Generalising induction, and coinduction

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Part III Seminar

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Generalising induction, and coinduction

- Recursive definitions and principle of induction away from $\mathbb{N}$
- Dualise: what is corecursive data?

- Universal algebra, model theory, automata, real analysis, theoretical computer science

Algebras of an endofunctor

Take an endofunctor $F : \mathscr{C} \to \mathscr{C}$

Definition (F-algebra)

An F-algebra is a pair $(A, \alpha : FA \to A)$ with $A \in \operatorname{ob} \mathscr{C}$

Definition (Algebra homomorphism)

A homomorphism of F-algebras $(A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ with

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha"] & FB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$\mathbf{Alg} F$ is the category of F-algebras

Coalgebras of an endofunctor

Take an endofunctor $F : \mathscr{C} \to \mathscr{C}$

Definition (F-coalgebra)

An F-coalgebra is a pair $(A, \alpha : A \to FA)$ with $A \in \operatorname{ob} \mathscr{C}$

Definition (Coalgebra homomorphism)

A homomorphism of F-coalgebras $(A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ with

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ FA & \xrightarrow{Ff} & FB \end{array}$$

Coalg$F$ is the category of F-coalgebras

"Monoids"

$$FX \coloneqq 1 + X \times X$$

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, "e"'] & 1+A\times A \arrow[d, dashed, "e+m"] & A \times A \arrow[l] \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
1+A\times A \arrow[r] \arrow[d, "e_A+m_A"'] & 1+B\times B \arrow[d, "e_B+m_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

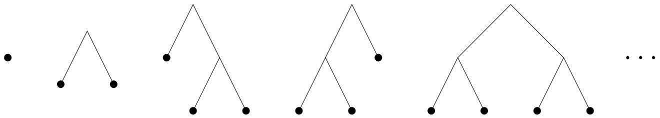
- An F-algebra gives an *interpretation*, not necessarily a model

$$f(e_A) = e_B$$
$$f(m_A(x,y)) = m_B(f(x), f(y))$$

- **Alg**F has a full subcategory isomorphic to **Mon**

Trees

- The set T of finite binary trees gives an F-algebra



$1 \to T$ gives empty tree, $T \times T \to T$ combines trees

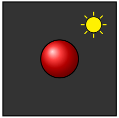
- All binary trees also works

More algebra examples

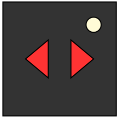
- $FX := 1 + X + X \times X$
  $FA \to A$ decomposes into $1 \to A$, $A \to A$, $A \times A \to A$
- $FX := X \times X$ has semigroups, $FX := 2 + X + 2 \times X \times X$ has rings
- $FX := \mathcal{P}X$, then $\mathcal{P}A \to A$ could be a point in A, or $\mathcal{P}\mathcal{P}B \to \mathcal{P}B$, or many things...
- If $F0 = 0$ then 0 has an F-algebra structure

Coalgebra examples

- $FX = 1 + X$, then $f : A \to 1 + A$



- $FX := B \times X$, then $A \to B \times A$ is a deterministic automaton with output in B (but no fixed input state)
- $FX := \mathcal{P}X$ models non-deterministic automata
- $FX := 1 + X \times X$



and trees

Initial algebras

Definition

An initial F-algebra is an initial object in the category of F-algebras

For any $FA \to A$, there is a unique morphism $i : I \to A$, with

$$\begin{tikzcd} FI \arrow[r, "Fi"] \arrow[d] & FA \arrow[d] \\ I \arrow[r, "i"] & A \end{tikzcd}$$

Definition

A terminal (or final) F-coalgebra is a terminal object in the category of F-coalgebras

Induction on $\mathbb{N}$

Take $FX = 1 + X$ on **Set**, then $\mathbb{N}$ forms an F-algebra

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, "0"'] & 1+\mathbb{N} \arrow[d, dashed, "0+s"] & \mathbb{N} \arrow[l] \arrow[dl, "s"] \\
& \mathbb{N} &
\end{tikzcd}$$

$$\begin{tikzcd}
1+\mathbb{N} \arrow[r, "1+\varphi"] \arrow[d, "0+s"] & 1+A \arrow[d, "x_0+f"] \\
\mathbb{N} \arrow[r, "\varphi"] & A
\end{tikzcd}$$

