Assignment 2: N-body problem

1. Code description

Structure

My code sets all values that can be changed between systems as global variables at the end of the code, just before the set-up function is called.

This set-up function, either two_bodies or three_bodies, finishes the setup of the 2 or 3 body systems respectively by calculating the initial positions, velocities, and accelerations. They, then both call the plotting function. As part of my validation steps, two_bodies additionally calls functions to calculate the semimajor axis of the system, the observational period, and the period calculated by Kepler's third law. Beyond the initial setup functions, the code works on any number of bodies and all types of system share the same functions.

The plotting function is where most of the functions are called from. It starts by calling energy_and_momentum for the initial conditions. This function calculates the total angular momentum and energy of the system, as specified in the relevant sub section. While the iteration is less than the one specified in the initial conditions, the code calls the velocity verlet which progresses the positions, velocity, and acceleration one time step, as specified in the subsection below. Then, energy_and_momentum is called again for the new variables.

Once all iterations have been performed, the list of energies is sent to a function that calculates the percentage error of the values given. Then, plotting produces three plots: the percentage error in total energy against time, the total angular momentum in the system against time, and the x against y positions of the bodies. Finally, this function returns the x and y values of the first body as well as the list of times; these are required to test Kepler's laws in the validation steps.

Velocity verlet

The velocity verlet algorithm uses a constant time step specified in the initial conditions to progress the velocities a half time step and positions a full step. Next, functions are called to calculate the new distance between the bodies, and then the new acceleration. From this the velocity is progressed the full time step. This is a second- order integrator.

Energy and angular momentum

For each particle, the kinetic energy is calculated, as $KE = \frac{1}{2} m \sum v_i^2$ where i is the x, y and z coordinates, m is the mass and v is the velocity. Once per pair of particles, the potential energy is calculated as $PE = \frac{m_j m_k}{r}$, where j and k are the two particles and r is the distance between them. These added together and summed over all particles to get the total energy of the system.

To calculate the angular momentum, first the centre of mass is calculated. The angular momentum for each particle is the cross product of the distance from the centre of mass to the particle, and the velocity. The resulting vector only has a z component. The angular momentum is summed over each particle to get the total angular momentum in the system.

2. Results

Part c – the 2-body problem

I have validated my code by demonstrating that Kepler's 2nd and 3rd laws are obeyed in a variety of different systems.

Kepler's 2nd Law

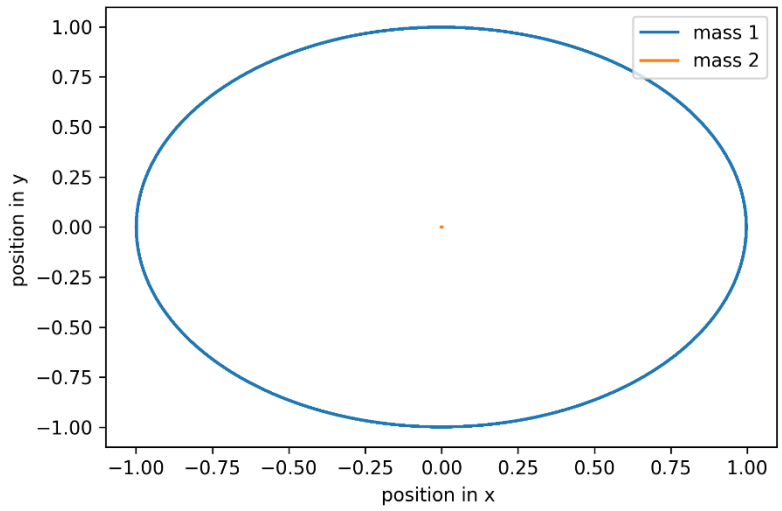
This law states that the line segment joining a planet and its host sweeps out equal areas during equal time intervals. This is a consequence of the conservation of angular momentum and energy. By demonstrating that these are conserved, it is shown that the system obeys Kepler's 2nd law.

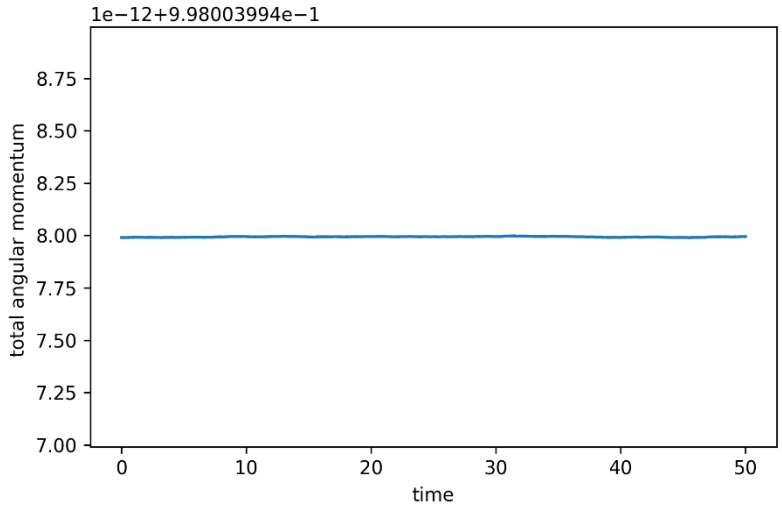
For each system I have plotted the z component of the angular momentum against time, the percentage error in energy against time and the x against y positions of the two orbits. Below I shall talk about a few of the systems. The plots for those not explicitly discussed are added in the appendix.

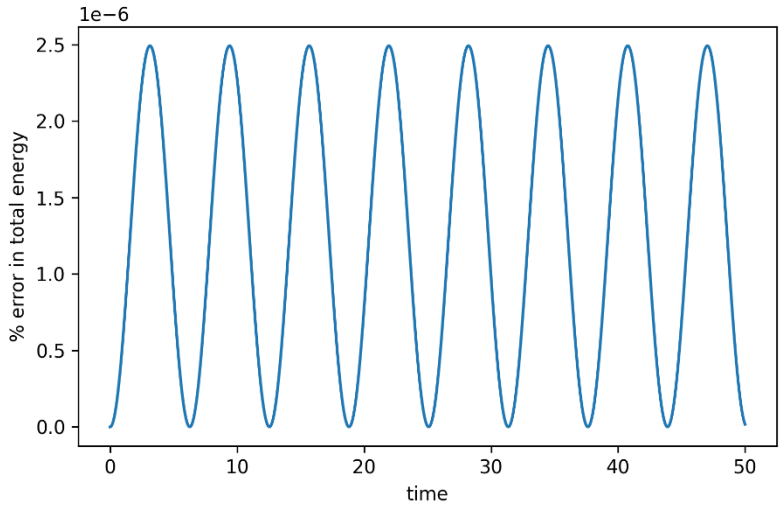
All systems conserve angular momentum to machine precision and total energy is also conserved.

System 0 (see table 1)

This system has a percentage error in energy which oscillates between 0 and 2.5 x 10^{-6} %. This is extremely low and demonstrates that energy is conserved, as expected. The total angular momentum is conserved to machine precision. It has a value of 0.998 and only varies in the 12th decimal place.

