

# The sphere eversion project

Patrick Massot

June 16, 2020

# Introduction

This project has two goals. First we want to check whether a proof assistant can do differential topology. Many people still think that formal mathematics are mostly suitable for algebra, combinatorics, or foundational studies. So we chose one of the most famous examples of geometric topology theorems associated to tricky geometric intuition: the existence of sphere eversions. Note however that we won't focus on any of the many videos of explicit sphere eversions. We will prove a general theorem which immediately implies the existence of sphere eversions.

The second goal of this project is to experiment using a formalization blueprint that evolves with the project until we get a proof that has very closely related formal and informal presentations.

In this introduction, we will describe the mathematical context of this project, the main definitions and statements, and outline the proof strategy.

Gromov observed that it's often fruitful to distinguish two kinds of geometric construction problems. He says that a geometric construction problem satisfies the  $h$ -principle if the only obstructions to the existence of a solution come from algebraic topology. In this case, the construction is called flexible, otherwise it is called rigid. This definition is purposely vague. We will see a rather general way to give it a precise meaning, but one must keep in mind that such a precise meaning will fail to encompass a number of situations that can be illuminated by the  $h$ -principle dichotomy point of view.

The easiest example of a flexible construction problem which is not totally trivial and is algebraically obstructed is the deformation of immersions of circles into planes. Let  $f_0$  and  $f_1$  be two maps from  $\mathbb{S}^1$  to  $\mathbb{R}^2$  that are immersions. Since  $\mathbb{S}^1$  has dimension one, this means that both derivatives  $f'_0$  and  $f'_1$  are nowhere vanishing maps from  $\mathbb{S}^1$  to  $\mathbb{R}^2$ . The geometric object we want to construct is a (smooth) homotopy of immersions from  $f_0$  to  $f_1$ , ie a smooth map  $F: \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{R}^2$  such that  $F|_{\mathbb{S}^1 \times \{0\}} = f_0$ ,  $F|_{\mathbb{S}^1 \times \{1\}} = f_1$ , and each  $f_p := F|_{\mathbb{S}^1 \times \{p\}}$  is an immersion. If such a homotopy exists then,  $(t, p) \mapsto f'_p(t)$  is a homotopy from  $f'_0$  to  $f'_1$  among maps from  $\mathbb{S}^1$  to  $\mathbb{R}^2 \setminus \{0\}$ . Such maps have a well defined winding number  $w(f'_i) \in \mathbb{Z}$  around the origin, the degree of the normalized map  $f'_i / \|f'_i\|: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ . So  $w(f'_0) = w(f'_1)$  is a necessary condition for the existence of  $F$ , which comes from algebraic topology. The Whitney–Graustein theorem states that this necessary condition is also sufficient. Hence this geometric construction problem is flexible. One can give a direct proof of this result, but it will also follow from general results proved in this project.

An important lesson from the above example is that algebraic topology can give us more than a necessary condition. Indeed the (one-dimensional) Hopf degree theorem ensures that, provided  $w(f'_0) = w(f'_1)$ , there exists a homotopy  $g_p$  of nowhere vanishing maps relating  $f'_0$  and  $f'_1$ . We also know from the topology of  $\mathbb{R}^2$  that  $f_0$  and  $f_1$  are homotopic, say using the straight-line homotopy  $p \mapsto f_p = (1 - p)f_0 + pf_1$ . But there is no a priori relation between  $g_p$  and the derivative of  $f_p$  for  $p \notin \{0, 1\}$ . So we can restate the crucial part of the Whitney–

Graustein theorem as: there is a homotopy of immersion from  $f_0$  to  $f_1$  as soon as there is (a homotopy from  $f_0$  to  $f_1$ ) and a homotopy from  $f'_0$  to  $f'_1$  among nowhere vanishing maps. The parenthesis in the previous sentence indicated that this condition is always satisfied, but it is important to keep in mind for generalizations. Gromov says that such a homotopy of uncoupled pairs  $(f, g)$  is a formal solution of the original problem.

One can generalize this discussion of uncoupled maps replacing a map and its derivative. This is pretty easy for maps from a manifold  $M$  to a manifold  $N$ . The so called 1-jet space  $J^1(M, N)$  is the space of triples  $(m, n, \varphi)$  with  $m \in M$ ,  $n \in N$ , and  $\varphi \in \text{Hom}(T_m M, T_n N)$ , the space of linear maps from  $T_m M$  to  $T_n N$ . One can define a smooth manifold structure on  $J^1(M, N)$ , of dimension  $\dim(M) + \dim(N) + \dim(M)\dim(N)$  which fibers over  $M$ ,  $N$  and their product  $J^0(M, N) := M \times N$ . Beware that the notation  $(m, n, \varphi)$  does not mean that  $J^1(M, N)$  is a product of three manifolds, the space where  $\varphi$  lives depends on  $m$  and  $n$ . Any smooth map  $f: M \rightarrow N$  gives rise to a section  $j^1 f$  of  $J^1(M, N) \rightarrow M$  defined by  $j^1 f(m) = (m, f(m), T_m f)$ . Such a section is called a *holonomic section* of  $J^1(M, N)$ . In the Whitney–Graustein example, we use the canonical trivialization of  $T\mathbb{S}^1$  and  $T\mathbb{R}^2$  to represent  $j^1 f$  has a pair of maps  $(f, f')$ . The role played by  $(f, g)$  in this example is played in general by sections of  $J^1(M, N) \rightarrow M$  which are not necessarily holonomic.

One can generalize this discussion to  $J^r(M, N)$  which remembers derivatives of maps up to order  $r$  for some given  $r \geq 0$ . One can also consider sections of an arbitrary bundle  $E \rightarrow M$  instead of functions from  $M$  to  $N$ , which are sections of the trivial bundle  $M \times N \rightarrow N$ . But the case of  $J^1(M, N)$  will be sufficient for this project.

**Definition.** A first order differential relation  $\mathcal{R}$  for maps from  $M$  to  $N$  is a subset of  $J^1(M, N)$ . A solution of  $\mathcal{R}$  is a function  $f: M \rightarrow N$  such that  $j^1 f(m)$  is in  $\mathcal{R}$  for all  $m$ . A formal solution of  $\mathcal{R}$  is a non-necessarily holonomic section of  $J^1(M, N) \rightarrow M$  which takes value in  $\mathcal{R}$ .

The partial differential relation  $\mathcal{R}$  satisfies the *h-principle* if any formal solution  $\sigma$  of  $\mathcal{R}$  is homotopic, among formal solutions, to some holonomic one  $j^1 f$ .

For instance, an immersion of  $M$  into  $N$  is a solution of

$$\mathcal{R} = \{(m, n, \varphi) \in J^1(M, N) \mid \varphi \text{ is injective}\}.$$

As we saw with the Whitney–Graustein problem, we are not only interested to individual solutions, but also in families of solutions. In differential topology, a smooth family of maps between manifolds  $X$  and  $Y$  is a smooth map  $h: P \times X \rightarrow Y$  seen as the collection of maps  $h_p: x \mapsto h(p, x)$ . Here  $P$  stands for “parameter space”. A smooth family of sections of  $E \rightarrow X$  is a smooth family of maps  $\sigma: P \times X \rightarrow E$  such that each  $\sigma_p$  is a section.

When the parameter space  $P$  has boundary, we will typically assume that formal solutions  $\sigma_p$  are holonomic for  $p$  in  $\mathcal{N}(\partial P)$ . This is an abbreviation meaning: “there is an unspecified neighborhood  $U$  of  $\partial P$  such that  $\sigma_p$  is holonomic for  $p$  in  $U$ ”. Note that an unspecified neighborhood can change from invocation to invocation. For instance in the next definition, the second unspecified neighborhood is typically smaller than the first one.

**Definition.** A partial differential relation  $\mathcal{R}$  satisfies the *parametric h-principle* if every family of formal solutions  $\sigma: M \times P \rightarrow J^1(M, N)$  which are holonomic for  $p$  in  $\mathcal{N}(\partial P)$  is homotopic, relative to  $\mathcal{N}(\partial P)$ , to a family of holonomic sections.

There are other variations on this definition. For instance a formal solution could be holonomic on  $\mathcal{N}(A)$  for some subset  $A$  of  $M$ , and we say that  $\mathcal{R}$  satisfies the relative *h-principle* if  $\sigma$  can be deformed to a holonomic solution without changing it on  $\mathcal{N}(A)$ .

One can also insist on the deformed solution to be  $C^0$ -close to the original one. In this case one talks about a  $C^0$ -dense  $h$ -principle. We are now ready to state our main goal.

**Theorem.** *The relation of immersions in positive codimension (ie immersions of  $M$  into  $N$  with  $\dim(N) > \dim(M)$ ) satisfies all forms of  $h$ -principles.*

This theorem covers the Whitney–Graustein theorem (in its second form, assuming the existence of a homotopy between derivatives). But there are much less intuitive applications. The most famous one is the existence of sphere eversions: one can “turn  $S^2$  inside-out among immersions of  $S^2$  into  $\mathbb{R}^3$ ).

