Part III – Ramsey Theory (Incomplete)

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Contents

0 Introduction

Lecture 1 If you liked Graph Theory, you'll almost certainly like Ramsey Theory. If you didn't like Graph Theory, you probably won't like Ramsey Theory. Ramsey theory is an unusual part of maths, in that it's all about answering one question. The basic question is:

Can we find some order in enough disorder?

As usual in discrete mathematics, the key ideas of the course are in the proofs rather than in the definitions.

The course is structured into three sections.

Chapter 1: Monochromatic systems (abstract and concrete)

Chapter 2: Partition regular equations (concrete)

Chapter 3: Infinite Ramsey Theory (abstract)

There are not many prerequisites to this course, only basic concepts of topology (compact spaces).

No single book covers all of the course, but there are two books which cover the relevant content:

- Bollobás, *Combinatorics*, C.U.P., 1986 (for chapter 3). An excellent survey of the material.
- Graham, Rothschild, Spencer, *Ramsey Theory*, Wiley, 1990 (for chapters 1,2).

As well as lots of nice proofs in the area, there are many open problems we will come across.

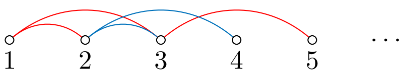
1 Monochromatic systems

1.1 Ramsey's Theorem

Write $\mathbb{N} = \{1, 2, 3, \dots\}$, and write $[n]$ for $\{1, \dots, n\}$. For any set X, write

$$X^{(r)} = \{A \subseteq X : |A| = r\}.$$

Suppose we have the natural numbers listed, and each pair of naturals is connected by an edge coloured either red or blue.



Formally, we have a 2-colouring c of $\mathbb{N}^{(2)}$, i.e. $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Can we always find an infinite set M that is **monochromatic**, i.e. c is constant on $M^{(2)}$?

Example.

(i) Colour ij red if $i+j$ even and blue if it is odd. Then we can find an M that works, by using the evens.

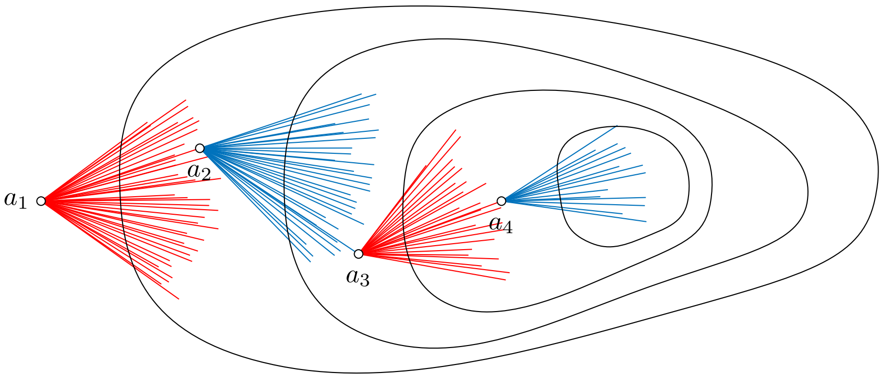
(ii) Colour ij red if $\max\{n : 2^n \mid i+j\}$ even, and blue otherwise.
Again yes, we can use $M = \{4^0, 4^1, 4^2, \dots\}$ or $M = \{x : x \equiv 1 \pmod 4\}$.

(iii) Colour ij red if $i+j$ has an even number of (distinct) prime factors, and blue if odd. Now the answer is less clear...

It turns out that the answer is always yes.

Theorem 1.1 (Ramsey's Theorem)**.** Let c be a 2-colouring of $\mathbb{N}^{(2)}$. Then c has an infinite monochromatic set.

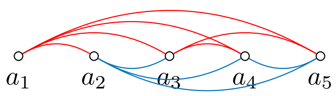
Proof.



Pick $a_1 \in \mathbb{N}$. There are infinitely many edges from a, so there is an infinite set $B_1 \subseteq \mathbb{N} - \{a_1\}$ such that all edges from a_1 to B_1 have the same colour, say C_1.

