Part II – Galois Theory

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0 Introduction

The primary motivation of this course is to study the solutions of polynomial equations in one variable to wander whether there is a formula involving roots, a solution by radicals. Quadratics were typically studied in school, while the solution in radicals for cubics and quartics has been known for a long time and studied in particular in 1770 by Lagrange.

In 1799, Ruffini claimed that there were some quintics that can't be solved by radicals, that is, there is no general formula, but it took until 1824 before Abel used existing ideas about permutations to produce the first accepted proof of insolubility, before dying in 1829. Galois' main contribution was in 1831, when he gave the first explanation as to why some polynomials are soluble by radicals and others are not. He made use of the group of permutations of the roots of a polynomial, and realised in particular the importance of *normal* subgroups.

Galois' work was not known generally in his lifetime - it was only published by Liouville in 1846, who realised that it tied in well with the work of Cauchy on permutations. Galois had submitted his work for various competitions and for entry into the Ecole Polytechnique in Paris. Unfortunately Galois died in a duel in 1832, leaving a six and a half page letter indicating his thoughts about the future development of his theory.

0.1 Course overview

Most of this course is Galois Theory, but presented in a more modern fashion - in terms of field extensions. Recall from GRM that if $f(t)$ is an irreducible polynomial in $k[t]$ where k is a field, then $k[t]/(f(t))$ is a field, where $(f(t))$ denotes the ideal of $k[t]$ generated by $f(t)$, and this new field contains k. In this way, we can see the field $k[t]/(f(t))$ as a field extension of k.

Prerequisites Quite a lot of the Groups, Rings and Modules course, but no modules except in one place where it's useful to know the structure of finite abelian groups. The DPMMS website has a Galois Theory page with a long history of example sheets and notes, in particular see Tony Scholl's 2013-4 course page.

1 Field Extensions

Definition 1.1 (Field extension)**.** A **field extension** $K \leq L$ is the inclusion of a field K into another field L with the same 0, 1, and where the restriction of $+$ and $\cdot$ (in L) to K gives the $+$ and $\cdot$ of K.

Example.

(i) $\mathbb{Q} \leq \mathbb{R}$

(ii) $\mathbb{R} \leq \mathbb{C}$

(iii) $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2}) = \{ \lambda + \mu\sqrt{2} \mid \lambda, \mu \in \mathbb{Q} \}$

(iv) $\{ \lambda + \mu i \mid \lambda, \mu \in \mathbb{Q} \} = \mathbb{Q}(i) \leq \mathbb{C}$

Suppose $K \leq L$ is a field extension. Then L is a K-vector space using the addition from the field structure and scalar multiplcation given by the multiplication in the field L.

Definition 1.2 (Degree)**.** The **degree** of L over K is $\dim_K L$, the K-vector space dimension of L. This may not be finite. We typically denote this by $|L : K|$. If $|L : K| < \infty$, then the extension is **finite**, otherwise the extension is **infinite**.

Example.

(i) $|\mathbb{C} : \mathbb{R}| = 2$, with $\mathbb{R}$-basis $1, i$

(ii) $|\mathbb{Q}(i) : \mathbb{Q}| = 2$, with $\mathbb{Q}$-basis $1, i$

(iii) $\mathbb{Q} \leq \mathbb{R}$ is an infinite extension.

Theorem 1.3 (Tower law)**.** Suppose $K \leq L \leq M$ are field extensions. Then $|M : K| = |M : L|\,|L : K|$.

Proof. Assume that $|M : L| < \infty$, and $|L : K| < \infty$. Take an L-basis of M, given by $\{ f_1, \ldots, f_b \}$, and a K-basis of L given by $\{ e_1, \ldots, e_a \}$. Take $m \in M$, so $m = \sum_{i=1}^{b} \mu_i f_i$ for some $\mu_i \in L$. Similarly, $\mu_i = \sum_{j=1}^{a} \lambda_{ij} e_j$ for some $\lambda_{ij} \in K$, so

$$m = \sum_{i=1}^{b} \sum_{j=1}^{a} \lambda_{ij} e_j f_i$$

Thus $\{ e_j f_i \mid 1 \leq j \leq a, 1 \leq i \leq a \}$ span M.

Linear independence: It's enough to show that if $0 = m = \sum\sum \lambda_{ij} e_j f_i$ then λ_{ij} are all zero. However if $m = 0$ the linear independence of f_i forces each $\mu_i = 0$. Then the linear indepedence of e_j forces λ_{ij} all to be zero, as required. $\square$

The Tower law will not be proved for infinite extensions, but observe that if M is an infinite extension of L then it is an infinite extension of K, and similarly if L is an infinite extension of K then the larger field M must also be an infinite extension of K.

1.1 Motivatory Example

Observe $\mathbb{Q} \leq \mathbb{Q}\left(\sqrt{2}\right) \leq \mathbb{Q}\left(\sqrt{2}, i\right)$

(i) $\mathbb{Q}\left(\sqrt{2}\right)$ has basis $1, \sqrt{2}$ over $\mathbb{Q}$.

(ii) $\mathbb{Q}\left(\sqrt{2}, i\right)$ has basis $1, i$ as a $\mathbb{Q}\left(\sqrt{2}\right)$–vector space.

(iii) $\mathbb{Q}\left(\sqrt{2}, i\right)$ has basis $1, \sqrt{2}, i, i\sqrt{2}$ over $\mathbb{Q}$.

$$\left|\mathbb{Q}\left(i, \sqrt{2}\right) : \mathbb{Q}\right| = 4 = 2 \cdot 2 = \left|\mathbb{Q}\left(i, \sqrt{2}\right) : \mathbb{Q}\left(\sqrt{2}\right)\right| \left|\mathbb{Q}\left(\sqrt{2}\right) : \mathbb{Q}\right|$$

Any intermediate field strictly between $\mathbb{Q}$ and $\mathbb{Q}\left(\sqrt{2}, i\right)$ must be of degree 2 by the Tower law.

What are these intermediate fields? There are $\mathbb{Q}\left(\sqrt{2}\right)$, $\mathbb{Q}(i)$ and $\mathbb{Q}\left(i\sqrt{2}\right)$, but are these all?

The Galois correspodence arising in the Fundamental Theorem of Galois theory gives an order reversing bijection between the lattice of intermediate subfields and the subgroups of a group of ring automorphisms of the big field (in this case $\mathbb{Q}\left(i, \sqrt{2}\right)$) that fix the smaller field elementwise. For instance, consider the ring automorphisms of $\mathbb{Q}\left(i, \sqrt{2}\right)$ that fix $\mathbb{Q}$:

$$\begin{aligned} e : \sqrt{2} &\longmapsto \sqrt{2} \\ i &\longmapsto i \\ g : \sqrt{2} &\longmapsto \sqrt{2} \\ i &\longmapsto -i \\ h : \sqrt{2} &\longmapsto -\sqrt{2} \\ i &\longmapsto i \\ gh : \sqrt{2} &\longmapsto -\sqrt{2} \\ i &\longmapsto -i \end{aligned}$$

Notice that i and $-i$ play the same role in the field $\mathbb{Q}\left(\sqrt{2}, i\right)$, both roots of $t^2 + 1 = 0$, similarly $\sqrt{2}$ and $-\sqrt{2}$ are both roots of $t^2 - 2 = 0$. The automorphism e is seen to be identity, and g is conjugation. These four form the group of order $4 = \left|\mathbb{Q}\left(\sqrt{2}, i\right) : \mathbb{Q}\right|$.

$$\begin{tikzcd}[column sep=small] & \mathbb{Q}\left(i, \sqrt{2}\right) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbb{Q}\left(\sqrt{2}\right) \arrow[dr, dash] & \mathbb{Q}(i) \arrow[d, dash] & \mathbb{Q}\left(i\sqrt{2}\right) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \{e\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \{e, g\} \arrow[dr, dash] & \{e, h\} \arrow[d, dash] & \{e, gh\} \arrow[dl, dash] \\ & \{e, g, h, gh\} & \end{tikzcd}$$

The recipe for producing an intermediate subfield from a subgroup is to take the elements of $\mathbb{Q}\left(i, \sqrt{2}\right)$ which are fixed by all elements of the subgroup. For instance, $\mathbb{Q}\left(i\sqrt{2}\right)$ is the field of elements fixed by both e and gh.

This correspondence doesn't always work for all finite field extensions. It works for Galois extensions. In the correspondence, normal extensions correspond to normal subgroups. In this example, all subgroups are normal and the extensions are normal. We'll also prove the Primitive Element Theorem, which in the context of finite extensions of $\mathbb{Q}$ tells us that they are necessarily of the form $\mathbb{Q}(\alpha)$ for some α, for instance $\mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(i + \sqrt{2})$.

1.2 Review of GRM

Definition 1.4 (Algebraic)**.** Suppose $K \leq L$ is a field extension. Take $\alpha \in L$ and define

$$I_\alpha = \{\, f \in K[t] \mid f(\alpha) = 0 \,\}$$

We say α is **algebraic** over K if $I_\alpha \neq 0$. Otherwise α is **transcendental**. We say L is algebraic over K if α is algebraic over K for all $\alpha \in L$.

Remark. We can see I_α is an ideal of $K[t]$ since it is the kernel of the ring homomorphism $K[t] \to L$ given by $f(t) \mapsto f(\alpha)$.

Example.

(i) $\sqrt{2}$ is algebraic over $\mathbb{Q}$

(ii) π is not algebraic over $\mathbb{Q}$

Lemma 1.5. Let $K \leq L$ be a finite field extension. Then L is algebraic over K.

Proof. Let $|L : K| = n$, and take $\alpha \in L$. Consider $1, \alpha, \alpha^2, \ldots, \alpha^n$, which must be linearly dependent in the n-dimensional K–vector space L. So, $\sum_{i=0}^{n} \lambda_i \alpha^i = 0$ for some $\lambda \in K$ not all zero, and hence α is a root of $f(t) = \sum_{i=0}^{n} \lambda_i t^i$, so α is algebraic over K. α was arbitrary, so L is algebraic over K. $\square$

Definition 1.6 (Minimal polynomial)**.** The non-zero ideal I_α (where α is algebraic over K) is principal since $K[t]$ is a principal ideal domain. In particular, we can say $I_\alpha = (f_\alpha(t))$ where $f_\alpha(t)$ can be assumed to be monic. Such a monic $f_\alpha(t)$ is the **minimal polynomial** of α over K.

Remark. Multiplication by α within the field L gives a K–linear map $L \to L$, an automorphism (if $\alpha \neq 0$). In GRM, we have seen the minimal polynomial of a linear map is unique.

Example.

(i) The minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $t^2 - 2$.

(ii) The minimal polynomial of $\sqrt{2}$ over $\mathbb{R}$ is $t - \sqrt{2}$.

Lemma 1.7. Suppose $K \leq L$ is a field extension, $\alpha \in L$ and α is algebraic over K. Then the minimal polynomial $f_\alpha(t)$ of α over K is irreducible in $K[t]$ and I_α is a prime ideal.

Proof. Suppose $f_\alpha(t) = p(t)q(t)$. We aim to show $p(t)$ or $q(t)$ is a unit in $K[t]$. But $0 = f_\alpha(\alpha) = p(\alpha)q(\alpha)$, so $p(\alpha) = 0$ or $q(\alpha) = 0$, without loss of generality take $p(\alpha) = 0$, thus $p(t) \in I_\alpha$.

But $I_\alpha = (f_\alpha(t))$, so $p(t) = f_\alpha(t)r(t)$, giving $f_\alpha(t) = f_\alpha(t)r(t)q(t)$ and so $r(t)q(t) = 1$ in $K[t]$, and $q(t)$ is a unit, as required. Recall from GRM that irreducible elements of $K[t]$ are prime and hence generate prime ideals of $K[t]$. So I_α is a prime ideal. □

Definition 1.8 (Simple extension)**.** Suppose $K \leq L$ is a field extension and $\alpha \in L$. $K(\alpha)$ is defined to be the smallest subfield of L containing K and α. It's called the field **generated** by K and α. We say that L is a **simple extension** if $L = K(\beta)$ for some $\beta \in L$.

Given $\alpha_1, \ldots, \alpha_n \in L$, $K \leq L$. $K(\alpha_1, \ldots \alpha_n)$ is the smallest field containing $\alpha_1, \ldots, \alpha_n$. It is the field generated by K and $\alpha_1, \ldots, \alpha_n$.

On the other hand $K[\alpha]$ is the ring generated by K and α, in particular the image of $K[t]$ under the map $f(t) \mapsto f(\alpha)$.

Theorem 1.9. Suppose $K \leq L$ is a field extension and $\alpha \in L$ is algebraic over K. Then

(i) $K(\alpha) = K[\alpha]$

(ii) $|K(\alpha) : K| = \deg f_\alpha(t)$ where $f_\alpha(t)$ is the minimal polynomial of α over K.

Proof.

(i) Clearly $K[\alpha] \leq K(\alpha)$. We aim to show that any non-zero element β of $K[\alpha]$ is a unit, so $K[\alpha]$ is a field.

By definition of $K[\alpha]$, we have $\beta = g(\alpha)$ for some $g(t) \in K[t]$. Since $\beta = g(\alpha) \neq 0$, $g(t) \notin I_\alpha = (f_\alpha(t))$. Thus $f_\alpha(t) \nmid g(t)$.

From Lemma 1.7, $f_\alpha(t)$ is irreducible and $K[t]$ is a PID, we know $\exists r(t), s(t) \in K[t]$ with

$$r(t)f_\alpha(t) + s(t)g(t) = 1 \in K[t].$$

Hence $s(\alpha)g(\alpha) = 1$ in $K[\alpha]$, and so $\beta = g(\alpha)$ is a unit, as required.

(ii) Let $n = \deg f_\alpha(t)$ We'll show that $T = \{ 1, \alpha, \alpha^2, \ldots, \alpha^{n-1} \}$ is a K–vector space basis of $K[\alpha]$.

Spanning: If $f_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ with $a_i \in K$, then $\alpha^n = -a_{n-1}\alpha^{n-1} - \cdots - a_0$. This implies α^n is a linear combination of $\{ 1, \alpha, \alpha^2, \ldots, \alpha^{n-1} \}$, and an easy induction shows that α^m for $m \geq n$ is likewise a linear combination of $\{ 1, \alpha, \alpha^2, \ldots, \alpha^{n-1} \}$, so we have spanning.

Linear independence: Suppose $\lambda_{n-1}\alpha^{n-1} + \ldots + \lambda_0 = 0$. Let $g(t) = \lambda_{n-1}t^{n-1} + \ldots + \lambda_0$. Since $g(\alpha) = 0$, we have $g(t) \in I_\alpha = (f_\alpha(t))$. So $g(t) = 0$ or $f_\alpha(t) \mid g(t)$. The latter is not possible since $\deg f_\alpha(t) > \deg g_\alpha(t)$ so $g(t) = 0$ in $K[t]$ and all the λ_i's are zero. □

Corollary 1.10. If $K \leq L$ is a field extension and $\alpha \in L$, then α is algebraic over K if and only if $K \leq K(\alpha)$ is finite.

Proof.

($\Rightarrow$) By Theorem 1.9, $|K(\alpha) : K| = \deg f_\alpha(t) \leq \infty$.

($\Leftarrow$) Lemma 1.5

□

Corollary 1.11. Let $K \leq L$ be a field extension with $|L : K| = n$. Let $\alpha \in L$, then $\deg f_\alpha(t) \mid n$.

Proof. Use the Tower law on $K \leq K(\alpha) \leq L$. We deduce that $|K(\alpha) : K|$ divides $|L : K|$. Theorem 1.9(ii) gives $\deg f_\alpha(t) = |K(\alpha) : K|$. □

1.3 Digression on (Non-)Constructibility

Schedules mention 'other classical problems' and we are now in a position to tackle some of these using Corollary 1.11.

A classical question from Greek geometry concerns the existence or otherwise of constructions using ruler and compasses (where a ruler refers to a single unmarked straight edge). If you're an expert you can divide a line betwen 2 points into arbitrarily many equal segments, you can bisect an angle, or you can produce parallel lines. Given a polygon you can produce a square of the same area or double the area. However,

1. You cannot duplicate the cube using ruler and compasses (given a cube you can't produce a cube of double the volume)
2. You cannot trisect the angle $\pi/3$ using ruler and compasses.
3. The circle cannot be squared using ruler and compasses (given a circle you can't construct a square of the same area)

Assume we're given a set P_0 of points in $\mathbb{R}^2$, and we can formalise our operations.

Ruler operation Draw a straight line through any two points in P_0.

Compass operation Draw a circle with the centre being a point in P_0 and radius the distance between a pair of points in P_0.

Definition 1.12 (Constructible)**.** The points of intersection of any two distinct lines or circles drawn using these operations are **constructible in one step** from P_0. More generally, a point $\mathbf{r} \in \mathbb{R}^2$ is **constructible** from P_0 if there is a finite sequence $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n} = \mathbf{r}$ such that $\mathbf{r_i}$ is constructible in one step from $P_0 \cup \{\mathbf{r_1}, \ldots, \mathbf{r_{i-1}}\}$.

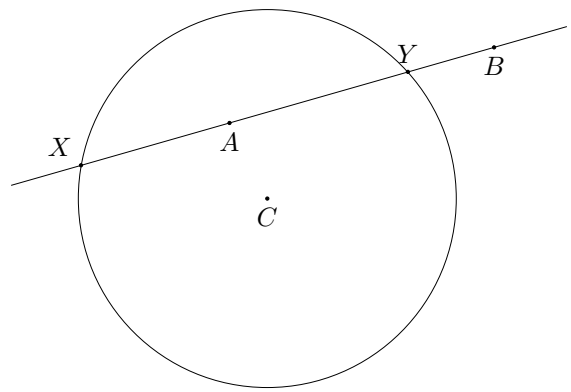
Exercise. Construct the midpoint of a line between two points.

Let K_0 be the subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and the co-ordinates of the points in P_0. Let $\mathbf{r_i} = (x_i, y_i)$ and set $K_i = K_{i-1}(x_i, y_i)$.

Thus $K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq \mathbb{R}$.