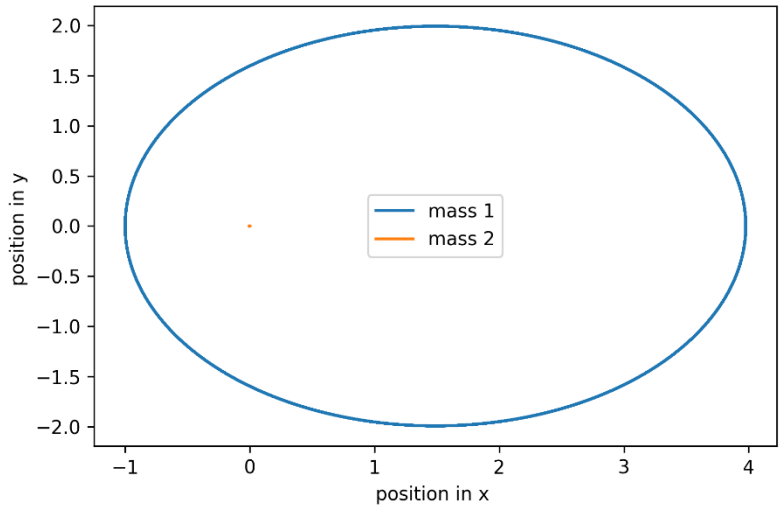


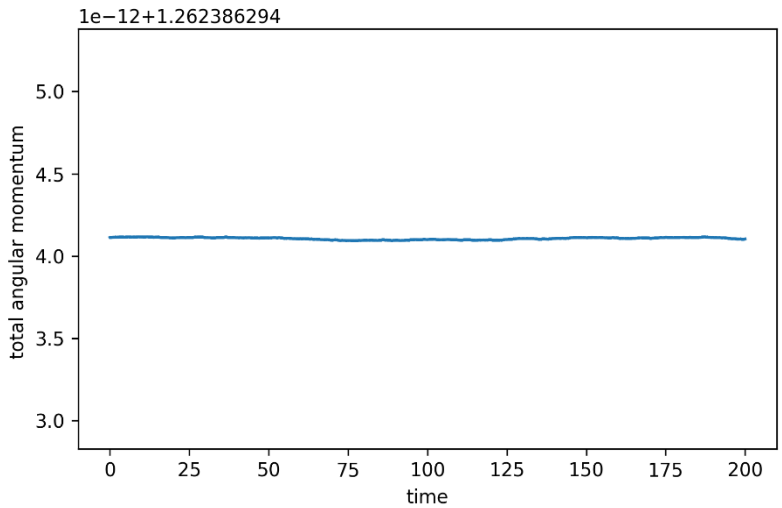


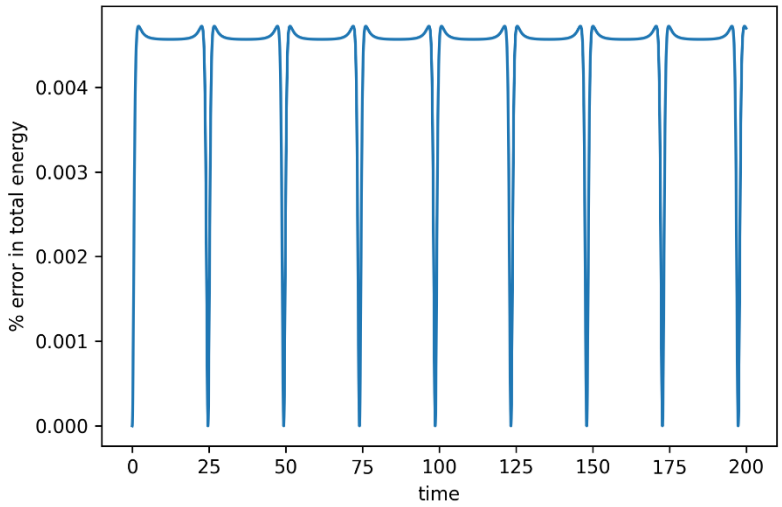


System 2

This system is similar to system 0 however it has an eccentricity of 0.6. As expected, this results in a slightly higher angular momentum of 1.26. However, it is still very well conserved to the 12th decimal place. The percentage error in energy oscillates with a different pattern to system 0. The error is dependent on the distance between the bodies. When the satellite body is closer to the host, it travels faster, and so the system no longer spends the same amount of time under conditions with both high and low error. This results in a sharp downwards spike in the error at 1 period intervals; the percentage error in energy is smaller when the two bodies are closer together.



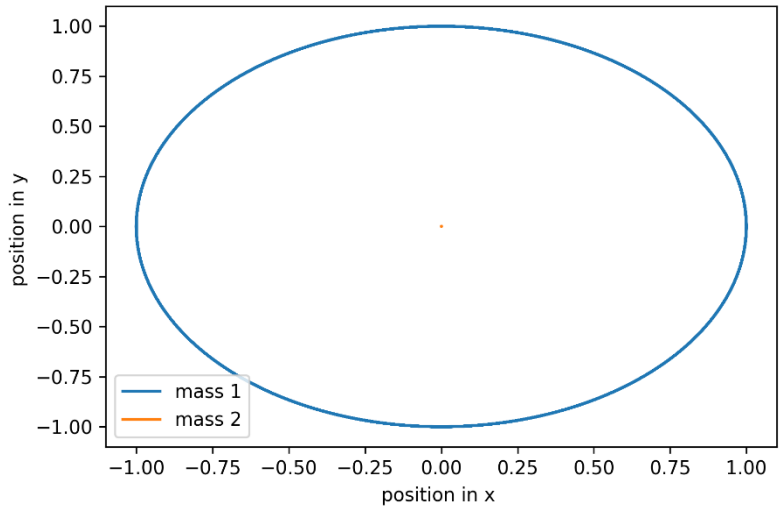


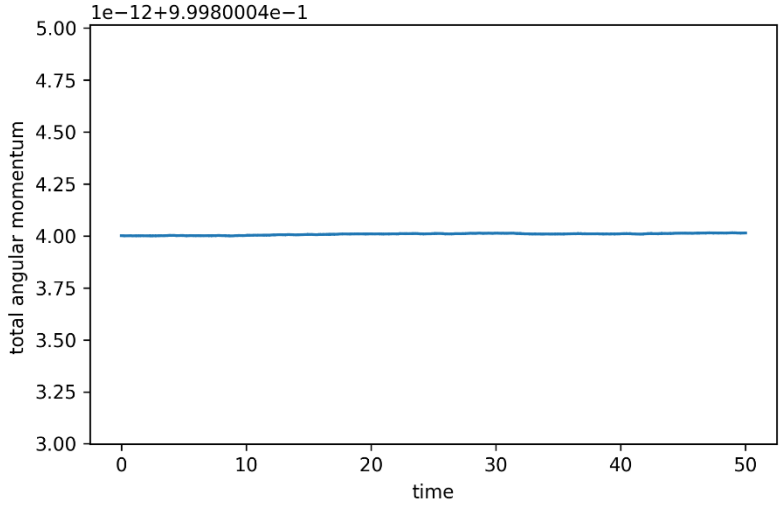


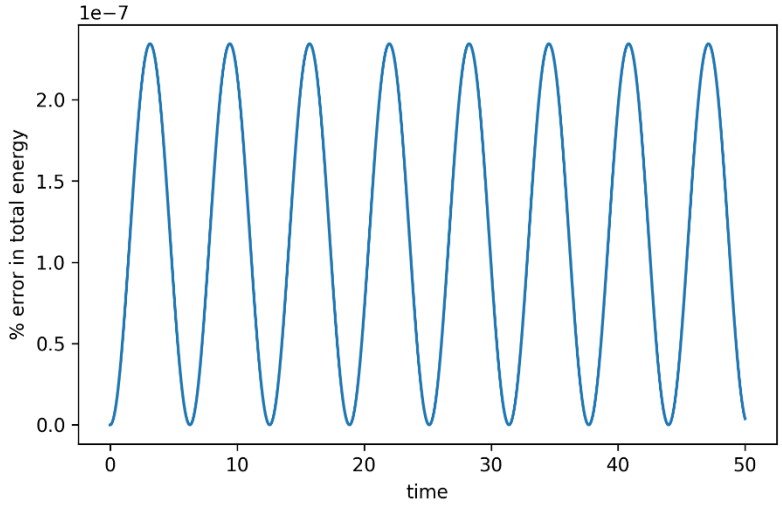
System 6

This system has an eccentricity of 0.0 (like system 0) and the satellite body is one order of magnitude smaller than systems 0 and 2. This system has the same angular momentum as system 0, 0.998 with fluctuations in the 12^{th} decimal. However, the error in total energy is one order of magnitude smaller (10^{-7} compared to 10^{-6}), proportional to the mass. Both angular momentum and energy are conserved as expected.

The initial setup assumes that the fist mass is much smaller than the second. As such, as long as this is obeyed, the percentage error in energy is prevented from becoming high enough that energy is no longer considered conserved.

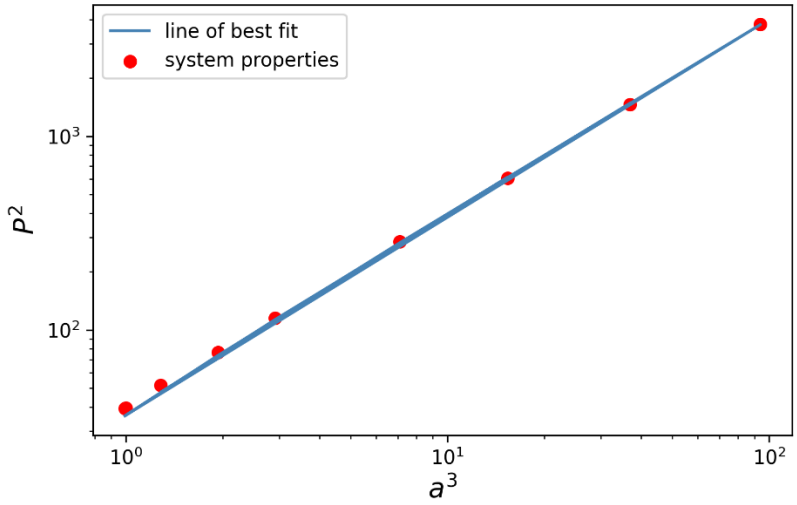






Kepler's 3rd Law

This law states that $P^2 = \frac{4\pi^2 a^3}{G(M_1+M_2)}$. The semimajor axis was calculated by finding the maximum and minimum x and y values and dividing them by two. I then used this to calculate the period using Kepler's 3rd law. This was compared to the observational period. This was calculated by finding the difference in the times the value turned from positive to negative. Both values for period can be seen in the table below. A plot of P^2 (observational) against a^3 for all systems tested is below. A line of best fit has been added which shows that $P^2 \propto a^3$ as expected from Kepler's 3rd law.



