Part III – Category Theory

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Contents

0 Introduction

Lecture 1 Category theory is like a language spoken by many different people, with many different dialects. Specifically, different parts of category theory are used in different branches of mathematics. In this course, we aim to speak the language of category theory, without an accent - a broad overview of all aspects of category theory. There will be many examples, some of which may not be understandable. As long as some examples make sense, it is not a point of concern that some examples seem unfamiliar.

1 Definitions and Examples

1.1 Definition (Category)**.** A **category** $\mathscr{C}$ consists of

(a) a collection $\mathscr{C}$ of **objects** $A, B, C, \ldots$

(b) a collection $\operatorname{mor}\mathscr{C}$ of **morphisms** $f, g, h, \ldots$

(c) two operations $\operatorname{dom}, \operatorname{cod}$ assigning to each $f \in \operatorname{mor}\mathscr{C}$ a pair of objects, its **domain** and **codomain**. We write $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'.

(d) an operation assigning to each $A \in \operatorname{ob}\mathscr{C}$ a morphism $A \xrightarrow{1_A} A$, called its **identity**.

(e) a partial binary operation **composition** $(f, g) \longmapsto fg$ on morphisms, such that fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod} f$ if fg is defined.

satisfying

(f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$

(g) $(fg)h = f(gh)$ whenever fg and gh are defined

1.2 Remark.

(a) This definition is independent of a model of set theory. If we're given a particular model of set theory, we call the category $\mathscr{C}$ **small** if $\operatorname{ob}\mathscr{C}$ and $\operatorname{mor}\mathscr{C}$ are sets.

(b) Some texts say fg means 'f followed by g', i.e. fg defined $\iff$ $\operatorname{cod} f = \operatorname{dom} g$.

(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the compositions are defined. So we could formulate the definition entirely in terms of morphisms.

1.3 Examples.

(a) The category **Set** has all sets as objects, and all functions between sets as morphisms. (Strictly, morphisms $A \longrightarrow B$ are pairs (f, B) where f is a set-theoretic function.)

(b) The category **Gp** has all groups as objects, and group homomorphisms as morphisms. Similarly, **Rng** is the category of rings, $\mathbf{Mod}_R$ the category of R-modules.

(c) The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** has uniform spaces and uniformly continuous functions, and **Mf** has manifolds and smooth maps.

(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given $\mathscr{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathscr{C}$ a **congruence** if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathscr{C}/\simeq$ with the same objects as $\mathscr{C}$, but congruence classes as morphisms.

(e) Given $\mathscr{C}$, the **opposite category** $\mathscr{C}^{\mathrm{op}}$ has the same objects and morphisms as $\mathscr{C}$, but dom and cod are interchanged, and fg in $\mathscr{C}^{\mathrm{op}}$ is gf in $\mathscr{C}$. This leads to the **Duality principle**: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

(f) A small category with one object is a **monoid**, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (f is an **isomorphism** if $\exists g$ such that fg and gf are identities).

(g) A **groupoid** is a category in which every morphism is an isomorphism. For a topological space X, the fundamental groupoid $\pi(X)$ has all points of X as objects and morphisms $x \longrightarrow y$ are homotopy classes rel $\{0,1\}$ of paths $u : [0,1] \longrightarrow X$ with $u(0) = x$, $u(1) = y$. (If you know how to prove that the fundamental group is a group, you can prove that $\pi(X)$ is a groupoid.)

(h) A **discrete** category is one whose only morphisms are identities. A **preorder** is a category $\mathscr{C}$ in which, for any pair (A, B) there is at most 1 morphism $A \longrightarrow B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as **Set**, but morphisms $A \longrightarrow B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define

$$S \circ R = \{\, (a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R \wedge (b, c) \in S) \,\}.$$

The identity $1_A : A \longrightarrow A$ is $\{\, (a, a) \mid a \in A \,\}$.

Similarly, the category **Part** of sets and partial functions (i.e. relations such that $(a, b) \in R, (a, b') \in R \implies b = b'$).

(j) Let K be a field. The category $\mathbf{Mat}_K$ has natural numbers as objects, and morphisms $n \longrightarrow p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.

1.4 Definition (Functor)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \longrightarrow \mathscr{D}$ consists of

(a) a mapping $A \longmapsto FA$ from $\operatorname{ob}\mathscr{C}$ to $\operatorname{ob}\mathscr{D}$

(b) a mapping $f \longmapsto Ff$ from $\operatorname{mor}\mathscr{C}$ to $\operatorname{mor}\mathscr{D}$

such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$ and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Lecture 2 **1.3 Examples** (*Continued*)**.**

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them.

1.5 Examples.

(a) We have **forgetful functors** $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, $\mathbf{Rng} \longrightarrow \mathbf{Set}$, $\mathbf{Top} \longrightarrow \mathbf{Set}$, $\mathbf{Rng} \longrightarrow \mathbf{AbGp}$ (forgetting $\times$), $\mathbf{Rng} \longrightarrow \mathbf{Mon}$ (forgetting $+$).

(b) Given a set A, the free group FA has the property: given any group G and any function $A \xrightarrow{f} UG$, there's a unique homomorphism $FA \xrightarrow{f} G$ extending f. F is a functor $\mathbf{Set} \longrightarrow \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$.

Functoriality follows from uniqueness: given $B \xrightarrow{g} C$, $F(gf)$ and $(Fg)(Ff)$ are both homoms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$. Call this the **free functor**.

(c) Given a set A, we write $\mathcal{P}A$ for the set of all subsets of A. We can make $\mathcal{P}$ into a functor $\mathbf{Set} \longrightarrow \mathbf{Set}$: given $A \xrightarrow{f} B$, we define $\mathcal{P}f(A') = \{ f(a) \mid a \in A' \}$ for $A' \subseteq A$. But we also have a functor $\mathcal{P}^* : \mathbf{Set} \longrightarrow \mathbf{Set}^{\mathrm{op}}$ defined on objects by $\mathcal{P}$, but $\mathcal{P}^* f(B') = \{ a \in A \mid f(a) \in B' \}$ for $B' \subseteq B$.

By a **contravariant** functor $\mathscr{C} \longrightarrow \mathscr{D}$, we mean a functor $\mathscr{C} \longrightarrow \mathscr{D}^{op}$ (or $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$). (A **covariant** functor is one that doesn't reverse arrows).

(d) Let K be a field. We have a functor $* : \mathbf{Mod}_K \longrightarrow \mathbf{Mod}_K^{\mathrm{op}}$ defined by $V^* = \{\text{linear maps } V \longrightarrow K\}$ and if $V \xrightarrow{f} W$, $f^*(\theta : W \longrightarrow K) = \theta f$.

(e) We have a functor $op : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ which is the 'identity' on morphisms. (Note that this is covariant).

(f) A functor between monoids is a monoid homomorphism.

(g) A functor between posets is an order-preserving map.

(h) Let G be a group. A functor $F : G \longrightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A, i.e. a permutation representation of G (where we use $*$ to refer to the unique object of the group). Similarly a functor $G \longrightarrow \mathbf{Mod}_K$ is a K-linear representation of G.

(i) The construction of a fundamental group $\pi_1(X, x)$ of a space X with basepoint x is a functor $\mathbf{Top}_* \longrightarrow \mathbf{Gp}$ where $\mathbf{Top}_*$ is the set of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \longrightarrow \mathbf{Gpd}$ where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

1.6 Definition (Natural transformation)**.** Let $\mathscr{C}, \mathscr{D}$ be categories and $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ two functors. A **natural transformation** $\alpha : F \longrightarrow G$ consists of an assignment $A \longmapsto \alpha_A$

from $\operatorname{ob}\mathscr{C}$ to $\operatorname{mor}\mathscr{D}$, such that $\operatorname{dom}\alpha_A = FA$ and $\operatorname{cod}\alpha_A = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathscr{C}$ the square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_a"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)\alpha_A$).

1.3 Examples (*Continued*)**.**

(l) Given categories $\mathscr{C}, \mathscr{D}$, we write $[\mathscr{C}, \mathscr{D}]$ for the category whose objects are functors $\mathscr{C} \longrightarrow \mathscr{D}$, and whose morphisms are natural transformations.

1.7 Examples.

(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \longrightarrow V^{**}$ given by

$$\alpha_V(v)(\theta) = \theta(v)$$

for $\theta \in V^*$. This is the V-component of a natural transformation

$$1_{\mathbf{Mod}_K} \longrightarrow {**} : \mathbf{Mod}_K \longrightarrow \mathbf{Mod}_K.$$

Lecture 3

(b) For any set A, we have a mapping $\sigma_A : A \longrightarrow \mathcal{P}A$ sending a to $\{a\}$. If $f : A \longrightarrow B$, then $\mathcal{P}f\{a\} = \{f(a)\}$, so σ is a natural transformation $1_{\mathbf{Set}} \longrightarrow \mathcal{P}$.

(c) Let $F : \mathbf{Set} \longrightarrow \mathbf{Gp}$ be the free group functor (Examples 1.5(b)) and $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \longrightarrow UFA$ form a natural transformation $1_{\mathbf{Set}} \longrightarrow UF$.

(d) Let G, H be groups and $f, g : G \longrightarrow H$ two homomorphisms. A natural transformation $\alpha : f \longrightarrow g$ corresponds to an element $h = \alpha_*$ of H such that $h.f(x) = g(x).h$ for all $x \in G$, or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.

(e) Let A, B be two G-sets regarded as functors $G \longrightarrow \mathbf{Set}$. A natural transformation $A \longrightarrow B$ is a function f satisfying $f(g.a) = g.f(a)$ for all $a \in A$, i.e. a G-equivariant map.

1.8 Lemma. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors, and $\alpha : F \longrightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathscr{C}, \mathscr{D}]$ iff each α_A is an isomorphism in $\mathscr{D}$.

Proof.

$\Longrightarrow$ trivial

$\Longleftarrow$ Suppose each α_A has an inverse β_A. Given $f : A \longrightarrow B$ in $\mathscr{C}$, we need to show that

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, "\beta_A"] & GB \arrow[d, "\beta_B"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

commutes.

But

$$\begin{aligned}(Ff)\beta_A &= \beta_B\alpha_B(Ff)\beta_A \\ &= \beta_B(Gf)\alpha_A\beta_A \\ &= \beta_B(Gf).\end{aligned}$$

□

1.9 Definition (Equivalent category)**.** Let $\mathscr{C}, \mathscr{D}$ be categories. By an **equivalence** between $\mathscr{C}$ and $\mathscr{D}$, we mean a pair of functors $F : \mathscr{C} \longrightarrow \mathscr{D}$, $G : \mathscr{D} \longrightarrow \mathscr{C}$ together with natural isomorphisms $\alpha : 1_{\mathscr{C}} \longrightarrow GF$ and $\beta : FG \longrightarrow 1_{\mathscr{D}}$. We write $\mathscr{C} \simeq \mathscr{D}$ if $\mathscr{C}$ and $\mathscr{D}$ are equivalent.

We say a property P of categories is a **categorical property** if whenever $\mathscr{C}$ has P and $\mathscr{C} \simeq \mathscr{D}$, then $\mathscr{D}$ has P.

For instance, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

1.10 Examples.

(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint-preserving functions). We define $F : \mathbf{Set}_* \longrightarrow \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$ and if $f : (A, a) \longrightarrow (B, b)$

$$Ff(x) = \begin{cases} f(x) & \text{if } f(x) \neq b \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $G : \mathbf{Part} \longrightarrow \mathbf{Set}_*$ by $G(A) = A^+ = A \cup \{A\}$ and if $f : A \longrightarrow B$ is a partial function, we define $Gf : A^+ \longrightarrow B^+$ by

$$Gf = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

The composite FG is the identity on **Part**, but GF is not the identity, however there's an isomorphism

$$(A, a) \longrightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism $\mathbf{Set}_* \longrightarrow \mathbf{Part}$ since **Part** has a 1-element isomorphism class $\{\varnothing\}$ and $\mathbf{Set}_*$ doesn't.

(b) The category $\mathbf{FdMod}_K$ of finite-dimensional vector spaces over K is equivalent to $\mathbf{FdMod}_K^{\mathrm{op}}$: the functors in both directions are $(-)^*$ and both isomorphisms are the natural transformations of Examples 1.7(a).

(c) $\mathbf{FdMod}_K$ is also equivalent to $\mathbf{Mat}_K$: We define $F : \mathbf{Mat}_K \longrightarrow \mathbf{FdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A with respect to the standard bases of K^n and K^p.

To define $G : \mathbf{FdMod}_K \longrightarrow \mathbf{Mat}_K$, choose a basis for each finite dimensional vector space,and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ as the matrix representing f with respect to the chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen basis gives isomorphisms $FG(V) = K^{\dim V} \longrightarrow V$ for each V, which form a natural isomorphism.

Lecture 4 **1.11 Definition** (Faithful, full, essentially surjective)**.** Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a functor.

(a) We say F is **faithful** if, given $f, f' \in \operatorname{mor}\mathscr{C}$ with $\operatorname{dom} f = \operatorname{dom} f'$, $\operatorname{cod} f = \operatorname{cod} f'$ and $Ff = Ff'$ then $f = f'$.

(b) We say F is **full** if, given $FA \xrightarrow{g} FB$ in $\mathscr{D}$, there exists $A \xrightarrow{f} B$ in $\mathscr{C}$ with $Ff = g$.

(c) We say F is **essentially surjective** if, for every $B \in \operatorname{ob}\mathscr{D}$, there exists $A \in \operatorname{ob}\mathscr{C}$ and an isomorphism $FA \longrightarrow B$ in $\mathscr{D}$.

We say a subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is **full** if the inclusion $\mathscr{C}' \longrightarrow \mathscr{C}$ is a full functor.

Example. **Gp** is a full subcategory of **Mon**, but **Mon** is not a full subcategory of the category **Sgp** of semigroups.

1.12 Lemma. Assuming the axiom of choice, a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is part of an equivalence $\mathscr{C} \simeq \mathscr{D}$ iff it is full, faithful and essentially surjective.

Proof.

$\implies$ Given G, α, β as in Definition 1.9, for each $B \in \operatorname{ob}\mathscr{D}$, β_B is an isometry $FGB \longrightarrow B$, so F is essentially surjective.

Given $A \xrightarrow{f} B$ in $\mathcal{C}$, we can recover f from Ff as the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B.$$

Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$.

Given $FA \xrightarrow{g} FB$, define f to be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$$

Then $GFf = \alpha_B f \alpha_A^{-1} = Gg$, and G is faithful for the same reason as F, so $Ff = g$.

$\impliedby$ For each $B \in \operatorname{ob}\mathscr{D}$, choose $GB \in \operatorname{ob}\mathscr{C}$ and an isomorphism $\beta_B : FGB \longrightarrow B$ in $\mathscr{D}$. Given

$$B \xrightarrow{g} B'$$

define $Gg : GB \longrightarrow GB'$ to be the unique morphism whose image under F is

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

Uniqueness implies functoriality: given

$$B' \xrightarrow{g'} B''$$

then note $(Gg')(Gg)$ and $G(g'g)$ have the same image under F, so they're equal.

By construction, β is a natural transformation $FG \longrightarrow 1_{\mathscr{D}}$.

Given $A \in \mathrm{ob}\,\mathscr{C}$, define $\alpha_A : A \longrightarrow GFA$ to be the unique morphism whose image under F is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA$$

α_A is an isomorphism, since β_{FA} also has a unique pre-image under F.

Also α is a natural transformation, since any naturality square for α is mapped by F to a commutative square, and F is faithful. □

1.13 Definition (Skeleton)**.** By a **skeleton** of a category $\mathscr{C}$, we mean a full subcategory $\mathscr{C}_0$ containing one object from each isomorphism class. We say $\mathscr{C}$ is **skeletal** if it's a skeleton of itself.

Example. $\mathbf{Mat}_K$ is skeletal, and the image of $F : \mathbf{Mat}_K \longrightarrow \mathbf{FdMod}_K$ of Examples 1.10(c) is a skeleton of $\mathbf{FdMod}_K$.

Warning. Almost any assertion about skeletons is equivalent to the axiom of choice. See question 2 on example sheet 1.

1.14 Definition (Monomorphism, epimorphism)**.** Let $A \xrightarrow{f} B$ be a morphism in $\mathscr{C}$

(a) We say f is a **monomorphism** (or f is **monic**) if, given any pair $C \overset{f}{\underset{g}{\rightrightarrows}} A$, $fg = fh$ implies $g = h$

(b) We say f is an **epimorphism** (or **epic**) if it's a monomorphism in C^{op} i.e. if $gf = hf$ implies $g = h$.

We denote monomorphisms by $A \overset{f}{\rightarrowtail} B$ and epimorphisms by $A \overset{f}{\twoheadrightarrow} B$

Any isomorphism is monic and epic: more generally if f has a left inverse (i.e. $\exists g$ such that gf is an identity) then it's monic. We call such monomorphisms **split**.

We say C is a **balanced** category if any morphism which is both monic and epic is an isomorphism.

1.15 Examples.

(a) In **Set**, mono $\iff$ injective ($\implies$ easy; for $\impliedby$ take $C = 1 = \{*\}$) and epi $\iff$ surjective ($\implies$ easy; for $\impliedby$ use two morphisms $B \longrightarrow 2 = \{0,1\}$). So **Set** is balanced.

(b) In **Gp** mono $\iff$ injective (for $\impliedby$ use homoms $\mathbb{Z} \longrightarrow A$) and epi $\iff$ surjective ($\impliedby$ uses free products with amalgamation). So **Gp** is balanced.

(c) In **Rng**, mono $\iff$ injective (proof much as for **Gp**) but the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \overset{f}{\underset{g}{\rightrightarrows}} R$ agree on all integers, they agree everywhere. So **Rng** isn't balanced.

(d) In **Top**, mono $\iff$ injective and epi $\iff$ surjective (proofs as in **Set**). But **Top** isn't balanced since a continuous bijection needn't have a continuous inverse.

2 The Yoneda Lemma

Lecture 5 **2.1 Definition** (Locally small)**.** We say a category $\mathscr{C}$ is **locally small** if, for any two objects A, B, the morphisms $A \longrightarrow B$ in $\mathscr{C}$ form a set $\mathscr{C}(A, B)$.

If we fix A and let B vary, the assignment $B \longmapsto \mathscr{C}(A, B)$ becomes a functor $\mathscr{C}(A, -) : \mathscr{C} \longrightarrow \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathscr{C}(A, f)$ is the mapping $g \longmapsto fg$. Similarly, $A \longmapsto \mathscr{C}(A, B)$ defines a functor $\mathscr{C}(-, B) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

2.2 Lemma (Yoneda Lemma)**.** Let $\mathscr{C}$ be a locally small category, $A \in \mathrm{ob}\,\mathscr{C}$ and $F : \mathscr{C} \longrightarrow \mathbf{Set}$ a functor.

(i) Then natural transformations $\mathscr{C}(A, -) \longrightarrow F$ are in bijection with elements of FA.

(ii) Moreover, this bijection is natural in both A and F.

