UNIT - V

Probability Theory and Random Process

Axiomatic construction of the theory of probability, independence, conditional probability, and basic formulae:

Elementary Properties of Probability:

1. $P(\overline{A}) = 1 - P(A)$
2. $P(A) \leq 1$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad iff \;\; A_i \cap A_j = \phi \;\; for \;\; i \neq j$
5. $P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i) \quad iff \;\; A_i's \;\; are \;\; indepedent \;\; events.$
6. $P(A/B) = \dfrac{P(A \cap B)}{P(B)} \;, \; P(B) > 0$
7. $P(B/A) = \dfrac{P(A \cap B)}{P(A)} \;, \; P(A) > 0$

Random Variable:

There are two types of variable. One which takes a value or values with certainty and the other one which takes any one value out of a set of values with some probabilities, such a variable is called random variable.

As an example, let X is the no. of telephone calls received in on hour. So X = 0, 1, 2, ………… It can not be said that X = 10, may be or may not be. There is some probability that X can be 10 (probability zero means it does not take that value and probability one means it takes that value certainly).

Continuous random variable and Discrete random variable:

If a random variable takes non-countable no. of values then that random variable is called as Continuous random variable.

Ex.: Let X = Temperature of any day. Generally the day temp. lies in (12^0, 32^0), say. So X can take any value which falls in (12^0, 32^0) which is non-countable (if elements can be put in one to one correspondence then and only then the elements are countable).

Ex.: Let X = time of receiving telephone calls in (10 pm, 11pm).

If a random variable takes countable no. of values then that random variable is called as Discrete random variable.

Ex.: X = No. of telephone calls received in a day = 0, 1, 2, ….

Ex.: X = No. of defective items in a packet of n = 0, 1, …,n.

Ex.: An experiments consists of observing the sum of the dice when two fair dice are thrown. Find (i) the probability that the sum is 7 and (b) the probability that the sum is greater than 10.

Solution: Let X and Y are the two random variables and stand for the outcomes of the first and the second dice respectively. Then S = X + Y, is another random variable which is the sum of X and Y.

(i) P(S = 7) = P[(x = 1 & Y = 6) or (x = 6 & Y = 1) or (X = 2 & Y = 5) or (X = 5 & Y = 2) or (X = 3 & Y = 4) or (X = 4 & Y = 3)]
= 3 (2/36) = 1/6

(ii) P(S > 10) = P{ (X = 5 & Y = 6) or (X = 6 & Y = 5) or (X = 6 & Y = 6)}
= 3(1/36) = 1/12

Ex.: A committee of 5 persons is to be selected randomly from a group of 5 men and 10 women. Then find the probability that

(i) the committee consists of 2 men and 3 women.
(ii) the committee consists of all women.

Solution: Let X and Y are the no. of men and women selected in the committee.

(i) $P(X = 2 , Y = 3) = \dfrac{{}^{5}C_{2} \times {}^{10}C_{3}}{{}^{15}C_{5}} = 0.4$

(ii) P(X = 0 , Y = 5) = $\dfrac{{}^5C_0 \times {}^{10}C_5}{{}^5C_{15}} = 0.084$

Ex.: If a fair coin is tossed repeatedly until the first head appears, find the probability that the first head appears on the k – th toss.

Solution: Let X_k is the result of the k – th toss. If first head appears on the k – th toss then that means upto (k – 1) – th toss every outcome was tail.

Hence,

$$P(X_k = H) = P(X_1 = T \ \& \ X_2 = T \ \& \ X_3 = T \ldots\ldots\ldots, X_{k-1} = T \ \& \ X_k = H)$$

$$= P(X_1 = T)P(X_2 = T)P(X_3 = T)\ldots.P(X_{k-1} = T)P(X_k = H) = \frac{1}{2^k}$$

EX.: Two manufacturing plants produce similar parts. Plant I produces 1000 parts, 100 of which are defective. Plant II produces 2000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant I?

Solution: Let X = selected defective part came from plant I and Y = a part selected randomly which is found defective. Then $P\left(X / Y\right) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{100/3000}{250/3000}$

Ex.: A committee consists of 9 students 2 of which from the first year, 3 from the second year and 4 from the third year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belongs to the same class and third to the different class (iii) the three belong to the same class?

Solution: Let X , Y and Z are the no. of students removed from the committee and belong to the First, Second and Third year respectively.