**Corollary** (Smale 1958). *There is a homotopy of immersion of  $S^2$  into  $\mathbb{R}^3$  from the inclusion map to the antipodal map  $a: q \mapsto -q$ .*

The reason why this is turning the sphere inside-out is that  $a$  extends as a map from  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$  by

$$\hat{a}: q \mapsto -\frac{1}{\|q\|^2}q$$

which exchanges the interior and exterior of  $S^2$ . More abstractly, one can say the normal bundle of  $S^2$  is trivial, hence one can extend  $a$  to a tubular neighborhood of  $S^2$  as an orientation preserving map. Since  $a$  is orientation reversing, any such extension will be reversing coorientation.

*Proof of the sphere eversion corollary.* We denote by  $\iota$  the inclusion of  $S^2$  into  $\mathbb{R}^3$ . We set  $j_t = (1-t)\iota + ta$ . This is a homotopy from  $\iota$  to  $a$  (but not an immersion for  $t = 1/2$ ). We need to check there is no obstruction to building a homotopy of formal solutions above those maps. One could show that the relevant homotopy group (replacing  $\pi_1(S^1)$  from the Whitney–Graustein example) is  $\pi_2(SO_3(\mathbb{R}))$ . This group is trivial, hence there is no obstruction. But actually we can write an explicit homotopy here, without using any algebraic topology. Using the canonical trivialization of the tangent bundle of  $\mathbb{R}^3$ , we can set, for  $(q, v) \in TS^2$ ,  $G_t(q, v) = \text{Rot}_{Oq}^{\pi t}(v)$ , the rotation around axis  $Oq$  with angle  $\pi t$ . The family  $\sigma: t \mapsto (j_t, G_t)$  is a homotopy of formal immersions relating  $j^1\iota$  to  $j^1a$ . The above theorem ensures this family is homotopic, relative to  $t = 0$  and  $t = 1$ , to a family of holonomic formal immersions, ie a family  $t \mapsto j^1f_t$  with  $f_0 = \iota$ ,  $f_1 = a$ , and each  $f_t$  is an immersion.  $\square$

The theorem above follows from a more general theorem which is slightly too technical for this introduction: the  $h$ -principle for open and ample first order differential relations. We will prove this theorem using a technique which is even more general: convex integration. For instance this technique also underlies the constructions of paradoxical isometric embeddings, which could be a nice follow-up project.

We’ll end this introduction by describing the key construction of convex integration, since it is very nice and elementary. Convex integration was invented by Gromov around 1970, inspired in particular by the  $C^1$  isometric embedding work of Nash and the original proof of flexibility of immersions. This term is pretty vague however, and there are several different implementations. The newest one, and by far the most efficient one, is Mélanie Theillièr’s corrugation process from 2017. And this is what we will use.

Let  $f$  be a map from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Say we want to turn  $f$  into a solution of some partial differential relation. For instance if we are interested in immersions, we want to make sure its differential (or equivalently its Jacobian matrix) is everywhere injective. We will ensure this by tackling each partial derivative in turn. In the immersion example, we first make sure  $\partial_1 f(x) := \partial f(x)/\partial x_1$  is non-zero for all  $x$ . Then we make sure  $\partial_2 f(x)$  is not colinear

to  $\partial_1 f(x)$ . Then we make sure  $\partial_3 f(x)$  is not in the plane spanned by the two previous derivatives, etc... until all  $n$  partial derivatives are everywhere linearly independent.

In general, what happens is that, for each number  $j$  between 1 and  $n$ , we wish  $\partial_j f(x)$  could live in some open subset  $\Omega_x \subset \mathbb{R}^m$ . Assume there is a smooth compactly supported family of loops  $\gamma: \mathbb{R}^n \times \mathbb{S}^1 \rightarrow \mathbb{R}^m$  such that each  $\gamma_x$  takes values in  $\Omega_x$ , and has average value  $\int_{\mathbb{S}^1} \gamma_x = \partial_j f(x)$ . Obviously such loops can exist only if  $\partial_j f(x)$  is in the convex hull of  $\Omega_x$ , hence the name convex integration, and we will see this condition is almost sufficient. In the immersion case, this convex hull condition will always be met because, from the above description, we see that  $\Omega_x$  will always be the complement of a linear subspace with codimension at least two.

For some large positive  $N$ , we replace  $f$  by the new map

$$x \mapsto f(x) + \frac{1}{N} \int_0^{Nx_j} [\gamma_x(s) - \partial_j f(x)] ds.$$

A wonderfully easy exercise shows that, provided  $N$  is large enough, we have achieved  $\partial_j f(x) \in \Omega_x$ , almost without modifying derivatives  $\partial_i f(x)$  for  $i \neq j$ , and almost without moving  $f(x)$ .

In addition, if we assume that  $\gamma_x$  is constant (necessarily with value  $\partial_j f(x)$ ) for  $x$  near some subset  $K$  where  $\partial_j f(x)$  was already good, then nothing changed on  $K$  since the integrand vanishes there. It is also easy to damp out this modification by multiplying the integral by a cut-off function. So this is a very local construction, and it isn't obvious how the absence of homotopical obstruction, embodied by the existence of a formal solution, should enter the discussion. The answer is that it essentially provides a way to coherently choose base points for the  $\gamma_x$  loops.

Chapter 1 provides the loops supply. Chapter 2 then discusses the local theory, including the key construction above, and Chapter 3 finally moves to manifolds, and proves the main theorem and its sphere eversion corollary.

# Chapter 1

## Loops

### 1.1 Introduction

In this chapter, we explain how to construct families of loops to feed into the corrugation process explained at the end of the introduction. A loop is a map defined on the circle  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . It can also freely be seen as 1-periodic maps defined on  $\mathbb{R}$ .

**Definition 1.1.** *The average of a loop  $\gamma$  is  $\bar{\gamma} := \int_{\mathbb{S}^1} \gamma(s) ds$ .*

Throughout this document,  $E$  and  $F$  will denote finite-dimensional real vector spaces. All of this chapter is devoted to proving the following proposition.

**Proposition 1.2.** *Let  $U$  be an open set in  $E$  and  $K \subseteq U$  a compact subset. Let  $\Omega$  be a set in  $E \times F$  such that, for each  $x$  in  $U$ ,  $\Omega_x := \Omega \cap (\{x\} \times F)$  is open and connected.*

*Let  $\beta$  and  $g$  be maps from  $E$  to  $F$  that are smooth on  $U$ . Assume that  $\beta(x) \in \Omega_x$  for all  $x$  in  $U$ , and  $g(x) = \beta(x)$  near  $K$ .*

*If, for every  $x$  in  $U$ ,  $g(x)$  is in the convex hull of  $\Omega_x$ , then there exists a smooth family of loops*

$$\gamma: E \times [0, 1] \times \mathbb{S}^1 \rightarrow F, (x, t, s) \mapsto \gamma_x^t(s)$$

*such that, for all  $x$  in  $U$ , and all  $(t, s) \in [0, 1] \times \mathbb{S}^1$*

- $\gamma_x^t(s) \in \Omega_x$
- $\gamma_x^0(s) = \beta(x)$
- $\bar{\gamma}_x^1 = g(x)$
- $\gamma_x^t(s) = \beta(x)$  if  $x$  is near  $K$ .

Let us briefly sketch the geometric idea behind the above proposition if we pretend there is only one point  $x$ , and drop it from the notation, and also focus only on  $\gamma^1$ . By assumption, there is a finite collection of points  $p_i$  in  $\Omega$  and  $\lambda_i \in [0, 1]$  such that  $g$  is the barycenter  $\sum \lambda_i p_i$ . Since  $\Omega$  is open and connected, there is a smooth loop  $\gamma_0$  which goes through each  $p_i$ . The claim is that  $g$  is the average value of  $\gamma = \gamma_0 \circ h$  for some self-diffeomorphism  $h$  of  $\mathbb{S}^1$ . The idea is to choose  $h$  such that  $\gamma$  rushes to  $p_1$ , stays there during a time roughly  $\lambda_1$ , rushes to  $p_2$ , etc. But, in order to achieve average exactly  $g$ , it seems like  $h$  needs to be a discontinuous piecewise constant map. The assumption that  $g$  is in the *interior* of the convex hull gives

enough slack to get away with a smooth  $h$ . Actually the conclusion would be false without this interior assumption.

In the previous proof sketch, there is a lot of freedom in constructing  $\gamma$ , which is problematic when trying to do it consistently when  $x$  varies.