Proof of Yoneda Lemma(i). Given $\alpha : \mathscr{C}(A, -) \longrightarrow F$, we define

$$\Phi(\alpha) = \alpha_A(1_A) \in FA.$$

Given $x \in FA$, we define $\Psi(x) : \mathscr{C}(A, -) \longrightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB.$$

$\Psi(x)$ is natural: given $g : B \longrightarrow C$, we have

$$\begin{aligned} \Psi(x)_C \mathscr{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x) \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x). \end{aligned}$$

$$\begin{tikzcd} \mathscr{C}(A, B) \arrow[r, "\mathscr{C}(A,g)"] \arrow[d, "\Psi(x)_B"'] & \mathscr{C}(A, C) \arrow[d, "\Psi(x)_C"] \\ FB \arrow[r, "Fg"'] & FC \end{tikzcd}$$

We also verify Ψ and Φ are inverse:

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x.$$

Given α,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathscr{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

so $\Psi\Phi(\alpha) = \alpha$. $\square$

2.3 Corollary. The assignment $A \longmapsto \mathscr{C}(A, -)$ defines a full and faithful functor $\mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{C}, \mathbf{Set}]$.

Proof. Put $F = \mathscr{C}(B,-)$ in Lemma 2.2(i): we get a bijection between $\mathscr{C}(B,A)$ and morphisms $\mathscr{C}(A,-) \longrightarrow \mathscr{C}(B,-)$ in $[\mathscr{C}, \mathbf{Set}]$. We need to verify this is functorial: but it sends $f : B \longrightarrow A$ to the natural transformation $g \longmapsto gf$. So functoriality follows from associativity. $\square$

We call this functor (or the functor $\mathscr{C} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$) sending A to $\mathscr{C}(-,A)$ the **Yoneda embedding** of $\mathscr{C}$, and denote it by Y.

Proof of Yoneda Lemma(ii). Suppose for the moment that $\mathscr{C}$ is small, so that $[\mathscr{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathscr{C} \times [\mathscr{C}, \mathbf{Set}] \longrightarrow \mathbf{Set}$: One sends (A, F) to FA, and the other is the composite

$$\mathscr{C} \times [\mathscr{C}, \mathbf{Set}] \xrightarrow{\quad Y \times 1 \quad} [\mathscr{C}, \mathbf{Set}]^{\mathrm{op}} \times [\mathscr{C}, \mathbf{Set}] \xrightarrow{[\mathscr{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

Yoneda Lemma(ii) says that these are naturally isomorphic.

We can translate this into an elementary statement, making sense even when $\mathscr{C}$ isn't small, given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathscr{C}(A,-) \longrightarrow F$ give the same result, namely

$$\alpha_B (Ff) \beta_A (1_A) = (GF) \alpha_A \beta_A (1_A)$$

which is equal to $\alpha_B \beta_B (f)$. $\square$

2.4 Definition. We say a functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$ is **representable** if it's isomorphic to $\mathscr{C}(A,-)$ for some A. By **representation** of F, we mean a pair (A,x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism. We also call x a **universal element** of F.

2.5 Corollary. If (A,x) and (B,y) are both representations of F, then there's a unique isomorphism $f : A \longrightarrow B$ such that $(Ff)(x) = y$.

Proof. Consider the composite

$$\mathscr{C}(B,-) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathscr{C}(A,-)$$

By Corollary 2.3, this is of the form $Y(f)$ for a unique isomorphism $f : A \longrightarrow B$ and the diagram

$$\begin{tikzcd}
\mathscr{C}(B,-) \arrow[rr, "Y(f)"] \arrow[dr, "\Psi(y)"'] & & \mathscr{C}(A,-) \arrow[dl, "\Psi(x)"] \\
& F &
\end{tikzcd}$$

commutes iff $(Ff)x = y$. $\square$

2.6 Examples.

(a) The forgetful functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$. Similarly, the forgetful functor $\mathbf{Rng} \longrightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}[x], x)$ and the forgetful functor $\mathbf{Top} \longrightarrow \mathbf{Set}$ is representable by $(\{*\}, *)$.

(b) The functor $\mathcal{P}^* : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (see Examples 1.5(c)) is representable by $(\{0,1\},\{1\})$: this is the bijection between subsets and characteristic functions.

(c) Let G be a group. The unique (up to isomorphism) representable functor $G(*,-) : G \longrightarrow \mathbf{Set}$ is the *Cayley representation* of G, i.e. the set UG with G acting by left multiplication.

(d) Let A, B be two objects of a locally small category $\mathscr{C}$. We have a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ sending C to $\mathscr{C}(C,A) \times \mathscr{C}(C,B)$. A representation of this, if it exists, is called a (categorical) **product** of A and B, and denoted

$$(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B)).$$

This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$ there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.

Products exist in many categories of interest: in $\mathbf{Set}, \mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}$ they are 'just' cartesian products, in posets they are binary meets.

Dually we have the notion of **coproduct** $(A+B, (A \xrightarrow{\nu_1} A+B, B \xrightarrow{\nu_2} A+B))$. These also exist in many categories of interest.

Lecture 6 (e) The dual-vector-space functor $\mathbf{Mod}_K^{\mathrm{op}} \longrightarrow \mathbf{Mod}_K$, when composed with the forgetful functor $\mathbf{Mod}_K \longrightarrow \mathbf{Set}$, is representable by $(K, 1_K)$.

(f) Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be morphisms in a locally small category $\mathscr{C}$. We have a functor $F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ defined by

$$F(C) = \{\, h \in \mathscr{C}(C,A) \mid fh = gh \,\}.$$

A representation of F, if it exists, is called an **equalizer** of (f,g). It consists of an objects E and a morphism $E \xrightarrow{e} A$ such $fe = ge$, and every h with $fh = gh$ factors uniquely through e. In $\mathbf{Set}$, we can take $E = \{\, x \in A \mid f(x) = g(x) \,\}$ and $e =$ inclusion. Similar constructions work in $\mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}, \ldots$.

Dually, we have the notion of **coequalizer**.

2.7 Remark. If e occurs as an equalizer, then it's a monomorphism, since any h factors through it in at most one way. We say a monomorphism is **regular** if it occurs as an equalizer.

Split monomorphisms are regular (c.f. question 6i on sheet 1). Note that regular mono + epi $\implies$ iso: if the equalizer e of (f,g) is epic, then $f = g$, so $e \cong 1_{\operatorname{cod} e}$.

2.8 Definition (Separating, detecting families)**.** Let $\mathscr{C}$ be a category, and $\mathscr{G}$ a class of objects of $\mathscr{C}$.

(a) We say $\mathscr{G}$ is a **separating family** for $\mathscr{C}$ if, given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathscr{G}$, then $f = g$ (i.e. the functors $\mathscr{C}(G,-)$, $G \in \mathscr{G}$ are collectively faithful).

(b) We say $\mathscr{G}$ is a **detecting family** for $\mathscr{C}$ if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathscr{G}$ factors uniquely through f, then f is an isomorphism.

If $\mathscr{G} = \{G\}$, we call G a **separator/detector**.

2.9 Lemma.

(i) If $\mathscr{C}$ is a balanced category, then any separating family is detecting.

(ii) If $\mathscr{C}$ has equalizers, then any detecting family is separating.

Proof.

(i) Suppose $\mathscr{G}$ is separating and $A \xrightarrow{f} B$ satisfies the condition of Definition 2.8(b). If $B \overset{g}{\underset{h}{\rightrightarrows}} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is epic.

Similarly if $D \overset{k}{\underset{l}{\rightrightarrows}} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorisations of fky through f. So $k = l$, i.e. f is monic.

(ii) Suppose $\mathscr{G}$ is detecting and $A \overset{f}{\underset{g}{\rightrightarrows}} B$ satisfies the condition of 2.8(a) . Then the equalizer $E \xrightarrow{e} A$ is an isomorphism, so $f = g$.

□

2.10 Examples.

(a) In $[\mathscr{C}, \mathbf{Set}]$ the family

$$\{\, \mathscr{C}(A, -) \mid A \in \operatorname{ob} \mathscr{C} \,\}$$

is both separating and detecting (this is just a restatement of Yoneda Lemma.)

(b) In $\mathbf{Set}$, $1 = \{*\}$ is both a separator and a detector since it represents the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$.

Similarly, $\mathbb{Z}$ is both in $\mathbf{Gp}$, since it represents the forgetful functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$.

And $2 = \{0, 1\}$ is a coseparator and a codetector in $\mathbf{Set}$, since it represents $\mathcal{P}^* : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

(c) In $\mathbf{Top}$, $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \longrightarrow \mathbf{Set}$, but not a detector. In fact, $\mathbf{Top}$ has no detecting *set* of objects:

For any infinite cardinal κ, let X be a discrete space of cardinality κ and let Y be the same set with 'co-$<\kappa$' topology, i.e. $F \subseteq Y$ closed $\iff$ $F = Y$ or $\operatorname{card} F < \kappa$. The identity $X \longrightarrow Y$ is continuous, but not a homeomorphism.

So if $\{\, G_i \mid i \in I \,\}$ is any set of spaces, taking $\kappa > \operatorname{card} G_i$ for all i yields an example to show that the set is not detecting.

(d) Let $\mathscr{C}$ be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous maps. JHC Whitehead proved that if $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \longrightarrow \pi_n(Y)$ for all n, then it's an isomorphism in $\mathscr{C}$. This says that $\{ S^n \mid n \geq 1 \}$ is a detecting set for $\mathscr{C}$.

But PJ Freyd showed there is no faithful functor $\mathscr{C} \longrightarrow \mathbf{Set}$, so no separating *set*: if $\{ G_i \mid i \in I \}$ were separating, then

$$x \longmapsto \coprod_{i \in I} \mathscr{C}(G_i, X)$$

would be faithful.

Note that any functor of the form $\mathscr{C}(A, -)$ preserves monomorphisms, but they don't normally preserve epimorphisms.

2.11 Definition (Projective)**.** We say an object P is **projective** if, given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

there exists $P \xrightarrow{g} A$ with $eg = f$. (If $\mathscr{C}$ is locally small, this says $\mathscr{C}(P, -)$ preserves epimorphisms).

Dually, an **injective** object of $\mathscr{C}$ is a projective object of $\mathscr{C}^{\mathrm{op}}$. Given a class $\mathscr{E}$ of epimorphisms, we say P is $\mathscr{E}$-projective if it satisfies the condition for all $e \in \mathscr{E}$.

2.12 Lemma. Representable functors are (pointwise) projective in $[\mathscr{C}, \mathbf{Set}]$.

Proof. Take

$$\begin{tikzcd} & \mathscr{C}(A, -) \arrow[d, "\beta"] \\ F \arrow[r, two heads, "\alpha"] & G \end{tikzcd}$$

where α is pointwise surjective. By Yoneda Lemma, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathscr{C}(A, -) \longrightarrow F$ corresponds to x then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Lecture 7 **3.1 Definition.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $\mathscr{C} \xrightarrow{F} \mathscr{D}$, $\mathscr{D} \xrightarrow{G} \mathscr{C}$ two functors. By an **adjunction** between F and G we mean a bijection between morphisms $FA \xrightarrow{\hat{f}} B$ in $\mathscr{D}$ and morphisms $A \xrightarrow{f} GB$ in $\mathscr{C}$ which is natural in A and B, i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, we have $h\hat{f}(Fg) = \widehat{(Gh)fg} : FA' \longrightarrow B'$.

We say F is **left adjoint** to G and write $F \dashv G$.

3.2 Examples.

(a) The free functor $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$ is left adjoint to the forgetful functor $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, since any function $f : A \longrightarrow UB$ extends uniquely to a homomorphism $\hat{f} : FA \longrightarrow B$. Naturality in B is easy, naturality in A follows from the definition of F as a functor.

(b) The forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology and a right adjoint I which equips a set A with the indiscrete topology $\{\varnothing, A\}$.

(c) The functor $\mathrm{ob} : \mathbf{Cat} \longrightarrow \mathbf{Set}$ has a left adjoint D sending A to the **discrete** category with $\mathrm{ob}(DA) = A$ and only identity morphisms. It also has a right adjoint I sending A to the (**indiscrete**) category with $\mathrm{ob}(IA) = A$ and one morphism $x \longrightarrow y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category $\mathscr{C}$ to its set of *connected components*, i.e. the quotient of $\mathrm{ob}\,\mathscr{C}$ by the smallest equivalent relation identifying $\mathrm{dom}\, f$ with $\mathrm{cod}\, f$ for all $f \in \mathrm{mor}\,\mathscr{C}$.

(d) Let $\mathscr{M}$ be the monoid $\{1, e\}$ with $e^2 = e$. An object of $[\mathscr{M}, \mathbf{Set}]$ is a pair (A, e) where $e : A \longrightarrow A$ satisfies $e^2 = e$.

We have a functor $G : [\mathscr{M}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ sending (A, e) to

$$\{\, x \in A \mid e(x) = x \,\} = \{\, e(x) \mid x \in A \,\}$$

and a functor $F : \mathbf{Set} \longrightarrow [\mathscr{M}, \mathbf{Set}]$ sending A to $(A, 1_A)$.

Claim $F \dashv G \dashv F$: given $f : (A, 1_A) \longrightarrow (B, e)$: it must take values in $G(B, e)$, and any $g : (B, e) \longrightarrow (A, 1_A)$ is determined by its values on the image of e.

(e) Let $\mathbf{1}$ be the discrete category with one object $*$. For any $\mathscr{C}$, there's a unique functor $\mathscr{C} \longrightarrow \mathbf{1}$: a left adjoint for this picks out an **initial object** of $\mathscr{C}$, i.e. an object I such that there exists a unique $I \longrightarrow A$ for each $A \in \mathrm{ob}\,\mathscr{C}$. Dually, a right adjoint for $\mathscr{C} \longrightarrow \mathbf{1}$ corresponds to a **terminal object** of $\mathscr{C}$.

(f) Let $A \xrightarrow{f} B$ be a morphism in $\mathbf{Set}$. We can regard $\mathcal{P}A$ and $\mathcal{P}B$ as posets, and we have functors

$$\begin{tikzcd} \mathcal{P}A \arrow[r, shift left, "\mathcal{P}f"] & \mathcal{P}B \arrow[l, shift left, "\mathcal{P}^* f"] \end{tikzcd}$$

Claim ($\mathcal{P}f \dashv \mathcal{P}^* f$): we have $\mathcal{P}f(A') \subseteq B' \iff f(x) \in B'$ for all $x \in A' \iff A' \subseteq \mathcal{P}^* f(B')$.

(g) Suppose given sets A, B and a relation $R \subseteq A \times B$. We define mappings $(-)^l, (-)^r$ between $\mathcal{P}A$ and $\mathcal{P}B$ by

$$\begin{aligned} S^r &= \{\, y \in B \mid (\forall x \in S)((x,y) \in R) \,\} \quad \text{for } S \subseteq A \\ T^l &= \{\, x \in A \mid (\forall y \in T)((x,y) \in R) \,\} \quad \text{for } T \subseteq B. \end{aligned}$$

These mappings are order-reversing (i.e. contravariant functors) and

$$T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l.$$

We say $(-)^r$ and $(-)^l$ are **adjoint on the right**.

(h) The functor $\mathcal{P}^* : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is self-adjoint on the right, since functions $A \longrightarrow \mathcal{P}B$ correspond bijectively to subsets of $A \times B$ and hence to functions $B \longrightarrow \mathcal{P}A$.

Definition (Comma category)**.** $(A \downarrow G)$ is the **comma category** with objects pairs (B, f) with $A \xrightarrow{f} GB$, and morphisms $(B, f) \longrightarrow (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ GB \arrow[rr, "Gg"'] & & GB' \end{tikzcd}$$

commutes.

3.3 Theorem. Let $G : \mathscr{D} \longrightarrow \mathscr{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \mathrm{ob}\,\mathscr{C}$.

Proof. Suppose we are given $F \dashv G$. Consider the morphism $\eta_A : A \longrightarrow GFA$ corresponding to $FA \xrightarrow{1} FA$. Then (FA, η_A) is an object of $(A \downarrow G)$. Moreover, given $g : FA \longrightarrow B$ and $f : A \longrightarrow GB$, the diagram

$$\begin{tikzcd} & A \arrow[dl, "\eta_A"'] \arrow[dr, "f"] & \\ GFA \arrow[rr, "Gg"'] & & GB \end{tikzcd}$$

commutes iff

$$\begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\hat{f}"] & \\ FA \arrow[rr, "g"'] & & B \end{tikzcd}$$

commutes, i.e. $g = \hat{f}$. So (FA, η_A) is initial in $(A \downarrow G)$.

Conversely, suppose we are given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $A \xrightarrow{f} A'$, we define $Ff : FA \longrightarrow FA'$ to be the unique morphism making

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{A'}"] & GFA' \end{tikzcd}$$

commute. Functoriality follows from uniqueness: given $f' : A' \longrightarrow A''$, both $F(f'f)$ and $(Ff')(Ff)$ are morphisms $(FA, \eta_A) \longrightarrow (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$.

Lecture 8

To show $F \dashv G$: given $A \xrightarrow{f} GB$, we define $\hat{f} : FA \longrightarrow B$ to be the unique morphism $(FA, \eta_A) \longrightarrow (B, f)$ in $(A \downarrow G)$. This is a bijection with inverse

$$(FA \xrightarrow{f} B) \longmapsto (A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB)$$

The latter mapping is natural in B since G is a functor, and in A since, by construction, η is a natural transformation $1_{\mathscr{C}} \longrightarrow GF$. □

3.4 Corollary. If F and F' are both left adjoint to $G : \mathscr{D} \longrightarrow \mathscr{C}$, then they are naturally isomorphic.

Proof. For any A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \longrightarrow (F'A, \eta'_A)$. In any naturality square for α, the two ways round are both morphisms in $(A \downarrow G)$ where the domain is initial, so they are equal. □

3.5 Lemma. Given

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathscr{E}$$

with $(F \dashv G)$ and $(H \dashv K)$ we have $(HF \dashv GK)$.

Proof. We have bijections between morphisms $A \longrightarrow GKC$, morphisms $FA \longrightarrow KC$ and morphisms $HFA \longrightarrow C$, which are both natural in A and C. □

3.6 Corollary. Given a commutative square

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{D} \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{F} \end{tikzcd}$$

of categories and functors, if the functors all have left adjoints, then the diagram of left adjoints commutes up to natural isomorphism.

Proof. By Lemma 3.5, both ways round the diagram of left adjoints are left adjoint to the composite $\mathscr{C} \longrightarrow \mathscr{F}$, so by Corollary 3.4 they are isomorphic. □

Given an adjunction $(F \dashv G)$, the natural transformation $\eta : 1_{\mathscr{C}} \longrightarrow GF$ emerging in the proof of Theorem 3.3 is called the **unit** of the adjunction. Dually, we have a natural transformation $\epsilon : FG \longrightarrow 1_{\mathscr{D}}$ such that $\epsilon_B : FGB \longrightarrow B$ corresponds to $GB \xrightarrow{1_{GB}} GB$ is called the **counit**.