Lemma 1.13. x_i, y_i are both roots in K_i of quadratic polynomials in $K_{i-1}[t]$.

Proof. There are three cases for $\mathbf{r_i}$: line meets line, line meets circle, circle meets circle. We do the second case only here.



The line is defined by two points $A = (p, q)$ and $B = (r, s)$ while the circle is defined with a centre $C = (t, u)$ and radius w. Then, points X and Y satisfy the equation of the line $\frac{x-p}{r-p} = \frac{y-q}{s-q}$, and the equation of the circle $(x-t)^2 + (y-u)^2 = w^2$. Solving these together gives coordinates of X and Y satisfying quadratic polynomials over K_{i-1}. The other two cases are left as an exercise for the reader. □

Theorem 1.14. If $\mathbf{r} = (x, y)$ is constructible from a set P_0 of points in $\mathbb{R}^2$ and if K_0 is the subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and the coordinates of the points in P_0, then the degrees $|K_0(x) : K_0|$ and $|K_0(y) : K_0|$ are powers of two.

Proof. Continue with the previous notation of $K_i = K_{i-1}(x_i, y_i)$. By the Tower law,

$$|K_i : K_{i-1}| = |K_{i-1}(x, y) : K_{i-1}(x)|\,|K_{i-1}(x) : K_{i-1}|$$

But Lemma 1.13 tells us that $|K_{i-1}(x) : K_{i-1}|$ must be 1 or 2 depending on whether the quadratic polynomial arising in the lemma is reducible or not, using Theorem 1.9(ii). Similarly, $|K_{i-1}(x, y) : K_{i-1}(x)|$ is 1 or 2.

So $|K_i : K_{i-1}| = 1, 2$ or 4, (but in fact 4 cannot happen), hence by the Tower law, $|K_n : K_0| = |K_n : K_{n-1}|\,|K_{n-1} : K_{n-2}| \ldots |K_1 : K_0|$ is a power of two.

If $r = (x, y)$ is constructible from P_0, then

$$x, y \in K_n \quad \text{and} \quad \begin{aligned} K_0 \le K_0(x) \le K_n \\ K_0 \le K_0(y) \le K_n \end{aligned}$$

and the Tower Law again gives that $|K_0(x) : K_0|$ and $|K_0(y) : K_0|$ are also powers of 2. □

To use this for proofs about non-constructibility we need to be reasonably expert at working out minimal polynomials.

Theorem 1.15. Let $f(t)$ be a primitive integral polynomial. Then $f(t)$ is irreducible in $\mathbb{Q}[t]$ if and only if it is irreducible in $\mathbb{Z}[t]$.

Proof. A special case of Gauss' lemma from GRM. □

Theorem 1.16 (Eisenstein's criterion)**.** Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0 \in \mathbb{Z}[t]$. Suppose there is a prime p such that

(i) $p \nmid a_n$

(ii) $p \mid a_{n-1}$, $p \mid a_{n-2}, \ldots, p \mid a_0$

(iii) $p^2 \nmid a_0$

Then $f(t)$ is irreducible in $\mathbb{Z}[t]$

Proof. Recall from GRM. □

Example. For p a prime, consider $f(t) = t^{p-1} + t^{p-2} + \cdots + 1$. This is irreducible over $\mathbb{Q}$ by considering $f(t+1)$ and using p as the prime in Eisenstein's criterion.

Another method is to consider an integral polynomial $f(t) \pmod p$. If $f(t)$ is irreducible in $\mathbb{Z}[t]$ then it is reducible over $\mathbb{Z}/p\mathbb{Z}$. So, if we find a prime p such that $f(t) \pmod p$ is irreducible then $f(t)$ is irreducible in $\mathbb{Z}[t]$.

Example. $t^3 + t + 1$ is irreducible mod 2. If it were reducible it would have a linear factor and so the polynomial would have a root mod 2. But $0, 1$ are not roots. So, $t^3 + t + 1$ is irreducible in $\mathbb{Z}[t]$, hence irreducible in $\mathbb{Q}[t]$.

Remark. On a later example sheet you'll meet an irreducible polynomial in $\mathbb{Z}[t]$ which is reducible mod p for all primes p.

Theorem 1.17. The cube cannot be duplicated by ruler and compasses.

Proof. The problem amounts to whether given a unit distance, one can construct points distance α apart, where α satisfies $t^3 - 2 = 0$. Starting with points $P_0 = \{(0,0), (1,0)\}$ can we produce $(\alpha, 0)$?

No. If we could, Theorem 1.14 would say $|\mathbb{Q}(\alpha) : \mathbb{Q}|$ is a power of 2. But $|\mathbb{Q}(\alpha) : \mathbb{Q}| = 3$ since $|\mathbb{Q}(\alpha) : \mathbb{Q}| = \deg f_\alpha(t)$ where $f_\alpha(t)$ is the minimal polynomial of α over $\mathbb{Q}$. α satisfies $t^3 - 2$, which is irreducible over $\mathbb{Z}$ by Eisenstein's criterion hence irreducible over $\mathbb{Q}$. So $t^3 - 2$ is the minimal polynomial $f_\alpha(t)$. □

Theorem 1.18. The circle cannot be squared using ruler and compasses.

Proof. Starting with $(0,0)$ and $(1,0)$, we must construct $(\sqrt{\pi}, 0)$ so that we have a square of side length $\sqrt{\pi}$ and hence area π. But π and hence $\sqrt{\pi}$ is transcendental over $\mathbb{Q}$ (Lindemann - not proved here). Theorem 1.14 tells us we can't do this construction. □

1.4 Return to theory development

Lemma 1.19. Let $K \leq L$ be a field extension. Then

(i) $\alpha_1, \ldots, \alpha_n \in L$ are algebraic over K if and only if $K \leq K(\alpha_1, \ldots \alpha_n)$ is a finite field extension.

(ii) If $K \leq M \leq L$ such that $K \leq M$ is finite, then there exist $\alpha_1, \ldots \alpha_n \in L$ such that $K(\alpha_1, \ldots \alpha_n) = M$.

Proof.

(i) By Corollary 1.10, α is algebraic over K if and only if $K \leq K[\alpha]$ is a finite field extension. α_i is algebraic over K and hence algebraic over $K(\alpha_1, \dots \alpha_{i-1})$ and so

$$|K(\alpha_1, \dots, \alpha_i) : K(\alpha_1, \dots, \alpha_{i-1})| < \infty.$$

By the Tower law applied to

$$K \leq K(\alpha_1) \leq K(\alpha_1, \alpha_2) \leq \dots \leq K(\alpha_1, \dots \alpha_n),$$

we get $|K(\alpha_1, \dots \alpha_n) : K| < \infty$.

Conversely, consider $K \leq K(\alpha_1) \leq K(\alpha_1, \dots \alpha_n)$. Then the tower law says that if $|K(\alpha_1, \dots, \alpha_n) : K| < \infty$ then $|K(\alpha_1) : K| < \infty$ and by Corollary 1.10, α is algebraic over K.

(ii) If $|M : K| = n$ then M is an n-dimensional K-vector space, so there exists a K-basis $\alpha_1, \dots, \alpha_n$ over M. Then $K(\alpha_1, \dots, \alpha_n) \leq M$. However, any element of M is a K-linear combination of $\alpha_1, \dots, \alpha_n$ and so lies in $K(\alpha_1, \dots, \alpha_n)$, so $M = K(\alpha_1, \dots, \alpha_n)$. □

Definition 1.20 (Homomorphism over a field)**.** Suppose $K \leq L$, $K \leq L'$ are field extensions. A K-**homomorphism** $\phi : L \to L'$ is a ring homomorphism such that $\phi|_K = \mathrm{id}$.

A K-homomorphism is a K-isomorphism if it is a ring isomorphism.

Lemma 1.21. Suppose $K \leq L$, $K \leq L'$ are field extensions. Then

(i) Any K-homomorphism $\phi : L \to L'$ is injective and $K \leq \phi(L)$ is a field extension.

(ii) If $|L : K| = |L' : K| < \infty$ then any K-homomorphism $\phi : L \to L'$ is a K-isomorphism.

Proof.

(i) L is a field and $\ker \phi$ is an ideal of L.

Note $1 \mapsto 1$ and so $\ker \phi$ can't be the whole of L, hence $\ker \phi = \{0\}$. So $\phi(L)$ is a field and $K \leq \phi(L)$ is a field extension.

(ii) ϕ is an injective K-linear map, so $|\phi(L) : K| = |L : K|$. In general, $|\phi(L) : K| \leq |L' : K|$, but since $|L : K| = |L' : K|$ by assumption, we have $|\phi(L) : K| = |L' : K|$, hence $\phi(L) = L'$ and ϕ is a K-isomorphism $L \to L'$. (If $L' = L$ then ϕ would be a K-automorphism also.) □

Definition 1.22 (Splitting)**.** Let $K \leq L$ be a field extension and $f(t) \in K[t]$. We say f **splits over** L if

$$f(t) = a(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)$$

where $a \in K$ and $\alpha_1, \dots, \alpha_n \in L$.

We say L is a **splitting field for** f **over** K if $L = K(\alpha_1, \dots, \alpha_n)$.

Remark. This is equivalent to saying that L is a splitting field for f over K iff

(i) f splits over L

(ii) if $K \leq M \leq L$ and f splits over M then $M = L$.

Example.

1. Consider $f(t) = t^3 - 2$ over $\mathbb{Q}$.

   $\mathbb{Q}(\sqrt[3]{2})$ is *not* a splitting field for f over $\mathbb{Q}$, but $\mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$ is a splitting field over $\mathbb{Q}$ where ω is a primitive cube root of unity $\omega = e^{2\pi i/3}$

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt[3]{2}, \omega) \arrow[dl, dash, "\leq 2"'] \arrow[dr, dash, "\leq 3"] & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[dr, dash, "3"'] & & \mathbb{Q}(\omega) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

   Starting in the bottom right, going clockwise:

   - The minimal polynomial of ω over $\mathbb{Q}$ is $t^2 + t + 1$, so $|\mathbb{Q}(\omega) : \mathbb{Q}| = 2$.
   - $t^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion, so $\left|\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\right| = 3$.
   - ω satisfies t^2+t+1 over $\mathbb{Q}$ and hence over $\mathbb{Q}(\sqrt[3]{2})$ and so $\left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\sqrt[3]{2})\right| \leq 2$.
   - $\sqrt[3]{2}$ satisfies $t^3 - 2$ over $\mathbb{Q}$ and hence over $\mathbb{Q}(\omega)$, so $\left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)\right| \leq 3$.

   Then by the Tower law, $2 \mid \left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}\right|$ and $3 \mid \left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}\right|$, so we deduce

$$\begin{aligned}
\left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)\right| &= 3 \\
\left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\sqrt[3]{2})\right| &= 2
\end{aligned}$$

2. Take $f(t) = (t^2 - 3)(t^3 - 1)$. The splitting field for f over $\mathbb{Q}$ is

$$\begin{aligned}
&\mathbb{Q}(\sqrt{3}, -\sqrt{3}, \omega, \omega^2, 1) \\
=&\mathbb{Q}(\sqrt{3}, \omega) = \mathbb{Q}(\sqrt{3}, i)
\end{aligned}$$

   since $\omega = \frac{-1+i\sqrt{3}}{2}$.

3. $t^2 - 3$ and $t^2 - 2t - 2$ have the same splitting field over $\mathbb{Q}$: $\mathbb{Q}(\sqrt{3})$.

4. Take $f(t) = t^2 + t + 1$ in $\mathbb{F}_2[t]$.

   $f(t)$ is irreducible over $\mathbb{F}_2$ since it has no roots in $\mathbb{F}_2$ and hence no linear factors in $\mathbb{F}_2[t]$. Hence, $\mathbb{F}_2[t]/(t^2 + t + 1)$ is a field.

   Set $\alpha = t + (t^2 + t + 1) \in \mathbb{F}_2[t]/(t^2 + t + 1)$, then

$$\frac{\mathbb{F}_2[t]}{(t^2 + t + 1)} = \mathbb{F}_2[\alpha].$$

The elements are $0, 1, \alpha, 1+\alpha$ noting that $\alpha^2 + \alpha + 1 = 0$ and we have characteristic 2, so $\alpha^2 = \alpha + 1$.

$f(t)$ splits over $\mathbb{F}_2[\alpha]$:

$$f(t) = (t - \alpha)(t - 1 - \alpha)$$

Thus $\mathbb{F}_2[\alpha]$ is the splitting field for f over $\mathbb{F}_2$.

We can use this final construction to produce splitting fields in general.

Theorem 1.23 (Existence of splitting fields)**.** Let K be a field and $f(t) \in K[t]$. Then there exists a splitting field for f over K.

Proof. If $\deg f = 0$ then K is the splitting field for f over K.

Suppose $\deg f > 0$ and pick an irreducible factor $g(t)$ of $f(t)$ in $K[t]$, noting that $K \leq K[t]/(g(t))$ is a field extension.

Take

$$\alpha_1 = t + (g(t)) \in K[t]/(g(t)),$$

then $K[t]/(g(t)) = K(\alpha_1)$ and $g(\alpha_1) = 0$ in $K(\alpha_1)$. Therefore $f(\alpha_1) = 0$ in $K(\alpha_1)$ and we can write $f(t) = (t - \alpha_1)h(t)$ in $K(\alpha_1)[t]$.

Repeat, noting that $\deg h(t) < \deg f(t)$ and so we get

$$f(t) = a(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)$$

where a is a constant in K. Thus, we have a factorisation of $f(t)$ in $K(\alpha_1, \ldots, \alpha_n)[t]$, and so $K(\alpha_1, \ldots, \alpha_n)$ is a splitting field for f over K. □

Theorem 1.24 (Uniqueness of splitting fields)**.** If K is a field and $f(t) \in K[t]$, then the splitting field for f over K is unique up to K-isomorphism, that is, if there are two such splitting fields L and L', there is a K-isomorphism $\phi : L \to L'$.

Proof. Suppose L and L' are splitting fields for $f(t) \in K[t]$ over K. We need to show that there is a K-isomorphism $L \to L'$.

Suppose $K \leq M \leq L$ and there exist M' with $K \leq M' \leq L'$ and a K-isomorphism $\psi : M \to M'$. Clearly some M exists (we can take $M = K$), so we pick M so that $|M : K|$ is maximal among all such M, M', ψ.

We must show $M = L$ and $M' = L'$. Note that if $M = L$ then $f(t)$ splits over M:

$$f(t) = a(t - \alpha_1) \cdots (t - \alpha_n) \in M[t]$$

Apply ψ, we get an induced map $M[t] \to M'[t]$.

$$f(t) = \psi(f(t)) = \psi(a)(t - \psi(\alpha_1)) \cdots (t - \psi(\alpha_n))$$

Thus $f(t)$ splits over $\psi(M) = M'$. But L' is a splitting field and $M' \leq L'$, so $M' = L'$.

So, suppose $M \neq L$ and we'll get a contradiction of maximality of M. Since $M \neq L$, there is a root α of $f(t)$ in L which isn't in M. Factorise $f(t) = g(t)h(t)$ in $M[t]$ so that $g(t)$ is irreducible in $M[t]$ while $g(\alpha) = 0$ in L. Then there exists a K-homomorphism $M[t]/(g(t)) \to L$ given by $t + (g(t)) \mapsto \alpha$ which has image $M(\alpha)$.

The K-isomorphism $M[t] \to M'[t]$ induced by ψ maps $g(t) \in M[t]$ to $\gamma(t) \in M'[t]$. $f(t) = g(t)h(t)$ in $M[t]$ yields $f(t) = \gamma(t)\delta(t)$ in $M'[t]$.

We have a field extension $M' \leq M'[t]/(\gamma(t))$ and there exists a M'-homomorphism $M'[t]/(\gamma(t)) \to L'$ given by $t + (\gamma(t))$ by picking a root α' of $\gamma(t)$ in L'. However $\gamma(t) \mid f(t)$ in $M'[t]$ and hence in $L'[t]$ and so α' is also a root of $f(t)$ in L'. The M'-homomorphism gives a K-isomorphism

$$M'[t]/(\gamma(t)) \to M'(\alpha')$$

and so we have a K-isomorphism $M(\alpha) \to M'(\alpha')$. This contradicts the maximality of M, since $M \lneqq M(\alpha)$. □

Definition 1.25 (Normal extension)**.** A field extension $K \leq L$ is **normal** if for every $\alpha \in L$ the minimal polynomial $f_\alpha(t)$ of α over K splits over L.

Theorem 1.26. Let $K \leq L$ be a finite field extension. Then $K \leq L$ is normal $\iff$ L is the splitting field for some $f(t) \in K[t]$.

Proof. Later. □

Example 1.27. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = m$. $\mathbb{F}$ has characteristic p for some p and $\mathbb{F}_p \leq \mathbb{F}$, therefore $m = p^r$ for some r. The non-zero elements form the multiplicative group of order $m - 1 = n$ and they satisfy $t^n - 1$: they are roots of $t^n - 1$. So,

$$t^n - 1 = (t - \alpha_1) \cdots (t - \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are the non-zero elements of $\mathbb{F}$. Thus $\mathbb{F}$ is the splitting field for $t^n - 1$ over $\mathbb{F}_p$. By Theorem 1.24, any other field with m elements is $\mathbb{F}_p$-isomorphic to $\mathbb{F}$. (We'll show later that there is a field of m elements.)

Theorem 1.28. Let G be a finite subgroup of the multiplicative group of a field K. Then G is cyclic. In particular, the multiplicative group of a finite field is cyclic.

Proof. Let $|G| = n$. By the structure theorem of finite abelian groups from GRM,

$$G \cong C_{q_1^{m_1}} \times C_{q_2^{m_2}} \times \cdots \times C_{q_r^{m_r}}$$

with q_i prime, not necessarily distinct. However if $q = q_i = q_j$ for some $i \neq j$, there are at least q^2 distinct solutions of $t^q - 1 = 0$ in K (since $C_q \times C_q \cong$ subgroup of G). But in a field (or even an integral domain), a polynomial of degree q has at most q roots, a contradiction. So all the q_i are distinct and hence G is cyclic, generated by $(g_1, \ldots, g_r)$ where g_i generates $C_{q_i^{m_i}}$ using the Chinese Remainder Theorem. □

2 Separable, normal and Galois extensions

Definition 2.1 (Separable polynomial)**.** Let K be a field and $f(t) \in K[t]$. Suppose $f(t)$ is irreducible in $K[t]$ and L is a splitting field for $f(t)$ over K. Then $f(t)$ is **separable** over K if $f(t)$ has no repeated roots in L.

