Part III – Model Theory (Ongoing course, rough)

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0 Introduction

Model theory is a part of logic that began by looking at algebraic objects such as groups and combinatorial objects such like graphs, described in formal language. The basic question in model theory is: 'how powerful is our description of these objects to pin them down'? In Logic and Set Theory, the focus was on what was provable from a theory and language, but here we focus on whether or not a model exists.

1 Languages and structures

Definition 1.1 (Language)**.** A **language** L consists of

(i) a set $\mathscr{F}$ of function symbols, and for each $f \in \mathscr{F}$ a positive integer m_f the **arity** of f.

(ii) a set $\mathscr{R}$ of relation symbols, and for each $R \in \mathscr{R}$, a positive integer m_R.

(iii) a set $\mathscr{C}$ of constant symbols.

Note: each of $\mathscr{F}, \mathscr{R}$ and $\mathscr{C}$ can be empty.

Example. Take $L = \{\{\cdot, ^{-1}\}, \{1\}\}$, for $\cdot$ a binary function and $^{-1}$ an unary function, 1 a constant. This is the language of groups, call it L_{gp}. Also, $L_{\text{lo}} = \{<\}$ a single binary relation, for linear orders.

Definition 1.2 (L-structure)**.** Given a language L, say, an L**-structure** consists of

(i) a set M, the **domain**

(ii) for each $f \in \mathscr{F}$, a function $f^{\mathcal{M}} : M^{m_f} \to M$.

(iii) for each $R \in \mathscr{R}$, a relation $R^{\mathcal{M}} \subseteq M^{m_R}$.

(iv) for each $c \in \mathscr{C}$, an element $c^{\mathcal{M}} \in M$.

f^M, R^M, c^M are the **interpretations** of f, R, c respectively.

Remark 1.3. We often fail to distinguish between the symbols in L and their interpretations in a structure, if the interpretations are clear from the context.

We may write $\mathcal{M} = \langle M, \mathscr{F}, \mathscr{R}, \mathscr{C} \rangle$.

Example 1.4.

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, ^{-1}\}, 1\rangle$ is an L_{gp}-structure.

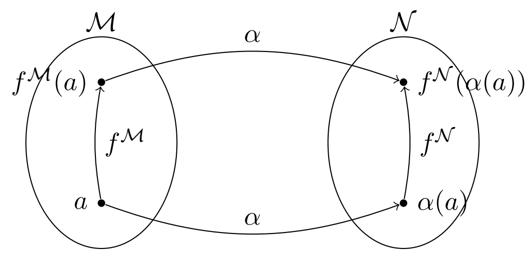
(b) $\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0\rangle$ is an L_{gp}-structure.

(c) $\mathcal{Q} = \langle \mathbb{Q}, <\rangle$ is an L_{lo}-structure.

Definition 1.5 (Embedding)**.** Let L be a language, let $\mathcal{M}, \mathcal{N}$ be L-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is a one-to-one mapping $\alpha : M \to N$ such that

(i) for all $f \in \mathscr{F}$, and $a_1, \ldots, a_{m_f} \in M$,

$$\alpha(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\alpha(a_1), \ldots, \alpha(a_{n_f}))$$



(ii) for all $R \in \mathscr{R}$, and $a_1, \ldots, a_{n_R} \in M$

$$(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}} \iff (\alpha(a_1), \ldots, \alpha(a_{n_R})) \in R^{\mathcal{N}}$$

(iii) for all $c \in \mathscr{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

An **isomorphism** of $\mathcal{M}$ into $\mathcal{N}$ is a surjective embedding (onto).

Exercise 1.6. Let G_1, G_2 be groups, regarded as L_{gp}-structures. Check that $G_1 \simeq G_2$ in the usual algebra sense if and only if there is an isomorphism $\alpha : G_1 \to G_2$ in the sense of Definition 1.5.

2 Review: Terms, formulae and their interpretations

In addition to the symbols of L, we also have

(i) infinitely many variables $\{x_i\}_{i \in I}$

(ii) logical connectives $\wedge, \neg$ (also expresses $\vee, \implies, \iff$)

(iii) quantifier $\exists$ (also expresses $\forall$)

(iv) (,)

(v) equality symbol $=$

Definition 2.1 (L-terms)**.** ***L*-terms** are defined recursively as follows:

- any variable x_i is a term
- any constant symbol is a term
- for any $f \in \mathscr{F}$, $f(t_1, \ldots, t_{m_f})$ for any terms $t_1, \ldots, t_{m_f}$ is a term
- nothing else is a term

Notation: we write $t(x_1, \ldots, x_m)$ to mean that the variables appearing in t are among $x_1, \ldots, x_m$.