3.7 Theorem. Given functors $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ specifying an adjunction $(F \dashv G)$ is equivalent to specifying natural transformations $\eta : 1_{\mathscr{C}} \longrightarrow GF$, $\epsilon : FG \longrightarrow 1_{\mathscr{D}}$ satisfying the commutative diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

called the **triangular identities**.

Proof. First suppose given $(F \dashv G)$. Define η and ϵ as in Theorem 3.3 and its dual; now consider the composite

$$FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA.$$

Under the adjunction this corresponds to

$$A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$$

but this also corresponds to 1_{FA}, so $\epsilon_{FA} \cdot F\eta_A = 1_{FA}$. The other identity is dual.

Conversely, suppose given η and ϵ satisfying the triangular identities. Given $A \xrightarrow{f} GB$, let $\Phi(f)$ be the composite

$$FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon B} B,$$

and given $FA \xrightarrow{g} B$, let $\Psi(g)$ be

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB.$$

Then Φ and Ψ are both natural; we need to show that $\Phi\Psi$ and $\Psi\Phi$ are identity mappings. But

$$\begin{aligned} \Psi\Phi\left(A \xrightarrow{f} GB \right) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f \end{aligned}$$

and dually $\Phi\Psi(g) = g$. $\square$

3.8 Lemma. Suppose given

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

and natural isomorphisms $\alpha : 1_{\mathscr{C}} \longrightarrow GF$, $\beta : FG \longrightarrow 1_{\mathscr{D}}$. Then there are isomorphisms $\alpha' : 1_{\mathscr{C}} \longrightarrow GF$, $\beta' : FG \longrightarrow 1_{\mathscr{D}}$ which satisfy the triangular identities, so $(F \dashv G)$ (and $(G \dashv F)$).

Proof. We define $\alpha' = \alpha$ and β' to be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha G)^{-1}} FG \xrightarrow{\beta} 1_{\mathscr{D}}.$$

Note that $FG\beta = \beta FG$ since

$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta FG"] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"] & 1_{\mathscr{D}} \end{tikzcd}$$

commutes by naturality of β and β is monic. Now $(\beta'_F)(F\alpha')$ is the composite

$$\begin{aligned}
&F \xrightarrow{\quad F\alpha \quad} FGF \xrightarrow{(\beta FGF)^{-1}} FGFGF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\quad \beta F \quad} F \\
=\ &F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{\quad FGF\alpha \quad} FGFGF \xrightarrow{(F\alpha GF)^{-1}} FGF \xrightarrow{\quad \beta F \quad} F \\
=\ &F \xrightarrow{(\beta F)^{-1}} FGF \xrightarrow{\quad \beta F \quad} F \ = 1_F
\end{aligned}$$

since $GF\alpha = \alpha_{GF}$. Similarly $(G\beta')(\alpha'_G)$ is

$$\begin{aligned}
&G \xrightarrow{\quad \alpha G \quad} GFG \xrightarrow{(GFG\beta)^{-1}} GFGFG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{\quad G\beta \quad} G \\
=\ &G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\quad \alpha GFG \quad} GFGFG \xrightarrow{(GF\alpha G)^{-1}} GFG \xrightarrow{\quad G\beta \quad} G \\
=\ &G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\quad G\beta \quad} G \ = 1_G.
\end{aligned}$$

□

3.9 Lemma. Suppose $G : \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint F with counit $\epsilon : FG \longrightarrow 1_{\mathscr{D}}$, then

(i) G is faithful iff ϵ is pointwise epic.

(ii) G is full and faithful iff ϵ is an isomorphism.

Proof.

(i) Given $B \xrightarrow{g} B'$, Gg corresponds under the adjunction to the composite

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'.$$

Hence the mapping $g \longmapsto Gg$ is injective on morphisms with domain B (and specified codomain) iff $g \longmapsto g\epsilon_B$ is injective, i.e. iff ϵ_B is epic.

(ii) Similarly, G is full and faithful iff $g \longmapsto g\epsilon_B$ is bijective. If $\alpha : B \longrightarrow FGB$ is such that $\alpha\epsilon_B = 1_{FGB}$, then $\epsilon_B\alpha\epsilon_B = \epsilon_B$, whence $\epsilon_B\alpha = 1_B$. So ϵ_B is an isomorphism for all B. □

Lecture 9 **3.10 Definition** (Reflection)**.** By a **reflection**, we mean an adjunction in which the right adjoint is full and faithful (equivalently, the counit is an isomorphism). We say a subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is **reflective** if the inclusion $\mathscr{C}' \longrightarrow \mathscr{C}$ has a left adjoint.

3.11 Examples.

(a) The category **AbGp** of abelian groups is reflective in **Gp**: the left adjoint sends a group G to its abelianization G/G', where G' is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$, $x, y \in G$. (The unit of the adjunction is the quotient map $G \longmapsto G/G'$.)

(b) Given an abelian group A, let A_t denote the torsion subgroup, i.e. the subgroup of elements of finite order. The assignment $A \longmapsto A/A_t$ gives a left adjoint to the inclusion **tfAbGp** $\longrightarrow$ **AbGp** where **tfAbGp** is the full subcategory of torsion-free abelian groups. And $A \longmapsto A_t$ is right adjoint to the inclusion **tfAbGp** $\longrightarrow$ **AbGp** so this subcategory is coreflective.

(c) Let **KHaus** $\subseteq$ **Top** be the full subcategory of compact Hausdorff spaces. The inclusion **KHaus** $\longrightarrow$ **Top** has a left adjoint β, the Stone-Čech compactification.

(d) Let X be a topological space. We say $A \subseteq X$ is sequentially closed if $x_n \longrightarrow x_\infty$ and $x_n \in A$ for all n implies $x_\infty \in A$. We say X is sequential if all sequentially closed sets are closed. Given a non-sequential space X, let X_s be the same set with topology given by the sequentially open sets in X; the identity $X_s \longrightarrow S$ is continuous, and defines the counit of an adjunction between the inclusion **Seq** $\longrightarrow$ **Top** and the functor $X \longmapsto X_s$.

(e) If X is a topological space, the poset CX of closed subsets of X is reflective in PX, with reflector given by closure, and the poset OX of open subsets is coreflective, with coreflector given by interior.

4 Limits

4.1 Definition.

(a) Let J be a category (almost always small, often finite). By a **diagram of shape J** in $\mathscr{C}$ we mean a functor $D : J \longrightarrow \mathscr{C}$. The objects $D(j)$, $j \in \operatorname{ob} J$ are called **vertices** of the diagram, and the morphisms $D(\alpha)$, $\alpha \in \operatorname{mor} J$ are called **edges** of D. For example, if J is the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with 4 objects and 5 non-identity morphisms, a diagram of shape J is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

If J is

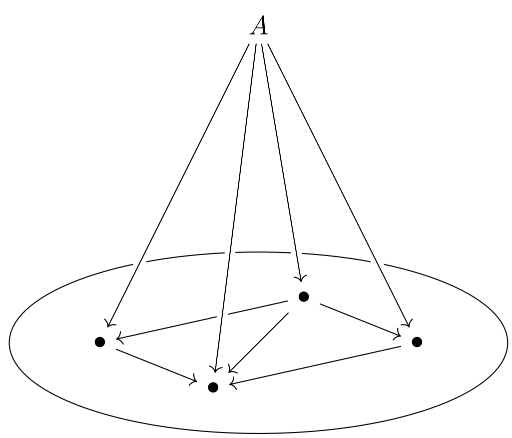
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

a diagram of shape J is a not-necessarily commutative square.

(b) Given $D : J \longrightarrow \mathscr{C}$, a **cone** over D consists of an object A of $\mathscr{C}$ (the **apex** of the cone) together with morphisms $A \xrightarrow{\lambda_j} D(j)$ for each $j \in \operatorname{ob} J$, such that

$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda'_j"] & \\ D(j) \arrow[rr, "D(\alpha)"'] & & D(j') \end{tikzcd}$$

commutes for all $j \xrightarrow{\alpha} j'$ in $\operatorname{mor} J$ (the λ_j are called the **legs** of the cone).

D 

Given cones $(A, (\lambda_j)_{j \in \mathrm{ob}\, J})$ and $(B, (\mu_j)_{j \in \mathrm{ob}\, J})$, a morphism of cones between them is a morphism $A \xrightarrow{f} B$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$$

commutes for all j.

We write $\mathbf{Cone}(D)$ for the category of cones over D.

(c) A **limit** for D is a terminal object of $\mathbf{Cone}(D)$, if this exists. Dually, we have the notion of **cone under a diagram**, and of **colimit** (initial cone under D).

Alternatively if $\mathscr{C}$ is locally small and J is small, we have a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ sending A to the set of cones with apex A. A limit for D is a representation of this functor.

If ΔA denotes the constant diagram of shape J with all vertices A and all edges 1_A, then a cone over D with apex A is the same thing as a natural transformation $\Delta A \longrightarrow D$. Δ is a functor $\mathscr{C} \longrightarrow [J, \mathscr{C}]$, and $\mathbf{Cone}(D)$ is the category $(\Delta \downarrow D)$ (a comma category, reversed). So to say that every diagram of shape J in $\mathscr{C}$ has a limit is equivalent to saying that Δ has a right adjoint. (We say $\mathscr{C}$ **has limits** of shape J). Dually, $\mathscr{C}$ **has colimits** of shape J iff $\Delta : \mathscr{C} \longrightarrow [J, \mathscr{C}]$ has a left adjoint.

4.2 Examples.

(a) Suppose $J = \varnothing$. There's a unique diagram of shape J in $\mathscr{C}$; a cone over it is just an object, and a morphism of cones is a morphism of $\mathscr{C}$. So a limit for this empty diagram is a terminal object of $\mathscr{C}$. (Dually, a colimit for it is an initial object).

Lecture 10

(b) Let J be the category

$$\bullet \qquad \bullet$$

A diagram of shape J is a pair of objects A, B; a cone over it is a **span**

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

and a limit for it is a product

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \end{tikzcd}$$

Dually, a colimit for it is a coproduct

$$\begin{tikzcd} A \arrow[dr, "\nu_1"'] & & B \arrow[dl, "\nu_2"] \\ & A + B & \end{tikzcd}$$

(c) More generally, if J is a small discrete category, a diagram of shape J is a J-indexed family $\{ A_j \mid j \in J \}$, and a limit for it is a product

$$\left\{ \prod_{j \in J} A_j \xrightarrow{\pi_j} A_j \;\middle|\; j \in J \right\}.$$

Dually, a colimit for it is a coproduct

$$\left\{ A_j \xrightarrow{\nu_j} \sum_{j \in J} A_j \;\middle|\; j \in J \right\},$$

also written $\coprod_{j \in J} A_j$.

(d) Let J be the category

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

A diagram of shape J is a parallel pair

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

a cone over this is

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[dr, "k"] & \\ A & & B \end{tikzcd}$$

satisfying $fh = k = gh$, or equivalently a morphism $C \xrightarrow{h} A$ satisfying $fh = gh$. A (co)limit for the diagram is a (co)equalizer.

(e) Let J be the category

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

A diagram of shape J is a cospan

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

a cone over it is

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, "r"] & A \\ B & C \end{tikzcd}$$

satisfying $fp = r = gq$, or equivalently a span (p, q) completing the diagram to a commutative square. A limit for the diagram is called a **pullback** of (f, g). In **Set**, the apex of the pullback is the 'fibre product'

$$A \times_C B = \{ (x, y) \in A \times B \mid f(x) = g(y) \}$$

Dually, colimits of shape J^{op} are called **pushouts**, given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

we 'push g along f' to get the right hand side of the colimit square.

(f) Let J be the poset of natural numbers. A diagram of shape J is a **direct system**

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

A colimit for this is called a **direct limit**: it consists of A_∞ equipped with morphisms $A_n \xrightarrow{g_n} A_\infty$ satisfying $g_n = g_{n+1} f_n$ for all n, and universal among such. Dually, we have **inverse system** and **inverse limit**.

4.3 Theorem.

(i) Suppose $\mathscr{C}$ has equalizers and all finite (resp. small) products. Then $\mathscr{C}$ has all finite (resp. small) limits.

(ii) Suppose $\mathscr{C}$ has pullbacks and a terminal object. Then $\mathscr{C}$ has all finite limits.

Proof.

(i) Suppose given $D : J \longrightarrow \mathscr{C}$. Form the products

$$P = \prod_{j \in \mathrm{ob}\, J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\operatorname{cod} \alpha).$$

We have morphisms $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ defined by $\pi_\alpha f = \pi_{\operatorname{cod}\alpha}$, $\pi_\alpha g = D(\alpha)\pi_{\operatorname{dom}\alpha}$ for all α.

Let $E \xrightarrow{e} P$ be an equalizer of (f, g). The composites $\lambda_j = \pi_j e : E \longrightarrow D(j)$ form a cone over D: given $\alpha : j \longrightarrow j'$ in J,

$$D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e = \lambda_{j'}.$$

Given any cone $(A, \{\, \mu_j \mid j \in \mathrm{ob}\, J \,\})$ over D, there's a unique $\mu : A \longrightarrow P$ with $\pi_j \mu = \mu_j$ for each j, and

$$\pi_\alpha f \mu = \mu_{\operatorname{cod}\alpha} = D(\alpha)\mu_{\operatorname{dom}\alpha} = \pi_\alpha g \mu$$

for all α, and hence $f\mu = g\mu$, so $\exists! \nu : A \longrightarrow E$ with $e\nu = \mu$. So $(E, \{\, \lambda_j \mid j \in \mathrm{ob}\, J \,\})$ is a limit cone.

(ii) It's enough to construct finite products and equalizers. But if 1 is the terminal object, then a pullback for

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

has the universal property of a product $A \times B$, and we can form $\prod_{i=1}^{n} A_i$ inductively as $A_1 \times (A_2 \times (A_3 \times \cdots (A_{n-1} \times A_n) \cdots))$.

Now, to form the equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$, consider the cospan

$$\begin{tikzcd} & A \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B. \end{tikzcd}$$

A cone over this consists of

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"] & A \\ A & \end{tikzcd}$$

satisfying $(1_A, f)h = (1_A, g)k$ or equivalently, $1_A h = 1_A k$ and $fh = gk$, or equivalently a morphism $P \xrightarrow{h} A$ satisfying $fh = gh$. So a pullback for $(1_A, f)$ and $(1_A, g)$ is an equalizer of (f, g). □

Definition (Complete)**.** We say a category $\mathscr{C}$ is **complete** if it has all small limits. (Dually, **cocomplete** = all small colimits).

Set is complete and cocomplete: products are cartesian products, coproducts are disjoint unions. Similarly, **Gp**, **AbGp**, **Rng**, $\mathbf{Mod}_R, \ldots$ are all complete and cocomplete. **Top** is also complete and cocomplete.

4.4 Definition. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

(a) We say F **preserves limits** of shape J if, given $D : J \longrightarrow \mathscr{C}$ and a limit cone $(L, \{\, \lambda_j \mid j \in \operatorname{ob} J \,\})$ in $\mathscr{C}$, $(FL, \{\, F\lambda_j \mid j \in \operatorname{ob} J \,\})$ is a limit for FD.

(b) We say F **reflects limits** of shape J if, given $D : J \longrightarrow \mathscr{C}$ and a cone $(L, (\lambda_j)_j)$ such that $(FL, (F\lambda_j)_j)$ is a limit for FD, then $(L, (\lambda_j)_j)$ is a limit for D.

(c) We say F **creates limits** of shape J if, given $D : J \longrightarrow \mathscr{C}$ and a limit $(M, (\mu_j)_j)$ for FD, there exists a cone $(L, (\lambda_j)_j)$ over D whose image under F is isomorphic to the limit cone, and any such that cone is a limit in $\mathscr{C}$.

Lecture 11 **4.5 Remark.**

(a) If $\mathscr{C}$ has limits of shape J, $F : \mathscr{C} \longrightarrow \mathscr{D}$ preserves them and F reflects isomorphisms, then F reflects limits of shape J.

(b) F reflects limits of shape $\mathbf{1} \iff F$ reflects isomorphisms.

(c) If $\mathscr{D}$ has limits of shape J and $F : \mathscr{C} \longrightarrow \mathscr{D}$ creates them, then F both preserves and reflects them.

(d) In any of the statements of Theorem 4.3, we may replace both instances of '$\mathscr{C}$ has' by either '$\mathscr{C}$ has and $F : \mathscr{C} \longrightarrow \mathscr{D}$ preserves' or '$\mathscr{D}$ has and $F : \mathscr{C} \longrightarrow \mathscr{D}$ creates'.

4.6 Examples.

(a) $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ (forgetful) creates all small limits: Given a family $\{ G_i \mid i \in I \}$ of groups, there's a unique group structure on $\prod_{i \in I} UG_i$ making the projections π_i into homomorphisms and this makes it into a product in $\mathbf{Gp}$. Similarly for equalizers.

But U doesn't preserve coproducts: $U(G * H) \ncong UG \sqcup UH$.

(b) $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ (forgetful) preserves all small limits and colimits, but doesn't reflect them: if L is a limit for $D : J \longrightarrow \mathbf{Top}$, and L is not discrete, there's another cone with apex L_d mapping to the limit in $\mathbf{Set}$.

(c) The inclusion functor $I : \mathbf{AbGp} \longrightarrow \mathbf{Gp}$ reflects coproducts but doesn't preserve them. The direct sum $A \oplus B$ (coproduct in $\mathbf{AbGp}$) is not normally isomorphic to the free product $A * B$; $A * B$ is not abelian unless either A or B is $\{e\}$, but if $A \cong \{e\}$ then $A * B \cong A \oplus B \cong B$.

4.7 Lemma. If $\mathscr{D}$ has limits of shape J, then so does the functor category $[\mathscr{C}, \mathscr{D}]$ for any $\mathscr{C}$, and the forgetful functor $[\mathscr{C}, \mathscr{D}] \longrightarrow \mathscr{D}^{\operatorname{ob}\mathscr{C}}$ creates them.

Proof. Suppose we are given a diagram of shape J in $[\mathscr{C}, \mathscr{D}]$; think of it as a functor $D : J \times \mathscr{C} \longrightarrow \mathscr{D}$. For each $A \in \operatorname{ob}\mathscr{C}$, let $(LA, \{ \lambda_{j,A} \mid j \in \operatorname{ob} J \})$ be a limit cone for the diagram $D(-, A) : J \longrightarrow \mathscr{D}$. Given $A \xrightarrow{f} B$ in $\mathscr{C}$, the composites

$$A \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, since the squares

$$\begin{tikzcd} D(j,A) \arrow[r, "D(j,f)"] \arrow[d, "D(\alpha,A)"] & D(j,B) \arrow[d, "D(\alpha,B)"] \\ D(j',A) \arrow[r, "D(j',f)"] & D(j',B) \end{tikzcd}$$

commute. So there's a unique $Lf : LA \longrightarrow LB$ making

$$\begin{tikzcd} LA \arrow[r, "\lambda_{j,A}"] \arrow[d, "Lf"] & D(j,A) \arrow[d, "D(j,f)"] \\ LB \arrow[r, "\lambda_{j,B}"] & D(j,B) \end{tikzcd}$$

commute for all j.

