Part III – Topics in Set Theory (Ongoing course)

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Lent 2019

Contents

0 Introduction

Lecture 1 The main 'topic in set theory' covered in this course will be one of the most important: solving the Continuum Problem. A priori, set theory does not seem intrinsically related to logic, but the continuum hypothesis showed that logic was a very important tool in set theory. In contrast to many other disciplines of mathematics, in set theory we typically try to prove things are *impossible*, rather than showing what is possible.

The second international congress of mathematicians in 1900 was in Paris, where Hilbert spoke. At that time, Hilbert was a 'universal' mathematician, and had worked in every major field of mathematics. He gave a list of problems for the century, the 23 Hilbert Problems. The first on this list was the Continuum Problem.

0.1 Continuum Hypothesis

Here is Hilbert's formulation of the Continuum hypothesis (CH): Every set of infinitely many real numbers is either equinumerous with the set of natural numbers or the set of real numbers. More formally, we might write

$$\forall X \subseteq \mathbb{R}, (X \text{ is infinite} \Rightarrow X \sim \mathbb{N} \text{ or } X \sim \mathbb{R})$$

In more modern terms, we write this as the claim $2^{\aleph_0} = \aleph_1$. These two statements are equivalent (in ZFC).

Assume that $2^{\aleph_0} > \aleph_1$, in particular $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \to \mathbb{R}$. Consider $X \coloneqq i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i|_{\aleph_1}$ is a bijection between $\aleph_1$ and X, so $X \sim \aleph_1$. But $\aleph_1 \nsim \mathbb{N}$ and $\aleph_1 \nsim \mathbb{R}$. Thus X refutes CH (in its earlier formulation). So: $2^{\aleph_0} \neq \aleph_1 \implies \neg\text{CH}$.

If $2^{\aleph_0} = \aleph_1$. Let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \to \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either $\aleph_0$, i.e. $X \sim \mathbb{N}$ or $\aleph_1$, i.e. $X \sim \mathbb{R}$. So, $2^{\aleph_0} = \aleph_1 \implies \text{CH}$.

0.2 History of CH

- 1938, Gödel: ZFC does not prove $\neg$CH.
- 1963, Cohen: ZFC does not prove CH.

Gödel's proof used the technique of inner models; Cohen's proof used forcing, sometimes referred to as outer models.

Gödel's Completeness Theorem:

$$\text{Cons}(T) \iff \exists (M, E)(M, E) \vDash T$$

From this, we might guess that Gödel's and Cohen's proof will show there is a model of ZFC + CH, and a model of ZFC + $\neg$CH, but by the incompleteness phenomenon, we cannot prove there is a model of ZFC! So, we are not going to be able to prove Cons(ZFC+CH), but instead

$$\text{Cons}(\mathsf{ZFC}) \to \text{Cons}(\mathsf{ZFC}+\text{CH})$$

Or, equivalently,

$$\text{if } M \vDash \mathsf{ZFC}, \text{ then there is } N \vDash \mathsf{ZFC} + \text{CH}.$$

1 Model theory of set theory

Let's assume for a moment that

$$(M, \in) \vDash \mathsf{ZFC}.$$

We refer to the canonical objects in M by the usual symbols, e.g., $0, 1, 2, 3, 4, \ldots, \omega, \omega + 1, \ldots$

What would an "inner model" be? Take $A \subseteq M$, and consider $(A, \in)$. This is a substructure of $(M, \in)$. Note: the language of set theory has no function or constant symbols. But we write down

$$X = \varnothing,\ X = \{Y\},\ X = \{Y, Z\},\ X = \bigcup Z,\ X = \mathcal{P}(Z)$$

which appear to use function or constant symbols. These are technically not part of the language of set theory; they are abbreviations:

$X = \varnothing$	abbreviates	$\forall w\ (\neg w \in X)$
$X = \{Y\}$	abbreviates	$\forall w\ (w \in X \leftrightarrow w = Y)$
$X \subseteq Y$	abbreviates	$\forall w\ (w \in X \to w \in Y)$

and so on.