Lecture 2 **Example.** Take $\mathcal{R} = \langle \mathbb{R}^*, \{\cdot, ^{-1}\}, 1 \rangle$. Then $\cdot(\cdot(x_1, x_2), x_3)$ is a term, usually written $(x_1 \cdot x_2) \cdot x_3$. Also, $(\cdot(1, x_1))^{-1}$ is a term, written $(1 \cdot x)^{-1}$

Definition 2.2. If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, \ldots, x_k)$ we assign a function a function $t^{\mathcal{M}} : M^k \to M$ defined as follows:

(i) If $t = x_i$, $t^{\mathcal{M}}[a_1, \ldots, a_k] = a_i$

(ii) If $t = c$, $t^{\mathcal{M}}[a_1, \ldots, a_k] = c^{\mathcal{M}}$.

(iii) If $t = f(t(x_1, \ldots, x_k), \ldots, t_{m_f}(x_1, \ldots, x_k))$,

$$t^{\mathcal{M}}(a_1, \ldots, a_k) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \ldots, a_k), \ldots, t_{m_f}^{\mathcal{M}}(a_1, \ldots, a_k))$$

Notice in L_{gp}, the term $x_2 \cdot x_3$ can be described as $t_1(x_1, x_2, x_3)$ or $t_2(x_1, x_2, x_3, x_4)$, or infinitely many other ways. Then t_1 is assigned to $t_1^{\mathcal{M}} : M^3 \to M$, with $(a_1, a_2, a_3) \mapsto (a_2, a_3)$, and t_2 is assigned to $t_2^{\mathcal{M}} : M^4 \to M$, with $(a_1, a_2, a_3, a_4) \mapsto a_2 \cdot a_3$.

Fact 2.3. Let $\mathcal{M}, \mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, \ldots, x_k)$ and $a_1, \ldots, a_k \in M$ we have

$$\alpha(t^{\mathcal{M}}(a_1, \ldots, a_k)) = t^{\mathcal{N}}(\alpha(a_1), \ldots, \alpha(a_k))$$

Proof. By induction on the complexity of t. Let $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{x} = (x_1, \ldots, x_k)$. Then

(i) if $t = x_i$, then $t^{\mathcal{M}}(\bar{a}) = a_i$, and $t^{\mathcal{N}}(\alpha(a_1), \ldots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds.

(ii) if $t = c$ a constant, then $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$, and $t^{\mathcal{N}}(\alpha(\bar{a})) = c^{\mathcal{N}}$, and $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$, as required.

(iii) if $t = f(t_1(\bar{x}), \ldots, t_{m_f}(\bar{x}))$, then

$$\alpha(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_f}^{\mathcal{M}}(\bar{a}))) = f^{\mathcal{N}}(\alpha(t_1^{\mathcal{M}}(\bar{a})), \ldots, \alpha(t_{m_f}^{\mathcal{M}}(\bar{a})))$$

since α is an embedding. $t_1(\bar{x}), \ldots, t_{m_f}(\bar{x})$ have lower complexity than t, so inductive hypothesis applies. □

Exercise 2.4. Exercise: conclude the proof of Fact 2.3.

Definition 2.5 (Atomic formula)**.** The set of **atomic formulas** of L is defined as follows

(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula

(ii) if R is a relation symbol and $t_1, \ldots, t_{m_R}$ are terms, then $R(t_1, \ldots, t_{m_R})$ is an atomic formula

(iii) nothing else is an atomic formula.

Definition 2.6 (Formula)**.** The set of L**-formulas** is defined as follows

(i) any atomic formula is an L-formula

(ii) if ϕ is an L-formula, then so is $\neg\phi$

(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$

(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i\ \phi$ is an L-formula

(v) nothing else is an L-formula

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$ and $x_1 \cdot x_2 = 1$ are atomic formulas, and $\exists x_1\ (x_1 \cdot x_2) = 1$ is an L_{gp}-formula.

A variable occurs freely in a formula if it does not occur within the scope of a quantifier $\exists$ (the variable is **free**). Otherwise the variable is **bound**. For instance, in $\exists x_1\ (x_1 \cdot x_2) = 1$, x_1 is bound and x_2 is free.

Important convention: no variable occurs both freely and as a bound variable in the same formula.

A **sentence** is a formula with no free variables. $\exists x_1 \exists x_2\ (x_1 \cdot x_2 = 1)$ is an L_{gp}-sentence. Notation: $\phi(x_1, \ldots, x_k)$ means that the free variables in ϕ are among $x_1, \ldots, x_k$.

