Part III – Algebraic Topology (Ongoing course, rough)

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Michaelmas 2018

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0 Introduction

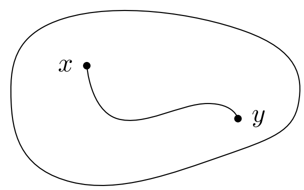
Algebraic topology concerns the connectivity properties of topological spaces. Recall a space X is **connected** if we cannot write $X = U \cup V$ where U, V are non-empty, open and disjoint.

Example. $\mathbb{R}$ is connected (with its Euclidean topology), $\mathbb{R} \setminus \{0\}$ is not connected.

Corollary (Intermediate value theorem)**.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0, f(y) < 0$, then there is some z lying between x, y such that $f(z) = 0$.

Proof. If $f(z) \neq 0$ for all z, then $\mathbb{R} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ is disconnected. $\square$

For nice spaces, connected $\iff$ path-connected. Recall: a space X is path-connected if $\forall x, y \in X, \exists \gamma : [0,1] \to X$ continuous such that $\gamma(0) = x, \gamma(1) = y$. Informally, any two maps of a point to X can be continuously deformed into one another.



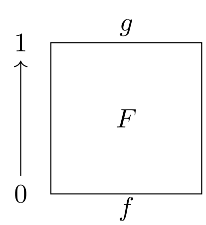
Definition (Homotopy)**.** If X, Y are topological spaces and $f, g : X \to Y$ are (continuous) maps, then f is **homotopic** to g if

$$\exists F : X \times [0,1] \to Y$$

continuous such that

$$F|_{X \times \{0\}} = f, F|_{X \times \{1\}} = g.$$

Write $f \simeq g$ or $f \simeq_F g$ and schematically



Definition (Simply connected)**.** A path-connected space X is **simply connected** if every two continuous maps $S^1 \to X$ are homotopic. Here $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$ is the n dimensional sphere, S^1 is the circle $\subseteq \mathbb{C}$.

Example. $\mathbb{R}^2$ is simply connected but $\mathbb{R}^2 \setminus \{0\}$ is not. In fact, continuous maps $S^1 \xrightarrow{\gamma} \mathbb{R}^2 \setminus \{0\}$ have a degree $\deg(\gamma) \in \mathbb{Z}$, invariant under homotopy. (If γ was differentiable, we could set $\deg(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \in \mathbb{Z}$.) If $\gamma_n : S^1 \to \mathbb{R}^2 \setminus \{0\}$ has $t \mapsto e^{2\pi i n t}$ then $\deg(\gamma_n) = n$.

Corollary (Fundamental theorem of algebra)**.** Every nonconstant complex polynomial has a root.

Proof. Let $f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ be a complex polynomial and suppose $f(z) \neq 0 \ \forall z \in \mathbb{C}$. Let $\gamma_R(t) = f(Re^{2\pi i t})$ so $\gamma_R : S^1 \to \mathbb{R}^2 \setminus \{0\}$. Now γ_0 is a constant map, so $\deg(\gamma_0) = 0$. By homotopy invariance of degree, $\deg(\gamma_R) = 0 \ \forall R$.

If $R \gg \sum_i |a_i|$, we can consider $f_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_n)$ for $0 \leq s \leq 1$, and on the circle $Re^{2\pi i t}$, f_s also takes values in $\mathbb{R}^2 \setminus \{0\}$. If $\gamma_{R,s}(t) = f_s(Re^{2\pi i t})$, then $\gamma_{R,1} = \gamma_R$ but $\gamma_{R,0} : z \mapsto z^n$, which has degree n. Now $\deg(\gamma_0) = \deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0})$, so $n = 0$ and f is constant. $\square$

Fact. Any two maps $S^n \to \mathbb{R}^{n+1}$ are homotopic but maps $S^n \xrightarrow{f} \mathbb{R}^{n+1} \setminus \{0\}$ have a degree $\deg(f) \in \mathbb{Z}$, invariant up to homotopy. Moreover, the degree of the constant map is 0 and the degree of inclusion is 1.

Corollary (Brouwer's fixed point theorem)**.** If $B^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$, any continuous map $f : B^n \to B^n$ has a fixed point.