$$\varphi(0) = x_0$$
$$\varphi(n+1) = f(\varphi(n))$$

Recall: Subobjects of an initial object are isomorphic to it

Streams

$FX = B \times X$, fixed set B; B^∞ is infinite sequences (streams) of B; $B^\infty \to B \times B^\infty$ is head and tail, $(x_n)_{n=0}^\infty \mapsto \langle x_0, (x_n)_{n=1}^\infty \rangle$

$$\begin{aligned} a_0 &\mapsto (b_1, a_2) \\ a_1 &\mapsto (b_3, a_0) \\ a_2 &\mapsto (b_4, a_1) \end{aligned}$$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"] & B^\infty \arrow[d] \\ B \times A \arrow[r, "1_B \times \varphi"] & B \times B^\infty \end{tikzcd}$$

$$\begin{aligned} \varphi(a_0) &= \langle b_1, \varphi(a_2) \rangle \\ &= \langle b_1, \langle b_4, \varphi(a_1) \rangle \rangle \\ &= \langle b_1, \langle b_4, \langle b_3, \varphi(a_0) \rangle \rangle \rangle \\ &= (b_1, b_4, b_3, b_1, b_4, \dots) \end{aligned}$$

Initial algebra is boring

Powerset functors

$\mathcal{P}_{\text{fin}}$ on **Set** has an initial algebra (V_ω, id), hereditarily finite sets

$$\begin{array}{ccc} \mathcal{P}_{\text{fin}} V_\omega & \xrightarrow{\mathcal{P}_{\text{fin}}\varphi} & \mathcal{P}_{\text{fin}} A \\ \Big\downarrow \text{id} & & \Big\downarrow f \\ V_\omega & \xrightarrow{\quad\varphi\quad} & A \end{array}$$

$$\varphi(x) = f(\{\varphi(s) : s \in x\})$$

$\mathcal{P}_\kappa =$ subsets of cardinality $< \kappa$ has V_κ. What about $\mathcal{P}$ itself?

A necessary condition

Lemma (Lambek)

If (A, α) is an initial F-algebra, then α is an isomorphism

Proof.

$$\begin{tikzcd}
FA \arrow[r, "Fi"] \arrow[d, "\alpha"] \arrow[dr, "F1_A"] & FFA \arrow[r, "F\alpha"] \arrow[d, "F\alpha"] & FA \arrow[d, "\alpha"] \\
A \arrow[r, "i"] \arrow[rr, bend right, "1_A"'] & FA \arrow[r, "\alpha"] & A
\end{tikzcd}$$

$\square$

Fixed points of an endofunctor

- If (A, α) is initial, then $FA \cong A$
- Dually, if (A, α) is a terminal coalgebra, then $A \cong FA$.

Definition (Fixed point)

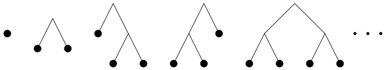
A fixed point of F is an isomorphism $FA \to A$ (equivalently, $A \to FA$)

- $\mathcal{P}$ has no fixed points, in particular no initial algebra (Cantor)
- If 0 is a fixed point of F, it is an initial algebra

Least, greatest fixed points

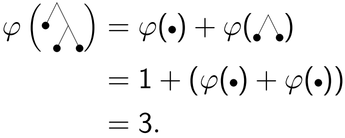
- $\mathrm{Fix}(F)$, a full subcategory of $\mathbf{Alg}F$ and of $\mathbf{Coalg}F$
- Initial algebra is initial in this subcategory
- Subobjects of an initial object are isomorphic to it
- So initial algebra is *least* fixed point
- Dually, terminal coalgebra is greatest fixed point

Trees



T is initial for $FX = 1 + X^2$

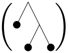
$$\begin{tikzcd}
1 + T^2 \arrow[r] \arrow[d] & 1 + \mathbb{N}^2 \arrow[d, "1+\mathrm{add}"] \\
T \arrow[u, bend left, dashed] \arrow[r, "\varphi"] & \mathbb{N}
\end{tikzcd}$$



Trees, continued

$$
\begin{tikzcd}
1 + T^2 \arrow[r] \arrow[d] & 1 + \mathbb{N}^2 \arrow[d, "0+f"] \\
T \arrow[u, bend left, dashed] \arrow[r, "\varphi"] & \mathbb{N}
\end{tikzcd}
$$

$$f(m, n) = 1 + \max\{m, n\}$$

$\varphi\Big($  $\Big) = 1 + \max\{\varphi(\bullet), \varphi($  $)\}$

$= 1 + (1 + \max\{\varphi(\bullet), \varphi(\bullet)\})$

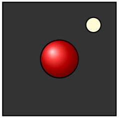
$= 2.$

Conaturals?

