Contents

1 Chapter 1

1.1 Section 1

1.1.1 Misc questions

- Remark in proposition 1.10, hy is the chain of closures maximal? Suppose $\exists i$ such that $\overline{Z_0} \subseteq \cdots \subseteq Z_{\overline{i+1}} \subseteq \overline{Z_{i+1}} \subseteq \cdots \subseteq \overline{Z_n}$

  $Z_{\overline{i+1}}$ is closed in Y, we will show it's irreducible later. Therefore, $\overline{Z_i} = Z_{i+1} \cap Y$ or $\overline{Z_{i+1}} = Z_{\overline{i+1}} \cap Y$. However, the latter would contradict the definition of closure for Z_{i+1} because $Z_{\overline{i+1}} \subset \overline{Z_i}$ yet $Z_{\overline{i+1}}$ contains Z_i

  So $\overline{Z_i} = Z_{\overline{i+1}} \cap Y$. But then $(Z_{\overline{i+1}} \backslash \overline{Z_i}) \cap Y = \emptyset$ for if it didn't, then $Z_{\overline{i+1}} \cap Y$ would contain elements not in Z_i. Therefore, $Z_{\overline{i+1}}$ consists of elements specifically in the closure of Y and the elements of $\overline{Z_i}$ so $Z_{\overline{i+1}} =)(\overline{Y} \backslash Y) \cap Z_{\overline{i+1}}) \cup \overline{Z_i}$ each of the two components in the union is a closed subset of $Z_{\overline{i+1}}$, a contradiction.

1.1.2 Exercises

1. 1.1a Let $Y = y - x^2$ Define $\varphi : k[x,y]/(y-x^2) \to k[z]$ to be $\varphi(ax+by+c) = az+bz^2+c$. This is clearly surjective as any element $a_n z^n + \cdots + a_0$ has in its preimage $a_n x^n + \cdots + a_0$. Now to show its injective: Suppose there was some non-zero polynomial $p(x,y) \in k[x,y]/(y-x^2)$ such that $\varphi(p(x,y)) = 0$ then the process of replacing all of the instances of y with x^2 renders the polynomial to be zero. However, the relation defined on $k[x,y]/(y-x^2)$ declares that $y \cong x^2$ which means that $p(x,y)$ was zero to begin with.

2. 1.1b Let $A = k[x,y]/(xy-1)$. Suppose $\exists \varphi : A \to k[x]$. Then, if this were to be an isomorphism, $1 = \varphi(1) = \varphi(xy) = \varphi(x)\varphi(y)$. However,

the only units of $k[x]$ are the elements of k. If x, y were to map to say, $f, f^{-1} \in k$ respectively, then f or f^{-1} needs to map to, say, x in order for surjectivity to hold (we could have also chosen y, but that has no influence on the proof). However, x has no inverse and therefore $1 = \varphi(ff^{-1}) = \varphi(f)\varphi(f^{-1}) = x\varphi(f^{-1}) \neq 1$ a contradiction.

3. 1.1c Skipped for now

4. 1.2 Let $A = k[x, y, z]\backslash(x^2 - y, x^3 - z)$ and define $\varphi : A \to k[w]$ to be $\varphi(x) \to w, \varphi(y) = w^2, \varphi(z) = w^3$. By an argument similar to $1.1a$, this is clearly surjective and to show it's injective, let $\varphi(p(x, y, z)) = 0$ with $p(x, y, z) \neq 0$. Then this means that replacing x with w, y with w^2, and z with w^3 makes $p(x, y, z)$ become zero. However, these are the same relations in A.

