Part III – Category Theory (Ongoing course)

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0 Introduction

Lecture 1

Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

1 Definitions and Examples

1.1 Definition (Category)**.** A **category** $\mathscr{C}$ consists of

(a) a collection $\mathscr{C}$ of **objects** $A, B, C, \ldots$

(b) a collection $\operatorname{mor}\mathscr{C}$ of **morphisms** $f, g, h, \ldots$

(c) two operations $\operatorname{dom}, \operatorname{cod}$ assigning to each $f \in \operatorname{mor}\mathscr{C}$ a pair of objects, its **domain** and **codomain**. We write $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'.

(d) an operation assigning to each $A \in \operatorname{ob}\mathscr{C}$ a morphism $A \xrightarrow{1_A} A$, called its **identity**.

(e) a partial binary operation **composition** $(f, g) \longmapsto fg$ on morphisms, such that fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod} f$ if fg is defined.

satisfying

(f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$

(g) $(fg)h = f(gh)$ whenever fg and gh are defined

1.2 Remark.

(a) This definition is independent of a model of set theory. If we're given a particular model of set theory, we call the category $\mathscr{C}$ **small** if $\operatorname{ob}\mathscr{C}$ and $\operatorname{mor}\mathscr{C}$ are sets.

(b) Some texts say fg means 'f followed by g', i.e. fg defined $\iff$ $\operatorname{cod} f = \operatorname{dom} g$.

(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

1.3 Examples.

(a) The category **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms $A \longrightarrow B$ are pairs (f, B) where f is a set-theoretic function.)

(b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, $\mathbf{Mod}_R$ the category of R-modules.

(c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.

(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given $\mathscr{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathscr{C}$ a **congruence** if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathscr{C}/\simeq$ with the same objects as $\mathscr{C}$, but congruence classes as morphisms.

(e) Given $\mathscr{C}$, the **opposite category** $\mathscr{C}^{op}$ has the same objects and morphisms as $\mathscr{C}$, but dom and cod are interchanged, and fg in $\mathscr{C}^{op}$ is gf in $\mathscr{C}$. This leads to the **Duality principle** if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

(f) A small category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (f is an **isomorphism** if $\exists g$ such that fg and gf are identities).

(g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space X, the fundamental groupoid $\pi(X)$ has all points of X as objects and morphisms $x \longrightarrow y$ are homotopy classes rel $\{0,1\}$ of paths $u : [0,1] \longrightarrow X$ with $u(0) = x$, $u(1) = y$. (If you know how to prove that the fundamental group is a group, you can prove that $\pi(X)$ is a groupoid.)

(h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category $\mathscr{C}$ in which, for any pair (A, B) there is at most 1 morphism $A \longrightarrow B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as **Set**, but morphisms $A \longrightarrow B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define

$$S \circ R = \{\, (a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R \wedge (b, c) \in S) \,\}\,.$$

The identity $1_A : A \longrightarrow A$ is $\{\, (a, a) \mid a \in A \,\}$.

Similarly, the category **Part** of sets and partial functions (i.e. relations such that $(a, b) \in R, (a, b') \in R \implies b = b'$).

(j) Let K be a field. The category $\mathbf{Mat}_K$ has natural numbers as objects, and morphisms $n \longrightarrow p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.

1.4 Definition (Functor)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \longrightarrow \mathscr{D}$ consists of

(a) a mapping $A \longmapsto FA$ from $\operatorname{ob}\mathscr{C}$ to $\operatorname{ob}\mathscr{D}$

(b) a mapping $f \longmapsto Ff$ from $\operatorname{mor}\mathscr{C}$ to $\operatorname{mor}\mathscr{D}$

such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$ and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Lecture 2 **1.3 Examples** (*Continued*)**.**

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them.

1.5 Examples.

(a) We have **forgetful functors** $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, $\mathbf{Rng} \longrightarrow \mathbf{Set}$, $\mathbf{Top} \longrightarrow \mathbf{Set}$, $\mathbf{Rng} \longrightarrow \mathbf{AbGp}$ (forgetting $\times$), $\mathbf{Rng} \longrightarrow \mathbf{Mon}$ (forgetting $+$).

(b) Given a set A, the free group FA has the property: given any group G and any function $A \xrightarrow{f} UG$, there's a unique homomorphism $FA \xrightarrow{f} G$ extending f. F is a functor $\mathbf{Set} \longrightarrow \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$.