(i) P(X = 1 , Y = 1, Z = 1)

$$= \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = 2/7$$

(ii) P{(X = 2 ,Y = 0, Z =1 or X = 2 ,Y = 1, Z = 0 or X = 0 ,Y = 2, Z =1 or X = 1 , Y = 2, Z =0 or X = 1 , Y = 0, Z =2 or X = 0 , Y = 1, Z = 2)}

$$= \frac{{}^2C_2 \times {}^3C_0 \times {}^4C_1 + {}^2C_2 \times {}^3C_1 \times {}^4C_0 + {}^2C_0 \times {}^3C_2 \times {}^4C_1 + {}^2C_1 \times {}^3C_2 \times {}^4C_0 + {}^2C_1 \times {}^3C_0 \times {}^4C_2 + {}^2C_0 \times {}^3C_1 \times {}^4C_2}{{}^9C_3}$$

Ex.: A box A contains 2 white and 4 black balls and another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white.

Solution:

Let us define two events.

E: a black ball is transferred from the box A to the box B and then a white ball is drawn from the box B.

F: a white ball is transferred from the box A to the box B and then a white ball is drawn from the box B.

So, we have to find P(E or F) = P(E) + P(F)

P(E) = P(a black ball is transferred from the box A to the box B and then a white ball is drawn from the box B)

= P(a black ball is transferred from the box A to the box B) P(then a white ball is drawn from the box B)

$$= \frac{{}^4C_1}{{}^6C_1} \times \frac{{}^5C_1}{{}^{13}C_1}$$

P(F) = P(a white ball is transferred from the box A to the box B and then a white ball is drawn from the box B)

= P(a white ball is transferred from the box A to the box B) P(then a white ball is drawn from the box B)

$$= \frac{{}^2C_1}{{}^6C_1} \times \frac{{}^6C_1}{{}^{13}C_1}$$

P(E or F) = P(E) + P(F) = $\frac{{}^4C_1}{{}^6C_1} \times \frac{{}^5C_1}{{}^{13}C_1} + \frac{{}^2C_1}{{}^6C_1} \times \frac{{}^6C_1}{{}^{13}C_1}$

Ex.: Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning.

Solution: P(A is winning) = P(H in first toss or T in first toss by A and T in second toss for B and H in third toss by A or T in first toss by A and T in second toss for B and T in third toss by A and T in fourth toss for B and H in fifth toss by A or so on)

$$= \frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} + ... = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + ... = \frac{1/2}{1-(1/2)^2} = \frac{2}{3}$$

Ex.: A problem in mechanics is given to three students A, B and C whose chances of solving it are ½, 1/3 and ¼ respectively. What is the probability that the problem will be solved?

Solution: P(problem is solved means either solved by any one or any two or all of them) = 1 – P(problem is not solved by any one) = 1 – P(A can not solve and B can not solve and C can not solve) = 1 – (1 – 1/2)(1 – 1/3)(1 – 1/4)

Ex.: A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance or winning.

Solution: In a throwing of a pair of dice, probability of getting 6 is 5/36 and that of 7 is 6/36. Since A starts the throwing, A can get 6 in the very first throw or A does not get 6 in the first throw, B does not get 7 in the second throw and A gets 6 in the third throw or so on.

= 5/36 + (31/36)(30/36)(5/36) + (31/36)(30/36)(31/36)(30/36)(5/36) + ……

$$= \frac{5}{36} + \left(\frac{31}{36}\frac{30}{36}\right)\frac{5}{36} + \left(\frac{31}{36}\frac{30}{36}\right)^2 \frac{5}{36} + ...$$

$$= \frac{5/36}{1-\left(\frac{31}{36}\frac{30}{36}\right)}$$

Bayes' theorem:

The events $A_1, A_2, A_3, ..., A_n \subseteq S$ *are called mutually exclusive and exhaustive if*

$$\bigcup_{i=1}^{n} A_i = S \quad \& \quad A_i \cap A_j = \phi \;\; for \;\; i \neq j \text{ then } P(A_i|S) = \frac{P(A_i)P(S|A_i)}{\sum_{i=1}^{n} P(A_i)P(S|A_i)}$$

Sometimes it is called as inverse probability theorem as by this we find probability when the event is finished.

Ex.: There are three bags: first containing 1 white, 2 red and 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red and 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn from the second bag.