For general $f(t)$ we say $f(t)$ is separable over K if every irreducible factor in $K[t]$ is separable over K.

All constant polynomials are separable.

Definition 2.2 (Formal differentiation)**.** If K is a field then **formal differentiation**

$$\begin{aligned} D : K[t] &\to K[t] \\ t^n &\mapsto nt^{n-1} \end{aligned}$$

is a K-linear map. We denote this by $D(f(t)) = f'(t)$.

Lemma 2.3. Let K be a field and $f(t), g(t) \in K[t]$. Then:

(a) $D(f(t)g(t)) = f'(t)g(t) + f(t)g'(t)$ (Leibniz' rule)

(b) Assume $f(t) \neq 0$. Then $f(t)$ has a repeated root in a splitting field L if and only if $f(t)$ and $f'(t)$ have a common irreducible factor in $K[t]$.

Proof.

(a) D is a K-linear map and so we only need to check for $f(t) = t^n$, $g(t) = t^m$. Left as an exercise.

(b) Let α be a repeated root in a splitting field L, then

$$\begin{aligned} f(t) &= (t-\alpha)^2 g(t) \in L[t] \\ f'(t) &= (t-\alpha)^2 g'(t) + 2(t-\alpha)g(t) \end{aligned}$$

and so $f'(\alpha) = 0$. Therefore the minimal polynomial $f_\alpha(t)$ of α in $K[t]$ divides both $f(t)$ and $f'(t)$ and thus $f_\alpha(t)$ is a common irreducible factor of $f(t)$ and $f'(t)$.

Conversely, let $h(t)$ be a common irreducible factor of $f(t)$ and $f'(t)$ in $K[t]$. Pick a root α in L of $h(t)$.

So $f(\alpha) = 0 = f'(\alpha)$, thus $f(t) = (t-a)g(t)$ in $L[t]$, and $f'(t) = (t-a)g'(t) + g(t)$. Since $f'(\alpha) = 0$ we have $(t-a) \mid f'(t)$. and so $(t-a) \mid g(t)$. Hence $(t-a)^2 \mid f(t)$ and we have a repeated root. □

Corollary 2.4. If K is a field and $f(t) \in K[t]$ is irreducible:

(i) If the characteristic of K is 0, then $f(t)$ is separable over K.

(ii) If the characteristic of K is greater than 0, then $f(t)$ is not separable if and only if $f(t) \in K[t^p]$.

Proof. By Lemma 2.3, $f(t)$ is not separable over K if and only if $f(t)$ and $f'(t)$ have a common irreducible factor. Since we're assuming $f(t)$ is irreducible, this is equivalent to saying $f'(t) = 0$.

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$$
$$f'(t) = n a_n t^{n-1} + \cdots + a_1$$

Thus $f'(t) = 0 \iff i a_i = 0$ for all $i > 0$.

(i) If $\operatorname{char} K = 0$ then $f'(t) \neq 0$ for any non-constant polynomial, so $f(t)$ is separable over K.

(ii) If $\operatorname{char} K = p > 0$ then if $f'(t) = 0$ we have $i a_i = 0$ for all $i > 0$, so $f(t)$ is not separable $\iff f(t) \in K[t^p]$. □

Definition 2.5 (Separable extension)**.** We say $\alpha \in L$ **is separable over** K if its minimal polynomial is separable over K.

L **is separable over** K if all $\alpha \in L$ are separable over K.

If $f_\alpha(t) = (t-\alpha)^n = t^n - \alpha^n$ where n is a power of $p (= \operatorname{char} K)$, we say that α is **purely inseparable over** K.

Example.

(1) Let $\mathbb{Q} \subseteq L$ be an algebraic field extension. Then L is separable over $\mathbb{Q}$.

(2) Let $L = \mathbb{F}_p(X)$ be the rational functions in X over $\mathbb{F}_p$, and $K = \mathbb{F}_p(X^p)$. Then $K \leq L$ is not separable. Observe that if $f(t) = t^p - X^p \in K[t]$ then $f'(t) = 0$. But $t^p - X^p = (t - X)^p$ in $L[t]$ since the binomial expansion gives other terms divisible by p and hence 0 since the characteristic is p.

However $f(t)$ is irreducible in $K[t]$: Suppose $f(t) = g(t)h(t)$ in $K[t]$ and hence in $L[t]$. So we get $g(t) = (t - X)^r$ for some $0 \leq r < p$ if the factorisation is non-trivial. But this would mean X^r was in K. However, $\exists$integers a, b such that $ar + bp = 1$. So $(X^r)^a (X^p)^b \in K$ and so $X \in K$, thus we'd have $X = \frac{u(X^p)}{v(X^p)}$, a contradiction.

Thus $f(t) = t^p - X^p$ is the minimal polynomial of X over K, thus X is purely inseparable over K and $K \leq L$ is not separable.

(3) Take $\mathbb{F}$ a finite field with order m, a power of p where $\operatorname{char} \mathbb{F} = p$. Consider $f(t) = t^n - 1$ where $n = m - 1$. This is separable over $\mathbb{F}_p$ since we saw that $f(t)$ has distinct linear factors in $\mathbb{F}[t]$.

Remark. It's useful to have an alternative approach to separability of field extensions without having to check separability of minimal polynomials for all elements of the larger field. This is where we start thinking about K-homomorphisms.

Lemma 2.6. Let $M = K(\alpha)$, where α is algebraic over K and let $f_\alpha(t)$ be the minimal polynomial of α over K.

Then, for any field extension $K \leq L$, the number of K-homomorphisms of M to L is equal to the number of distinct roots of $f_\alpha(t)$ in L. Thus this number is $\leq \deg f_\alpha(t) = |K(\alpha) : K| = |M : K|$.

$$\begin{tikzcd}
 & & L \\
M = K(\alpha) \arrow[dr, dash] & & \\
 & K \arrow[uur, dash] &
\end{tikzcd}$$

Proof. We saw in Lemma 1.21 that any K-homomorphism $M \to L$ is injective, and we have

$$K(\alpha) \cong \frac{K[t]}{(f_\alpha(t))}.$$

For any root β of $f_\alpha(t)$ in L we can define a K-homomorphism

$$\begin{aligned} \frac{K[t]}{(f_\alpha(t))} &\to L \\ t + (f_\alpha(t)) &\mapsto \beta \end{aligned}$$

Thus we get a K-homomorphism $M \to L$.

Conversely, for any K-homomorphism $\phi : M \to L$ the image $\phi(\alpha)$ must satisfy

$$f_\alpha(\phi(\alpha)) = 0.$$

These processes are inverse to each other, giving a 1-1 correspondence

$$\{K \text{ homomorphisms } M \to L\} \longleftrightarrow \{\text{roots of } f_\alpha(t) \in L\}.$$ □

Example. Take $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{2})$, and use $\alpha = \sqrt[3]{2}$, with minimal polynomial $f_\alpha(t) = t^3 - 2$.

Then, there is only one K-homomorphism $M = K(\sqrt[3]{2}) = L \to L$, the identity map.

Corollary 2.7. The number of K-homomorphisms $K(\alpha) \to L = \deg f_\alpha(t) \iff L$ is large enough, in particular L contains a splitting field for $f_\alpha(t)$ and α is separable over K.

Proof. Immediate from Lemma 2.6. □

Lemma 2.8. Let $K \leq M$ be a field extension and $M_1 = M(\alpha_1)$ (where α_1 is algebraic over M). Let $f(t)$ be the minimal polynomial of α_1 over M and let $K \leq L$. Let $\phi : M \to L$ be a K-homomorphism. Then there is a correspondence

$$\{\text{Extensions } \phi_1 : M_1 \to L \text{ of } \phi\} \longleftrightarrow \{\text{roots of } \phi(f(t)) \in L\}.$$

$$\begin{tikzcd}
 & & L \\
M_1 \arrow[dr, dash] & & \\
 & M \arrow[dr, dash] & \\
 & & K \arrow[uuur, dash]
\end{tikzcd}$$

Remark. Lemma 2.6 is the special case $M = K$ and $\phi =$ inclusion of K in L.

Proof. $f(t)$ is irreducible in $M[t]$, so $\phi(f(t))$ is irreducible in $\phi(M)[t]$. Any extension $\phi_1 : M \to L$ of ϕ produces a root $\phi_1(\alpha_1)$ of $\phi(f(t))$.

Conversely, given a root γ of $\phi(f(t))$ in L,

$$M_1 = M(\alpha_1) \cong \frac{M[t]}{(f(t))} \cong \frac{\phi(M)[t]}{(\phi(f(t)))} \cong \phi(M)(\phi) \leq L.$$

Thus we get an extension ϕ_1 of ϕ as required. □

Corollary 2.9. If L is large enough, the number of ϕ_1 which extend ϕ is equal to the number of distinct roots of $f(t)$ in L. This is equal to $|M_i : M| \iff \alpha$ is separable over M.

Proof. Immediate from Lemma 2.8. □

Corollary 2.10. Let $K \leq M \leq N$ be finite field extensions, $K \leq L$. Let $\phi : M \to L$ be a K-homomorphism. Then the number of extensions of ϕ to maps $\theta : N \to L$ is $\leq |N : M|$. Moreover, such a θ exists if L is large enough.

Proof. Pick $\alpha_1, \ldots, \alpha_r$ so that $N = M(\alpha_1, \ldots, \alpha_r)$ and set $M_i = M(\alpha_1, \ldots, \alpha_i)$. Then we've got

$$M \leq M_1 \leq M_2 \leq \cdots \leq M_r = N.$$

Using Lemma 2.8, there are

$$\begin{aligned}
&\leq |M_1 : M| \text{ extensions } \phi_1 : M_1 \to L \text{ of } \phi \\
&\leq |M_2 : M_1| \text{ extensions } \phi_2 : M_2 \to L \text{ of } \phi \\
&\vdots \\
&\leq |M_r : M_{r-1}| \text{ extensions } \phi_r : M_r \to L \text{ of } \phi
\end{aligned}$$

By the Tower law, the number of extensions $\theta : N \to L$ (recall $N = M_r$) of $\phi : M \to L$ is

$$\leq |M_r : M_{r-1}|\,|M_{r-1} : M_{r-2}| \cdots |M_1 : M| = |N : M|$$

where the last part comes from the proof of Lemma 2.8 - we need L to contain roots. □

Remark 2.11. This proof together with Corollary 2.9 shows that the number of extensions θ of ϕ is equal to $|N : M|$ iff L is large enough (so that everything splits) and α_i is separable over $M(\alpha_1, \ldots, \alpha_{i-1})$.

Lemma 2.12. Let $K \leq N$ be a field extension with $|N : K| = n$ and $N = K(\alpha_1, \ldots, \alpha_r)$ say. Then the following are equivalent:

(i) N is separable over K.

(ii) Each α_i is separable over $K(\alpha_1, \ldots, \alpha_n)$.

(iii) If $K \leq L$ is large enough there are exactly n distinct K-homomorphisms $N \to L$.

Proof. (i) $\Rightarrow$ (ii). N is separable over $K \implies \alpha_i$ is separable over K. The minimal polynomial of α_i over $K(\alpha_1, \ldots, \alpha_{i-1})$ divides the minimal polynomial of α_i over K (in $K(\alpha_1, \ldots, \alpha_{i-1})[t]$).

So if the latter has distinct roots in a splitting field then the former does. So α_i separable over $K \implies \alpha_i$ separable over $K(\alpha_1, \ldots, \alpha_{i-1})$.

(ii) $\Rightarrow$ (iii) follows from Remark 2.11.

(iii) $\Rightarrow$ (i). Assume (iii) is false and (i) true, aiming for a contradiction. So, $\exists \beta \in N$ that is not separable over K, so there are $\lneqq |K(\beta) : K|$ K-homomorphisms $\phi : K(\beta) \to L$ by Corollary 2.7.

By Corollary 2.10, ϕ extends to $\leq |N : K(\beta)|$ extensions $\theta : N \to L$, and so there are $\lneqq |N : K(\beta)|\,|K(\beta) : K|$ K-homomorphisms $N \to L$, contradiction. □

Definition 2.13 (Separably generated)**.** We say $M = K(\alpha_1, \ldots, \alpha_r)$ is **separably generated** by $\alpha_1, \ldots, \alpha_r$ over K if each α_i is separable over K.

Corollary 2.14. A finite extension is separable $\iff$ it is separably generated.

Proof. Lemma 2.12. □

Lemma 2.15. If $K \leq M \leq L$ finite field extensions, $M \leq L$, then

$$K \leq M, \quad M \leq L \text{ are both separable} \iff K \leq L \text{ is separable}$$

Proof. Example sheet. □

Example 2.16. Take $\mathbb{F}$ a finite field, $|\mathbb{F}| = m$. Multiplicative group of order $n = m - 1$ is cyclic. Take a generator α, then $\mathbb{F} = \mathbb{F}_p(\alpha)$, and so the minimal polynomial of α divides $t^n - 1$. Observe this has distinct roots, which are all of $\mathbb{F} \setminus \{0\}$ since $\alpha^n = 1$. So the minimal polynomial of α is separable, and $\mathbb{F} = \mathbb{F}_p(\alpha)$ is separable over $\mathbb{F}_p$.

Theorem 2.17 (Primitive Element Theorem)**.** Any finite separable extension $K \leq M$ is a simple extension, that is, $M = K(\alpha)$ for some α, called a primitive element.

Proof. First deal with the case where K is a finite field. Then M is also finite and we can take α to be a generator of the multiplicative group of M, which is cyclic.

Now assume K is an infinite field.

Since $K \leq M$ is a finite extension, $M = K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some α_i. It is enough to show that any field $M = K(\alpha, \beta)$ with β separable over K is of the form $K(\gamma)$.

Take $f(t)$ and $g(t)$ to be the minimal polynomials of α and β over K and let L be the splitting field for $f(t)g(t)$ over $K(\alpha, \beta)$. Say the distinct zeros of $f(t)$ in L are $\alpha = \alpha_1, \ldots, \alpha_a$ and of $g(t)$ are $\beta = \beta_1, \ldots, \beta_b$.

By separability, $b = \deg g(t)$. Choose $\lambda \in K$ such that all $\alpha_i + \lambda\beta_j$ are distinct, which is possible since K is infinite. Set $\gamma = \alpha + \lambda\beta$.

Let $F(t) = f(\gamma - \lambda t) \in K(\gamma)[t]$. We have $g(\beta) = 0$ and $F(\beta) = f(\alpha) = 0$. Thus $F(t)$ and $g(t)$ have a common zero.

Any other common zero would have to be β_j for some $j > 1$. But then $F(\beta_j) = f(\alpha + \lambda(\beta - \beta_j))$. By assumption, $\alpha + \lambda(\beta - \beta_j)$ is never an α_i and so $F(\beta_j) \neq 0$. Separability of $g(t)$ says its linear factors are all distinct, so $(t - \beta)$ is a highest common factor of $F(t)$ and $g(t)$ in $L[t]$.

However the minimal polynomial $h(t)$ of β over $K(\gamma)$ then divides $F(t)$ and $g(t)$ in $K(\gamma)[t]$ and hence in $L[t]$. This implies $h(t) = t - \beta$ and so $\beta \in K(\gamma)$. Therefore $\alpha = \gamma - \lambda\beta \in K(\gamma)$ and so $K(\alpha, \beta) \subset K(\gamma)$ and equality holds since $\gamma \in K(\alpha, \beta)$. □

Example. From our example in (link) chapter 1, $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2}, i)$, we had intermediate subfields.

$$\begin{tikzcd}
 & \mathbb{Q}\left(i, \sqrt{2}\right) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}\left(\sqrt{2}\right) \arrow[dr, dash] & \mathbb{Q}(i) \arrow[d, dash] & \mathbb{Q}\left(i\sqrt{2}\right) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

If we follow the procedure of the proof of Theorem 2.17, $\alpha = \sqrt{2}$, $\beta = i$, $f(t) = t^2 - 2$, $g(t) = t^2 + 1$. Consider $\sqrt{2} + \lambda i$ where $\pm\sqrt{2} \pm \lambda i$ are distinct. For instance $\lambda = 1$, so $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

2.1 Trace and Norm

Definition 2.18 (Trace and norm)**.** Let $K \leq M$ be a finite field extension, and $\alpha \in M$. Multiplication by α gives a K-linear map $\theta_\alpha : M \to M$.

Then we define

Trace of α over K is given by $\mathrm{Tr}_{M/K}(\alpha) = \text{trace of } \theta_\alpha \in K$.

Norm of α over K is given by $N_{M/K}(\alpha) = \text{determinant of } \theta_\alpha \in K$.

Note these are dependent on the field extension.

Theorem 2.19. With the above notation, suppose $f_\alpha(t) = t^s + a_{s-1}t^{s-1} + \cdots + a_0$ is the minimal polynomial for α over K. Let $r = |M : K(\alpha)|$, then the characteristic polynomial of θ_α is $(f_\alpha(t))^r$.

Note

$$|M : K| = |M : K(\alpha)|\,|K(\alpha) : K| = rs.$$

Then $\mathrm{Tr}_{M/K}(\alpha) = -ra_{s-1}$ and $N_{M/K} = ((-1)^s a_0)^r$.

Proof. Regard M as a $K(\alpha)$-vector space with basis $1 = \beta_1, \ldots, \beta_r$. Now take the K-vector space basis $1, \alpha, \alpha^2, \ldots, \alpha^{s-1}$ of $K(\alpha)$. So, $1, \alpha, \alpha^2, \ldots, \alpha^{s-1}, \beta_2, \beta_2\alpha, \ldots, \beta_2\alpha^{s-1}, \beta_3, \ldots$ is a K-vector space basis for M. Multiplication by α in $K(\alpha)$ is represented by matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ 0 & 0 & 1 & \ldots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -a_{s-1} \end{pmatrix}$$

an $s \times s$ matrix whose characteristic polynomial is $f_\alpha(t)$.