Functoriality follows from uniqueness: given $B \xrightarrow{g} C$, $F(gf)$ and $(Fg)(Ff)$ are both homoms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$.

(c) Given a set A, we write PA for the set of all subsets of A. We can make P into a functor $\mathbf{Set} \longrightarrow \mathbf{Set}$: given $A \xrightarrow{f} B$, we define $Pf(A') = \{ f(a) \mid a \in A' \}$ for $A' \subseteq A$. But we also have a functor $P^* : \mathbf{Set} \longrightarrow \mathbf{Set}^{op}$ defined on objects by P, but $P^* f(B') = \{ a \in A \mid f(a) \in B' \}$ for $B' \subseteq B$.

By a **contravariant** functor $\mathscr{C} \longrightarrow \mathscr{D}$, we mean a functor $\mathscr{C} \longrightarrow \mathscr{D}^{op}$ (or $\mathscr{C}^{op} \longrightarrow \mathscr{D}$). (A **covariant** functor is one that doesn't reverse arrows).

(d) Let K be a field. We have a functor $* : \mathbf{Mod}_K \longrightarrow \mathbf{Mod}_K^{op}$ defined by $V^* = \{\text{linear maps } V \longrightarrow K\}$ and if $V \xrightarrow{f} W$, $f^*(\theta : W \longrightarrow K) = \theta f$.

(e) We have a functor $op : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ which is the 'identity' on morphisms. (Note that this is covariant).

(f) A functor between monoids is a monoid homomorphism.

(g) A functor between posets is an order-preserving map.

(h) Let G be a group. A functor $F : G \longrightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A, i.e. a permutation representation of G (where we use $*$ to refer to the unique object of the group). Similarly a functor $G \longrightarrow \mathbf{Mod}_K$ is a K-linear representation of G.

(i) The construction of a fundamental group $\pi_1(X, x)$ of a space X with basepoint x is a functor $\mathbf{Top}_* \longrightarrow \mathbf{Gp}$ where $\mathbf{Top}_*$ is the set of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \longrightarrow \mathbf{Gpd}$ where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

1.6 Definition (Natural transformation)**.** Let $\mathscr{C}, \mathscr{D}$ be categories and $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ two functors. A **natural transformation** $\alpha : F \longrightarrow G$ consists of an assignment $A \longmapsto \alpha_A$

from ob$\mathscr{C}$ to mor$\mathscr{D}$, such that dom $\alpha_A = FA$ and cod $\alpha_A = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathscr{C}$ the square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_a"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)\alpha_A$).

1.3 Examples (*Continued*)**.**

(l) Given categories $\mathscr{C}, \mathscr{D}$, we write $[\mathscr{C}, \mathscr{D}]$ for the category whose objects are functors $\mathscr{C} \longrightarrow \mathscr{D}$, and whose morphisms are natural transformations.

1.7 Examples.

(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \longrightarrow V^{**}$ given by

$$\alpha_V(v)(\theta) = \theta(v)$$

for $\theta \in V^*$. This is the V-component of a natural transformation

$$1_{\mathbf{Mod}_K} \longrightarrow ** : \mathbf{Mod}_K \longrightarrow \mathbf{Mod}_K.$$

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(b) For any set A, we have a mapping $\sigma_A : A \longrightarrow PA$ sending a to $\{a\}$. If $f : A \longrightarrow B$, then $Pf\{a\} = \{f(a)\}$, so σ is a natural transformation $1_{\mathbf{Set}} \longrightarrow P$.

(c) Let $F : \mathbf{Set} \longrightarrow \mathbf{Gp}$ be the free group functor (Examples 1.5(b)) and $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \longrightarrow UFA$ form a natural transformation $1_{\mathbf{Set}} \longrightarrow UF$.

(d) Let G, H be groups and $f, g : G \longrightarrow H$ two homomorphisms. A natural transformation $\alpha : f \longrightarrow g$ corresponds to an element $h = \alpha_*$ of H such that $h.f(x) = g(x).h$ for all $x \in G$, or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.

(e) Let A, B be two G-sets regarded as functors $G \longrightarrow \mathbf{Set}$. A natural transformation $A \longrightarrow B$ is a function f satisfying $f(g.a) = g.f(a)$ for all $a \in A$, i.e. a G-equivariant map.

1.8 Lemma. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors, and $\alpha : F \longrightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathscr{C}, \mathscr{D}]$ iff each α_A is an isomorphism in $\mathscr{D}$.

Proof.