5. 1.3 Lemma: $Z(a + b) = Z(a) \cap Z(b)$. By prop 1.2.a, $Z(a) \supseteq Z(a + b)$ and also $Z(b) \supset Z(a+b)$ so every element of $Z(a+b)$ is an element of both $Z(a)$ and $Z(b)$ so $Z(a + b) \subseteq Z(a) \cap Z(b)$. Now, take an element of $Z(a) \cap Z(b)$ call it z. $\forall i \in a$, $i(z) = 0$. Similarly, $\forall j \in b$, $j(z) = 0$ so $(i + j)(z) = 0$ so $Z(a) \cap Z(b) \supseteq Z(a + b)$

   Therefore $Z(x^2 - yz, xz - x) = Z(x^2 - yz) \cap Z(xz - x) = Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) = (Z(x^2-yz) \cap Z(x)) \cup (Z(x^2-yz) \cap Z(z-1))$. Notably, $Z(x^2 - yz)$ intersects $Z(x)$ precisely where $x = 0$ and with $Z(z - 1)$ where $z = 1$ so the first component reduces to $Z(-yz) = Z(y) \cup Z(z)$ and the second component reduces to $Z(x^2-y)$. Putting it all together, $Z(y) \cup Z(z) \cup Z(x^2 - y)$

6. 1.4 The prime ideal x corresponds to the infinite set of the points where $x = 0$. A^2 only has finit sets as closed sets.

7. 1.5 $\Rightarrow$ The affine coordinate ring of Y is of the form $A = k[x_1, \cdots, x_n]/ I(Z(T)) = k[x_1, \cdots, x_n]/\sqrt{(T)}$ for some $T \subset k[x_1, \cdots, x_n]$ which has nilradical 0 by the definition of the radical. Because A is noetherian $\sqrt{(T)}$ is finitely generated and, because $k[x_1, \cdots, x_n]$ is finitely generate, A is finitely generated as well. Therefore A is a finitely generate k algebra with no nilpotent elements. So if B is isomorphic to A, it must also be a finitely generated k algebra with no nilpotent elements.

   $\Leftarrow$ Enumerate the generators of B as $x_1, \cdots, x_n$ which we may do because B is finitely generated and let R be the set of relations. Let us define $\varphi : B \to k[x_1, \cdots, x_n]/R$ where $x_i \mapsto x_i$ Similarly to 1.1a and 1.2, this is surjective and, because the relations of the two rings are the same, it's injective as well. It only remains to show that R is of the form $I(Z(T))$ or, in other words, radical. However, this is equivalent to saying that the nilradical of B is zero, which is one of our assumptions.

8. 1.6 Let $Y \subseteq X$ be open. If $Y = Y_1 \cup Y_2$ in the induced topology, then $X = (X \backslash Y) \cup Y_1 \cup Y_2$, each of which is a closed, proper subset of X.

   Similarly, let $Y \subseteq X$ be open. If $\overline{Y} = Y_1 \cup Y_2$ then $X = \overline{Y} \cup (X \backslash Y)$.

   Let $Y \subseteq X$ be irreducible. If $\overline{Y} = Y_1 \cup Y_2$, then $(Y \cap Y_1) \cup (Y \cap Y_2) = Y$ then $Y \cap Y_1 = Y$ or $Y \cap Y_2 = Y$ but that would contradict that $\overline{Y}$ is the smallest closed set containing Y.

9. 1.7a Note: We take "family of x sets" (e.g., a family of closed sets) to mean a set whose elements are in turn x sets (e.g., closed). This is consistent with his use of the term "family" in proposition 1.1 (that algebraic sets form a topology)