Definition. If φ is a formula in n free variables. We say

(1) φ is **upwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (A, \in) \vDash \varphi(a_1, \ldots, a_n) \implies (M, \in) \vDash \varphi(a_1, \ldots, a_n)$$

(2) φ is **downwards absolute** between A and M if

$$\text{for all } a_1, \ldots, a_n \in A, \quad (M, \in) \vDash \varphi(a_1, \ldots, a_n) \implies (A, \in) \vDash \varphi(a_1, \ldots, a_n)$$

(3) φ is **absolute** between A and M if it is upwards absolute and downwards absolute.

Definition. We say that a formula is Σ_1 if it is of the form

$$\exists x_1 \ldots \exists x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free}$$

or Π_1 if it is of the form

$$\forall x_1 \ldots \forall x_n\ \varphi(x_1, \ldots, x_n) \text{ where } \varphi \text{ is quantifier-free.}$$

Remark.

(a) If φ is quantifier-free, then φ is absolute between A and M.

(b) If φ is Π_1, then it's downward absolute

(c) If φ is Σ_1, then it's upward absolute

Lecture 2 Under our assumption that $(M, \in) \vDash \mathsf{ZFC}$, which subsets $A \subseteq M$ give a model of ZFC? Using standard model theory, we observed that if φ is quantifier-free, then φ is absolute between $(A, \in)$ and $(M, \in)$, but hardly anything is quantifier-free:

$$x = \varnothing \iff \forall w\ (w \notin x) =: \Phi_0(x)$$

For instance, suppose $A \coloneqq M \setminus \{1\}$ (recall $0, 1, 2, \dots$ refer to the ordinals in M). In A, we have $0, 2, \{1\}$. Clearly $(M, \in) \vDash \Phi_0(0)$. $\Phi_0(x)$ is a Π_1 formula, so by Π_1-downwards absoluteness, $(A, \in) \vDash \Phi_0(0)$.

In reality, $2 = \{0, 1\}$, but 1 is not in A, so informally in A, the object 2 has only one element. Similarly, in A, $\{1\}$ has no elements, since 1 is missing from A. Thus

$$(A, \in) \vDash \Phi_0(\{1\}).$$

Clearly $(M, \in) \nvDash \Phi_0(\{1\})$, so Φ_0 is not absolute between A and M. As a corollary, we get $(A, \in) \nvDash$ Extensionality, since 0 and $\{1\}$ have the same elements in A, but are not equal.

(Remark: We could go on, defining formulas $\Phi_1(x), \Phi_2(x)$ etc. to analyse which of the elements correspond to the natural numbers in A.)

Definition. We call A **transitive** in M, if for all $a \in A$ and $x \in M$ such that $(M, \in) \vDash x \in a$, we have $x \in A$.

Proposition. If A is transitive, then Φ_0 is absolute between A and M.

Proof. Since Φ_0 is Π_1, we only need to show upwards absoluteness. Suppose $a \in A$, such that $(A, \in) \vDash \Phi_0(a)$. Suppose $a \neq 0$. Thus there is some $x \in a$. By transitivity, $x \in A$. So $(A, \in) \vDash x \in a$ and so $(A, \in) \nvDash \Phi_0(a)$, contradiction. □

(Similarly, if Φ_n is the formula describing the natural number n, and there is $a \in A$ such that $(A, \in) \vDash \Phi_n(a)$ and A is transitive, then $a = n$.)

Proposition. If A is transitive in M, then

$$(A, \in) \vDash \mathsf{Ext}.$$

Proof. Take $a, b \in A$ with $a \neq b$. By Extensionality in $(M, \in)$, find without loss of generality some $c \in a \setminus b$. Since $c \in a \in A$, by transitivity, $c \in A$. Thus

$$(A, \in) \vDash c \in a$$
$$(A, \in) \vDash c \notin b,$$

so a and b do not satisfy the assumptions of Extensionality. □

Consider now $A \coloneqq \omega + 2 \subseteq M$, the ordinal consisting of $\{0, 1, 2, \dots, \omega, \omega + 1\}$. This is a transitive subset of M (since it's an ordinal). So

$$(A, \in) \vDash \mathsf{Ext}.$$

Consider the formula $x = \mathcal{P}(y)$, which we can informally define as $x = \{z \mid z \subseteq y\}$, but this is not good enough. More properly, we try

$$\mathcal{P}(x) = \forall w\ (w \in x \leftrightarrow w \subseteq y).$$

This still includes the symbol $\subseteq$, so still needs improving.