Pick $a_2 \in B_1$. There are infinitely maby edges from a_2 inside B_1, so there is an infinite set $B_2 \subset B_1 - \{a_2\}$ such that all edges from a_2 to B_2 have same colour, say C_2.

Continue inductively. We obtain distinct points $a_1, a_2, \dots$ and colours $C_1, C_2, \dots$ such that $a_i a_j$ (for $i < j$) has colour C_i.



We must have $C_{i_1} = C_{i_2} = C_{i_3} = \cdots$ for some $i_1 < i_2 < \cdots$ (as there are only two colours), so $\{a_{i_1}, a_{i_2}, \dots\}$ is monochromatic. $\square$

Remark.

(i) This is called a two-pass proof.

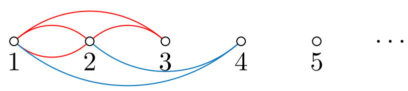
(ii) In example 3, no explicit example is known.

(iii) What about a k-colouring? (i.e. $c : \mathbb{N}^{(2)} \to [k]$). The same proof would show there is an infinite monochromatic set. Alternatively, we can deduce this from Ramsey's Theorem, by 'turquoise spectacles': view our colouring as a 2-colouring by colours '1' and '2 or 3 or ... or k' and apply Ramsey's Theorem and induction.

(iv) Asking for an infinite monochromatic set is much more than asking for arbitrarily large finite monochromatic sets, e.g. in we have no infinite red set, but arbitrarily large finite red sets.

Example. Any sequence $x_1, x_2, \dots$ in $\mathbb{R}$ (or in any totally ordered set) has a monotone subsequence. Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving ij (for $i < j$) colour up if $x_i < x_j$ and colour down if $x_i \geq x_j$ and apply Ramsey's Theorem.

Lecture 2 What if we 2-colour $\mathbb{N}^{(r)}$, for some $r = 3, 4, \dots$? Do we always get an infinite monochromatic set?



For example, colour ijk (for $i < j < k$) red if $i | j + k$, and blue if not. In this case we can find a such a set, e.g. take $M = \{2^0, 2^1, 2^2, \dots\}$.

Let's try to prove this in general.

Theorem 1.2 (Ramsey for r-sets)**.** Whenever $\mathbb{N}^{(r)}$ is 2-coloured, there is an infinite monochromatic set.

Proof. Induction on r. $r = 1$ is easy, by pigeonhole. Alternatively, we could say $r = 2$ is true by Theorem 1.1.

Given r and a 2-colouring $c : \mathbb{N}^{(r)} \to \{1, 2\}$, pick $a_1 \in \mathbb{N}$. Induce a 2-colouring c' of $(\mathbb{N} - \{a_1\})^{(r-1)}$ by

$$c'(F) = c(F \cup \{a_1\})$$

for each $F \in (\mathbb{N} - \{a_1\})^{(r-1)}$.

By induction, there is an infinite monochromatic set B_1, for c', i.e. $c(F \cup \{a_1\}) = c_1$, for each $F \in B_1^{(r-1)}$. Repeat: choose $a_2 \in B_1$ and obtain an infinite set $B_2 \subset B_1 - \{a_2\}$ such that $c(F \cup \{a_2\}) = c_2$, for each $F \in B_2^{(r-1)}$. Continue inductively, we obtain distinct $a_1, a_2, \dots \in \mathbb{N}$ and colours $c_1, c_2, \dots$ such that $c(a_{i_1}, \dots, a_{i_r}) = c_{i_1}$, for $i_1 < \cdots < i_r$.

We must have $c_{i_1} = c_{i_2} = \cdots$ for some sequence $i_1, i_2, \dots$ and so $\{a_{i_1}, a_{i_2}, \dots\}$ is monochromatic, as required. $\square$

We saw that, given $(1, x_1), (2, x_2), (3, x_3) \in \mathbb{R}^2$ there is a subsequence for which the induced (piecewise-linear) function is monotone. We could actually insist that the induced function is convex ($\smile$) or concave ($\frown$). Indeed, 2-colour $\mathbb{N}^{(3)}$ by colouring ijk according to whether $(i, x_i), (j, x_j), (k, x_k)$ are or and apply Theorem 1.2 for $r = 3$.