   - X Noetherian $\Rightarrow$ family of closed sets has a minimal element. Let Y be a family of closed sets. Consider an element $Y_1 \in Y$. If there is no other element $Y_2 \in Y$ such that $Y_1 \supset Y_2$, then Y_1 is minimal. Otherwise, we have the start of a chain $Y_1 \supset Y_2$; because X is noetherian, we can iteratively continue this process of finding closed sets Y_{i+1} that are subsets of Y_i and that we'll eventually stabilize for some integer n (which is to say, $\forall N > n, Y_N = Y_n$) so Y_n is a minimal element.
   - Family of closed sets has a minimal element $\Rightarrow$ X noetherian Let $\tilde{X} = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ be a (possibly infinite) sequence of closed subsets. By assumption, $\tilde{X}$ has a minimal element, call it X_i. Because $\tilde{X}$ may be infinite, this means that $\forall I > i, X_I = X_i$ because, by the construction of $\tilde{X}$, $j > i \Rightarrow X_i \supseteq X_j$.
   - X satisfies a.c.c. on open sets $\Rightarrow$ X noetherian Let $X_1 \supseteq X_2 \supseteq \cdots$ be an arbitrary chain of closed subsets of X. Then $(X \backslash X_1) \subseteq (X \backslash X_2) \subseteq \cdots$ is a chain of open sets. By assumption, the chain of open sets has some set $X \backslash X_i$ such that, $\forall I > i, (X \backslash X_I) = (X \backslash X_i)$. Therefore, $\forall I > i, X_I = X_i$.
   - X noetherian $\Rightarrow$ X satisfies a.c.c. on open sets Let $X_0 \subseteq X_1 \subseteq \cdots$ be an arbitrary chain of open subsets of X. Then $(X \backslash X_1) \subseteq (X \backslash X_2) \subseteq \cdots$ is a chain of closed sets. By assumption, the chain of closed sets has some set $X \backslash X_i$ such that, $\forall I > i, (X \backslash X_I) = (X \backslash X_i)$. Therefore, $\forall I > i, X_I = X_i$.

- Every non-empty family of closed sets of X has a maximal element $\Rightarrow$ every non-empty family of open sets of X has a maximal element Let Y be a family of open sets of X. Consider the family of closed sets $\tilde{Y}$ consisting of the complement of each set in Y with X. By assumption, there is some minimal element $\tilde{Y_i}$. Therefore, the corresponding set Y_i is a maximal element of Y.
- Every non-empty family of open sets of X has a maximal element $\Rightarrow$ Every non-empty family of closed sets of X has a maximal element Let Y be a family of closed sets of X. Consider the family of open sets $\tilde{Y}$ consisting of the complement of each set in Y with X. By assumption, there is some maximal element $\tilde{Y_i}$. Therefore, the corresponding set Y_i is a minimal element of Y.

10. 1.7b Show that X noetherian $\Rightarrow$ (quasi-)compact

    Let $\{U\}_\alpha$ be an open cover of X indexed by some set α. Using the axiom of choice, construct choice functions $f_1, f_2 \cdots$ such that $\forall i \in \mathbb{N}, f_i(\{U\}_\alpha)\backslash(\cup_{j<i} f_j(\{U\}_\alpha)) \neq \emptyset$. Now construct a series of closed sets $X \supseteq (X\backslash f_1(\{U\}_\alpha) \supseteq (X\backslash(f_1(\{U\}_\alpha) \cup f_2(\{U\}_\alpha))) \supseteq \cdots$. Because X is noetherian, we know that this eventually terminates after some number of iterations n. However, the way that we've constructed our choice function, this means that we're no longer able to find an open cover that has elements distinct from those covered by our previous choices of open sets. However, because $\{U\}_\alpha$ is an open cover, this can only happen once we've covered the whole space. Therefore our open sets $\{f_i(\{U\}_\alpha)\}_{i=1}^n$ is an open cover.

    I wonder if there's a more elegant solution that doesn't necessarily rely on the axiom of choice.

1.2 Section 3

1.2.1 Misc Questions

- Why is a function (on an affine variety) defined to be regular at a point if there is some open set U containing P such that $f = \frac{g}{h}$ for some polynomials $g, h \in k[x_1, ..., x_n]$ and then a regular function one that is regular at each point (implying that they are, in general quotients) when it turns out that regular functions are defined to be equal to the affine coordinate ring?

  It's completely the right definition for being local at a point unambiguously and mirrors localizing the coordinate ring at a point. One reason is that it ties together functions that are regular at a point and regular on the whole variety as "the same thing" (subrings of the same "overarching ring").

Furthermore, the local ring of a point is geometrically motivated and it makes proofs easier. A crucial part of theorem 3.2 is showing that $A(Y)_{m_p}$ is isomorphic to the ring of regular functions at p; making the "algebraic part" (as opposed to the "topological part" that is the open subsets) be a fraction (since they're represented as equivalence classes of a regular function and an open set) makes the proof very simple.