Proof. Suppose f has no fixed point. Let $\gamma_R : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be the map $v \mapsto R_v - f(R_v)$ for $0 \leq R \leq 1$. So γ_0 is a constant, so has degree 0. Hence $\deg(\gamma_1) = 0$.

Let $\gamma_{1,s}(v) \coloneqq v - sf(v)$, for $0 \leq s \leq 1$ and $v \in S^{n-1}$. Note $\gamma_{1,s}$ has image in $\mathbb{R}^n \setminus \{0\}$: if $s = 1$, this is because $v \neq f(v) \ \forall v$ and if $s < 1$ then $|v| > |sf(v)|$. Therefore $\gamma_1 = \gamma_{1,1}$ has the same degree as $\gamma_{1,0}$ which is the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$, a contradiction. $\square$

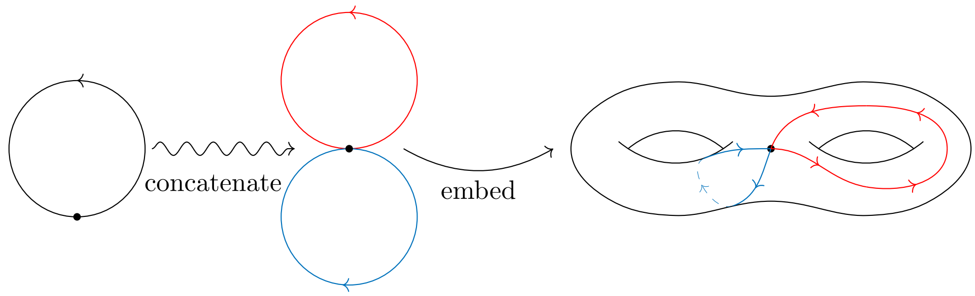
Definition (Homotopy equivalence)**.** We say spaces X, Y are **homotopy equivalent** if $\exists$ maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$.

Example.

- Trivial case: if X, Y are homeomorphic, i.e. $X \cong Y$, then clearly $X \simeq Y$.
- $\mathbb{R}^n \simeq \{0\}$, the single point. A space homotopy equivalent to a point is sometimes called contractible.
- $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. If $i : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ inclusion, and $p : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is projection $v \mapsto \frac{v}{\|v\|}$, then $p \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ p \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy
$$\begin{aligned} F : \mathbb{R}^n \setminus \{0\} \times [0,1] &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ (v,t) &\longmapsto tv + (1-t)\frac{v}{\|v\|} \end{aligned}$$

Algebraic topology is the study of the set of spaces up to homotopy equivalence via the set of groups up to isomorphism.

The first naive attempt would be homotopy groups: Loops (continuous maps $S^1 \to X$) with a common base-point can be concatenated and this induces a group structure on the set of homotopy classes of maps $(S^1, *) \to (X, x_0)$. Recall this refers to continuous maps $S^1 \to X$ taking $* \mapsto x_0$ and a based homotopy $F : f \simeq g$ of two such is one such that $F|_{S \times \{t\}}$ sends $*$ to x_0 $\forall t$.



Again, there is a group structure on the set of based homotopy classes of maps $(S^n, *) \to (X, x_0)$ calld $\pi_n(X, x_0)$, the n-th homotopy group of X.

Fact. $\{\pi_n(S^2, x)\}_{n\geq 1}$ is not known. Indeed, there is no simply connected manifold of dimension > 0 for which all π_n are known.

Instead, we will focus on homology theory, more precisely singular (co)homology. We will obtain invariants of spaces in a two-step process:

(a) Associate to X a chain complex (or cochain complex).

(b) take (co)homology of that complex.

This will be rather computable for simple spaces. In this course, we will mostly focus on studying manifolds.

Definition (Chain complex, cochain complex)**.** A **chain complex** (C_*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \ldots$$

(indexed by $\mathbb{N}$ or $\mathbb{Z}$) with the key property that $\forall n\, d_{n-1} \circ d_n = 0$ Then $\mathrm{image}(d_{n+1}) \subseteq \ker(d_n)$ and the n-th homology group $H_n(C_*, d)$ of the chain complex is the quotient

$$H_n(C_*, d) \coloneqq \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})}.$$

A **cochain complex** (C^*, d) is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \longrightarrow \ldots$$

such that $d^n \circ d^{n-1} \equiv 0\ \forall n$ The n-th cohomology group $H^n(C^*, d)$ is the quotient

