

# Part II – Analysis of Functions

Based on lectures by Prof C. Mouhot

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# 1 Lebesgue theory

**Exercise.** Show pointwise limit of Riemann-integrable functions is not necessarily Riemann-integrable. (Hint: Dirichlet function).

## 1.1 Recap of measure theory

Consider a set  $X$  and  $\mathcal{P}(X)$  subsets of  $X$ .

**Definition** (Algebra).  $\mathcal{A} \subset \mathcal{P}(X)$  is an **algebra** if it is

- (i) stable under finite union
- (ii) stable under absolute difference
- (iii)  $X \in \mathcal{A}$ .

**Definition** ( $\sigma$ -algebra).  $\mathcal{A} \subset \mathcal{P}(X)$  is a  **$\sigma$ -algebra** if it is

- (i) stable under countable union
- (ii) stable under absolute difference
- (iii)  $X \in \mathcal{A}$ .

**Remark.** Topologies  $\mathcal{T} \subset \mathcal{P}(X)$  are (i) stable under *any* union, (ii) finite intersection, (ii) include  $X$  and  $\emptyset$ .

**Remark.** The property of being a  **$\sigma$ -algebra** is stable under intersection. The smallest  $\sigma$ -algebra containing some topology  $\mathcal{T}$  has elements called **Borel** sets, written  $\mathcal{B}(X)$ .

**Definition.** Consider  $(X, \mathcal{A})$ , where  $\mathcal{A}$  is a  **$\sigma$ -algebra**. A **measure**  $\mu$  is a function  $\mathcal{A} \rightarrow [0, +\infty]$  such that  $\mu(\emptyset) = 0$ . It is  **$\sigma$ -additive** if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then  $(X, \mathcal{A}, \mu)$  is a **measure space**. It is called **complete** if  $A \in \mathcal{A}$  with  $B \subset A$  and  $\mu(A) = 0$ , then  $B \in \mathcal{A}$  and  $\mu(B) = 0$ .

**Exercise.** Show  **$\sigma$ -additivity** is implied by either of the following properties:

- finite additivity and continuity from below
- finite  $\mu(X) < +\infty$  and finite additivity and continuity from above at  $\emptyset$

where

- continuity from below:

$$A_n \in \mathcal{A}, \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n \rightarrow +\infty} \mu\left(\bigcup_{k=1}^n A_k\right)$$

- continuity from above

$$A_n \in \mathcal{A}, \mu(A_1) < +\infty, \mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{n \rightarrow +\infty} \mu\left(\bigcap_{k=1}^n A_k\right)$$

**Exercise.** Find the cardinality of  $\mathcal{T}(\mathbb{R})$ ,  $\mathcal{B}(\mathbb{R})$ ,  $\mathcal{L}(\mathbb{R})$  where  $\mathcal{L}(\mathbb{R})$  are the Lebesgue sets, defined by adding all subsets of null sets to  $\mathcal{B}(\mathbb{R})$ .

**Theorem.** There is a unique **measure** on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  such that

$$\mu\left(\prod_{i=1}^n [a_i, b_i]\right) = \prod_{i=1}^n (b_i - a_i) \quad a_i \leq b_i \in \mathbb{R}$$

called the Lebesgue measure.

*Proof.* See Probability and Measure. □

**Remark.** **Lebesgue measure** is  $\sigma$ -finite:  $\exists$  a countable increasing sequence of sets with finite measure covering  $\mathbb{R}^n$ .

**Definition** (Measurable function). Take  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  two spaces with  **$\sigma$ -algebras**. A function  $f : X \rightarrow Y$  is said to be **measurable** if  $\forall B \in \mathcal{B}$ ,  $f^{-1}(B) \in \mathcal{A}$ .

**Proposition.** Take  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  two spaces with  **$\sigma$ -algebras** where  $Y$  is a metric space and  $\mathcal{B}$  is the collection of **Borel sets**. Let  $f_k : X \rightarrow Y$  be a sequence of **measurable functions** which converge pointwise to  $f : X \rightarrow Y$ . Then  $f$  is measurable.

*Proof.* Since  $B$  is formed from open sets through countable union/intersection and difference, it is enough to prove that  $\forall U \in \mathcal{T}(Y)$ ,  $f^{-1}(U) \in \mathcal{A}$ . (Exercise: Check.)

Let

$$U_n = \left\{ y \in Y \mid d(y, Y \setminus U) > \frac{1}{n} \right\}$$

$$F_n = \left\{ y \in Y \mid d(y, Y \setminus U) \geq \frac{1}{n} \right\}$$

so that

$$U_n \subset F_n \subset U_{n+1} \subset \dots \subset U$$

and  $F_n$  are closed.

We can see  $U = \bigcup_{n \geq 1} U_n = \bigcup_{n \geq 1} F_n$  because  $U$  is open. Hence,

$$f^{-1}(U) = f^{-1}\left(\bigcup_{n \geq 1} U_n\right) = \bigcup_{n \geq 1} f^{-1}(U_n) \subset \bigcup_{n \geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(U_n).$$

We used the fact that

$$f^{-1}(U_n) \subset \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(U_n)$$

To show this, take  $x \in f^{-1}(U_n)$ , so  $f(x) = y \in U_n$ . We know

$$f_k(x) \xrightarrow{k \rightarrow \infty} f(x).$$

Since  $U_n$  open,  $\exists l_x \geq 1$  such that  $\forall k \geq l_x, f_k(x) \in U_n$  giving  $x \in \bigcap_{k \geq l} f_k^{-1}(U_n)$ .

Continuing,

$$\begin{aligned} f^{-1}(U) &\subset \bigcup_{n \geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(U_n). \\ &\subset \bigcup_{n \geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(F_n). \end{aligned}$$

$F_n$  closed, so

$$\bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(F_n) \subset f^{-1}(F_n).$$

In particular, if  $x \in \text{LHS}$ ,  $\exists l \geq 1$  such that  $\forall k \geq l, f_k(x) \in F_n$ . Pass to the limit, and  $f_n$  closed gives  $f(x) \in F_n, x \in f^{-1}(F_n)$ .

In conclusion,

$$\begin{aligned} f^{-1}(U) &\subset \bigcup_{n \geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(U_n) \subset \bigcup_{n \geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(F_n) \\ &\subset \bigcup_{n \geq 1} f^{-1}(F_n) = f^{-1} \left( \bigcup_{n \geq 1} F_n \right) = f^{-1}(U). \end{aligned}$$

So, all inclusions are equality:  $f^{-1}(U)$  is formed of countable intersections and unions of preimages of sets in  $\mathcal{B}$ , hence  $f^{-1}(U) \in \mathcal{A}$ .  $\square$

## 1.2 Lebesgue integration

The important result from measures is the existence of Lebesgue measure, and that the ‘theory’ is closed for pointwise convergence.

We move now from Riemann integration to Lebesgue integration. In Riemann’s theory of integration, we approximate the integral with Darboux sums, by dividing the domain. We require the domain to have a total order, while the codomain must be a Banach space. Conversely, in Lebesgue integration we divide the codomain (again, needing a total order) while the domain must have a [measure](#) defined on it.

**Definition** (Simple).  $f : (X, \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B}(R))$  is **simple** if it is [measurable](#) and takes a finite number of values in  $[0, +\infty)$ .

From here on in, when working on the real line, subsets thereof, or the extended real line the  $\sigma$ -algebra will be the [Borel sets](#).

**Remark.**  $A \subset X, \chi_A$  is [measurable](#) iff  $A \in \mathcal{A}$  is [simple](#). The general form of a simple function is  $s = \sum \alpha_i \chi_{A_i}$ .

**Notation.**  $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$ , so the neighbourhoods of  $\infty$  are  $(a, +\infty]$ , and we can have a metric  $d(x, y) = |\arctan x - \arctan y|$ .

**Proposition.** Let  $f : (X, \mathcal{A}) \rightarrow [0, +\infty]$  [measurable](#). There is  $(s_k)$ , a sequence of [simple](#) functions  $s_{k+1} \geq s_k$  converging pointwise to  $f$ .