$$\mathcal{P}(x) = \forall w\ (w \in x \leftrightarrow (\forall v\ (v \in w \to v \in y)))$$

In A, what is $\mathcal{P}(0)$?

$$(A, \in) \vDash \omega + 1 = \mathcal{P}(\omega)$$

1.1 Bounded quantification

We define

$$\exists (v \in w)\ \varphi :\Longleftrightarrow \exists v\ (v \in w \wedge \varphi)$$
$$\forall (v \in w)\ \varphi :\Longleftrightarrow \forall v\ (v \in w \to \varphi).$$

Definition. A formula φ is called Δ_0 if it is in the smallest set of formulas with the following properties

1. All quantifier-free formulas are in S.
2. If $\varphi, \psi \in S$ then so are
   (a) $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$
   (b) $\neg\varphi$
   (c) $\exists (v \in w)\ \varphi$, $\forall (v \in w)\ \varphi$.

Theorem. If φ is Δ_0 and A is transitive, then φ is absolute between A and M.

Proof. We already knew that quantifier free formulas are absolute. Absoluteness is obviously preserved under propositional connectives. So, let's deal with (2c): Let's just do

$$\varphi \mapsto \exists (v \in w)\ \varphi = \exists v\ (v \in w \wedge \varphi).$$

So suppose φ is absolute. We need to deal with downwards absoluteness.

$$(M, \in) \vDash \exists (v \in a)\ \varphi(v, a) \quad \text{for some } a \in A$$
$$(M, \in) \vDash \exists v\ (v \in a \wedge \varphi(v, a)).$$

Let's find $m \in M$ such that

$$(M, \in) \vDash m \in a \wedge \varphi(m, a).$$

Transitivity gives $m \in A$. By absoluteness of φ, we get

$$(A, \in) \vDash m \in a \wedge \varphi(m, a) \implies (A, \in) \vDash \exists (v \in a)\ \varphi(v, a). \qquad \square$$

Definition. Let T be any 'set theory'. Then we say that φ is Δ_0^T if there is a Δ_0 formula ψ such that $T \vdash \phi \leftrightarrow \psi$.

- φ is called Σ_1^T if it is T-equivalent to $\exists v_1 \dots \exists v_n\ \psi$ where ψ is Δ_0.
- φ is called Π_1^T if it is T-equivalent to $\forall v_1 \dots \forall v_n\ \psi$ where ψ is Δ_0.

Corollary. If A is transitive in M and both $(M, \in)$ and $(A, \in)$ are models of T, then Δ_0^T formulas are absolute between A and M, and Σ_1^T, (Π_1^T) formulas are upwards (downwards) absolute between A and M.

Lecture 3 **Definition.** A formula is called Δ_1^T if it is both Σ_1^T and Π_1^T.

Corollary. If A is transitive, $A, M \vDash T$ and φ is Δ_1^T, then φ is absolute between A and M.

1.2 'Set theory'

What do we mean by a 'set theory'? The usual theories we care about are:

FST_0	Extensionality Pairing Union PowerSet Separation (Aussonderung)	FST	FST_0 + Foundation (Regularity)
Z_0	FST_0 + Infinity	Z	Z_0 + Foundation
ZF_0	Z_0 + Replacement (Ersetzung)	ZF	ZF_0 + Foundation
ZFC_0	ZF_0 + Choice	ZFC	ZFC_0 + Foundation

The subscript 0 denotes the absence of Foundation.

1.3 A long list of Δ_0^T formulas

We noted earlier that there are very few Δ_0 formulas, so can we find any Δ_0^T formulas?

1. $x \in y$ (in fact, Δ_0)

2. $x = y$ (in fact, Δ_0)

3. $x \subseteq y$. This is an abbreviation, so we have to define what it means:

$$\forall w\ (w \in x \to w \in y).$$

   But note this is $\forall (w \in x)\ (w \in y)$, so Δ_0.

4.

$$\begin{aligned} \Phi_0(t) :\Longleftrightarrow&\ \forall w\ (w \notin x) \\ \Longleftrightarrow&\ \forall w\ (\neg w \in x) \\ \Longleftrightarrow&\ \forall w\ (w \in x \to \neg x = x) \end{aligned}$$

   so it's Δ_0 in predicate logic.

