# Part II – Analysis of Functions

Based on lectures by Prof C. Mouhot
Notes taken by Bhavik Mehta

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## Contents



# 1 Lebesgue theory

**Exercise.** Show pointwise limit of Riemann-integrable functions is not necessarily Riemann-integrable. (Hint: Dirichlet function).

## 1.1 Recap of measure theory

Consider a set $X$ and $\mathscr{P}(X)$ subsets of $X$.

**Definition** (Algebra)**.** $\mathscr{A} \subset \mathscr{P}(X)$ is an **algebra** if it is

(i) stable under finite union

(ii) stable under absolute difference

(iii) $X \in \mathscr{A}$.

**Definition** ($\sigma$-algebra)**.** $\mathscr{A} \subset \mathscr{P}(X)$ is a $\sigma$**-algebra** if it is

(i) stable under countable union

(ii) stable under absolute difference

(iii) $X \in \mathscr{A}$.

**Remark.** Topologies $\mathscr{T} \subset \mathscr{P}(X)$ are (i) stable under *any* union, (ii) finite intersection, (ii) include $X$ and $\varnothing$.

**Remark.** The property of being a $\sigma$-algebra is stable under intersection. The smallest $\sigma$-algebra containing some topology $\mathscr{T}$ has elements called **Borel** sets, written $\mathcal{B}(X)$.

**Definition.** Consider $(X, \mathscr{A})$, where $\mathscr{A}$ is a $\sigma$-algebra. A **measure** $\mu$ is a function $\mathscr{A} \to [0, +\infty]$ such that $\mu(\varnothing) = 0$. It is **$\sigma$-additive** if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then $X, \mathscr{A}, \mu)$ is a **measure space**. It is called **complete** if $A \in \mathscr{A}$ with $B \subset A$ and $\mu(A) = 0$, then $B \in \mathscr{A}$ and $\mu(B) = 0$.

**Exercise.** Show $\sigma$-additivity is implied by either of the following properties:

- finite additivity and continuity from below
- finite $\mu(X) < +\infty$ and finite additivity and continuity from above at $\varnothing$

where

- continuity from below:

$$A_n \in \mathscr{A}, \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n \to +\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right)$$

- continuity from above

$$A_n \in \mathscr{A}, \mu(A_1) < +\infty, \mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{n \to +\infty} \mu\left(\bigcap_{k=1}^{n} A_k\right)$$

**Exercise.** Find the cardinality of $\mathscr{T}(\mathbb{R})$, $\mathcal{B}(\mathbb{R})$, $\mathscr{L}(\mathbb{R})$ where $\mathscr{L}(\mathbb{R})$ are the Lebesgue sets, defined by adding all subsets of null sets to $\mathcal{B}(\mathbb{R})$.

**Theorem.** There is a unique measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\mu\left(\prod_{i=1}^{n} [a_i, b_i]\right) = \prod_{i=1}^{n} (b_i - a_i) \quad a_i \leq b_i \in \mathbb{R}$$

called the Lebesgue measure.

*Proof.* See Probability and Measure. □

**Remark.** Lebesgue measure is $\sigma$-finite: $\exists$ a countable increasing sequence of sets with finite measure covering $\mathbb{R}^n$.

**Definition** (Measurable function)**.** Take $(X, \mathscr{A})$, $(Y, \mathscr{B})$ two spaces with $\sigma$-algebras. A function $f : X \to Y$ is said to be **measurable** if $\forall B \in \mathscr{B}$, $f^{-1}(B) \in \mathscr{A}$.

**Proposition.** Take $(X, \mathscr{A}), (Y, \mathscr{B})$ two spaces with $\sigma$-algebras where $Y$ is a metric space and $\mathscr{B}$ is the collection of Borel sets. Let $f_k : X \to Y$ be a sequence of measurable functions which converge pointwise to $f : X \to Y$. Then $f$ is measurable.

*Proof.* Since $B$ is formed from open sets through countable union/intersection and difference, it is enough to prove that $\forall U \in \mathscr{T}(Y)$, $f^{-1}(U) \in \mathscr{A}$. (Exercise: Check.)