Solution: we have to find P(the drawn balls come from the second bag such that one white and one red ball are drawn from a randomly selected bag)

i.e., to find $P\left(A_i \middle| S\right)$

Let S is the event of drawing of one white ball and one red ball from a randomly chosen bag and

A_1 = First bag is chosen,

A_2 = Second bag is chosen,

A_3 = Third bag is chosen

Then $P\left(A_i \middle| S\right) = \dfrac{P(A_i)P(S|A_i)}{\sum_{i=1}^{n} P(A_i)P(S|A_i)}$

Now, $P(A_i)$ = 1/3, $P(S|A_i)$ = P(one red and one white ball are drawn from a bag such that i – the bag is selected).

Hence, $P(S|A_1)$ = P(one red and one white ball are drawn from the first bag)

= $\dfrac{{}^1C_1 \times {}^2C_1}{{}^6C_2}$ = 2/15

Similarly, $P(S|A_2)$ = $\dfrac{{}^3C_1 \times {}^2C_1}{{}^6C_2}$ = 2/5 and $P(S|A_3)$ = $\dfrac{{}^3C_1 \times {}^1C_1}{{}^6C_2}$ = 1/5

$$P\left(A_2 \middle| S\right) = \frac{P(A_2)P(S|A_2)}{\sum_{i=1}^{n} P(A_i)P(S|A_i)} = \frac{(1/3)(2/5)}{(1/3)(2/15) + (1/3)(2/5) + (1/3)(1/5)} = 6/11$$

Random variables, binomial, poisson and normal random variable, probability distributions:

- **Random Variable:** There are two types of variable. One which takes a value or values with certainty and the other one which takes any one value out of a set of values with some probabilities, such a variable is called random variable. As an example, let X is the no. of telephone calls received in on hour. So X = 0, 1, 2, ………… It can not be said that X = 10, may be or may not be. There is some probability that X can be 10 (probability zero means it does not take that value and probability one means it takes that value certainly).

Few definitions:

I. <u>Density Function</u>**:** The density function is constructed from the data with variable and corresponding frequencies (known as the frequency curve). It is denoted by f(x).

If X is a discrete random variable, then

$P(X = x_k) = f(x_k)$

But if X is a continuous random variable then

$P(a \le X \le b) = \int_a^b f(x)dx$ = area bounded by the curve, the x-axis, between a and b.

Clearly, $f(x) \ge 0 \;\; and \;\; \sum_x f(x) = 1, \;\; \int_{-\infty}^{\infty} f(x)dx = 1$