We say that an operation $x_1, \ldots, x_k \mapsto F(x_1, \ldots, x_n)$ is defined by a formula in class Γ (where Γ is any class of formulas) in the theory T if there is a formula $\Phi \in \Gamma$ such that

(1) $T \vdash \forall x_1 \cdots \forall x_n\ \exists y\ \Phi(x_1, \ldots, x_n, y)$

(2) $T \vdash \forall x_1 \cdots \forall x_n\ \forall y, z\ \Phi(x_1, \ldots, x_n, y) \wedge \Phi(x_1, \ldots, x_n, z) \to y = z$

(3) $\Phi(x_1, \ldots, x_n, y)$ if and only if $y = F(x_1, \ldots, x_n)$.

Example.

$$\begin{aligned} x &\mapsto \{x\} \\ x, y &\mapsto \{x, y\} \end{aligned}$$

These are operations in FST_0!

5. $x \mapsto \{x\}$. The formula to express this is

$$\begin{aligned}
`z = \{x\}\text{'} &\iff \Phi(x,z) \\
&\iff \forall w\ (w \in z \leftrightarrow w = x) \\
&\iff \forall w\ ((w \in z \to w = x) \wedge (w = x \to w \in z)) \\
&\iff \exists (w \in z)\ (w = w) \wedge \forall (w \in z)\ (w = x)
\end{aligned}$$

   So this is Δ_0 in some weak set theory.

6. $x, y \mapsto \{x, y\}$
7. $x, y \mapsto x \cup y$
8. $x, y \mapsto x \cap y$
9. $x, y \mapsto x \setminus y$
10. $x, y \mapsto (x, y)$, where $(x, y) = \{\{x\}, \{x, y\}\}$ which is the combination of earlier formulas

If two operations f, g_1, g_2 are defined by Δ_0^T-formulas, then so is the operation

$$x_1, \dots, x_n \mapsto f(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n))$$

11. $x \mapsto x \cup \{x\} =: S(x)$. By the previous fact from 5. and 7.
12. $x \mapsto \bigcup x$
13. the formula describing 'x is transitive'
14. the formula describing 'x is an ordered pair' (the quantifiers for the two components of x are bounded by $\bigcup x$)
15. $x, y \mapsto x \times y$
16. the formula 'x is a binary relation' (again, the quantifiers can be made bounded)
17. $x \mapsto \mathrm{dom}(x) := \{y | \exists p \in x; (p \text{ is an ordered pair},\ p = (v, w),\ y = v)\}$
18. $x \mapsto \mathrm{ran}(x) := \{y | \exists p \in x; (p \text{ is an ordered pair},\ p = (v, y))\}$
19. the formula 'x is a function'
20. the formula 'x is injective'
21. the formula 'x is function from A to B'
22. the formula 'x is a surjection from A to B'
23. the formula 'x is a bijection from A to B'

What is an ordinal?

Definition. α is an **ordinal** if α is a transitive set well-ordered by $\in$, i.e. it is totally ordered (several axioms, all Δ_0^T) and well-founded ($\forall X\ (X \subseteq \alpha \to X$ has a $\in$ -least element)).

Observe '(X, R) is a well-founded relation' is not obviously absolute since the bound for the $\forall Z\ (Z \subseteq X \dots)$ quantifier is the power set, so this is Π_1. However, in models with the Axiom of Foundation, well-foundedness is automatic, so α is an ordinal iff α is transitive and totally ordered by $\in$, which is Δ_0^T.

Lecture 4

24. 'x is an ordinal' is Δ_0^T (with the right choice of T)
25. 'x is a successor ordinal' $\iff$ 'x is an ordinal' +

$$\exists(y \in x)\ (y \text{ is the } \in\text{-largest element of } x)$$

26. 'x is a limit ordinal' (is an ordinal, not 0 and not a successor)
27. '$x = \omega$' (is the $\in$-minimal limit ordinal), similarly, $x = \omega + \omega$, $x = \omega + 1$, $x = \omega + \omega + 1$, $x = \omega^2$, $x = \omega^3$, $x = \omega^4$, $x = \omega^\omega$

1.4 Absoluteness of well-foundedness

If (X, R) is well-founded, we can define a rank function

$$\mathrm{rk} : X \to \alpha$$

where α is some ordinal such that rk is order-preserving between (X, R) and $(\alpha, \in)$.