$\Longrightarrow$ trivial

$\Longleftarrow$ Suppose each α_A has an inverse β_A. Given $f : A \longrightarrow B$ in $\mathscr{C}$, we need to show that

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, "\beta_A"] & GB \arrow[d, "\beta_B"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

commutes.

But

$$\begin{aligned}(Ff)\beta_A &= \beta_B\alpha_B(Ff)\beta_A \\ &= \beta_B(Gf)\alpha_A\beta_A \\ &= \beta_B(Gf).\end{aligned}$$

□

1.9 Definition (Equivalent category)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. By an **equivalence** between $\mathscr{C}$ and $\mathscr{D}$, we mean a pair of functors $F : \mathscr{C} \longrightarrow \mathscr{D}$, $G : \mathscr{D} \longrightarrow \mathscr{C}$ together with natural isomorphisms $\alpha : 1_{\mathscr{C}} \longrightarrow GF$ and $\beta : FG \longrightarrow 1_{\mathscr{D}}$. We write $\mathscr{C} \simeq \mathscr{D}$ if $\mathscr{C}$ and $\mathscr{D}$ are equivalent.

We say a property P of categories is a **categorical property** if whenever $\mathscr{C}$ has P and $\mathscr{C} \simeq \mathscr{D}$, then $\mathscr{D}$ has P.

For instance, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

1.10 Examples.

(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint-preserving functions). We define $F : \mathbf{Set}_* \longrightarrow \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$ and if $f : (A, a) \longrightarrow (B, b)$

$$Ff(x) = \begin{cases} f(x) & \text{if } f(x) \neq b \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $G : \mathbf{Part} \longrightarrow \mathbf{Set}_*$ by $G(A) = A^+ = A \cup \{A\}$ and if $f : A \longrightarrow B$ is a partial function, we define $Gf : A^+ \longrightarrow B^+$ by

$$Gf = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

The composite FG is the identity on **Part**, but GF is not the identity, however there's an isomorphism

$$(A, a) \longrightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism $\mathbf{Set}_* \longrightarrow \mathbf{Part}$ since **Part** has a 1-element isomorphism class $\{\varnothing\}$ and $\mathbf{Set}_*$ doesn't.

(b) The category $\mathbf{FdMod}_K$ of finite-dimensional vector spaces over K is equivalent to $\mathbf{FdMod}_K^{op}$: the functors in both directions are $(-)^*$ and both isomorphisms are the natural transformations of Examples 1.7(a).

(c) $\mathbf{FdMod}_K$ is also equivalent to $\mathbf{Mat}_K$: We define $F : \mathbf{Mat}_K \longrightarrow \mathbf{FdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A with respect to the standard bases of K^n and K^p.

To define $G : \mathbf{FdMod}_K \longrightarrow \mathbf{Mat}_K$, choose a basis for each finite dimensional vector space,and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ as the matrix representing f with respect to the chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen basis gives isomorphisms $FG(V) = K^{\dim V} \longrightarrow V$ for each V, which form a natural isomorphism.

Lecture 4 **1.11 Definition** (Faithful, full, essentially surjective)**.** Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor.

(a) We say F is **faithful** if, given $f, f' \in \operatorname{mor}\mathscr{C}$ with $\operatorname{dom} f = \operatorname{dom} f'$, $\operatorname{cod} f = \operatorname{cod} f'$ and $Ff = Ff'$ then $f = f'$.

(b) We say F is **full** if, given $FA \xrightarrow{g} FB$ in $\mathscr{D}$, there exists $A \xrightarrow{f} B$ in $\mathscr{C}$ with $Ff = g$.

(c) We say F is **essentially surjective** if, for every $B \in \operatorname{ob}\mathscr{D}$, there exists $A \in \operatorname{ob}\mathscr{C}$ and an isomorphism $FA \longrightarrow B$ in $\mathscr{D}$.

We say a subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is **full** if the inclusion $\mathscr{C}' \longrightarrow \mathscr{C}$ is a full functor.

Example. **Gp** is a full subcategory of **Mon**, but **Mon** is not a full subcategory of the category **Sgp** of semigroups.

1.12 Lemma. Assuming the axiom of choice, a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is part of an equivalence $\mathscr{C} \simeq \mathscr{D}$ iff it is full, faithful and essentially surjective.

Proof.

$\Longrightarrow$ Given G, α, β as in Definition 1.9, for each $B \in \operatorname{ob}\mathscr{D}$, β_B is an isometry $FGB \longrightarrow B$, so F is essentially surjective.