Uniqueness implies functoriality: given $g : B \longrightarrow C$, $L(gf)$ and $(Lg)(Lf)$ are factorizations of the same cone through the limit LC. And this is the unique functor structure on $(A \longmapsto LA)$ making the $\lambda_{j,-}$ into natural transformations.

The cone $(L, \{\lambda_{j,-} \mid j \in \operatorname{ob} J\})$ is a limit: suppose given another cone $(M, \{\mu_{j,-} \mid j \in \operatorname{ob} J\})$, then for each A, $(MA, \{ \mu_{j,A} \mid j \in \operatorname{ob} J \})$ is a cone over $D(-, A)$, so induces a unique $\alpha_A : MA \longrightarrow LA$. Naturality of α follows from uniqueness of factorisations through a limit. So $(M, (\mu_j))$ factors uniquely through $(L, (\lambda_j))$. $\square$

4.8 Remark. In any category, a morphism $A \xrightarrow{f} B$ is monic iff

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback. Hence any functor which preserves pullbacks preserves monomorphisms. In particular, if $\mathscr{D}$ has pullbacks, then monomorphisms in $[\mathscr{C}, \mathscr{D}]$ are just pointwise monomorphisms.

4.9 Theorem. Suppose $G : \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint F. Then G preserves all limits which exist in $\mathscr{D}$.

Proof 1. Suppose $\mathscr{C}$ and $\mathscr{D}$ both have limits of shape J. We have a commutative diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\Delta"] & \mathscr{D} \arrow[d, "\Delta"] \\ {[J, \mathscr{C}]} \arrow[r, "{[J,F]}"] & {[J, \mathscr{D}]} \end{tikzcd}$$

and all the functors in it have right adjoints (in particular $[J, F] \dashv [J, G]$). So by Corollary 3.6, the diagram of right adjoints

$$\begin{tikzcd} \mathscr{D} \arrow[r, "G"] & \mathscr{C} \\ {[J, \mathscr{D}]} \arrow[r, "{[J,G]}"] \arrow[u, "\lim_J"] & {[J, \mathscr{C}]} \arrow[u, "\lim_J"] \end{tikzcd}$$

commutes up to isomorphism, i.e. G preserves limits of shape J. $\square$

Proof 2. Suppose given $D : J \longrightarrow \mathscr{D}$ and a limit cone $(L, \{ L \xrightarrow{\lambda_j} D(j) \mid j \in \operatorname{ob} J \})$. Given a cone $(A, \{ A \xrightarrow{\alpha_j} GD(j) \mid j \in \operatorname{ob} J \})$ over GD the morphisms $FA \xrightarrow{\hat{\alpha}_j} D(j)$ form a cone over D, so they induce a unique $FA \xrightarrow{\hat{\beta}} L$ such that $\lambda_j \hat{\beta} = \hat{\alpha}_j$ for all j. Then $A \xrightarrow{\beta} GL$ is the unique morphism satisfying $(G\lambda_j)\beta = \alpha_j$ for all j. So $(GL, \{ G\lambda_j \mid j \in \operatorname{ob} J \})$ is a limit cone in $\mathscr{C}$. $\square$

Lecture 12 The 'Primeval' Adjoint Functor Theorem says that the converse of Theorem 4.9 is true: if $\mathscr{D}$ has and $G : \mathscr{D} \longrightarrow \mathscr{C}$ preserves *all* limits, then G has a left adjoint.

4.10 Lemma. Suppose $\mathscr{D}$ has and $G : \mathscr{D} \longrightarrow \mathscr{C}$ preserves limits of shape J. Then for any $A \in \operatorname{ob} \mathscr{C}$, the arrow category $(A \downarrow G)$ has limits of shape J, and the forgetful functor $U : (A \downarrow G) \longrightarrow \mathscr{D}$ creates them.

Proof. Suppose given $D : J \longrightarrow (A \downarrow G)$, write $D(j)$ as $(UD(j), f_j)$. Let $(L, (\lambda_j : L \longrightarrow UD(j))_{j \in \operatorname{ob} J})$ be a limit for UD; then $(GL, (G\lambda_j)_{j \in \operatorname{ob} J})$ is a limit for GUD.

Since the edges of UD are morphisms in $(A \downarrow G)$, the f_j form a cone over GUD. So there's a unique $h : A \longrightarrow GL$ such that $(G\lambda_j)h = f_j$ for all j, i.e. there's a unique h such that the λ_j are all morphisms $(L, h) \longrightarrow (UD(j), f_j)$ in $(A \downarrow G)$.

If $((C,k)(\mu_j)_{j\in \mathrm{ob}\, J})$ is any cone over D, then $(C,(\mu_j)_{j\in \mathrm{ob}\, J})$ is a cone over UD, so there's a unique $l : C \longrightarrow L$ with $\lambda_j l = \mu_j$ for all j. We need to show $(Gl)k = h$: but $(G\lambda_j)(Gl)k = (G\mu_j)k = f_j = (G\lambda_j)h$ for all j so $(Gl)k = h$ by uniqueness of factorizations through limits. $\square$

4.11 Lemma. A category $\mathscr{C}$ has an initial object iff $1_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}$, regarded as a diagram of shape $\mathscr{C}$ in $\mathscr{C}$, has a limit.

Proof. Suppose $\mathscr{C}$ has an initial object I. Then the unique morphisms $\{\, I \longrightarrow A \mid A \in \mathrm{ob}\,\mathscr{C} \,\}$ form a cone over $1_{\mathscr{C}}$; and given any cone $\left\{\, C \xrightarrow{\lambda_A} A \,\middle|\, A \in \mathrm{ob}\,\mathscr{C} \,\right\}$, then for any A the triangle

$$\begin{tikzcd} C \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d] \\ & A \end{tikzcd}$$

commutes, so λ_I is the unique factorization of $\{\, \lambda_A \mid A \in \mathrm{ob}\,\mathscr{C} \,\}$ through the cone

$$\{\, I \longrightarrow A \mid A \in \mathrm{ob}\,\mathscr{C} \,\}\,.$$

Conversely, suppose $(I, \{\, \lambda_A : I \longrightarrow A \mid A \in \mathrm{ob}\,\mathscr{C} \,\})$ is a limit. Then, for any $I \xrightarrow{f} A$, the diagram

$$\begin{tikzcd} I \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d, "f"] \\ & A \end{tikzcd}$$

commutes. In particular, putting $f = \lambda_A$, we see that λ_I is a factorization of the limit cone through itself, so $\lambda_I = 1_I$. Hence every $f : I \longrightarrow A$ satisfies $f = \lambda_A$. $\square$

The primeval Adjoint Functor Theorem now follows immediately from Lemma 4.10, Lemma 4.11 and Theorem 3.3. However, it only applies to functors between preorders (cf question 6 on example sheet 2).

4.12 Theorem (General Adjoint Functor Theorem)**.** Suppose that $\mathscr{D}$ is locally small and complete. Then $G : \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint if and only if G satisfies the **solution set condition**: the solution set condition says that G preserves all small limits and, for each $A \in \mathrm{ob}\,\mathscr{C}$ there exists a set of morphisms $\{\, A \xrightarrow{f_i} GB_i \mid i \in I \,\}$ such that every $A \xrightarrow{h} GC$ factors as $A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$ for some i and some $g : B_i \longrightarrow C$.

Proof. $(\Rightarrow)$. If $F \dashv G$, G preserves limits by Theorem 4.9, and $\{A \xrightarrow{\eta_A} GFA\}$ is a singleton solution set, by Theorem 3.3.

$(\Leftarrow)$. By Lemma 4.10, $(A \downarrow G)$ is complete, and it inherits local smallness from $\mathscr{D}$. So we need to show: if $\mathscr{A} := (A \downarrow G)$ is complete and locally small, and has a weakly initial set of objects $\{\, B_i \mid i \in I \,\}$, then $\mathscr{A}$ has an initial object. (An object is **weakly initial** if it has a (not necessarily unique) morphism to any other object.)

First form $P = \prod_{i\in I} B_i$, then P is weakly initial. Now form the limit of

$$P \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow\end{array} P \qquad (*)$$

where edges are all the endomorphisms of P: denote the limit $I \xrightarrow{i} P$. I is also weakly initial in $\mathscr{A}$: suppose given $I \underset{g}{\overset{f}{\rightrightarrows}} C$. Form the equalizer $E \xrightarrow{e} I$ of (f, g), then there exists $P \xrightarrow{h} E$ since P is weakly initial. $ieh : P \longrightarrow P$, and 1_P are edges of the diagram $(*)$ so $i = iehi$. But i is monic, so $ehi = 1_I$, so e is split epic, so $f = g$. Hence I is initial. $\square$

4.13 Examples.

(a) Consider the forgetful functor $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$. By Examples 4.6(a), U creates all small limits, so $\mathbf{Gp}$ has them and U preserves them. $\mathbf{Gp}$ is locally small, given a set A, any $f : A \longrightarrow UG$ factors as $A \longrightarrow UG' \longrightarrow UG$, where G' is the subgroup generated by $\{ f(x) \mid x \in A \}$ and $\operatorname{card} G' \leq \max\{\aleph_0, \operatorname{card} A\}$.

Let B be a set of this cardinality; consider all subsets $B' \subseteq B$, all group structures on B', and all mappings $A \longrightarrow B'$. These give us a solution set at A.

(b) Consider the category $\mathbf{CLat}$ of complete lattices (posets with all meets and joins). Again $U : \mathbf{CLat} \longrightarrow \mathbf{Set}$ creates all small limits. But A. W. Hales (1964) showed that, for any cardinal κ, there exist complete lattices of card $\geq \kappa$ generated by three elements, so the solution set condition fails at $A = \{x, y, z\}$, and U doesn't have a left adjoint.

Lecture 13 **4.14 Definition** (Subobject)**.** By a **subobject** of an object A of $\mathscr{C}$, we mean a monomorphism $A' \rightarrowtail A$. Dually, we have quotient objects. The subobjects of A are preordered by $A'' \leq A'$ if there exists a factorization

$$\begin{tikzcd} A'' \arrow[rr] \arrow[dr, tail] & & A' \arrow[dl, tail] \\ & A. & \end{tikzcd}$$

We say $\mathscr{C}$ is **well-powered** if each $A \in \operatorname{ob}\mathscr{C}$ has a set of subobjects $\{ A_i \rightarrowtail A \mid i \in I \}$ such that every subobject of A is isomorphic to some A_i (e.g. in $\mathbf{Set}$ we can take the inclusions $\{ A' \hookrightarrow A \mid A' \in \mathcal{P}A \}$).

If C^{op} is well-powered, we say $\mathscr{C}$ is **well-copowered** (not cowell-powered).

4.15 Lemma. Suppose given a pullback square

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"] & A \arrow[d, tail, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

with f monic. Then k is monic.

Proof. Suppose $D \overset{x}{\underset{y}{\rightrightarrows}} P$ satisfy $kx = ky$. Then $fhx = gkx = gky = fhy$, but f is monic so $hx = hy$. So x and y are factorizations of the same cone through the limit cone (h, k). $\square$

4.16 Theorem (Special adjoint functor theorem)**.** Suppose $\mathscr{C}$ and $\mathscr{D}$ are both locally small, and that $\mathscr{D}$ is complete and well-powered and has a coseparating set. Then a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ has a left adjoint iff it preserves all small limits.

Proof. ($\Rightarrow$) by Theorem 4.9.

($\Leftarrow$). For any $A \in \operatorname{ob}\mathscr{C}$, $(A \downarrow G)$ is complete by Lemma 4.10, locally small, and well-powered since the subobjects of (B, f) in $(A \downarrow G)$ are just those subobjects $B' \rightarrowtail B$ in $\mathscr{D}$ for which f factors through $GB' \rightarrowtail GB$.

Also, if $\{ S_i \mid i \in I \}$ is a coseparating set for $\mathscr{D}$, then the set

$$\{ (s_i, f) \mid i \in I, f \in \mathscr{C}(A, GS_i) \}$$

is coseparating in $(A \downarrow G)$: given $(B, f) \overset{g}{\underset{h}{\rightrightarrows}} (B', f')$ in $(A \downarrow G)$ with $g \neq h$, there exists $k : B' \longrightarrow S_i$ for some i with $kg \neq kh$, and then k is also a morphism $(B', f') \longrightarrow (S_i, (Gk)f')$ in $(A \downarrow G)$.

So we need to show that if $\mathscr{A}$ is complete, locally small and well powered, and has a coseparating set $\{ S_i \mid i \in I \}$ then $\mathscr{A}$ has an initial object. Form the product $P = \prod_{i \in I} S_i$. Now consider the diagram

$$\begin{tikzcd}
 & & P_i \arrow[dd, tail] \\
 & P_j \arrow[dr, tail] & \\
P' \arrow[ur, dotted, dash] \arrow[rr, tail] & & P
\end{tikzcd}$$

whose edges are a representative set of subobjects of P, and form its limit

$$\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] & & P_i \arrow[dd, tail] \\
 & P_j \arrow[dr, tail] & \\
P' \arrow[ur, dotted, dash] \arrow[rr, tail] & & P
\end{tikzcd}$$

By the argument of Lemma 4.15, the legs of this cone are all monic; in particular $I \rightarrowtail P$ is monic, and it's a least subobject of P. Hence I has no proper subobjects, so, given $I \overset{f}{\underset{g}{\rightrightarrows}} A$, their equalizer is an isomorphism and hence $f = g$.

Now let A be any object of $\mathscr{A}$; form the product

$$Q = \prod_{\substack{i \in I \\ f \in \mathscr{A}(A, S_i)}} S_i.$$

There's an obvious $h : A \longrightarrow Q$ defined by $\pi_{i,f} h = f$; and h is monic, since the S_i are a coseparating set. We also have a morphism $k : P \longrightarrow Q$ defined by $\pi_{i,f} k = \pi_i$.

Now form the pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d, tail, "h"] \\ P \arrow[r, "k"] & Q \end{tikzcd} ;$$

by Lemma 4.15, $B \longrightarrow P$ is monic, so B is a subobject of P. Hence there exists

$$\begin{tikzcd} I \arrow[rr] \arrow[dr, tail] & & B \arrow[dl, tail] \\ & P & \end{tikzcd}$$

and hence a morphism $I \longrightarrow B \longrightarrow A$. □

4.17 Examples. Consider the inclusion $\mathbf{KHaus} \xrightarrow{I} \mathbf{Top}$, where **KHaus** is the full subcategory of compact Hausdorff spaces. **KHaus** has and I preserves small products (by Tychonoff's Theorem) and equalizers (since equalizers of pairs $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with Y Hausdorff are closed subspaces). Both categories are locally small and **KHaus** is well-powered (subobjects of X are all isomorphic to closed subspaces). The closed interval $[0, 1]$ is a coseparator in **KHaus** by Urysohn's Lemma. So by Theorem 4.16, I has a left adjoint β.

4.18 Remark.

(a) Čech's construction of β: given X form $P = \prod_{f : X \longrightarrow [0,1]} [0, 1]$ and define $h : X \longrightarrow P$ by $\pi_f h = f$. Define βX to be the closure of the image of h.

Čech's proof that this works is essentially the same as Theorem 4.16.

(b) We could have used General Adjoint Functor Theorem to construct β: we get a solution set at X by considering all continuous $X \xrightarrow{f} Y$ with Y compact Hausdorff, and $\operatorname{im} f$ dense in Y and such Y have cardinality $\leq 2^{2^{\operatorname{card} X}}$.

5 Monads

Lecture 14 Suppose given

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

with $(F \dashv G)$. How much of this structure can we describe without mentioning $\mathscr{D}$?

We have the functor $T = GF : \mathscr{C} \longrightarrow \mathscr{C}$, and the unit $\eta : 1_{\mathscr{C}} \longrightarrow T = GF$ and the natural transformation

$$\mu = G\epsilon_F : TT = GFGF \longrightarrow GF = T.$$

These satisfy the commutative diagrams

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & T \arrow[l, "\eta_T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

by triangular identities and

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"] \arrow[dr, phantom, "(3)"] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd}$$

by naturality of ϵ.

5.1 Definition (Monad)**.** A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ consists of a functor $T : \mathscr{C} \longrightarrow \mathscr{C}$ and natural transformation $\eta : 1_{\mathscr{C}} \longrightarrow T$, $\mu : TT \longrightarrow T$ satisfying equations (1)-(3). η and μ are called the **unit** and **multiplication** of $\mathbb{T}$.

5.2 Examples.

(a) Any adjunction $(F \dashv G)$ induces both a monad $(GF, \eta, G\epsilon_F)$ on $\mathscr{C}$ and a comonad $(FG, \epsilon, F\eta_G)$ on $\mathscr{D}$.

(b) Let M be a monoid. The functor $(M \times -) : \mathbf{Set} \longrightarrow \mathbf{Set}$ has a monad structure with unit given by $\eta_A(a) = (1_M, a)$ and multiplication $\mu_A(m, m', a) = (mm', a)$. The monad identities follow from the monoid ones.

(c) Let $\mathscr{C}$ be any category with finite products, $A \in \mathrm{ob}\,\mathscr{C}$. The functor $(A \times -) : \mathscr{C} \longrightarrow \mathscr{C}$ has a comonad structure with counit $\epsilon_B : A \times B \longrightarrow B$ given by π_2 and comultiplication $\delta_B : A \times B \longrightarrow A \times A \times B$ given by (π_1, π_1, π_2).

Does every monad arise from an adjunction?

In Examples 5.2(b) we have the category $[M, \mathbf{Set}]$. Its forgetful functor to **Set** has a left adjoint, sending A to $M \times A$ with M acting by multiplication on the left factor. This adjunction gives rise to the monad of Examples 5.2(b).

5.3 Definition (Eilenberg-Moore category)**.** Let $\mathbb{T}$ be a monad on $\mathscr{C}$. A **$\mathbb{T}$-algebra** is a pair (A, α) with $A \in \operatorname{ob}\mathscr{C}$ and $TA \xrightarrow{\alpha} A$ satisfying the commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \arrow[dl, phantom, "(4)"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "(5)"] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A. \end{tikzcd}$$

A **homomorphism** $f : (A, \alpha) \longrightarrow (B, \beta)$ is a morphism $A \xrightarrow{f} B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] \arrow[dr, phantom, "(6)"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes. The category of $\mathbb{T}$-algebras is denoted $\mathscr{C}^{\mathbb{T}}$, and called the **Eilenberg-Moore category**.

5.4 Lemma. The forgetful functor $G^{\mathbb{T}} : \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ has a left adjoint $F^{\mathbb{T}}$ and the adjunction induces $\mathbb{T}$.

Proof. We define $F^{\mathbb{T}}A = (TA, \mu_A)$ (an algebra by (2) and (3)) and $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$ (a homomorphism by naturality of μ).

Clearly $G^{\mathbb{T}}F^{\mathbb{T}} = T$, the unit of the adjunction is η. We define the counit

$$\epsilon_{(A,a)} = \alpha : (TA, \mu_A) \longrightarrow (A, \alpha)$$

(a homomorphism by (5)) ϵ is natural by (6); for the triangular identities, $\epsilon_{FA}(F\eta_A) = 1_{FA}$ is (1), $G\epsilon_{(A,\epsilon)}\eta_A = 1_A$ is (4).