This theorem is proved using the right instances of Replacement. In particular, ZF proves:

$$\begin{aligned} &(X, R) \text{ is well founded} \\ \iff &\exists\alpha\,\exists f\ \alpha \text{ is an ordinal and } f \text{ is an order-preserving function from } (X, R) \text{ to } (\alpha, \in) \end{aligned}$$

But note that the left hand side is Π_1^{ZF}, while the right hand side is Σ_1^{ZF}. Thus, for sufficiently strong T, '(X, R) is wellfounded' is Δ_1^T and hence absolute for transitive models of T. Generalise this to concepts defined by transfinite recursion.

Recall the method of transfinite recursion: Let (X, R) be well-founded. Let F be 'functional', so for every x there is unique y such that $x = F(y)$. Then there is a unique f with domain X and for all $x \in X$,

$$f(x) = F(f \restriction \mathrm{IS}_R(x))$$

where $\mathrm{IS}_R(x) \coloneqq \{z \in X \mid zRx\}$

Proposition. Let T be a set theory that is strong enough to prove the transfinite recursion theorem for F. Let F be absolute for transitive models of T. Let (X, R) be in A. Then f defined by transfinite recursion is absolute between A and M.

Example. Let $\mathscr{L}$ be any first-order language whose symbols are all in A. Then the set of $\mathscr{L}$-formulas and the set of $\mathscr{L}$-sentences are in A. (Assumptions on A are suppressed, it needs to be transitive and strong enough to prove transfinite recursion and have natural numbers, e.g. ZF).

The relation $S \vDash \varphi$ is defined by recursion and thus is absolute between A and M. So: If S is an $\mathscr{L}$-structure, $S \in A$ then

$$(A, \in) \vDash \text{“}S \vDash \varphi\text{”} \iff (M, \in) \vDash \text{“}S \vDash \varphi\text{”}$$

1.4.1 Gödel's Incompleteness Theorem

For a theory T, if T is consistent, then $T \nvDash \mathrm{Cons}(T)$. Of course, this requires some restrictions on T. Examples for strong enough T include PA, Z, ZF, ZFC, $\mathsf{ZFC} + \varphi$. In particular,

$$\mathsf{ZFC}^* \coloneqq \mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC})$$

cannot prove its own consistency.

By Gödel's Completeness Theorem,

$$\mathrm{Cons}(T) \iff \exists M\ (M \vDash T).$$

Now write β for 'there is a transitive set A such that $(A, \in) \vDash \mathsf{ZFC}$'. Clearly, $\beta \Rightarrow \mathrm{Cons}(\mathsf{ZFC})$.

Theorem. If $\mathsf{ZFC^*}$ is consistent, then $\mathsf{ZFC^*} \nvdash \beta$.

Proof. Let $(M, \in) \vDash \mathsf{ZFC^*}$. Suppose $\mathsf{ZFC} \vdash \beta$. So $(M, \in) \vDash \beta$. Thus in M, find A transitive such that $(A, \in) \vDash \mathsf{ZFC}$. By assumption, $(M, \in) \vDash \mathrm{Cons}(\mathsf{ZFC})$. By absoluteness, $(A, \in) \vDash \mathrm{Cons}(\mathsf{ZFC})$. Thus, $(A, \in) \vDash \mathsf{ZFC^*}$. So we proved $\mathrm{Cons}(\mathsf{ZFC^*})$, contradiction. □

In particular, it is stronger to say that there are transitive models of ZFC than that ZFC is consistent. Assuming β is not an obvious assumption, so we need to study under which (natural) assumptions β is true.

So, let us investigate transitive models A inside M.

1.5 Concrete transitive models of ZFC

The two most basic constructions:

1. von Neumann hierarchy (cumulative hierarchy)
2. hereditarily small sets

Lecture 5

Start with the von Neumann hierarchy.

$$\begin{aligned} V_0 &:= \varnothing \\ V_{\alpha+1} &:= \mathcal{P}(V_\alpha) \\ V_\lambda &:= \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

Proposition. V_α is transitive for all α.