Let

$$U_n = \left\{ y \in Y \,\middle|\, d(y, Y \setminus U) > \frac{1}{n} \right\}$$
$$F_n = \left\{ y \in Y \,\middle|\, d(y, Y \setminus U) \geq \frac{1}{n} \right\}$$

so that

$$U_n \subset F_n \subset U_{n+1} \subset \cdots \subset U$$

and $F_n$ are closed.

We can see $U = \bigcup_{n\geq 1} U_n = \bigcup_{n \geq 1} F_n$ because $U$ is open. Hence,

$$f^{-1}(U) = f^{-1}\left( \bigcup_{n\geq 1} U_n \right) = \bigcup_{n \geq 1} f^{-1}(U_n) \subset \bigcup_{n\geq 1} \bigcup_{l \geq 1} \bigcap_{k \geq l} f_k^{-1}(U_n).$$

We used the fact that

$$f^{-1}(U_n) \subset \bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(U_n)$$

To show this, take $x \in f^{-1}(U_n)$, so $f(x) = y \in U_n$. We know

$$f_k(x) \xrightarrow{k\to\infty} f(x).$$

Since $U_n$ open, $\exists l_x \geq 1$ such that $\forall k \geq l_x$, $f_k(x) \in U_n$ giving $x \in \bigcap_{k\geq l} f_k^{-1}(U_n)$.

Continuing,

$$\begin{aligned} f^{-1}(U) &\subset \bigcup_{n\geq 1} \bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(U_n). \\ &\subset \bigcup_{n\geq 1} \bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(F_n). \end{aligned}$$

$F_n$ closed, so

$$\bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(F_n) \subset f^{-1}(F_n).$$

In particular, if $x \in$ LHS, $\exists l \geq 1$ such that $\forall k \geq l$, $f_k(x) \in F_n$. Pass to the limit, and $f_n$ closed gives $f(x) \in F_n$, $x \in f^{-1}(F_n)$.

In conclusion,

$$\begin{aligned} f^{-1}(U) &\subset \bigcup_{n\geq 1} \bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(U_n) \subset \bigcup_{n\geq 1} \bigcup_{l\geq 1} \bigcap_{k\geq l} f_k^{-1}(F_n) \\ &\subset \bigcup_{n\geq 1} f^{-1}(F_n) = f^{-1}\left( \bigcup_{n\geq 1} F_n \right) = f^{-1}(U). \end{aligned}$$

So, all inclusions are equality: $f^{-1}(U)$ is formed of countable intersections and unions of preimages of sets in $\mathscr{B}$, hence $f^{-1}(U) \in \mathscr{A}$. $\square$

## 1.2 Lebesgue integration

The important result from measures is the existence of Lebesgue measure, and that the 'theory' is closed for pointwise convergence.

We move now from Riemann integration to Lebesgue integration. In Riemann's theory of integration, we approximate the integral with Darboux sums, by dividing the domain. We require the domain to have a total order, while the codomain must be a Banach space. Conversely, in Lebesgue integration we divide the codomain (again, needing a total order) while the domain must have a measure defined on it.

**Definition** (Simple)**.** $f : (X, \mathscr{A}) \to (\mathbb{R}, \mathcal{B}(R))$ is **simple** if it is measurable and takes a finite number of values in $[0, +\infty)$.

From here on in, when working on the real line, subsets thereof, or the extended real line the $\sigma$-algebra will be the Borel sets.

**Remark.** $A \subset X$, $\chi_A$ is measurable iff $A \in \mathscr{A}$ is simple. The general form of a simple function is $s = \sum \alpha_i \chi_{A_i}$.

**Notation.** $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$, so the neighbourhoods of $\infty$ are $(a, +\infty]$, and we can have a metric $d(x, y) = |\arctan x - \arctan y|$.

**Proposition.** Let $f : (X, \mathscr{A}) \to [0, +\infty]$ measurable. There is $(s_k)$, a sequence of simple functions $s_{k+1} \geq s_k$ converging pointwise to $f$.