## 1.2 Preliminaries

In this section,  $E$  is a real vector space with (finite) dimension  $d$ . We'll need the Carathéodory lemma:

**Lemma 1.3** (Carathéodory's lemma). *If a point  $x$  of  $E$  lies in the convex hull of a set  $P$ , then  $x$  can be written as the convex combination of at most  $d + 1$  points in  $P$ .*

*Proof.* By assumption, there is a finite set of points  $t_i$  in  $P$  and weights  $f_i$  such that  $x = \sum f_i t_i$ , each  $f_i$  is non-negative and  $\sum f_i = 1$ . The goal is to reduce the number of these points until reaching at most  $d + 1$ . It suffices to prove that one can get rid of one point as long as there are at least  $d + 2$  points. In this case there is some vanishing combination  $\sum g_i t_i$  with  $\sum g_i = 0$  and not all  $g_i$  vanish. Let  $S = \{i | g_i > 0\}$ . Let  $i_0$  in  $S$  be an index minimizing  $f_i/g_i$ . We define new weights  $k_i = f_i - g_i f_{i_0}/g_{i_0}$ . Those weights sum to  $\sum f_i - (\sum g_i) f_{i_0}/g_{i_0} = 1$  and  $k_{i_0} = 0$ . Each  $k_i$  is non-negative, thanks to the choice of  $i_0$  if  $i$  is in  $S$  or using that  $f_i, -g_i$  and  $f_{i_0}/g_{i_0}$  are all non-negative when  $i$  is not in  $S$ . It remains to compute

$$\begin{aligned} \sum_{i \neq i_0} k_i t_i &= \sum_i k_i t_i \\ &= \sum_i (f_i - g_i f_{i_0}/g_{i_0}) t_i \\ &= \sum_i f_i t_i - \left( \sum_i g_i t_i \right) f_{i_0}/g_{i_0} \\ &= x \end{aligned}$$

where we use  $k_{i_0} = 0$  in the first equality.  $\square$

**Lemma 1.4.** *If a point  $x$  of  $E$  lies in the convex hull of a set  $P$ , then  $x$  can be written as the convex combination of at most  $d + 1$  affinely independent points in  $P$  with positive coefficients.*

*Proof.* Lemma 1.3 gives points  $p_i$  in  $P$  and non-negative weights  $w_i$  such that  $x = \sum w_i p_i$  for  $0 \leq i \leq k$  where  $k \leq d$ . We first discard every point with vanishing weights. Now suppose the  $p_i$ 's are not affinely independent. Then there is some relation  $p_{i_0} = \sum \lambda_j p_j$  with positive weights  $\lambda_j$ . We can then discard  $p_{i_0}$  and replace each remaining  $w_j$  by  $w_j + w_{i_0} \lambda_j$ .  $\square$

**Definition 1.5.** *Let  $E$  be a real vector space with dimension  $d$ . A point  $x$  in  $E$  is surrounded by points  $p_0, \dots, p_d$  if those points are affinely independent and there exist weights  $w_i \in (0, 1)$  such that  $x = \sum_i w_i p_i$ .*

Note that, in the above definition, the number of points  $p_i$  is fixed by the dimension  $d$  of  $E$ .

**Lemma 1.6.** *If a point  $x$  of  $E$  lies in the convex hull of an open set  $P$ , then it is surrounded by some collection of points belonging to  $P$ .*

*Proof.* Lemma 1.4 gives affinely independent points  $p_0, \dots, p_k$  in  $P$  such that  $x$  is the convex combination of these points with positive weights  $w_i$  (here  $k \geq 0$ ). In particular, the family of vectors  $p_i - p_0$ ,  $1 \leq i \leq k$  is free. So we can complete it to a basis by appending vectors  $f_{k+1}, \dots, f_d$ . We set  $p_i = p_0 + \varepsilon f_i$  for  $i \geq k+1$ . This belongs to  $P$  for  $\varepsilon > 0$  small enough since  $p_0$  is in  $P$  and  $P$  is open. Note that the extend family of points is still affinely independent since the vectors  $p_i - p_0$  are non-zero multiple of our basis elements.

The next idea is to expand the simplex spanned by the  $p_i$ 's around its center of mass  $b = 1/(d+1) \sum p_i$ . So we set:

$$p'_i = b + (1 + \varepsilon)(p_i - b).$$

Those point are all in  $P$  for  $\varepsilon$  small enough because  $P$  is open, and there are still affinely independent since they are the image of our original points by a homothety. Then we have:

$$x = \sum_i w_i p_i = \sum_i \frac{w_i + \varepsilon/(d+1)}{1 + \varepsilon} p'_i$$

where all coefficients are positive □

**Lemma 1.7.** *For every  $x$  in  $E$  and every collection of points  $p \in E^{d+1}$  surrounding  $x$ , there is a positive  $\varepsilon$  and a function  $w : E \times E^{d+1} \rightarrow \mathbb{R}^{d+1}$  such that, for every  $(y, q)$  in  $B_\varepsilon(x) \times \Pi_i B_\varepsilon(p_i)$ ,*

- $w$  is smooth at  $(y, q)$
- $w(y, q) \in (0, 1)$
- $y = \sum_{i=0}^d w_i(y, q) q_i$

*Proof.* If  $d = 0$  then there is nothing to prove. Hence we will assume  $d \geq 1$ . Components of elements of  $E^{d+1}$  or  $\mathbb{R}^{d+1}$  will always be numbered from 0 to  $d$ . By assumption, the family of points  $p_i$  is affinely independent and there are weights  $w_0, \dots, w_d$  such that  $x = \sum_i w_i p_i$  where each  $w_i$  is in  $(0, 1)$  and their sum is one. In particular

$$\begin{aligned} x - p_0 &= \sum_{i=0}^d w_i p_i - \sum_{i=0}^d w_i p_0 \\ &= \sum_{i=0}^d w_i (p_i - p_0) \\ &= \sum_{i=1}^d w_i (p_i - p_0). \end{aligned}$$

For  $q$  in  $E^{d+1}$  and  $i \in \{1, \dots, d\}$ , we set  $e_i(q) = q_i - q_0$ . Since  $p$  is a collection of  $d+1$  affinely independent points, the family  $e_i(p)$  is a basis of  $e$ . By continuity of the determinant, this stays true for  $q$  in  $\Pi_i B_\delta(p_i)$  for some positive  $\delta$ . Let  $e_i^*(q)$  denote the elements of the dual basis. In order to prove smoothness of these maps and define them for every  $q$ , we fix a basis  $B$  of  $E$  and the corresponding determinant  $\det_B : E^d \rightarrow \mathbb{R}$ . We set  $\delta(q) = \det_B(e(q))$  and define

$$e_i^*(q) = v \mapsto \det_B(e_1(q), \dots, e_{i-1}(q), v, e_{i+1}(q), \dots, e_d(q)) / \delta(q).$$



which should be interpreted as the zero linear form if  $\delta(q) = 0$  (this interpretation is automatic if division by zero in  $\mathbb{R}$  is defined as zero, as it should be). The map  $q \mapsto e_i^*(q)$  is smooth on  $\Pi_i B_\delta(p_i)$  where  $\delta$  does not vanish since the determinant is polynomial. We set  $w_i(y, q) = e_i^*(q)(y - q_0)$ . The computation of  $x - p_0$  above proves that  $w_i(x, p) = w_i$ . We have  $y - q_0 = \sum_{i=1}^d w_i(y, q)(q_i - q_0)$ . Hence

$$y = \left(1 - \sum_{i=1}^d w_i(y, q)\right) q_0 + \sum_{i=1}^d w_i(y, q) q_i.$$

We denote by  $w_0(y, q)$  the coefficient in front of  $q_0$  in the above formula. Hence we have  $y = \sum_{i=0}^d w_i(y, q) q_i$  with  $w_i(x, p) = w_i$  hence each  $w_i(y, q)$  is in  $(0, 1)$  if  $(y, q)$  is sufficiently close to  $(x_0, p)$ .  $\square$

## 1.3 Constructing loops

### 1.3.1 Surrounding families

It will be convenient to introduce some more vocabulary.

**Definition 1.8.** *We say a loop  $\gamma$  surrounds a vector  $v$  if  $v$  is surrounded by a collection of points belonging to the image of  $\gamma$ . Also, we fix a base point  $0$  in  $\mathbb{S}^1$  and say a loop is based at some point  $b$  if  $0$  is sent to  $b$ .*

Thanks to Carathéodory's lemma, constructing *one* such loop with values in some open  $O$  is easy as soon as  $v$  belongs to the convex hull of  $O$ .

**Lemma 1.9.** *If a vector  $v$  is in the convex hull of a connected open subset  $O$  then, for every base point  $b \in O$ , there is a continuous family of loops  $\gamma: [0, 1] \times \mathbb{S}^1 \rightarrow E, (t, s) \mapsto \gamma^t(s)$  such that, for all  $t$  and  $s$ :*

- $\gamma^t$  is based at  $b$
- $\gamma^0(s) = b$
- $\gamma^t(s) \in O$
- $\gamma^1$  surrounds  $v$

*Proof.* Since  $O$  is open, Lemma 1.6 gives points  $p_i$  in  $O$  surrounding  $x$ . Since  $O$  is open and connected, it is path connected. Let  $\lambda: [0, 1] \rightarrow \Omega_x$  be a continuous path starting at  $b$  and going through the points  $p_i$ . We can concatenate  $\lambda$  and its opposite to get  $\gamma^1$ , say  $\gamma^1(s) = \lambda((1 - \cos 2\pi s)/2)$ . This is a round-trip loop: it back-tracks when it reaches  $\lambda(1)$  at  $s = 1/2$ . We then define  $\gamma^t$  as the round-trip that stops at  $s = t/2$ , stays still until  $s = 1 - t/2$  and then backtracks.  $\square$

The first main task in proving Proposition 1.2 is to construct suitable families of loops  $\gamma_x$  surrounding  $g(x)$ , by assembling local families of loops. Those will then be reparametrized to get the correct average in the next section. In this section, we will work only with *continuous* loops. This will make constructions easier and we will smooth those loops in the end, taking advantage of the fact that  $\Omega$  and the surrounding condition are open.