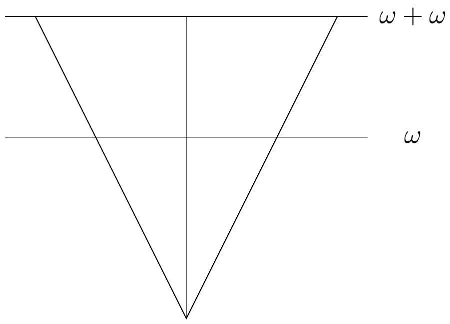
Proof sketch. Induction, the key lemma is: If X is transitive, then $\mathcal{P}(X)$ is transitive. □

Start with V_α. We know:

1. If λ is a limit ordinal, then $V_\lambda \vDash \mathsf{FST}$.
2. If $\lambda > \omega$ and λ a limit ordinal, $V_\lambda \vDash \mathsf{Z}$ (on example sheet 1).

The critical axiom here is Replacement. Take, as an example, $\lambda = \omega + \omega$. Replacement says: if $F : V_{\omega+\omega} \to V_{\omega+\omega}$ is a function definable in $V_{\omega+\omega}$ and $x \in V_{\omega+\omega}$, then

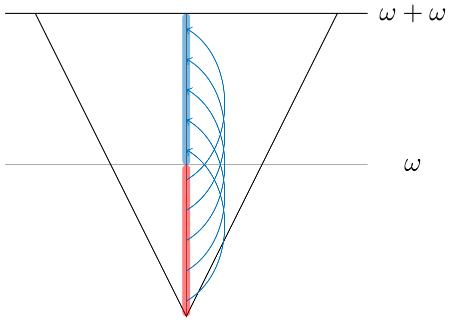
$$\{F(y) \mid y \in x\} \in V_{\omega+\omega}.$$



Idea: Take $x = \omega$, and

$$F : \begin{cases} n \mapsto \omega + n \\ y \mapsto 0 & \text{if } y \notin \omega \end{cases}$$

$\omega + n$ is definable (in Z): the unique ordinal which contains ω and $n-1$ elements above ω. Let $Y := \{F(n) \mid n \in \omega\}$. Then $Y \subseteq V_{\omega+\omega}$, but $Y \notin V_{\omega+\omega}$: it is not bounded in $V_{\omega+\omega}$.



This example shows concretely that

$$V_{\omega+\omega} \vDash \neg\mathsf{Repl}.$$

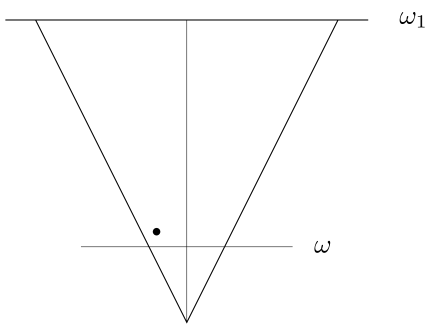
Similarly, if α is any ordinal such that there is a definable function $f : \omega \to \alpha$ such that the range of f is unbounded in α, then $V_\alpha \vDash \neg\mathsf{Repl}$. Even more generally, if $\beta < \alpha$ and a definable function $f : \beta \to \alpha$ with unbounded image, then $V_\alpha \vDash \neg\mathsf{Repl}$.

Definition. We call a cardinal κ **regular** if there is no partition

$$\kappa = \bigcup_{i \in I} A_i$$

such that $|I|, |A_i| < \kappa$ for all $i \in I$. Equivalently, for every $\alpha < \kappa$, there is no unbounded function $f : \alpha \to \kappa$.

We know, e.g. that $\aleph_1$ is regular. Moreover, for any α, $\aleph_{\alpha+1}$ is regular. So our next candidate is $\alpha = \aleph_1$. $\mathcal{P}(\omega) \in V_{\omega+2} \subseteq V_{\omega+1}$.



Clearly, there is a surjection

$$s : \mathcal{P}(\omega) \to \omega_1.$$

so the range of s is unbounded in ω_1. Thus: $V_{\omega_1} \vDash \neg\mathsf{Repl}$.

Definition. A cardinal α is called **inaccessible** if

(a) κ is regular

(b) $\forall \lambda < \kappa$, $|\mathcal{P}(\lambda)| < \kappa$.

That is, just take the two problems we had, negate them and make a definition.

Remark. We know every successor cardinal is regular, and our simple examples of limit cardinals are all not regular: they were defined as unions. So, we can ask: 'Are there regular limit cardinals?' (Such cardinals are sometimes called weakly inaccessible). Under GCH ($\forall \kappa\ 2^\kappa = k^+$), we have:

$$\kappa \text{ is inaccessible} \iff \kappa \text{ is a regular limit cardinal.}$$

Let's assume that $\kappa > \alpha$ is inaccessible.

