-separator} \,\}.$$

Then G contains k vertex-disjoint AB-paths.

Corollary 35 (Menger's Theorem)**.** Let G be an incomplete k-connected graph and let $a, b \in V(G)$, $a \neq b$. Then G contains k independent ab-paths.

4.3 Edge connectivity

Corollary 36 (Edge Menger)**.** Let G be l-edge connected and $a, b \in V(G)$ be distinct. Then G has l edge-disjoint ab-paths.

5 Probabilistic Techniques

5.1 The Probabilistic Method

Theorem 37 (Erdős)**.**

$$R(s) = \Omega(\sqrt{2}^s)$$

5.2 Modifying a Random Graph

Theorem 38. If $t \geq 2$ then $z(n,t) = \Omega(n^{2-\frac{2}{t+1}})$.

Theorem 39. Let $g \geq 3$, $k \geq 2$. Then there is a graph G with no cycles of length $\leq g$ and $\chi(G) \geq k$.

5.3 The Structure of Random Graphs

Proposition 40. $p = \frac{1}{n}$ is a sharp threshold for $G \in \mathcal{G}(n,p)$ to contain a $\triangle$, in the sense that:

- if $p = o(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has no $\triangle$, whereas
- if $p = \omega(\frac{1}{n})$ then almost every $G \in \mathcal{G}(n,p)$ has a $\triangle$.

Theorem 41. There exists a function $d : \mathbb{N} \to \mathbb{N}$ such that a.e. $G \in \mathcal{G}(n,p)$ has $\omega(G) \in \{d-1, d, d+1\}$ (where $d = d(n)$).

Corollary 42. Almost every $G \in \mathcal{G}(n,p)$ has

$$\chi(G) \geq (1 + o(1)) \frac{n \log \frac{1}{q}}{2 \log n}$$

where $q = 1 - p$.

6 Algebraic Methods

6.1 The Chromatic Polynomial

Theorem 43 (Cut-fuse relation)**.** Let G be a graph, $e \in E(G)$, $k \geq 1$. Then $f_G(k) = f_{G-e}(k) - f_{G/e}(k)$.

Corollary 44. Let G be a graph. Then f_G is a polynomial.

Corollary 45. If $|G| = n$, $e(G) = m$ then

$$f_G(X) = X^n - mX^{n-1} + \dots$$

6.2 Eigenvalues

Theorem 46. Let G be a graph, $\Delta(G) = \Delta$, λ an eigenvalue of G. Then $|\lambda| \leq \Delta$. Moreover if G is connected then Δ is an eigenvalue $\iff$ G is Δ-regular; in this case Δ has multiplicity 1 and eigenvector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$.

6.3 Strongly Regular Graphs

Theorem 47 (Rationality condition)**.** Let G be (k, a, b)-strongly regular. Then

$$\frac{1}{2}\left\{(n-1) \pm \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2 - 4(b-k)}}\right\} \in \mathbb{Z}.$$