**Definition 1.10.** A continuous family of loops  $\gamma: E \times [0, 1] \times \mathbb{S}^1 \rightarrow F, (x, t, s) \mapsto \gamma_x^t(s)$  surrounds a map  $g: E \rightarrow F$  with base  $\beta: E \rightarrow F$  on  $U \subseteq E$  if, for every  $x$  in  $U$ , every  $t \in [0, 1]$  and every  $s \in \mathbb{S}^1$ ,

- $\gamma_x^t$  is based at  $\beta(x)$
- $\gamma_x^0(s) = \beta(x)$
- $\gamma_x^1$  surrounds  $g(x)$ .

The space of such families will be denoted by  $\mathcal{L}(g, \beta, U)$ .

Families of surrounding loops are easy to construct locally.

**Lemma 1.11.** Assume  $\Omega$  is open and connected over some neighborhood of  $x_0$ . If  $g(x)$  is the convex hull of  $\Omega_x$  for  $x$  near  $x_0$  then there is a continuous family of loops defined near  $x_0$ , based at  $\beta$ , taking value in  $\Omega$  and surrounding  $g$ .

*Proof.* In this proof we don't mention the  $t$  parameter since it plays no role, but it is still there. Lemma 1.9 gives a loop  $\gamma$  based at  $\beta(x_0)$ , taking values in  $\Omega_{x_0}$  and surrounding  $g(x_0)$ . We set  $\gamma_x(s) = \beta(x) + (\gamma(s) - \beta(x_0))$ . Each  $\gamma_x$  takes values in  $\Omega_x$  because  $\Omega$  is open over some neighborhood of  $x_0$ . Lemma 1.7 guarantees that this loop surrounds  $g(x)$  for  $x$  close enough to  $x_0$ .  $\square$

The difficulty in constructing global families of surrounding loops is that there are plenty of surrounding loops and we need to choose them consistently. The key feature of the above definition is that the  $t$  parameter not only allows us to cut out the corrugation process in the next chapter, but also brings a “satisfied or refund” guarantee, as explained in the next lemma.

**Lemma 1.12.** Each  $L(g, \beta, U)$  is path connected: for every  $\gamma_0$  and  $\gamma_1$  in  $\mathcal{L}(g, \beta, U)$ , there is a continuous map  $\gamma: [0, 1] \times E \times [0, 1] \times \mathbb{S}^1 \rightarrow F, (\tau, x, t, s) \mapsto \gamma_{\tau, x}^t(s)$  which interpolates between  $\gamma_0$  and  $\gamma_1$  in  $L(g, \beta, U)$ .

*Proof.* Let  $\rho$  be the piecewise affine map from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $\rho(\tau) = 1$  if  $\tau \leq 1/2$ ,  $\rho$  is affine on  $[1/2, 1]$ ,  $\rho(\tau) = 0$  if  $\tau \geq 1$ . We set

$$\gamma_{\tau, x}^t(s) = \begin{cases} \gamma_{0, x}^{\rho(\tau)t} \left( \frac{1}{1-\tau} s \right) & \text{if } s < 1 - \tau \text{ and } \tau < 1 \\ \beta(x) & \text{if } s = 1 - \tau \\ \gamma_{1, x}^{\rho(1-\tau)t} \left( \frac{1}{\tau} (s - (1 - \tau)) \right) & \text{if } s > 1 - \tau \text{ and } \tau > 0 \end{cases}$$

There is no surprise and no fun in checking that this is a well-defined continuous homotopy of families of loops based at  $\beta$  interpolating between  $\gamma_0$  and  $\gamma_1$ .

The beautiful observation motivating the above formula is why each  $\gamma_{\tau, x}^1$  surrounds  $g(x)$ . The key is that the image of  $\gamma_{\tau, x}^1$  contains the image of  $\gamma_{0, x}^1$  when  $\tau \leq 1/2$ , and contains the image of  $\gamma_{1, x}^1$  when  $\tau \geq 1/2$ . Hence the interior of the convex hull of the image always contains  $g(x)$ .  $\square$

**Corollary 1.13.** Let  $U_0$  and  $U_1$  be open sets in  $E$ . Let  $K_0 \subseteq U_0$  and  $K_1 \subseteq U_1$  be compact subsets. For any  $\gamma_0 \in \mathcal{L}(U_0, g, \beta)$  and  $\gamma_1 \in \mathcal{L}(U_1, g, \beta)$ , there exists  $U \in \mathcal{N}(K_0 \cup K_1)$  and there exists  $\gamma \in \mathcal{L}(U, g, \beta)$  which coincides with  $\gamma_0$  near  $K_0$ .

*Proof.* Let  $U'_0$  be an open neighborhood of  $K_0$  whose closure  $\bar{U}'_0$  is compact in  $U_0$ . Since  $\bar{U}'_0$  and  $K'_1 := K_1 \setminus (K_1 \cap U_0)$  are disjoint compact subsets of  $E$ , there is some continuous cut-off  $\rho: E \rightarrow [0, 1]$  which vanishes on  $U'_0$  and equals one on some neighborhood  $U'_1$  of  $K'_1$ .

Lemma 1.12 gives a homotopy of loops  $\gamma_\tau$  from  $\gamma_0$  to  $\gamma_1$  on  $U_0 \cap U_1$ . On  $U'_0 \cup (U_0 \cap U_1) \cup U'_1$ , which is a neighborhood of  $K_0 \cup K_1$ , we set

$$\gamma_x = \begin{cases} \gamma_{0,x} & \text{for } x \in U'_0 \\ \gamma_{\rho(x),x} & \text{for } x \in U_0 \cap U_1 \\ \gamma_{1,x} & \text{for } x \in U'_1 \end{cases}$$

which has the required properties.  $\square$

**Lemma 1.14.** *In the setup of Proposition 1.2, assume we have a continuous family  $\gamma$  of loops defined near  $K$  which is based at  $\beta$ , surrounds  $g$  and such that each  $\gamma_x^1$  takes values in  $\Omega_x$ . Then there is such a family which is defined on all of  $U$  and agrees with  $\gamma$  near  $K$ .*

*Proof.* Let  $U_0$  be an open set containing  $K$  and contained in the domain of  $\gamma$ . Let  $U'_0$  be an open neighborhood of  $K$  with compact closure in  $U_0$ . Let  $U_i, i \geq 1$  be a local finite covering of  $U \setminus U'_0$  by open subsets not intersecting  $K$  and where the preceding observations gives families of loops  $\gamma^i$ . We also set  $\gamma^0 = \gamma|_{U_0}$ . In particular the open sets  $U_i, i \geq 0$  cover the whole of  $U$ , and only  $U_0$  intersects  $K$ . Let  $K_i, 0 \leq i \leq N$ , be a family of compact sets with  $K_i \subset U_i$  which covers  $U$ . We repeatedly apply Corollary 1.13 to  $K_i$  and  $K_{i+1}$ , in this order, to get a family  $\gamma'$  defined over all  $U$ . Since each step preserves the family on  $\text{Op } K_i$  and only  $U_0$  intersects (in fact contains)  $K$ , we do have  $\gamma' = \gamma$  on  $\text{Op } K$ .  $\square$

### 1.3.2 The reparametrization lemma

The second ingredient needed to prove Proposition 1.2 is a parametric reparametrization lemma.

**Lemma 1.15.** *Let  $\gamma: E \times \mathbb{S}^1 \rightarrow F$  be a smooth family of loops surrounding a map  $g$  with base  $\beta$  over some  $U \subseteq E$ . There is a family of circle diffeomorphisms  $h: U \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that each  $\gamma_x \circ h_x$  has average  $g(x)$  and  $h_x(0) = 0$ .*

*Proof.* For any fixed  $x$ , since  $\gamma_x$  strictly surrounds  $g(x)$ , there are points  $s_1, \dots, s_{n+1}$  in  $\mathbb{S}^1$  such that  $g(x)$  is surrounded by the corresponding points  $\gamma_x(s_j)$ .

Let  $\mu_1, \dots, \mu_{n+1}$  be smooth positive probability measures very close to the Dirac measures on  $s_j$  (ie.  $\mu_j = f_j ds$  for some smooth positive function  $f_j$  and, for any function  $h$ ,  $\int h d\mu_j$  is almost  $h(s_j)$ ). We set  $p_j = \int \gamma_x d\mu_j$ , which is almost  $\gamma_x(s_j)$  so that  $g(x) = \sum w_j p_j$  for some weights  $w_j$  in the open interval  $(0, 1)$  according to Lemma 1.7.