Given $A \xrightarrow{f} B$ in $\mathcal{C}$, we can recover f from Ff as the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B.$$

Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$.

Given $FA \xrightarrow{g} FB$, define f to be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$$

Then $GFf = \alpha_B f \alpha_A^{-1} = Gg$, and G is faithful for the same reason as F, so $Ff = g$.

$\Longleftarrow$ For each $B \in \operatorname{ob}\mathscr{D}$, choose $GB \in \operatorname{ob}\mathscr{C}$ and an isomorphism $\beta_B : FGB \longrightarrow B$ in $\mathscr{D}$. Given

$$B \xrightarrow{g} B'$$

define $Gg : GB \longrightarrow GB'$ to be the unique morphism whose image under F is

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

Uniqueness implies functoriality: given

$$B' \xrightarrow{g'} B''$$

then note $(Gg')(Gg)$ and $G(g'g)$ have the same image under F, so they're equal.

By construction, β is a natural transformation $FG \longrightarrow 1_{\mathscr{D}}$.

Given $A \in \text{ob}\,\mathscr{C}$, define $\alpha_A : A \longrightarrow GFA$ to be the unique morphism whose image under F is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA$$

α_A is an isomorphism, since β_{FA} also has a unique pre-image under F.

Also α is a natural transformation, since any naturality square for α is mapped by F to a commutative square, and F is faithful. $\square$

1.13 Definition (Skeleton)**.** By a **skeleton** of a category $\mathscr{C}$, we mean a full subcategory $\mathscr{C}_0$ containing one object from each isomorphism class. We say $\mathscr{C}$ is **skeletal** if it's a skeleton of itself.

Example. $\mathbf{Mat}_K$ is skeletal, and the image of $F : \mathbf{Mat}_K \longrightarrow \mathbf{FdMod}_K$ of Examples 1.10(c) is a skeleton of $\mathbf{FdMod}_K$.

Warning. Almost any assertion about skeletons is equivalent to the axiom of choice. See question 2 on example sheet 1.

1.14 Definition (Monomorphism, epimorphism)**.** Let $A \xrightarrow{f} B$ be a morphism in $\mathscr{C}$

(a) We say f is a **monomorphism** (or f is **monic**) if, given any pair $C \underset{g}{\overset{f}{\rightrightarrows}} A$, $fg = fh$ implies $g = h$

(b) We say f is an **epimorphism** (or **epic**) if it's a monomorphism in C^{op} i.e. if $gf = hf$ implies $g = h$.

We denote monomorphisms by $A \overset{f}{\rightarrowtail} B$ and epimorphisms by $A \overset{f}{\twoheadrightarrow} B$

Any isomorphism is monic and epic: more generally if f has a left inverse (i.e. $\exists g$ such that gf is an identity) then it's monic. We call such monomorphisms **split**.

We say C is a **balanced** category if any morphism which is both monic and epic is an isomorphism.

1.15 Examples.

(a) In **Set**, mono $\iff$ injective ($\implies$ easy; for $\impliedby$ take $C = 1 = \{*\}$) and epi $\iff$ surjective ($\implies$ easy; for $\impliedby$ use two morphisms $B \longrightarrow 2 = \{0, 1\}$). So **Set** is balanced.

(b) In **Gp** mono $\iff$ injective (for $\impliedby$ use homoms $\mathbb{Z} \longrightarrow A$) and epi $\iff$ surjective ($\impliedby$ uses free products with amalgamation). So **Gp** is balanced.

(c) In **Rng**, mono $\iff$ injective (proof much as for **Gp**) but the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} R$ agree on all integers, they agree everywhere. So **Rng** isn't balanced.

(d) In **Top**, mono $\iff$ injective and epi $\iff$ surjective (proofs as in **Set**). But **Top** isn't balanced since a continuous bijection needn't have a continuous inverse.

2 The Yoneda Lemma

Lecture 5 **2.1 Definition** (Locally small)**.** We say a category $\mathscr{C}$ is **locally small** if, for any two objects A, B, the morphisms $A \longrightarrow B$ in $\mathscr{C}$ form a set $\mathscr{C}(A, B)$.

If we fix A and let B vary, the assignment $B \longmapsto \mathscr{C}(A, B)$ becomes a functor $\mathscr{C}(A, -) : \mathscr{C} \longrightarrow \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathscr{C}(A, f)$ is the mapping $g \longmapsto fg$. Similarly, $A \longmapsto \mathscr{C}(A, B)$ defines a functor $\mathscr{C}(-, B) : \mathscr{C}^{op} \longrightarrow \mathbf{Set}$.