Rather unexpectedly, the infinite Ramsey implies the finite Ramsey.

Theorem 1.3 (Finite Ramsey)**.** For any m, r there is an n such that whenever $[n]^{(r)}$ is 2-coloured, there is a monochromatic m-set.

Proof. Suppose not: so for some fixed m, r we have for *each* n, a 2-colouring $c_n : [n]^{(r)} \to \{1, 2\}$ with no monochromatic m-set.

We'll obtain a 2-colouring of $\mathbb{N}^{(r)}$ with no monochromatic m-set, contradicting Theorem 1.2. [We want to 'put the c_n together' - but can only do this if the c_n are *nested*, meaning $c_{n+1}|[n]^{(r)} = c_n$ for every n.]

There are only finitely many ways (2, in fact) to colour $[r]^{(r)}$, so infinitely many of the c_n agree on $[r]^{(r)}$: say $c_n|[r]^{(r)} = d_r$ for every $n \in A_1$ (A_1 infinite, d_r is a 2-colouring of $[r]^{(r)}$).

There are only finitely many ways to 2-colour $[r+1]^{(r)}$, so infinitely many of the $c_n, n \in A_1$ agree on $[r+1]^{(r)}$, say $c_n|[r+1]^{(r)} = d_{r+1}$ for every $n \in A_2$. (There is an infinite $A_2 \subseteq A_1$, d_{r+1} is a 2-colouring of $[r+1]^{(r)}$).

Continue. We obtain $d_r, d_{r+1}, \dots$ (where $d_n : [n]^{(r)} \to \{1, 2\}$) such that

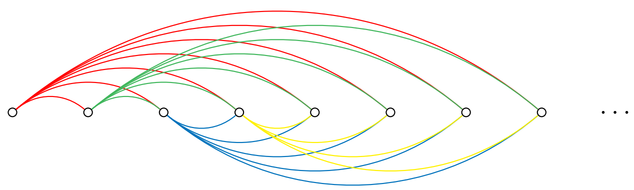
- no d_n has a mono m-set ($d_n = c_{n'}|[n]^{(r)}$ for some n').
- The d_n are nested (because $A_2 \subset A_1$ etc.)

so can find $c : \mathbb{N}^{(r)} \to \{1, 2\}$ by setting $c(F) = d_n(F)$ for any $n \geq \max F$. Then c has no monochromatic m-set, a contradiction. $\square$

Remark.

1. The proof gives absolutely no information about the least possible value of $n = n(m, r)$, but there are other proofs that do give bounds.
2. This proof is often called a 'compactness' proof. We actually showed that the space of all infinite sequences from $\{1, 2\}$ with the topology given by the metric
$$d(f, g) = \frac{1}{\min\{n : f_n \neq g_n\}}$$
is compact. (This topology is also called the product topology.)

What about infinitely many colours? What if we have $c : \mathbb{N}^{(2)} \to X$ for some arbitrary set X. Do we get an infinite monochromatic set? Definitely not, we could just give every edge a different colour. Could it be that we can always find an infinite set on which c is either constant or injective?



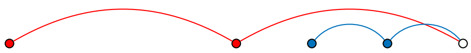
No.

Lecture 3 **missing**

Lecture 4 Write $W(m,k)$ for least n (if it exists) such that whenever $[n]$ is k-coloured, there is a monochromatic AP of length m.

Proposition 1.4. For every k, there is n such that whenever $[n]$ is k-coloured, there is a monochromatic AP of length 3.

Remark. Proposition 5 is included in Theorem 6.