If  $x'$  is in a sufficiently small neighborhood of  $x$ , Lemma 1.7 gives smooth weight functions  $w_j$  such that  $g(x') = \sum w_j(x') p_j(x')$ . Let  $U^i, i \geq 1$  be a locally finite cover of  $U$  by such neighborhoods, with corresponding measures  $\mu_j^i$ , moving points  $p_j^i$  and weight functions  $w_j^i$ . Let  $(\rho_i)$  be a partition of unity associated to this covering. For every  $x$ , we set

$$\mu_x = \sum_{i=1}^{\infty} \sum_{j=1}^{n+1} \rho_i(x) w_j^i(x) \mu_j^i$$

so that:

$$\begin{aligned}
\int \gamma_x d\mu_x &= \sum_i \rho_i(x) \sum_{j=1}^{n+1} w_j^i(x) \int \gamma_x d\mu_j^i \\
&= \sum_i \rho_i(x) \sum_{j=1}^{n+1} w_j^i(x) p_j^i(x) \\
&= \sum_i \rho_i(x) g(x) = g(x).
\end{aligned}$$

We now set  $h_x^{-1}(t) = \int_0^t d\mu_x$  so that  $g(x) = \overline{\gamma_x \circ h_x}$  for all  $x$ . □

### 1.3.3 Proof of the loop construction proposition

We finally assemble the ingredients from the previous two sections.

*Proof of Proposition 1.2.* Let  $\gamma^*$  be a family of loops surrounding the origin in  $F$ , constructed using Lemma 1.11. For  $x$  in some neighborhood  $U^*$  of  $K$  where  $g = \beta$ , we set  $\gamma_x = g(x) + \varepsilon \gamma^*$  where  $\varepsilon > 0$  is sufficiently small to ensure the image of  $\gamma_x$  and its convex hull are contained in  $\Omega_x$  (recall  $\Omega$  is open and  $K$  is compact). Lemma 1.14 extends this family to a continuous family of surrounding loops  $\gamma_x$  for all  $x$  (this is not yet our final  $\gamma$ ).

We then need to approximate this continuous family by a smooth one. Some care is needed to ensure that it stays based at  $\beta$ . For instance, we can first compose each loop by some fixed surjective continuous map from  $\mathbb{S}^1$  to itself that sends a neighborhood of 0 to 0. This way each loop becomes constant near 0, and a convolution smoothing will then keep the value at 0. If the smoothing is sufficiently  $C^0$  small then the new  $\gamma$  is still surrounding and takes values in  $\Omega$ .

Then Lemma 1.15 gives a family of circle diffeomorphisms  $h_x$  such that  $\gamma_x^1 \circ h_x$  has average  $g(x)$ .

Finally we choose a cut-off function  $\chi$  which vanishes on  $\text{Op } K$  and equals one on  $\text{Op } U \setminus U^*$ . In  $U^*$ , we replace  $\gamma_x \circ h_x = g(x) + \gamma^* \circ h_x$  by  $g(x) + \chi(x) \gamma^* \circ h_x$ . This operation does not change the average values of these loops, because it rescales them around their average value, but makes them constant on  $\text{Op } K$ . Also, those loops stay in  $\Omega$ , thanks to our choice of  $\varepsilon$ . □

## Chapter 2

# Local theory of convex integration

### 2.1 Key construction

The goal of this chapter is to explain the local aspects of (Theillière's implementation of) convex integration, the next chapter will cover global aspects.

The elementary step of convex integration modifies the derivative of a map in one direction. Let  $E$  and  $F$  be finite dimensional real normed vector spaces. Let  $f: E \rightarrow F$  be a smooth map with compact support.

**Definition 2.1.** *A dual pair on  $E$  is a pair  $(\pi, v)$  where  $\pi$  is a linear form on  $E$  and  $v$  a vector in  $E$  such that  $\pi(v) = 1$ .*

Say we wish  $Df(x)v$  could live in some open subset  $\Omega_x \subset F$ . Assume there is a smooth compactly supported family of loops  $\gamma: E \times \mathbb{S}^1 \rightarrow F$  such that each  $\gamma_x$  takes values in  $\Omega_x$ , and has average value  $\int_{\mathbb{S}^1} \gamma_x = Df(x)v$ . Obviously such loops can exist only if  $Df(x)v$  is in the convex hull of  $\Omega_x$ , and we saw in the previous chapter that this is almost sufficient (and we'll see this is sufficiently almost sufficient for our purposes).

**Definition 2.2.** *The map obtained by corrugation of  $f$  in direction  $(\pi, v)$  using  $\gamma$  with oscillation number  $N$  is*

$$x \mapsto f(x) + \frac{1}{N} \int_0^{N\pi(x)} [\gamma_x(s) - Df(x)v] ds.$$

In the above definition, we mostly think of  $N$  as a large natural number. But we don't actually require it, any positive real number will do.

The next proposition implies that, provided  $N$  is large enough, we have achieved  $Df'(x)v \in \Omega_x$ , almost without modifying derivatives in the other directions of  $\ker \pi$ , and almost without moving  $f(x)$ . In addition, if we assume that  $\gamma_x$  is constant (necessarily with value  $Df(x)v$ ) for  $x$  in some closed subset  $K$  where  $Df(x)v$  was already good, then the modification is relative to  $K$ .

**Lemma 2.3** (Theillière 2018). *The corrugated function  $f'$  satisfies, uniformly in  $x$ :*

1.  $Df'(x)v = \gamma(x, N\pi(x)) + O\left(\frac{1}{N}\right)$ , and the error vanishes whenever  $\gamma_x$  is constant.

2.  $Df'(x)w = Df(x)w + O\left(\frac{1}{N}\right)$  for  $w \in \ker \pi$
3.  $f'(x) = f(x) + O\left(\frac{1}{N}\right)$
4.  $f'(x) = f(x)$  whenever  $\gamma_x$  is constant.

*Proof.* We set  $\Gamma_x(t) = \int_0^t (\gamma_x(s) - Df(x)v) ds$ , so that  $f'(x) = f(x) + \Gamma_x(N\pi(x))/N$ . Because each  $\Gamma_x$  is 1-periodic, and everything has compact support in  $E$ , all derivatives of  $\Gamma$  are uniformly bounded. Item 3 in the statement is then obvious. Item 2 also follows since  $\partial_i f'(x) = \partial_i f(x) + \partial_i \Gamma(x, N\pi(x))/N$ . In order to prove Item 1, we compute:

$$\begin{aligned} Df'(x)v &= Df(x)v + \frac{1}{N} \partial_j \Gamma(x, N\pi(x)) + \frac{N}{N} \partial_t \Gamma(x, N\pi(x)) \\ &= Df(x)v + O\left(\frac{1}{N}\right) + \gamma(x, N\pi(x)) - Df(x)v \\ &= \gamma(x, N\pi(x)) + O\left(\frac{1}{N}\right). \end{aligned}$$

Item 4 is obvious since  $\Gamma_x$  vanishes identically when  $\gamma_x$  is constant.  $\square$

## 2.2 The main inductive step

**Definition 2.4.** Let  $E'$  be a linear subspace of  $E$ . A map  $\mathcal{F} = (f, \varphi) : E \rightarrow F \times \text{Hom}(E, F)$  is  $E'$ -holonomic if, for every  $v$  in  $E'$  and every  $x$ ,  $Df(x)v = \varphi(x)v$ .

**Definition 2.5.** A first order differential relation for maps from  $E$  to  $F$  is a subset  $\mathcal{R}$  of  $E \times F \times \text{Hom}(E, F)$ .

Until the end of this section,  $\mathcal{R}$  will always denote a first order differential relation for maps from  $E$  to  $F$ .

**Definition 2.6.** A formal solution of a differential relation  $\mathcal{R}$  over  $U \subset E$  is a map  $\mathcal{F} = (f, \varphi) : E \rightarrow F \times \text{Hom}(E, F)$  such that, for every  $x$  in  $U$ ,  $(x, f(x), \varphi(x))$  is in  $\mathcal{R}$ .

The first component of a map  $\mathcal{F} : E \rightarrow F \times \text{Hom}(E, F)$  will sometimes be denoted by  $\text{bs } \mathcal{F} : E \rightarrow F$  and called the base map of  $\mathcal{F}$ .

**Definition 2.7.** A homotopy of formal solutions over  $U$  is a map  $\mathcal{F} : \mathbb{R} \times E \rightarrow F \times \text{Hom}(E, F)$  which is smooth over  $[0, 1] \times U$  and such that each  $x \mapsto \mathcal{F}(t, x)$  is a formal solution over  $U$  when  $t$  is in  $[0, 1]$ .

Typically,  $x \mapsto \mathcal{F}(t, x)$  will be denoted by  $\mathcal{F}_t$ .

We'll use the notation  $\text{Conn}_w A$  to denote the connected component of  $A$  that contains  $w$ , or the empty set if  $w$  doesn't belong to  $A$ .