*Proof.* For $n \geq 1$, define

$$\begin{aligned} B_n &= \{\, x \mid f(x) \geq n \,\} \\ A_n^i &= \left\{ x \;\middle|\; f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right] \right\} \qquad i = 0, \dots, n2^n \end{aligned}$$

Also, set

$$s_n = \begin{cases} \frac{i-1}{2^n} & \text{on } A_n^i \\ n & \text{on } B_n \end{cases}$$

Check

- $s_{n+1} \geq s_n$ $[A_n^i = A_{n+1}^{2i} \cup A_{n+1}^{2i+1}]$
- on $B_n$, $|s_n - f| \leq \frac{1}{2^n}$
- on $x \in \bigcap_{k \geq 1} B_n$, $s_n(x) \to +\infty$. $\square$

Observe this construction is simply

$$s_n = \max\left(n, 2^{-n}\left\lfloor \frac{x}{2^n} \right\rfloor\right)$$

as used in Probability and Measure.

**Definition** (Integral of simple function)**.** Take $(X, \mathscr{A}, \mu)$ and $s$ a simple function on it given by $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, $\alpha_i \in [0, \infty)$. For $E \in \mathscr{A}$, define

$$\int_E s \, d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap E).$$

**Remark.** This induces a new measure on $\mathscr{A}$, sending $E \mapsto \int_E s\, d\mu$.

**Definition** (Integral)**.** Take $f : (X, \mathscr{A}, \mu) \to [0, \infty]$ measurable and $E \in \mathscr{A}$. Then define

$$\int_E f\, d\mu := \sup \left\{ \int_E s\, d\mu \;\middle|\; s \le f \right\} \in [0, +\infty]$$

**Remark.** This integral always makes sense. Also, $\int_E f\, d\mu = 0$ if $\mu(E) = 0$.

**Exercise.** Check linearity of the integral. Show Chebyshev's inequality:

$$\mu(\{\, x \mid f(x) \ge \alpha \,\}) \le \alpha^{-1} \int_X f\, d\mu \quad \forall \alpha > 0.$$

Also show that for $f$ measurable, $X \to [0, \infty]$

$$\int_X f\, d\mu < \infty \implies \mu(\{\, x \mid f(x) = \infty \,\}) = 0.$$

**Theorem** (Beppo-Levi monotone convergence)**.** Take $f_k : (X, \mathscr{A}, \mu) \to [0, \infty]$ measurable, converging pointwise to $f$ with $f_k \le f_{k+1}$. Then $\forall E \in \mathscr{A}$,

$$\int_E f_k\, d\mu \xrightarrow{k \to \infty} \int_E f\, d\mu.$$

*Proof.* Reduce to $E = X$ by consdering $f_k \chi_E, f \chi_E$. Then $\left(\int_X f_k\, d\mu\right)_{k \ge 1}$ is a sequence in $[0, \infty]$, non-decreasing.

By monotonicity, $f_k \nearrow f$, so $\int_X f_k\, d\mu \le \int_X f\, d\mu$. Let

$$\alpha := \lim_{k \to +\infty} \int_X f_k\, d\mu \le \int_X f\, d\mu.$$

Consider a simple function $s \le f$ and $c \in (0, 1)$.

$$E_k = \{\, x \in X \mid f_k(x) \ge cs(x) \,\} \in \mathscr{A} \text{ (using that } f_k,\ s \text{ are measurable)}$$
$$E_k \subset E_{k+1}, \bigcup_{k \ge 1} E_k = X \text{ (by pointwise convergence)}$$

Thus $\int_X s\, d\mu = \lim_{k \to \infty} \int_{E_k} s\, d\mu$ (by continuity from below of $\mu$).

$$\int_X f_k\, d\mu \ge \int_{E_k} f_k\, d\mu \ge c \int_{E_k} s\, d\mu$$

Take $k \to +\infty$. $\alpha \ge c \int_X s\, d\mu$, and let $c \nearrow 1$, giving $\int_X s\, d\mu$. Taking the supremum over $s \le f$ for $s$ simple,

$$\alpha \ge \int_X f\, d\mu.$$

□

**Exercise.** Taking $f_k$ as above, show

$$\int_X \left( \sum_{k \ge 1} f_k \right) d\mu = \sum_{k \ge 1} \int_X f_k\, d\mu.$$

Let $\nu : A \in \mathscr{A} \mapsto \int_A f\, d\mu$ (for $f : X \to [0, +\infty]$ measurable). Show that for any measure $g : (X, \mathscr{A}, \mu) \to [0, +\infty]$, $\int_X g\, d\mu = \int_X fg\, d\mu$.