II. <u>Distribution function:</u>

It is defined as $F(x) = P(X \le x) = P(-\infty \le X \le x)$

$F(-\infty) = 0, F(\infty) = 1$

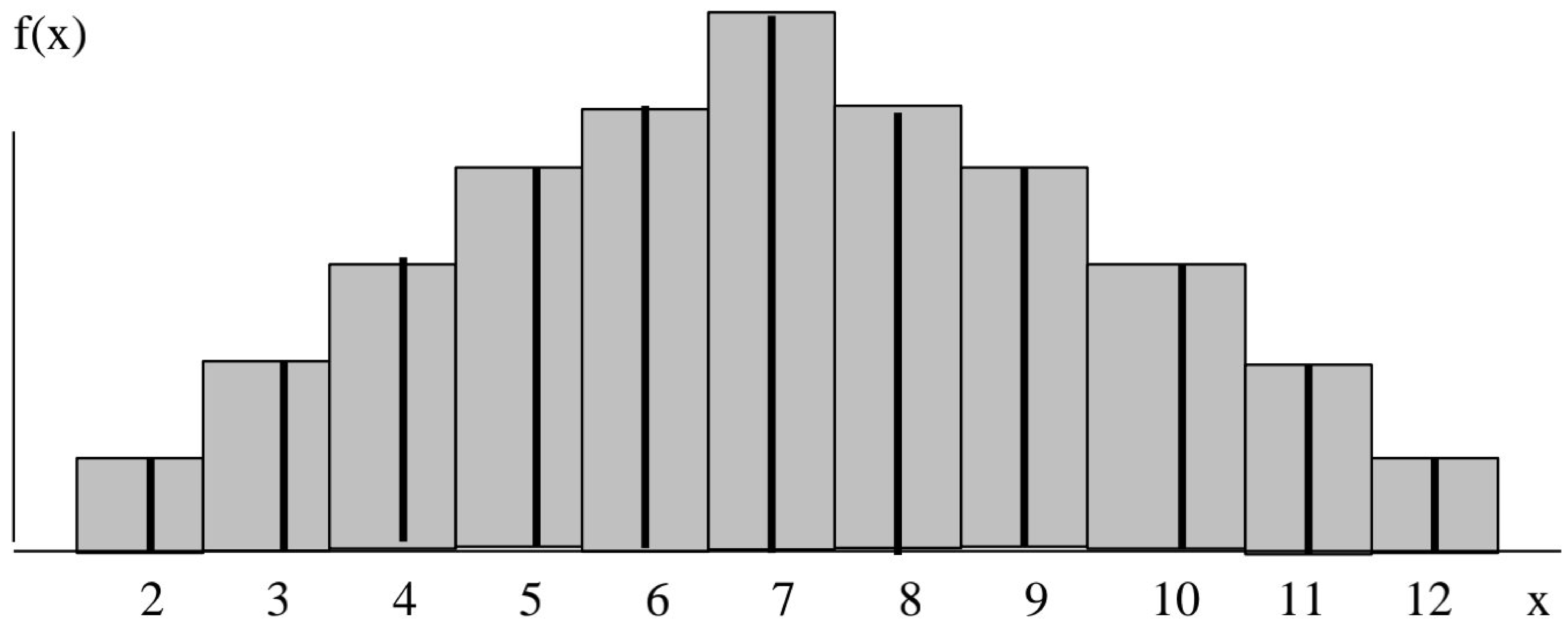
<u>Relation between them</u>: In continuous case: $f(x) = \frac{d}{dx}\{F(x)\}$ & $F(x) = \int_{-\infty}^{x} f(x)dx$

& $\int_{-\infty}^{\infty} f(x)dx = 1$ as $\int_{-\infty}^{\infty} f(x)dx = P(-\infty \le X \le \infty) = 1$

Ex.: Suppose a pair of dice are tossed and let the random variable X is the sum of the points.

In this case, the p. d. f. or probability function is

x	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36



The Distribution function F(x) is

$$F(x)=\begin{cases}0, & -\infty \le x<2 \\ 1/36, & 2 \le x<3 \\ 2/36, & 3 \le x<4 \\ 3/36, & 4 \le x<5 \\ --- & \\ 1, & 12 \le x<\infty\end{cases}$$

<u>Mathematical Expectation or Mean</u>: Suppose X is a random variable whose possible values are $x_1,\ x_2,\ x_3, \ldots.\ x_n$ then it is the mean and denoted as E(X) and defined as

$$E(x)=\sum_{j=1}^{n} x_j P(X=x_j)$$

For continuous case,

$$E(x)=\mu=\int_{-\infty}^{\infty} xf(x)dx$$

$$Variance=\sigma^2=E(X^2)-\mu^2$$

- **Continuous random variable and Discrete random variable:**

If a random variable takes non-countable no. of values then that random variable is called as Continuous random variable.

Ex.: Let X = Temperature of any day. Generally the day temp. lies in (12^0, 32^0), say. So X can take any value which falls in (12^0, 32^0) which is non-countable (if elements can be put in one to one correspondence then and only then the elements are countable).

Ex.: Let X = time of receiving telephone calls in (10 pm, 11pm).

If a random variable takes countable no. of values then that random variable is called as Discrete random variable.

Ex.: X = No. of telephone calls received in a day = 0, 1, 2, ….

Ex.: X = No. of defective items in a packet of n = 0, 1, …,n.

♦ Binomial Distribution

Let E is an experiment which is repeated n times. In a single experiment only two outcomes are possible, success or failure. Let p = probability of success in a single experiment and q = probability of failure in a single experiment. Always p + q = 1.

As an example, let E is the coin tossing experiment, tossed thrice(n = 3). If the outcome is head then it is success = p = ½ and If the outcome is tail then it is failure = q = ½. Let us define X as X = No. of heads after three tosses.

Let us find the probability of X = 2 i.e., symbolically, P(X=2).

Since the coin is tossed three times, we can have $X = 2$ i.e., getting two heads in three tosses. There are three alternatives E_1, E_2 and E_3 for the above case as given below:

E_1 : HHT i.e., head in the first toss and head in the second toss and tail in the last toss.