**Definition 2.8.** For every  $\sigma = (x, y, \varphi)$ , the slice of  $\mathcal{R}$  at  $\sigma$  with respect to  $(\pi, v)$  is:

$$\mathcal{R}(\sigma, \pi, v) = \text{Conn}_{\varphi(v)} \{w \in F \mid (x, y, \varphi + (w - \varphi(v)) \otimes \pi) \in \mathcal{R}\}.$$

**Lemma 2.9.** The linear map  $\varphi + (w - \varphi(v)) \otimes \pi$  coincides with  $\varphi$  on  $\ker \pi$  and sends  $v$  to  $w$ . If  $\sigma$  belongs to  $\mathcal{R}$  then  $\varphi(v)$  belongs to  $\{w \in F, (x, y, \varphi + (w - \varphi(v)) \otimes \pi) \in \mathcal{R}\}$ .

*Proof.* This is direct check.  $\square$

**Definition 2.10.** A formal solution  $\mathcal{F}$  of  $\mathcal{R}$  over  $U$  is  $(\pi, v)$ -short if, for every  $x$  in  $U$ ,  $Df(x)v$  belongs to the interior of the convex hull of  $\mathcal{R}((x, f(x), \varphi(x)), \pi, v)$ .

**Lemma 2.11.** Let  $\mathcal{F}$  be a formal solution of  $\mathcal{R}$  over an open set  $U$ . Let  $K_1 \subset U$  be a compact subset, and let  $K_0$  be a compact subset of the interior of  $K_1$ . Let  $C$  be a subset of  $U$ . Let  $E'$  be a linear subspace of  $E$  contained in  $\ker \pi$ . Let  $\varepsilon$  be a positive real number.

Assume  $\mathcal{R}$  is open over  $U$ . Assume that  $\mathcal{F}$  is  $E'$ -holonomic near  $K_0$ ,  $(\pi, v)$ -short over  $U$ , and holonomic near  $C$ . Then there is a homotopy  $\mathcal{F}_t$  such that:

1.  $\mathcal{F}_0 = \mathcal{F}$  ;
2.  $\mathcal{F}_t$  is a formal solution of  $\mathcal{R}$  over  $U$  for all  $t$  ;
3.  $\mathcal{F}_t(x) = \mathcal{F}(x)$  for all  $t$  when  $x$  is near  $C$  or outside  $K_1$  ;
4.  $d(\text{bs } \mathcal{F}_t(x), \text{bs } \mathcal{F}(x)) \leq \varepsilon$  for all  $t$  and all  $x$  ;
5.  $\mathcal{F}_1$  is  $E' \oplus \mathbb{R}v$ -holonomic near  $K_0$ .

*Proof.* We denote the components of  $F$  by  $f$  and  $\varphi$ . Since  $\mathcal{F}$  is short over  $U$ , Proposition 1.2 applied to  $g: x \mapsto Df(x)v$ ,  $\beta: x \mapsto \varphi(x)v$ ,  $\Omega_x = \mathcal{R}(\mathcal{F}(x), \pi, v)$ , and  $K = C \cap K_1$  gives us a smooth family of loops  $\gamma: E \times [0, 1] \times \mathbb{S}^1 \rightarrow F$  such that, for all  $x$  in  $U$ :

- $\forall t s, \gamma_x^t(s) \in \mathcal{R}(\mathcal{F}(x), \pi, v)$
- $\forall s, \gamma_x^0(s) = \varphi(x)v$
- $\bar{\gamma}_x^1 = Df(x)v$
- if  $x$  is near  $C$ ,  $\forall t s, \gamma_x^t(s) = \varphi(x)v$

Let  $\rho: E \rightarrow \mathbb{R}$  be a smooth cut-off function which equals one on a neighborhood of  $K_0$  and whose support is contained in  $K_1$ .

Let  $N$  be a positive real number. Let  $\bar{f}$  be the corrugated map constructed from  $f$ ,  $\gamma^1$  and  $N$ . Lemma 2.3 ensures that, for all  $x$  in  $U$ ,

$$D\bar{f}(x) = Df(x) + [\gamma_x^1(N\pi(x)) - Df(x)v] \otimes \pi + \frac{1}{N}B_x$$

for some bounded map  $B$  which vanishes whenever  $\gamma_x$  is constant, hence vanishes near  $C$ .

We set  $\mathcal{F}_t(x) = (f_t(x), \varphi_t(x))$  where:

$$f_t(x) = f(x) + \frac{t\rho(x)}{N} \int_0^{N\pi(x)} [\gamma_x^t(s) - Df(x)v] ds$$

and

$$\varphi_t(x) = \varphi(x) + [\gamma_x^{t\rho(x)}(N\pi(x)) - \varphi(x)v] \otimes \pi + \frac{t\rho(x)}{N}B_x.$$

We now prove that  $\mathcal{F}_t$  has the announced properties, starting with the obvious ones. The fact that  $\mathcal{F}_0 = \mathcal{F}$  is obvious since  $\gamma_x^0(s) = \varphi(x)v$  for all  $s$ .

When  $x$  is near  $C$ ,  $Df(x) = \varphi(x)$  since  $\mathcal{F}$  is holonomic near  $C$ . In addition,  $\gamma_x^t(s) = \varphi(x)v$  for all  $s$  and  $t$ , hence  $B_x$  vanishes. Hence  $\mathcal{F}_t(x) = \mathcal{F}(x)$  for all  $t$  when  $x$  is near  $C$ .

Outside of  $K_1$ ,  $\rho$  vanishes. Hence  $f_t(x) = f(x)$  for all  $t$ , and  $\gamma_x^{t\rho(x)}(s) = \varphi(x)v$  for all  $s$  and  $t$ , and  $\varphi_t(x) = \varphi(x)$ .

The distance between  $f(x)$  and  $f_t(x)$  is zero outside of  $K_1$  which is compact, and  $O(1/N)$ , so it is less than  $\varepsilon$  for  $N$  large enough.

We now turn to the interesting parts. The first one is that each  $\mathcal{F}_t$  is a formal solution of  $\mathcal{R}$  over  $U$ . We already know that  $\mathcal{F}_t$  coincides with  $\mathcal{F}$ , which is a formal solution, outside of the compact set  $K_1$ . We set

$$\mathcal{F}'_t(x) = (f(x), \varphi(x) + [\gamma_x^{t\rho(x)}(N\pi(x)) - \varphi(x)v] \otimes \pi).$$

Since  $\mathcal{R}$  is open over  $U$ , and  $K_1 \times [0, 1]$  is compact and  $\mathcal{F}_t$  is within  $O(1/N)$  of  $\mathcal{F}'_t$ , it suffices to prove that  $\mathcal{F}'_t$  is a formal solution for all  $t$ . This is guaranteed by the definition of the slice  $\mathcal{R}(\mathcal{F}(x), \pi, v)$  to which  $\gamma_x^{t\rho(x)}(N\pi(x))$  belongs.

Finally, let's prove that  $\mathcal{F}_1$  is  $E' \oplus \mathbb{R}v$ -holonomic near  $K_0$ . Since  $\rho = 1$  near  $K_0$ , we have, for  $x$  near  $K_0$ ,

$$Df_1(x) = Df(x) + [\gamma_x^1(N\pi(x)) - Df(x)v] \otimes \pi + \frac{1}{N}B_x,$$

and

$$\varphi_1(x) = \varphi(x) + [\gamma_x^1(N\pi(x)) - \varphi(x)v] \otimes \pi + \frac{1}{N}B_x.$$

Let  $p$  be the projection of  $E$  onto  $\ker \pi$  along  $v$ , so that  $\text{Id}_E = p + v \otimes \pi$ . We can rewrite the above formulas as

$$Df_1(x) = Df(x) \circ p + \gamma_x^1(N\pi(x)) \otimes \pi + \frac{1}{N}B_x,$$

and

$$\varphi_1(x) = \varphi(x) \circ p + \gamma_x^1(N\pi(x)) \otimes \pi + \frac{1}{N}B_x.$$

So we see the difference is  $Df(x) \circ p - \varphi(x) \circ p$  which vanishes on  $E'$  since  $\mathcal{F}$  is  $E'$ -holonomic near  $K_0$ , and vanishes on  $v$  since  $p(v) = 0$ .  $\square$

## 2.3 Ample differential relations

**Definition 2.12.** A subset  $\Omega$  of a real vector space  $E$  is ample if the convex hull of each connected component of  $\Omega$  is the whole  $E$ .