$FX = 1 + X$. Take $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and

$$\begin{aligned} \alpha : \overline{\mathbb{N}} &\longrightarrow 1 + \overline{\mathbb{N}} \\ 0 &\longmapsto * \\ n &\longmapsto n - 1 \\ \infty &\longmapsto \infty \end{aligned}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \overline{\mathbb{N}} \arrow[d] \\ 1 + A \arrow[r] & 1 + \overline{\mathbb{N}} \end{tikzcd}$$



Finite lists

$FX = 1 + B \times X$, B^* are the finite sequences of B (finite lists)

$$\begin{tikzcd} 1+\mathbb{Z}\times\mathbb{Z}^* \arrow[r] \arrow[d, "\varepsilon+\langle\cdot,\cdot\rangle"'] & 1+\mathbb{Z}\times\mathbb{Q} \arrow[d, "f"] \\ \mathbb{Z}^* \arrow[r, "\varphi"] & \mathbb{Q} \end{tikzcd}$$

$$f(*) = 1$$
$$f(n, q) = n \times q$$

$$\begin{aligned} \varphi([3, -4, 2]) &= f(3, \varphi([-4, 2])) \\ \varphi([-4, 2]) &= f(-4, \varphi([2])) \\ \varphi([2]) &= f(2, \varphi([])) \\ \varphi([]) &= f(*) = 1 \\ \varphi([3, -4, 2]) &= 3 \times (-4 \times (2 \times 1)) \\ &= -24 \end{aligned}$$

More coalgebras

Terminal coalgebra of $FX = 1 + B \times X$ is (potentially infinite) lists, B^ω

$$\begin{aligned} \beta : B^\omega &\to 1 + B \times B^\omega \\ \varepsilon &\mapsto * \\ \langle b_0, \mathbf{b} \rangle &\mapsto (b_0, \mathbf{b}) \end{aligned}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "f"] & \mathbb{Z}^\omega \arrow[d, "\beta"] \\ 1 + \mathbb{Z} \times \mathbb{Z} \arrow[r] & 1 + \mathbb{Z} \times \mathbb{Z}^\omega \arrow[u, dashed, bend left] \end{tikzcd}$$

$$\begin{aligned} f(0) &= * \\ f(n) &= (n, n-1) \end{aligned}$$

$$\begin{aligned} \varphi(2) &= \langle 2, \varphi(1) \rangle \\ &= \langle 2, \langle 1, \varphi(0) \rangle \rangle \\ &= \langle 2, \langle 1, \varepsilon \rangle \rangle \\ &= [2, 1] \end{aligned}$$

$\varphi(-5)$ is infinite: $[-5, -6, -7, \dots]$

Our examples

Functor	Initial algebra	Terminal coalgebra
$1 + X$	naturals	conaturals
$1 + X^2$	finite binary trees	binary trees
$1 + B \times X$	finite lists	lists
$B \times X$	empty	streams
$\mathcal{P}_{\text{fin}}$	V_ω	finitely branching trees

- Dyadic rationals in $[0, 1]$ as a coalgebra
- (Freyd) $[0, 1]$ itself as a coalgebra (as a set, poset, totally ordered set or topologically)
- (Leinster) $L^1[0, 1]$ as a coalgebra

Recursion vs corecursion

- Recursion allows defining a map out of a structure by reducing to easier cases
- Induction defines what to do on constructors
- Corecursion allows defining a map to a complex structure by building up from a seed
- Coinduction defines what destructors do

Sufficient conditions?

- Can have a least fixed point and no initial algebra
- But with some conditions on $\mathscr{C}$ and F, F has a fixed point iff an initial F-algebra exists
  - for example, if F preserves monomorphisms in **Set**, **Top**
- But F can preserve monomorphisms and have a fixed point, but no terminal coalgebra

Theorem (Adamek)

Let 0 *be initial in* $\mathscr{C}$*, and suppose*

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \dots$$

has a colimit, which is preserved by F*. Then the colimit carries an initial algebra.*