

# Part III – Ramsey Theory (Incomplete)

Based on lectures by Professor I. Leader

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## 0 Introduction

*Lecture 1* If you liked Graph Theory, you'll almost certainly like Ramsey Theory. If you didn't like Graph Theory, you probably won't like Ramsey Theory. Ramsey theory is an unusual part of maths, in that it's all about answering one question. The basic question is:

*Can we find some order in enough disorder?*

As usual in discrete mathematics, the key ideas of the course are in the proofs rather than in the definitions.

The course is structured into three sections.

**Chapter 1:** Monochromatic systems (abstract and concrete)

**Chapter 2:** Partition regular equations (concrete)

**Chapter 3:** Infinite Ramsey Theory (abstract)

There are not many prerequisites to this course, only basic concepts of topology (compact spaces).

No single book covers all of the course, but there are two books which cover the relevant content:

- Bollobás, *Combinatorics*, C.U.P., 1986 (for chapter 3). An excellent survey of the material.
- Graham, Rothschild, Spencer, *Ramsey Theory*, Wiley, 1990 (for chapters 1,2).

As well as lots of nice proofs in the area, there are many open problems we will come across.

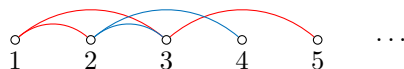
# 1 Monochromatic systems

## 1.1 Ramsey's Theorem

Write  $\mathbb{N} = \{1, 2, 3, \dots\}$ , and write  $[n]$  for  $\{1, \dots, n\}$ . For any set  $X$ , write

$$X^{(r)} = \{A \subseteq X : |A| = r\}.$$

Suppose we have the **natural numbers** listed, and each pair of naturals is connected by an edge coloured either **red** or **blue**.



Formally, we have a 2-colouring  $c$  of  $\mathbb{N}^{(2)}$ , i.e.  $c : \mathbb{N}^{(2)} \rightarrow \{\text{red}, \text{blue}\}$ . Can we always find an infinite set  $M$  that is **monochromatic**, i.e.  $c$  is constant on  $M^{(2)}$ ?

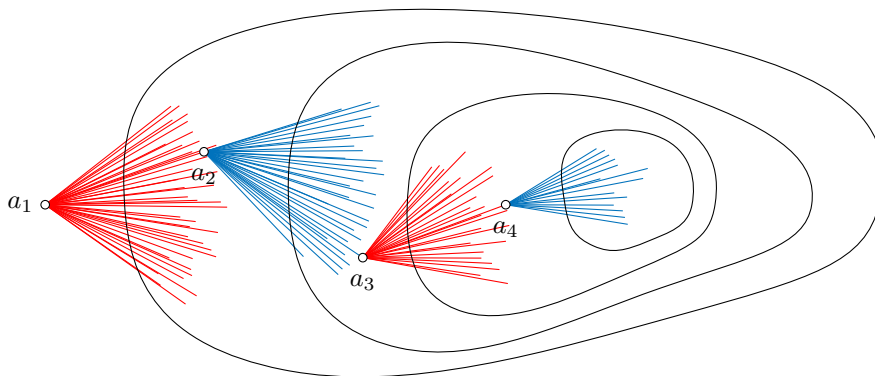
**Example.**

- (i) Colour  $ij$  **red** if  $i + j$  even and **blue** if it is odd. Then we can find an  $M$  that works, by using the evens.
- (ii) Colour  $ij$  **red** if  $\max\{n : 2^n \mid i + j\}$  even, and **blue** otherwise.  
Again yes, we can use  $M = \{4^0, 4^1, 4^2, \dots\}$  or  $M = \{x : x \equiv 1 \pmod{4}\}$ .
- (iii) Colour  $ij$  **red** if  $i + j$  has an even number of (distinct) prime factors, and **blue** if odd. Now the answer is less clear...

It turns out that the answer is always yes.

**Theorem 1.1** (Ramsey's Theorem). Let  $c$  be a 2-colouring of  $\mathbb{N}^{(2)}$ . Then  $c$  has an infinite monochromatic set.

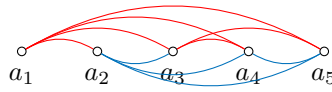
*Proof.*



Pick  $a_1 \in \mathbb{N}$ . There are infinitely many edges from  $a_1$ , so there is an infinite set  $B_1 \subseteq \mathbb{N} - \{a_1\}$  such that all edges from  $a_1$  to  $B_1$  have the same colour, say  $C_1$ .

Pick  $a_2 \in B_1$ . There are infinitely many edges from  $a_2$  inside  $B_1$ , so there is an infinite set  $B_2 \subseteq B_1 - \{a_2\}$  such that all edges from  $a_2$  to  $B_2$  have same colour, say  $C_2$ .