The monad induced by $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ has functor T and unit η, and $G^{\mathbb{T}}\epsilon_{F^{\mathbb{T}}A} = \mu_A$ by definition of $F^{\mathbb{T}}A$. $\square$

Kleisli took a 'minimalist' approach: if

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

induces $\mathbb{T}$, then so does

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D}' \arrow[l, shift left, "G|_{\mathscr{D}'}"] \end{tikzcd}$$

where $\mathscr{D}'$ is the full subcategory of $\mathscr{D}$ on objects FA.

So in trying to construct $\mathscr{D}$, we may assume F is surjective (or indeed bijective) on objects. But then morphisms $FA \longrightarrow FB$ correspond bijectively to morphisms $A \longrightarrow GFB = TB$ in $\mathscr{C}$.

5.5 Definition (Kleisli category)**.** Given a monad $\mathbb{T}$ on $\mathscr{C}$, the **Kleisli category** $\mathscr{C}_{\mathbb{T}}$ has $\operatorname{ob}\mathscr{C}_{\mathbb{T}} = \operatorname{ob}\mathscr{C}$, and morphisms $A \longrightarrow B$ are morphisms $A \longrightarrow TB$ in $\mathscr{C}$. The composite $A \xrightarrow{f} B \xrightarrow{g} C$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$$

and the identity $A \longrightarrow A$ is $A \xrightarrow{\eta_A} TA$.

To verify associativity, suppose given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$. Then

$$\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "Tg"] & TTC \arrow[r, "TTh"] \arrow[d, "\mu_C"] & TTTD \arrow[r, "T\mu_D"] \arrow[d, "\mu_{TD}"] \arrow[dr, phantom, "(3)"] & TTD \arrow[d, "\mu_D"] \\
& & TC \arrow[r, "Th"] & TTD \arrow[r, "\mu_D"] & TD
\end{tikzcd}$$

commutes by naturality: the upper way round is $(hg)f$ and the lower is $h(gf)$.

The unit laws for the category similarly follow from

$$\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "T\eta_B"] \arrow[dr, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \arrow[dl, phantom, "(1)"] \\
& & TB
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"] & TB \arrow[d, "\eta_{TB}"'] \arrow[dr, "1_{TB}"] & \\
TA \arrow[r, "Tf"'] & TTB \arrow[r, "\mu_B"'] \arrow[ur, phantom, "(2)"] & TB.
\end{tikzcd}$$

Lecture 15 **5.6 Lemma.** There exists an adjunction

$$\begin{tikzcd}
\mathscr{C} \arrow[r, shift left, "F_{\mathbb{T}}"] & \mathscr{C}_{\mathbb{T}} \arrow[l, shift left, "G_{\mathbb{T}}"]
\end{tikzcd}$$

inducing the monad $\mathbb{T}$.

Proof. We define $F_{\mathbb{T}}A = A$,

$$F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB.$$

$F_{\mathbb{T}}$ preserves identities by definition; for composites, consider $A \xrightarrow{f} B \xrightarrow{g} C$. We get

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "\eta_B"] \arrow[d, "g"] & TB \arrow[d, "Tg"] & \\
& C \arrow[r, "\eta_C"] & TC \arrow[r, "T\eta_c"] \arrow[dr, "1_{TC}"'] & TTC \arrow[d, "\mu_C"] \arrow[dl, phantom, "(1)"] \\
& & & TC.
\end{tikzcd}$$

We define $G_{\mathbb{T}}A = TA$,

$$G_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB.$$

$G_{\mathbb{T}}$ preserves identities by (1); for composites, consider $A \xrightarrow{f} B \xrightarrow{g} C$. We get

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "TTg"] \arrow[d, "\mu_B"] & TTTC \arrow[r, "T\mu_C"] \arrow[d, "\mu_{TC}"] \arrow[dr, phantom, "(3)"] & TTC \arrow[d, "\mu_C"] \\
& TB \arrow[r, "Tg"] & TTC \arrow[r, "\mu_C"] & TC
\end{tikzcd}$$

We have

$$G_{\mathbb{T}}F_{\mathbb{T}}A = TA$$
$$G_{\mathbb{T}}F_{\mathbb{T}}f = \mu_B(T\eta_B)Tf = Tf$$

so we take $\eta : 1_{\mathbb{C}} \longrightarrow T$ as the unit of $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$. The counit $TA \xrightarrow{\epsilon_A} A$ is 1_{TA}. To verify naturality, consider the square

$$\begin{tikzcd}
TA \arrow[r, "F_{\mathbb{T}}G_{\mathbb{T}}f"] \arrow[d, "\epsilon_A"] & TB \arrow[d, "\epsilon_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

This expands to

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "\mu_B"] & TB \arrow[r, "\eta_{TB}"] \arrow[dr, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \arrow[dl, phantom, "(2)"] \\
& & & TB
\end{tikzcd}$$

so ϵ is natural.

$$G_{\mathbb{T}}(TA \xrightarrow{\epsilon_A} A) = \mu_A, \text{ so } G_{\mathbb{T}}(\epsilon_A)\eta_{G_{\mathbb{T}}A} = \mu_A \cdot \eta_{TA} = 1_{TA}$$

and $(\epsilon_{F_{\mathbb{T}}A})(F_{\mathbb{T}}\eta_A)$ is

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] & TA \arrow[r, "\eta_{TA}"] \arrow[dr, "1_{TA}"'] & TTA \arrow[d, "\mu_A"] \arrow[dl, phantom, "(1)"] \\
& & TA
\end{tikzcd}$$

which is $1_{F_{\mathbb{T}}A}$. Also $G_{\mathbb{T}}(\epsilon_{F_{\mathbb{T}}A}) = \mu_A$, so $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ induces $\mathbb{T}$. □

5.7 Theorem. Given a monad $\mathbb{T}$ on $\mathscr{C}$, let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are the adjunctions $\left(\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}\right)$ inducing $\mathbb{T}$, and whose morphisms

$$\left(\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}\right) \longrightarrow \left(\mathscr{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathscr{D}'\right)$$

are functors $H : \mathscr{D} \longrightarrow \mathscr{D}'$ satisfying $HF = F'$ and $G'H = G$. Then the Kleisli adjunction is an initial object of $\mathbf{Adj}(\mathbb{T})$, and the Eilenberg-Moore adjunction is terminal.

Proof. Let $\left(\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}\right)$ be an object of $\mathbf{Adj}(\mathbb{T})$. We define $K : \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ (the **Eilenberg-Moore comparison functor**) by $KB = (GB, G\epsilon_B)$ where ϵ is the counit of $(F \dashv G)$; note this is an algebra by one of the triangular identities for $(F \dashv G)$ and naturality of ϵ and $K(B \xrightarrow{g} B') = Gg$ (a homomorphism by naturality of ϵ). Clearly $G^{\mathbb{T}}K = G$ and

$$KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A,$$
$$KF(A \xrightarrow{f} A') = Tf = F^{\mathbb{T}}f.$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Suppose $K' : \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ is another such; then since $G^{\mathbb{T}}K' = G$, we know $K'B = (GB, \beta_B)$ where β is a natural transformation $GFG \longrightarrow G$. Also, since $K'F = F^{\mathbb{T}}$, we have $\beta_{FA} = \mu_A = G\epsilon_{FA}$.

Now, given any $B \in \operatorname{ob} \mathscr{D}$, consider the diagram

$$\begin{tikzcd}
GFGFGB \arrow[r, "GFG\epsilon_B"] \arrow[d, shift right, "G\epsilon_{FGB}"'] \arrow[d, shift left, "\beta_{FGB}"] & GFGB \arrow[d, shift right, "G\epsilon_B"'] \arrow[d, shift left, "\beta_B"] \\
GFGB \arrow[r, "G\epsilon_B"] & GB
\end{tikzcd}$$

Both squares commute so $G\epsilon_B$ and β_B have the same composite with $GFG\epsilon_B$. But this is split epic, with splitting $GF\eta_{GB}$, so $\beta = G\epsilon$. Hence $K' = K$.

We now define the **Kleisli comparison functor** $L : \mathscr{C}_{\mathbb{T}} \longrightarrow \mathscr{D}$ by $LA = FA$,

$$L(A \xrightarrow{f} B) = \quad FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB.$$

L preserves identities by one of the triangular identities for $(F \dashv G)$; given $A \xrightarrow{f} B \xrightarrow{g} C$, we have

$$\begin{tikzcd}
FA \arrow[r, "Ff"] & FGFB \arrow[r, "FGFg"] \arrow[d, "\epsilon_{FB}"] & FGFGFC \arrow[r, "FG\epsilon_{FC}"] \arrow[d, "\epsilon_{FGFC}"] & FGFC \arrow[d, "\epsilon_{FC}"] \\
& FB \arrow[r, "Fg"] & FGFC \arrow[r, "\epsilon_{FC}"] & FC
\end{tikzcd}$$

$GLA = TA = G_{\mathbb{T}}A$,

$$GL(A \xrightarrow{f} B) = (G\epsilon_{FB})(GFf) = \mu_B(Tf).$$

$LF_{\mathbb{T}}A = FA$,

$$LF_{\mathbb{T}}(A \xrightarrow{f} B) = (\epsilon_{FB})(F\eta_B)(Ff) = Ff = G_{\mathbb{T}}f.$$

Note that L is full and faithful; its effect on morphisms (with given domain and codomain) is that of transposition across $(F \dashv G)$. Suppose $L' : \mathscr{C}_{\mathbb{T}} \longrightarrow \mathscr{D}$ is a morphism of $\mathbf{Adj}(\mathbb{T})$. We must have $L'A = FA$, and L' maps the counit $TA \longrightarrow A$ to the counit $FGFA \xrightarrow{\epsilon_{FA}} FA$. For any $A \xrightarrow{f} B$, we have $f = 1_{TA}(F_{\mathbb{T}}f)$ so $L'(f) = \epsilon_{FA} \cdot (Ff) = Lf$. $\square$

Lecture 16 If $\mathscr{C}$ has coproducts, then so does $\mathscr{C}_{\mathbb{T}}$, since $F_{\mathbb{T}}$ preserves them. But in general, it has few other limits or colimits. In contrast, we have

5.8 Theorem.

(i) The forgetful functor $G : \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ creates all limits which exist in $\mathscr{C}$.

(ii) If $\mathscr{C}$ has colimits of shape J, then $G : \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ creates them $\iff$ T preserves them.

Proof.

(i) Suppose given $D : J \longrightarrow \mathscr{C}^{\mathbb{T}}$; write $D(j) = (GD(j), \delta_j)$ and suppose

$$(L, \{\, \mu_j : L \longrightarrow GD(j) \mid j \in \operatorname{ob} J \,\})$$

is a limit cone for GD. Then the composites

$$TL \xrightarrow{T\mu_j} TGD(j) \xrightarrow{\delta_j} GD(j)$$

form a cone over GD, since the edges of GD are homomorphisms, so they induce a unique $\lambda : TL \longrightarrow L$ such that $\mu_j \lambda = \delta_j(T\mu)$ for all j. The fact that λ is a $\mathbb{T}$-algebra structure on L follows from the fact that the δ_j are algebra structures and uniqueness of factorisations through limits.

So $((L, \lambda), \{\, \mu_j \mid j \in \operatorname{ob} J \,\})$ is the unique lifting of the limit cone over GD to a cone over D; and it's a limit, since given a cone over D with apex (A, α), we get a unique factorisation $A \xrightarrow{f} L$ in $\mathscr{C}$ and F is an algebra homomorphism by uniqueness of factorisations through L.

(ii) ($\Rightarrow$) $F : \mathscr{C} \longrightarrow \mathscr{C}^{\mathbb{T}}$ preserves colimits since it's a left adjoint, so $T = GF$ preserves colimits of shape J.

($\Leftarrow$) Suppose given $D : J \longrightarrow \mathscr{C}^{\mathbb{T}}$ as in (i), and a colimit cone

$$\{\, GD(j) \xrightarrow{\mu_j} L \mid j \in \operatorname{ob} J \,\}$$

in $\mathscr{C}$. Then

$$\{\, TGD(j) \xrightarrow{T\mu_j} TL \mid j \in \operatorname{ob} J \,\}$$

is also a colimit cone, so the composites

$$TGD(j) \xrightarrow{\delta_j} GD(j) \xrightarrow{\mu_j} L$$

induce a unique $\lambda : TL \longrightarrow L$. The rest of the argument is like (i). $\square$

5.9 Definition (Monadic adjunction)**.** Given an adjunction $\left(\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}\right)$, with $(F \dashv G)$ we say the adjunction (or the functor G) is **monadic** if the comparison functor $K : \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ is part of an equivalence of categories.

(Note that, since the Kleisli comparison $\mathscr{C}_{\mathbb{T}} \longrightarrow \mathscr{D}$ is full and faithful, it's part of an equivalence if and only if it (equivalently, F) is essentially surjective on objects.)

Remark. Given any adjunction $(F \dashv G)$, for each object B of $\mathscr{D}$ we have a diagram

$$FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB \xrightarrow{\quad \epsilon_B \quad} B$$

with equal composites. The 'primeval monadicity theorem' asserts that $\mathscr{C}^{\mathbb{T}}$ is characterised in $\mathbf{Adj}(\mathbb{T})$ by the fact that these diagrams are all coequalizers.

5.10 Definition (Reflexive coequalizers)**.**

(a) We say a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is **reflexive** if there exists $B \xrightarrow{r} A$ such that $fr = gr = 1_B$. (Note that $FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB$ is reflexive, with $r = F\eta_{GB}$). We say $\mathscr{C}$ has **reflexive coequalizers** if it has coequalizers of all reflexive pairs (equivalently, colimits of shape J where $J = $
$$\begin{tikzcd} \bullet \arrow[loop above] \arrow[loop below] \arrow[r, shift left=2] \arrow[r, shift right=2] & \bullet \arrow[l] \end{tikzcd}$$
)

(b) By a **split coequalizer** diagram, we mean a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=40, "t"] \arrow[r, "h"] & C \arrow[l, bend left=40, "s"] \end{tikzcd}$$

satisfying
$$hf = hg \quad hs = 1_C \quad gt = 1_B \quad ft = sh.$$
These equations imply that h is a coequalizer of (f,g); if $B \xrightarrow{x} D$ satisfies $xf = xg$ then $x = xgt = xft = xsh$, so x factors through h, and the factorisation is unique since h is split epic. Note that split coequalizers are preserved by all functors.

(c) Given a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$, a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is called G-**split** if there exists a split coequalizer diagram

$$\begin{tikzcd} GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[l, bend left=40, "t"] \arrow[r, "h"] & C \arrow[l, bend left=40, "s"] \end{tikzcd}$$

in $\mathscr{C}$.

Note that $FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB$ is G-split, since

$$\begin{tikzcd} GFGFGB \arrow[r, shift left, "GFG\epsilon_B"] \arrow[r, shift right, "G\epsilon_{FGB}"'] & GFGB \arrow[l, bend left=40, "\eta_{GFGB}"] \arrow[r, "G\epsilon_B"] & GB \arrow[l, bend left=40, "\eta_{GB}"] \end{tikzcd}$$

is a split coequalizer.

5.11 Lemma. Suppose given an adjunction $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ where $F \dashv G$, inducing a monad $\mathbb{T}$ on $\mathscr{C}$. Then $K : \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ has a left adjoint provided, for every $\mathbb{T}$-algebra (A, α), the pair $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ has a coequalizer in $\mathscr{D}$.

Proof. We define $L : \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{D}$ by taking $FA \longrightarrow L(A, \alpha)$ to be a coequalizer for $(F\alpha, \epsilon_{FA})$. Note that this is a functor $\mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{D}$.

Recall that K is defined by $KB = (GB, G\varepsilon_B)$. For any B, morphisms $LA \longrightarrow B$ correspond bijectively to morphisms $FA \xrightarrow{f} B$ satisfying $f(F\alpha) = f(\epsilon_{FA})$. These correspond to morphisms $A \xrightarrow{\check{f}} GB$ satisfying

$$\check{f}\alpha = Gf = G(\epsilon_B(F\check{f})) = (G\epsilon_B)(T\check{f})$$

i.e. to algebra homomorphisms $(A, \alpha) \longrightarrow KB$. And these bijections are natural in (A, α) and in B. $\square$

Lecture 17 **5.12 Theorem** (Precise Monadicity Theorem)**.** $G : \mathscr{D} \longrightarrow \mathscr{C}$ is monadic iff G has a left adjoint and creates coequalizers of G-split pairs.

5.13 Theorem (Refined/Reflexive Monadicity Theorem)**.** Suppose $\mathscr{D}$ has and $G : \mathscr{D} \longrightarrow \mathscr{C}$ preserves reflexive coequalizers and that G reflects isomorphisms and has a left adjoint. Then G is monadic.

Proof of Theorem 5.12 $\Rightarrow$. It is sufficient to show that $G^{\mathbb{T}} : \mathscr{C}^{\mathbb{T}} \longrightarrow \mathscr{C}$ creates coequalizers of $G^{\mathbb{T}}$ split-pairs. But this follows from the argument of Theorem 5.8(ii), since if $(A, \alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B, \beta)$ is a $G^{\mathbb{T}}$-split pair, the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is preserved by T and TT. $\square$

Proof of Theorem 5.12 $\Leftarrow$ and Theorem 5.13. Let $\mathbb{T}$ be the monad induced by $(F \dashv G)$. For any $\mathbb{T}$-algebra (A, α), the pair $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ is both reflexive and G-split, so has a coequalizer in $\mathscr{D}$, and hence by Lemma 5.11, $K : \mathscr{D} \longrightarrow \mathscr{C}^{\mathbb{T}}$ has a left adjoint L.

The unit of $(L \dashv K)$ at an algebra (A, α): the coequalizer defining $L(A, \alpha)$ is mapped by K to the diagram

$$\begin{tikzcd}
F^{\mathbb{T}}TA \arrow[r, shift left, "F^{\mathbb{T}}A"] \arrow[r, shift right, "\mu_A"'] & F^{\mathbb{T}}A \arrow[rr] \arrow[dr, "\alpha"'] & & KL(A,\alpha) \\
& & (A,\alpha) \arrow[ur, dashed, "\iota_{A,\alpha}"'] &
\end{tikzcd}$$

and $\iota_{A,\alpha}$ is the factorisation of this through the ($G^{\mathbb{T}}$-split) coequalizer α. But either set of hypotheses implies that G preserves the coequalizer defining $L(A, \alpha)$ so $\iota_{(A,\alpha)}$ is an isomorphism.