*Proof.* For  $n \geq 1$ , define

$$B_n = \{x \mid f(x) \geq n\}$$

$$A_n^i = \left\{x \mid f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right\} \quad i = 0, \dots, n2^n$$

Also, set

$$s_n = \begin{cases} \frac{i-1}{2^n} & \text{on } A_n^i \\ n & \text{on } B_n \end{cases}$$

Check

- $s_{n+1} \geq s_n$  [ $A_n^i = A_{n+1}^{2i} \cup A_{n+1}^{2i+1}$ ]
- on  $B_n$ ,  $|s_n - f| \leq \frac{1}{2^n}$
- on  $x \in \bigcap_{k \geq 1} B_k$ ,  $s_n(x) \rightarrow +\infty$ . □

Observe this construction is simply

$$s_n = \max(n, 2^{-n} \lfloor 2^n f(x) \rfloor)$$

as used in Probability and Measure.

**Definition** (Integral of simple function). Take  $(X, \mathcal{A}, \mu)$  and  $s$  a **simple** function on it given by  $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$ ,  $\alpha_i \in [0, \infty)$ . For  $E \in \mathcal{A}$ , define

$$\int_E s \, d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap E).$$

**Remark.** This induces a new **measure** on  $\mathcal{A}$ , sending  $E \mapsto \int_E s \, d\mu$ .

**Definition** (Integral). Take  $f : (X, \mathcal{A}, \mu) \rightarrow [0, \infty]$  **measurable** and  $E \in \mathcal{A}$ . Then define

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu \mid s \leq f \right\} \in [0, +\infty]$$

**Remark.** This **integral** always makes sense. Also,  $\int_E f \, d\mu = 0$  if  $\mu(E) = 0$ .

**Exercise.** Check linearity of the **integral**. Show Chebyshev's inequality:

$$\mu(\{x \mid f(x) \geq \alpha\}) \leq \alpha^{-1} \int_X f \, d\mu \quad \forall \alpha > 0.$$

Also show that for  $f$  **measurable**,  $X \rightarrow [0, \infty]$

$$\int_X f \, d\mu < \infty \implies \mu(\{x \mid f(x) = \infty\}) = 0.$$

**Theorem** (Beppo-Levi monotone convergence). Take  $f_k : (X, \mathcal{A}, \mu) \rightarrow [0, \infty]$  **measurable**, converging pointwise to  $f$  with  $f_k \leq f_{k+1}$ . Then  $\forall E \in \mathcal{A}$ ,

$$\int_E f_k \, d\mu \xrightarrow{k \rightarrow \infty} \int_E f \, d\mu.$$

*Proof.* Reduce to  $E = X$  by considering  $f_k \chi_E, f \chi_E$ . Then  $(\int_X f_k d\mu)_{k \geq 1}$  is a sequence in  $[0, \infty]$ , non-decreasing.

By monotonicity,  $f_k \nearrow f$ , so  $\int_X f_k d\mu \leq \int_X f d\mu$ . Let

$$\alpha := \lim_{k \rightarrow +\infty} \int_X f_k d\mu \leq \int_X f d\mu.$$

Consider a **simple** function  $s \leq f$  and  $c \in (0, 1)$ .

$$E_k = \{x \in X \mid f_k(x) \geq cs(x)\} \in \mathcal{A} \text{ (using that } f_k, s \text{ are measurable)}$$

$$E_k \subset E_{k+1}, \bigcup_{k \geq 1} E_k = X \text{ (by pointwise convergence)}$$

Thus  $\int_X s d\mu = \lim_{k \rightarrow \infty} \int_{E_k} s d\mu$  (by continuity from below of  $\mu$ ).

$$\int_X f_k d\mu \geq \int_{E_k} f_k d\mu \geq c \int_{E_k} s d\mu$$

Take  $k \rightarrow +\infty$ .  $\alpha \geq c \int_X s d\mu$ , and let  $c \nearrow 1$ , giving  $\int_X s d\mu$ . Taking the supremum over  $s \leq f$  for  $s$  simple,

$$\alpha \geq \int_X f d\mu. \quad \square$$

**Exercise.** Taking  $f_k$  as above, show

$$\int_X \left( \sum_{k \geq 1} f_k \right) d\mu = \sum_{k \geq 1} \int_X f_k d\mu.$$

Let  $\nu : A \in \mathcal{A} \mapsto \int_A f d\mu$  (for  $f : X \rightarrow [0, +\infty]$  **measurable**). Show that for any **measure**  $g : (X, \mathcal{A}, \mu) \rightarrow [0, +\infty]$ ,  $\int_X g d\mu = \int_X fg d\mu$ .

**Theorem** (Fatou's lemma). Take  $f_k$  as above, then

$$\int_X (\liminf f_k) d\mu \leq \liminf \left( \int_X f_k d\mu \right)$$

*Proof.* Let  $F_k = \inf \{f_l \mid l \geq k\}$ , non-decreasing, valued in  $[0, +\infty]$ . These are measurable:  $\{F_k \geq a\} = \bigcap_{l \geq k} \{f_l \geq a\}$ . Observe that  $\int \min(f, g) d\mu \leq \min(\int f d\mu, \int g d\mu)$ . Now, by ,

$$\begin{aligned} \int_X (\liminf f_k) d\mu &= \int_X (\lim F_k) d\mu = \lim_{k \rightarrow +\infty} \left( \int_X F_k d\mu \right) \\ &= \lim_{k \rightarrow \infty} \left( \int_X \left( \inf_{l \geq k} f_l \right) d\mu \right) \\ &\leq \lim_{k \rightarrow \infty} \inf_{l \geq k} \left( \int_X f_l d\mu \right) \\ &\leq \liminf_{k \rightarrow \infty} \int_X f_k d\mu. \end{aligned} \quad \square$$

**Definition** (Integrable).  $f : (X, \mathcal{A}, \mu) \rightarrow \mathbb{C}$  measurable is integrable if  $|f| : X \rightarrow [0, +\infty)$  satisfies  $\int_X |f| d\mu < +\infty$ .

Compute by splitting  $f$  into real and imaginary parts, and each into nonnegative and nonpositive parts.

**Theorem** (Lebesgue's Dominated Convergence). Take  $f_k : (X, \mathcal{A}, \mu) \rightarrow \mathbb{C}$  where

- convergence:  $f_k$  converges pointwise to  $f$
- domination:  $\exists g$  integrable such that  $|f_k| \leq g \ \forall k \geq 1$ .

Then  $f_k, f$  integrable and  $\int_X |f_k - f| d\mu \rightarrow 0$ .

*Proof.* Let  $h_k = 2g - |f - f_k|$ , taking values in  $[0, +\infty)$ . Then  $h_k \rightarrow 2g$  pointwise.

$$\begin{aligned} \int_X 2g d\mu &= \int_X (\lim h_k) d\mu \leq \liminf_{k \rightarrow \infty} \left( \int_X h_k d\mu \right) \\ &= \int_X 2g d\mu - \limsup_{k \rightarrow \infty} \int_X |f_k - f| d\mu \\ \implies \int_X 2g d\mu &\leq \int_X 2g d\mu - \limsup_{k \rightarrow \infty} \int_X |f_k - f| d\mu. \quad \square \end{aligned}$$

### 1.3 Lebesgue spaces

**Definition** ( $L^p$  space). Let  $p \in [1, +\infty]$ , and  $(X, \mathcal{A}, \mu)$  a measure space.  $L^p(X)$  is the set of equivalence classes for almost everywhere equality of functions  $f : X \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) such that  $|f|^p$  is integrable (for  $p \in [1, +\infty)$ , or  $f$  essentially bounded i.e. bounded outside null sets for  $p = \infty$ ).

**Theorem** (Riesz-Fischer).  $(L^p(X), \|\cdot\|_{L^p})$  is a Banach space for  $p \in [1, +\infty]$ , where

$$\|f\|_{L^p} := \left( \int_X |f|^p \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} := \inf \{ M \geq 0 \mid \mu\{|f(x)| \geq M\} = 0 \}.$$

**Exercise.** Take  $\mu(X) < \infty$ , so the domain of  $f$  is a finite measure space. Take  $f \in L^\infty(X)$ , then show  $f \in L^p(X)$  for any  $p \in [1, \infty)$  and

$$\lim_{p \rightarrow \infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}.$$

*Proof.* Vector space axioms and triangular inequality (here called Minkowski) left as an exercise - see Linear Analysis. Instead, we focus on completeness. Start with  $p \in [1, +\infty)$ .