System	Smaller mass	eccentricity	Time step	Number of iterations	Semi-major axis	Observational period	Period from Kepler's 3rd law	% difference in period
0	0.0010	0.0000	0.0050	10,000	0.9980	6.2707	6.2613	0.1501
1	0.0010	0.2000	0.0050	10,000	1.2469	8.7575	8.7439	0.1555
2	0.0010	0.6000	0.0100	20,000	2.4877	24.677	24.641	0.1461
3	0.0100	0.1000	0.0050	10,000	1.0870	7.1917	7.0850	1.5060
4	0.0100	0.5000	0.0100	10,000	1.9231	16.924	16.674	1.4993
5	0.0100	0.8000	0.0100	30,000	4.5459	61.507	60.597	1.5017
6	0.0001	0.0000	0.0050	10,000	0.9998	6.2821	6.2810	0.0175
7	0.0001	0.3000	0.0100	10,000	1.4282	10.725	10.724	0.0093
8	0.0001	0.7000	0.0100	30,000	3.3313	38.208	38.202	0.0157

Table 1: properties of each system

The tested values vary the mass ratio of the two bodies by three orders of magnitude, and the eccentricity from 0.0 to 0.8. An increase in eccentricity resulted in larger semi major axis and longer periods. This had to be compensated for by increasing the timestep and number of iterations in order to get a similar number of orbits across each system. The observational period, and the period calculated using Kepler's 3rd law were remarkably similar. This error was proportional to the mass of the system; increasing the smaller mass by one order of magnitude also increased the percentage error in the period by one order of magnitude. As discussed in the section on Kepler's 2nd law, the percentage error in total energy also scales with the mass. This suggests that the main cause of error the period is due to the error in the energy.

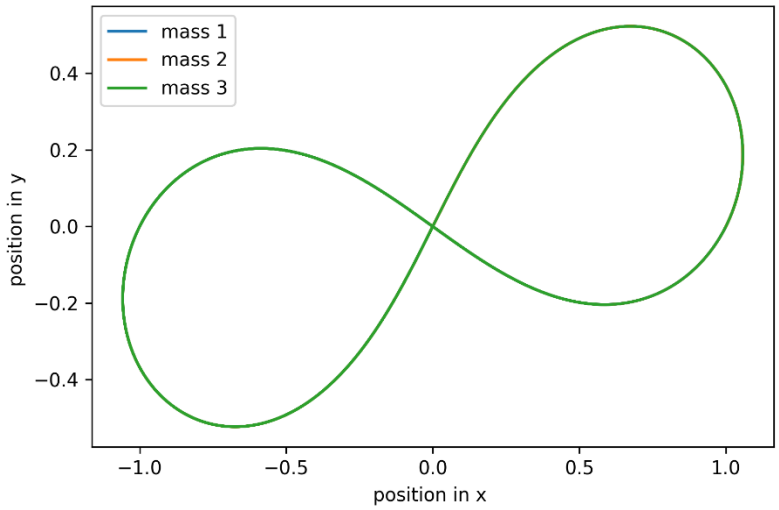
Part d – the 3-body problem

I recreated four stable orbits of the 3-body problem. All systems had equal mass bodies of 1, initial positions $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and initial velocities $\begin{bmatrix} p1 & p2 & 0 \\ p1 & p2 & 0 \\ -2p1 & -2p2 & 0 \end{bmatrix}$. The exact values of p1 and p2 are specified for each system. Each system should have a total angular momentum of zero. The total energy varied from system to system. To get stable orbits, the gravitational softening constant, timestep and number of iterations had to be fine tuned to the system. This was more demanding in systems where the planets got closer together.

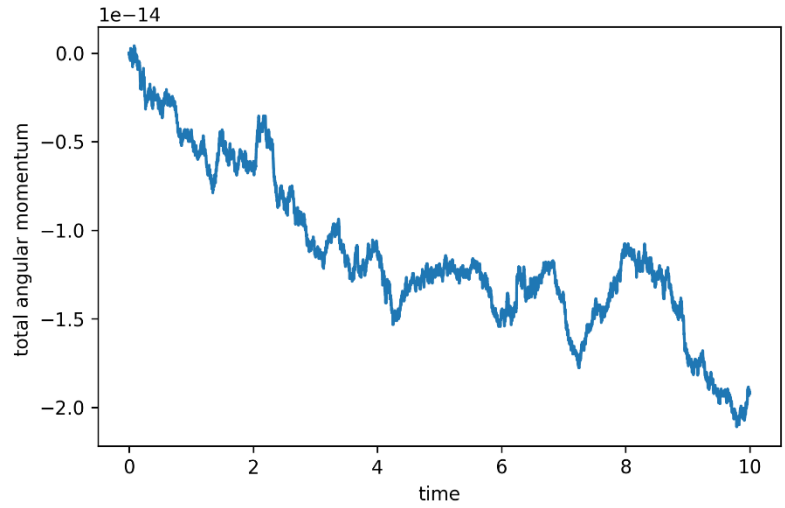
Figure of eight

Initial conditions	
P1	0.347111
P2	0.532728
Gravitational softening constant	1×10^{-10}
Time step	0.001
Number of iterations	10,000

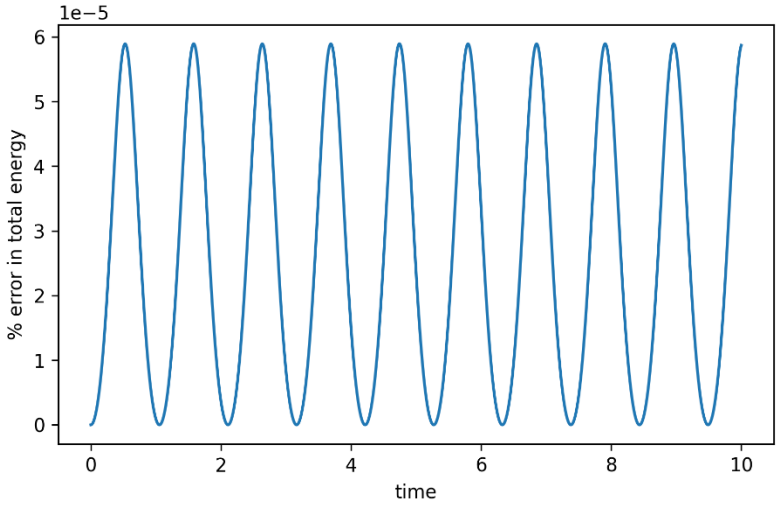
I was able to perfectly recreate this system. As the three bodies take the exact same path, only the plot of the green mass is visible. The system is stable with each body taking the same path as its previous orbit.



The angular momentum of this system is supposed to be conserved at zero. In my code, the angular momentum ranged from 0 to -2×10^{-14}. While the angular momentum sometimes increases, the general trend in downward. This is a very small variation, and so angular momentum in this system is conserved. This variation is two orders of magnitude less than the variation in the two body systems.



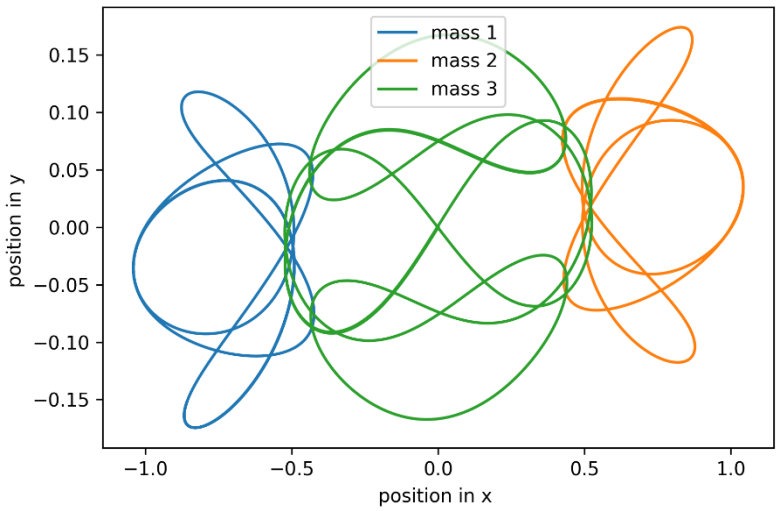
The total energy of this system should be -1.287146. The energy value for my final datapoint is -1.287145. This only differs in the sixth decimal place. The error in the total energy of this system is periodic much like the two body systems with low eccentricities. This is because the distance between the bodies, never significantly changes in either case.