Definition 2.7 ($\vDash$)**.** Let $\phi(x_1, \ldots, x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = (a_1, \ldots, a_k)$ be elements of M. We define $\mathcal{M} \vDash \phi(\bar{a})$ as follows.

(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

(ii) if ϕ is $R(t_1, \ldots, t_{m_k})$ then $\mathcal{M} \vDash \phi(\bar{a})$ iff

$$(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_k}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}.$$

(iii) if ϕ is $\psi \wedge \chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$.

(iv) if $\phi = \neg\psi$ then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$. (this is well-defined since $\psi(\bar{a})$ is shorter than $\phi(\bar{a})$)

(v) if ϕ is $\exists x_j : \chi(x_1, \ldots, x_k, x_j)$ (where $x_j \neq x_i$ for $i = 1, \ldots, k$). Then $\mathcal{M} \vDash \phi(\bar{a})$ iff there is $b \in \mathcal{M}$ such that $\mathcal{M} \vDash \chi(a_1, \ldots, a_k, b)$.

Example. For $\mathcal{R} = \langle \mathbb{R}^*, \cdot, ^{-1}, 1 \rangle$, if $\phi(x_1) = \exists x_2\ (x_2 \cdot x_2) = x_1$ then $\mathcal{R} \vDash \phi(1)$ but $\mathcal{R} \nvDash \phi(-1)$.

Notation 2.8 (Useful abbreviations)**.** We write

– $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$

– $\phi \to \psi$ for $\neg\phi \vee \psi$

– $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$

– $\forall x_i\ \phi$ for $\neg\exists x_i\ (\neg\phi)$

Proposition 2.9. Let $\mathcal{M}, \mathcal{N}$ be L-structures, let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. Let $\phi(\bar{x})$ be atomic and $\bar{a} \in M^{|\bar{x}|}$, then

$$M \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a})).$$

Question: If ϕ is an L-formula, not necessarily atomic, does Proposition 2.9 hold?

Lecture 3

Proof of Proposition 2.9. Cases:

(i) $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. (Exercise: complete this case, using Fact 2.3)

(ii) $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \ldots, t_{m_R}(\bar{x}))$. Then $\mathcal{M} \vDash R(t_1(\bar{a}), \ldots, t_{m_R}(\bar{a}))$ if and only if... (Exercise: complete this case)

□

Exercise 2.10. Show that Proposition 2.9 holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

Example 2.11. Do embeddings preserve *all* formulas? No. Take $\mathcal{Z} = (\mathbb{Z}, <)$ and $\mathcal{Q} = (\mathbb{Q}, <)$ an L_{lo}-structure Then $\alpha : \mathbb{Z} \to \mathbb{Q}$ (inclusion) is an embedding, but

$$\phi(x_1, x_2) = \exists x_3\, (x_1 < x_3 \wedge x_3 < x_2).$$
$$\mathcal{Q} \vDash \phi(1, 2) \text{ but } \mathcal{Z} \nvDash \phi(1, 2).$$

Fact 2.12. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{M} \vDash \phi(\alpha(\bar{a})).$$

Proof. Exercise. □

3 Theories and elementarity

Throughout, L is a language, $\mathcal{M}, \mathcal{N}$ are L-structures.

Definition 3.1 (L-theory)**.** An L**-theory** T is a set of L-sentences. $\mathcal{M}$ is a **model** of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$. The class of all the models of T is written $\mathrm{Mod}(T)$. The theory of $\mathcal{M}$ is the set

$$\mathrm{Th}(\mathcal{M}) = \{ \sigma \mid \sigma \text{ is an } L\text{-structure and } \mathcal{M} \vDash \sigma \} .$$

Example 3.2. Let T_{gp} be the set of L_{gp}-sentences

(i) $\forall x_1 x_2 x_3 \, (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$

(ii) $\forall x_1 \, (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$

(iii) $\forall x_1 \, (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$

Clearly for a group G, $G \vDash T_{\mathrm{gp}}$. For a specific G, clearly $\mathrm{Th}(G)$ is larger than T_{gp}!

Definition 3.3 (Elementarily equivalent)**.** Say $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$. Clearly if $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$ but if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as L_{lo}-structures.

Definition 3.4 (Elementary substructure)**.**

(i) an embedding $\beta : \mathcal{M} \to \mathcal{N}$ is **elementary** if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a}))$$

(ii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is said to be a **substructure** of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$.