2.2 Lemma (Yoneda Lemma)**.** Let $\mathscr{C}$ be a locally small category, $A \in \operatorname{ob}\mathscr{C}$ and $F : \mathscr{C} \longrightarrow \mathbf{Set}$ a functor.

(i) Then natural transformations $\mathscr{C}(A, -) \longrightarrow F$ are in bijection with elements of FA.

(ii) Moreover, this bijection is natural in both A and F.

Proof of Lemma 2.2(i). Given $\alpha : \mathscr{C}(A, -) \longrightarrow F$, we define

$$\Phi(\alpha) = \alpha_A(1_A) \in FA.$$

Given $x \in FA$, we define $\Psi(x) : \mathscr{C}(A, -) \longrightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB.$$

$\Psi(x)$ is natural: given $g : B \longrightarrow C$, we have

$$\begin{aligned} \Psi(x)_C \mathscr{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x) \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x). \end{aligned}$$

$$\begin{tikzcd} \mathscr{C}(A,B) \arrow[r, "\mathscr{C}(A,g)"] \arrow[d, "\Psi(x)_B"'] & \mathscr{C}(A,C) \arrow[d, "\Psi(x)_C"] \\ FB \arrow[r, "Fg"'] & FC \end{tikzcd}$$

We also verify Ψ and Φ are inverse:

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x.$$

Given α,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathscr{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

so $\Psi\Phi(\alpha) = \alpha$. □

2.3 Corollary. The assignment $A \longmapsto \mathscr{C}(A, -)$ defines a full and faithful functor $\mathscr{C}^{op} \longrightarrow [\mathscr{C}, \mathbf{Set}]$.

Proof. Put $F = \mathscr{C}(B, -)$ in Lemma 2.2(i): we get a bijection between $\mathscr{C}(B, A)$ and morphisms $\mathscr{C}(A, -) \longrightarrow \mathscr{C}(B, -)$ in $[\mathscr{C}, \mathbf{Set}]$. We need to verify this is functorial: but it sends $f : B \longrightarrow A$ to the natural transformation $g \longmapsto gf$. So functoriality follows from associativity. $\square$

We call this functor (or the functor $\mathscr{C} \longrightarrow [\mathscr{C}^{op}, \mathbf{Set}]$) sending A to $\mathscr{C}(-, A)$ the **Yoneda embedding** of $\mathscr{C}$, and denote it by Y.

Proof of Lemma 2.2(ii). Suppose for the moment that $\mathscr{C}$ is small, so that $[\mathscr{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathscr{C} \times [\mathscr{C}, \mathbf{Set}] \longrightarrow \mathbf{Set}$: One sends (A, F) to FA, and the other is the composite

$$\mathscr{C} \times [\mathscr{C}, \mathbf{Set}] \xrightarrow{\quad Y \times 1 \quad} [\mathscr{C}, \mathbf{Set}]^{op} \times [\mathscr{C}, \mathbf{Set}] \xrightarrow{[\mathscr{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

Lemma 2.2(ii) says that these are naturally isomorphic.

We can translate this into an elementary statement, making sense even when $\mathscr{C}$ isn't small, given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathscr{C}(A, -) \longrightarrow F$ give the same result, namely

$$\alpha_B (Ff) \beta_A (1_A) = (GF) \alpha_A \beta_A (1_A)$$

which is equal to $\alpha_B \beta_B (f)$. $\square$

2.4 Definition. We say a functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if it's isomorphic to $\mathscr{C}(A, -)$ for some A. By **representation** of F, we mean a pair (A, x) where $x \in FA$ is such that $\Phi(x)$ is an isomorphism. We also call x a **universal element** of F.

2.5 Corollary. If (A, x) and (B, y) are both representations of F, then there's a unique isomorphism $f : A \longrightarrow B$ such that $(Ff)(x) = y$.

Proof. Consider the composite

$$\mathscr{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathscr{C}(A, -)$$

By Corollary 2.3, this is of the form $Y(f)$ for a unique isomorphism $f : A \longrightarrow B$ and the diagram

$$\begin{tikzcd}
\mathscr{C}(B,-) \arrow[rr, "Y(f)"] \arrow[dr, "\Psi(y)"'] & & \mathscr{C}(A,-) \arrow[dl, "\Psi(x)"] \\
& F &
\end{tikzcd}$$

commutes iff $(Ff)x = y$. $\square$

2.6 Examples.

(a) The forgetful functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$. Similarly, the forgetful functor $\mathbf{Rng} \longrightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}[x], x)$ and the forgetful functor $\mathbf{Top} \longrightarrow \mathbf{Set}$ is representable by $(\{*\}, *)$.