E_2 : HTH i.e., head in the first toss and tail in the second toss and head in the last toss.

E_3 : THH i.e., tail in the first toss and head in the second toss and head in the last toss.

$P(X = 2) = P(\text{happening of any one of } E_1, E_2 \text{ and } E_3)$

$= P(E_1 \text{ or } E_2 \text{ or } E_3)$

$= P(E_1) + P(E_2) + P(E_3)$ since E_1, E_2 and E_3 are independent.

$= p^2q + p^2q + p^2q$

$= 3\,p^2q$

Here the interesting thing is that $P(E_1) = P(E_2) = P(E_3)$ and in

$3\,p^2q$, 3 is there because 3 alternative cases were there by which the event $X = 2$ was possible.

If we consider the coin tossing n times and we want to know the probability of $X = r$ then we have take the product of prob. of one alternative and no. of alternatives.

One alternative of happening of $X = r$ can be taken as:

HHHHHH…….TTTTTT…..

r Heads in (n – r) tails in

first r tosses remaining tosses

Prob. of such alternative is $p^r q^{n-r}$ and no. of alternatives are ${}^nC_r$ as it can be thought of filling up r empty blocks out of n empty blocks.

Hence, $P(X = r) = {}^nC_r p^r q^{n-r}$.

Such a prob. distribution is called as binomial distribution.

How to proceed?

First read the given question carefully and decide whose prob. is to be found. Then declare X. Always X must have the name as:

X = No. of__________ out of _________. Fill up the blanks.

(Success) (n trials)

↓

Then define p **as p = prob. of ________ in a single trial or experiment.**

Keep in mind **p + q = 1.**

↓

Then, after the values of n, p and q we can find

P(X = r) by the formula:

$$P(X = r) = {}^nC_r p^r q^{n-r}.$$

Ex. 1.: If the prob. that a screw is defective is 0.1 then find the prob. that there are exactly 4 defective screws in a lot of 6.

Sol.: Here the question is *find the prob. that there are exactly 4 defective screws in a lot of 6.* If we try to write in symbolic form then:

Prob.(*exactly 4 defective screws in a lot of 6*) = ?

i.e., P(*exactly 4 defective screws in a lot of 6*) = ?

i.e., P(No. of *defective screws* in a lot of *6* = 4.) = ?. This is written in this form so that we can introduce X as **X = No. of__________ out of _________.**

(Success) (n trials)

Comparing them we decide

X = No. of defective screws out of 6 .

In this question we are considering success as defectiveness.

Hence p = prob. of defectiveness of a single screw (as **p = prob. of _______ in a single trial or experiment**) which is given in the question as 0.1.

So, the conclusion is that: n = 6, p = 0.1 & q = 0.9 as p + q =1.

Hence, $P(X = 4) = {}^6C_4 p^4 q^{6-4} = ????$

Suppose it is predicted that the prob. of raining of any day in the month of July'04 is 0.3. How many rainy days one can expect if the prediction is correct?

How we find prob. is no. of favorable cases divided by total no. of cases. So, prob. of raining of any day equals to 0.3 means 3/10 which can be interpreted as out of 10 days only 3 days are found raining.

So, in 10 days only 3 days are rainy days.

$\Rightarrow$ in 30 days $30 \times \dfrac{3}{10}$ days will be rainy days.

$\Rightarrow$ No. of rainy days in a month = No. of total days in a month $\times$ Prob. of raining of a single day.

Thus if the question is to find how many ……then :

How many **<u>days/items</u>** in/out of a month/packet = No. of total **<u>days/items</u>** in or out of a month/packet $\times$ Prob. of single **<u>day/item.</u>**

Ex.: *Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys (ii) 5 girls (iii) either two 2 or 3 boys? Assume equal probabilities for boys and girls.*

Sol.: If we read the question carefully then for (i) the question can be put in the following form:

How many **families have 3 boys** out of **800 family** = No. of total **<u>families</u>** $\times$ Prob. of **<u>single family with 3 boys.</u>**

= 800 $\times$ Prob. of **<u>single family with 3 boys.</u>** (i)

Now, since we know how to find $P(X = 3)$ we will like to put the sentence "Prob. of **<u>single family with 3 boys</u>** " as $P(X = 3)$.

The sentence **"**Prob. of **<u>single family with 3 boys"</u>** can be put as "prob. of no. of boy child in a single family = 3".

Thus X = No. of boy child in a single family.