(iii) if $M \subseteq N$ and $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is said to be an **elementary substructure** of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example 3.5. Consider $\mathcal{M} = [0,1] \subseteq \mathbb{R}$, an L_{lo}-structure, where $<$ is the usual order, and $\mathcal{N} = [0,2] \subseteq \mathbb{R}$ in the same way. Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo}-structures.

Is $\mathcal{M} \equiv \mathcal{N}$? Yes: they are isomorphic!

Is $\mathcal{M} \subseteq \mathcal{N}$? Yes (the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$.)

But $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if $\phi(x) = \exists y \, (x < y)$, then

$$\mathcal{N} \vDash \phi(1) \quad \text{and} \quad \mathcal{M} \nvDash \phi(1).$$

Definition 3.6. Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, then

$$L(A) \coloneqq L \cup \{ c_a \mid a \in A \}$$

for c_a each constant symbols. An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way ($c_a^{\mathcal{M}} = a$). The elements of A are called **parameters**. If $\mathcal{M}, \mathcal{N}$ are L-structures and $A \subseteq M \cap N$, then $\mathcal{M} \equiv_A \mathcal{N}$ when $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$ sentences.

Lecture 4 **Exercise 3.7.** $\mathcal{M} \preccurlyeq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of $\mathcal{M}$).

Lemma 3.8 (Tarski-Vaught test)**.** Let $\mathcal{N}$ be an L-structure, let $A \subseteq N$. The following are equivalent:

(i) A is the domain of a structure $\mathcal{M}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$.

(ii) if $\phi(x) \in L(A)$, if $\mathcal{N} \vDash \exists x\ \phi(x)$, then $\mathcal{N} \vDash \phi(b)$ for some $b \in A$.

Proof.

(i) $\Rightarrow$ (ii) Suppose $\mathcal{N} \vDash \phi(x)$. Then by elementarity, $\mathcal{M} \vDash \exists x\ \phi(x)$, and so $\mathcal{M} \vDash \exists x\ \phi(x)$ for $b \in \mathcal{M}$, so again by elementarity $\mathcal{N} \vDash \phi(b)$.

(ii) $\Rightarrow$ (i) First we prove that A is the domain $\mathcal{M} \subseteq \mathcal{N}$. By exercise 4 on sheet 1, it is enough to check:

(a) for each constant c, $c^{\mathcal{N}} \in A$.

(b) for each function symbol f, $f^{\mathcal{N}}(\bar{a}) \in A$ (for all $\bar{a} \in A^{m_f}$).

For (a), use property (ii) with $\exists x\ (x = c)$. For (b) use property (ii) with $\exists x\ (f(\bar{a}) = x)$.

So we now have $\mathcal{M} \subseteq \mathcal{N}$, and the domain of $\mathcal{M}$ is A. Let $\chi(\bar{x})$ be an L-formula. We show that for $\bar{a} \in A^{|\bar{x}|}$,

$$\mathcal{M} \vDash \chi(\bar{a}) \iff \mathcal{N} \vDash \chi(\bar{a}). \tag{$*$}$$

By induction on the complexity of $\chi(\bar{x})$:

– if $\chi(\bar{x})$ is atomic ($*$) follows from $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M}$ is a substructure).

– if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$: straightforward induction.

– if $\chi(\bar{x}) = \exists y\ \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L-formula, suppose that $\mathcal{M} \vDash \chi(\bar{a})$. Then $\mathcal{M} \vDash \exists y\ \psi(\bar{a}, y)$, hence $\mathcal{M} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom} \mathcal{M}$. But then $\mathcal{N} \vDash \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \vDash \chi(\bar{a})$. Now let $\mathcal{N} \vDash \chi(\bar{a})$, i.e. $\mathcal{N} \vDash \exists y\ \psi(\bar{a}, y)$. By property (ii), $\mathcal{N} \vDash \psi(\bar{a}, b)$ for some $b \in A = \operatorname{dom}(\mathcal{M})$. By inductive hypothesis, $\mathcal{M} \vDash \psi(\bar{a}, b)$ and so $\mathcal{M} \vDash \chi(\bar{a})$. ☐

Remark 3.9. Assume the set of variables is countably infinite. Then

– the cardinality of the set of L-formulas is $|L| + \omega$. (We abuse notation and write ω for the ordinal and cardinal, and define the cardinality of L as the # of symbols in it: $|L_{\mathrm{gp}}| = 3$, $|L_{\mathrm{lo}}| = 1$).

– if A is a set of parameters in some structure, the cardinality of the set $L(A)$-formulas is $|A| + |L| + \omega$.

