Part III – Topics in Set Theory (Ongoing course)

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0 Introduction

Lecture 1 The main 'topic in set theory' covered in this course will be one of the most important: solving the Continuum Problem. A priori, set theory does not seem intrinsically related to logic, but the continuum hypothesis showed that logic was a very important tool in set theory. In contrast to many other disciplines of mathematics, in set theory we typically try to prove things are *impossible*, rather than showing what is possible.

The second international congress of mathematicians in 1900 was in Paris, where Hilbert spoke. At that time, Hilbert was a 'universal' mathematician, and had worked in every major field of mathematics. He gave a list of problems for the century, the 23 Hilbert Problems. The first on this list was the Continuum Problem.

0.1 Continuum Hypothesis

Here is Hilbert's formulation of the Continuum hypothesis (CH): Every set of infinitely many real numbers is either equinumerous with the set of natural numbers or the set of real numbers. More formally, we might write

$$\forall X \subseteq \mathbb{R}, (X \text{ is infinite} \Rightarrow X \sim \mathbb{N} \text{ or } X \sim \mathbb{R})$$

In more modern terms, we write this as the claim $2^{\aleph_0} = \aleph_1$. These two statements are equivalent (in ZFC).

Assume that $2^{\aleph_0} > \aleph_1$, in particular $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \to \mathbb{R}$. Consider $X \coloneqq i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i|_{\aleph_1}$ is a bijection between $\aleph_1$ and X, so $X \sim \aleph_1$. But $\aleph_1 \nsim \mathbb{N}$ and $\aleph_1 \nsim \mathbb{R}$. Thus X refutes CH (in its earlier formulation). So: $2^{\aleph_0} \neq \aleph_1 \implies \neg$CH.

If $2^{\aleph_0} = \aleph_1$. Let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \to \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either $\aleph_0$, i.e. $X \sim \mathbb{N}$ or $\aleph_1$, i.e. $X \sim \mathbb{R}$. So, $2^{\aleph_0} = \aleph_1 \implies$ CH.

0.2 History of CH

- 1938, Gödel: ZFC does not prove $\neg$CH.
- 1963, Cohen: ZFC does not prove CH.

Gödel's proof used the technique of inner models; Cohen's proof used forcing, sometimes referred to as outer models.

Gödel's Completeness Theorem:

$$\text{Cons}(T) \iff \exists (M, E)(M, E) \vDash T$$

From this, we might guess that Gödel's and Cohen's proof will show there is a model of ZFC + CH, and a model of ZFC + $\neg$CH, but by the incompleteness phenomenon, we cannot prove there is a model of ZFC! So, we are not going to be able to prove Cons(ZFC+CH), but instead

$$\text{Cons(ZFC)} \to \text{Cons(ZFC+CH)}$$

Or, equivalently,

$$\text{if } M \vDash \text{ZFC, then there is } N \vDash \text{ZFC + CH.}$$

1 Model theory of set theory

Let's assume for a moment that

$$(M, \in) \models \mathrm{ZFC}.$$

We refer to the canonical objects in M by the usual symbols, e.g., $0, 1, 2, 3, 4, \ldots, \omega, \omega+1, \ldots$

What would an "inner model" be? Take $A \subseteq M$, and consider $(A, \in)$. This is a substructure of $(M, \in)$.

Note: the language of set theory has no function or constant symbols. But we write down

$$X = \varnothing,\ X = \{Y\},\ X = \{Y, Z\},\ X = \bigcup Z,\ X = \mathcal{P}(Z)$$

which appear to use function or constant symbols. These are technically not part of the language of set theory; they are abbreviations:

$X = \varnothing$	abbreviates	$\forall w\ (\neg w \in X)$
$X = \{Y\}$	abbreviates	$\forall w\ (w \in X \leftrightarrow w = Y)$
$X \subseteq Y$	abbreviates	$\forall w\ (w \in X \to w \in Y)$

and so on.

Definition. If φ is a formula in n free variables. We say

(1) φ is **upwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (A, \in) \models \varphi(a_1, \ldots, a_n) \implies (M, \in) \models \varphi(a_1, \ldots, a_n)$$

(2) φ is **downwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (M, \in) \models \varphi(a_1, \ldots, a_n) \implies (A, \in) \models \varphi(a_1, \ldots, a_n)$$

(3) φ is **absolute** between A and M if it is upwards absolute and downwards absolute.

Definition. We say that a formula is Σ_1 if it is of the form

$$\exists x_1 \ldots \exists x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free}$$

or Π_1 if it is of the form

$$\forall x_1 \ldots \forall x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free.}$$

Remark.

(a) If φ is quantifier-free, then φ is absolute between A and M.

(b) If φ is Π_1, then it's downward absolute

(c) If φ is Σ_1, then it's upward absolute

Lecture 2 Under our assumption that $(M, \in) \vDash \text{ZFC}$, which subsets $A \subseteq M$ give a model of ZFC? Using standard model theory, we observed that if φ is quantifier-free, then φ is absolute between $(A, \in)$ and $(M, \in)$, but hardly anything is quantifier-free:

$$x = \varnothing \iff \forall w (w \notin x) =: \Phi_0(x)$$

For instance, define $A := M \setminus \{1\}$ (recall $0, 1, 2, \dots$ refer to the ordinals in M). In A, we have $0, 2, \{1\}$. Clearly $(M, \in) \vDash \Phi_0(0)$. $\Phi_0(x)$ is a Π_1 formula, so by Π_1-downwards absoluteness, $(A, \in) \vDash \Phi_0(0)$.