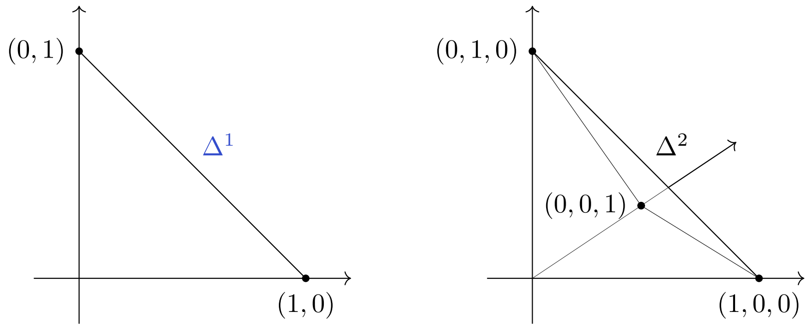
$$H^n(C^*, d) \coloneqq \frac{\ker(d^n)}{\mathrm{im}(d^{n-1})}.$$

0.1 Singular (co)chains

Definition (Simplex)**.** A **simplex** in a topological space X is defined as follows.

- An n-simplex is the convex hull of $(n+1)$ ordered points $v_0, \ldots, v_n$ in $\mathbb{R}^m$ such that $\{\, v_i - v_0 \mid 1 \leq i \leq n \,\}$ are linearly independent. Write this as $[v_0, \ldots, v_n] = \sigma$.

The **standard n-simplex** is $\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \ \forall i \}$, e.g.



Note *any* n-simplex is canonically the image of Δ^n under a linear homeomorphism $\Delta^n \to \sigma$ given by $(t_i) \mapsto \Sigma t_i v_i \in \sigma$.

An n simplex in X is a continuous map $\sigma : \Delta^n \to X$, or from any n-simplex to X. Note any n-simplex has faces $\Delta_i^{n-1} \subseteq \Delta^n$ is defined by $\{t_i = 0\}$ and then this defines a corresponding face of any σ via the map $\Delta^n \to \sigma$. Write ith face of σ as $[v_0, \dots, \hat{v}_i, \dots, v_n] \subseteq [v_0, \dots, v_n]$ (so a hat over a vertex means omit it).

Note the edges of any simplex are canonically oriented via '$v_i \to v_j$ if $i < j$'.

Definition (Singular chain complex)**.** If X is a space, the **singular chain complex** $C_*(X; \mathbb{Z})$ or just $C_*(X)$ is defined as follows:

$$\left\{ \sum_{i=1}^N h_i \sigma_i \;\middle|\; N \in \mathbb{N}_{\geq 0}, h_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ an } n\text{-simplex in } X \right\}$$

the free abelian group on n-simplices in X.

Definition (Boundary map)**.** The boundary map $d : C_n(X) \to C_{n-1}(X)$ is defined by

$$d\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

on $\sigma = [v_0 \cdots v_n]$ and extend to $C_n(X)$ by linearity.

Example. For the simplex $\sigma = [v_0 v_1 v_2]$, $d(\sigma) = [v_0 v_1] - [v_0 v_2][v_1 v_2]$.

Lemma. $d^2 = 0$, i.e. $d_{n-1} \circ d_n = 0 \ \forall n$.

Proof.

$$\begin{aligned} d \circ d(\sigma) &= d(\sum_{i=0}^n (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}) \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma|_[v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n] \\ &+ \sum_{j>i} (-1)^i (-1)^(j+1) \sigma|_[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]. \end{aligned}$$

Exchanging i and j, these two terms exactly cancel. $\square$

The resulting homology theory $H_*(X)$ or $H_*(X;\mathbb{Z})$ is called **singular homology**.

The $\mathbb{Z}$ keeps track of the fact that $h_i \in \mathbb{Z}$, we could similarly define $C_*(X,G)$ and $H_*(X,G)$ for any abelian group G. Note $H_*(X,\mathbb{Z})$ is tautologically a homeomorphism invariant of X.

The idea is that d takes the boundary of a region covered by simplices.

Elements of $\ker(d : C_i(X) \to C_{i-1}(X))$ are called cycles or i-cycles. Elements in $\mathrm{im}(d)$ are called boundaries.