Initially, prove this auxiliary result. **Claim:** Take  $p \in [1, +\infty)$ , and consider a sequence  $(g_k) \in L^p(X)$  such that  $\sum_{k \geq 1} \|g_k\|_{L^p(X)} = M < \infty$ , then there exists  $G \in L^p(X)$  such that  $\sum_{k=1}^n g_k$  converges to  $G$  in  $L^p(X)$  and almost everywhere.

**Proof of the claim:** Let

$$h_n := \sum_{k=1}^n |g_k|, \quad h := \sum_{k=1}^{\infty} |g_k| \in [0, +\infty].$$

$(h_n)$  is a non-decreasing sequence since  $h_n \leq h_{n+1}$ , and  $h_n \rightarrow h$  pointwise. Hence, Beppo-Levi gives that

$$\int_X h_n^p d\mu \rightarrow \int_X h^p d\mu \in [0, +\infty].$$

By the assumption on  $(g_k)$ ,

$$\begin{aligned} \Rightarrow \|h_n\|_{L^p(X)} &\leq \sum_{k=1}^n \|g_k\|_{L^p(X)} \leq M < +\infty \\ \Rightarrow \|h\|_{L^p(X)} &\text{ is finite and less than } M. \end{aligned}$$

Hence  $h$  finite almost everywhere, and  $\sum g_k$  is absolutely convergent almost everywhere, so convergent almost everywhere.

Let us call  $G = \lim_{k \rightarrow \infty} \sum_{k=1}^n g_k$  (almost everywhere).

$$|G(x)| \leq \left| \sum_{k=1}^n g_k \right| \leq \sum_{k=1}^{\infty} |g_k| = h(x)$$

so  $G \in L^p(X)$ .

Using the Dominated Convergence Theorem,

$$\int_X \left| G(x) - \sum_{k=1}^n g_k \right|^p d\mu \rightarrow 0$$

since the integrand converges pointwise to 0, and the domination is given by

$$\left| G(x) - \sum_{k=1}^n g_k \right|^p \leq 2^p h(x)^p,$$

where  $h^p$  integrable since  $G \in L^p(X)$  and  $\sum_{k=1}^m g_k \rightarrow G$  in  $L^p(X)$  and almost everywhere. This proves the claim, so we go back to the main proof.

Let  $(f_k)$  be a Cauchy sequence in  $L^p(X)$ . Build a subsequence  $(f_{\phi(k)})$  such that  $g_k := f_{\phi(k+1)} - f_{\phi(k)}$  satisfies  $\|g_k\|_{L^p(X)} \leq \frac{1}{2^k}$ , so  $g_k$  satisfies the assumptions of the claim. Hence,  $\exists G \in L^p(X)$  such that  $\sum_{k=1}^n g_k \rightarrow G$  almost everywhere and in  $L^p(X)$ . But  $g_k = f_{\phi(n+1)} - f_{\phi(n)}$ . So,

$$f_{\phi(n)} \longrightarrow f_{\phi(1)} + G =: F$$

where the convergence is in  $L^p(X)$  and almost everywhere. But  $(f_n)$  Cauchy in  $L^p(X)$ , so  $f_n \rightarrow F$  in  $L^p(X)$ .  $\square$

**Remark.** Take  $(f_n)$  convergent in  $L^p(X)$ ,  $f_n \rightarrow F$  in  $L^p(X)$ . Then (we proved)  $\exists$  subsequence  $f_{\phi(n)} \rightarrow F$  converging almost everywhere.

**Exercise.**

- 1) Find a sequence  $(f_n)$  converging in  $L^p(\mathbb{R})$  and not converging almost everywhere to its  $L^p$  limit (for  $p \in [1, \infty)$ ). This shows passing to a subsequence was necessary, and not a defect of the argument.
- 2) Complete the proof of [Riesz-Fischer](#) theorem in the  $p = \infty$  case.

**Theorem** (Abstract density result). Take a measure space  $(X, \mathcal{A}, \mu)$  and  $p \in [1, \infty]$ . Then [simple functions](#) that belong to  $L^p(X)$  are dense in  $L^p(X)$ .



*Proof.* For  $f$  real or complex, split into real/imaginary parts and positive/negative parts to reduce to approximating  $f \geq 0$  by [simple](#) functions. Define

$$S_n = \begin{cases} n & \text{on } B_n = \{f \geq n\} \\ \frac{i-1}{2^n} & \text{on } A_n^i = \{f \in [\frac{i-1}{2^n}, \frac{i}{2^n})\}, 1 \leq i \leq n2^n \end{cases}$$

our usual approximation. In the case  $p = \infty$ ,  $\|f\|_{L^\infty(X)} < \infty$ , thus for  $n > \|f\|_{L^\infty(X)}$ ,  $|S_n - f| \leq \frac{1}{2^n}$  almost everywhere. Then  $\|S_n - f\|_{L^\infty(X)} \leq \frac{1}{2^n} \rightarrow 0$ . In the case  $p \in [1, \infty)$ ,  $0 \leq S_n \leq f$  and  $S_{n+1} \geq S_n$ , with  $S_n \rightarrow f$  pointwise. By [Beppo-Levi monotone convergence](#),  $\|S_n - f\|_{L^p(X)} \rightarrow 0$ .  $\square$

**Theorem** (Density-separability of  $L^p(\mathbb{R}^n)$ ,  $p \in [1, +\infty)$ ).

- (1) For  $\mathcal{O}$  an open set in  $\mathbb{R}^n$ ,  $L^p(\mathcal{O})$  is separable (has a countable dense subset).
- (2) Smooth functions compactly supported in  $\mathcal{O}$  are dense in  $L^p(\mathcal{O})$ .

We need and admit a theorem from Probability and Measure (this statement is non-examinable).

**Definition** (Regular measure). A regular [measure](#) on a topological space  $X$  with  $\sigma$ -algebra  $\mathcal{A}$  of [measurable](#) sets is a measure such that every measurable set can be approximated from above by open measurable sets and from below by compact measurable sets.

**Theorem** (Regularity of the Lebesgue measure). The [Lebesgue measure](#) on  $\mathbb{R}^n$  is regular for the Lebesgue sets.

Observe this implies that any Lebesgue set of finite [measure](#) in  $\mathbb{R}^n$  is squeezed between two [Borel sets](#) with the same measure.

*Proof.*

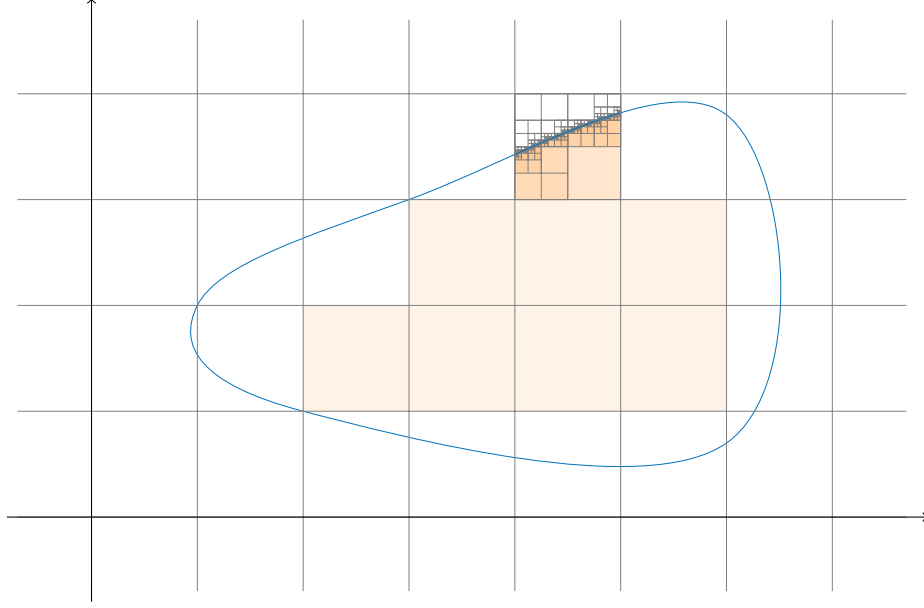
- (1) Let  $\mathcal{C} = \{\text{open sets of the form } \prod_{i=1}^n (a_i, b_i), a_i, b_i \in \mathbb{Q}\}$ , a countable set.