**Lemma 2.13.** The complement of a linear subspace of codimension at least 2 is ample.

**Definition 2.14.** A first order differential relation  $\mathcal{R}$  is ample if all its slices are ample.

**Lemma 2.15.** The relation of immersions in positive codimension is open and ample.

*Proof.* For every  $\sigma = (x, y, \varphi)$  in the immersion relation  $\mathcal{R}$ , and for every dual pair  $(\pi, v)$ , the slice  $\mathcal{R}(\sigma, \pi, v)$  is the set of  $w$  which do not belong to the image of  $\ker \pi$  under  $\varphi$ . Since  $\dim F > \dim E$ , this image has codimension at least 2 in  $F$ , and Lemma 2.13 concludes.  $\square$

**Lemma 2.16.** Let  $\mathcal{F}$  be a formal solution of  $\mathcal{R}$  over an open set  $U$ . Let  $K_1 \subset U$  be a compact subset, and let  $K_0$  be a compact subset of the interior of  $K_1$ . Assume  $\mathcal{F}$  is holonomic near a subset  $C$  of  $U$ . Let  $\varepsilon$  be a positive real number.

If  $\mathcal{R}$  is open and ample over  $U$  then there is a homotopy  $\mathcal{F}_t$  such that:

1.  $\mathcal{F}_0 = \mathcal{F}$



2.  $\mathcal{F}_t$  is a formal solution of  $\mathcal{R}$  over  $U$  for all  $t$  ;
3.  $\mathcal{F}_t(x) = \mathcal{F}(x)$  for all  $t$  when  $x$  is near  $C$  or outside  $K_1$ .
4.  $d(\text{bs } \mathcal{F}_t(x), \text{bs } \mathcal{F}(x)) \leq \varepsilon$  for all  $t$  and all  $x$  ;
5.  $\mathcal{F}_1$  is holonomic near  $K_0$ .

*Proof.* This is a straightforward induction using Lemma 2.11. Let  $(e_1, \dots, e_n)$  be a basis of  $E$ , and let  $(\pi_1, \dots, \pi_n)$  be the dual basis. Let  $E'_i$  be the linear subspace of  $E$  spanned by  $(e_1, \dots, e_i)$ , for  $1 \leq i \leq n$ , and let  $E'_0$  be the zero subspace of  $E$ . Each  $(\pi_i, e_i)$  is a dual pair and the kernel of  $\pi_i$  contains  $E'_{i-1}$ .

Lemma 2.11 allows to build a sequence of homotopies of formal solutions, each homotopy relating a formal solution which is  $E'_i$ -holonomic to one which is  $E'_{i+1}$ -holonomic (always near  $K_0$ ). The shortness condition is always satisfies because  $\mathcal{R}$  is ample over  $U$ . Each homotopy starts where the previous one stopped, stay at  $C^0$  distance at most  $\varepsilon/n$ , and is relative to  $C$  and the complement of  $K_1$ .

It then suffices to do a smooth concatenation of theses homotopies. We first pre-compose with a smooth map from  $[0, 1]$  to itself that fixes 0 and 1 and has vanishing derivative to all orders at 0 and 1. Then we precompose by affine isomorphisms from  $[0, 1]$  to  $[i/n, (i+1)/n]$  before joining them.  $\square$

## Chapter 3

# Global theory of open and ample relations

### 3.1 Preliminaries

#### 3.1.1 Vector bundles operations

**Definition 3.1.** For every bundle  $p : E \rightarrow B$  and every map  $f : B' \rightarrow B$ , the pull-back bundle  $f^*E \rightarrow B'$  is defined by  $f^*E = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$  with the obvious projection to  $B'$ .

The case of vector bundles.

**Definition 3.2.** Let  $E \rightarrow B$  and  $F \rightarrow B$  be two vector bundles over some smooth manifold  $B$ . The bundle  $\text{Hom}(E, F) \rightarrow B$  is the set of linear maps from  $E_b$  to  $F_b$  for some  $b$  in  $B$ , with the obvious project map.

Set-theoretically, one can define  $\text{Hom}(E, F)$  as the set of subsets  $S$  of  $E \times F$  such that there exists  $b$  such that  $S \subset E_b \times F_b$  and  $S$  is the graph of a linear map. But the type theory formalization will use other tricks here. The facts that really matter are listed in Lemma 3.5.

#### 3.1.2 Jets spaces

**Definition 3.3.** Let  $M$  and  $N$  be smooth manifolds. Denote by  $p_1$  and  $p_2$  the projections of  $M \times N$  to  $M$  and  $N$  respectively.

The space  $J^1(M, N)$  of 1-jets of maps from  $M$  to  $N$  is  $\text{Hom}(p_1^*TM, p_2^*TN)$

We will use notations like  $(m, n, \varphi)$  to denote an element of  $J^1(M, N)$ , but one should keep in mind that  $J^1(M, N)$  is not a product, since  $\varphi$  lives in  $\text{Hom}(T_m M, T_n N)$  which depends on  $m$  and  $n$ .

**Definition 3.4.** The 1-jet of a smooth map  $f : M \rightarrow N$  is the map from  $m$  to  $J^1(M, N)$  defined by  $j^1 f(m) = (m, f(m), T_m f)$ .

The composition of a section  $\mathcal{F} : M \rightarrow J^1(M, N)$  with the projection onto  $N$  will sometimes be denoted by  $\text{bs } \mathcal{F} : M \rightarrow N$  and called the base map of  $\mathcal{F}$ .

**Lemma 3.5.** *For every smooth map  $f: M \rightarrow N$ ,*

1.  $j^1 f$  *is smooth*
2.  $j^1 f$  *is a section of  $J^1(M, N) \rightarrow M$*
3.  $j^1 f$  *composed with  $J^1(M, N) \rightarrow N$  is  $f$ .*

*Proof.* This is obvious by construction...  $\square$

**Definition 3.6.** *A section  $\mathcal{F}$  of  $J^1(M, N) \rightarrow M$  is called holonomic if it is the 1-jet of its base map. Equivalently,  $\mathcal{F}$  is holonomic if there exists  $f: M \rightarrow N$  such that  $\mathcal{F} = j^1 f$ , since such a map is necessarily  $\text{bs } \mathcal{F}$ .*

## 3.2 First order differential relations

**Definition 3.7.** *A first order differential relation for maps from  $M$  to  $N$  is a subset  $\mathcal{R}$  of  $J^1(M, N)$ .*

**Definition 3.8.** *A formal solution of a differential relation  $\mathcal{R} \subseteq J^1(M, N)$  is a section of  $J^1(M, N) \rightarrow M$  taking values in  $\mathcal{R}$ . A solution of  $\mathcal{R}$  is a map from  $M$  to  $N$  whose 1-jet extension is a formal solution.*

**Definition 3.9.** *A homotopy of formal solutions of  $\mathcal{R}$  is a family of sections  $\mathcal{F}: \mathbb{R} \times M \rightarrow J^1(M, N)$  which is smooth over  $[0, 1] \times M$  and such that each  $m \mapsto \mathcal{F}(t, m)$  is a formal solution when  $t$  is in  $[0, 1]$ .*

**Definition 3.10.** *A first order differential relation  $\mathcal{R} \subseteq J^1(M, N)$  satisfies the  $h$ -principle if every formal solution of  $\mathcal{R}$  is homotopic to a holonomic one. It satisfies the parametric  $h$ -principle if, for every manifold with boundary  $P$ , every family  $\mathcal{F}: P \times M \rightarrow J^1(M, N)$  of formal solutions which are holonomic for  $p$  in  $\mathcal{N}(\partial P)$  is homotopic to a family of holonomic ones relative to  $\mathcal{N}(\partial P)$ . It satisfies the parametric  $h$ -principle if, for every manifold with boundary  $P$ , every family  $\mathcal{F}: P \times M \rightarrow J^1(M, N)$  of formal solutions is homotopic to a family of holonomic ones.*

**Lemma 3.11.** *The above definitions translate to the definitions of the previous chapter in local charts. (We'll need more precise statements...)*

### Parametricity for free

In many cases, relative parametric  $h$ -principles can be deduced from relative non-parametric ones with a larger source manifold. Let  $X$ ,  $P$  and  $Y$  be manifolds, with  $P$  seen as a parameter space. Denote by  $\Psi$  the map from  $J^1(X \times P, Y)$  to  $J^1(X, Y)$  sending  $(x, p, y, \psi)$  to  $(x, y, \psi \circ \iota_{x,p})$  where  $\iota_{x,p}: T_x X \rightarrow T_x X \times T_p P$  sends  $v$  to  $(v, 0)$ .

To any family of sections  $F_p: x \mapsto (f_p(x), \varphi_{p,x})$  of  $J^1(X, Y)$ , we associate the section  $\bar{F}$  of  $J^1(X \times P, Y)$  sending  $(x, p)$  to  $\bar{F}(x, p) := (f_p(x), \varphi_{p,x} \oplus \partial f / \partial p(x, p))$ .