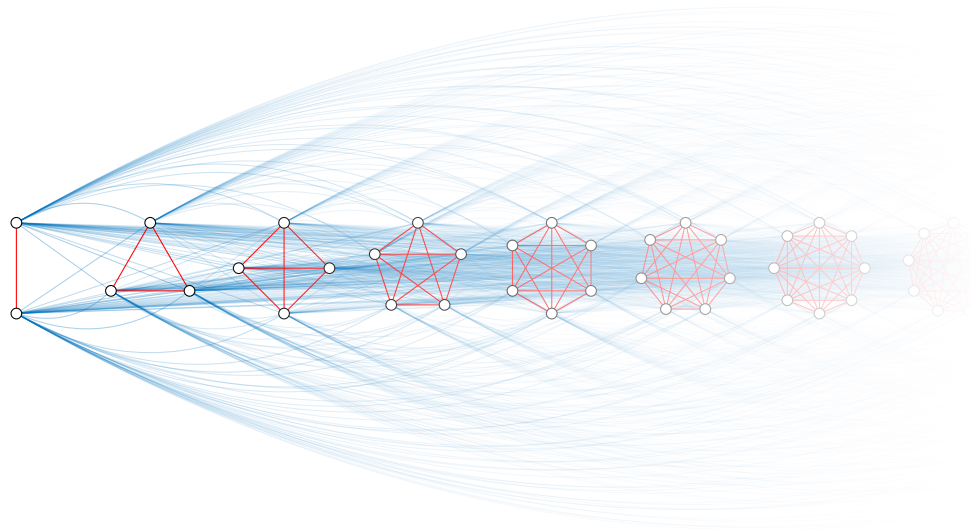
Continue inductively. We obtain distinct points  $a_1, a_2, \dots$  and colours  $C_1, C_2, \dots$  such that  $a_i a_j$  (for  $i < j$ ) has colour  $C_i$ .



We must have  $C_{i_1} = C_{i_2} = C_{i_3} = \dots$  for some  $i_1 < i_2 < \dots$  (as there are only two colours), so  $\{a_{i_1}, a_{i_2}, \dots\}$  is monochromatic.  $\square$

**Remark.**

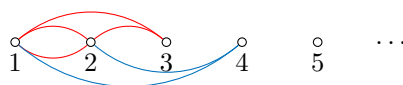
- (i) This is called a two-pass proof.
- (ii) In example 3, no explicit example is known.
- (iii) What about a  $k$ -colouring? (i.e.  $c : \mathbb{N}^{(2)} \rightarrow [k]$ ). The same proof would show there is an infinite monochromatic set. Alternatively, we can deduce this from [Ramsey's Theorem](#), by 'turquoise spectacles': view our colouring as a 2-colouring by colours '1' and '2 or 3 or ... or  $k$ ' and apply Ramsey's Theorem and induction.
- (iv) Asking for an infinite monochromatic set is much more than asking for arbitrarily large finite monochromatic sets, e.g. in



we have no infinite red set, but arbitrarily large finite red sets.

**Example.** Any sequence  $x_1, x_2, \dots$  in  $\mathbb{R}$  (or in any totally ordered set) has a monotone subsequence. Indeed, 2-colour  $\mathbb{N}^{(2)}$  by giving  $ij$  (for  $i < j$ ) colour **up** if  $x_i < x_j$  and colour **down** if  $x_i \geq x_j$  and apply [Ramsey's Theorem](#).

*Lecture 2* What if we 2-colour  $\mathbb{N}^{(r)}$ , for some  $r = 3, 4, \dots$ ? Do we always get an infinite monochromatic set?



For example, colour  $ijk$  (for  $i < j < k$ ) **red** if  $i|j+k$ , and **blue** if not. In this case we can find a such a set, e.g. take  $M = \{2^0, 2^1, 2^2, \dots\}$ .

Let's try to prove this in general.

**Theorem 1.2** (Ramsey for  $r$ -sets). Whenever  $\mathbb{N}^{(r)}$  is 2-coloured, there is an infinite monochromatic set.

*Proof.* Induction on  $r$ .  $r = 1$  is easy, by pigeonhole. Alternatively, we could say  $r = 2$  is true by [Theorem 1.1](#).

Given  $r$  and a 2-colouring  $c : \mathbb{N}^{(r)} \rightarrow \{1, 2\}$ , pick  $a_1 \in \mathbb{N}$ . Induce a 2-colouring  $c'$  of  $(\mathbb{N} - \{a_1\})^{(r-1)}$  by

$$c'(F) = c(F \cup \{a_1\})$$

for each  $F \in (\mathbb{N} - \{a_1\})^{(r-1)}$ .