Claim: Any open set  $\mathcal{U} \subset \mathbb{R}^n$  can be covered with a countable union of elements of  $\mathcal{C}$  with disjoint interiors.

Use an inductive procedure: Split into  $\mathbb{Z}^n$ .

1. Keep cubes that are fully inside, discard ones that are fully outside.
2. For borderline cubes, divide into  $2^n$  cubes evenly and go back to step 1.

Check that  $\mathcal{U} = \bigcup_{n,k} C_{n,k}$ , and note that the hypercubes have dyadic, hence rational coordinates. The figure shows a few steps for an example in  $\mathbb{R}^2$ .



Take  $f \in L^p(\mathcal{O})$ ,  $s_k = \sum_{i=1}^k \alpha \chi_{A_i}$  simple functions such that  $s_k \nearrow f$ .

Each  $A_i$  (measurable) can be approximated by  $U_i$  open sets with  $\frac{\epsilon}{k}$  error in measure (outer regularity). Each  $U_i$  is covered by rational cubes  $C_{l,i}$  by the claim:

$$(\infty < \mu(A_i) + \epsilon >) \mu(U_i) = \sum_{e \geq 1} \mu(C_{l,i}).$$

Pick up enough (finitely many)  $(C_{l,i})_{l=1}^{m_i}$  such that

$$\left| \mu(U_i) - \sum_{i=1}^{m_i} \mu(C_{l,i}) \right| \leq \frac{\epsilon}{k}.$$

Putting together all cubes  $(C_{l,i})_{i=1,l=1,\dots,m_i}^k$  with error in measure less than  $2\epsilon$ ,

$$\begin{aligned} \tilde{s}_k &= \sum_{i=1}^k \alpha_i \left( \sum_{l=1}^{m_i} \chi_{C_{l,i}} \right) \\ \tilde{\tilde{s}}_k &= \sum_{i=1}^k \tilde{\alpha}_i \left( \sum_{l=1}^{m_i} \chi_{C_{l,i}} \right) \end{aligned}$$

where  $|\alpha_i - \tilde{\alpha}_i| \leq \frac{\epsilon}{k}$ , and  $\tilde{\alpha}_i \in \mathbb{Q}$ , so  $\tilde{\tilde{s}}_k$  belongs to the target set.

One more approximation step:  $L^\infty$  bound on coefficients.

- (2) On each  $C_{l,i}$  approximate  $\chi_{C_{l,i}}$ . In  $C^0$  we can do this easily with affine functions. To approximate with  $C^\infty$  functions, we can use translations and copies from a smooth compactly supported functions, for instance  $e^{-\frac{1}{x^2}}$ . This requires on the continuity of

the translation operator in  $L^p$ . In particular, with the translation operator  $\tau_h f = f(\cdot + h_n)$ , we need

$$\|\tau_h f - f\|_{L^p} \xrightarrow{h \rightarrow 0} 0.$$

We can show this by proving for simple functions, then approximate. □

**Exercise.** Prove  $L^\infty(\mathbb{R}^n)$  is not separable. (Hint: consider  $(\chi_{B(0,r)})_{r>0}$ ).

## 1.4 How regular are integrable/measurable functions

Take  $f \in L^1(\mathbb{R})$ , and let

$$\int_0^x f(y) dy =: F(x).$$

Is  $F$  differentiable?

**Definition** (Lebesgue point). Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be [measurable](#).  $x \in \mathbb{R}^n$  is a **Lebesgue point** if

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy \xrightarrow{r \rightarrow 0} 0.$$

**Remark.**

$$\frac{1}{r} \left( \int_0^{x+r} f - \int_0^x f \right) = \frac{1}{r} \int_x^{x+r} f \xrightarrow{?} f(x) = F'(x).$$

**Exercise** (Worked example). Prove that points of continuity are [Lebesgue points](#):

$$x \text{ a point of continuity} \implies \forall \epsilon > 0, \exists r_0 > 0 \text{ s.t. } \forall y \in B(x, r_0) |f(x) - f(y)| < \epsilon.$$

Hence for  $r < r_0$ ,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| d\mu < \epsilon.$$

True for any  $\epsilon > 0$ , so we have a Lebesgue point.

**Theorem.** For  $f \in L^1(\mathbb{R}^n)$ :

- (1) Differentiation: almost every  $x \in \mathbb{R}^n$  is a Lebesgue point.
- (2) For any  $E$  a [Borel set](#) of  $\mathbb{R}^n$ , for almost every  $x \in \mathbb{R}^n$ ,

$$\frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \xrightarrow{r \rightarrow 0} \chi_E(x).$$

*Proof* (1)  $\Rightarrow$  (2). Consider  $|x| \leq M, r < 1$ .

$$\frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = \frac{\mu(E \cap B(0, M+1) \cap B(x,r))}{\mu(B(x,r))}.$$

Apply (1) to  $f := \chi_{E \cap B(0, M+1)} \in L^1(\mathbb{R}^n)$ . Do that for any  $M \in \mathbb{N}_{\geq 0}$ , (2) true for almost every  $x \in B(0, M)$ ,  $\forall M \in \mathbb{N}_{\geq 0}$ , i.e. outside  $N_M$ .  $N_0 = \bigcup_{M \geq 1} N_M$  is a null set, (2) true on  $\mathbb{R}^n \setminus N_0$ . □

*Proof of (1).* Note that this is immediately true if  $f$  is continuous, from the exercise earlier. So, aim to write  $f = g + h$  where  $g$  continuous and  $\|h\|_{L^1(\mathbb{R}^n)}$  is as small as is needed.

**Step 1.** Define the Hardy-Littlewood operator

$$Mf(x) := \sup_{r \geq 0} mf(x, r), \quad mf(x, r) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu$$

and

$$E_a := \{x \in \mathbb{R}^n \mid Mf(x) > a\} \quad \text{for } a > 0$$

Claim:  $E_a$  is an open set, hence measurable. Take  $x \in E_a$ .  $Mf(x) > a$ , so  $\exists r > 0$  such that  $mf(x, r) > a$  and  $\frac{a}{mf(x, r)} < 1$ . Pick  $\epsilon > 0$  small enough such that  $(\frac{r}{r+\epsilon}) > \frac{a}{mf(x, r)}$ . Consider  $y \in B(x, \epsilon)$ , so  $B(x, r) \subset B(y, r + \epsilon)$  by triangle inequality.

$$\begin{aligned} mf(y, r + \epsilon) &= \frac{1}{\mu(B(y, r + \epsilon))} \int_{B(y, r + \epsilon)} |f| \, d\mu \\ &\geq \left(\frac{r}{r + \epsilon}\right)^n \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu \\ &\geq \left(\frac{r}{r + \epsilon}\right)^n mf(x, r) > a \end{aligned}$$

Conclude that  $B(x, \epsilon) \subset E_a$ , hence open.

Now take  $x \in E_a$ . Relate the volume of a local ball to  $\int |f|$ :

**Step 2.** Vitali's covering lemma. Take  $X \subset \mathbb{R}^n$ , and  $X \subset \bigcup_{i=1}^N B(x_i, r_i)$ . Then  $\exists J \subset \{1, \dots, N\}$  subset of indices such that

(a)  $(B(x_i, r_i))_{i \in J}$  are pairwise disjoint

(b)

$$X \subset \bigcup_{i \in J} B(x_i, 3r_i)$$

Proof: WLOG, take  $r_1 \geq r_2 \geq \dots \geq r_N$ . Consider  $B(x_1, r_1)$ . All balls that intersect it are entirely included in  $B(x_1, 3r_1)$ . Remove these intersecting balls, to give a set of indices  $J_1$ . By induction, start the argument again on the second largest radius. The induction preserves the covering property, so we are done.