Multiplication by α in M is represented by the $rs \times rs$ matrix

$$\begin{pmatrix} \mathbf{A} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{A} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{A} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{A} \end{pmatrix}$$

whose characteristic polynomial is $(f_\alpha(t))^r$.

Look at the terms of this characteristic polynomial to get the trace and norm. □

Theorem 2.20. Let $K \leq M$ be a finite separable field extension and $|M : K| = n$, $\alpha \in M$. Let $K \leq L$ be large enough so that there are n distinct K-homomorphisms

$$\sigma_1, \sigma_2, \dots, \sigma_n : M \longrightarrow L.$$

Then the characteristic polynomial of $\theta_\alpha : M \to M$ (the multiplication map) is

$$\prod_{i=1}^{n} (t - \sigma_i(\alpha))$$

hence

$$\mathrm{Tr}_{M/K}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha) \qquad \text{and} \qquad N_{M/K}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

Proof. Write

$$\begin{aligned} f_\alpha(t) &= (t - \alpha_1) \dots (t - \alpha_s) \in L[t] \\ &= t^s + a_{s-1} t^{s-1} + \dots + a_0 \end{aligned}$$

the minimal polynomial of α over K (where L large enough implies $f_\alpha(t)$ splits in L). There are s K-homomorphisms $K[\alpha] \to L$ corresponding to maps sending α to α_i.

Each of these extends in $|M : K(\alpha)|$ ways to give K-homomorphisms $M \to L$ (by separability and Corollary 2.9).

However each of these extensions of a map sending $\alpha \to \alpha_i$ still sends $\alpha \to \alpha_i$. Set $r = |L : K(\alpha)|$. Thus there are r maps sending $\alpha \to \alpha_i$ for each i. Thus if the $n (= rs)$ distinct K-homomorphisms $M \to L$ are $\sigma_1, \dots, \sigma_n$, then

$$\begin{aligned} \sum_{i=1}^{n} \sigma_i(\alpha) &= r(\alpha_1 + \alpha_2 + \dots + \alpha_s) = -r a_{s-1} = \mathrm{Tr}_{M/K}(\alpha) \\ \prod_{i=1}^{n} \sigma_i(\alpha) &= ((-1)^s a_0)^n = N_{M/K}(\alpha). \end{aligned}$$

□

Theorem 2.21. Let $K \leq M$ be a finite separable extension. Then we define a K-bilinear form

$$\begin{aligned} T : M \times M &\to K \\ (x, y) &\longmapsto \mathrm{Tr}_{M/K}(xy). \end{aligned}$$

Then this is non-degenerate and in particular the K-linear map $\mathrm{Tr}_{M/K} : M \to K$ is non-zero, and hence surjective.

Remark. If $K \leq M$ is a finite extension which is not separable then $\mathrm{Tr}_{M/K} : M \to K$ is always zero and so $T : M \times M \to K$ is degenerate (see example sheet).

Proof. Separability and finiteness give $M = K(\alpha)$ for some α, by Theorem 2.17. We have a K-basis $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ of $K(\alpha)$ where $n = |M : K|$. The K-bilinear form is represented by

$$A = \begin{pmatrix} \mathrm{Tr}_{M/K}(1) & \mathrm{Tr}_{M/K}(\alpha) & \ldots \\ \mathrm{Tr}_{M/K}(\alpha) & \mathrm{Tr}_{M/K}(\alpha^2) & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Let L be the splitting field of the minimal polynomial $f_\alpha(t)$ of α over K.

Thus $f_\alpha(t) = (t - \alpha_1) \cdots (t - \alpha_n)$ with $\alpha_1, \ldots, \alpha_n \in L$. The entries in A are of the form $\mathrm{Tr}_{M/K}(\alpha^e)$ which is $\alpha_1^e + \cdots + \alpha_n^e$ using Theorem 2.20.

Now consider $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)$, the discriminant of V:

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{pmatrix}.$$

Observe that $VV^T = A$, and $0 \neq \Delta^2 = \left|VV^T\right| = |A|$, so A is non-singular and therefore the bilinear form T is non-degenerate. $\square$

Remark. We'll meet Δ again shortly, as it is the discriminant of the polynomial $f_\alpha(t)$.

2.2 Normal extensions

We restate Definition 1.25 and Theorem 1.26:

Definition 1.25 (Normal extension)**.** An extension $K \leq L$ is **normal** if for every $\alpha \in L$ the minimal polynomial $f_\alpha(t)$ of α over K splits over L.

Theorem 1.26. Let $K \leq L$ be a finite field extension. Then $K \leq L$ is normal $\iff$ L is the splitting field for some $f(t) \in K[t]$.

Proof. Assume $K \leq M$ is normal. Pick $\alpha_1, \ldots, \alpha_r \in M$ so that $M = K(\alpha_1, \ldots, \alpha_r)$. Let $f_{\alpha_i}(t)$ be the minimal polynomial for α_i over K.

Let

$$f(t) = f_{\alpha_1}(t) f_{\alpha_2}(t) \ldots f_{\alpha_r}(t).$$

By normality, each $f_{\alpha_i}(t)$ splits over M and therefore $f(t)$ splits over M. M is the splitting field of $f(t)$ over K since if $\beta_1, \ldots, \beta_m$ are the roots of $f(t)$ then $M = K(\beta_1, \ldots, \beta_m)$.

Conversely, suppose M is a splitting field for $f(t)$ over K. Thus $M = K(\beta_1, \ldots, \beta_m)$ where the β_j are the roots of $f(t)$ in M.

Take $\alpha \in M$. Let $f(t)$ be the minimal polynomial of α over K. Let $M \leq L$ large enough so that $f_\alpha(t)$ splits in L and consider K-homomorphisms $\phi : M \to L$. $\phi(\beta_j)$ is also a root of $f(t)$ and is therefore one of the β_js. Injectivity of K-homomorphisms (Lemma 1.21) implies that ϕ generate the β_j.

$M = K(\beta_1, \ldots, \beta_m)$ and so ϕ is determined by the images of the β_j and thus $\phi(M) = M$. However if α_i is a root of $f_\alpha(t)$ in L, there is a K-homomorphism

$$\begin{aligned} K(\alpha) &\longrightarrow K(\alpha_i) \leq L \\ \alpha &\longmapsto \alpha_i. \end{aligned}$$

This extends by Corollary 2.10 to a K-homomorphism $\phi : M \to L$ with $\phi(\alpha) = \alpha_i$. But $\phi(M) = M$, so $\alpha_i \in M$. Thus M is normal over K. □

Remark. As for separability, the property of 'normality' is equivalent to 'normally generated'. In particular, a finite extension $K \leq L$ is normal if and only if $L = K(\alpha_1, \ldots, \alpha_r)$, with each $f_{\alpha_i}(t)$ splitting over L.

Definition 2.22 (Automorphism group)**.** Let $K \leq M$ be a finite field extension. Its **K-automorphism** group is $\mathrm{Aut}_K(M) = \{\, \phi \mid \phi \text{ a } K\text{-homomorphism } M \to M \,\}$.

From Lemma 1.21 we know that such K-homomorphisms are isomorphisms and thus have inverses. Composition gives the group operation.

Lemma 2.23.

$$\mathrm{Aut}_K(M) \leq |M : K|\,.$$

Proof. Corollary 2.10. □

Theorem 2.24. Let $K \leq M$ be a finite field extension. Then $|\mathrm{Aut}_K(M)| = |M : K|$ iff the extension is both normal and separable.

Definition 2.25 (Galois extension)**.** A finite field extension that is normal and separable is a **Galois extension**.

Definition 2.26 (Galois group)**.** Let $K \leq M$ be a Galois extension. Then, the K-automorphism group of M is the **Galois group** of M over K. Write this as $\mathrm{Gal}(M/K)$.

Remark. Some authors use 'Galois group' for the automorphism group even when the extension is not Galois.

Proof of Theorem 2.24. ($\Rightarrow$). Suppose $|\mathrm{Aut}_K(M)| = |M : K| = n$. Let L be large enough containing M.

The n distinct K-homomorphisms $\phi : M \to M \leq L$ give us n K-homomorphisms $\phi : M \to L$ and Lemma 2.12 says that M is separable over K. For normality, pick $\alpha \in M$ with minimal polynomial $f_\alpha(t)$ over K.

Take $M = K(\alpha_1, \ldots, \alpha_m)$ as in the proof of Corollary 2.10 with $\alpha = \alpha_1$ and $L = M$. We only get $|M : K|$ extensions of the inclusion $K \hookrightarrow M$ if each inequality in the proof is an equality. In particular we need the number of K-homomorphisms $K(\alpha_1) \to M$ to be $|K(\alpha_1) : K|$.

But then Lemma 2.6 says we have $|K(\alpha) : K|$ distinct roots of $f_\alpha(t)$ in M. Thus $f_\alpha(t)$ splits over M.

Conversely, suppose $K \leq M$ is separable and normal. Then for $K \leq M \leq L$ with L large enough, separability implies there are $|M : K|$ K-homomorphisms $\phi : M \to L$ by Lemma 2.12. However since $K \leq M$ is normal, it is the splitting field for some polynomial $f(t) \in K[t]$ (Theorem 1.26) and thus $M = K(\alpha_1, \ldots, \alpha_n)$, where $f(t) = (t - \alpha_1) \cdots (t - \alpha_n)$. Note that $\phi(a_j)$ is also a root of $\phi(f(t)) = f(t)$ and is therefore one of the α_js. Thus $\phi(M) = M$. Thus we have $|M : K|$ K-homomorphisms $\phi : M \to M$. □

Remark 2.27. In the proof we have shown that if $K \leq M \leq L$ and $\phi : M \to L$ is a K-homomorphism and $K \leq M$ is normal, then $\phi(M) = M$.

Example.

(1) $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2}, i)$ is Galois. $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ has 4 elements:

$$\begin{aligned} \sigma : \sqrt{2} &\longmapsto \pm\sqrt{2} \\ i &\longmapsto \pm i \end{aligned}$$

In particular, $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}) \cong C_2 \times C_2$, all non-identity elements have order 2.

(2) Consider $f(t) = t^3 - 2$. The splitting field over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$, where ω is a primitive cube root of 1.

Thus $\mathbb{Q} \leq \mathbb{Q}(\sqrt[3]{2}, \omega)$ is Galois and $\left|\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}\right| = 6$ (we saw this in the example before Theorem 1.23).

$$\sigma_1 : \begin{cases} \sqrt[3]{2} & \longmapsto \sqrt[3]{2} \\ \omega & \longmapsto \omega \end{cases} \qquad \text{identity}$$

$$\sigma_2 : \begin{cases} \sqrt[3]{2} & \longmapsto \omega\sqrt[3]{2} \\ \omega & \longmapsto \omega \end{cases} \qquad \text{order 3}$$

$$\sigma_3 : \begin{cases} \sqrt[3]{2} & \longmapsto \omega^2\sqrt[3]{2} \\ \omega & \longmapsto \omega \end{cases} \qquad \text{order 3}$$

$$\sigma_4 : \begin{cases} \sqrt[3]{2} & \longmapsto \sqrt[3]{2} \\ \omega & \longmapsto \omega^2 \end{cases} \qquad \begin{array}{l} \text{complex conjugation} \\ \text{order 2} \end{array}$$

$$\sigma_5 : \begin{cases} \sqrt[3]{2} & \longmapsto \omega^2\sqrt[3]{2} \\ \omega & \longmapsto \omega^2 \end{cases} \qquad \text{order 2}$$

$$\sigma_6 : \begin{cases} \sqrt[3]{2} & \longmapsto \omega\sqrt[3]{2} \\ \omega & \longmapsto \omega^2 \end{cases} \qquad \text{order 2}$$

Observe also $\sigma_4\sigma_2\sigma_4^{-1} = \sigma_3$, so this is the dihedral group.

3 Fundamental Theorem of Galois Theory

3.1 Artin's Theorem

Definition 3.1 (Fixed field)**.** Let $K \leq L$ be a field extension and $H \leq \mathrm{Aut}_K(L)$. The **fixed field** of H is,

$$L^H := \{\, \alpha \in L \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H \,\}$$

Exercise. Check it is a field and $K \leq L^H \leq L$.

Theorem 3.2 (Fundamental Theorem of Galois Theory)**.** Let $K \leq L$ be a finite Galois extension. Then

(i) there is a 1 to 1 correspondence

$$\begin{aligned} \{\text{intermediate subfields } K \leq M \leq L\} &\longleftrightarrow \{\text{subgroups } H \text{ of } \mathrm{Gal}(L/K)\} \\ M &\longmapsto \mathrm{Aut}_M(L) \\ L^H &\longleftarrow\!\shortmid H \end{aligned}$$

This is called the Galois correspondence.

(ii) H is a normal subgroup of $\mathrm{Gal}(L/K)$ iff $K \leq L^H$ is normal iff $K \leq L^H$ is Galois.

(iii) If $H \lhd \mathrm{Gal}(L/K)$ then the map

$$\theta : \mathrm{Gal}(L/K) \longrightarrow \mathrm{Gal}(L^H/K)$$

given by restriction to L^H is a surjective group homomorphism with kernel H.

Remark.

(i) Observe that $M \leq L$ is Galois and so we could have written $\mathrm{Gal}(L/M)$ instead of $\mathrm{Aut}_M(L)$ in (i). To see this: separability follows from Lemma 2.15. Normality: If $\alpha \in L$, the minimal polynomial of α over M divides the minimal polynomial of α over K. But the latter splits over L.

(ii) If $K \leq M$ is normal then Remark 2.27 says that if $\sigma : L \to L$ then $\sigma(M) = M$ and so we can talk about the restriction of α to M giving an automorphism of M.

Example.

(i) $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2}, i)$. We saw in Section 1 the lattice of intermediate fields and subgroups $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}) \cong C_2 \times C_2$ abelian. All subgroups are normal and intermediate subfields are normal extensions of $\mathbb{Q}$.

(ii) $\mathbb{Q} \leq \mathbb{Q}(\sqrt[3]{2}, \omega)$:

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt[3]{2}, \omega) \arrow[dl, dash] \arrow[dl, dash] \arrow[dl, dash] \arrow[dl, dash] \\
\mathbb{Q}(\omega) \arrow[dr, dash] & \mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash] & \mathbb{Q}(\omega\sqrt[3]{2}) \arrow[d, dash] & \mathbb{Q}(\omega^2\sqrt[3]{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \{e\} \arrow[dl, dash] \arrow[dl, dash] \arrow[dl, dash] \arrow[dl, dash] \\
\{\sigma_1, \sigma_2, \sigma_3\} \arrow[dr, dash] & \{\sigma_1, \sigma_4\} \arrow[d, dash] & \{\sigma_1, \sigma_5\} \arrow[d, dash] & \{\sigma_1, \sigma_6\} \arrow[dl, dash] \\
 & D_6 \cong \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\} &
\end{tikzcd}$$

The subgroup H of order 3 is normal but those of order 2 are not.

$$\mathbb{Q} \leq \mathbb{Q}(\omega) \text{ is normal} \longleftrightarrow H \text{ of order } 3$$

$$\begin{aligned} \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) &\longmapsto \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \\ D_6 &\longmapsto C_2 \end{aligned}$$

This final mapping has kernel H, and the image is generated by conjugation.

Theorem 3.3 (Artin's Theorem)**.** Let $K \leq L$ be a field extension and H a finite subgroup of $\mathrm{Aut}_K(L)$. Let $M = L^H$. Then $M \leq L$ is a finite Galois extension, and $H = \mathrm{Gal}(L/M)$.

Remark. This implies some of the Galois correspondence

$$H \longrightarrow L^H \longrightarrow \mathrm{Gal}(L/L^H)$$

we get back to H.

Proof of Artin's Theorem. Take $\alpha \in L$.
First step: Show that $|M(\alpha) : M| \leq |H|$. Let

$$\underbrace{\{\alpha_1, \ldots, \alpha_n\}}_{\text{all distinct}} = \{\, \phi(\alpha) \mid \phi \in H \,\}.$$

Define $g(t) = \prod_{i=1}^n (t - \alpha_i)$. Each ϕ induces a homomorphism $L[t] \to L[t]$ that sends $g(t)$ to itself, since ϕ is permuting the α_i. So the coefficients of $g(t)$ are fixed by all $\phi \in H$ and thus they all lie in $L^H = M$. Thus $g(t) \in M[t]$.

By definition, $g(\alpha) = 0$ since α is one of the α_i. Hence the minimal polynomial $f_\alpha(t)$ of α over M divides $g(t)$. Thus $|M(\alpha) : M| = \deg f_\alpha(t) \leq \deg g(t) \leq |H|$. We've shown that α is algebraic over M. Moreover, $f_\alpha(t)$ is separable since $g(t)$ is. Thus $M \leq L$ is a separable extension.

Next step: Show that $M \leq L$ is a simple extension. Pick $\alpha \in L$ with $|M(\alpha) : M|$ maximal. We'll show that $L = M(\alpha)$ for this choice of α. Suppose $\beta \in L$. Then $M \leq M(\alpha, \beta)$ is finite and is separably generated and hence is a finite separable extension by Lemma 2.12.

By the Primitive Element Theorem, $M(\alpha, \beta) = M(\gamma)$ for some γ. But $M \leq M(\alpha) \leq M(\gamma)$. The maximality of $|M(\alpha) : M|$ forces $M(\alpha) = M(\gamma)$. Thus $\beta \in M(\gamma) = M(\alpha)$ and so $L = M(\alpha)$ so $|L : M| \leq |H|$.

Finally,

$$|L : M| = |M(\alpha) : M| \leq |H| \leq |\mathrm{Aut}_M(L)| \underset{\substack{\uparrow \\ \text{Lemma 2.23}}}{\leq} |L : M|$$

We must have equality throughout, and so $|L : M| = |\mathrm{Aut}_M(L)| = |H|$. Hence by Theorem 2.24 we have $M \leq L$ is a finite Galois extension and $H = \mathrm{Gal}(L/M)$. □

Theorem 3.4. Let $K \leq L$ be a finite field extension. Then the following are equivalent:

(i) $K \leq L$ is Galois

(ii) $L^H = K$ when $H = \mathrm{Aut}_K(L)$

Remark. The theorem allows some authors yet another alternative definition of a 'Galois extension'.