Definition (Cochain complex)**.** The singular **cochain** complex of a space X, $C^*(X,\mathbb{Z})$ or $C^*(X)$, has cochain groups $C^n(X) \coloneqq \mathrm{Hom}(C_n(X),\mathbb{Z})$ and coboundary map $d^* : C^n(X) \to C^{n+1}(X)$ by $(d^*\psi)(\sigma) \coloneqq \psi(d\sigma)$. Here $\sigma \in C_{n+1}(X)$ and $d\sigma \in C_n(X)$, i.e. $d\sigma = d_{n+1}\sigma$.

Observe $d^*(d^*\psi)(\sigma) = d^*(\psi|_{d\sigma}) = \psi|_{d\circ d(\sigma)}$ and $d \circ d(\sigma) = 0$, so $(d^*)^2 = 0$. So indeed $(C^*(X), d^*)$ is a cochain complex and the cohomology $H^*(X,\mathbb{Z})$ or $H^*(X)$ is called **singular cohomology**.

Note $H^*(X,\mathbb{Z}) \neq \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),\mathbb{Z})$ in general. Observe if $f : X \to Y$ is continuous and $\sigma : \Delta^n \to X$ is continuous, then I get $f\circ\sigma : \Delta^n \to Y$ is an n-simplex in Y, so I get

$$\begin{aligned} f_* &: C_*(X) \to C_*(Y) \\ f_* &: C_n(X) \to C_n(Y)\ \forall n \end{aligned}$$

are group homomorphisms.

Key observation: $df_* = f_*d$ since $f \circ (\sigma|_{[v_0\cdots\hat{v_i}\cdots v_n]}) = (f\circ\sigma)|_{[v_0\cdots\hat{v_i}\cdots v_n]}$ i.e. a continuous map $f : X \to Y$ induces a **chain map** of chain complexes

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1}(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_n(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_{n-1}(X) \arrow[r, "d"] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r] & C_{n+1}(Y) \arrow[r, "d"] & C_n(Y) \arrow[r, "d"] & C_{n-1}(Y) \arrow[r, "d"] & \dots \end{tikzcd}$$

and each square commutes.

Lemma. If C_* and D_* are chain complexes and $f_* : C_* \to D_*$ is a chain map, then f_* indexes homomorphisms $f_* : H_i(C_*) \to H_i(D_*)$ for every i.

Proof. Let $a \in H_i(C_*) = \frac{\ker(d_i : C_i \to C_{i-1})}{\mathrm{im}(d_{i+1} : C_{i+1} \to C_i)}$, so a is represented by some i-cycle $\alpha \in C_i$, where $d\alpha = 0$.

Then $f_*(d\alpha) = 0 = d(f_*\alpha) \implies f_*(\alpha) \in D_i$ is a cycle in the D_*-chain complex, and hence defines an element in $H_i(D_*) = \frac{\ker(d : D_i \to D_{i-1})}{\mathrm{im}(d : D_{i+1} \to D_i)}$. Call this element b and set $f_*(a) = b$. This is well-defined. If α' also repesents $a \in H_i(C_*)$, then $\alpha - \alpha'$ is a boundary, i.e. $\alpha - \alpha_i = d_{i+1}(\gamma)$ for some $\gamma \in C_{i+1}$. Then $f_*(\alpha) - f_*(\alpha') = f_*(d_{i+1}\gamma) = d_{i+1}f_*(\gamma)$ so $f_*(\alpha')$ and $f_*(\alpha)$ differ by a boundary, so define some element $b \in H_i(D_*)$. It is an easy exercise to check that this map $f_* : H_i(C_*) \to H_i(D_*)$ is indeed a homomorphism of groups. ☐

The upshot of this is that if $f : X \to Y$ is a continuous map of spaces, it induces maps $f_* : H_i(X) \to H_i(Y)$ for all i.

Lemma. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g\circ f)_* = g_* \circ f_*$, $id_* = id$.

Proof. Exercise. ☐

In category-theoretic language, the association $X \mapsto H_*(X)$ is a functor from the category of topological spaces to the set of graded abelian groups. Observe $f : X \to Y$ $f_* : C_*(X) \to C_*(Y)$ and this has an adjoint $f^* : C^*(Y) \to C^*(X)$. Note this goes the other way. This again induces a map $H^*(Y) \to H^*(X)$.