For the counit $\zeta_B : LKB \longrightarrow B$, we have a coequalizer

$$\begin{tikzcd} FGFGB \arrow[r, shift left, "FG\epsilon_B"] \arrow[r, shift right, "\epsilon_{FGB}"'] & FGB \arrow[r] \arrow[dr, "\epsilon_B"'] & LKB \arrow[d, dashed, "\zeta_B"] \\ & & B \end{tikzcd}$$

Again, either set of hypotheses implies that ϵ_B is a coequalizer of $(FG\epsilon_B, \epsilon_{FGB})$ so ζ_B is an isomorphism. $\square$

5.14 Examples.

(a) The forgetful functors $\mathbf{Gp} \longrightarrow \mathbf{Set}$, $\mathbf{Rng} \longrightarrow \mathbf{Set}$, $\mathbf{Mod}_R \longrightarrow \mathbf{Set}, \ldots$ all satisfy the hypotheses of Theorem 5.13; for the reflexive coequalizers, use question 3 on example sheet 4 which shows that if

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$$

is a reflexive coequalizer diagram in $\mathbf{Set}$, then so is

$$A^n \underset{g^n}{\overset{f^n}{\rightrightarrows}} B^n \xrightarrow{h^n} C^n.$$

(b) Any reflection is monadic: this follows from q2 on example sheet 3, but also can be proved using Theorem 5.12. Let $\mathscr{D}$ be a reflective (full) subcategory of $\mathscr{C}$ and suppose a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathscr{D}$ fits into a split coequalizer diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=60, "t"] \arrow[r, "h"] & C \arrow[l, bend left=60, "s"] \end{tikzcd}$$

in $\mathscr{C}$. Then t and $ft = sh$ belong to $\mathscr{D}$ since $\mathscr{D}$ is full, and hence s is in $\mathscr{D}$ since it's an equalizer of $(1_B, sh)$ and $\mathscr{D}$ is closed under limits in $\mathscr{C}$. Hence also $h \in \operatorname{mor} \mathscr{D}$.

(c) Consider the composite adjunction

$$\mathbf{Set} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{AbGp} \underset{I}{\overset{L}{\rightleftarrows}} \mathbf{tfAbGp}.$$

The two factors are monadic by (a) and (b) respectively, but the composite isn't, since the monad it induces on $\mathbf{Set}$ is isomorphic to that induced by $(F \dashv U)$.

(d) Consider the forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$. This is faithful and has both left and right adjoints (so preserves all coequalizers), but the monad induced on $\mathbf{Set}$ is $(1, 1, 1)$ and the category of algebras is $\mathbf{Set}$.

(e) Consider the composite adjunction

$$\mathbf{Set} \underset{U}{\overset{D}{\rightleftarrows}} \mathbf{Top} \underset{I}{\overset{\beta}{\rightleftarrows}} \mathbf{KHaus}.$$

We'll show that this satisfies the hypotheses of Theorem 5.12. Let

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend left, "t"] \arrow[r, "h"] & Z \arrow[l, bend left, "s"]
\end{tikzcd}$$

be a split coequalizer in **Set**, where X, Y have compact Hausdorff topologies and f, g are continuous. Note that the quotient topology on $Z \cong Y/R$ is compact, so it's the only possible candidate for a compact Hausdorff topology making h continuous.

We use the lemma from general topology: if Y is compact Hausdorff, then a quotient Y/R is Hausdorff $\iff$ $R \subseteq Y \times Y$ is closed. We note

$$\begin{aligned} R = \{\, (y, y') \mid h(y) = h(y') \,\} &= \{\, (y, y') \mid sh(y) = sh(y') \,\} \\ &= \{\, (y, y') \mid ft(y) = ft(y') \,\}. \end{aligned}$$

So if we define $S \subseteq X \times X = \{\, (x, x') \mid f(x) = f(x') \,\}$ then $R \subseteq (g \times g)(S)$, but the reverse inclusion also holds. But $S \longrightarrow X \times X \overset{f\pi_1}{\underset{f\pi_2}{\rightrightarrows}} Y$ is an equalizer, Y is Hausdorff, so S is closed in $X \times X$ and hence compact. So $R = (g \times g)(S)$ is compact and hence closed in $Y \times Y$.

Lecture 18 **5.15 Definition** (Monadic tower)**.** Let $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ be an adjunction, and suppose $\mathscr{D}$ has reflexive coequalizers. The **monadic tower** of $F \dashv G$ is the diagram

$$\begin{tikzcd}
\mathscr{D} \arrow[drrr, shift left, "K'"] \arrow[ddrr, shift left, "K"] \arrow[dddr, shift left, "G"] & & & & \\
 & & & (\mathscr{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[ulll, shift left, "L'"] \arrow[dl, shift left] \arrow[ur, dotted, no head] & \\
 & & \mathscr{C}^{\mathbb{T}} \arrow[uull, shift left, "L"] \arrow[ur, shift left] \arrow[dl, shift left, "G^{\mathbb{T}}"] & & \\
 & \mathscr{C} \arrow[uuul, shift left, "F"] \arrow[ur, shift left, "F^{\mathbb{T}}"] & & &
\end{tikzcd}$$

where $\mathbb{T}$ is the monad induced by $(F \dashv G)$, K is as in Theorem 5.7, L as in Lemma 5.11, $\mathbb{S}$ is the monad induced by $(L \dashv K)$, and so on.

We say $(F \dashv G)$ has **monadic length** n if we reach an equivalence after n steps.

For example, the adjunction of Examples 5.14(c) has monadic length 2, and the adjunction of Examples 5.14(d) has monadic length ∞.

5.16 Theorem. Suppose given an adjunction $\mathscr{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathscr{D}$ and monads $\mathbb{T}, \mathbb{S}$ on $\mathscr{C}, \mathscr{D}$ respectively, and a functor $\bar{R} : \mathscr{D}^{\mathbb{S}} \longrightarrow \mathscr{C}^{\mathbb{T}}$ such that

$$\begin{tikzcd} \mathscr{D}^{\mathbb{S}} \arrow[r, "\bar{R}"] \arrow[d, "G^{\mathbb{S}}"] & \mathscr{C}^{\mathbb{T}} \arrow[d, "G^{\mathbb{T}}"] \\ \mathscr{D} \arrow[r, "R"] & \mathscr{C} \end{tikzcd}$$

commutes up to isomorphism.

Suppose also that $\mathscr{D}^{\mathbb{S}}$ has reflexive coequalizers. Then $\bar{R}$ has a left adjoint $\bar{L}$.

Proof. Note that if $\bar{L}$ exists, we must have $\bar{L}F^{\mathbb{T}} \cong F^{\mathbb{S}}L$, by Corollary 3.6. So we'd expect $\bar{L}(A, \alpha)$ to be a coequalizer of two morphisms

$$F^{\mathbb{S}}LTA \underset{?}{\overset{F^{\mathbb{S}}L\alpha}{\rightrightarrows}} F^{\mathbb{S}}LA$$

To construct the second morphism, note first that we can assume wlog $G^{\mathbb{T}}\bar{R} = RG^{\mathbb{S}}$, by transporting $\mathbb{T}$-algebra structures along the isomorphism $G^{\mathbb{T}}R(B, \beta) \longrightarrow RB$. We obtain $\theta : TR \longrightarrow RS$ by

$$R \xrightarrow{R\iota} RS = RG^{\mathbb{S}}F^{\mathbb{S}} = G^{\mathbb{T}}\bar{R}F^{\mathbb{S}}$$
$$F^{\mathbb{T}}R \longrightarrow \bar{R}F^{\mathbb{S}}$$
$$TR = G^{\mathbb{T}}F^{\mathbb{T}}R \xrightarrow{\theta} G^{\mathbb{T}}\bar{R}F^{\mathbb{S}} = RG^{\mathbb{S}}F^{\mathbb{S}} = RS$$

Convert it into $\varphi : LT \longrightarrow SL$ by

$$LT \xrightarrow{LT\gamma} LTRL \xrightarrow{L\theta_L} LRSL \xrightarrow{\delta_{SL}} SL$$

where γ and δ are the unit and counit of $(L \dashv R)$. Transposing across $(F^{\mathbb{S}} \dashv G^{\mathbb{S}})$, we get $F^{\mathbb{S}}LT \xrightarrow{\bar{\varphi}} F^{\mathbb{S}}L$. The pair $(F^{\mathbb{S}}L\alpha, \bar{\varphi}_A)$ is reflexive, with common splitting $F^{\mathbb{S}}L\eta$.

It can be verified that the coequalizer of this pair has the universal property we require for $\bar{L}(A, \alpha)$. $\square$

6 Cartesian Closed Categories

6.1 Definition (Exponentiable object)**.** Let $\mathscr{C}$ be a category with finite products. We say $A \in \operatorname{ob}\mathscr{C}$ is **exponentiable** if the functor $(-) \times A : \mathscr{C} \longrightarrow \mathscr{C}$ has right adjoint $(-)^A$. If every object of $\mathscr{C}$ is exponentiable, we say $\mathscr{C}$ is **cartesian closed**.

6.2 Examples.

(a) **Set** is cartesian closed, with $B^A = \mathbf{Set}(A, B)$. A function $f : C \times A \longrightarrow B$ corresponds to $\bar{f} : C \longrightarrow B^A$.

(b) **Cat** is cartesian closed, with $\mathscr{D}^{\mathscr{C}} = [\mathscr{C}, \mathscr{D}]$.

(c) In **Top**, if an exponential Y^X exists, its points must be the continuous maps $X \longrightarrow Y$. The compact-open topology on $\mathbf{Top}(X, Y)$ has the universal property of an exponential iff X is locally compact.

Note that finite products of exponentiable objects are exponentiable: since

$$(-) \times (A \times B) \cong (- \times A) \times B,$$

we have $(-)^{A \times B} \cong ((-)^B)^A$. However, even if X, Y are locally compact, Y^X needn't be, so the exponentiable objects don't form a cartesian closed full subcategory.

(d) A cartesian closed poset is called a **Heyting semilattice**: it's a poset with finite meets and a binary operation $\Rightarrow$ satisfying $a \leq (b \Rightarrow c)$ iff $a \wedge b \leq c$. For example, a complete poset is a Heyting semilattice iff it satisfies the infinite distributive law

$$a \wedge \bigvee \{ b_i \mid i \in I \} = \bigvee \{ a \wedge b_i \mid i \in I \} .$$

For any topological space X, the lattice $\mathcal{O}(X)$ of open subsets satisfies this condition, since $\wedge$ and $\bigvee$ coincide with $\cap$ and $\bigcup$.

Recall from example sheets that, if $B \in \operatorname{ob}\mathscr{C}$, we define the **slice category** $\mathscr{C}/B$ to have objects which are morphisms

$$\begin{tikzcd} A \arrow[d] \\ B \end{tikzcd}$$

in $\mathscr{C}$ and morphisms are commutative triangles

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A' \arrow[dl] \\ & B. & \end{tikzcd}$$

The forgetful functor $\mathscr{C}/B \longrightarrow \mathscr{C}$ will be denoted Σ_B. If $\mathscr{C}$ has finite products, Σ_B has a right adjoint B^* which sends A to $\left(\begin{tikzcd} A \times B \arrow[d, "\pi_2"] \\ B \end{tikzcd}\right)$, since morphisms

$$\begin{tikzcd} C \arrow[rr, "{(f,g)}"] \arrow[dr, "g"'] & & A \times B \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

correspond to morphisms $C = \Sigma_B g \xrightarrow{f} A$.

6.3 Lemma. If $\mathscr{C}$ has all finite limits, then an object B is exponentiable iff $B^* : \mathscr{C} \longrightarrow \mathscr{C}/B$ has a right adjoint Π_B.

Proof.

$\Leftarrow$ The composite $\Sigma_B B^*$ is equal to $(-) \times B$, so we take $(-)^B$ to be $\Pi_B B^*$.

$\Rightarrow$ If B is exponentiable, for any $A \xrightarrow{f} B$ we define $\Pi_B(f)$ to be the pullback

$$\begin{tikzcd} \Pi_B(f) \arrow[r] \arrow[d] & A^B \arrow[d, "f^B"] \\ 1 \arrow[r, "\overline{\pi_2}"] & B^B. \end{tikzcd}$$

The morphisms $C \longrightarrow \Pi_B(f)$ correspond to morphisms $C \longrightarrow A^B$ making

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A^B \arrow[d, "f^B"] \\ 1 \arrow[r, "\overline{\pi_2}"] & B^B \end{tikzcd}$$

i.e. to homomorphisms $C \times B \longrightarrow A$ making

$$\begin{tikzcd} C \times B \arrow[r] \arrow[dr, "\pi_2"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commute. $\square$

Lecture 19 **6.4 Lemma.** Suppose $\mathscr{C}$ has finite limits. If A is exponentiable in $\mathscr{C}$, then B^*A is exponentiable in $\mathscr{C}/B$ for any B. Moreover, B^* preserves exponentials.

Proof. Given an object $\begin{tikzcd} C \arrow[d, "f"] \\ B \end{tikzcd}$, form the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f^{(B^*A)}"'] & C^A \arrow[d, "f^A"] \\ B \arrow[r, "\overline{\pi_1}"'] & B^A. \end{tikzcd}$$

Then, for any $\begin{tikzcd} D \arrow[d, "g"] \\ B \end{tikzcd}$, morphisms $g \longrightarrow f^{B^*A}$ in $\mathscr{C}/B$ correspond to morphisms $D \xrightarrow{\bar{h}} C^A$

making

$$\begin{tikzcd} D \arrow[r, "\bar{h}"] \arrow[d, "g"'] & C^A \arrow[d, "f^A"] \\ B \arrow[r, "\overline{\pi_1}"'] & B^A \end{tikzcd}$$

commute, and hence to morphisms $D \times A \xrightarrow{h} C$ making

$$\begin{tikzcd} D \times A \arrow[r, "h"] \arrow[dr, "g\pi_1"'] & C \arrow[d, "f"] \\ & B \end{tikzcd}$$

commute. But

$$\begin{tikzcd} D \times A \arrow[r, "g \times 1_A"] \arrow[d, "\pi_1"'] & B \times A \arrow[d, "\pi_1"] \\ D \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback in $\mathscr{C}$, i.e. a product in $\mathscr{C}/B$.

For the second assertion, note that if $\begin{tikzcd} C \arrow[d, "f"] \\ B \end{tikzcd}$ is of the form $\begin{tikzcd} B \times E \arrow[d, "\pi_1"] \\ B \end{tikzcd}$, then the pullback defining f^{B^*A} becomes

$$\begin{tikzcd} B \times E^A \arrow[r, "\overline{\pi_1} \times 1"] \arrow[d, "\pi_1"'] & B^A \times E^A \arrow[d, "\pi_1"] \\ B \arrow[r, "\overline{\pi_1}"'] & B^A \end{tikzcd}$$

so $f^{B^*A} \cong B^*(E^A)$. □

Remark. $\mathscr{C}/B$ is isomorphic to the category of coalgebras for the monad structure on $(-) \times B$ (Examples 5.2(a)); so the first part of Lemma 6.4 could have been proved using Theorem 5.16.

6.5 Definition (Locally cartesian closed)**.** We say $\mathscr{C}$ is **locally cartesian closed** if it has all finite limits and each $\mathscr{C}/B$ is cartesian closed. (Note that this includes the fact that $\mathscr{C} \cong \mathscr{C}/1$ is cartesian closed).

6.6 Examples.

(a) **Set** is locally cartesian closed, since $\mathbf{Set}/B \cong \mathbf{Set}^B$ for any B.

(b) For any small category $\mathscr{C}$, $[\mathscr{C}, \mathbf{Set}]$ is cartesian closed: by Yoneda,

$$\begin{aligned} G^F(A) &\cong [\mathscr{C}, \mathbf{Set}](\mathscr{C}(A, -), G^F) \\ &\cong [\mathscr{C}, \mathbf{Set}](\mathscr{C}(A, -) \times F, G), \end{aligned}$$

so we take the right hand side as a definiition of $G^F(A)$ and define G^F on morphisms $A \xrightarrow{f} B$ by composition with $\mathscr{C}(f,-) \times 1_F$. Note that the class of functors H for which we have

$$[\mathscr{C}, \mathbf{Set}](H, G^F) \cong [\mathscr{C}, \mathbf{Set}](H \times F, G)$$

is closed under colimits; but every functor $\mathscr{C} \longrightarrow \mathbf{Set}$ is a colimit of representables.

In fact $[\mathscr{C}, \mathbf{Set}]$ is locally cartesian closed, since all its slice categories $[\mathscr{C}, \mathbf{Set}]/F$ are of the same form (see question 6 on example sheet 4).

(c) Any Heyting semilattice H is locally cartesian closed, since $H/B \cong \downarrow(b)$, the poset of elements $\leq b$, and $b^* = (-) \wedge b$ is surjective.

(d) **Cat** is cartesian closed and not locally cartesian closed, since not all strong epis are regular (cf question 6 on example sheet 3).

Note that, given $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ in $\mathscr{C}/B$, the iterated slice $(\mathscr{C}/B)/f$ is isomorphic to $\mathscr{C}/A$, and this identifies

$$f^* : \mathscr{C}/B \longrightarrow (\mathscr{C}/B)/f$$

with the operation of pulling back morphisms along f. So by Lemma 6.3, $\mathscr{C}$ is locally cartesian closed iff it has finite limits and $f^* : \mathscr{C}/B \longrightarrow \mathscr{C}/A$ has a right adjoint Π_f for every $A \xrightarrow{f} B$ in $\mathscr{C}$.

6.7 Theorem. Suppose $\mathscr{C}$ is locally cartesian closed and has reflexive coequalizers. Then every morphism $A \xrightarrow{f} B$ factors as

$$\begin{tikzcd} & I \arrow[dr, tail, "m"] & \\ A \arrow[ur, two heads, "q"] \arrow[rr, "f"] & & B \end{tikzcd}$$

where q is regular epic and m is monic.

Proof. First form the pullback

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "b"] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

and then form the coequalizer

$$R \overset{a}{\underset{b}{\rightrightarrows}} A \xrightarrow{q} I.$$

Since $fa = fb$, f factors as $A \xrightarrow{q} I \xrightarrow{m} B$.

Suppose given $D \overset{g}{\underset{h}{\rightrightarrows}} I$ with $mg = mh$. Form the pullback

$$\begin{tikzcd} E \arrow[r, "n"] \arrow[d, "{(k,l)}"'] & D \arrow[d, "{(g,h)}"] \\ A \times A \arrow[r, "q \times q"'] & I \times I. \end{tikzcd}$$

Since $q \times q$ factors as $(q \times 1_I)(1_A \times q)$ and both factors are pullbacks of q, it's an epimorphism, and hence so is n. Now $fk = mqk = mgn = mhn = mql = fl$, so $\exists E \xrightarrow{p} R$ with $ap = k$, $bp = l$. Now $qk = qap = qbp = ql$, i.e. $gn = hn$. But n is epic, so $g = h$. Hence m is monic. □

Note that this implies any strong epimorphism $A \xrightarrow{f} B$ is regular, since the monic part of its image factorization is an isomorphism. In particular, regular epimorphisms are stable under composition.

6.8 Definition (Cartesian closed functor)**.** If $\mathscr{C}$ and $\mathscr{D}$ are cartesian closed categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ preserves finite products, then for each pair of objects A, B of $\mathscr{C}$ we get a natural morphism

$$\theta : F(B^A) \longrightarrow FB^{FA},$$

namely the transpose of

$$F(B^A) \times FA \cong F(B^A \times A) \xrightarrow{F(\mathrm{ev})} FB$$

where ev is the counit of $((-) \times A \dashv (-)^A)$. We say F is a **cartesian closed functor** if θ is an isomorphism for every pair (A, B).