**Lemma 3.12.** *In the above setup, we have:*

- $\bar{F}$  *is holonomic at  $(x, p)$  if and only if  $F_p$  is holonomic at  $x$ .*
- $F$  *is a family of formal solutions of some  $\mathcal{R} \subset J^1(X, Y)$  if and only if  $\bar{F}$  is a formal solution of  $\mathcal{R}^P := \Psi^{-1}(\mathcal{R})$ .*

*Proof.* TODO... □

**Lemma 3.13.** *Let  $\mathcal{R}$  be a first order differential relation for maps from  $M$  to  $N$ . If, for every manifold with boundary  $P$ ,  $\mathcal{R}^P$  satisfies the  $h$ -principle then  $\mathcal{R}$  satisfies the parametric  $h$ -principle. Likewise, the  $C^0$ -dense and relative  $h$ -principle for all  $\mathcal{R}^P$  imply the parametric  $C^0$ -dense and relative  $h$ -principle for  $\mathcal{R}$ .*

*Proof.* This obviously follows from Lemma 3.12. □

### 3.3 The $h$ -principle for open and ample differential relations

In this chapter,  $X$  and  $Y$  are smooth manifolds and  $\mathcal{R}$  is a first order differential relation on maps from  $X$  to  $Y$ :  $\mathcal{R} \subset J^1(X, Y)$ . For any  $\sigma = (x, y, \varphi)$  in  $\mathcal{R}$  and any dual pair  $(\lambda, v) \in T_x^*X \times T_x X$ , we set:

$$\mathcal{R}_{\sigma, \lambda, v} = \text{Conn}_{\varphi(v)} \{ w \in T_y Y ; (x, y, \varphi + (w - \varphi(v)) \otimes \lambda) \in \mathcal{R} \}$$

where  $\text{Conn}_a A$  is the connected component of  $A$  containing  $a$ . In order to decipher this definition, it suffices to notice that  $\varphi + (w - \varphi(v)) \otimes \lambda$  is the unique linear map from  $T_x X$  to  $T_y Y$  which coincides with  $\varphi$  on  $\ker \lambda$  and sends  $v$  to  $w$ . In particular,  $w = \varphi(v)$  gives back  $\varphi$ .

Of course we will want to deal with more than one point, so we will consider a vector field  $V$  and a 1-form  $\lambda$  such that  $\lambda(V) = 1$  on some subset  $U$  of  $X$ , a formal solution  $F$  (defined at least on  $U$ ), and get the corresponding  $\mathcal{R}_{F, \lambda, v}$  over  $U$ .

One easily checks that  $\mathcal{R}_{\sigma, \kappa^{-1}\lambda, \kappa v} = \kappa \mathcal{R}_{\sigma, \lambda, v}$  hence the above definition only depends on  $\ker \lambda$  and the direction  $\mathbb{R}V$ .

**Definition 3.14.** *A relation  $\mathcal{R}$  is ample if, for every  $\sigma = (x, y, \varphi)$  in  $\mathcal{R}$  and every  $(\lambda, v)$ , the slice  $\mathcal{R}_{\sigma, \lambda, v}$  is ample in  $T_y Y$ .*

**Lemma 3.15.** *If a relation is ample then it is ample in the sense of Definition 2.14 when seen in local charts.*

*Proof.* This follows from the fundamental properties of the tangent bundle. □

**Lemma 3.16.** *The relation of immersions of  $M$  into  $N$  in positive codimension is open and ample.*

*Proof.* This obviously follows from Lemma 2.15. □

**Theorem 3.17** (Gromov). *If  $\mathcal{R}$  is open and ample then it satisfies the relative and parametric  $C^0$ -dense  $h$ -principle.*

We first explain how to get rid of parameters, using the relation  $\mathcal{R}^P$  for families of solutions parametrized by  $P$ .

**Lemma 3.18.** *If  $\mathcal{R}$  is ample then, for any parameter space  $P$ ,  $\mathcal{R}^P$  is also ample.*

*Proof.* We fix  $\sigma = (x, y, \psi)$  in  $\mathcal{R}^P$ . For any  $\lambda = (\lambda_X, \lambda_P) \in T_x^*X \times T_p^*P$  and  $v = (v_X, v_P) \in T_x X \times T_p P$  such that  $\lambda(v) = 1$ , we need to prove that  $\text{Conv } \mathcal{R}_{\sigma, \lambda, v}^P = T_y Y$ . Unfolding the definitions gives:

$$\mathcal{R}_{\sigma, \lambda, v}^P = \text{Conn}_{\psi(v)} \{ w \in T_y Y ; (x, y, \psi \circ \iota_{x, p} + (w - \psi(v)) \otimes \lambda_X) \in \mathcal{R} \}.$$

A degenerate but easy case is when  $\lambda_X = 0$ . Then the condition on  $w$  becomes  $\psi \circ \iota_{x,p} \in \mathcal{R}$ , which is true by definition of  $\mathcal{R}^P$ , so  $\mathcal{R}_{\sigma,\lambda,v}^P = T_y Y$ .

We now assume  $\lambda_X$  is not zero and choose  $u \in T_x X$  such that  $\lambda_X(u) = 1$ . We then have  $\mathcal{R}_{\sigma,\lambda,v}^P = \mathcal{R}_{\psi\sigma,\lambda_X,u} + \psi(v) - \psi \circ \iota_{x,p}(u)$ . Because  $\mathcal{R}$  is ample and taking convex hull commutes with translation, we get that  $\text{Conv } \mathcal{R}_{\sigma,\lambda,v}^P = T_y Y$ .  $\square$

*Proof of Theorem 3.17.* Lemmas 3.13 and 3.18 prove we can assume there are no parameters. So we start with a single formal solution  $F$  of  $\mathcal{R}$ , which is holonomic near some closed subset  $A \subset X$ .

We first assume  $X$  is closed, and will then explain the proof adjustments needed in the non-compact case. By compactness, there are finite many compact subsets  $(K_i)_{1 \leq i \leq N}$  contained in coordinate charts  $U_i$  and such that  $\bigcup K_i = X$ .

We prove by induction on  $i$  from 0 to  $N$  that there are formal solutions  $F_i$ , starting with  $F_0 = F$ , that are homotopic to  $F$  relative to  $A$ , holonomic on  $K_{\leq i} := \bigcup_{j \leq i} K_j$  and whose base maps are  $(1 - 2^{-i})\varepsilon$ -close to that of  $F$  (this contrived bound will be convenient for the non-compact case).

Assume  $F_i$  has been constructed for some  $i < N$ . We now want to construct it on  $U_{i+1}$ . Lemma 3.15 ensures the pull-back of  $\mathcal{R}$  in the chart corresponding to  $U_{i+1}$  is ample in the sense of the preceding chapter. Hence Lemma 2.16 can be applied to construct  $F_{i+1}$ , using the image of  $K_{i+1}$  as  $K_0$  and the image of  $A \cap K_{i+1} \cap K_{\leq i}$  as  $C$ . It is holonomic on  $K_{\leq i+1}$  because it agrees with  $F_i$  near  $K_{\leq i}$  and is holonomic near  $K_{i+1}$ .

Once all  $F_i$  are constructed, we define  $F^t$  to be the concatenation of all homotopies relating  $F_i$  to  $F_{i+1}$ .

If  $X$  is not compact, then one can use a countable family of subsets  $(U_i, K_i)$  which is locally finite (ie. every point  $x$  has a neighborhood intersecting only finitely many  $U_i$ ). The way we have chosen  $C^0$ -bounds ensures that each  $\text{bs } F_i$  is still at distance at most  $\varepsilon$  from  $\text{bs } F$ . We now have countably many homotopies to concatenate, so we need to reparametrize the  $i$ -th homotopy by an interval of length  $2^{-i}$ . This give a family of sections of  $J^1(X, Y)$  parametrized by  $t \in [0, 1)$ . But our local finiteness assumption implies that, for each  $x$ , there is some  $t_0 < 1$  and some neighborhood  $U$  of  $x$  such that our family is  $t$ -independent on  $U$  for  $t \geq t_0$ . So we can extend to  $t = 1$ . The resulting family is smooth since smoothness is a local condition in both  $x$  and  $t$ .  $\square$

**Theorem 3.19** (Smale 1958). *There is a homotopy of immersions of  $\mathbb{S}^2$  into  $\mathbb{R}^3$  from the inclusion map to the antipodal map  $a : q \mapsto -q$ .*

*Proof.* We denote by  $\iota$  the inclusion of  $\mathbb{S}^2$  into  $\mathbb{R}^3$ . We set  $j_t = (1 - t)\iota + ta$ . This is a homotopy from  $\iota$  to  $a$  (but not an immersion for  $t = 1/2$ ). Using the canonical trivialization of the tangent bundle of  $\mathbb{R}^3$ , we can set, for  $(q, v) \in T\mathbb{S}^2$ ,  $G_t(q, v) = \text{Rot}_{Oq}^{\pi t}(v)$ , the rotation around axis  $Oq$  with angle  $\pi t$ . The family  $\sigma : t \mapsto (j_t, G_t)$  is a homotopy of formal immersions relating  $j^1 \iota$  to  $j^1 a$ . It is homotopic by reparametrization to a homotopy of formal immersions relating  $j^1 \iota$  to  $j^1 a$  which are holonomic for  $t$  near the 0 and 1.

The above theorem ensures this family is homotopic, relative to  $t = 0$  and  $t = 1$ , to a family of holonomic formal immersions, ie a family  $t \mapsto j^1 f_t$  with  $f_0 = \iota$ ,  $f_1 = a$ , and each  $f_t$  is an immersion.  $\square$