Proof. **(i) ⇒ (ii):** Let $M = L^H$ where $H = \mathrm{Aut}_K(L)$. By Artin's Theorem, $M \leq L$ is a Galois extension, and $|L : M| = |\mathrm{Gal}(L/M)|$ and $H = \mathrm{Gal}(L/M)$.

However if $K \leq L$ is Galois then $|H| = |\mathrm{Aut}_K(L)| = |L : K|$ by Theorem 2.24. Thus $|L : M| = |L : K|$ and so $M = K$.

(ii) ⇐ (i): Use Theorem 3.3. □

Proof of Fundamental Theorem of Galois Theory.

(i) Composing the maps $H \to L^H$ and $M \to \mathrm{Gal}(L/M)$ gives $H \to H$ by Theorem 3.3. Also $M \longrightarrow \mathrm{Gal}(L/M) \longrightarrow L^H$ where $H = \mathrm{Gal}(L/M)$ yields M since $M \leq L^H$ where $H = \mathrm{Gal}(L/M)$ and

$$\left|L : L^H\right| \underset{\substack{(2.24)\\(3.3)}}{=} |H| = |\mathrm{Gal}(L/M)| \underset{(2.24)}{=} |L : M|$$

So $M = L^H$.

(ii) Take $H \leq \mathrm{Gal}(L/K)$, then $L^{\phi H \phi^{-1}} = \phi(L^H)$ when $\phi \in \mathrm{Gal}(L/K)$. So by (i), H is normal iff $\phi(L^H) = L^H$. Set $M = L^H$.

We'll show that $K \leq M$ is normal iff $\phi(M) = M \quad \forall \phi \in \mathrm{Gal}(L/K)$. $K \leq M$ is normal $\implies \phi(M) = M$ by remark 2 after the statement of Fundamental Theorem of Galois Theory.

Conversely if $\phi(M) = M \quad \forall \phi \in \mathrm{Gal}(L/K)$, pick $\alpha \in M$ and let $f_\alpha(t)$ be its minimal polynomial over K. Take β to be a root of $f_\alpha(t)$ in L (possible by normality). Then there is a K-homomorphism

$$\begin{aligned} K(\alpha) \cong \tfrac{K[t]}{(f_\alpha(t))} &\longrightarrow K(\beta) \cong \tfrac{K[t]}{(f_\alpha(t))} \leq L \\ \alpha &\longmapsto \beta. \end{aligned}$$

This extends to a K-homomorphism $\phi : L \to L$.

However we are assuming $\phi(M) = M$ and so $\phi(\alpha) = \beta \in M$. Thus $K \leq M$ is normal. Note that $K \leq L^H$ is separable since $K \leq L^H \leq L$ and $K \leq L$ separable.

(iii) By remark 2 after statement of Theorem 3.2, the restriction map

$$\theta : \mathrm{Gal}(L/K) \to \mathrm{Gal}(L^H/K)$$

is defined. Surjectivity follows from being able to extend a K-homomorphism $L^H \to L^H \leq L$ to a K-homomorphism $L \to L$ by Corollary 2.10. Clearly $H \leq \mathrm{Ker}\,\theta$. However

$$\begin{aligned} \frac{|L:K|}{|\mathrm{Ker}\,\theta|} &= \frac{\mathrm{Gal}(L/K)}{|\mathrm{Ker}\,\theta|} \\ &= \left|\mathrm{Gal}(L^H/K)\right| \quad \text{by surjectivity of } \theta \\ &= \left|L^H : K\right| \quad \text{since } K \leq L^H \text{ is Galois} \\ &= \frac{|L:K|}{|L:L^H|} \quad \text{by Tower law} \end{aligned}$$

So $|\mathrm{Ker}\,\theta| = \left|L : L^H\right| = \left|\mathrm{Gal}(L/L^H)\right| = |H|$ by Theorem 3.3, so $H = \mathrm{Ker}\,\theta$. □

3.2 Galois groups of polynomials

Definition 3.5 (Galois group of polynomial)**.** Let $f(t)$ be a separable polynomial $\in K[t]$ and let $K \leq L$ with L a splitting field for $f(t)$. Then the **Galois group of** $f(t)$ over K is

$$\mathrm{Gal}(f) \coloneqq \mathrm{Gal}(L/K).$$

Since L is a splitting field for $f(t)$, $L = K(\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are the roots of $f(t)$ in L. Observe that if $\phi \in \mathrm{Gal}(L/K)$ it maps

$$\{\text{roots of } f(t)\} \longrightarrow \{\text{roots of } f(t)\}$$

Thus ϕ permutes the α_i. Moreover, if ϕ fixes each α_i, it also fixes all elements of L, and so is the identity map. Thus $\mathrm{Gal}(f)$ may be regarded as a permutation group of the n roots, so $\mathrm{Gal}(f) \leq S_n$.

Lemma 3.6. Suppose $f(t)$ is separable, $f(t) = g_1(t) \cdots g_s(t)$ with $g_i(t)$ irreducible in $K[t]$ is a factorisation in $K[t]$. Then the orbits of $\mathrm{Gal}(f)$ on the roots of $f(t)$ correspond to the factors $g_j(t)$.

$$\text{Two roots are in the same orbit} \iff \text{they are roots of the same } g_j(t).$$

In particular, if $f(t)$ is irreducible in $K[t]$ there is one orbit, i.e., $\mathrm{Gal}(f)$ acts transitively on the roots of $f(t)$.

Proof. Let α_k, α_l be in the same orbit under $\mathrm{Gal}(f)$. Thus there is $\phi \in \mathrm{Gal}(f)$ with $\alpha_l = \phi(\alpha_k)$. But if α_k is a root of $g_j(t)$ then $\phi(\alpha_k) = \alpha_l$ is also a root of $g_j(t)$.

Conversely, if α_k, α_l are roots of $g_j(t)$ then

$$\begin{tikzcd} K(\alpha_k) \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "\phi_0"'] & \frac{K[t]}{(g_j(t))} \arrow[r, phantom, "\cong"] & K(\alpha_l) \arrow[r, phantom, "\leq"] & L \end{tikzcd}$$

with $\phi_0(\alpha_k) = \alpha_l$. ϕ_0 extends to a $\phi : L \to L \in \mathrm{Gal}(L/K)$, thus α_k, α_l are in the same orbit. □

Since the Galois group of an irreducible polynomial acts transitively, and is a subgroup of S_n, it is useful to know what the transitive subgroups of S_n are.

Lemma 3.7. The transitive subgroups of S_n for $n \leq 5$ are

$$\begin{array}{ll} n = 2: & S_2 \ (\cong C_2) \\ n = 3: & A_3 \ (\cong C_3),\ S_3 \\ n = 4: & C_4,\ V_4,\ D_8,\ A_4,\ S_4 \\ n = 5: & C_5,\ D_{10},\ H_{20},\ A_5,\ S_5 \end{array}$$

where H_{20} is generated by a 5-cycle and a 4-cycle.

Proof. Exercise. □

Theorem 3.8. Let p be a prime, and $f(t)$ irreducible $\in \mathbb{Q}[t]$ of degree p. Suppose $f(t)$ has exactly 2 non-real roots in $\mathbb{C}$. Then $\mathrm{Gal}(f)$ over $\mathbb{Q} \cong S_p$.

Proof. $\mathrm{Gal}(f)$ acts on the p distinct roots of $f(t)$ in a splitting field L of $f(t)$ (in $\mathbb{C}$). By Lemma 3.6, the irreducibility of $f(t)$ implies that $\mathrm{Gal}(f)$ is acting transitively on the p roots. By the orbit-stabiliser theorem, $p \mid |\mathrm{Gal}(f)|$ but $|\mathrm{Gal}(f)| \leq |S_p| = p!$ and so $\mathrm{Gal}(f)$ has a Sylow p-subgroup of order p, necessarily cyclic. Thus, $\mathrm{Gal}(f)$ contains a p-cycle.

The supposition that we have precisely 2 non-real roots gives that complex conjugation yields a transposition in $\mathrm{Gal}(f)$. The p-cycle and transposition generate the whole of S_p. □

Example 3.9. Consider $f(t) = t^5 - 6t + 3 \in \mathbb{Q}[t]$. Then $\mathrm{Gal}(f) \cong S_5$.

Proof. $f(t)$ is irreducible by Eisenstein's criterion with $p = 3$. We want to show that $f(t)$ has three real roots, two non-real ones and apply Theorem 3.8.

$$f(-2) = -17,\ f(-1) = 8,\ f(1) = -2,\ f(2) = 23$$

and $f'(t) = 5t^4 - 6$ which has two real roots. From the intermediate value theorem, f has at least three real roots, and by Rolle's theorem there are at most three real roots, so we are done. □

Definition 3.10 (Discriminant)**.** Let $f(t) \in K[t]$ with distinct roots $\alpha_1, \ldots, \alpha_n$ in a splitting field (with $f(t)$ not necessarily irreducible). Let

$$\Delta = \prod_{i<j} (\alpha_i - \alpha_j).$$

Then the **discriminant** $D = D(f)$ of f is

$$\begin{aligned} D = \Delta^2 &= \prod_{i<j} (\alpha_i - \alpha_j)^2 \\ &= (-1)^{\frac{n(n-1)}{2}} \prod_{i \neq j} (\alpha_i - \alpha_j). \end{aligned}$$

Remark. We've already met this in the proof of Theorem 2.21.

Lemma 3.11. Let $f(t)$ be separable $\in K[t]$ of degree n with $\operatorname{char} K \neq 2$. Then

$$\mathrm{Gal}(f) \leq A_n \iff D(f) \text{ is a square in } K.$$

Proof. Let L be a splitting field of $f(t)$ over K. Then $D(f) \neq 0$ and is fixed by all elements of $G = \mathrm{Gal}(L/K)$ as the latter permutes the roots. Thus $D \in K$, since $L^G = K$ (by Galois correspondence).

On the other hand, if $\sigma \in G$ then $\sigma(\Delta) = (\mathrm{sgn}\sigma)\Delta$ where we're regarding G as a subgroup of S_n and the signature of σ:

$$\mathrm{sgn}\sigma = \begin{cases} +1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$$

(This is where we need $\operatorname{char} K \neq 2$).

Thus if $G \leq A_n$ we get that Δ is fixed by all $\sigma \in G$. Thus $\Delta \in K = L^G$. Otherwise if $G \not\leq A_n$, we get $\sigma(A) = -\Delta$ if σ is an odd permutation, and so $\Delta \notin K = L^G$. Note that if D does have square roots, they must be $\pm\Delta$. □

Example 3.12. For $n = 2$, take $f(t) = t^2 + bt + c$. Then $D(f) = b^2 - 4c$.

For $n = 3$, consider $f(t) = t^3 + ct + d$. Then we can check that $D(f) = -4c^3 - 27d^2$. Any general monic cubic can be written in the above form by suitable substitutions.

Take $f(t) = t^3 - t - 1 \in \mathbb{Q}[t]$. f is irreducible since it is irreducible in $\mathbb{Z}[t]$ since it is irreducible mod 2. $D(f) = -23$, which is not a square in $\mathbb{Q}$, so $\mathrm{Gal}(f) \cong S_3$.

$f(t) = t^3 - 3t - 1 \in \mathbb{Q}[t]$ is irreducible since it is irreducible mod 2. Now $D(f) = 81$, so $\mathrm{Gal}(f) \cong A_3$.

For irreducible quartics, we saw that the possible Galois groups are C_4, V_4, D_8, A_4, S_4. Those that are subgroups of A_4 are V_4 and A_4. From looking at the discriminant, one gets information as to whether the group is one of V_4, A_4 or is C_4, D_8, S_4. We need further methods to pin down which group we are dealing with.

Theorem 3.13 (Mod p reduction)**.** Let $f(t) \in \mathbb{Z}[t]$ be monic of degree n with n distinct roots in a splitting field. Let p be a prime such that $\overline{f}(t)$, the reduction of $f(t)$ mod p also has n distinct roots in a splitting field. Let $\overline{f}(t) = \overline{g_1}(t) \cdots \overline{g_s}(t)$ be the factorisation into irreducibles in $\mathbb{F}_p[t]$ with $n_j = \deg \overline{g_j}(t)$. Then $\mathrm{Gal}(\overline{f}) \hookrightarrow \mathrm{Gal}(f)$ and has an element of cycle type $(n_1, n_2, \ldots, n_s)$.

Proof. We will talk about the last sentence after thinking about Galois groups of finite fields. The fact that $\mathrm{Gal}(\overline{f}) \hookrightarrow \mathrm{Gal}(f)$ is from Number Fields - see Tony Scholl's teaching page on Galois. □

Example 3.14. For $f(t) = t^4 + dt + e$, we have $D(f) = -27d^4 + 256e^3$, not proved here. Consider $f(t) = t^4 - t - 1$, which is irreducible since it is irreducible mod 2. Then $D(f) = -283$, which is not a square in $\mathbb{Q}$.

Now take $p = 7$.

$$\begin{aligned}\overline{f}(t) &= t^4 - t - 1 \\ &= (t+4)(t^3 + 3t^2 + 2t + 5) \pmod 7\end{aligned}$$

and the latter factor is irreducible over $\mathbb{F}_7$ as it is a cubic which has no roots in $\mathbb{F}_7$.

By Theorem 3.13, $\mathrm{Gal}(f)$ contains an element of cycle type $(1, 3)$, i.e. a 3-cycle. We deduce $\mathrm{Gal}(f) \cong S_4$, as the other possibilities that contain an odd permutation do not contain 3-cycles.

3.3 Galois Theory of Finite Fields

Recall what we already know from Section 1. From Example 1.27, a finite field $\mathbb{F}$ is of characteristic $p > 0$, p necessarily prime, and $|\mathbb{F}| = p^r$ for some r. The multiplicative group of $\mathbb{F}$ is cyclic (Theorem 1.28). It is a splitting field for $t^n - 1$ over $\mathbb{F}_p$ where $n = p^r - 1$. By the Uniqueness of splitting fields, this is unique. Observe we could also describe $\mathbb{F}$ as the splitting field of $t^{p^r} - t$ over $\mathbb{F}_p$. What we haven't shown yet is that for any p^r, there is a field $\mathbb{F}$ with $\mathbb{F} = p^r$.

Definition 3.15 (Frobenius automorphism)**.** Let $\mathbb{F}$ be a finite field of characteristic p. Then the **Frobenius automorphism** of $\mathbb{F}$ is

$$\begin{aligned}\phi : \mathbb{F} &\longrightarrow \mathbb{F} \\ \alpha &\longmapsto \alpha^p.\end{aligned}$$

Remark.

(i) $(\alpha + \beta)^p = \alpha^p + \beta^p$ since all other terms in binomial expansion are divisible by p.

(ii) $\mathbb{F}_p$ is fixed under this, so it is an $\mathbb{F}_p$-automorphism. Since t^{p^r} splits as a product of distinct linear factors $(t - \alpha)$ in $\mathbb{F}$, we have that $\mathbb{F}_p \leq \mathbb{F}$ is a Galois extension and so we consider $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) = G$. It is of order r since $|\mathbb{F} : \mathbb{F}_p| = r$.

Theorem 3.16 (Galois groups of finite fields)**.** Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^r$. Then $\mathbb{F}_p \leq \mathbb{F}$ is a Galois extension with $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) = G$, a cyclic group with the Frobenius automorphism as generator.

Proof. It remains to show that the order of the Frobenius automorphism is r. Suppose $\phi^s = \mathrm{id}$. Then $\alpha^{p^s} = \alpha \ \forall \alpha \in \mathbb{F}$. But $t^{p^s} - t$ has at most p^s roots in $\mathbb{F}$, so we deduce that $s \geq r$. Observe that $\phi^r = \mathrm{id}$ since $\alpha^{p^n} = \alpha$, $\forall \alpha \in \mathbb{F}$.

Now apply the Fundamental Theorem of Galois Theory:

$$\{\mathbb{F}_p \leq M \leq \mathbb{F} \text{ intermediate fields } M\} \longleftrightarrow \{\text{subgroups } H \leq G\}$$

where $G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ is cyclic.

But we know all about subgroups of a cyclic group with generator ϕ of order r. There is exactly one subgroup of order s for each $s \mid r$ generated by $\phi^{\frac{r}{s}}$. The corresponding

intermediate subfields are the fixed fields $\mathbb{F}^{\langle \phi^{\frac{r}{s}} \rangle}$, and $\left|\mathbb{F} : \mathbb{F}^{\langle \phi^{\frac{r}{s}} \rangle}\right| = s$. By the Tower Law, $\left|\mathbb{F}^{\langle \phi^{\frac{r}{s}} \rangle} : \mathbb{F}_p\right| = \frac{r}{s}$. Observe that all subgroups of cyclic groups are normal and therefore all our intermediate fields are normal extensions of $\mathbb{F}_p$.

By Theorem 3.2 part (iii), $\mathrm{Gal}(\mathbb{F}^{\langle \phi^{\frac{r}{s}} \rangle}/F_p) \cong \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)/H$ where $H = \langle \phi^{\frac{r}{s}} \rangle$. □

Corollary 3.17. Let $\mathbb{F}_p \leq M \leq \mathbb{F}$ be finite fields. Then $\mathrm{Gal}(\mathbb{F}/M)$ is cyclic, generated by ϕ^u, where ϕ is the Frobenius automorphism and $|M| = p^u$ and M is the fixed field of $\langle \phi^u \rangle$.

Proof. Set $n = \frac{r}{s}$. □

Theorem 3.18 (Existence of finite fields)**.** Let p be a prime and $u \geq 1$. Then there is a field of order p^u, unique up to isomorphism.