Note that the second part of Lemma 6.4 says that if $\mathscr{C}$ is locally cartesian closed then $f^* : \mathscr{C}/B \longrightarrow \mathscr{C}/A$ is a cartesian closed functor for any $A \xrightarrow{f} B$.

6.9 Theorem. Let $\mathscr{C}$ and $\mathscr{D}$ be cartesian closed categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor having a left adjoint L. Then F is cartesian closed iff the canonical morphism

$$L(B \times FA) \xrightarrow{(L\pi_1, L\pi_2)} LB \times LFA \xrightarrow{1_{LB} \times \epsilon_A} LB \times A$$

is an isomorphism for all $A \in \mathrm{ob}\,\mathscr{D}$, $B \in \mathrm{ob}\,\mathscr{C}$.

This condition is called **Frobenius reciprocity**. Note that if $\mathscr{C}$ is locally cartesian closed, then $f^* : \mathscr{C}/B \longrightarrow \mathscr{C}/A$ has a left adjoint Σ_f given by composition with f and it's easy to verify that $\Sigma_f(g \times f^* h) \cong \Sigma_f g \times h$.

Proof.

Lecture 20 (⇒) Given an inverse for $\theta : F(C^A) \longrightarrow FC^{FA}$, we define $\varphi^{-1}_{A,B}$ to be the composite

$$\begin{array}{ccccc} LB \times A & \xrightarrow{L\lambda \times 1} & L((B \times FA)^{FA}) & \xrightarrow{L(\eta^{FA}) \times 1} & L(FL(B \times FA)^{FA}) \times A \\ & & & & \downarrow{\scriptstyle L\theta^{-1} \times 1} \\ L(B \times FA) & \xleftarrow[\mathrm{ev}]{} & L(B \times FA)^A \times A & \xleftarrow[\epsilon \times 1]{} & LF(L(B \times FA)^A) \times A \end{array}$$

(⇐) Given an inverse for φ, we define θ^{-1} to be

$$\begin{tikzcd}
F((L(FC^{FA}) \times A)^A) \arrow[d, "F(\varphi^{-1 A})"'] & FL(FC^{FA}) \arrow[l, "F\lambda"'] & FC^{FA} \arrow[l, "\eta"'] \\
F((L(FC^{FA} \times FA))^A) \arrow[r, "F\left((L(\mathrm{ev}))^A\right)"'] & F((LFC)^A) \arrow[r, "F(\epsilon^A)"'] & F(C^A).
\end{tikzcd}$$

We omit checking that these work. □

6.10 Corollary. Suppose $\mathscr{C}$ and $\mathscr{D}$ are cartesian closed, and $F : \mathscr{C} \longrightarrow \mathscr{D}$ has a left adjoint L which preserves finite products. Then F is cartesian closed $\iff$ F is full and faithful.

Proof.

(⇒) L preserves 1, so if we substitute $B = 1$ in the definition of φ, we get

$$LFA \xrightarrow{\epsilon_A} A.$$

But ϵ an isomorphism $\iff$ F is full and faithful by Lemma 3.9.

(⇐) If L preserves binary products and ϵ is isomorphic, then both factors in the definition of φ are isomorphisms. □

6.11 Definition (Exponential ideal)**.** Let $\mathscr{C}$ be a cartesian closed category. By an **exponential ideal** in $\mathscr{C}$, we mean a class of objects (or a full subcategory) $\mathscr{E}$ such that $B \in \mathscr{E} \implies B^A \in \mathscr{E}$ for all $A \in \mathrm{ob}\,\mathscr{C}$.

6.12 Examples.

(a) We say A is **subterminal** if $A \longrightarrow 1$ is monic. In any cartesian closed category $\mathscr{C}$, the class $\mathrm{Sub}_{\mathscr{C}}(1)$ of subterminal objects is an exponential ideal, since

$$\begin{aligned}
A \text{ subterminal} &\iff \exists \leq 1 \quad B \longrightarrow A, \text{ for any B} \\
&\implies \exists \leq 1 \quad C \times B \longrightarrow A, \text{ for any } B \text{ and } C, \\
&\iff \exists \leq 1 \quad C \longrightarrow A^B \\
&\iff A^B \text{ subterminal}
\end{aligned}$$

More generally, if $\mathscr{C}$ is locally cartesian closed, then $\mathrm{Sub}_{\mathscr{C}}(A)$ is an exponential ideal in $\mathscr{C}/A$, for any A. If $\mathscr{C}$ also satisfies the hypotheses of Theorem 6.7, then $\mathrm{Sub}_{\mathscr{C}}(A)$ is reflective in $\mathscr{C}/A$.

(b) Let X be a topological space. By a **presheaf** on X, we mean a functor

$$\mathcal{O}(X)^{\mathrm{op}} \xrightarrow{F} \mathbf{Set},$$

where $\mathcal{O}(X)$ is the poset of open subsets of X. So F has sets $F(U)$ for each open U, and restriction maps $x \longmapsto x|_V : F(U) \longrightarrow F(V)$ whenever $V \subseteq U$. We say F is a **sheaf** if, whenever $U = \bigcup_{i \in I} U_i$ and we're given $x_i \in F(U_i)$ for each i such that

$$x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$$

for all (i,j), there exists a unique $x \in F(U)$ such that $x|_{U_i} = x_i$ for all i.

We write $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ for the full subcategory of sheaves. We'll show $\mathbf{Sh}(X)$ is an exponential ideal:

Given presheaves F, G, $G^F(U)$ is the set of natural transformations

$$F \times \mathcal{O}(X)(-, U) \longrightarrow G$$

or equivalently the set of natural transformations $F|_U \longrightarrow G|_U$, where

$$F|_U : \mathcal{O}(U)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is the presheaf obtained by restricting F to open sets $\subseteq U$.

Now suppose G is a sheaf, suppose $U = \bigcup_{i \in I} U_i$ and suppose given $\alpha_i : F|_{U_i} \longrightarrow G|_{U_i}$ for each, such that

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

for all i, j. Given $x \in F(V)$ where $V \subseteq U$, write $V_i = V \cap U_i$, then $V = \bigcup_{i \in I} V_i$.

The elements $x|_{V_i}$, $i \in I$, satisfy the compatibility condition, and hence so do the elements

$$(\alpha_i)_{V_i}(x|_{V_i}) \in G(V_i).$$

So there's a unique $y \in G(V)$ such that $y|_{V_i} = (\alpha_i)_{V_i}(x|_{V_i})$ for all i, and we define this to be $\alpha_V(x)$. This defines a natural transformation $\alpha : F|_U \longrightarrow G|_U$, and it's the unique transformation where restriction to U_i is α_i, for each i.

Note that, since $\mathbf{Sh}(X)$ is closed under finite products (and in fact all limits) in $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$, it is itself cartesian closed.

6.13 Lemma. Suppose $\mathscr{C}$ is cartesian closed, $\mathscr{D} \subseteq \mathscr{C}$ is a (full) reflective subcategory, with reflector $L : \mathscr{C} \longrightarrow \mathscr{D}$. Then $\mathscr{D}$ is an exponential ideal $\iff$ L preserves binary products.

Proof.

($\Rightarrow$) Suppose $A, B \in \operatorname{ob}\mathscr{C}$, $C \in \operatorname{ob}\mathscr{D}$. Then we have bijections

$$\begin{aligned} A \times B &\longrightarrow C \\ A &\longrightarrow C^B \\ LA &\longrightarrow C^B \\ LA \times B &\longrightarrow C \\ B &\longrightarrow C^{LA} \\ LB &\longrightarrow C^{LA} \\ LA \times LB &\longrightarrow C \end{aligned}$$

so $LA \times LB$ has the universal property of $L(A \times B)$.

($\Leftarrow$) Suppose $B \in \operatorname{ob} \mathscr{D}$, $A, C \in \operatorname{ob} \mathscr{C}$. We have bijections

$$\begin{aligned} C &\longrightarrow B^A \\ C \times A &\longrightarrow B \\ LC \times LA \cong L(C \times A) &\longrightarrow B \\ L(LC \times A) &\longrightarrow B \\ LC \times A &\longrightarrow B \\ LC &\longrightarrow B^A \end{aligned}$$

So every $C \longrightarrow B^A$ factors through $C \longrightarrow LC$, hence $B^A \in \operatorname{ob} \mathscr{D}$. $\square$

7 Toposes

Lecture 21 Grothendieck (around 1963) introduced toposes as categories of 'generalized sheaves'. J. Giraud gave a characterization of such categories by (set-theoretic) categorical properties.

F. W. Lawvere and M. Tierney (1969-70) investigated the elementary categorical properties of these categories and came up with the elementary definition. In fact a Grothendieck topos is exactly a Lawvere-Tierney topos which is (co)complete and locally small, and has a separating set of objects.

7.1 Definition (Subobject classifier, topos)**.**

(a) Let $\mathscr{E}$ be a category with finite limits. A **subobject classifier** for $\mathscr{E}$ is a monomorphism $\Omega' \overset{\top}{\rightarrowtail} \Omega$ such that for every monomorphism $A' \overset{m}{\rightarrowtail} A$ in $\mathscr{E}$, there is a unique $\chi_m : A \longrightarrow \Omega$ for which there is a pullback square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail, "m"] & \Omega' \arrow[d, tail, "\top"] \\ A \arrow[r, "\chi_m"] & \Omega \end{tikzcd}$$

Note that, for any A, there's a unique $A \longrightarrow \Omega$ which factors through $\Omega' \overset{\top}{\rightarrowtail} \Omega$, so the domain of $\top$ is actually a terminal object.

If $\mathscr{E}$ is well-powered, we have a functor

$$\mathrm{Sub}_{\mathscr{E}}(-) : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

sending A to the set of (isomorphism classes of) subobjects of A and acting on morphisms by pullback, and a subobject classifier is representation of this functor.

(b) A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

(c) If $\mathscr{E}$ and $\mathscr{F}$ are toposes, a **logical functor** $F : \mathscr{E} \longrightarrow \mathscr{F}$ is one which preserves finite limits, exponentials and the subobject classifier.

7.2 Examples.

(a) **Set** is a topos, with $\Omega = \{0, 1\}$ and $\top = 1 : 1 \longrightarrow \{0, 1\}$. Here

$$\chi_m(a) = \begin{cases} 1 & \text{if } a \in A' \\ 0 & \text{if } a \notin A'. \end{cases}$$

So also is the category $\mathbf{Set}_f$ of finite sets, or the category $\mathbf{Set}_\kappa$ of sets of cardinality $< \kappa$, where κ is an infinite cardinal such that $\lambda < \kappa \implies 2^\lambda < \kappa$.

(b) For any small category $\mathscr{C}$, $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos: we've seen that it's cartesian closed, and Ω is determined by Yoneda Lemma:

$$\Omega(A) \cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(-, A), \Omega) \cong \{\text{subfunctors of } \mathscr{C}(-, A)\}$$

So we define $\Omega(A)$ to be the set of **sieves** on A, i.e. sets of morphisms with cod $= A$, such that $f \in R \implies fg \in R$ for any g.

Given $B \xrightarrow{f} A$ and a sieve R on A, we define f^*R to be the set of g with codomain B such that $fg \in R$. This makes Ω into a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, as follows. $\top : 1 \longrightarrow \Omega$ is defined by $\top_A(*) = \{$all morphisms with codomain $A\}$. Given a subfunctor $F' \overset{m}{\rightarrowtail} F$, we define $\chi_m : F \longrightarrow \Omega$ by

$$(\chi_m)_A(x) = \{\, f : B \longrightarrow A \mid Ff(x) \in F'(B) \,\} .$$

This is the unique natural transformation making

$$\begin{tikzcd} F' \arrow[r] \arrow[d, tail, "m"] & 1 \arrow[d, tail, "\top"] \\ F \arrow[r] & \Omega \end{tikzcd}$$

a pullback.

(c) For any space X, $\mathbf{Sh}(X)$ is a topos. It's cartesian closed by Examples 6.12(ii); for the subobject classifier, we take

$$\Omega(U) = \{\, V \in \mathcal{O}(X) \mid V \subseteq U \,\} ,$$

$\Omega(U' \longrightarrow U)$ is the map $V \longmapsto V \cap U'$ and Ω is a sheaf since if we have $U = \bigcup_{i \in I} U_i$, and $V_i \subseteq U_i$ such that $V_i \cap U_j = V_j \cap U_i$ for each i, j, then $V = \bigcup_{i \in I} V_i$ is the unique open subset of U with $V \cap U_i = V_i$ for each i.

If $F' \overset{m}{\rightarrowtail} F$ is a subsheaf, then for any $x \in F(U)$, the sieve $\{\, V \subseteq U \mid x|_V \in F'(V) \,\}$ has a greatest element since F' is a sheaf, so we define $\chi_m : F \longrightarrow \Omega$ to send x to this element.

(d) Let $\mathscr{C}$ be a group G. The topos structure on $[G, \mathbf{Set}]$ is particularly simple: B^A is the set of all G-equivariant maps $A \times G \xrightarrow{f} B$, but such an f is determined by its values at elements of the form $(a, 1)$ since $f(a, g) = g \cdot f(g^{-1} \cdot a, 1)$, and this restriction can be any mapping $A \times \{1\} \longrightarrow B$. So we can take B^A to be the set of functions $A \longrightarrow B$, with G acting by $(g \cdot f)(a) = g(f(g^{-1} \cdot a))$, and $\Omega = \{0, 1\}$ with trivial G-action.

So the forgetful functor $[G, \mathbf{Set}] \longrightarrow \mathbf{Set}$ is logical, as is the functor which equips a set A with trivial G-action. Moreover, even if G is infinite, $[G, \mathbf{Set}_f]$ is a topos, and the inclusion $[G, \mathbf{Set}_f] \longrightarrow [G, \mathbf{Set}]$ is logical. Similarly, if $\mathcal{G}$ is a large group, then $[\mathcal{G}, \mathbf{Set}]$ is a topos.

(e) Let $\mathscr{C}$ be a category such that every slice $\mathscr{C}/A$ is equivalent to a finite category. Then $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}_f]$ is a topos; similarly if $\mathscr{C}$ is large but all $\mathscr{C}/A$ are small then $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos. In particular, $[\mathbf{On}^{\mathrm{op}}, \mathbf{Set}]$ is a topos, but it's not locally small.

Lecture 22 **7.3 Lemma.** Suppose $\mathscr{E}$ has finite limits and a subobject classifier. Then every monomorphism in $\mathscr{E}$ is regular. In particular, $\mathscr{E}$ is balanced.

Proof. The universal monomorphism $1 \overset{\top}{\rightarrowtail} \Omega$ is split and hence regular. But any pullback of a regular monomorphism is regular: if f is an equalizer of (g, h) then $k^*(f)$ is an equalizer of (gk, hk). The second assertion follows since regular monomorphism together with epimorphism implies isomorphism. □

Given an object A in a topos $\mathscr{E}$, we write PA for the exponential Ω^A, and $\ni_A \rightarrowtail PA \times A$ for the subobject corresponding to $PA \times A \xrightarrow{\mathrm{ev}} \Omega$. This has the property that, for any B and any $R \overset{m}{\rightarrowtail} B \times A$ there is a unique $\ulcorner m \urcorner : B \longrightarrow PA$ such that

$$\begin{tikzcd}
R \arrow[r] \arrow[d, tail, "m"] & \ni_A \arrow[d, tail] \\
B \times A \arrow[r, "\ulcorner m \urcorner \times 1_A"] & PA \times A
\end{tikzcd}$$

is a pullback.

7.4 Definition (Power object, weak topos)**.** By a **power object** for A in a category $\mathscr{E}$ with finite limits, we mean an object PA equipped with $\ni_A \rightarrowtail PA \times A$ satisfying the above.

We say $\mathscr{E}$ is a **weak topos** if every $A \in \operatorname{ob}\mathscr{E}$ has a power-object. Similarly, we say $F : \mathscr{E} \longrightarrow \mathscr{F}$ is **weakly logical** if $F(\ni_A) \rightarrowtail F(PA) \times FA$ is a power-object for FA, for every $A \in \operatorname{ob}\mathscr{E}$.

7.5 Lemma. P is a functor $\mathscr{E}^{\mathrm{op}} \longrightarrow \mathscr{E}$. Moreover, it is self-adjoint on the right.

Proof. Given $A \xrightarrow{f} B$, we define $PB \xrightarrow{Pf} PA$ to correspond to the pullback

$$\begin{tikzcd}
E_f \arrow[r] \arrow[d, tail] & \ni_B \arrow[d, tail] \\
PB \times A \arrow[r, "1 \times f"] & PB \times B.
\end{tikzcd}$$

For any $C \xrightarrow{\ulcorner m \urcorner} PB$, it's easy to see that $(Pf)\ulcorner m \urcorner$ corresponds to $(1_C \times f)^*(m)$; hence $f \longmapsto Pf$ is functorial. For any A, B, we have a bijection between subobjects of $A \times B$ and of $B \times A$, given by composition with $(\pi_2, \pi_1) : A \times B \longrightarrow B \times A$; this yields a (natural) bijection between morphisms $A \longrightarrow PB$ and $B \longrightarrow PA$. $\square$

We write $\{\}_A : A \longrightarrow PA$ for the morphism corresponding to $A \overset{(1_A, 1_A)}{\rightarrowtail} A \times A$.

7.6 Lemma. Given $A \xrightarrow{f} B$, $\{\}_B f$ corresponds to $A \overset{(1_A, f)}{\rightarrowtail} A \times B$ and $(Pf)\{\}_B$ corresponds to $A \overset{(f, 1_A)}{\rightarrowtail} B \times A$.

Proof. The square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "(1,f)"] & B \arrow[d, tail, "(1,1)"] \\
A \times B \arrow[r, "f \times 1"] & B \times B
\end{tikzcd}$$

is a pullback. Similarly for the second assertion. $\square$

7.7 Corollary.

(i) $\{\} : A \longrightarrow PA$ is monic.

(ii) P is faithful.

Proof.

(i) If $\{\}f = \{\}g$, then $(1_A, f)$ and $(1_A, g)$ are isomorphic as subobjects of $A \times B$, which forces $f = g$.

(i) Similarly, if $Pf = Pg$ then $(Pf)\{\} = (Pg)\{\}$, so we again deduce $f = g$. $\square$

Given a monomorphism $A \xrightarrow{f} B$ in $\mathscr{E}$, we define $\exists f : PA \longrightarrow PB$ to correspond to the composite

$$\ni_A \rightarrowtail PA \times A \xrightarrow{1 \times f} PA \times B$$

Then, for any $C \xrightarrow{\ulcorner m \urcorner} PA$, $(\exists f)\ulcorner m \urcorner$ corresponds to

$$R \overset{m}{\rightarrowtail} C \times A \xrightarrow{1 \times f} C \times B.$$

So $f \longmapsto \exists f$ is a functor $\mathbf{Mono}(\mathscr{E}) \longrightarrow \mathscr{E}$.