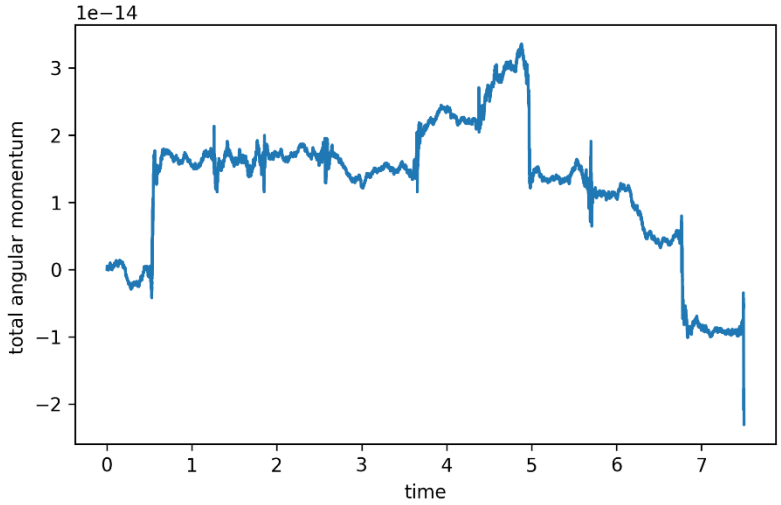
Butterfly 1

Initial conditions	
P1	0.306893
P2	0.125507
Gravitational softening constant	1×10^{-13}
Time step	0.00005
Number of iterations	150,000

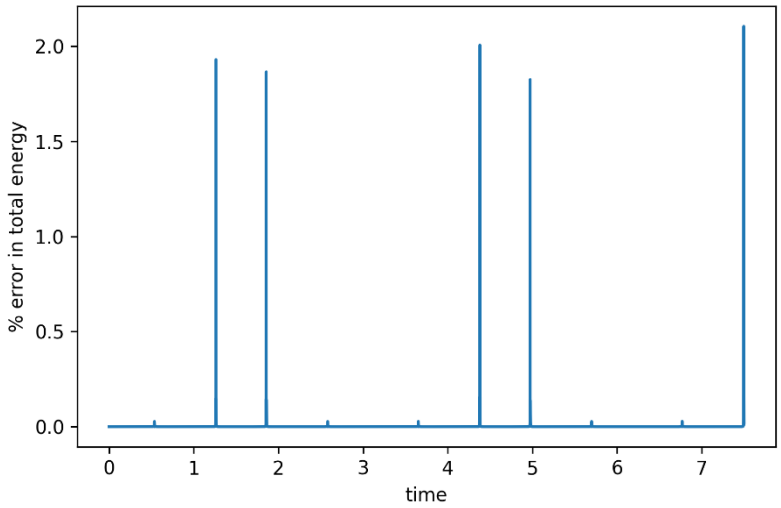
This system required a lot of fine tuning in the gravitational softening constant, time step and number of iterations to achieve a stable orbit. This is because the bodies get very close to each other. The plot of the positions of the three bodies perfectly matches the expected pattern.



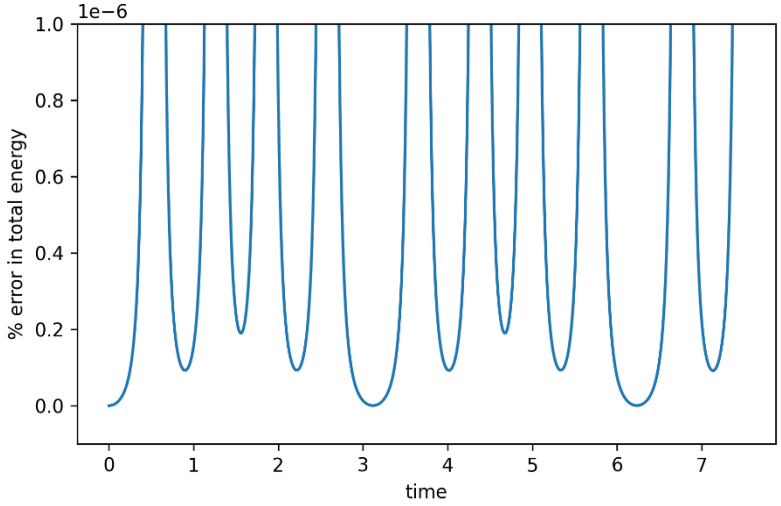
The angular momentum of this system should be zero. My recreation has angular momentum between 3×10^{-14} and -2×10^{-14}. Much like the figure of eight, this is a small enough variation that angular momentum is considered conserved and at zero.



The percentage error in energy is usually around zero, however there are significant peaks (up to 2%) at times where the bodies get close together. The energy value of my final data point is -2.169700, the slight difference is due to bodies 2 and 3 being very close together at the final data point.



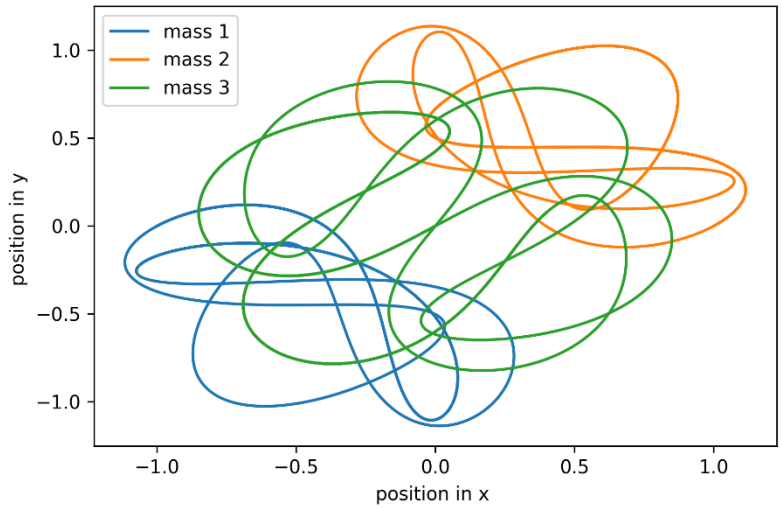
Zooming in around 0%, it can be seen that the percentage error is periodic and there is no increasing trend with time.



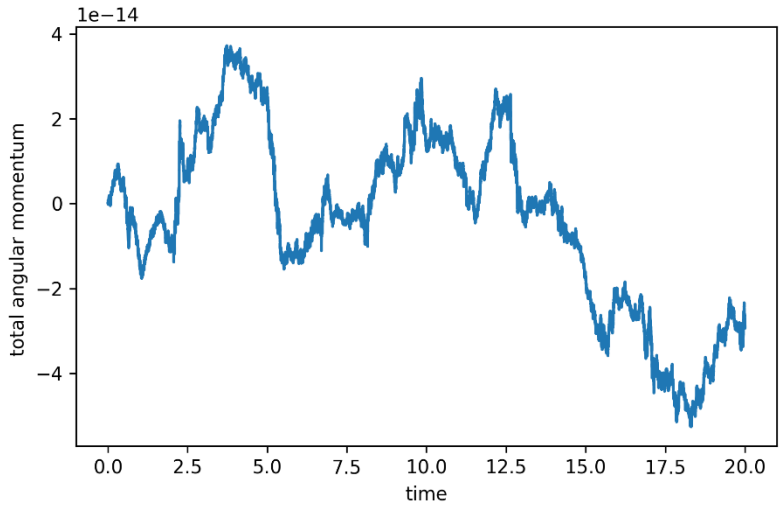
Moth 1

Initial conditions	
P1	0.464445
P2	0.396060
Gravitational softening constant	1×10^{-12}
Time step	0.0001
Number of iterations	200,000

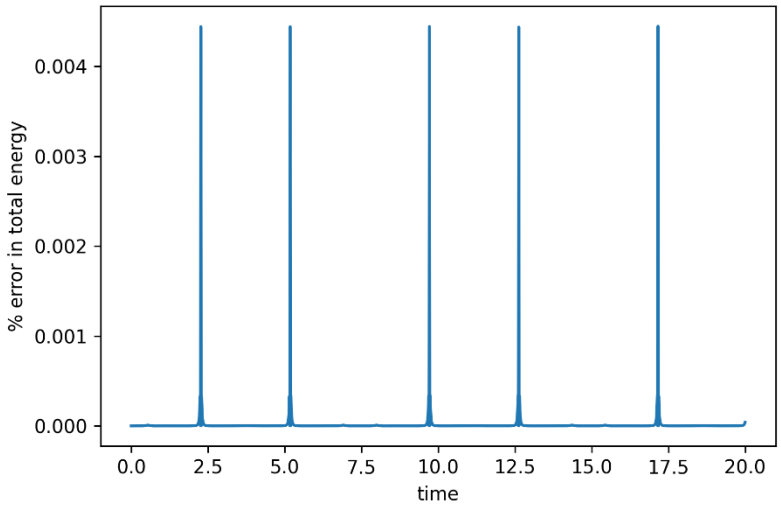
In this system the bodies do not get quite as close together as butterfly 1 and so did not require as much fine tuning in the initial conditions, with an increase in the timestep possible. The number of iterations still had to be increased due to the increased period. This orbit is stable with all three bodies taking the same path on their second orbit, as their first. The figure perfectly matches the expected pattern.