**Step 3.** Claim:  $\mu(E_a) \leq \frac{3^n}{a} \|f\|_{L^1(\mathbb{R}^n)}$ .

Let  $K \subset E_a$  be compact. For each  $x \in K$   $\exists r_x$  with  $mf(x, r_x) > a$ .

$$\begin{aligned} K &\subset \bigcup_{x \in K} B(x, r_x) \\ &\subset \bigcup_{i=1}^N B(x_i, r_i) \\ &\subset \bigcup_{i \in J} B(x_i, 3r_i) \end{aligned}$$

using compactness to get to the second line, and Vitali's covering lemma for the third. Now,

$$\begin{aligned}
\mu(K) &\leq \sum_J \mu(B(x_i, 3r_i)) \\
&= 3^n \sum_J \mu(B(x_i, r_i)) \\
&\leq \frac{3^n}{a} \left( \sum_J \int_{B(x_i, r_i)} |f| \, d\mu \right) \\
&\leq \frac{3^n}{a} \int_{\bigcup_J B(x_i, r_i)} |f| \, d\mu \\
&\leq \frac{3^n}{a} \int_{\mathbb{R}^n} |f| \, d\mu.
\end{aligned}$$

This bound is independent of  $K$ , and by inner regularity

$$\mu(E_a) = \sup \{ \mu(K) \mid K \subset E_a \text{ compact} \},$$

giving the required result:

$$\mu(E_a) \leq \frac{3^n}{a} \int |f| \, d\mu = \frac{3^n}{a} \|f\|_{L^1}.$$

**Step 4.** Say  $f = g + h$  for  $g \in L^1(\mathbb{R}^n)$  continuous, and  $h \in L^1(\mathbb{R}^n)$ ,  $\|h\|_{L^1} < \epsilon$ . Define

$$tf(x, r) = \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, d\mu(y).$$

So,

$$tf(x, r) \leq tg(x, r) + \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |h(y)| \, dy + |h(x)|.$$

Define  $Tf(x) = \limsup_{r \rightarrow 0} tf(x, r)$ , so  $Tf(x) \leq Tg(x) + Mh(x) + |h(x)|$ , and  $Tg(x) = 0$  as  $g$  is continuous. Take  $k \in \mathbb{N}$ , then

$$\begin{aligned}
\mu \left( \left\{ x \mid Tf > \frac{1}{k} \right\} \right) &\leq \mu \left( \left\{ x \mid Mh > \frac{1}{k} \right\} \right) + \mu \left( \left\{ x \mid |h| > \frac{1}{2k} \right\} \right) \\
&\leq 2k3^n \|h\|_{L^1} + 2k \|h\|_{L^1} \\
&\leq 2k(3^n + 1)\epsilon
\end{aligned}$$

using [Chebyshev's inequality](#) and Step 3. This holds  $\epsilon > 0$ , since continuous functions are dense in  $L^1$  so  $\mu(\{x \mid Tf > \frac{1}{k}\}) = 0$ . Take the (countable) union for all  $k \geq 1$ , giving  $\mu(\{x \mid Tf > 0\}) = 0$ . So, almost every  $x$  is a [Lebesgue point](#).  $\square$

Pointwise convergence is 'almost' uniform convergence.

**Theorem** (Egorov's Theorem). Take  $f_n : \mathbb{R}^d \rightarrow \mathbb{C}$  a sequence of measurable functions, and  $A \subset \mathbb{R}^d$  a [Borel set](#) of finite measure,  $f_n \rightarrow f$  pointwise on  $A$ . Then  $\forall \epsilon > 0$ ,  $\exists A_\epsilon \subset A$ ,  $A_\epsilon$  a Borel set with  $\mu(A \setminus A_\epsilon) \leq \epsilon$ , and  $f_n$  converges uniformly to  $f$  on  $A_\epsilon$ .

*Proof.* Define for  $k, n \geq 1$ ,

$$E_n^{(k)} = \bigcap_{p \geq n} \{x \in A \mid |f_p(x) - f(x)| \leq \frac{1}{k}\}.$$

Check  $E_n^{(k)} \subset E_{n+1}^{(k)}$ ,  $E_n^{(k+1)} \subset E_n^{(k)}$  and for fixed  $k \geq 1$   $A = \bigcup_{n \geq 1} E_n^{(k)}$  (pointwise convergence). Using continuity from below,  $\mu(A) = \lim_n \mu(E_n^{(k)})$ , so  $\exists n_k \geq 1$  such that  $\Delta_k = A \setminus E_{n_k}^{(k)}$  has measure  $\mu(\Delta_k) \leq \frac{\epsilon}{2^k}$ . Then  $\Delta := \bigcup_{k \geq 1} \Delta_k$  has measure  $\mu(\Delta) \leq \sum \mu(\Delta_k) \leq \epsilon$ , and let  $A_\epsilon = A \setminus \Delta$ .

Observe there is uniform convergence on  $A_\epsilon$ :  $\forall k \geq 1, \exists n_k \geq 1$  such that  $A_\epsilon \subset E_{n_k}^{(k)}$ . This means:  $\sup_{x \in A_\epsilon} |f_p(x) - f(x)| \leq \frac{1}{k} \forall p \geq n_k$ .  $\square$

**Theorem** (Lusin version 1). Take  $f : \mathbb{R} \rightarrow \mathbb{C}$  measurable. Given any  $\epsilon > 0$ ,  $\exists E \subset \mathbb{R}$  measurable and  $\mu(E) < \epsilon$  such that  $f|_{\mathbb{R} \setminus E}$  is continuous.

**Remark.** Notice this is different from ‘ $f$  continuous at all points in  $\mathbb{R} \setminus E$ ’.

*Sketch of long proof.* Build a sequence  $s_n$  of step functions converging to  $f$ , use [Egorov’s Theorem](#) and conclude... See full details in [official notes](#) (Theorem 1.34).  $\square$

*Short proof.* It is enough to do it for  $f : F \rightarrow \mathbb{C}$  (where  $F = [l, l+1)$ ) for any  $l \in \mathbb{Z}$ , and  $E = \bigcup_{l \in \mathbb{Z}} E_l$ ,  $\mu(E_l) \leq \frac{\epsilon}{2^{l+1}}$ . Take  $(V_n)_{n \geq 1}$  an enumeration of open intervals with rational endpoints (intersected with  $F$ ). By inner regularity, with  $f^{-1}(V_n)$  measurable and  $F \setminus f^{-1}(V_n)$  measurable, choose  $K_n, K'_n$  compact sets such that  $K_n \subset f^{-1}(V_n)$  and  $K'_n \subset F \setminus f^{-1}(V_n)$  and  $\mu(F \setminus (K_n \cup K'_n)) < \frac{\epsilon}{2^n}$ . Fix open sets  $U_n$  such that  $K_n \subset U_n$ ,  $U_n \cap K'_n = \emptyset$  (using compactness). Now,  $K := \bigcap_{n \geq 1} (K_n \cup K'_n)$  satisfies

$$\mu(F \setminus K) \leq \sum_{n \geq 1} \mu(F \setminus (K_n \cup K'_n)) \leq \epsilon.$$

Given  $x \in K$ , for any  $n \geq 1$  such that  $f(x) \in V_n$ ,  $\Rightarrow x \in K_n$ . Moreover,  $K_n \subset U_n$ ,  $U_n$  open hence  $f(U_n \cap K) \subset V_n$ .  $\square$

**Theorem** (Lusin version 2). Take  $f : \mathbb{R} \rightarrow \mathbb{C}$  measurable. Given any  $\epsilon > 0$ ,  $\exists G \subset \mathbb{R}$  with  $\mu(G) < \epsilon$  and  $g : \mathbb{R} \rightarrow \mathbb{C}$  such that  $f = g$  on  $\mathbb{R} \setminus G$ ,  $g$  continuous.