7.8 Lemma (Beck-Chevalley condition)**.** Suppose

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, tail, "k"] & A \arrow[d, tail, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

is a pullback with f monic. Then the diagram

$$\begin{tikzcd} PA \arrow[r, "\exists f"] \arrow[d, "Ph"] & PC \arrow[d, "Pg"] \\ PD \arrow[r, "\exists k"] & PB \end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, tail] & \ni_A \arrow[d, tail] \\ PA \times D \arrow[r, "1\times h"] \arrow[d, tail, "1\times k"] & PA \times A \arrow[d, tail, "1\times f"] \\ PA \times B \arrow[r, "1\times g"] & PA \times C \end{tikzcd}$$

The lower square is a pullback, so the upper square is a pullback if and only if the composite is a pullback. $\square$

7.9 Theorem (Paré)**.** The functor $P : \mathscr{E}^{\text{op}} \longrightarrow \mathscr{E}$ is monadic.

Proof. It has a left adjoint $P : \mathscr{E} \longrightarrow \mathscr{E}^{\mathrm{op}}$ by Lemma 7.5. It's faithful by Corollary 7.7(ii) and hence reflects isomorphisms by Lemma 7.3. $\mathscr{E}^{\mathrm{op}}$ has coequalizers, since $\mathscr{E}$ has equalizers. Suppose

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l, "r"' description] \end{tikzcd}$$

is a coreflexive pair in $\mathscr{E}$; then f and g are (split) monic, and the equalizer $E \xrightarrow{e} A$ makes

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, tail, "e"] & A \arrow[d, tail, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

a pullback square since any cone over

$$\begin{tikzcd} & A \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

has both legs equal. So by Lemma 7.8 we have $(Pf)(\exists g) = (\exists e)(Pe)$; but we also have $(Pg)(\exists g) = 1_{PA}$ since

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[d, "g"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

is a pullback, and similarly $(Pe)(\exists e) = 1_{PE}$. So

$$\begin{tikzcd} PB \arrow[r, shift left, "Pf"] \arrow[r, shift right, "Pg"'] & PA \arrow[r, "Pe"] \arrow[l, bend left, "\exists g"] & PE \arrow[l, bend left, "\exists e"] \end{tikzcd}$$

is a split coequalizer, and in particular a coequalizer. Hence by Theorem 5.13, P is monadic. □

7.10 Corollary.

(i) A weak topos has finite colimits. Moreover, if it has any infinite limits, then it has the corresponding colimits.

(ii) If a weakly logical functor has a left adjoint, then it has a right adjoint.

Proof.

(i) P creates all limits which exist, by Theorem 5.8.

(ii) By definition, if F is weakly logical then

$$\begin{tikzcd} \mathscr{E}^{\mathrm{op}} \arrow[r, "F"] \arrow[d, "P"] & \mathscr{F}^{\mathrm{op}} \arrow[d, "P"] \\ \mathscr{E} \arrow[r, "F"] & \mathscr{F} \end{tikzcd}$$

commutes up to isomorphism. So this follows from Theorem 5.16.

$\square$

Lecture 23 **7.11 Lemma.** Let $\mathscr{E}$ be a category with finite limits, suppose $A \in \operatorname{ob}\mathscr{E}$ has a power object PA. Then for any B, $B^*(PA)$ is a power object for B^*A in $\mathscr{E}/B$.

Proof. Given $\begin{array}{c} C \\ \big\downarrow{\scriptstyle g} \\ B \end{array}$, we have a pullback square

$$\begin{tikzcd} C \times A \arrow[r, "g\times 1"] \arrow[d, "\pi_1"] & B \times A \arrow[d, "\pi_1"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

So $\Sigma_B(g \times B^*A) \cong C \times A$. Hence $\mathrm{Sub}_{\mathscr{E}/B}(g \times B^*A) \cong \mathrm{Sub}_{\mathscr{E}}(C \times A)$, but if $C \xrightarrow{h} PA$ corresponds to $\begin{array}{c} R \\ \big\downarrow \\ C \times A \end{array}$, then the upper square of the diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[d, tail] & & B\times \ni_A \arrow[d, tail] \\ C \times A \arrow[rr, "(g,h)\times 1_A"] \arrow[dr, "g\pi_1"'] & & B \times PA \times A \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

is a pullback. So $\begin{array}{c} B \times PA \\ \big\downarrow{\scriptstyle \pi_1} \\ B \end{array}$, equipped with $B^*(\ni_A) \rightarrowtail B^*(PA \times A)$ is a power object for B^*A. $\square$

7.12 Theorem. Suppose $\mathscr{E}$ is a weak topos. Then for any $B \in \operatorname{ob}\mathscr{E}$, $\mathscr{E}/B$ is a weak topos and $B^* : \mathscr{E} \longrightarrow \mathscr{E}/B$ is weakly logical.

Proof. The second assertion follows from Lemma 7.11. For the first, we need to construct a power object for an arbitrary $\begin{array}{c} A \\ \big\downarrow{\scriptstyle f} \\ B \end{array}$ in $\mathscr{E}/B$. Then the pullback

$$\begin{tikzcd} \Sigma_B(g \times f) \arrow[r] \arrow[d] & f \arrow[d, "f"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

is a subobject of $C \times A$, namely the equalizer of

$$C \times A \underset{g\pi_1}{\overset{f\pi_2}{\rightrightarrows}} B.$$

Define $\wedge : PA \times PA \longrightarrow PA$ to correspond to the intersection of $\pi_{13}^*(\ni_A \rightarrowtail PA \times A)$ and $\pi_{23}^*(\ni_A \rightarrowtail PA \times A)$ and define $P_1A \rightarrowtail PA \times PA$ to be the equalizer of $PA \times PA \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} PA$. Then, for any C,

$$C \xrightarrow{(\ulcorner m \urcorner, \ulcorner n \urcorner)} PA \times PA$$

factors through P_1A iff $m \leq n$ in $\mathrm{Sub}_{\mathscr{E}}(C \times A)$. Now form the pullback

$$\begin{tikzcd}
Q \arrow[rr] \arrow[d, tail, "(h,k)"] & & P_1A \arrow[d, tail] \\
PA \times B \arrow[r, "1\times\{\}"] & PA \times PB \arrow[r, "1\times Pf"] & PA \times PA.
\end{tikzcd}$$

Given any $\begin{array}{c} C \\ \downarrow g \\ B \end{array}$, morphisms $g \xrightarrow{l} k$ in $\mathscr{E}/B$ correspond to morphisms $C \xrightarrow{hl} PA$ such that the subobject named by hl is contained in that named by $(Pf)(\{\})g$. But the latter is indeed $\Sigma_B(g \times f) \rightarrowtail C \times A$. So k is a power-object for f in $\mathscr{E}/B$. □

7.13 Corollary. A weak topos is locally cartesian closed (in particular, it's a topos).

Proof. For any $f : A \longrightarrow B$ in $\mathscr{E}$, we can identify $(\mathscr{E}/B)/f$ with $\mathscr{E}/A$, and $f^* : \mathscr{E}/B \longrightarrow \mathscr{E}/A$ with pullback along f. Hence all such functors are weakly logical.

But f^* has a left adjoint Σ_f, so by Corollary 7.10(ii) it has a right adjoint Π_f. Hence by Lemma 6.3, $\mathscr{E}/B$ is cartesian closed for any B. □

Remark. It can be shown that a weakly logical functor is cartesian closed (and hence logical).

7.14 Corollary.

(i) Any epimorphism in a topos is regular.

(ii) Any $A \xrightarrow{f} B$ in a topos factors uniquely up to isomorphism as

$$A \overset{q}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} B.$$

Proof. For a topos $\mathscr{E}$, it is locally cartesian closed by Corollary 7.13 and has coequalizers by Corollary 7.10(i), so by Theorem 6.7 every f factors uniquely as regular epi + mono. If f itself is epic, then the monic part of this factorization is isomorphic by Lemma 7.3, so f is regular epic. □

*Sheaves and local operators

Recall that $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ is a full subcategory closed under limits: in fact it's reflective and the reflector $L : [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Sh}(X)$ preserves finite limits. This suggests considering reflective subcategories $\mathscr{D} \subseteq \mathscr{E}$ for which the reflector preserves finite limits (equivalently, pullbacks).

7.15 Lemma. Given such a reflective subcategory, and a monomorphism $A' \rightarrowtail A$ in $\mathscr{E}$, define $c(A') \rightarrowtail A$ by the pullback diagram

$$\begin{tikzcd} c(A') \arrow[r] \arrow[d, tail] & LA' \arrow[d, tail] \\ A \arrow[r, "\eta_A"] & LA. \end{tikzcd}$$

Then $A' \longmapsto c(A')$ is a closure operation on $\mathrm{Sub}_{\mathscr{E}}(A)$, and commutes with pullback along a fixed morphism of $\mathscr{E}$. (A **closure** operation is an order-preserving inflationary idempotent operator.)

Proof. Since

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail, "\eta_{A'}"'] & LA' \arrow[d, tail] \\ A \arrow[r, "\eta_A"] & LA. \end{tikzcd}$$

commutes, we have $A' \leq c(A')$ and $A' \leq A''$ in $\mathrm{Sub}(A)$ implies $LA' \leq LA''$ in $\mathrm{Sub}(LA)$ and hence $c(A') \leq c(A'')$. Since $L\eta$ is an isomorphism,

$$\begin{tikzcd} LA' \arrow[r] \arrow[d, tail, "L\eta_{A'}"'] & LLA' \arrow[d, tail] \\ LA \arrow[r, "L\eta_A"] & LLA. \end{tikzcd}$$

is a pullback, and since L preserves pullbacks we deduce $Lc(A') \cong LA'$ in $\mathrm{Sub}(LA)$. Hence $c(c(A')) \cong c(A')$. For stability under pullback, suppose

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback. Then in the cube

$$\begin{tikzcd} c(A') \arrow[rr] \arrow[dr] \arrow[dd] & & LA' \arrow[dr] \arrow[dd] & \\ & c(B') \arrow[rr] \arrow[dd] & & LB' \arrow[dd] \\ A \arrow[rr, "\eta_A" near end] \arrow[dr, "f"'] & & LA \arrow[dr, "Lf"] & \\ & B \arrow[rr, "\eta_B"] & & LB \end{tikzcd}$$

the front, back and right faces are pullbacks, whence the left face is too. $\square$

7.16 Definition (Local operator)**.** Let $\mathscr{E}$ be a topos. By a **local operator** on $\mathscr{E}$, we mean a morphism $j : \Omega \longrightarrow \Omega$ satisfying the commutative diagrams

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega \arrow[d, "j"] & \Omega \arrow[l] \arrow[dl, "j"] \\ & \Omega & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Omega_1 \arrow[r] \arrow[d, tail] & \Omega_1 \arrow[d, tail] \\ \Omega \times \Omega \arrow[r, "j \times j"] & \Omega \times \Omega \end{tikzcd}$$

where Ω_1 is the order-relation in Ω, defined as in Theorem 7.12.

Given a closure operator on subobjects as in Lemma 7.15, define $J \rightarrowtail \Omega$ to be the closure of $1 \overset{\top}{\rightarrowtail} \Omega$ and $j : \Omega \longrightarrow \Omega$ to be the classifying map of $J \rightarrowtail \Omega$. Then for any $A' \rightarrowtail A$ with classifying map $\chi_m : A \longrightarrow \Omega$, the composite $j\chi_m$ classifies $c(A') \rightarrowtail A$.

Lecture 24 Given a pullback-stable operator c on subobjects, we say $A' \rightarrowtail A$ is **dense** if $c(A') \rightarrowtail A$ is an isomorphism, and **closed** if $A' \rightarrowtail c(A')$ is an isomorphism.

7.17 Lemma. Suppose given a commutative square

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, tail, "n"] & A' \arrow[d, tail, "m"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

with n dense and m closed. Then there is a unique $B \xrightarrow{g} A'$ with $mg = f$ (and $gn = f'$).

Proof. We have $n \leq f^*(m)$ in $\mathrm{Sub}(B)$, so

$$1_B \cong c(n) \leq f^*(c(m)) \cong f^*(m).$$

So we define g as $B \xrightarrow{\cong} f^*(A') \longrightarrow A'$. $\square$

Note that $c(A')$ may be characterised as the unique (up to isomorphism) subobject A'' such that $A' \rightarrowtail A''$ is dense and $A'' \rightarrowtail A'$ is closed.

7.18 Lemma. Suppose c is induced as in Lemma 7.15 by a reflector $L : \mathscr{E} \longrightarrow \mathscr{D}$ preserving finite limits. Then an object A of $\mathscr{E}$ belongs to $\mathscr{D}$ (up to isomorphism) iff, given any diagram

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, tail, "m"] & A \\ B & \end{tikzcd}$$

with m dense, there exists a unique $B \xrightarrow{f} A$ such that $fm = f'$.

Proof. Note first that m is dense $\iff$ Lm is an isomorphism: $\Leftarrow$ follows from the definition; $\Rightarrow$ follows, since by the proof of Lemma 7.15, we know $L(B')$ and $L(c(B'))$ are isomorphic in $\mathrm{Sub}(B)$.

Given this, if A is in $\mathscr{D}$ then the given diagram extends uniquely to

$$\begin{tikzcd} B' \arrow[r, "\eta_{B'}"] \arrow[d, tail] & LB' \arrow[r] \arrow[d, tail, "\cong"] & A. \\ B \arrow[r, "\eta_B"] & LB \arrow[ur] & \end{tikzcd}$$

Conversely, suppose A satisfies the condition. Let

$$R \overset{a}{\underset{b}{\rightrightarrows}} A$$

be the kernel-pair of $A \xrightarrow{\eta_A} LA$ and $d : A \rightarrowtail R$ the factorisation of $(1_A, 1_A)$ through (a, b). Since $L\eta_A$ is an isomorphism and L preserves pullbacks, Ld is an isomorphism, so d is dense. This forces $a = b$, so η_A is monic. And η_A is dense, so we get a unique $r : LA \longrightarrow A$ with $r\eta_A = 1_A$. Now $\eta_A r\eta_A = \eta_A$ and since LA satisfies the condition, we have $\eta_A r = 1_{LA}$. □

We say A is a **sheaf** (for c, or for j) if it satisfies the condition in Lemma 7.18. Given a local operator j on $\mathscr{E}$, we write $\mathbf{sh}_j(\mathscr{E})$ for the full subcategory of j-sheaves in $\mathscr{E}$.

7.19 Lemma. $\mathbf{sh}_j(\mathscr{E})$ is closed under limits in $\mathscr{E}$, and is an exponential ideal.

Proof. The first assertion follows since the definition involves only morphisms with codomain A. For the second, note if $B' \overset{m}{\rightarrowtail} B$ is dense then so is $B' \times C \overset{(m,1)}{\rightarrowtail} B \times C$ for any C (since it is $\pi_1^*(m)$) so if A is a sheaf then any morphism $B' \longrightarrow A^C$ extends uniquely to a morphism $B \longrightarrow A^C$. □

7.20 Lemma. If A is a sheaf, then a subobject $A' \overset{m}{\rightarrowtail} A$ in $\mathscr{E}$ is a sheaf iff it is closed.

Proof.

- $\Leftarrow$ Immediate by Lemma 7.17.
- $\Rightarrow$ Consider

  $$A' \overset{p}{\rightarrowtail} c(A') \overset{q}{\rightarrowtail} A$$

  with p dense, so if A is a sheaf, we get a unique $r : c(A') \longrightarrow A'$ with $rp = 1_{A'}$. But $c(A')$ is a sheaf and $prp = p$, so deduce $pr = 1_{c(A')}$.

□

We define $\Omega_j \rightarrowtail \Omega$ to be the equalizer of

$$\Omega \overset{j}{\underset{1}{\rightrightarrows}} \Omega.$$

Then, for any A, morphisms $A \longrightarrow \Omega_j$ correspond to closed subobjects of A (a 'closed subobject' classifier).

7.21 Lemma. Ω_j is a j-sheaf.

Proof. We have to show that if $B \overset{m}{\rightarrowtail} A$ is dense, then the pullback along m yields a bijection from closed subobjects of A to closed subobjects of B. If $A' \overset{n}{\rightarrowtail} A$ is closed, then in the pullback

$$\begin{tikzcd} B' \arrow[r, tail, "m'"] \arrow[d, tail, "n'"] & A' \arrow[d, tail, "n"] \\ B \arrow[r, tail, "m"] & A \end{tikzcd}$$

m' is dense, so $A' \rightarrowtail A$ is the closure of $B' \rightarrowtail B \rightarrowtail A$.

It remains to show that if $B' \rightarrowtail B$ is closed, it is isomorphic to the pullback of its closure in A. But (writing $A' \rightarrowtail A$ for the closure) we have a factorisation $B' \longrightarrow f^*A'$ which is dense since $B' \longrightarrow A'$ is dense, and closed since $B' \longrightarrow B$ is closed. □

7.22 Theorem. For any local operator j on $\mathscr{E}$, $\mathbf{sh}_j(\mathscr{E})$ is a topos. Moreover, it's reflective in $\mathscr{E}$ and the reflector preserves finite limits.

Proof. We have that $\mathbf{sh}_j(\mathscr{E})$ is cartesian closed by Lemma 7.19 and has subobject classifier Ω_j by Lemma 7.20 and Lemma 7.21. To construct the reflector, consider the composite

$$f : A \overset{\{\}}{\rightarrowtail} \Omega^A \xrightarrow{j^A} \Omega_j^A.$$

This corresponds to the closure $\overline{A} \overset{(a,b)}{\rightarrowtail} A \times A$ of the diagonal subobject $A \overset{(1_A,1_A)}{\rightarrowtail} A \times A$.

Claim: $\overline{A} \overset{a}{\underset{b}{\rightrightarrows}} A$ is the kernel-pair of f (proof omitted). Hence any morphism $g : A \longrightarrow B$ for sheaf B satisfies $ga = gb$. So if we form the image

$$A \overset{q}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} \Omega_j^A$$

of f, any such g factorises uniquely through q.

Now Ω_j^A is a sheaf by Lemma 7.19 and Lemma 7.21 so if we form the closure $LA \rightarrowtail \Omega_j^A$ of m, we get a morphism $A \longrightarrow LA$, through which any morphism from A to a sheaf factors uniquely. Hence L becomes a functor $\mathscr{E} \longrightarrow \mathbf{sh}_j(\mathscr{E})$, left adjoint to inclusion.

By Lemma 6.13, we know L preserves finite products. In fact it preserves equalizers as well (omitted). □

7.23 Theorem. Let $\mathscr{E}$ be a category. The following are equivalent:

(i) $\mathscr{E}$ is a topos, complete, locally small with a separating set of objects.

(ii) $\exists$ a small category $\mathscr{C}$ and a local operator on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ such that

$$\mathscr{E} \cong \mathbf{sh}_j([\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]).$$

Proof. (ii) $\Rightarrow$ (i): since $\mathbf{sh}_j([\mathscr{C}^{\mathrm{op}}, \mathbf{Set}])$ has given properties. (i) $\Rightarrow$ (ii): take $\mathscr{C}$ the full subcategory of $\mathscr{E}$ on the separating set and consider $\mathscr{E} \xrightarrow{Y} [\mathscr{E}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. □

Index