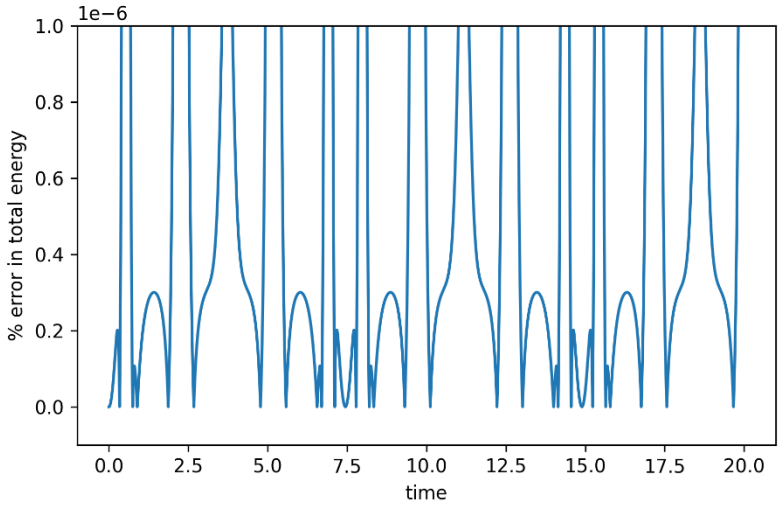


The angular momentum of this system is supposed to be conserved at zero. In my code, the angular momentum ranged from around 4×10^{-14} to -6×10^{-14}. This is a very small variation, and so angular momentum in this system is conserved.



The error in energy in this system is usually around zero with peaks of just over 0.004%. This is significantly lower than the peaks of butterfly 1, as the bodies do not get as close together. The zoomed in plot shows a periodic oscillation in the error that has no upwards trend. The energy of this system is supposed to be -1.38228; the final data point of my recreation has an energy of -1.38228. This is the exact same energy showing that, excluding times where the bodies are very close to one another, energy is conserved.

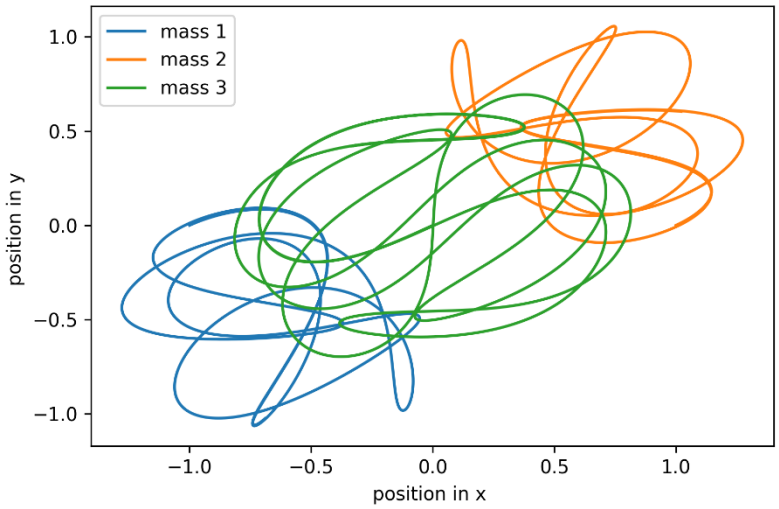




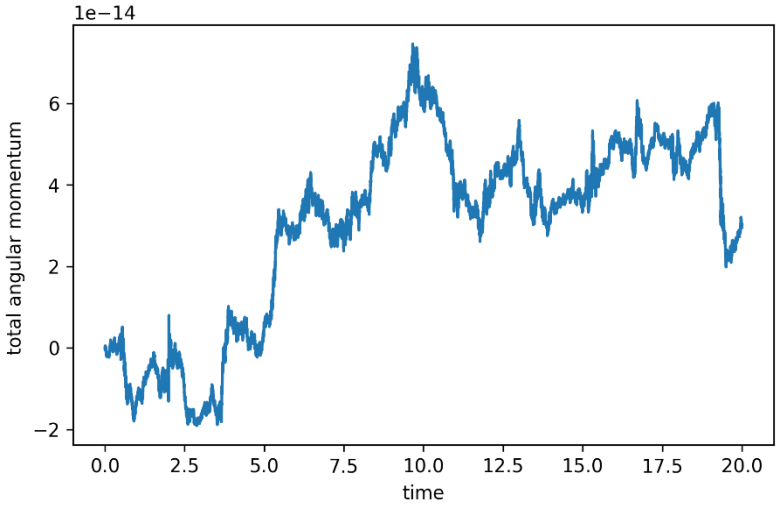
Yin-Yang 1a

Initial conditions	
P1	0.513938
P2	0.304736
Gravitational softening constant	1×10^{-12}
Time step	0.0001
Number of iterations	200,000

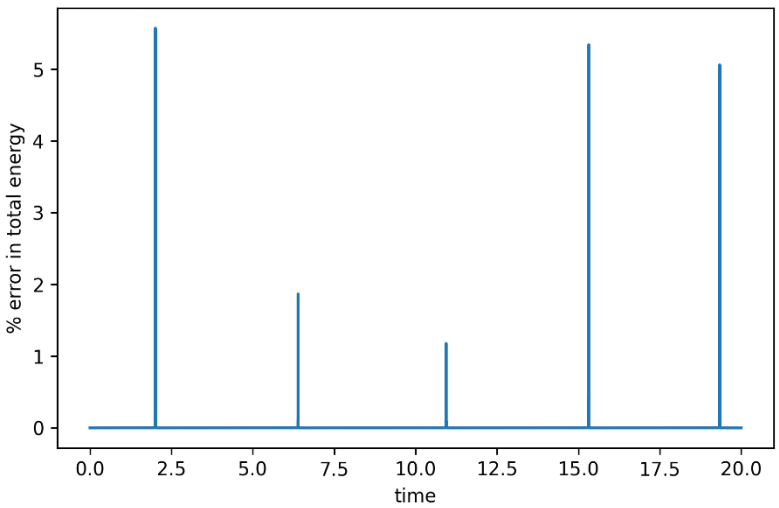
The final system I tested was also well recreated with the pattern of the bodies matching the expected pattern. The system is stable with the positions of the second orbit overlapping the first.

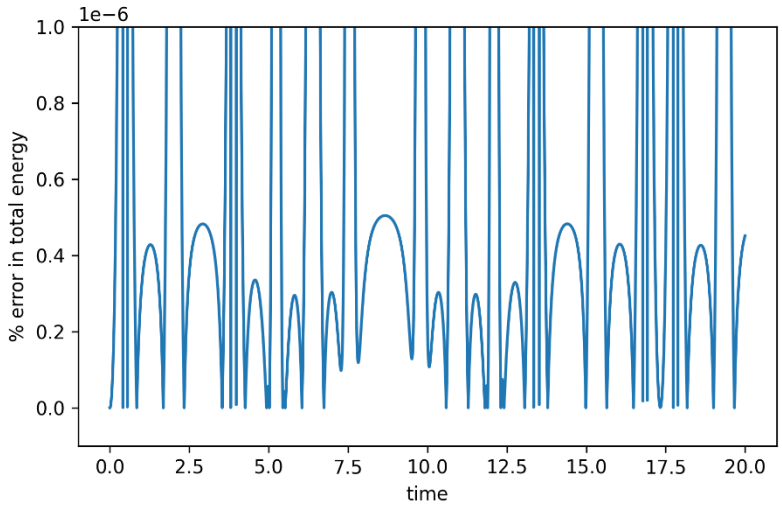


The angular momentum of this system should be zero. My recreation has angular momentum ranging from -2×10^{-14} to almost 8×10^{-14}, with an upwards trend. This variation is very small showing that angular momentum is conserved.