(b) The functor $P^* : \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$ (see Examples 1.5(c)) is representable by $(\{0,1\},\{1\})$: this is the bijection between subsets and characteristic functions.

(c) Let G be a group. The unique (up to isomorphism) representable functor $G(*,-) : G \longrightarrow \mathbf{Set}$ is the *Cayley representation* of G, i.e. the set UG with G acting by left multiplication.

(d) Let A, B be two objects of a locally small category $\mathscr{C}$. We have a functor $\mathscr{C}^{op} \longrightarrow \mathbf{Set}$ sending C to $\mathscr{C}(C,A) \times \mathscr{C}(C,B)$. A representation of this, if it exists, is called a (categorical) **product** of A and B, and denoted

$$(A \times B, (A \times B \xrightarrow{\pi_1} A), (A \times B \xrightarrow{\pi_2} B)).$$

This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$ there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.

Products exist in many categories of interest: in $\mathbf{Set}, \mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}$ they are 'just' Cartesian products, in posets they are binary meets.

Dually we have the notion of **coproduct** $(A + B, A \xrightarrow{\nu_1} A + B, B \xrightarrow{\nu_2} A + B)$.

These also exist in many categories of interest.

Lecture 6 (e) The dual-vector-space functor $\mathbf{Mod}_K^{op} \longrightarrow \mathbf{Mod}_K$, when composed with the forgetful functor $\mathbf{Mod}_K \longrightarrow \mathbf{Set}$, is representable by $(K, 1_K)$.

(f) Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be morphisms in a locally small category $\mathscr{C}$. We have a functor $F : \mathscr{C}^{op} \longrightarrow \mathbf{Set}$ defined by

$$F(C) = \{\, h \in \mathscr{C}(C,A) \mid fh = gh \,\}.$$

A representation of F, if it exists, is called an **equalizer** of (f,g). It consists of an objects E and a morphism $E \xrightarrow{e} A$ such $fe = ge$, and every h with $fh = gh$ factors uniquely through e.

In **Set**, we take $E = \{\, x \in A \mid f(x) = g(x) \,\}$ and $e =$ inclusion.

Similar constructions work in $\mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}, \ldots$.

Dually, we have the notion of **coequalizer**.

2.7 Remark. If e occurs as an equalizer, then it's a monomorphism, since any h factors through it in at most one way. We say a mono is **regular** if it occurs as an equalizer.

Split monomorphisms are regular (c.f. question 6i on sheet 1). Note that regular mono + epi $\implies$ iso: if the equalizer e of (f,g) is epic, then $f = g$, so $e \cong 1_{\operatorname{cod} e}$.

2.8 Definition (Separating, detecting families)**.** Let $\mathscr{C}$ be a category, $\mathscr{G}$ a class of objects of $\mathscr{C}$.

(a) We say $\mathscr{G}$ is a **separating family** for $\mathscr{C}$ if, given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathscr{G}$, then $f = g$. (i.e. the functors $\mathscr{C}(G,-)$, $G \in \mathscr{G}$ are collectively faithful).

(b) We say $\mathscr{G}$ is a **detecting family** for $\mathscr{C}$ if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} b$ with $G \in \mathscr{G}$ factors uniquely through f, then f is an isomorphism.

If $\mathscr{G} = \{G\}$, we call G a **separator/detector**.

2.9 Lemma.

(i) If $\mathscr{C}$ is a balanced category, then any separating family is detecting.

(ii) If $\mathscr{C}$ has equalizers, then any detecting family is separating.

Proof.

(i) Suppose $\mathscr{G}$ is separating and $A \xrightarrow{f} B$ satisfies the condition of 2.8(b). If $B \overset{g}{\underset{h}{\rightrightarrows}} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is epic. Similarly if $D \overset{k}{\underset{l}{\rightrightarrows}} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorisations of fky through f. So $k = l$, i.e. f is monic.

(ii) Suppose $\mathscr{G}$ is detecting and $A \overset{f}{\underset{g}{\rightrightarrows}} B$ satisfies the condition of 2.8(a) . Then the equalizer $E \xrightarrow{e} A$ is an isomorphism, so $f = g$.