**Theorem** (Fatou's lemma)**.** Take $f_k$ as above, then

$$\int_X (\liminf f_k)\, d\mu \le \liminf \left( \int_X f_k \, d\mu \right)$$

*Proof.* Let $F_k = \inf \{ f_l \mid l \ge k \}$, non-decreasing, valued in $[0, +\infty]$. These are measurable: $\{F_k \ge a\} = \bigcap_{l \ge k} \{f_l \ge a\}$. Observe that $\int \min(f, g)\, d\mu \le \min(\int f \, d\mu, \int g \, d\mu)$. Now, by ,

$$\begin{aligned}
\int_X (\liminf f_k)\, d\mu = \int_X (\lim F_k)\, d\mu &= \lim_{k \to +\infty} \left( \int_X F_k \, d\mu \right) \\
&= \lim_{k \to \infty} \left( \int_X \left( \inf_{l \ge k} f_l \right) d\mu \right) \\
&\le \lim_{k \to \infty} \inf_{l \ge k} \left( \int_X f_l \, d\mu \right) \\
&\le \liminf_{k \to \infty} \int f_k \, d\mu.
\end{aligned}$$

□

**Definition** (Integrable)**.** $f : (X, \mathscr{A}, \mu) \to \mathbb{C}$ measurable is integrable if $|f| : X \to [0, +\infty)$ satisfies $\int_X |f| \, d\mu < +\infty$.

Compute by splitting $f$ into real and imaginary parts, and each into nonnegative and nonpositive parts.

**Theorem** (Lebesgue's Dominated Convergence)**.** Take $f_k : (X, \mathscr{A}, \mu) \to \mathbb{C}$ where

- convergence: $f_k$ converges pointwise to $f$
- domination: $\exists g$ integrable such that $|f_k| \le g \; \forall k \ge 1$.

Then $f_k, f$ integrable and $\int_X |f_k - f| \, d\mu \to 0$.

*Proof.* Let $h_k = 2g - |f - f_k|$, taking values in $[0, +\infty)$. Then $h_k \to 2g$ pointwise.

$$\int_X 2g \, d\mu = \int_X (\lim h_k)\, d\mu \le \liminf_{k \to \infty} \underbrace{\left( \int_X h_k \, d\mu \right)}_{= \int_X 2g \, d\mu - \int_X |f_k - f| \, d\mu}.$$

$$\implies \int_X 2g \, d\mu \le \int_X 2g \, d\mu - \limsup_{k \to \infty} \int_X |f_k - f| \, d\mu.$$

□

## 1.3 Lebesgue spaces

**Definition** ($L^p$ space)**.** Let $p \in [1, +\infty]$, and $(X, \mathscr{A}, \mu)$ a measure space. $\boldsymbol{L^p(X)}$ is the set of equivalence classes for almost everywhere equality of functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) such that $|f|^p$ is integrable (for $p \in [1, +\infty)$), or $f$ essentially bounded i.e. bounded outside null sets for $p = \infty$).

**Theorem** (Riesz-Fischer)**.** $(L^p(X), \|\cdot\|_{L^p})$ is a Banach space for $p \in [1, +\infty]$, where

$$\|f\|_{L^p} := \left( \int_X |f|^p \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} := \inf \{ M \ge 0 \mid \mu\{|f(x)| \ge M\} = 0 \}.$$

**Exercise.** Take $\mu(X) < \infty$, so the domain of $f$ is a finite measure space. Take $f \in L^\infty(X)$, then show $f \in L^p(X)$ for any $p \in [1, \infty)$ and

$$\lim_{p \to \infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}.$$

*Proof.* Vector space axioms and triangular inequality (here called Minkowski) left as an exercise - see Linear Analysis. Instead, we focus on completeness. Start with $p \in [1, +\infty)$.

Initially, prove this auxiliary result. **Claim**: Take $p \in [1, +\infty)$, and consider a sequence $(g_k) \in L^p(X)$ such that $\sum_{k \geq 1} \|g_k\|_{L^p(X)} = M < \infty$, then there exists $G \in L^p(X)$ such that $\sum_{k=1}^n g_k$ converges to $G$ in $L^p(X)$ and almost everywhere.