Lemma.

$$\forall \lambda < \kappa \quad |V_\lambda| < \kappa.$$

Proof. Clearly $|V_\omega| = \aleph_0$, so $|V_\omega| < \kappa$.

By induction, suppose $|V_\lambda| < \kappa$. Then $V_{\lambda+1} = \mathcal{P}(V_\lambda)$.

$$|V_{\lambda+1}| = |\mathcal{P}(V_\lambda)| < \kappa$$

by (b).

Now let $\lambda < \kappa$ be a limit ordinal. Then

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

Suppose for contradiction that $|V_\lambda| = \kappa$. But $|V_\alpha| < \kappa$ for all $\alpha < \kappa$, so you can write κ as a union of λ many things of smaller cardinality. This contradicts regularity. □

Theorem. If κ is inaccessible, then $V_\kappa \vDash \mathsf{Repl}$.

Proof. Take any $F : V_\kappa \to V_\kappa$ and any $x \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$. Thus, find $\alpha \in \kappa$ such that $x \in V_\alpha$. Since V_α is transitive, $x \subseteq V_\alpha$. So $|x| \le |V_\alpha| < \kappa$ (by the lemma).

Now consider $X := \{F(y) \mid y \in x\}$. For each $y \in x$, consider $\rho(F(y)) :=$ least α such that $F(y) \in V_{\alpha+1} \setminus V_\alpha$. By assumption, $\rho(F(y)) < \kappa$. Consider $\{\rho(F(y)) | y \in x\} =: R$, then $|R| \le |x| < \kappa$. By regularity, $\alpha := \bigcup R < \kappa$. But $\forall y \in x\ F(y) \in V_{\alpha+1}$. So $X \subseteq V_{\alpha+1}$, $X \in V_{\alpha+2}$. This proves Replacement. □

Note we didn't even use that F was definable: we showed a statement stronger than Replacement. As a consequence, we get that the existence of inaccessible cardinals cannot be proved in ZFC.

Lecture 6 Write IC for the axiom 'there is an inaccessible cardinal'. If κ is inaccessible, then $V_\kappa \vDash \mathsf{ZFC}$. V_κ is a transitive model of ZFC, so,

$$\mathsf{ZFC} + \mathsf{IC} \vdash \underbrace{\text{'there is a transitive set that is a model of } \mathsf{ZFC}\text{'}}_{\beta}$$

Recall that $\mathsf{ZFC} + \mathrm{Cons}\,\mathsf{ZFC} \nvdash \beta$, so $\mathsf{ZFC} + \mathrm{Cons}\,\mathsf{ZFC} \nvdash \mathsf{IC}$.

1.5.1 Model-theoretic reminders

1. Löwenheim-Skolem theorem: If S is any structure in some countable first-order language $\mathcal{L}$ and $X \subseteq S$ is any subset, then there is a **Skolem hull** of X in S, $X \subseteq \mathcal{H}^S(X) \subseteq S$ such that

   (a) $\mathcal{H}^S(X) \preccurlyeq S$ Recall $\preccurlyeq$ means elementary substructure, meaning that

   $$\forall \varphi\ \forall h_1, \dots, h_n \in \mathcal{H}^S(X), \quad \mathcal{H}^S(X) \vDash \varphi(h_1, \dots, h_n) \iff S \vDash \varphi(h_1, \dots, h_n)$$

   (b) $|\mathcal{H}^S(X)| \leq \max(\aleph_0, |X|)$

   Proof sketch. The key ingredient for this theorem is the Tarski-Vaught criterion, which says that for $Z \subseteq S$, we have $Z \preccurlyeq S$ iff for every φ and all $z_1, \dots, z_n$,

   $$S \vDash \exists x\ \varphi(x, z_1, \dots, z_n) \implies Z \vDash \exists x\ \varphi(x, z_1, \dots, z_n).$$

   Observe there are $\max(\aleph_0, |X|)$ many possible $\varphi(x, z_1, \dots, z_n)$, so for each formula which we need to satisfy, take a witness in S and add it into X. But this introduces new z_i, so we need to add more witnesses, so repeat this process and take a union. Specifically,

   $$\begin{aligned} Z_0 &:= X \\ Z_1 &:= Z_0 \cup \text{the witnesses for all tuples } \varphi, z_1, \dots, z_n \text{ where } z_1, \dots, z_n \in Z_0 \\ Z_{n+1} &:= Z_n \cup \text{the witnesses for all tuples } \varphi, z_1, \dots, z_n \text{ where } z_1, \dots, z_n \in Z_n \\ Z &:= \bigcup_{n \in \mathbb{N}} Z_n. \end{aligned}$$

   Z is the required model.