Proof. Consider the splitting field L of $f(t) = t^{p^u} - t$ over $\mathbb{F}_p$. It is a finite Galois extension $\mathbb{F}_p \leq L$. However the roots of $f(t)$ form a field, the fixed field of ϕ^u. Set $L = \mathbb{F}$ and $|\mathbb{F} : \mathbb{F}_p| = u$. □

Remark (About Theorem 3.13)**.** We'll discover in Number Fields that $\mathrm{Gal}(\overline{f}) \hookrightarrow \mathrm{Gal}(f)$ if $f(t) \in \mathbb{Z}[t]$. We factorised $\overline{f}(t) = \overline{g_1}(t) \cdots \overline{g_s}(t)$, a product of irreducibles. We know from Lemma 3.6 that the orbits of $\mathrm{Gal}(\overline{f})$ correspond to the factorsiation.

We now know $\mathrm{Gal}(\overline{f})$ is cyclic generated by the Frobenius map, which must have cycle type $(n_1, \ldots, n_s)$ where $n_j = \deg \overline{g_j}(t)$

4 Cyclotomic and Kummer extensions

4.1 Cyclotomic extensions

Definition 4.1 (Cyclotomic extension)**.** Suppose $\operatorname{char} K = 0$ or p prime where $p \nmid m$. The mth **cyclotomic extension** of K is the splitting field L of $t^m - 1$.

Remark. $f(t) = t^m - 1$ and $f'(t) = mt^{m-1}$ have no common roots, and so the roots of $f(t)$ are distinct, the mth roots of unity. They form a finite subgroup μ_m of $L^\times$, and hence by Theorem 1.28 a cyclic group $\langle \xi \rangle$. Thus, $L = K(\xi)$ is a simple extension.

Definition 4.2 (Primitive mth root of unity)**.** An element $\xi \in \mu_m$ is a **primitive mth root of unity** if $\mu_m = \langle \xi \rangle$.

Choosing a primitive mth root of unity determines $\mu_m \to \mathbb{Z}/m\mathbb{Z}$ an isomorphism. Note that ξ^i is a generator of μ_m iff $(i, m) = 1$ and so the primitive mth roots of unity correspond to elements of $(\mathbb{Z}/m\mathbb{Z})^\times$.

Now consider Galois groups of cyclotomic extensions: we'll see they must be abelian. Observe $f(t) = t^m - 1$ is separable and so the extension $K \leq L$ is Galois. Let $G = \operatorname{Gal}(L/K)$. An element $\sigma \in G$ sends a primitive mth root of unity ξ to a primitive mth root of unity ξ^i where $(i, m) = 1$. Then $\xi \mapsto \xi^i$ determines a K-homomorphism $K^L(\xi) \to K^L(\xi)$ with $\xi \mapsto \xi^i$. Thus we get an injective map:

Definition 4.3 (Group of roots of unity)**.**

$$\theta : G \longrightarrow (\mathbb{Z}/m\mathbb{Z})^\times$$

This is a group homomorphism: If $\sigma(\xi) = \xi^c, \phi(\xi) = \xi^j$ then $(\sigma\phi)(\xi) = \sigma(\xi^j) = \xi^{ij}$. Hence G is abelian.

Thus we regard G as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$.

Definition 4.4 (Cyclotomic polynomial)**.** The mth **cyclotomic polynomial** is

$$\Phi_m(t) = \prod_{i \in (\mathbb{Z}/m\mathbb{Z})^\times} (t - \xi^i),$$

the product of the linear factors of $t^m - 1$ corresponding to the primitive mth roots of unity.

Remark.

$$f(t) = t^m - 1 = \prod_{i \in \mathbb{Z}/m\mathbb{Z}} (t - \xi^i) = \prod_{d|m} \Phi_d(t).$$

Example. Take $K = \mathbb{Q}$.

$$\Phi_1(t) = t - 1,\ \Phi_2(t) = t + 1,\ \Phi_3(t) = t^2 + t + 1,\ \Phi_4(t) = t^2 + 1,\ \Phi_8(t) = t^4 + 1$$

since $t^8 - 1 = (t - 1)(t + 1)(t^2 + 1)(t^4 - 1)$.

Lemma 4.5. $\Phi_m(t) \in \mathbb{Z}[t]$ if $\operatorname{char} K = 0$ (with $\mathbb{Q} \hookrightarrow K$, prime subfield). $\Phi_m(t) \in \mathbb{F}_p[t]$ if $\operatorname{char} K = p$ (with $\mathbb{F}_p \hookrightarrow K$, prime subfield).

Proof. Induct on m. $m = 1$ is clearly true.

For $m > 1$, consider

$$f(t) = t^m - 1 = \Phi_m(t) \left(\prod_{\substack{d|m \\ d \neq m}} \Phi_d(t) \right).$$

Note that $\prod_{\substack{d|m \\ d\neq m}} \Phi_d(t)$ is monic and is defined in $\mathbb{Z}[t]$ or $\mathbb{F}_p[t]$ by induction.

If $\operatorname{char} K = 0$, we deduce $\Phi_m(t) \in \mathbb{Q}[t]$ by division of polynomials and by Gauss' Lemma it is in $\mathbb{Z}[t]$. If $\operatorname{char} K = p > 0$, we deduce by division that $\Phi_m(t) \in \mathbb{F}_p[t]$. □

Lemma 4.6. The homomorphism $\theta : G \to (\mathbb{Z}/m\mathbb{Z})^\times$ defined in Definition 4.3 is an isomorphism iff $\Phi_m(t)$ is irreducible.

Proof. We know from Lemma 3.6 that the orbits of $G = \mathrm{Gal}(L/K)$ correspond to the factorisation of $f(t)$ in $K[t]$. In particular, the primitive mth roots of unity form one orbit iff $\Phi_m(t)$ is irreducible. Then θ is surjective iff $\Phi_m(t)$ is irreducible. □

Theorem 4.7. Let L be the mth cyclotomic extension of finite field $\mathbb{F} = \mathbb{F}_q$ where $q = p^n$. Then the Galois group $G = \mathrm{Gal}(L/\mathbb{F})$ is isomorphic to the cyclic subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ generated by q.

Proof. We know from Corollary 3.17 that G is generated by $\alpha \mapsto \alpha^{p^n} = \alpha^q$ so $\theta(G) = \langle q \rangle \leq (\mathbb{Z}/m\mathbb{Z})^\times$. □

Remark. Thus if $(\mathbb{Z}/m\mathbb{Z})^\times$ is not cyclic then θ is not surjective for any finite field $\mathbb{F}$ and $\Phi_m(t)$ is reducible over $\mathbb{F}$.

Example. Take $\mathbb{F} = \mathbb{F}_3$, then $\Phi_8(t) = t^4 + 1 = (t^2 + t - 1)(t^2 - t - 1)$. $t^8 - 1$ factorises as a product of linear and quadratic polynomials mod 3, so the splitting field $L = \mathbb{F}_9$ is the unique field of order 9 whose multiplicative group is cyclic C_8.

$$(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\} \cong C_2 \times C_2$$

We have $|L : \mathbb{F}_3| = 2$, so $|\mathrm{Gal}(L/\mathbb{F}_3)| = 2$ hence $\mathrm{Gal}(L/\mathbb{F}_3)$ cyclic order 2, so θ is not surjective.

What happens when $K = \mathbb{Q}$?

Theorem 4.8. For all $m > 0$, $\Phi_m(t)$ is irreducible in $\mathbb{Z}[t]$ and hence in $\mathbb{Q}[t]$. Thus θ in Definition 4.3 is an isomorphism and thus $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ where ξ = primitive mth root of unity.

Remark. We already know this when $m = p$ prime by substitution and Eisenstein's criterion.

Proof of Theorem 4.8. Gauss' Lemma gives us that irreducibility in $\mathbb{Z}[t]$ implies irreducibility in $\mathbb{Q}[t]$. From Lemma 4.5, irreducibility corresponds to surjectivity of θ. It's left to show that $\Phi_m(t)$ is irreducible in $\mathbb{Z}[t]$.

Suppose not, and $\Phi_m(t) = g(t)h(t)$ in $\mathbb{Z}[t]$ with $g(t)$ irreducible. monic and $\deg g(t) \lneq \deg \Phi_m(t)$. Let $\mathbb{Q} \leq L$ be the mth cyclotomic extension and ξ be a root of $g(t)$, ξ primitive mth root of unity.

Claim: if $p \nmid m$, p prime, then ξ^p is also a root of $g(t)$ in L. Suppose not. Then ξ^p is also a primitive mth root of 1, since $p \nmid m$, as a root of $\Phi_m(t)$. By the supposition, ξ^p is a root of $h(t)$. Define $r(t) = h(t^p)$. Then $r(\xi) = 0$ but $g(t)$ is the minimal polynomial of ξ over $\mathbb{Q}$. So $g(t) \mid r(t)$ in $\mathbb{Q}[t]$.

By Gauss' Lemma, $r(t) = g(t)s(t)$ with $s(t) \in \mathbb{Z}[t]$. Now reduce mod p. $\overline{r}(t) = \overline{g}(t)\overline{s}(t)$. But $\overline{r}(t) = \overline{h}(t^p) = (\overline{h}(t))^p$. If $\overline{a}(t)$ is any irreducible factor of $\overline{g}(t)$ in $\mathbb{F}_p[t]$ then $\overline{a}(t) \mid (\overline{h}(t))^p$ and so $\overline{a}(t) \mid \overline{h}(t)$. But then $(\overline{a}(t))^2 \mid \overline{g}(t)\overline{h}(t) = \overline{\Phi_m}(t)$. Hence $\overline{\Phi_m}(t)$ has a repeated root and thus $t^m - 1$ has repeated root mod p. Contradiction, since $p \nmid m$, so claim is true.

Now consider a root γ of $h(t)$. Then it is also a primitive root of 1 and so $\gamma = \xi^i$ for some i with $(i, m) = 1$. Write $i = p_1 \cdots p_k$ factorisation with p_j prime, not necessarily distinct, $p_j \nmid m$. Applying the claim repeatedly we get that γ is a root of $g(t)$, and so $\Phi_m(t)$ has a repeated root.

Hence $\Phi_m(t)$ is irreducible over $\mathbb{Q}$. $\square$

Definition 4.9 (Cyclic, abelian extension)**.** An extension $K \leq L$ is **cyclic** if the extension is Galois and $\mathrm{Gal}(L/K)$ is cyclic. Similarly, it is called **abelian** if $\mathrm{Gal}(L/K)$ is abelian.

Example. In Corollary 3.17 we saw that for finite fields $\mathbb{F} \leq L$ is cyclic and from Definition 4.3 cyclotomic extensions are abelian. However Theorem 4.8 says that $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ and so if $m = 8$, $\mathbb{Q} \leq \mathbb{Q}(\xi)$ is an abelian, non-cyclic extension, where ξ primitive 8th root of unity.

4.2 Kummer Theory

We consider Galois extensions $K \leq L$ where L is a splitting field of a polynomial of the form $t^m - \lambda$ with $\lambda \in K$.

Theorem 4.10. Let $f(t) = t^m - \lambda \in K[t]$ and $\operatorname{char} K \nmid m$. Then the splitting field L of $f(t)$ over K contains a primitive mth root of unity ξ and $\mathrm{Gal}(L/K(\xi))$ is cyclic of order dividing m. Moreover $f(t)$ is irreducible over $K(\xi)$ iff $|L : K(\xi)| = m$.

Remark.

$$\begin{tikzcd}
L \arrow[d, dash] & \{e\} \arrow[d, dash, "\text{cyclic}"] \\
K(\xi) \arrow[d, dash] & \mathrm{Gal}(L/K(\xi)) \arrow[d, dash, "\text{abelian}"] \\
K & \mathrm{Gal}(L/K)
\end{tikzcd}$$

As $\mathrm{Gal}(L/K(\xi))$ is cyclic, $\mathrm{Gal}(L/K(\xi)) \lhd \mathrm{Gal}(L/K)$.

By Theorem 3.2 part (iii), $\mathrm{Gal}(L/K)/\mathrm{Gal}(L/K(\xi)) \cong \mathrm{Gal}(K(\xi)/K)$ is abelian.

Proof of Theorem 4.10. Since $t^m - \lambda$ and mt^{m-1} are coprime, we know that $t^m - \lambda$ has distinct roots $\alpha_1, \ldots, \alpha_m$ in the splitting field L. Since $(\alpha_i \alpha_j^{-1})^m = \lambda\lambda^{-1} = 1$, the elements $1 = \alpha_1\alpha_1^{-1}, \alpha_2\alpha_1^{-1}, \ldots, \alpha_m\alpha_1^{-1}$ are m distinct mth roots of unity in L and so

$$t^m - \lambda = (t - \beta)(t - \xi\beta)(t - \xi^2\beta) \cdots (t - \xi^{m-1}\beta) \in L[t]$$

where $\beta = \alpha_1$ and ξ primitive mth root of unity.

So $L = K(\xi, \beta)$. Let $\sigma \in \mathrm{Gal}(L/K(\xi))$, which is determined by its action on β. Note that $\sigma(\beta)$ is another root of $t^m - \lambda$ and so $\sigma(\beta) = \xi^{j(\sigma)}\beta$, where $0 \leq j(\sigma) < m$. Also, if $\sigma, \tau \in \mathrm{Gal}(L/K(\xi))$ then

$$\tau\sigma(\beta) = \tau(\xi^{j(\sigma)}\beta) = \xi^{j(\sigma)}\tau(\beta) = \xi^{j(\sigma)}\xi^{j(\tau)}\beta$$

since ξ is fixed by τ. Thus $\sigma \to j(\sigma)$ gives a group homomorphism

$$\theta : \mathrm{Gal}(L/K(\xi)) \to \mathbb{Z}/m\mathbb{Z}.$$

Note that $j(\sigma) = 1$, only if σ is the identity and so θ is injective. Hence $\mathrm{Gal}(L/K(\xi)) \cong$ subgroup of $\mathbb{Z}/m\mathbb{Z}$. Finally $|L : K(\xi)| = |\mathrm{Gal}(L/K(\xi))| \leq m$ with equality exactly when the action of $\mathrm{Gal}(L/K(\xi))$ is transitive on the roots, i.e. when $t^m - 1$ is irreducible over $K(\xi)$ by Lemma 3.6. $\square$

Example. Consider $f(t) = t^6 + 3$. If ξ is a primitive 6th root of unity, $\xi = -\omega$, with ω a primitive cube root of 1.

$$\mathbb{Q}(\xi) = \mathbb{Q}(\omega) = \mathbb{Q}\left(\frac{1}{2}(1 + \sqrt{-3})\right) = \mathbb{Q}(\sqrt{-3})$$

$f(t)$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion with $p = 3$. However over $\mathbb{Q}(\xi) = \mathbb{Q}(\sqrt{-3})$ it factorises as

$$f(t) = (t^3 - \sqrt{-3})(t^3 + \sqrt{-3})$$

Let L be the splitting field of $f(t)$. From Theorem 4.10 since $f(t)$ factorises over $K(\xi)$, $\mathrm{Gal}(L/K(\xi)) \lneqq \mathbb{Z}/6\mathbb{Z}$.

$$\begin{tikzcd}
L \arrow[d, dash, "3"'] & \{e\} \arrow[d, dash, "3"] \\
\mathbb{Q}(\xi) \arrow[d, dash, "2"'] & \mathrm{Gal}(L/\mathbb{Q}(\xi)) \arrow[d, dash, "2"] \\
\mathbb{Q} & \mathrm{Gal}(L/\mathbb{Q})
\end{tikzcd}$$

$\mathrm{Gal}(L/\mathbb{Q})$ is dihedral of order 6, and $\mathrm{Gal}(L/\mathbb{Q}(\xi)) \hookrightarrow \mathbb{Z}/6\mathbb{Z}$ is cyclic of order 3. The element of order 2 is given by complex conjugation. Take β a root of $f(t)$. Then, the roots are

$$\begin{tikzcd}
\beta \arrow[d, dash] & \xi\beta \arrow[d, dash] & \xi^2\beta \arrow[d, dash] & \xi^3\beta \arrow[d, dash] & \xi^4\beta \arrow[d, dash] & \xi^5\beta \arrow[d, dash] \\
\beta & -\omega\beta & \omega^2\beta & -\beta & \omega\beta & \omega^2\beta
\end{tikzcd}$$

A 3-cycle is given by permuting $\beta, \omega^2\beta, \omega\beta$. Conjugating by complex conjugation yields the inverse of the 3-cycle - a dihedral relation.

We have a converse of Theorem 4.10:

Theorem 4.11. Suppose $K \leq M$ is a cyclic extension with $|L : K| = m$, where $\operatorname{char} K \nmid m$ and that K contains a primitive mth root of unity. Then $\exists \lambda \in K$ such that $t^m - \lambda$ is irreducible over K and K is the splitting field of $t^m - \lambda$ over K. If β is a root of $t^m - \lambda$ in L, then $L = K(\beta)$.

Definition 4.12 (Kummer extension)**.** A cyclic extension $K \leq L$ with $|L : K| = m$, where $\operatorname{char} K \nmid m$ and K contains a primitive mth root of unity is a **Kummer extension**.

The proof of Theorem 4.11 uses

Lemma 4.13. Let $\phi_1, \ldots, \phi_n$ be embeddings of a field K into a field L. Then there do not exist $\lambda_1, \ldots, \lambda_n$ not all zero such that $\lambda_1\phi_1(x) + \cdots + \lambda_n\phi_n(x) = 0 \; \forall x \in K$.

Proof. Example sheet 2, question 10. □

Proof of Theorem 4.11. Let $\mathrm{Gal}(L/K) = \langle \sigma \rangle$ of order m. Observe that $1, \sigma, \sigma^2, \ldots, \sigma^{m-1}$ are distinct maps $L \to L$, and we can apply Lemma 4.13. There exists $\alpha \in L$ such that

$$\beta = \alpha + \xi\sigma(\alpha) + \cdots + \xi^{m-1}\sigma^{m-1}(\alpha) \neq 0$$

where ξ is a primitive mth root of unity. Observe that $\sigma(\beta) = \xi^{-1}\beta \neq \beta$ and so $\beta \notin K$, the fixed field of $\mathrm{Gal}(L/K)$.