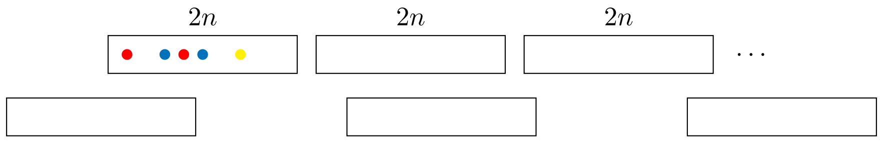
Proof. Claim: For every $r \leq k$, there is n such that whenever $[n]$ is k-coloured there is either

- a monochromatic AP of length 3, or
- r colour-focused APs of length 2

Once we have the claim, we are done: put $r = k$ and look at the colour of the focus.

Let's now prove the claim by induction on r. $r = 1$ is easy, by taking $n = k + 1$. Given n suitable for $r - 1$, we'll show that $(k^{2n} + 1)2n$ is suitable for r.

So, given a k-colouring of $[(k^{2n} + 1)2n]$ with no monochromatic AP of length 3.



Break up $[(k^{2n}+1)2n]$ into $k^{2n}+1$ blocks of length $2n$, called $B_1, B_2, \dots, B_{k^{2n}+1}$ where $B_i = [2n(i-1)+1, 2ni]$. The number of patterns for a block is k^{2n}, as we have k colours, so we must have two blocks B_s, B_{s+t} identically coloured.

Now, B_s contains $r - 1$ colour-focused APs of length 2 (by definition of n), together with their focus (as the length is $2n$): say we have $\{a_1, a_1 + d_1\}, \{a_2, a_2 + d_2\}, \dots, \{a_{r-1}, a_{r-1} + d_{r-1}\}$ focused at f. But then the $r - 1$ APs $\{a_1, a_1 + d_1 + 2nt\}, \dots, \{a_{r-1}, a_{r-1}, d_{r-1} + 2nt\}$ are colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is mono, of a different colour to those. Thus have r colour-focused APs of length 2. $\square$

Remark.
1. The idea of looking at the number of ways to colour a block is called a **product argument**.
2. The proof shows

$$W(3,k) \leq \underbrace{k^{k^{k^{\cdot^{\cdot^{\cdot^{k^{4k}}}}}}}}_{k}.$$

Theorem 1.5 (van der Waerden's Theorem)**.** For every m, k there is n such that whenever $[n]$ is k-coloured, there is a monochromatic AP of length m.

Proof. Induction on m. $m = 1$ is immediate. Alternatively, $m = 2$ is pigeonhole, or $m = 3$ is Prop 5. Given m, we may assume by induction that $W(m-1,k)$ exists for all k.

Claim: For every $r \leq k$, there is n such that whenever $[n]$ is k-coloured, there is either

- a monochromatic AP of length m, or

- r colour-focused APs of length $m-1$

Once we have this claim, we are done: put $r=k$ and look at the focus.

Proof of claim: Induction on r: $r=1$ is easy by taking $n=W(m-1,k)$. Given n suitable for $r-1$, we'll show that $n'=2nW(m-1,k^{2n})$ is suitable for r.

So, given a k-colouring of $[n']$ with no monochromatic AP of length m,

Break up $[n']$ into $W(m-1,k^{2n})$ blocks of length $2n$: call them $B_1,B_2,\dots,B_{W(m-1,k^{2n})}$.

Now, the number of patterns for a block is k^{2n}.

So, by definition of $W(m-1,k^{2n})$, there are blocks $B_s,B_{s+t},B_{s+2t},\dots,B_{s+(m-2)t}$ that are coloured identically.

Inside B_s, have $r-1$ colour-focused APs of length $m-1$ (by definition of n) say $A_1,\dots,A_{r-}$ where A_i has first term a_i and common difference d_i, focused at f, as the length is $2n$. But now the APs $A_1',\dots,A_{r-1}'$, where A_i' has first term a_i and common difference d_i+2nt are colour-focused at $f+2nt(m-1)$. Also, $\{f,f+2nt,f+4nt,\dots,f+2nt(m-2)\}$ is monochromatic, of a different colour to $A_1',\dots,A_{r-1}'$. $\square$

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