**Proof of the claim**: Let

$$h_n := \sum_{k=1}^n |g_k|, \quad h := \sum_{k=1}^\infty |g_k| \in [0, +\infty].$$

$(h_n)$ is a non-decreasing sequence since $h_n \leq h_{n+1}$, and $h_n \to h$ pointwise. Hence, Beppo-Levi gives that

$$\int_X h_n^p \, d\mu \to \int_X h^p \, d\mu \in [0, +\infty].$$

By the assumption on $(g_k)$,

$$\begin{aligned} &\implies \|h_n\|_{L^p(X)} \leq \sum_{k=1}^n \|g_k\|_{L^p(X)} \leq M < +\infty \\ &\implies \|h\|_{L^p(X)} \text{ is finite and less than } M. \end{aligned}$$

Hence $h$ finite almost everywhere, and $\sum g_k$ is absolutely convergent almost everywhere, so convergent almost everywhere.

Let us call $G = \lim_{k \to \infty} \sum_{k=1}^n g_k$ (almost everywhere).

$$|G(x)| \leq \left| \sum_{k=1}^n g_k \right| \leq \sum_{k=1}^\infty |g_k| = h(x)$$

so $G \in L^p(X)$.

Using the Dominated Convergence Theorem,

$$\int_X \left| G(x) - \sum_{k=1}^n g_k \right|^p \, d\mu \to 0$$

since the integrand converges pointwise to 0, and the domination is given by

$$\left| G(x) - \sum_{k=1}^n g_k \right|^p \leq 2^p h(x)^p,$$

where $h^p$ integrable since $G \in L^p(X)$ and $\sum_{k=1}^m g_k \to G$ in $L^p(X)$ and almost everywhere. This proves the claim, so we go back to the main proof.

Let $(f_k)$ be a Cauchy sequence in $L^p(X)$. Build a subsequence $(f_{\phi(k)})$ such that $g_k := f_{\phi(k+1)} - f_{\phi(k)}$ satisfies $\|g_k\|_{L^p(X)} \leq \frac{1}{2^k}$, so $g_k$ satisfies the assumptions of the claim. Hence,

$\exists G \in L^p(X)$ such that $\sum_{k=1}^n g_k \to G$ almost everywhere and in $L^p(X)$. But $g_k = f_{\phi(n+1)} - f_{\phi(n)}$. So,

$$f_{\phi(n)} \longrightarrow f_{\phi(1)} + G =: F$$

where the convergence is in $L^p(X)$ and almost everywhere. But $(f_n)$ Cauchy in $L^p(X)$, so $f_n \to F$ in $L^p(X)$. □

**Remark.** Take $(f_n)$ convergent in $L^p(X)$, $f_n \to F$ in $L^p(X)$. Then (we proved) $\exists$ subsequence $f_{\phi(n)} \to F$ converging almost everywhere.

**Exercise.**

1) Find a sequence $(f_n)$ converging in $L^p(\mathbb{R})$ and not converging almost everywhere to its $L^p$ limit (for $p \in [1, \infty)$). This shows passing to a subsequence was necessary, and not a defect of the argument.

2) Complete the proof of Riesz-Fischer theorem in the $p = \infty$ case.

**Theorem** (Abstract density result)**.** Take a measure space $(X, \mathscr{A}, \mu)$ and $p \in [1, \infty]$. Then simple functions that belong to $L^p(X)$ are dense in $L^p(X)$.