*Proof.* The previous theorem implies that  $\exists E$  measurable such that  $\mu(E) < \frac{\epsilon}{2}$  where  $f|_{\mathbb{R} \setminus E}$  is continuous. Outer regularity gives  $\exists G$  open,  $G \supset E$ ,  $\mu(G) < \epsilon$  where  $G = \bigsqcup_{n \geq 1} I_n$ , for  $I_n$  open intervals. Define

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus G \\ f(a_n) + \frac{(x-a_n)}{b_n-a_n} (f(b_n) - f(a_n)) & x \in (a_n, b_n) = I_n. \end{cases} \quad \square$$

**Exercise.** Reprove from this that continuous functions are dense in  $L^1(\mathbb{R})$ .

## 2 Vector spaces of functions

### 2.1 Recalls on vector spaces

Omitted from lectures - see [official notes](#) for details. These are recalls from previous courses, mostly Linear Analysis.

### 2.2 Separating points and Hahn-Banach theorem

We restate the Hahn-Banach theorem from Linear Analysis, and give consequences. The consequences are important, but proofs thereof are non-examinable.

**Theorem** (Hahn-Banach). Take a normed vector space  $E$ .

- (i) Take  $E_0$  subspace of  $E$ ,  $F : E_0 \rightarrow \mathbb{R}$  linear continuous. Then,  $\exists \tilde{F} : E \rightarrow \mathbb{R}$  continuous linear which extends  $F$ , and

$$\|F\|_{E'_0} := \sup \{ F(f) \mid f \in E_0, \|f\| \leq 1 \} = \|\tilde{F}\|_{E'} = \sup \{ \tilde{F}(f) \mid f \in E, \|f\| \leq 1 \}.$$

The letter  $f$  is used to encourage thinking about function spaces.

- (ii) Let  $A$  be an open convex non-empty in  $E$  and  $B$  convex non-empty in  $E$  with  $A \cap B = \emptyset$ . Then  $\exists F : E \rightarrow \mathbb{R}$  linear continuous such that

$$F \leq \alpha \text{ on } A$$

$$F \geq \alpha \text{ on } B$$

for some  $\alpha \in \mathbb{R}$ .

- (iii) Take  $A$  closed convex non-empty  $\subset E$ ,  $B$  a compact convex non-empty  $\subset E$  with  $A \cap B = \emptyset$ . Then  $\exists F : E \rightarrow \mathbb{R}$  linear continuous such that

$$F \leq \alpha - \epsilon \text{ on } A$$

$$F \geq \alpha + \epsilon \text{ on } B$$

for some  $\alpha \in \mathbb{R}$ ,  $\epsilon > 0$ .

### 2.3 Duality and weak topologies

We know that the Bolzano-Weierstrass theorem holds for  $\mathbb{R}$ , for  $\mathbb{R}^n$  and more generally for  $\mathbb{C}^n$ . But, the theorem fails in infinite dimensional vector spaces over  $\mathbb{R}$ . For example, take the basis vectors of  $\ell^2$ . How can we restore this? In order to get *more* compact sets, we should try taking *fewer* open sets, so try to weaken the topologies.

**Definition** (Initial topologies). Take  $X$ ,  $(Y_i)_{i \in I}$  topological spaces and a family of functions  $\phi_i : X \rightarrow Y_i$ . Then the **initial topology** is the weakest topology on  $X$  that makes all  $(\phi_i)_{i \in I}$  continuous.

**Remark.** There is also a notion of final topology, with  $\phi_i : Y_i \rightarrow X$ , taking the finest topology such that they are all continuous.

**Exercise.**

- (1) Construct the smallest topology  $\mathcal{T}$  (subsets of  $X$ ) that is stable under finite intersection and general union, that contains a given family of subsets  $\mathcal{F}$ .
- (2) Check that  $\mathcal{T}$  above is realised by first taking finite intersections of elements of  $\mathcal{F}$  and second by taking any union of what you have obtained. (Hint: Prove the resulting  $\mathcal{T}$  is stable by finite intersection).
- (3) Check that if you reverse these two operations, you need to do a third union again.

**Definition** (Neighbourhood). Take  $X$  a topological space.

- A **neighbourhood**  $V$  of  $x \in X$  is a subset of  $X$  that contains  $U \in \mathcal{T}_X$ , such that  $x \in U \subset V$ .
- Call  $\mathcal{V}(x)$  the **neighbourhood system** of all such subsets.
- Say  $\mathcal{B}(x)$  is a **neighbourhood basis** if any  $V \in \mathcal{V}(x)$  contains a  $B \in \mathcal{B}(x)$ .

**Proposition.** Consider  $X$ ,  $(Y_i)_{i \in I}$  topological spaces, and  $\phi_i : X \rightarrow Y_i$  functions. Take  $\mathcal{T}$  the **initial topology** on  $X$  as above,  $x \in X$ , then a possible **neighbourhood basis** is

$$\mathcal{B}(x) := \left\{ \bigcap_{\text{finite}} \phi_i^{-1}(\omega_i) \mid \omega_i \text{ open set of } Y_i \text{ containing } \phi_i(x) \right\}$$

*Proof.* Use the 2 steps from the exercise. Consider any  $V \in \mathcal{V}(x)$ ,  $x \in U \subset V$  for  $U$  open.

$$U = \bigcup_{j \in J} \left( \bigcap_{n=1}^{N_j} \phi_{i_n}^{-1}(\omega_{j,i_n}) \right)$$

Hence  $\exists j_0 \in J$  such that

$$x \in \bigcap_{n=1}^{N_{j_0}} \phi_{i_n}^{-1}(\omega_{j_0,i_n}) \text{ open.} \quad \square$$

**Exercise.**

- Prove that  $x_n \rightarrow x$  in  $(X, \mathcal{T}_{\text{initial}})$  iff

$$\phi_i(x_n) \rightarrow \phi_i(x) \text{ in } Y_i, \text{ for all } i \in I$$

- $Z$  another topological space,  $\phi : Z \rightarrow X$  is continuous ( $X$  has the initial topology) iff  $\phi_i \circ \phi : Z \rightarrow Y_i$  is continuous for all  $i \in I$ .

**Remark.** The product topology on  $(\bigotimes_{i \in I} X_i)$  where each  $(X_i, \mathcal{T}_i)$  a topological space, is the **initial topology** with  $Y_i = X_i$  each with topology  $\mathcal{T}_i$  and  $\phi_i$  as the canonical projection

$$\phi_i : (x_j)_{j \in I} \in \bigotimes_{j \in I} X_j \rightarrow x_i$$

Describe the  $B((x_i)_{i \in I})$  as above.

The induced (subspace) topology is the initial topology for the canonical embedding, and  $Y_1$  the whole space.

The quotient topology can be seen as a **final topology**.



**Definition.** Take  $E$  a normed vector space. Then  $E'$  is the space of linear continuous forms to  $\mathbb{R}$ :

$$F : E \rightarrow \mathbb{R}$$

The **weak topology** on  $E'$ , denoted  $\sigma(E, E')$  is the **initial topology** for the family of functions  $E'$ .

It is the coarsest topology that makes all dual forms continuous.

**Proposition.** The topology  $\sigma(E, E')$  on  $E$  is Hausdorff: distinct points can be separated by disjoint open sets.

*Proof.* Take the third formulation of the **Hahn-Banach theorem**: Given  $f \neq g$  in  $E$ , set  $A = \{f\}$  which is closed convex non-empty and  $B = \{g\}$  compact convex. Clearly  $A \cap B = \emptyset$ . Hence  $\exists F \in E, \alpha \in \mathbb{R}$  such that  $f \in U_1 := \{F < \alpha\}$  and  $g \in U_2 := \{F > \alpha\}$ ,  $U_1 \cap U_2 = \emptyset$  by definition.  $\square$

**Remark.**

- Prove  $\sigma(E, E')$  is not more refined than the strong (original) topology.
- In finite dimension, prove  $\sigma(E, E') = \text{strong topology}$ . Give a counter-example in infinite dimensions.
- Show strong/weak convergences can still agree, e.g. on  $C^1(\mathbb{R})$ .