Definition 3.10. Let λ be an ordinal. Then **a chain of length** λ of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$. A **chain of L-structures** is a sequence $\langle \mathcal{M}_i : i < \lambda \rangle$ such that $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i \leq j < \lambda$.

The **union** of this chain is the L-structure $\mathcal{M}$ is defined as follows:

– the domain of $\mathcal{M}$ is $\bigcup_{i<\lambda} M_i$

– $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$ (c is a constant).

- if f is a function symbol, $\bar{a} \in M^{m_f}$, $f^{\mathcal{M}}\bar{a} = f^{\mathcal{M}_i}\bar{a}$ where i is such that $\bar{a} \in M_i^{m_f}$.
- if R is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i<\lambda} R^{\mathcal{M}_i}$

Theorem 3.11 (Downward Löwenheim-Skolem)**.** Let $\mathcal{N}$ be an L-structure, and $|N| \geq |L| + \omega$. Let $A \subseteq N$. Then for any cardinal λ such that $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preccurlyeq \mathcal{N}$ such that

(i) $A \subseteq M$

(ii) $|\mathcal{M}| = \lambda$.

(It helps to think about the case $|L| \leq \omega$, $|A| = \omega$ and $|N|$ is uncountable).

For instance, think of $(\mathbb{C}, +, \cdot, -, ^{-1}, 0, 1)$ as a field. Then $\mathbb{Q} \subseteq \mathbb{C}$, it is a subset and a substructure. In particular, the property of being algebraically closed is in the theory of $\mathbb{C}$. Thus Theorem 3.11 gives a algebraically closed field, which is countable and contains $\mathbb{Q}$ - the algebraic closure of $\mathbb{Q}$.

Proof. We build a chain $\langle A_i : i < \omega \rangle$, with $A_i \subseteq N$, such that $|A_i| = \lambda$. (Our goal is to define $M = \bigcup_{i<\omega} A_i$).

Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$. At stage $i+1$, we assume that A_i has been built, with $|A_i| = \lambda$. Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$-formulas such that $\mathcal{N} \vDash \phi_k(x)$. Let a_k be such that $\mathcal{N} \vDash \phi_k(a_k)$ and let $A_{i+1} = A_i \cup \{ a_k : k < \lambda \}$. Then $|A_{i+1}| = \lambda$.

Now let $M = \cup_{i<\omega} A_i$. We use Lemma 3.8 to show that M is the domain of $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|M| = \lambda$: Let $\mathcal{N} \vDash \exists x \psi(x, \bar{a})$, where $\bar{a}$ is a tuple in M. Then $\bar{a}$ is a *finite* tuple, so there is an i such that $\bar{a}$ is in A_i. Then A_{i+1}, by construction, contains b such that $\mathcal{N} \vDash \phi(b, \bar{a})$. But $A_{i+1} \subseteq M$, $b \in M$. $\square$

4 Two relational structures

Lecture 5 **Definition 4.1** (Dense linear orders)**.** A **linear order** is an $L_{\text{lo}} = \{<\}$-structure such that

(i) $\forall x\, \neg(x < x)$

(ii) $\forall xyz((x < y \wedge y < z) \to x < z)$

(iii) $\forall xy((x < y) \wedge (y < x) \vee (x = y))$.

A linear order is **dense** if it also satisfies

(iv) $\exists xy(x < y)$

(v) $\forall xy(x < y \to \exists z(x < z < y))$ (density)

A linear order has no endpoints if

(vi) $\forall x(\exists y(x < y) \wedge \exists z(z < x))$

T_{dlo} is the theory that includes axioms (i) to (vi), T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark: (iv) and (v) imply that if $\mathcal{M} \vDash T_{\text{dlo}}$ then $|\mathcal{M}| \geq \omega$.

Definition 4.2 ((Finite) Partial embedding)**.** If $\mathcal{M}, \mathcal{N} \vDash T_{\text{lo}}$, then an injective map $p : A \subseteq M \to N$ is a **partial embedding** if

$$\mathcal{M} \vDash a < b \implies \mathcal{N} \vDash p(a) < p(b).$$

If $|\operatorname{dom}(p)| < \omega$, then p is a **finite partial embedding**.

Lemma 4.3 (Extension lemma)**.** Suppose $\mathcal{M} \vDash T_{\text{lo}}$, $\mathcal{N} \vDash T_{\text{dlo}}$, let $p : M \to N$ be a finite partial embedding. Then if $c \in M$, there is a finite partial embedding $\hat{p}$ such that $p \subseteq \hat{p}$ and $c \in \operatorname{dom}(\hat{p})$.