This system has the largest peaks in energy error at just above 5%. This is much larger than Moth 1 with an energy peak of just over 0.004%, suggesting that the bodies get much closer together in Yin-Yang 1a. The energy of this system is supposed to be -1.429011, my final data point has energy -1.429011, which is exactly as expected. The zoomed in plot shows no upwards trend in the energy error. Energy is conserved.

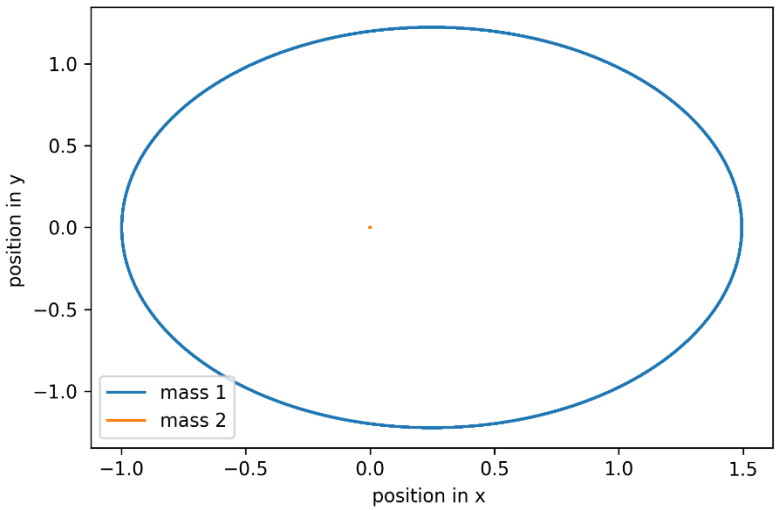


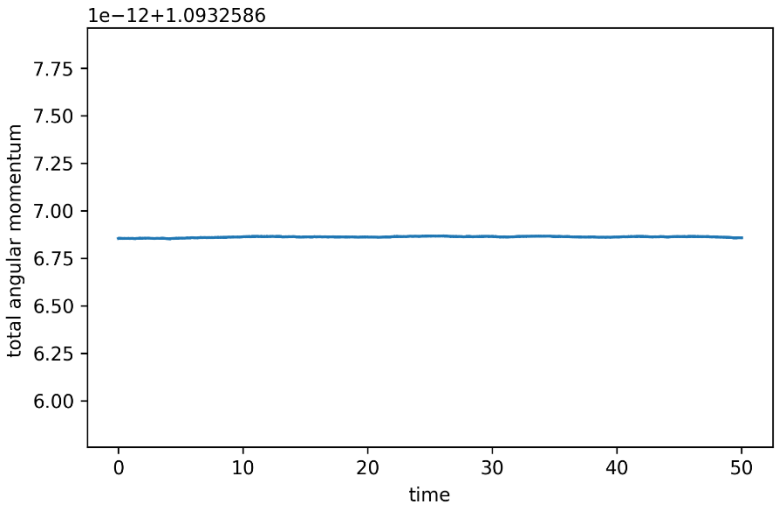


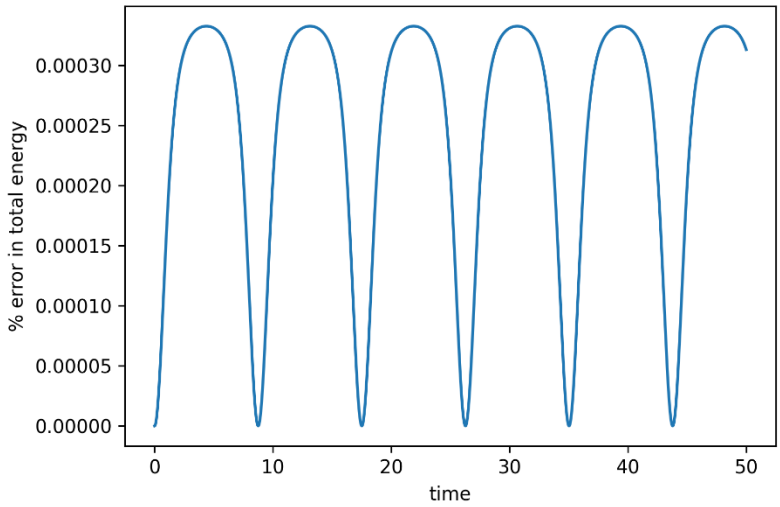
1. Appendix

Below I have added my plots for the systems not directly discussed in the section on Kepler's 2nd law. For each system as detailed in table 1, there are plots of orbital position, total angular momentum in the system against time, and percentage error in energy against time. These show the same trends as those discussed above and angular momentum and energy are conserved.

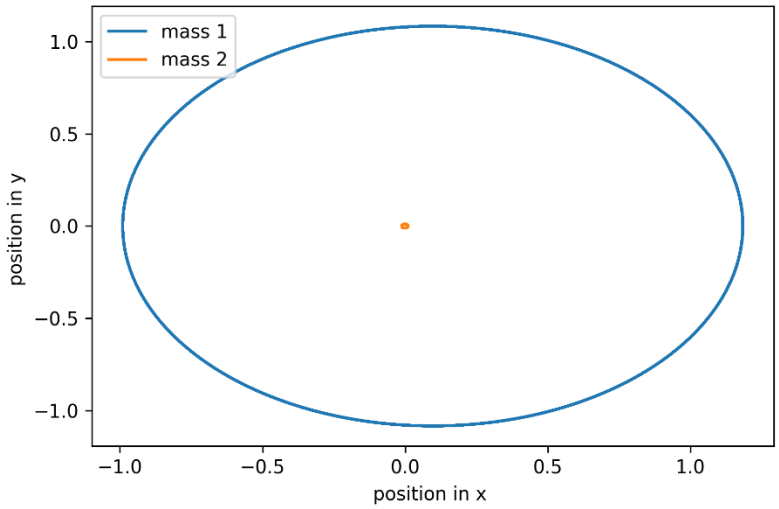
System 1

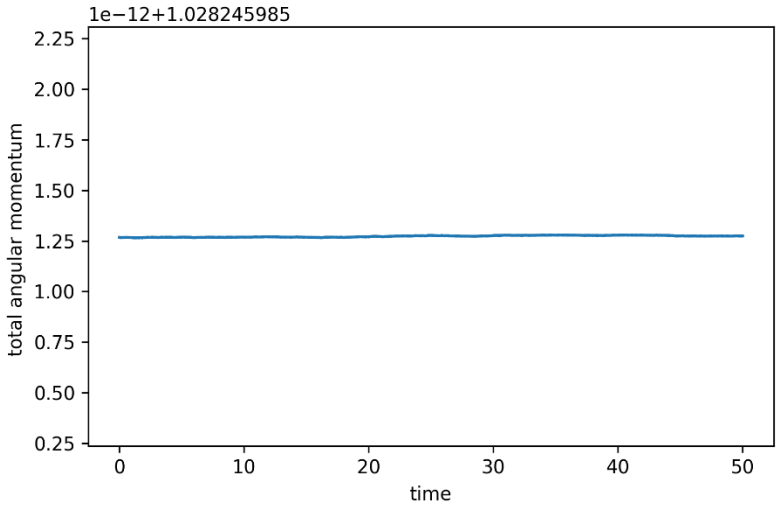


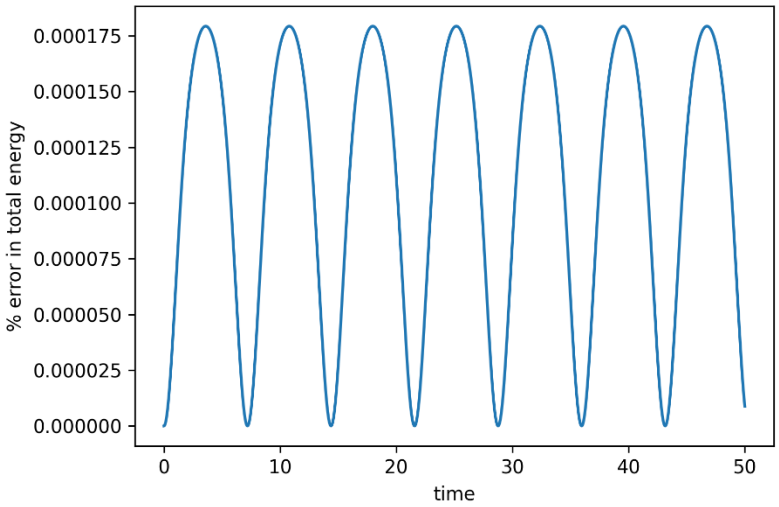



% error in total energy
0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0
10
20
30
40
50
time