$\sigma(\beta^m) = (\sigma(\beta))^m = \beta^m$. Let $\lambda = \beta^m \in K$. But $t^m - \lambda = (t - \beta)(t - \xi\beta) \cdots (t - \xi^{m-1}\beta)$ in $L[t]$, and so $K(\beta)$ is the splitting field of $t^m - \lambda$ over K (recall $\xi \in K$). Observe that $1, \sigma, \ldots, \sigma^{m-1}$ are distinct K-automorphisms of $K(\beta)$ and so $|K(\beta) : K)| \geq m$.

So $L = K(\beta) = K(\xi\beta)$ since $\xi \in K$. $t^m - \lambda$ is the minimal polynomial of β over K and hence is irreducible. □

Definition 4.14 (Extension by radicals)**.** A field extension $K \leq L$ is an **extension by radicals** if $\exists K = L_0 \leq L_1 \leq \cdots \leq L_n = L$ such that each $L_i \leq L_{i+1}$ is either cyclotomic or Kummer extension. A polynomial $f(t) \in K[t]$ is **soluble by radicals** if its splitting field lies in an extension by radicals.

4.3 Cubics

We've already seen that if $f(t)$ is a monic irreducible cubic in $K[t]$ with L its splitting field over K,

$$\mathrm{Gal}(f) = \mathrm{Gal}(L/K) = G \text{ is } A_3 \text{ or } S_3$$

(since irreducibility $\implies$ action on roots is transitive, and transitive subgroups of S_3 are A_3 or S_3).

$$\begin{tikzcd}
L \arrow[d, dash] & \{e\} \arrow[d, dash] \\
K(\Delta) \arrow[d, dash, "1 \text{ or } 2"] & G \cap A_3 \arrow[d, dash, "1 \text{ or } 2"] \\
K & G
\end{tikzcd}$$

where $\Delta^2 = D(f)$ the discriminant of f. But to see that we can solve f by radicals we want to make use of Theorem 4.11, and so we need to adjoin the appropriate roots of unity.

Now we get a bigger picture, denoting ω a primitive cube root of 1.

$$\begin{tikzcd}
 & L(\omega) \arrow[dl, dash] \arrow[dr, dash, "1 \text{ or } 2"] & \\
K(\Delta,\omega) \arrow[dr, dash, "1 \text{ or } 2"'] & & L \arrow[dl, dash, "3"] \\
 & K(\Delta) \arrow[d, dash, "1 \text{ or } 2"] & \\
 & K &
\end{tikzcd}$$

From the Tower law, $|L(\omega) : K(\Delta, \omega)| = 3$. Hence $\mathrm{Gal}(L(\omega)/K(\Delta, \omega)) \cong C_3$. We can apply Theorem 4.11 to see that $L(\omega) = K(\Delta, \omega)(\beta)$, where β is a root of an irreducible polynomial $t^3 - \lambda \in K(\delta, \omega)[t]$. In fact, from the proof of Theorem 4.11 we see that $\beta = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$ where $\alpha_1, \alpha_2, \alpha_3$ roots of $f(t)$.

Now all the extensions $K \leq K(\Delta) \leq K(\Delta, \omega) \leq L(\omega)$ are cyclotomic or Kummer. So $f(t)$ is soluble by radicals.

In practice

Given an irreducible cubic

$$\begin{aligned} f(t) &= t^3 + at^2 + bt + c \\ &= (t - \alpha_1)(t - \alpha_2)(t - \alpha_3) \end{aligned}$$

we have $\alpha_1 + \alpha_2 + \alpha_3 = -a$.

Replace the α_is by $\alpha_i' = \alpha_i + \frac{a}{3}$ so that $\alpha_1' + \alpha_2' + \alpha_3' = 0$. They are roots of a polynomial $t^3 + pt + q$ and $K(\alpha_1, \alpha_2, \alpha_3) = K(\alpha_1', \alpha_2', \alpha_3')$. We have $D(g) = -4p^3 - 27q^2$.

Set $\beta = \alpha_1' + \omega\alpha_2' + \omega^2\alpha_3'$, $\gamma = \alpha_1' + \omega^2\alpha_2' + \omega\alpha_3'$. Then

$$\begin{aligned} \beta\gamma &= \alpha_1' + \alpha_2' + \alpha_3' + (\omega + \omega^2)(\alpha_1'\alpha_2' + \alpha_1'\alpha_3' + \alpha_2'\alpha_3') \\ &= (\alpha_1' + \alpha_2' + \alpha_3')^2 - 3(\alpha_1'\alpha_2' + \alpha_1'\alpha_3' + \alpha_2'\alpha_3') \\ &= -3p \end{aligned}$$

and so $\beta^3\gamma^3 = -27p^3$.

$$\begin{aligned} \beta^3 + \gamma^3 &= (\alpha_1 + \omega\alpha_2' + \omega^2\alpha_3')^3 + (\alpha_1' + \omega^2\alpha_2' + \omega\alpha_3')^3 + (\alpha_1' + \alpha_2' + \alpha_3')^3 \\ &= 3(\alpha_1'^3 + \alpha_2'^3 + \alpha_3'^3) + 18\alpha_1'\alpha_2'\alpha_3' \\ &= -27q \end{aligned}$$

since $\alpha_i'^3 = -p\alpha_i' - q$ and so $\alpha_1'^3 + \alpha_2'^3 + \alpha_3'^3 = -3q$.

So β^3 and γ^3 are roots of a quadratic $t^2 + 27qt - 27p^3$, hence they are

$$-\frac{27}{2}q \pm \frac{3\sqrt{-3}}{2}\sqrt{-27q^3 - 4p^3} = -\frac{27}{2}q \pm \frac{3\sqrt{3}}{2}\sqrt{D}.$$

We can solve for β^3 and γ^3 in $K(\sqrt{-3}, \sqrt{D}) = K(\omega, \Delta)$. We can get β by adjoining a cube root of β^3, and then set $\gamma = \frac{-3p}{\beta}$. Finally we solve in $L(\omega)$ for $\alpha_1', \alpha_2', \alpha_3'$ (remembering $\alpha_1' + \alpha_2' + \alpha_3' = 0$), using

$$\alpha_1' = \frac{1}{3}(\beta + \gamma), \quad \alpha_2' = \frac{1}{3}(\omega^2\beta + \omega\gamma), \quad \alpha_3' = \frac{1}{3}(\omega\beta + \omega^2\gamma).$$

4.4 Quartics

As with the cubics by making a substitution of the form $\alpha_i' = \alpha_i + \frac{a}{4}$ we may assume the sum of the roots to be zero, and so the t^3 term is zero.

$$\begin{aligned} f(t) &= t^4 + bt^2 + ct + d \quad \text{monic irreducible} \\ &= (t - \alpha_1)(t - \alpha_2)(t - \alpha_3)(t - \alpha_4) \end{aligned}$$

Let $L = K(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the splitting field for $f(t)$ over K and set $M = L^{G \cap V_4}$. The Fundamental Theorem of Galois Theory gives $\mathrm{Gal}(M/K) \cong \frac{G}{G \cap V_4}$.

$$\begin{tikzcd}
 & L \arrow[d, dash] & \{e\} \arrow[d, dash] \\
\text{normal extension of } K & M = L^{G \cap V_4} \arrow[d, dash] & G \cap V_4 \lhd G \arrow[d, dash] \\
 & K(\Delta) \arrow[d, dash] & G \cap A_4 \arrow[d, dash] \\
 & K & G = \mathrm{Gal}(L/K) \leq S_4
\end{tikzcd}$$

Take $\theta : S_4 \to S_3$ with $\ker\theta = V_4$, so $S_4/V_4 \cong S_3$. Consider $\theta|_G : G \to S_3$, so $\ker\theta|_G = G \cap V_4$, and $\frac{G}{G \cap V_4} \cong \operatorname{im}\theta|_G \leq S_3$.

We therefore go looking for a cubic for which M is the splitting field, called the resolvent cubic.

Set $x = \alpha_1 + \alpha_2$, $\gamma = \alpha_1 + \alpha_3$, $z = \alpha_1 + \alpha_4$ and so

$$\alpha_1 = \frac{1}{2}(x + y + z), \alpha_2 = \frac{1}{2}(x - y - z), \alpha_3 = \frac{1}{2}(-x + y - z), \alpha_4 = \frac{1}{2}(-x - y + z).$$

Thus $K(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = K(x, y, z)$.

$$\begin{aligned} x^2 &= (\alpha_1 + \alpha_2)^2 = -(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\ y^2 &= (\alpha_1 + \alpha_3)^2 = -(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) \\ z^2 &= (\alpha_1 + \alpha_4)^2 = -(\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) \end{aligned}$$

These are distinct, e.g. if $y^2 = z^2$ then $y = \pm z$ and so either $\alpha_3 = \alpha_4$ or $\alpha_1 = \alpha_2$. The x^2, y^2, z^2 are permuted by G and are fixed by $G \cap V_4$ by observation. So $K(x^2, y^2, z^2) \leq M = L^{G \cap V_4}$.

Claim: We have equality $M = K(x^2, y^2, z^2)$. Consider the **resolvent cubic** $g(t) = (t - x^2)(t - y^2)(t - z^2) \in K[t]$. Note that its coefficients are fixed by G and so lie in K.

Observe $D(f) = D(g)$, (check on example sheet). So $K(\Delta) \leq K(x^2, y^2, z^2)$. Now observe that $\mathrm{Gal}(L/K(x^2, y^2, z^2)) = \mathrm{Gal}(K(x, y, z)/K(x^2, y^2, z^2))$.

$$K(x^2, y^2, z^2) \leq K(x, y^2, z^2) \leq K(x, y, z^2) \leq K(x, y, z)$$

where each extension is degree 1 or 2. So $\left|K(x, y, z) : K(x^2, y^2, z^2)\right|$ divides 8. So, the elements of $\mathrm{Gal}(L/K(x^2, y^2, z^2))$ have order dividing 8. But $\mathrm{Gal}(L/K(x^2, y^2, z^2)) \leq G \cap A_4$, so $\mathrm{Gal}(L/K(x^2, y^2, z^2)) \leq G \cap V_4$. By the Fundamental Theorem of Galois Theory, $M = K(x^2, y^2, z^2)$. $\square$

Consider the coefficients of $g(t)$: x^2, y^2, z^2 are permuted by G and so the coefficients of $g(t)$ are fixed by G and therefore in K.

$$\begin{aligned} x^2 + y^2 + z^2 &= -2b, \\ x^2y^2 + x^2z^2 + y^2z^2 &= b^2 - 4d, \\ xyz = -c, \quad x^2y^2z^2 &= c^2, \\ \text{So } g(t) &= t^3 + 2bt^2 + (b^2 - 4d)t - c^2 \end{aligned}$$

We know how to solve cubics and so we can solve for x^2, y^2, z^2. Then we can solve for x, y, z. Then use the formulae from earlier to find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

$$\begin{tikzcd}
 & L \arrow[d, dash] & \{e\} \arrow[d, dash] \\
\text{normal extension of } K & M \arrow[d, dash, "1 \text{ or } 3"'] & G \cap V_4 \lhd G \arrow[d, dash, "1 \text{ or } 3"'] \\
 & K(\Delta) \arrow[d, dash] & G \cap A_4 \arrow[d, dash] \\
 & K & G = \mathrm{Gal}(L/K) \leq S_4
\end{tikzcd}$$

Example. For $f(t) = t^4 + 4t^2 + 2$, we have $g(t) = t^3 + 8t^2 + 8t$ reducible.

4.5 Solubility by radicals

Now suppose we have a Galois extension $K \leq L$ with

$$K = L_0 \leq L_1 \leq \cdots \leq L_m = L$$

such that $L_i \leq L_{i+1}$ is either cyclotomic or Kummer extension.

Let $G = \mathrm{Gal}(L/K)$. There is a corresponding chain of subgroups of G

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{e\}$$

with $G_i = \mathrm{Gal}(L/L_i)$, $L_i = L^{G_i}$ from Theorem 3.2.

However each extension $L_i \leq L_{i+1}$ is Galois and we know

$$G_{i+1} = \mathrm{Gal}(L/L_{i+1}) \lhd \mathrm{Gal}(L/L_i) = G_i$$

hence $G_i/G_{i+1} \cong \mathrm{Gal}(L_{i+1}/L_i)$, which is abelian if $L_i \leq L_{i+1}$ is cyclotomic and cyclic if $L_i \leq L_{i+1}$ is Kummer.

Definition 4.15 (Soluble group)**.** A group is **soluble** if there is a chain of subgroups

$$\{e\} = G_m \lhd G_{m-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

with G_i/G_{i+1} abelian.

Example.

- S_3 is soluble: $\{e\} \lhd \langle(123)\rangle \lhd S_3$.
- S_4 is soluble: $\{e\} \lhd V_4 \lhd A_4 \lhd S_4$ and $A_4/V_4 \cong C_3$, $S_4/A_4 \cong C_2$.
- A_5 is simple, and therefore any normal subgroup is A_5 and so any chain would have a non-abelian quotient. So, A_5 is not soluble.

Lemma 4.16. A finite group G is soluble if and only if we have

$$\{e\} = G_m \lhd G_{m-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

with G_i/G_{i+1} cyclic.

Proof. ($\Leftarrow$) is immediate. ($\Rightarrow$). We know about the structure of finite abelian groups. If A abelian then there is a chain

$$\{e\} = A_r \lhd A_{r-1} \lhd \cdots \lhd A_0 = A$$

with A_r/A_{r+1} cyclic. Thus if we have a chain with abelian factors G_i/G_{i+1} we can refine it to have cyclic factors. $\square$

Definition 4.17 (Derived subgroup)**.** The **derived subgroup** G' of a group G is the subgroup generated by all the commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1 g_2 \in G$.

Lemma 4.18. Let $K \lhd G$. Then G/K abelian $\iff$ $G' \leq K$.

Proof.

$$\begin{aligned} G/K \text{ abelian} &\iff Kg_1Kg_2Kg_1^{-1}Kg_2^{-1} = K \quad \forall g_1, g_2 \in G \\ &\iff g_1g_2g_1^{-1}g_2^{-1} \in K \\ &\iff G' \leq K. \end{aligned}$$

□

Remark.

(1) Next term in Representation Theory, we will prove Burnside's theorem: If $|G| = p^a q^b$ for p, q distinct primes then G is soluble.

(2) The Feit-Thompson theorem says that if $|G|$ is odd, then G is soluble.

(3) There's an analogue of Sylow's theorems due to Philip Hall. For all $|G| = mn$ with m, n coprime there is a subgroup of order m iff G is soluble.

Definition 4.19 (Derived series)**.** The **derived series** $\{G^{(m)}\}$ of G is defined inductively:

$$\begin{aligned} G^{(0)} &= G \\ G^{(1)} &= G' \\ G^{(2)} &= (G')' \\ G^{(j+1)} &= (G^{(j)})' \end{aligned}$$

Thus $G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \rhd \cdots$ with $G^{(j)}/G^{(j+1)}$ abelian.

Lemma 4.20. For G finite, G is soluble $\iff$ $G^{(m)} = \{e\}$ for some m.

Proof. If $G^{(m)} = \{e\}$ then the derived series gives a chain in the definition of solubility.
Conversely if there is such a chain

$$G \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_m = \{e\}$$

with G_i/G_{i+1} abelian then an easy induction shows that $G^{(j)} \leq G_j$ and so $G^{(m)} = \{e\}$. □

Lemma 4.21.

(i) Let $H \leq G$, G soluble. Then H soluble.

(ii) Let $H \lhd G$, then G soluble $\iff$ H and G/H both soluble.

Proof.

(i) G soluble $\implies$ $G^{(m)} = \{e\}$ by Lemma 4.20. But $H^{(m)} \leq G^{(m)}$ and so H soluble by Lemma 4.20.

(ii) Let $H \lhd G$, then G soluble $\implies$ H soluble by (i). G soluble $\implies$ $G^{(m)} = \{e\}$, say. Observe that

$$\left(\frac{G}{H}\right)' = \frac{G'H}{H} \leq \frac{G}{H}.$$

Similarly,

$$(\frac{G}{H})^{(j)} = \frac{G^{(j)}H}{H} \leq \frac{G}{H}.$$

Thus $(G/H)^{(m)} = H/H$, a trivial subgroup of G/H and so G/H soluble.

Now consider the converse. Suppose that H and G/H are soluble. $H^(r) = \{e\}$ and $(G/H)^{(s)} = H/H$. But

$$\left(\frac{G}{H}\right)^{(s)} = \frac{G^{(s)}H}{H}$$

so $G^{(s)}H = H$ thus $G^{(s)} \leq H$. Hence $G^{(r+s)} \leq H^{(r)} = \{e\}$. Thus G is soluble by Lemma 4.20.

□

Example. S_5 is not soluble since its subgroup A_5 is not soluble.

Theorem 4.22. Let K be a field and $f(t) \in K[t]$. Assume $\operatorname{char} K = 0$. Then $f(t)$ is soluble by radicals over $K \iff \operatorname{Gal} f$ over K is soluble.

Remark. We don't need to restrict to $\operatorname{char} K = 0$. What we need to do for a particular $f(t)$ is to avoid a finite number of bad characteristics (avoid characteristics $\leq \deg f(t)$).

Corollary 4.23. If $f(t)$ is a monic irreducible polynomial $\in K[t]$ with $\operatorname{Gal}(f) \cong A_5$ or S_5 then $f(t)$ is not soluble by radicals (with $\operatorname{char} K = 0$).

Example. In Example 3.9, we considered $f(t) = t^5 - 6t + 3 \in \mathbb{Q}[t]$. $\operatorname{Gal}(f)$ over $\mathbb{Q}$ is S_5, and so f is not soluble by radicals.

Lemma 4.24. If $K \leq N$ is an extension by radicals then $\exists N'$ with $N \leq N'$ with $K \leq N'$ is an extension by radicals, with $K \leq N'$ a Galois extension.

Proof of Theorem 4.22. Suppose $f(t)$ is soluble by radicals. Thus if L is the splitting field of $f(t)$ over K then L lies in an extension of K by radicals

$$K = L_0 \leq L_1 \leq \cdots \leq L_m$$

with each $L_i \leq L_{i+1}$ cyclotomic or Kummer.