Proof. Split into three cases:

1. $c > a$ for all $a \in \operatorname{dom}(p)$. Then choose $d \in \mathcal{N}$ so that $d > b$ for all $b \in \operatorname{img}(p)$.
2. $a_i < c < a_{i+1}$ for some $a_i, a_{i+1} \in \operatorname{dom}(p)$. Then $\mathcal{N} \vDash p(a_i) < p(a_{i+1})$, so by density, $\mathcal{N} \vDash p(a_i) < d < p(a_{i+1})$.
3. $c < a$ for all $a \in \operatorname{dom} p$. Similar to case 1. □

Theorem 4.4. Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$ such that $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \to N$ be a finite partial embedding. Then there is $\pi : \mathcal{M} \to \mathcal{N}$, an isomorphism such that $p \subseteq \pi$.

Proof. Enumerate M, N. Say $M = \langle a : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ sequences of elements. We define inductively a chain of finite partial embeddings $\langle p_i : i < \omega \rangle$ (idea: $\pi = \bigcup_{i<\omega} p_i$).

Let $p_0 = p$. At stage $i + 1$, p_i is given. We want to include a_i in $\operatorname{dom}(p_{i+1})$, and b_i in $\operatorname{img}(p_{i+1})$.

Forward step: By Lemma 4.3, extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \operatorname{dom}(p_{i+\frac{1}{2}})$. Backward step: By Lemma 4.3 applied to $p^{-1}_{i+\frac{1}{2}}$ to include $b_i \in \operatorname{dom}(p_{i+\frac{1}{2}})$ (i.e. in the range of p_{i+1}). Then p_{i+1} extends p_i as required.

Let $\pi = \bigcup_{i<\omega} p_i$. Then (check) π is an isomorphism (i.e. order-preserving bijection). □

Definition 4.5 (Consistent, complete)**.** An L-theory is **consistent** if there is $\mathcal{M}$ such that $\mathcal{M} \vDash T$. If T is a theory in L and ϕ is an L-sentence, then we write $T \vdash \phi$ if for all $\mathcal{M}$ such that $\mathcal{M} \vDash T$, also $\mathcal{M} \vDash \phi$. An L-theory T is **complete** if for all L-sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$.

Is T_{dlo} complete?

Lecture 6 **Definition 4.6** (ω-categorical)**.** A theory T in a countable language with a countably infinite model is ω-categorical if any two countable models of T are isomorphic.

Corollary 4.7. of Theorem 4.4: T_{dlo} is ω-categorical.

Proof. If $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$, $\mathcal{M} = \mathcal{N} = \omega$. Then $\varnothing$ (the empty map) is a finite partial embedding. By Theorem 4.4, $\mathcal{M} \simeq \mathcal{N}$. (Can also use any $\{\langle a, b\rangle\}$ where $a \in \mathcal{M}, b \in \mathcal{N}$ as initial finite partial embedding). □

Theorem 4.8. If T is an ω-categorical theory in a countable language, and T has no finite models then T is complete.

Proof. Let $\mathcal{M} \vDash T$ and ϕ be an L-sentence.

If $\mathcal{M} \vDash \phi$, suppose $\mathcal{N} \vDash T$. Then by Theorem 3.11, there are $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$ such that $|\mathcal{M}'| = |\mathcal{N}'| = \omega$. By $\mathcal{M}' \simeq \mathcal{N}'$ (by ω-categoricity), so in particular $\mathcal{M}' \equiv \mathcal{N}'$ and so $\mathcal{N}' \vDash \phi$.

If $\mathcal{M} \vDash \neg\phi$, similar. □

Corollary 4.9. T_{dlo} is complete.

Definition 4.10 ((Partial) elementary map)**.** If $\mathcal{M}, \mathcal{N}$ are L-structures, a map f such that $\text{dom}(f) \subseteq M$ and $\text{img}(f) \subseteq N$ is a **(partial) elementary map** if for all L-formulae $\phi(\bar{x})$ and $\bar{a} \in (\text{dom}(f))^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(f(\bar{a}))$$

Remark 4.11. A map f is elementary iff every finite restriction of f is elementary.

Proof.