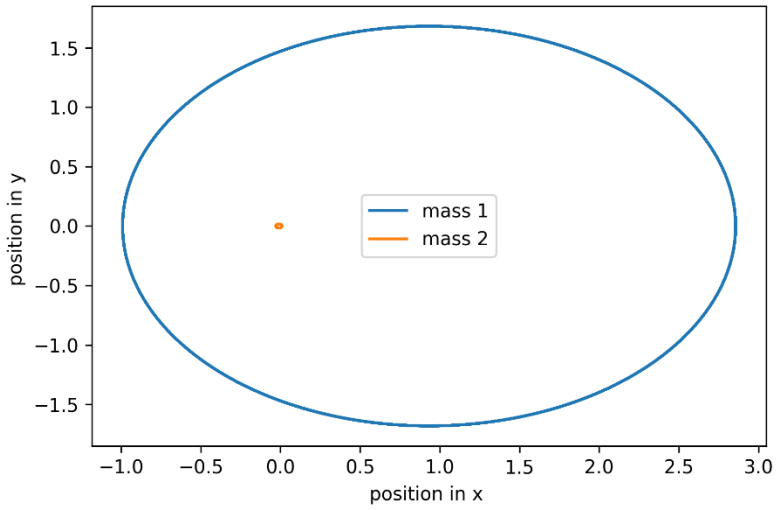
System 3

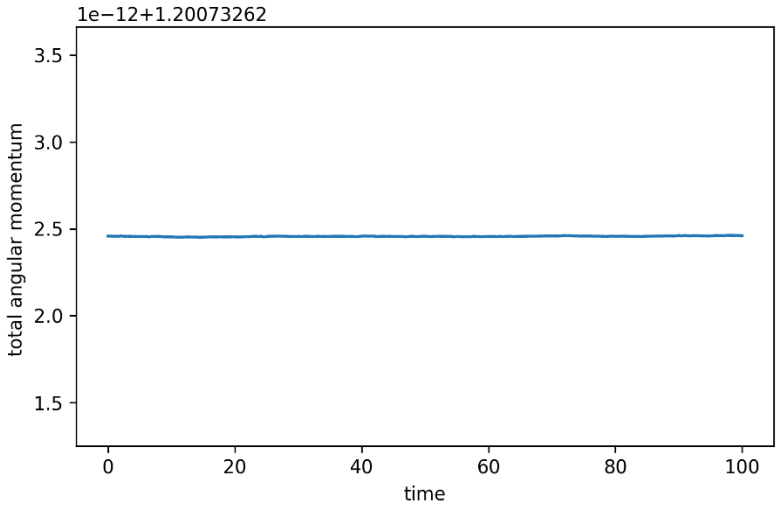

mass 1
mass 2
position in y
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
position in x

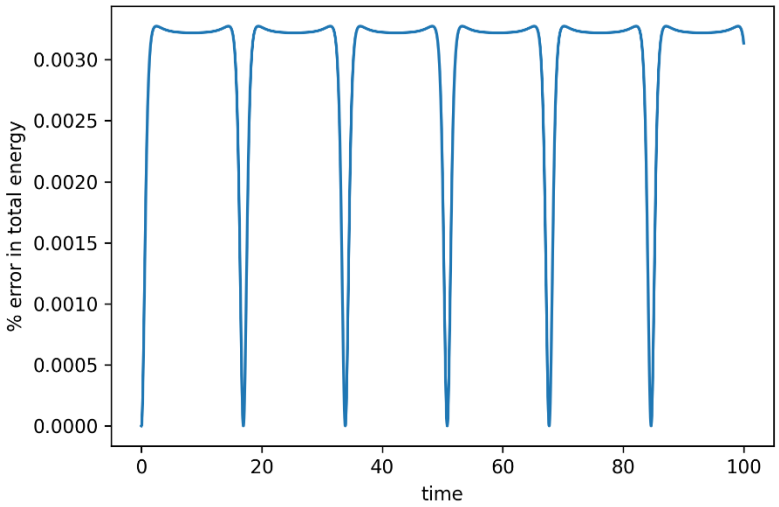

1e−12+1.028245985
total angular momentum
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0
10
20
30
40
50
time


0.000175
0.000150
0.000125
0.000100
0.000075
0.000050
0.000025
0.000000
% error in total energy
0
10
20
30
40
50
time

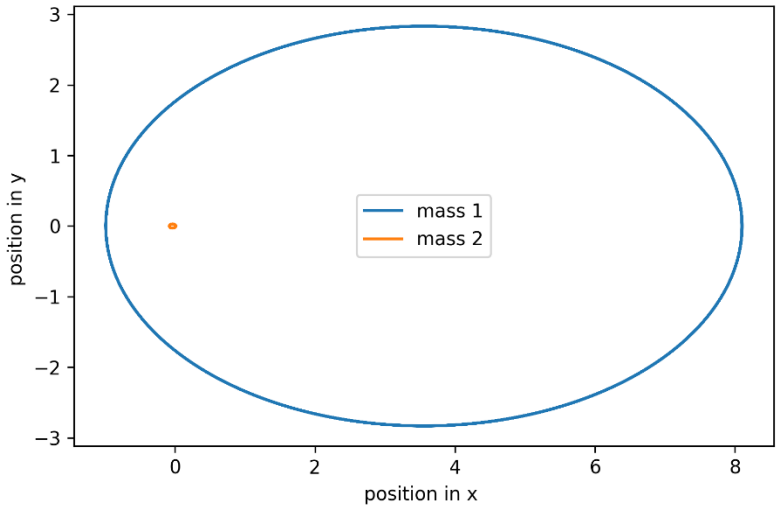
System 4

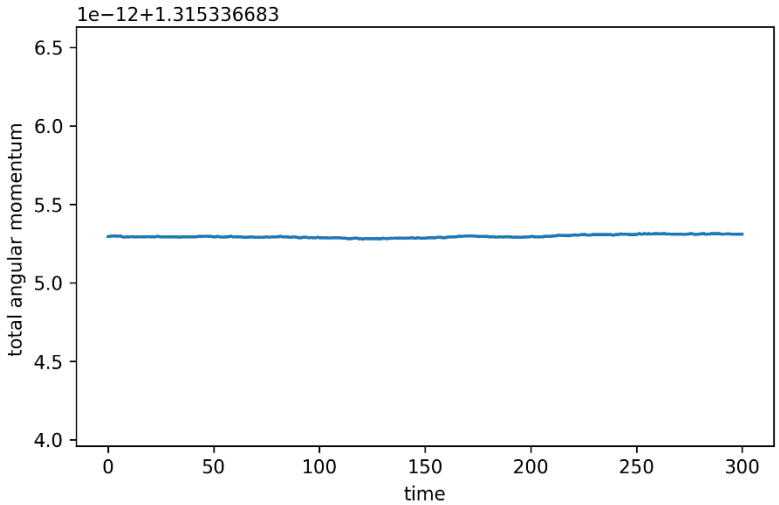

1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
position in y
mass 1
mass 2
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
position in x

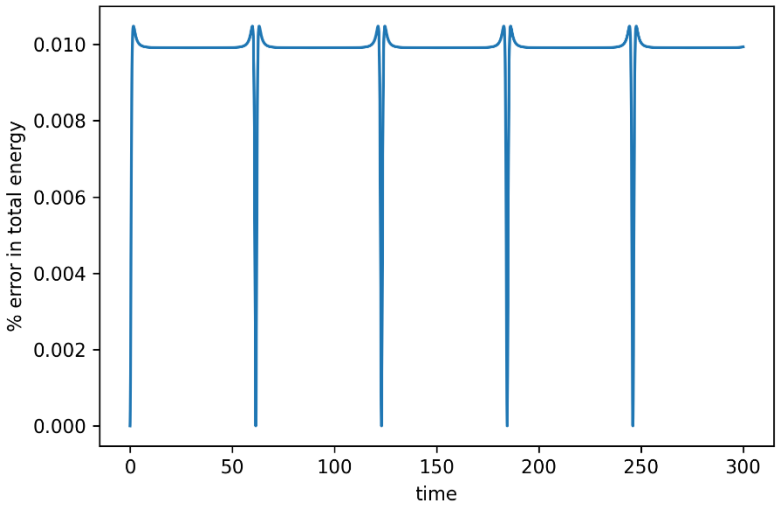

1e−12+1.20073262
3.5
3.0
2.5
2.0
1.5
total angular momentum
0
20
40
60
80
100
time


% error in total energy
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
0
20
40
60
80
100
time