With Lemma 4.24, we may assume L_m is Galois over K. By Fundamental Theorem of Galois Theory there is a corresponding chain of subgroups of $\operatorname{Gal}(L_m/K)$. Our previous discussion at the beginning of this section (before Lemma 4.13) we know that $\operatorname{Gal}(L_m/K)$ is soluble.

But $F \leq L \leq L_m$ with $K \leq L$ Galois. By Fundamental Theorem of Galois Theory, $\operatorname{Gal}(L/K) \cong \operatorname{Gal}(L_m/K)/\operatorname{Gal}(L_m/L)$.

But quotients of soluble groups are soluble, so $\operatorname{Gal}(L/K)$ is soluble. □

Proof of Lemma 4.24. We have $K = L_0 \leq L_1 \leq \cdots \leq L_m$ with each $L_i \leq L_{i+1}$ cyclotomic or Kummer, and we want to embed this into a Galois extension of the same form.

Assume $\operatorname{char} K = 0$. By the primitive element theorem, $L_m = K(\alpha_1)$ for some α_1. Let $g(t)$ be the minimal polynomial of α_1 over K with splitting field M. Thus $M = K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are roots of $g(t)$.

There are K-homomorphisms

$$\begin{aligned} \phi_i : M &\longrightarrow M \\ \alpha_1 &\longmapsto \alpha_i \end{aligned}$$

extending the K-homs $K(\alpha_1) \to K(\alpha_i) \leq M$.

The tower $K \leq \phi_i(K) \leq \phi_i(L_1) \leq \cdots \leq \phi_i(L_m) = K(\alpha_i)$ with cyclotomic or Kummer extensions as before, Consider $L_m = K(\alpha_1) \leq \phi_2(L_1)(\alpha_1) \leq \phi_2(L_2)(\alpha_1) \leq \cdots \leq \phi_2(L_m)(\alpha_1) = K(\alpha_1, \alpha_2)$.

Consider the extension $\phi_2(L_j)(\alpha_1) \leq \phi_2(L_{j+1})(\alpha_1)$:

> **if $L_j \leq L_{j+1}$ is cyclotomic** then all the roots of unity adjoined are now in $L_m = K(\alpha_1)$ and so $\phi_2(L_j)(\alpha_1) = \phi_2(L_{j+1})(\alpha_1)$.
>
> **if $L_j \leq L_{j+1}$ is Kummer** then we obtain L_{j+1} by adjoining roots of an element of L_j and so we obtain $\phi_2(L_{j+1})$ by adjoining roots of an element in $\phi_2(L_j)$. Hence we get from $\phi_2(L_j)(\alpha_1)$ to $\phi_2(L_{j+1})(\alpha_1)$ by adjoining roots of an element of $\phi_2(L_j)$. So it's a Kummer extension.

Now continue to get suitable chain $K(\alpha_1, \alpha_2) \leq \cdots \leq K(\alpha_1, \alpha_2, \alpha_3)$.

Thus we get a suitable chain from K to $K(\alpha_1, \ldots, \alpha_n) = M$. Observe that $K \leq M$ is Galois. □

Converse of Theorem 4.22. Suppose $G = \mathrm{Gal}(f)$ over K is soluble (and $\operatorname{char} K = 0$). Let L be the splitting field of $f(t)$ over K and so $|G| = |L : K| = n$. Set $m = n!$ and let ξ be a primitive root of unity and consider $L(\xi)$.

$$\begin{tikzcd}
 & L(\xi) \arrow[dl, dash, "\leq n"'] \arrow[dr, dash] & \\
K(\xi) \arrow[dr, dash] & & L \arrow[dl, dash, "n"] \\
 & K &
\end{tikzcd}$$

Our proof is similar to that used for cubics. Observe that $|L(\xi) : K(\xi)| \leq n$. By the Primitive Element Theorem $L = K(\alpha)$ for some α with minimal polynomial $g(t)$ say of degree n. Then $L(\xi) = K(\xi)(\alpha)$ and the minimal polynomial of α over $K(\xi)$ divides $g(t)$ and so is of degree $\leq n$.

Then $\mathrm{Gal}(L(\xi)/K)$ is soluble since $\mathrm{Gal}(L(\xi)/L)$ is soluble and $\mathrm{Gal}(L/K) \cong \frac{\mathrm{Gal}(L(\xi)/K)}{\mathrm{Gal}(L(\xi)/L)}$ soluble by Fundamental Theorem of Galois Theory and Lemma 4.21. Then the subgroup $\mathrm{Gal}(L(\xi)/K(\xi)) \leq \mathrm{Gal}(L(\xi)/K)$ is soluble by Lemma 4.21.

Thus there is a chain of subgroups

$$\mathrm{Gal}(L(\xi)/K(\xi)) = G_0 \rhd G_1 \rhd \cdots \rhd G_m = \{e\},$$

with G_i/G_{i+1} cyclic (using Lemma 4.16).

Now use the Fundamental Theorem of Galois Theory to get a corresponding chain of fields $K(\xi) \leq K_1 \leq \cdots \leq K_m = L(\xi)$, with each $K_i \leq K_{i+1}$ Galois, with cyclic Galois group. By Theorem 4.11, all these extensions are Kummer (not all the extensions are of degree $\leq n$ and so we have the appropriate roots of unity). Thus we've embedded L in an extension of K by radicals. $\square$

Example.

(1) Take $f(t) = t^4 + 4t^2 + 2$, irreducible over $\mathbb{Q}$ by Eisenstein's criterion, and it has roots $\pm\sqrt{-2 \pm \sqrt{2}}$ (two conjugate pairs). Complex conjugation gives double transposition. The resolvent cubic is $g(t) = t^3 + 8t^2 + 8t$ which is reducible with roots $0, -4 \pm 2\sqrt{2}$.

We have $L = K(\sqrt{-4 + 2\sqrt{2}}, \sqrt{-4 - 2\sqrt{2}})$, and $M = K(-4 + 2\sqrt{2}, -4 - 2\sqrt{2}) = K(\sqrt{2})$. So $|L : K| = 8$, and the Galois group is dihedral.

(2) $f(t) = t^4 + 2t + 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. Its discriminant is $101 \cdot 4^2$, not a square, and the resolvent cubic is $g(t) = t^3 - 8t - 4$, irreducible since it is $t^3 + 2t + 1$ mod 5, irreducible. So there is a 3-cycle in the Galois group.

$\mathrm{Gal}(f)$ is a transitive subgroup of S_4 and not of A_4 containing a 3-cycle, so $\mathrm{Gal}(f)$ is S_4.

(3) $f(t) = t^5 - t - 1$ is irreducible over $\mathbb{Q}$ since it is irreducible mod 5, and so the Galois group contains a 5-cycle and is a transitive subgroup of S_5. Mod 2, it factorises as a product of an irreducible cubic and quadratic: $(t^3 + t^2 + 1)(t^2 + t + 1)$. So $\mathrm{Gal}(\bar{f})$ is generated by an element of cycle type $3, 2$, so $\mathrm{Gal}(f)$ contains an element of cycle type $3, 2$, so g^3 is a transposition. A subgroup of S_5 containing a 5-cycle and a transposition must be S_5 itself. So $\mathrm{Gal}(f) = S_5$ over $\mathbb{Q}$.

5 Final Thoughts

5.1 Algebraic closure

Definition 5.1 (Algebraically closed)**.** A field L is **algebraically closed** if any $f(t) \in L[t]$ splits into a product of linear factors in $L[t]$.

Remark. This is equivalent to saying that any $f(t) \in L[t]$ has a root in L, or that any algebraic extension of L is L itself.

Definition 5.2 (Algebraic closure)**.** An extension $K \leq L$ is an algebraic closure of K if $K \leq L$ is algebraic and L is algebraically closed.

Lemma 5.3. If $K \leq L$ is algebraic and every polynomial in $K[t]$ splits completely over L, then L is an algebraic closure of K.

Proof. We need to show L is algebraically closed. Suppose $L \leq L(\alpha)$ is a finite extension, and $f_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ is the minimal polynomial of α over L. Let $M = K(a_0, a_1, \ldots, a_{n-1})$. Then $M \leq M(\alpha)$ is a finite extension. But each a_i is algebraic over K and so $|M : K| < \infty$. Hence $|M(\alpha) : K| < \infty$ by Tower law and so α is algebraic over K. The minimal polynomial over K must split over L, and so $\alpha \in L$. Thus any algebraic extension of L is L itself. $\square$

Example. $\mathbb{A} = \{\, \alpha \in C \mid \alpha \text{ algebraic over } \mathbb{Q} \,\}$ It is a subfield of $\mathbb{C}$: If α, β are algebraic over $\mathbb{Q}$ then $|\mathbb{Q}(\alpha, \beta) : \mathbb{Q}| < \infty$ so if $\gamma = \alpha + \beta, \alpha - \beta, \alpha\beta$ or α/β (if $\beta \neq 0$) we get $\mathbb{Q}(\gamma) \leq \mathbb{Q}(\alpha, \beta)$ and so $|\mathbb{Q}(\gamma) : \mathbb{Q} < \infty$. So γ is algebraic over $\mathbb{Q}$ so $\gamma \in \mathbb{A}$. Therefore $\mathbb{A} = \overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$.

If we want to prove existence and uniqueness of algebraic closures in general then we need to appeal to Zorn's Lemma. This appears in Logic and Set Theory, and it's equivalent to the Axiom of Choice and to the Well Ordering Principle.

Definition 5.4 (Partial order)**.** $(\mathcal{S}, \leq)$ is a **partial order** on $\mathcal{S}$ if

(i) $\forall x \in \mathcal{S}\ x \leq x$

(ii) $x \leq y$ and $y \leq z \implies x \leq z$.

(iii) if $x \leq y$ and $y \leq x$ then $x = y$.

$\mathcal{S}$ is **totally ordered** if for any $x, y \in S$ either $x \leq y$ or $y \leq x$. A **chain** is a partially ordered set $(\mathcal{S}, \leq)$ that is a totally ordered subset.

Lemma 5.5 (Zorn's Lemma)**.** Let $(\mathcal{S}, \leq)$ be a non-empty partially ordered set. Suppose that any chain has an upper bound in $\mathcal{S}$. Then $\mathcal{S}$ has a maximal element.

Lemma 5.6. Let R be a ring. Then R has a maximal ideal.

Proof. Let $\mathcal{S}$ be the set of proper ideals of R. This is is non-empty, since (0) is proper. Partially order $\mathcal{S}$ by inclusion. Any ideal I is proper $\iff 1 \notin I$. Any chain of proper ideals has an upper bound in $\mathcal{S}$, namely the union of the chain. Zorn's Lemma gives that $\mathcal{S}$ has a maximal element, i.e. a maximal ideal of R. $\square$

Theorem 5.7 (Existence of algebraic closures)**.** For any field K there is an algebraic closure.

Proof. Let

$$\mathcal{S} = \{\, (f(t), j) \mid f(t) \text{ irreducible, monic in } K[t], 1 \le j \le \deg f \,\}$$

For each pair $s = (f(t), j) \in \mathcal{S}$ we introduce an indeterminate $X_s = X_{f,j}$. Consider the polynomial ring $K[X_s : s \in \mathcal{S}]$ and set

$$\tilde{f}(t) = f(t) - \prod_{j=1}^{\deg g} (t - X_{f,j}) \in K[X_s : s \in \mathcal{S}][t].$$

Let $I \lhd K[X_s : s \in \mathcal{S}]$ generated by all the coefficients of all the $\tilde{f}(t)$. Denote the coefficents of $\tilde{f}(t)$ by $a_{f,l}$ for $0 \le l \le \deg f$.

Claim: $I \neq K[X_s : s \in \mathcal{S}]$. Proof: Suppose $1 \in I$ and aim for a contradiction.

$$b_1 a_{f_1,l_1} + \cdots + b_N a_{f_N,l_N} = 1 \quad \text{in } K[X_s : s \in \mathcal{S}]. \tag{+}$$

Let L be a splitting field for $f_1(t) \cdots f_N(t)$.

For each i, f_i splits over L. $f_i(t) = \prod_{j=1}^{\deg f_i} (t - a_{ij})$. Define a K-linear ring homomorphism, identity on K,

$$\begin{aligned} \theta : K[X_s : s \in \mathcal{S}] &\longrightarrow L \\ X_{f_i,j} &\longmapsto \alpha_{ij} \\ X_s &\longmapsto 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

This induces a map $K[X_s : s \in \mathcal{S}] \to L[t]$. Then

$$\begin{aligned} \theta(\tilde{f}_i(t)) &= \theta(f_i(t)) - \prod_{j=1}^{\deg f_i} \theta(t - X_{f_i,j}) \\ &= f_i(t) - \prod_{j=1}^{\deg f_i} (t - \alpha_{i,j}) = 0. \end{aligned}$$

But then $\theta(a_{f_i,j}) = 0$ since $a_{f_i,j}$ are the coefficients of $\tilde{f}_i(t)$. But applying θ to (+) we get $0 = 1$.

Then I is a proper ideal of $K[X_s : s \in \mathcal{S}]$. By Zorn's Lemma there is a maximal ideal P of $K[X_s : s \in \mathcal{S}]$ containing I. Set $L_1 = K[X_s : s \in \mathcal{S}]/P$, a field. Thus we have a field extension $K \le L_1$.

Claim: L_1 is an algebraic closure of K. First show $K \le L_1$ is algebraic: L_1 is generated by the maps $x_{f,j}$ of the $X_{f,j}$. However $\tilde{f}(t)$ has coefficients in I and so its image $L_1[t]$ is the zero polynomial. Thus in $L_1[t]$,

$$f(t) = \prod (t - x_{f,j}) \tag{$*$}$$

and so $f(x_{f,j}) = 0$. Thus the $x_{f,j}$ are algebraic.

Any element of L_1 involves only finitely many of the $x_{i,j}$ and so is algebraic over K. Moreover from ($*$) any $f(t) \in K[t]$ splits completely over L_1.

The result follows from Lemma 5.3. □

Theorem 5.8. Suppose $\theta : K \to L$ is a ring homomorphism and L is algebraically closed. Suppose $K \leq M$ is an algebraic extension. Then θ can be extended to a homomorphism $\theta : M \to L$ (i.e. $\phi|_K = \theta$).

Proof. Let

$$\xi = \{ (N, \phi) \mid K \leq N \leq M, \phi \text{ a homomorphism } N \to L \text{ extending } \theta \}.$$

Partially order ξ with $(N_1, \phi_1) \leq (N_2, \phi_2)$ if $N_1 \leq N_2$ and $\phi_2|_{N_2} = \phi_1$. ξ is non-empty since $(K, \theta) \in \xi$.

If there is a chain $(N_1, \phi_1) \leq \cdots$ then set $N = \bigcup N_\lambda$. This is a subfield of M, and we can define $\psi : N \to L$ as follows: if $\alpha \in N$ then $\alpha \in N_\lambda$ for some λ and we set $\psi(\alpha) = \phi_\lambda(\alpha)$. This is well defined.

Then (N, ψ) is an upper bound for our chain ξ.

Zorn's Lemma applies and gives a maximal element of ξ, (N, ϕ). We now show $N = M$. Given $\alpha \in M$, it is algebraic over K, and hence over N. Let $f_\alpha(t)$ be its minimal polynomial over N. But $\phi f(t)$ is in $L[t]$ and so splits completely over L, since L is algebraically closed.

So $\phi f(t) = (t - \beta_1) \cdots (t - \beta_r)$, say. Since $\phi f(B_\gamma) = 0$ then there is a map

$$\begin{array}{ccc} N(\alpha) \cong \frac{N[t]}{(f\alpha(t))} & \longrightarrow & L \\ \\ \alpha & \longmapsto & \beta_1 \qquad \text{extending} \phi \end{array}$$

Maximality of (N, ϕ) implies that $N(\alpha) = N$. So $\alpha \in N$, so $N = M$. □

Theorem 5.9 (Uniquness of algebraic closures)**.** If $K \leq L_1$, $L \leq L_2$ are two algebraic closures of K then there exists an isomorphism $\phi : L_1 \to L_2$.

Proof. By Theorem 5.8 there is a homomorphism $\phi : L_1 \to L_2$ extending the embedding of K into L_2. Since $K \leq L_2$ is algebraic, so too is $\phi(L_1)$. But L_1 is algebraically closed and so $\phi(L_1)$ is algebraically closed. So $L_2 = \phi(L_1)$ and ϕ is an isomorphism. □

5.2 Symmetric polynomials and invariant theory

In the buildup of the Fundamental Theorem of Galois Theory we met Artin's Theorem: Let $K \leq L$ and H a finite group of $\mathrm{Aut}_K(L)$. Let $M = L^H$. Then $M \leq L$ is a Galois extension and $H = \mathrm{Gal}(L/M)$.

Example. $L = K(X_1, \ldots, X_n)$. Let S_n permute the variables. These permutations induce K-automorphisms of L. By Artin's Theorem, if $M = L^{S_n} \leq L$ then it is Galois and $\mathrm{Gal}(L/M) = S_n$. Thus S_n is a Galois group of some field extension.

We know that we can regard any finite group G as a subgroup of some S_n, and so we see that any finite group is a Galois group of some field extension (using Fundamental Theorem of Galois Theory). Let

$$\begin{aligned} f(t) &= (t - X_1)(t - X_2) \cdots (t - X_n) \in M[t] \\ &= t^n - s_1 t^{n-1} + s_2 t^{n-2} + \cdots + (-1)^n s_n. \end{aligned}$$

Thus $s_1 = X_1 + \cdots + X_n$, $s_2 = \sum_{i<j} X_i X_j$, and $s_n = X_1 X_2 \cdots X_n$.

Definition 5.10 (Elementary symmetric polynomials)**.** These s_i are the elementary symmetric polynomials.

Theorem 5.11. The fixed field $M = L^{s_n} = K(s_1, \ldots, s_n)$ and the $s_1, \ldots, s_n$ are algebraically independent over K (in L).

Definition 5.12. $\alpha_1, \ldots, \alpha_n$ are **algebraically independent** over K if the ring homomorphism $K[Y_1, \ldots, Y_n] \to K[\alpha_1, \ldots, \alpha_n] \leq L$ is an isomorphism where $K[Y_1, \ldots, Y_n]$ is the polynomial ring in $Y_1, \ldots, Y_n$.