- ($\Leftarrow$) If $f_0 \subseteq f$ is a finite restriction that is not elementary, then for some $\phi(\bar{x})$, $\bar{a} \in \text{dom}(f_0)$, $\mathcal{M} \vDash \phi(\bar{a}) \not\Longleftrightarrow \mathcal{N} \vDash \phi(f_0(\bar{a}))$. Then f is not elementary.
- ($\Rightarrow$) Clear. □

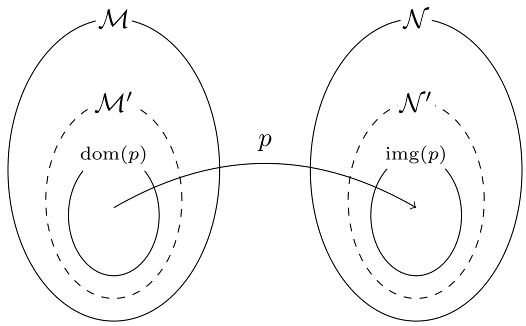
Proposition 4.12. Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{dlo}}$ and let $p : A \subseteq M \to N$ be a partial embedding. Then p is elementary.

Proof. By Remark 4.11, it suffices to consider p finite. By Downward Löwenheim-Skolem, we choose $\mathcal{M}', \mathcal{N}'$ such that

(i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.

(ii) $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$

(iii) $\text{dom}(p) \subseteq \mathcal{M}', \text{img}(p) \subseteq \mathcal{N}'$



Now p is a finite partial embedding between countable models, so p extends to an isomorphism $\pi : \mathcal{M}' \to \mathcal{N}'$ by Theorem 4.4. In particular, π is an elementary map between $\mathcal{M}$ and $\mathcal{N}$. ☐

Corollary 4.13. $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

Proof. Use Proposition 4.12 with $\mathrm{id} : \mathbb{Q} \to \mathbb{R}$. ☐

Definition 4.14 (Random graph)**.** Let $L_{\mathrm{gph}} = \{R\}$, a binary relation symbol. An L_{gph}-structure is a **graph** if

(i) $\forall x\, \neg R(x, x)$

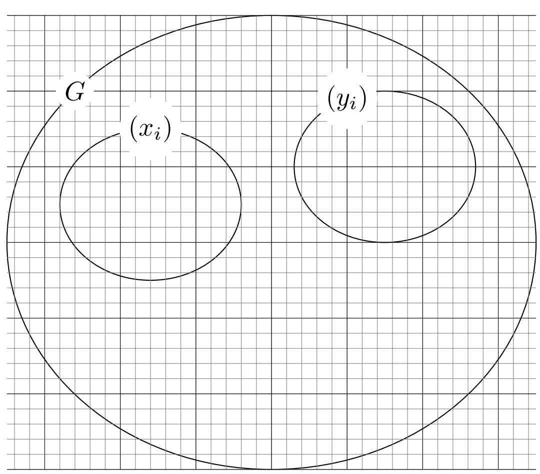
(ii) $\forall xy\, (R(x, y) \leftrightarrow R(y, x))$

An L_{gph}-structure is a **random graph** if it is a graph such that, for all $n \in \omega$, axiom (r_n) holds:

$$\forall x_0 \dots x_n, y_0 \dots y_n \left(\bigwedge_{i,j=0}^{n} x_i \neq y_j \to \exists z \left(\bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii) $\exists xy\, (x \neq y)$.

Axiom (r_n) effectively says that for disjoint subsets (x_i) and (y_i) each of size n, there is a (different) node z connected to each x_i and none of the y_i.



Remark. A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

Fact 4.15. There is a random graph.

Proof. Let the domain be ω, let $i, j \in \omega$ such that $i < j$. Write j as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff 2^i appears in the sum. $\square$

Exercise. Prove that ω with this definition of R is a random graph.

Definition 4.16 (Graph theories, partial embedding)**.** T_{gph} consists of the axioms (i),(ii) above, and $T_{\text{rg}} = T_{\text{gph}} \cup \{(\text{iii}), (r_n) : n \in \omega\}$. If $\mathcal{M}, \mathcal{N} \vDash T_{\text{gph}}$, a **partial embedding** is an injective map $p : A \subseteq M$ to N such that

$$\mathcal{M} \vDash R(a, b) \iff \mathcal{N} \vDash R(p(a), p(b))$$

for all a, b in the domain. Just as before, if $|\operatorname{dom}(p)| < \omega$ then p is called a **finite partial embedding**.

Lemma 4.17. Let $\mathcal{M} \vDash T_{\text{gph}}$, $\mathcal{N} \vDash T_{\text{rg}}$, let $p : A \subseteq M \to N$ be a finite partial embedding, and let $c \in M$. Then there is $\hat{p} : \hat{A} \subseteq M \to N$ such that $\hat{p}$ is a partial embedding, $c \in \operatorname{dom}(\hat{p})$, $p \subseteq \hat{p}$.