   Now, work in $\mathsf{ZFC} + \mathsf{IC}$. Suppose $(M, \in) \vDash \mathsf{ZFC} + \mathsf{IC}$, which contains $V_\kappa \vDash \mathsf{ZFC}$ ($V_\kappa \subseteq M$). Apply Löwenheim-Skolem to V_κ with $X := \varnothing$. Then

   $$H := \mathcal{H}^{V_\kappa}(\varnothing) \preccurlyeq V_\kappa$$

   and $\mathcal{H}^{V_\kappa}(\varnothing)$ has cardinality $\leq \aleph_0$, and $H \vDash \mathsf{ZFC}$.

   There is a formula φ such that $\varphi(x)$ iff x is the least uncountable cardinal. We have $V_\kappa \vDash \exists x \varphi(x)$, but the only element that satisfies φ in V_κ is $\aleph_1$. So in the Skolem hull construction,

   $$\aleph_1 \in Z_1 \subseteq H$$

   This implies H can't be transitive, since $\aleph_1$ has uncountably many elements, but H has only countably many.

2. Mostowki Collapse Theorem: If X is any set and $R \subseteq X \times X$ such that R is well-founded and extensional, then there is a transitive set T such that $(T, \in) \cong (X, R)$.

   Consider $(H, \in) \vDash \mathsf{ZFC}$. Since $(H, \in) \vDash \mathsf{ZFC}$, $\in$ is extensional on H. Since $\in$ (in M) is well-founded, $\in$ is well-founded on H. So, let T be the Mostowski collapse of H: T is transitive and

   $$(T, \in) \cong (H, \in).$$

   But this is an isomorphism, so $(T, \in) \vDash \mathsf{ZFC}$. It is a bijection also, so $|T| = |H| \leq \aleph_0$.

   Together: there is a countable transitive model of ZFC.

Notice that

$$\begin{aligned}\varphi(x) &\coloneqq \text{`x is countable'}\\ &= \exists f\ (f : x \to \mathbb{N}, f \text{ is injective})\end{aligned}$$

is Σ_1^{ZFC}, so is upwards absolute. But this formula is not downwards absolute: If $\alpha \in \mathrm{Ord}$, $\alpha \in T$ then $V_\kappa \models \alpha$ is countable. But since $(T, \in) \models \mathsf{ZFC}$, there is some $\alpha \in T$ such that $(T, \in) \models \alpha$ is uncountable, so V_κ and T disagree about the truth value of $\varphi(\alpha)$.

Consider now

$$\begin{aligned}\psi(x) &\coloneqq \text{`x is a cardinal'}\\ &= \forall\alpha\ (\alpha < x \to \text{there is no injection from } x \text{ to } \alpha).\end{aligned}$$

This is Π_1^{ZFC}. In $(T, \in)$, take α least such that $(T, \in) \models\neq \varphi(\alpha)$. Then $(T, \in) \models \alpha$ is a cardinal. Clearly, $V_\kappa \models \alpha$ is not a cardinal.

Note that if λ is an uncountable cardinal in V_κ, then $\lambda \notin T$, so the downwards absoluteness of ψ is not very interesting.

Instead of building $\mathcal{H}^{V_\kappa}(\varnothing)$, build $H^* \coloneqq \mathcal{H}^{V_\kappa}(\omega_1 + 1)$. Clearly $\omega_1 \in H$ and $\omega_1 \subseteq H$, so $\omega_1 \subseteq T^*$ and $\omega_1 \in T^*$. We have $|H^*| = \aleph_1$. Now we have $V_\kappa \models \omega_1$ is a cardinal, so by downwards absoluteness of ψ, so $T^* \models \omega_1$ is a cardinal.

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