**Remark.** In functional analysis, the term ‘strong topology’ refers to the topology induced by the norm. Given a normed vector space  $E$ , we constructed a dual space  $E'$  (in previous courses called  $E^*$ ) consisting of the linear continuous forms. This also gives the dual norm,

$$\|F\|_{E'} := \sup \{ |F(f)| \mid \|f\|_E \leq 1 \}.$$

This gives a new normed vector space, and so we could continue onto  $E''$ .

In infinite dimensions, the Bolzano-Weierstrass theorem fails, and we would like to restore it by creating a weaker topology  $\sigma(E, E')$ . We will soon see the Banach-Alaoglu-Bolzano theorem gives us that the unit ball in the topology  $\sigma(E'', E')$  is compact, and try to identify  $E''$  and  $E$  (reflexivity).

**Definition** (Weak-\* topology). Take  $E'$  a normed vector space dual to  $E$ , with the norm as given above. The **weak-\* topology** on  $E'$ , called  $\sigma(E', E)$  is generated by the linear maps

$$\begin{aligned} \phi_f : E' &\longrightarrow \mathbb{R} \\ F &\longmapsto F(f) \end{aligned}$$

for all  $f \in E$  (**initial topology**).

**Exercise.**

- (1) A simple consequence of **Hahn-Banach (i)** is that

$$\|f\|_E = \max \{ |F(f)| \mid F \in E', \|F\|_{E'}' \leq 1 \}.$$

Proof: The RHS  $\leq$  LHS by definition of  $\|F\|_{E'} \leq 1$ . Take  $F_0 : \mathbb{R}f \rightarrow \mathbb{R}$ , given by  $F_0(tf) = t\|f\|$ , and extend  $F_0$  to  $F \in E'$  without increasing the  $\|\cdot\|_{E'}$ .

(2)  $(E', \|\cdot\|_{E'})$  is complete (Hint:  $\mathbb{R}$  is complete).

(3) Describe a basis of neighbourhoods for the (recall how we did it for the weak topology).

**Proposition.**  $(E', \sigma(E', E))$  is Hausdorff.

*Proof.* Let  $F_1 \neq F_2$ , so  $\exists f \in E$  where  $F_1(f) \neq F_2(f)$ , and wlog  $F_1(f) < \alpha < F_2(f)$  for  $\alpha \in \mathbb{R}$ . Then define

$$U_1 := \varphi_f^{-1}((-\infty, \alpha))$$

$$U_2 := \varphi_f^{-1}((\alpha, +\infty)).$$

These separate the points and are open. □

**Remark.** We now have three topologies on  $E'$ :

$$\text{weak-}^* \subset \text{weak} \subset \text{strong}.$$

Exercise: prove these inclusions. Hint: for the left inclusion, observe that  $\phi_f \in E''$ .

**Theorem** (Banach-Alaoglu(-Bourbaki)). The closed unit ball of  $E'$  is weak-\* compact.

Unpacking this, the closed unit ball of  $E'$  refers to the subset

$$B_{E'}(0, 1) = \{ F \in E' \mid \|F\|_{E'} \leq 1 \}$$

and weak-\* refers to the topology  $\sigma(E', E)$ .

*Sketch.* The proof is non-examinable. The Tychonoff theorem says: Product of compact topological spaces is compact for the product topology. Now, consider

$$D = \prod_{f \in E} D_f = \prod_{f \in E} [-\|f\|_E, \|f\|_E]$$

This is compact for the product topology. Now, embed

$$F \in B_{E'}(0, 1) \xrightarrow{\Psi} (F(f))_{f \in E}$$

$\Psi$  is injective and bicontinuous (from  $\sigma(E', E) \rightarrow$  product topology) with closed image. □

**Remark.** This illustrates our idea that fewer open sets can give more compact sets. What if  $E$  is separable? Now we can use a weaker argument than Tychonoff's theorem, which required AC. Instead, Cantor's diagonal argument suffices.

## 2.4 Reflexivity

**Aim.** We have BAB theorem on  $(E'', \sigma(E'', E'))$ , can we bring this back to  $(E, \sigma(E, E'))$ , the weak topology?

**Definition.** A normed vector space  $E$  is called **reflexive** if

$$\Phi : E \longrightarrow E'$$

$$f \longmapsto \varphi_f$$

is surjective.

**Exercise.** Prove that  $\Phi$  is isometric (using Hahn-Banach). To talk about isometry, we need to say which topology on  $E''$  is of interest. Here, endow  $E''$  with

$$\|\varphi\|_{E''} = \sup \varphi(F) | F \in E', \|F\|_{E'} \leq 1$$

Prove also that finite-dimensional normed vector spaces are reflexive.

**Theorem** (Kakutani). A Banach space  $E$  is reflexive iff its closed unit ball is weakly compact. (Compact in  $\sigma(E, E')$ ).

*Proof.* ( $\Rightarrow$ ).  $B_E(0, 1)$  is  $\Phi^{-1}(B_{E''}(0, 1))$  so it is enough to prove that  $\Phi^{-1}$  is continuous:

$$\Phi^{-1} : (E'', \sigma(E'', E')) \longrightarrow (E, \sigma(E, E')).$$

Then  $\Phi^{-1}(B_{E''}(0, 1))$  is weakly compact since BAB gives  $B_{E''}(0, 1)$  is weak-\* compact.

This continuity property is equivalent to proving that  $\forall F \in E'$ ,

$$F \circ \Phi^{-1} : (E'', \sigma(E'', E')) \rightarrow \mathbb{R}.$$

For such an  $F \in E'$ , consider  $\varphi \in E''$ ,

$$\begin{aligned} F \circ \Phi^{-1}(\varphi) &= F \circ \Phi^{-1}(\varphi_f) \\ &= F(f) \\ &= \varphi_f(F) = \varphi(F) \end{aligned}$$

The map  $\psi_F : \varphi \in E'' \mapsto \varphi(F)$  is one of the maps generating  $\sigma(E', E')$ , hence continuous in this topology.

( $\Leftarrow$ ). The reverse implication relies on a lemma by Goldstine:

**Lemma.** For a Banach space  $E$ ,  $\Phi(B_E(0, 1))$  is dense in  $B_{E''}(0, 1)$  for the weak-\* topology  $\sigma(E'', E')$ .

Assuming the lemma,  $\Phi(B_E(0, 1)) \subset B_{E''}(0, 1)$  is dense with topology  $(\sigma(E'', E'))$ . If closed we are done, and we prove the stronger fact that it is compact. It is enough to have that  $\Phi(E, \sigma(E, E')) \rightarrow (E'', \sigma(E'', E'))$  is continuous. First we show  $\Phi : (E, \sigma(E, E')) \rightarrow (E'', \sigma(E'', E'''))$  is continuous. This is equivalent to  $\zeta \circ \Phi : (E, \sigma(E, E')) \rightarrow \mathbb{R}$  is continuous  $\forall \zeta \in E'''$ . But  $\zeta \circ \Phi : (E, \|\cdot\|_E) \rightarrow \mathbb{R}$  is continuous, hence by definition  $\zeta \circ \Phi$  is continuous in  $\sigma(E, E')$ . Moreover, since the weak-\* topology is weaker than the weak topology, this implies that  $\Phi : (E, \sigma(E, E')) \rightarrow (E'', \sigma(E'', E'))$  is continuous.

It now only remains to prove the lemma.

Consider  $\varphi \in B_{E''}(0, 1)$  and  $V$  a neighbourhood of  $\varphi$  for  $\sigma(E'', E')$  that we take to be (wlog)

$$V = \{ \psi \in E'' \mid |\psi(F_i) - \varphi(F_i)| < \epsilon \} \quad \text{for some } \epsilon > 0, F_1, \dots, F_k \in E'$$

Our goal is to find  $f \in B_E$  such that  $|\varphi_f(F_i) - \varphi(F_i)| < \epsilon, i = 1, \dots, k$ . (end of argument coming soon).  $\square$

**Corollary.**

- (i) A closed subspace of a reflexive Banach space is reflexive.
- (ii) A Banach space  $E$  is reflexive iff  $E'$  is reflexive.