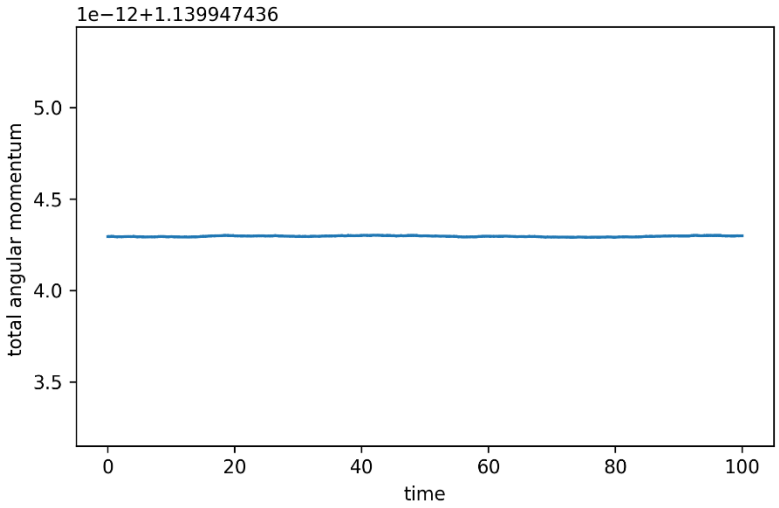
System 5

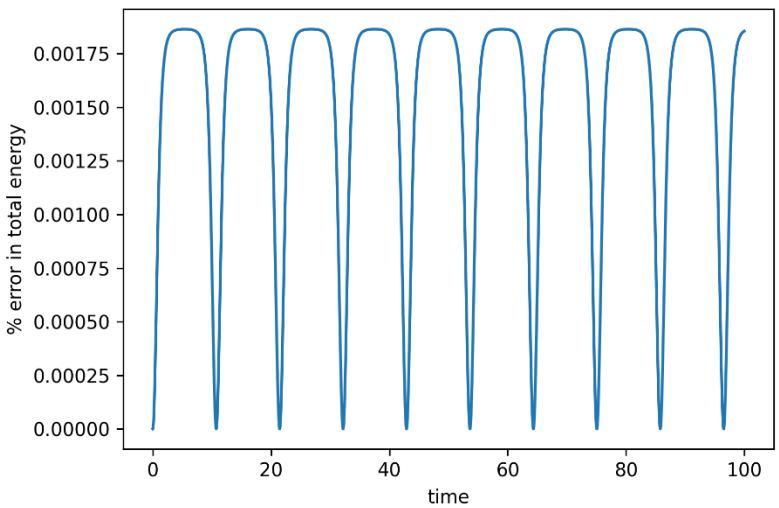

position in y
3
2
1
0
−1
−2
−3
mass 1
mass 2
0
2
4
6
8
position in x

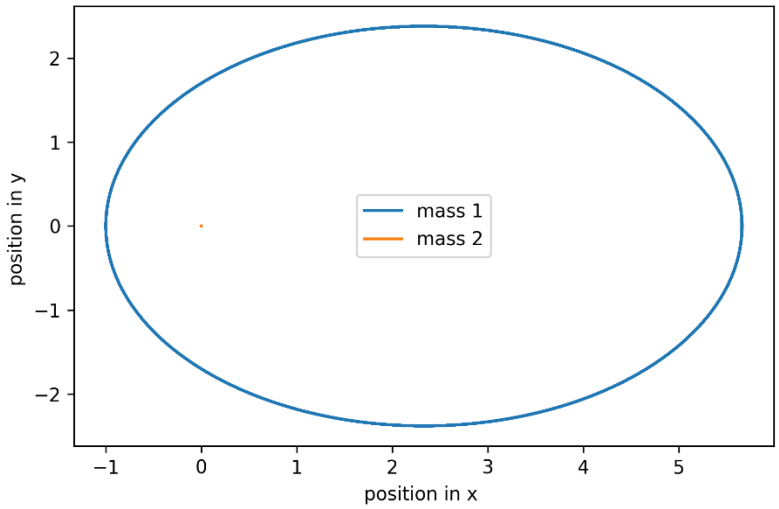

1e−12+1.315336683
total angular momentum
6.5
6.0
5.5
5.0
4.5
4.0
0
50
100
150
200
250
300
time


0.010
0.008
0.006
0.004
0.002
0.000
% error in total energy
0
50
100
150
200
250
300
time

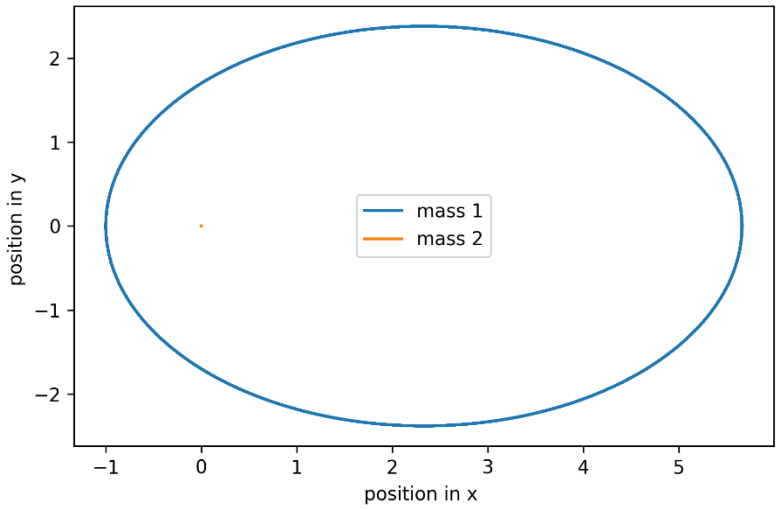
System 7

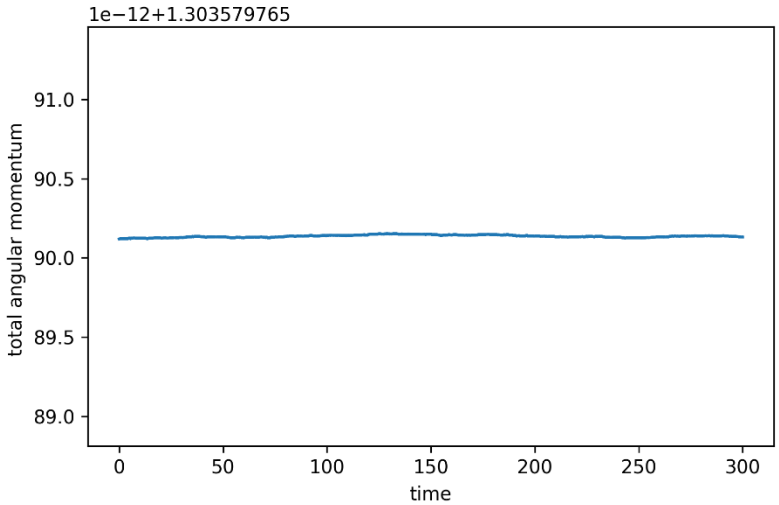

1e−12+1.139947436
5.0
4.5
4.0
3.5
total angular momentum
0
20
40
60
80
100
time

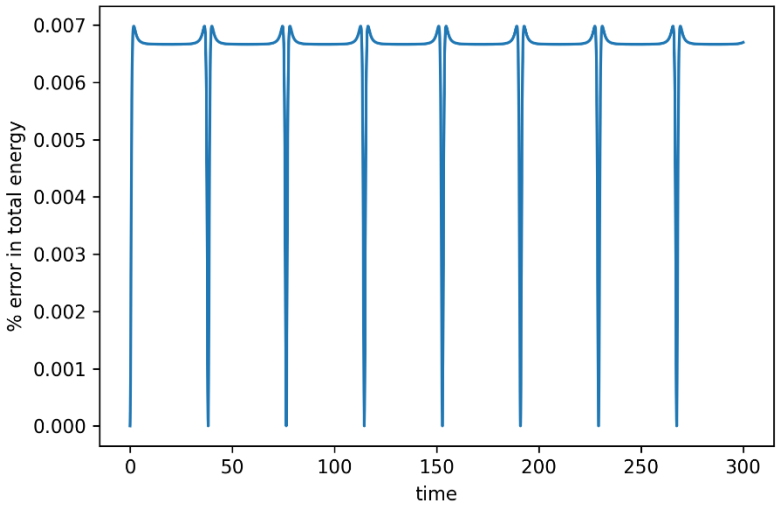

0.00175
0.00150
0.00125
0.00100
0.00075
0.00050
0.00025
0.00000
% error in total energy
0
20
40
60
80
100
time


mass 1
mass 2
position in x
position in y

System 8


mass 1
mass 2
position in x
position in y


1e−12+1.303579765
time
total angular momentum


0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
% error in total energy
0
50
100
150
200
250
300
time