Lecture 7 *Proof.* Take $c \in M$, $c \notin \operatorname{dom}(p)$.

empty diagram

Find $d \in N$ such that $N \vDash R(d, p(a)) \iff M \nvDash R(c, a)$. $\square$

Theorem 4.18. Let $\mathcal{M}, \mathcal{N} \vDash T_{\text{rg}}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $p : A \subset M \to N$ is a finite partial embedding. Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends p.

Proof. Same as proof of Theorem 4.4, but with Lemma 4.17 instead of Lemma 4.3. $\square$

Corollary 4.19. T_{rg} is ω-categorical and complete. Moreover, every finite partial embedding between models of T_{rg} is an elementary map.

Remark 4.20. The unique (up to isomorphism) model of T_{rg} is *the* countable random graph, or the **Rado graph**. It is universal with respect to finite and countable graphs (i.e. it embeds them all). It is **ultrahomogeneous** i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.

5 Compactness

Definition 5.1. Take an L-theory T.

(i) T is **finitely satisfiable** if every finite subset of sentences in T has a model.

(ii) T is **maximal** if for all L-sentences σ, either $\sigma \in T$ or $\neg\sigma \in T$.

(iii) T has the **witness property** (WP) if for all $\phi(x)$ (L-formula with one free variable) there is a constant $c \in \mathscr{C}$ such that

$$(\exists x\, \phi(x) \to \phi(c)) \in T.$$

Lemma 5.2. If T is maximal and finitely satisfiable and ϕ is an L-sentence, and $\Delta \overset{\text{finite}}{\subseteq} T$ and $\Delta \vDash \phi$, then $T \vDash \phi$.

Lemma 5.3. Let T be a maximal, finitely satisfiable theory with the witness property. Then T has a model. Moreover, if λ is a cardinal and $|\mathscr{C}| \leq \lambda$, then T has a model of size at most λ.

Proof. Let $c, d \in \mathscr{C}$, define $c \sim d$ iff $c = d \in T$.

Claim: $\sim$ is an equivalence relation. **Proof:** For transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$, so $c = e \in T$ (by Lemma 5.2), and so $c \sim e$. ■

We denote $[c] \in \mathscr{C}/\sim$ by c^*

Now, define a structure $\mathcal{M}$ whose domain is $\mathscr{C}/\sim\, = M$. Clearly, $|M| \leq \lambda$ if $|\mathscr{C}| \leq \lambda$. We must define interpretations in $\mathcal{M}$ for symbols of L.

- If $c \in \mathscr{C}$, then $c^{\mathcal{M}} = c^*$.
- If $R \in \mathscr{R}$, define
$$R^{\mathcal{M}} := \{\, (c_1^*, \ldots, c_{n_R}^*) \mid R(c_1, \ldots, c_n) \in T \,\}$$
Claim: $R^{\mathcal{M}}$ is well defined. **Proof:** Suppose $\bar{c}, \bar{d} \in \mathscr{C}^{n_R}$ and suppose $c_i \sim d_i$. That is, $c_i = d_i \in T$ for $i = 1, \ldots, n_R$
$$R(\bar{c}) \in T \iff R(\bar{d}) \in T.$$
This proves that $R^{\mathcal{M}}$ is well defined. ■
- If $f \in \mathscr{F}$, and $\bar{c} \in \mathscr{C}^{n_R}$, then $f\bar{c} = d \in T$ for some $d \in \mathscr{C}$. (This is because $\exists x\, (f(\bar{c}) = x) \in T$ by maximality and finite satisfiability.)
Then define $f^{\mathcal{M}}(\bar{c}^*) = d^*$. Exercise: Check $f^{\mathcal{M}}(\bar{c}^*)$ is well-defined!

Claim: if $t(x_1, \ldots, x_n)$ is an L-term and $c_1, \ldots, c_n, d \in \mathscr{C}$, then

$$t(c_1, \ldots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*.$$

Proof: ($\Rightarrow$) by induction on the complexity of t. ($\Leftarrow$) Assume $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. Then

$$t(c_1, \ldots, c_n) = e \in T$$

for some constant e. Use $\Rightarrow$ to get that $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. But then $d^* = e^*$, i.e. $d = e \in T$. Then $t(c_1, \ldots, c_n) = d \in T$. ■

Claim: For all L-formulas $\phi(\bar{x})$, and $\bar{c} \in \mathscr{C}^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{c}) \iff \phi(\bar{c}) \in T.$$

Proof: By induction on $\phi(\bar{x})$. (Exercise: Fill in the details). ■ □

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