*Proof.*

- (i) Take  $M$  a closed subspace of  $E$ , then check (exercise)  $\sigma(E, E') \cap M = \sigma(M, M')$ , and that  $\overline{B}_M$  is  $\sigma(E, E')$  closed. By the Kakutani theorem,  $\overline{B}_E$  is  $\sigma(E, E')$  compact, so  $\overline{B}_M$  is precompact hence compact in  $\sigma(M, M')$ . Again using Kakutani, this gives that  $M$  is reflexive.
- (ii) First show that  $E$  reflexive  $\implies E'$  is reflexive, the identification  $\Phi$  of  $E$  with  $E''$  shows  $\sigma(E', E) = \sigma(E', E'')$ . Thus BAB shows  $\overline{B}'_E$  is compact in  $\sigma(E', E'')$  so  $\sigma(E', E)$  compact. Hence by Kakutani,  $E'$  is reflexive. Conversely, if  $E'$  is reflexive then by the previous case,  $E''$  is reflexive, and  $\Phi(E)$  is a closed subspace of  $E''$  hence reflexive.  $E$  is isometric to it, so is reflexive.

□

**Definition.** A Banach space  $E$  is called uniformly convex  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\forall f, g \in \overline{B}_E, \|f - g\|_E > \epsilon \implies \left\| \frac{f+g}{2} \right\|_E < 1 - \delta$ .

**Example.**

**Theorem.** Any **uniformly convex** Banach space is **reflexive**.

*Proof.* Since  $\Phi(\overline{B}_E)$  is closed in  $(E'', \|\cdot\|_{E''})$ , it is enough to prove that it is dense in the strong topology. Wlog  $\varphi \in E'', \|\varphi\|_{E''} = 1$ . Consider  $\epsilon > 0$  so  $\exists \delta$  with

$$\|f - g\|_E > \epsilon \implies \left\| \frac{f+g}{2} \right\|_E < 1 - \delta.$$

By definition of  $E'', \exists F \in \overline{B}_{E'}$  such that  $|\varphi(F)| > 1 - \frac{\delta}{2}$ .

An open neighbourhood in  $\sigma(E'', E')$  can be written as  $V = \{ \psi \in E'' \mid |\psi(F) - \varphi(F)| < \frac{\delta}{2} \}$ . Recall Goldstine's lemma ( $\Phi(\overline{B}_E)$  is dense in  $\overline{B}_{E''}$  for  $\sigma(E'', E')$ ).  $\exists f \in \overline{B}_E$  such that  $\Phi(f) \in V$ . If  $\|\Phi(f) - \varphi\|_{E''} \leq \epsilon$ , we are done. If not,  $\varphi \in W := (\Phi(f) + \epsilon \overline{B}_{E''})^c$  which is open in  $\sigma(E'', E')$ . So,  $V \cap W$  is open and non-empty (contains  $\varphi$ ). Using Goldstine,  $\exists g \in \overline{B}_E, \Phi(g) \in W \cap V$ .

$$\begin{aligned} \left| \underbrace{\Phi(f)(F) - \varphi(F)}_{F(f)} \right| &< \frac{\delta}{2} \\ \left| \underbrace{\Phi(g)(F) - \varphi(F)}_{F(g)} \right| &< \frac{\delta}{2} \\ \|f - g\| &> \epsilon \end{aligned}$$

The first two give  $2\varphi(F) \leq F(f+g) + \delta$ , so  $|\varphi(F)| \leq \left\| \frac{f+g}{2} \right\|_E + \delta$  and  $\left\| \frac{f+g}{2} \right\|_E \geq 1 - \delta$ . But this together with  $\|f - g\| > \epsilon$  contradict uniform convexity. □

**Notation.** For a sequence  $(f_n)$  in  $E, f_n \rightharpoonup f$  means  $f_n$  converges to  $f$  in the weak ( $\sigma(E, E')$ ) topology.

**Proposition.** Take  $E$  a **uniformly convex** Banach space, then a sequence  $f_n$  of  $E$  converges strongly to  $f \in E$  iff  $f_n \rightharpoonup f$  and  $\|f_n\| \rightarrow \|f\|$ .

*Proof.* ( $\Rightarrow$ ) left as an exercise. ( $\Leftarrow$ ) Consider  $f_n \xrightarrow{w} f$  with  $\|f_n\| \rightarrow \|f\|$ . If  $f = 0$  we are done. Assume  $f \neq 0$  and  $n \geq n_0$ ,  $n_0$  large enough such that  $\|f_n\| \neq 0$ . For  $n \geq n_0$ : take

$$g_n := \frac{f_n}{\|f_n\|}, \quad g := \frac{f}{\|f\|}$$

and we aim to show  $\|g_n - g\| \rightarrow 0$ . Observe that  $\frac{g_n + g}{2} \in \overline{B_E}$ , and  $g_n \xrightarrow{w} g$ .  $\forall F \in E'$ ,  $F(g_n) \rightarrow F(g)$ .

$$F\left(\frac{f_n}{\|f_n\|}\right) = \frac{1}{\|f_n\|} F(f_n) \longrightarrow \frac{1}{\|f\|} F(f) = F(g)$$

since these are just converging sequences.

Thus  $\frac{g_n + g}{2} \xrightarrow{w} g$  (linearity). Claim: Then  $\|g\| \leq \liminf_n \|\frac{g_n + g}{2}\|$ . Hence  $\|\frac{g_n + g}{2}\| \rightarrow 1$ , and UC implies that  $\|g_n - g\| \rightarrow 0$  and finally  $\|f_n - f\| \rightarrow 0$ .  $\square$

## 2.5 Separability

**Definition.** A topological space  $X$  is separable if it contains a dense countable subset.

**Exercise.**  $Y \subset X$  and  $X$  a separable topology space, prove  $Y$  is separable in the induced topology.

**Proposition.** Take a Banach space  $E$  with  $(E', \|\cdot\|'_E)$  separable. Then  $(E, \|\cdot\|_E)$  is separable.

*Proof.* Take  $(F_n)$  dense in  $(E', \|\cdot\|'_E)$ , and for each  $n \geq 1$  pick  $f_n \in B_E$  such that  $F_n(f_n) \geq \frac{1}{2} \|F_n\|_{E'}$  (by definition of *something*).  $L = \{(f_n)\}$  is separable (linear combinations with rational coefficients). Let us prove  $L$  is dense in  $(E, \|\cdot\|_E)$ , which is enough to conclude. Claim: Assume  $F \in E'$  is zero on  $L$ , let us prove that  $F = 0$ . Given  $\epsilon > 0$ ,  $\exists n_0 \geq 1$  such that  $\|F - F_{n_0}\|_{E'} \leq \epsilon$ .

$$\|F_{n_0}\|_{E'} \leq 2F_{n_0}(f_{n_0}) = 2(F_{n_0} - F)(f_{n_0}) \leq 2\epsilon$$

where the first inequality comes from the construction of  $f_n$ . Thus  $\|F_{n_0}\|_{E'} \leq 2\epsilon$  and so  $\|F\|_{E'} \leq 3\epsilon$ . But this holds  $\forall \epsilon > 0$ , so  $F = 0$ .

It implies that  $L$  is dense in  $E$ : Otherwise use Hahn-Banach (III) to separate  $A = \|L\|$ ,  $B = \{f_0\}$  with  $f_0 \in E \setminus I$ . This gives  $G \in E'$ ,  $\alpha \in \mathbb{R}$  such that  $\alpha f \in L$ ,  $G(f) < \alpha < G(f_0)$ . This implies  $G|_L \equiv 0$ , so  $G \equiv 0$ , a contradiction.  $\square$

**Exercise.**  $E$  is Banach reflexive separable iff  $E'$  is Banach reflexive separable.

**Proposition.** A Banach space  $E$  is separable iff  $(\overline{B_{E'}}, \sigma(E', E))$  is metrisable.

**Proposition** ('Countable BAB'). (i) Take  $E$  a Banach separable space, then closed bounded subsets of  $E'$  are sequentially weak-\* compact  $(\sigma(E', E))$ .

(ii) Take  $E$  a Banach reflexive, then closed bounded subsets of  $E$  are sequentially weakly compact.