By induction, there is an infinite monochromatic set  $B_1$ , for  $c'$ , i.e.  $c(F \cup \{a_1\}) = c_1$ , for each  $F \in B_1^{(r-1)}$ . Repeat: choose  $a_2 \in B_1$  and obtain an infinite set  $B_2 \subset B_1 - \{a_2\}$  such that  $c(F \cup \{a_2\}) = c_2$ , for each  $F \in B_2^{(r-1)}$ . Continue inductively, we obtain distinct  $a_1, a_2, \dots \in \mathbb{N}$  and colours  $c_1, c_2, \dots$  such that  $c(a_{i_1}, \dots, a_{i_r}) = c_{i_1}$ , for  $i_1 < \dots < i_r$ .

We must have  $c_{i_1} = c_{i_2} = \dots$  for some sequence  $i_1, i_2, \dots$  and so  $\{a_{i_1}, a_{i_2}, \dots\}$  is monochromatic, as required.  $\square$

We saw that, given  $(1, x_1), (2, x_2), (3, x_3) \in \mathbb{R}^2$  there is a subsequence for which the induced (piecewise-linear) function is monotone. We could actually insist that the induced function is convex ( $\smile$ ) or concave ( $\frown$ ). Indeed, 2-colour  $\mathbb{N}^{(3)}$  by colouring  $ijk$  according to whether  $(i, x_i), (j, x_j), (k, x_k)$  are  $\times \times \times$  or  $\times \times \times$  and apply [Theorem 1.2](#) for  $r = 3$ .

Rather unexpectedly, the infinite Ramsey implies the finite Ramsey.

**Theorem 1.3** (Finite Ramsey). For any  $m, r$  there is an  $n$  such that whenever  $[n]^{(r)}$  is 2-coloured, there is a monochromatic  $m$ -set.

*Proof.* Suppose not: so for some fixed  $m, r$  we have for *each*  $n$ , a 2-colouring  $c_n : [n]^{(r)} \rightarrow \{1, 2\}$  with no monochromatic  $m$ -set.

We'll obtain a 2-colouring of  $\mathbb{N}^{(r)}$  with no monochromatic  $m$ -set, contradicting [Theorem 1.2](#). [We want to 'put the  $c_n$  together' - but can only do this if the  $c_n$  are *nested*, meaning  $c_{n+1}|[n]^{(r)} = c_n$  for every  $n$ .]

There are only finitely many ways (2, in fact) to colour  $[r]^{(r)}$ , so infinitely many of the  $c_n$  agree on  $[r]^{(r)}$ : say  $c_n|[r]^{(r)} = d_r$  for every  $n \in A_1$  ( $A_1$  infinite,  $d_r$  is a 2-colouring of  $[r]^{(r)}$ ).

There are only finitely many ways to 2-colour  $[r+1]^{(r)}$ , so infinitely many of the  $c_n, n \in A_1$  agree on  $[r+1]^{(r)}$ , say  $c_n|[r+1]^{(r)} = d_{r+1}$  for every  $n \in A_2$ . (There is an infinite  $A_2 \subseteq A_1$ ,  $d_{r+1}$  is a 2-colouring of  $[r+1]^{(r)}$ ).

Continue. We obtain  $d_r, d_{r+1}, \dots$  (where  $d_n : [n]^{(r)} \rightarrow \{1, 2\}$ ) such that

- no  $d_n$  has a mono  $m$ -set ( $d_n = c_{n'}|[n]^{(r)}$  for some  $n'$ ).
- The  $d_n$  are nested (because  $A_2 \subset A_1$  etc.)

so can find  $c : \mathbb{N}^{(r)} \rightarrow \{1, 2\}$  by setting  $c(F) = d_n(F)$  for any  $n \geq \max F$ . Then  $c$  has no monochromatic  $m$ -set, a contradiction.  $\square$

**Remark.**

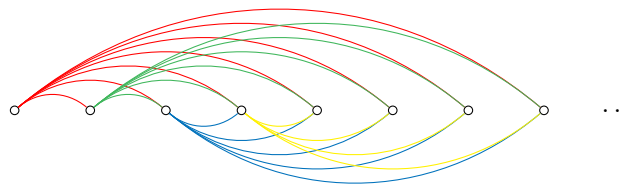
1. The proof gives absolutely no information about the least possible value of  $n = n(m, r)$ , but there are other proofs that do give bounds.

2. This proof is often called a ‘compactness’ proof. We actually showed that the space of all infinite sequences from  $\{1, 2\}$  with the topology given by the metric

$$d(f, g) = \frac{1}{\min\{n : f_n \neq g_n\}}$$

is compact. (This topology is also called the product topology.)

What about infinitely many colours? What if we have  $c : \mathbb{N}^{(2)} \rightarrow X$  for some arbitrary set  $X$ . Do we get an infinite monochromatic set? Definitely not, we could just give every edge a different colour. Could it be that we can always find an infinite set on which  $c$  is either constant or injective?



No.

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