*Proof.* For $f$ real or complex, split into real/imaginary parts and positive/negative parts to reduce to approximating $f \geq 0$ by simple functions. Define

$$S_n = \begin{cases} n & \text{on } B_n = \{ f \geq n \} \\ \frac{i-1}{2^n} & \text{on } A_n^i = \{ f \in [\frac{i-1}{2^n}, \frac{i}{2^n}) \}, 1 \leq i \leq n2^n \end{cases}$$

our usual approximation. In the case $p = \infty$, $\|f\|_{L^\infty(X)} < \infty$, thus for $n > \|f\|_{L^\infty(X)}$, $|S_n - f| \leq \frac{1}{2^n}$ almost everywhere. Then $\|S_n - f\|_{L^\infty(X)} \leq \frac{1}{2^n} \to 0$. In the case $p \in [1, \infty)$, $0 \leq S_n \leq f$ and $S_{n+1} \geq S_n$, with $S_n \to f$ pointwise. By Beppo-Levi monotone convergence, $\|S_n - f\|_{L^p(X)} \to 0$. □

**Theorem** (Density-separability of $L^p(\mathbb{R}^n)$, $p \in [1, +\infty)$)**.**

(1) For $\mathcal{O}$ an open set in $\mathbb{R}^n$, $L^p(\mathcal{O})$ is separable (has a countable dense subset).

(2) Smooth functions compactly supported in $\mathcal{O}$ are dense in $L^p(\mathcal{O})$.

We need and admit a theorem from Probability and Measure (this statement is non-examinable).

**Definition** (Regular measure)**.** A regular measure on a topological space $X$ with $\sigma$-algebra $\mathscr{A}$ of measurable sets is a measure such that every measurable set can be approximated from above by open measurable sets and from below by compact measurable sets.

**Theorem** (Regularity of the Lebesgue measure)**.** The Lebesgue measure on $\mathbb{R}^n$ is regular for the Lebesgue sets.

Observe this implies that any Lebesgue set of finite measure in $\mathbb{R}^n$ is squeezed between two Borel sets with the same measure.

*Proof.*

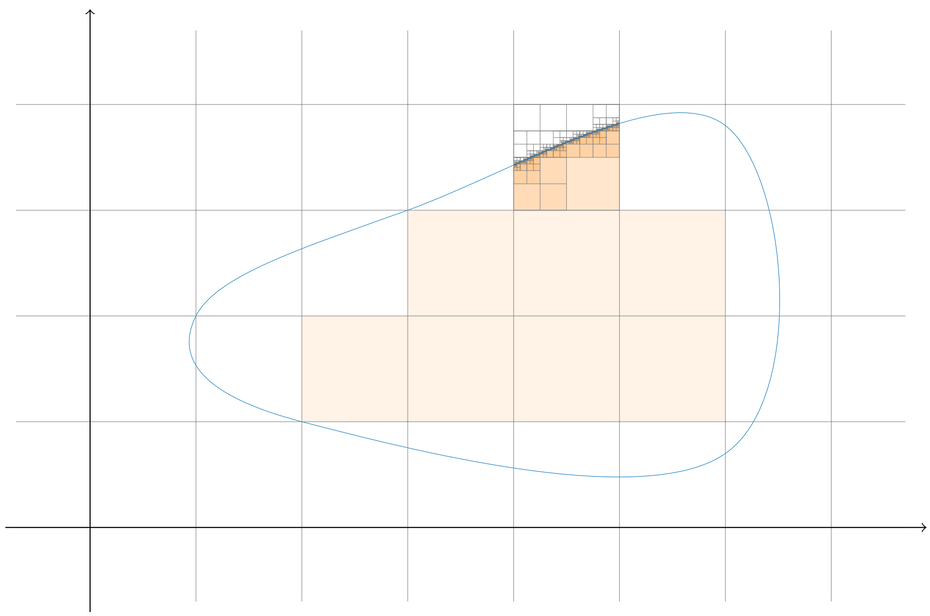
(1) Let $\mathcal{C} = \{\text{open sets of the form } \prod_{i=1}^{n}(a_i, b_i), a_i, b_i \in \mathbb{Q}\}$, a countable set.

Claim: Any open set $\mathcal{O} \subset \mathbb{R}^n$ can be covered with a countable union of elements of $\mathcal{C}$ with disjoint interiors.

Use an inductive procedure: Split into $\mathbb{Z}^n$.

1. Keep cubes that are fully inside, discard ones that are fully outside.
2. For borderline cubes, divide into $2^n$ cubes evenly and go back to step 1.

Check that $\mathcal{O} = \bigcup_{n,k} C_{n,k}$, and note that the hypercubes have dyadic, hence rational coordinates. The figure shows a few steps for an example in $\mathbb{R}^2$.



$\square$