□

2.10 Examples.

(a) In $[\mathscr{C}, \mathbf{Set}]$ the family
$$\{\, \mathscr{C}(A, -) \mid A \in \operatorname{ob}\mathscr{C} \,\}$$
is both separating and detecting (this is just a restatement of Yoneda Lemma.)

(b) In **Set**, $1 = \{*\}$ is both a separator and a detector since it represents the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$.

Similarly, $\mathbb{Z}$ is both in **Gp**, since it represents the forgetful functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$.

And $2 = \{0, 1\}$ is a coseparator and a codetector in **Set**, since it represents $P^* : \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$.

(c) In **Top**, $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \longrightarrow \mathbf{Set}$, but not a detector. In fact, **Top** has no detecting *set* of objects.

For any infinite cardinal κ, let X be a discrete space of cardinality κ and let Y be the same set with 'co-$<\kappa$' topology, i.e. $F \subseteq Y$ closed $\iff$ $F = Y$ or $\operatorname{card} F < \kappa$. The identity $X \longrightarrow Y$ is continuous, but not a homeomorphism.

So if $\{\, G_i \mid i \in I \,\}$ is any set of spaces, taking $\kappa > \operatorname{card} G_i$ for all i yields an example to show that the set is not detecting.

(d) Let $\mathscr{C}$ be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous maps. JHC Whitehead proved that if $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \longrightarrow \pi_n(Y)$ for all n, then it's an isomorphism in $\mathscr{C}$. This says that $\{ S^n \mid n \geq 1 \}$ is a detecting set for $\mathscr{C}$.

But PJ Freyd showed there is no faithful functor $\mathscr{C} \longrightarrow \mathbf{Set}$, so no separating *set*: if $\{ G_i \mid i \in I \}$ were separating, then

$$x \longmapsto \coprod_{i \in I} \mathscr{C}(G_i, X)$$

would be faithful.

Note that any functor of the form $\mathscr{C}(A, -)$ preserves monos, but they don't normally preserve epis.

2.11 Definition. We say an object P is **projective** if, given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

there exists $P \xrightarrow{g} A$ with $eg = f$. (If $\mathscr{C}$ is locally small, this says $\mathscr{C}(P, -)$ preserves epis).

Dually, an **injective** object of $\mathscr{C}$ is a projective object of $\mathscr{C}^{op}$. Given a class $\mathscr{E}$ of epimorphisms, we say P is $\mathscr{E}$-projective if it satisfies the condition for all $e \in \mathscr{E}$.

2.12 Lemma. Representable functors are (pointwise) projective in $[\mathscr{C}, \mathbf{Set}]$.

Proof.

$$\begin{tikzcd} & \mathscr{C}(A,-) \arrow[d, "f"] \\ F \arrow[r, two heads, "\alpha"] & G \end{tikzcd}$$

where α is pointwise surjective. By Yoneda Lemma, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathscr{C}(A, -) \longrightarrow F$ corresponds to x then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Lecture 7 **3.1 Definition.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $\mathscr{C} \xrightarrow{F} \mathscr{D}$, $\mathscr{D} \xrightarrow{G} \mathscr{C}$ two functors. By an **adjunction** between F and G we mean a bijection between morphisms $FA \xrightarrow{\hat{f}} B$ in $\mathscr{D}$ and morphisms $A \xrightarrow{f} GB$ in $\mathscr{C}$ which is natural in A and B, i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, we have $h\hat{f}(Fg) = \widehat{(Gh)fg} : FA' \longrightarrow B'$.

We say F is **left adjoint** to G and write $F \dashv G$.

3.2 Examples.

(a) The free functor $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$ is left adjoint to the forgetful functor $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, since any function $f : A \longrightarrow UB$ extends uniquely to a homomorphism $\hat{f} : FA \longrightarrow B$. Naturality in B is easy, naturality in A follows from the definition of F as a functor.

(b) The forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology and a right adjoint I which equips a set A with the indiscrete topology $\{\varnothing, A\}$.

(c) The functor $\mathrm{ob} : \mathbf{Cat} \longrightarrow \mathbf{Set}$ has a left adjoint D sending A to the *discrete* category with $\mathrm{ob}(DA) = A$ and only identity morphisms and a right adjoint I sending A to the category with $\mathrm{ob}(IA) = A$ and one morphism $x \longrightarrow y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category $\mathscr{C}$ to its set of *connected components*, i.e. the quotient of $\mathrm{ob}\,\mathscr{C}$ by the smallest equivalent relation identifying $\mathrm{dom}\, f$ with $\mathrm{cod}\, f$ for all $f \in \mathrm{mor}\,\mathscr{C}$.