In reality, $2 = \{0, 1\}$, but 1 is not in A, so informally in A, the object 2 has only one element. Similarly, in A, $\{1\}$ has no elements, since 1 is missing from A. Thus

$$(A, \in) \vDash \Phi_0(\{1\}).$$

Clearly $(M, \in) \nvDash \Phi_0(\{1\})$, so Φ_0 is not absolute between A and M. As a corollary, we get $(A, \in) \nvDash$ Extensionality, since 0 and $\{1\}$ have the same elements in A, but are not equal.

(Remark: We could go on, defining formulas $\Phi_1(x), \Phi_2(x)$ etc. to analyse which of the elements correspond to the natural numbers in A.)

Definition. We call A **transitive** in M, if for all $a \in A$ and $x \in M$ such that $(M, \in) \vDash x \in a$, we have $x \in A$.

Proposition. If A is transitive, then Φ_0 is absolute between A and M.

Proof. Since Φ_0 is Π_1, we only need to show upwards absoluteness. Suppose $a \in A$, such that $(A, \in) \vDash \Phi_0(a)$. Suppose $a \neq 0$. Thus there is some $x \in a$. By transitivity, $x \in A$. So $(A, \in) \vDash x \in a$ and so $(A, \in) \nvDash \Phi_0(a)$, contradiction. □

(Similarly, if Φ_n is the formula describing the natural number n, and there is $a \in A$ such that $(A, \in) \vDash \Phi_n(a)$ and A is transitive, then $a = n$.)

Proposition. If A is transitive in M, then

$$(A, \in) \vDash \text{Extensionality}.$$

Proof. Take $a, b \in A$ with $a \neq b$. By Extensionality in $(M, \in)$, find without loss of generality some $c \in a \setminus b$. Since $c \in a \in A$, by transitivity, $c \in A$. Thus

$$\begin{aligned} (A, \in) &\vDash c \in a \\ (A, \in) &\vDash c \notin b, \end{aligned}$$

so a and b do not satisfy the assumptions of Extensionality. □

Consider now $A := \omega + 2 \subseteq M$, the ordinal consisting of $\{0, 1, 2, \dots, \omega, \omega + 1\}$. This is a transitive subset of M (since it's an ordinal). So

$$(A, \in) \vDash \text{Extensionality}.$$

Consider the formula $x = \mathcal{P}(y)$, which we can informally define as $x = \{z \mid z \subseteq y\}$, but this is not good enough. More properly, we try

$$\mathcal{P}(x) = \forall w \ (w \in x \leftrightarrow w \subseteq y).$$

This still includes the symbol $\subseteq$, so still needs improving.

$$\mathcal{P}(x) = \forall w \ (w \in x \leftrightarrow (\forall v \ (v \in w \to v \in y)))$$

In A, what is $\mathcal{P}(0)$?

$$(A, \in) \vDash \omega + 1 = \mathcal{P}(\omega)$$

1.1 Bounded quantification

We define

$$\begin{aligned}\exists(v\in w)\ \varphi &:\Longleftrightarrow \exists v\ (v\in w\wedge\varphi)\\ \forall(v\in w)\ \varphi &:\Longleftrightarrow \forall v\ (v\in w\to\varphi).\end{aligned}$$

Definition. A formula φ is called Δ_0 if it is in the smallest set of formulas with the following properties

1. All quantifier-free formulas are in S.
2. If $\varphi, \psi \in S$ then so are
   (a) $\varphi\wedge\psi$, $\varphi\vee\psi$, $\varphi\to\psi$, $\varphi\leftrightarrow\psi$
   (b) $\neg\varphi$
   (c) $\exists(v\in w)\ \varphi$, $\forall(v\in w)\ \varphi$.

Theorem. If φ is Δ_0 and A is transitive, then φ is absolute between A and M.

Proof. We already knew that quantifier free formulas are absolute. Absoluteness is obviously preserved under propositional connectives. So, let's deal with (2c): Let's just do

$$\varphi\mapsto\exists(v\in w)\ \varphi=\exists v\ (v\in w\wedge\varphi).$$

So suppose φ is absolute. We need to deal with downwards absoluteness.

$$\begin{aligned}&(M,\in)\vDash\exists(v\in a)\ \varphi(v,a)\quad\text{for some } a\in A\\ &(M,\in)\vDash\exists v\ (v\in a\wedge\varphi(v,a)).\end{aligned}$$

Let's find $m\in M$ such that

$$(M,\in)\vDash m\in a\wedge\varphi(m,a).$$

Transitivity gives $m\in A$. By absoluteness of φ, we get

$$(A,\in)\vDash m\in a\wedge\varphi(m,a)\implies(A,\in)\vDash\exists(v\in a)\ \varphi(v,a).\qquad\square$$

Definition. Let T be any 'set theory'. Then we say that φ is Δ_0^T if there is a Δ_0 formula ψ such that $T\vdash\phi\leftrightarrow\psi$.

- φ is called Σ_1^T if it is T-equivalent to $\exists v_1\ldots\exists v_n\ \psi$ where ψ is Δ_0.
- φ is called Π_1^T if it is T-equivalent to $\forall v_1\ldots\forall v_n\ \psi$ where ψ is Δ_0.

Corollary. If A is transitive in M and both $(M,\in)$ and $(A,\in)$ are models of T, then Δ_0^T formulas are absolute between A and M, and $\Sigma_1^T,(\Pi_1^T)$ formulas are upwards (downwards) absolute between A and M.

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