(d) Let M be the monoid $\{1, e\}$ with $e^2 = e$. An object of $[M, \mathbf{Set}]$ is a pair (A, e) where $e : A \longrightarrow A$ satisfies $e^2 = e$.

We have a functor $G : [M, \mathbf{Set}] \longrightarrow \mathbf{Set}$ sending (A, e) to

$$\{\, x \in A \mid e(x) = x \,\} = \{\, e(x) \mid x \in A \,\}$$

and a functor $F : \mathbf{Set} \longrightarrow [M, \mathbf{Set}]$ sending A to $(A, 1_A)$.

Claim $F \dashv G \dashv F$: given $f : (A, 1_A) \longrightarrow (B, e)$: it must take values in $G(B, e)$, and any $g : (B, e) \longrightarrow (A, 1_A)$ is determined by its values on the image of e.

(e) Let $\mathbf{1}$ be the discrete category with one object $*$. For any $\mathscr{C}$, there's a unique functor $\mathscr{C} \longrightarrow \mathbf{1}$: a left adjoint for this picks out an initial object of $\mathscr{C}$, i.e. an object I such that there exists a unique $I \longrightarrow A$ for each $A \in \mathrm{ob}\,\mathscr{C}$. Dually, a right adjoint for $\mathscr{C} \longrightarrow \mathbf{1}$ corresponds to a terminal object of $\mathscr{C}$.

(f) Let $A \xrightarrow{f} B$ be a morphism in $\mathbf{Set}$. We can regard PA and PB as posets, and we have functors

$$PA \underset{P^*f}{\overset{Pf}{\rightleftarrows}} PB$$

Claim $(Pf \dashv P^*f)$: we have $Pf(A') \subseteq B' \iff f(x) \in B'$ for all $x \in A' \iff A' \subseteq P^*f(B')$.

(g) Suppose given sets A, B and a relation $R \subseteq A \times B$. We define mappings $(-)^l, (-)^r$ between PA and PB by

$$S^r = \{ y \in B \mid (\forall x \in S)((x,y) \in R) \} \quad \text{for } S \subseteq A$$
$$T^l = \{ x \in A \mid (\forall y \in T)((x,y) \in R) \} \quad \text{for } T \subseteq B.$$

These mappings are order-reversing (i.e. contravariant functors) and

$$T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l.$$

We say $(-)^r$ and $(-)^l$ are adjoint on the right.

(h) The functor $P^* : \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$ is self-adjoint on the right, since functions $A \longrightarrow PB$ correspond bijectively to subsets of $A \times B$ and hence to functions $B \longrightarrow PA$.

3.3 Theorem. Let $G : \mathscr{D} \longrightarrow \mathscr{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \mathrm{ob}\,\mathscr{C}$, where $(A \downarrow G)$ has objects pairs (B, f) with $A \xrightarrow{f} GB$, and morphisms $(B, f) \longrightarrow (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f'"'] & & GB \arrow[dl, "Gg"] \\ & GB' & \end{tikzcd}$$

commutes.

Proof. Suppose given $F \dashv G$. Consider the morphism $\eta_A : A \longrightarrow GFA$ corresponding to $FA \xrightarrow{1} FA$. Then FA, η_A is an object of $(A \downarrow G)$. Moreover, given $g : FA \longrightarrow B$ and $f : A \longrightarrow GB$, the diagram

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f'"'] & & GFA \arrow[dl, "Gg"] \\ & GB & \end{tikzcd}$$

commutes iff

$$\begin{tikzcd} FA \arrow[rr, "1_A"] \arrow[dr, "\hat{f}"'] & & FA \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

commutes, i.e. $g = \hat{f}$. So (FA, η_A) is initial in $(A \downarrow G)$.

Conversely, suppose given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $A \xrightarrow{f} A'$, we define $Ff : FA \longrightarrow FA'$ to be the unique morphism making

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{A'}"] & GFA' \end{tikzcd}$$

commute. Functoriality follows from uniqueness: given $f' : A' \longrightarrow A''$, both $F(f'f)$ and $(Ff')(Ff)$ are morphisms $(FA, \eta_A) \longrightarrow (FA'', \eta_{A''})$ in $(A \downarrow G)$. $\square$

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