= **No. of boy child out of 5 child** since every family has exactly 5 children.

Comparing with **X = No. of__________ out of _______** we conclude

(Success) (n trials)

that "Success" $\cong$ "Boy child" and "n = 5".

Thus p = Prob. of boy child

= ½ since it is said to assume equal probability for boy and girl.

Therefore,

Prob. of **<u>single family with 3 boys</u>** = P(X = 3)

$$= {}^5C_3 p^3 q^{5-3} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

Substituting in (i), we get

How many **families have 3 boys** out of **800 family**

$$= 800 \times {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

<u>Mean and Variance of a Binomial Distribution are **np** and **npq** resp.</u> **(prove it !!!)**

Fitting of Binomial Distribution:

In statistics or in data analysis it is required to fit a binomial distribution to a random variable.

Let us take an example, an experiment consists of tossing of 5 coins together. This experiment is repeated 100 times. If X is the no. of heads in a single experiment then X = 0, 1, …,5.

So, what we will observe is that out of 100 times few times will be all tails i.e., X = 0 few times; one head few times i.e., X = 1 few times and so on. This way X can have maximum value as 5. The above result can be arranged as:

X:	0	1	2	…….	5
f (the no of times):	4	12	20	…….	7(total = 100)

If we wish to fit a binomial distribution with the above data then do the following steps:

Find mean from the given data by the formula: $\mu = \dfrac{\sum_i f_i x_i}{\sum_i f_i}$ (1)

↓

Always in Binomial distribution, X = 0, 1, 2,……n. So, identify n as the last value of X. In binomial distribution μ = np (2)

Equate (1) & (2) to get p = ?

After knowing n, p and q, calculate P(X = 0), P(X = 1),……, P(X = n). If N = $\sum f_i$ then find N×P(X = 0), N×P(X = 1),……, N×P(X = n).

At last write the answer as:

The theoretical frequencies are as given below:

X:	0	1	2	………	n
f:	N×P(X = 0)	N×P(X = 1)	N×P(X = 2)	………	N×P(X = n)

♦ Poi[illegible]n

Since [illegible]on we predict the happening of an event which is correct upto certain degree, accuracy is the most important factor. When the p, the probability of success in a single experiment, is very small (tends to zero) and simultaneously n tends to infinity then binomial distribution does not give accurate prediction. In this case, we improve the binomial distribution to Poisson Distribution by the following method:

$$P(X = r) = {}^{n}C_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r} = \frac{n(n-1).......(n-r+1)}{r!} p^r q^{n-r}$$

$$= \frac{np(np-p).......(np-rp+p)}{r!(1-p)^r}(1-p)^n$$

Let λ = np where **n→∞ & p → 0.**

$$\text{Then } P(X = r) = \frac{\lambda(\lambda-p).......(\lambda-rp-p)}{r!(1-p)^r}\left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^r}{r!}\left(1-\frac{\lambda}{n}\right)^n \qquad (1)$$

Now let us find out $\underset{n\to\infty}{\text{Lt}}\left(1-\frac{\lambda}{n}\right)^n$

Let $L = \underset{n\to\infty}{\text{Lt}}\left(1-\frac{\lambda}{n}\right)^n \Rightarrow \log L = \underset{n\to\infty}{\text{Lt}}\frac{\log\left(1-\frac{\lambda}{n}\right)}{\frac{1}{n}} \quad \left[\frac{0}{0}\right]$

$= \underset{n\to\infty}{\text{Lt}}\frac{\left(\frac{n}{n-\lambda}\right)\left(\frac{\lambda}{n^2}\right)}{-\frac{1}{n^2}} = -\underset{n\to\infty}{\text{Lt}}\frac{\lambda n}{n-\lambda} = -\lambda$

$\Rightarrow \log L = -\lambda \Rightarrow L = e^{-\lambda}$

Substituting in (1), we get

Then $P(X = r) = \frac{\lambda^r}{r!}e^{-\lambda}$

Thus when $p \to 0$ (simultaneously $n \to \infty$), we consider Poisson distribution i.e.,

$P(X = r) = \frac{\lambda^r}{r!}e^{-\lambda}$, λ is the mean.

Since Poisson distribution is a case of Binomial distribution, **λ = np.**

Ex.: Prove that the mean and the variance of the poisson distribution are same and equal to λ.

♦ Normal Distribution

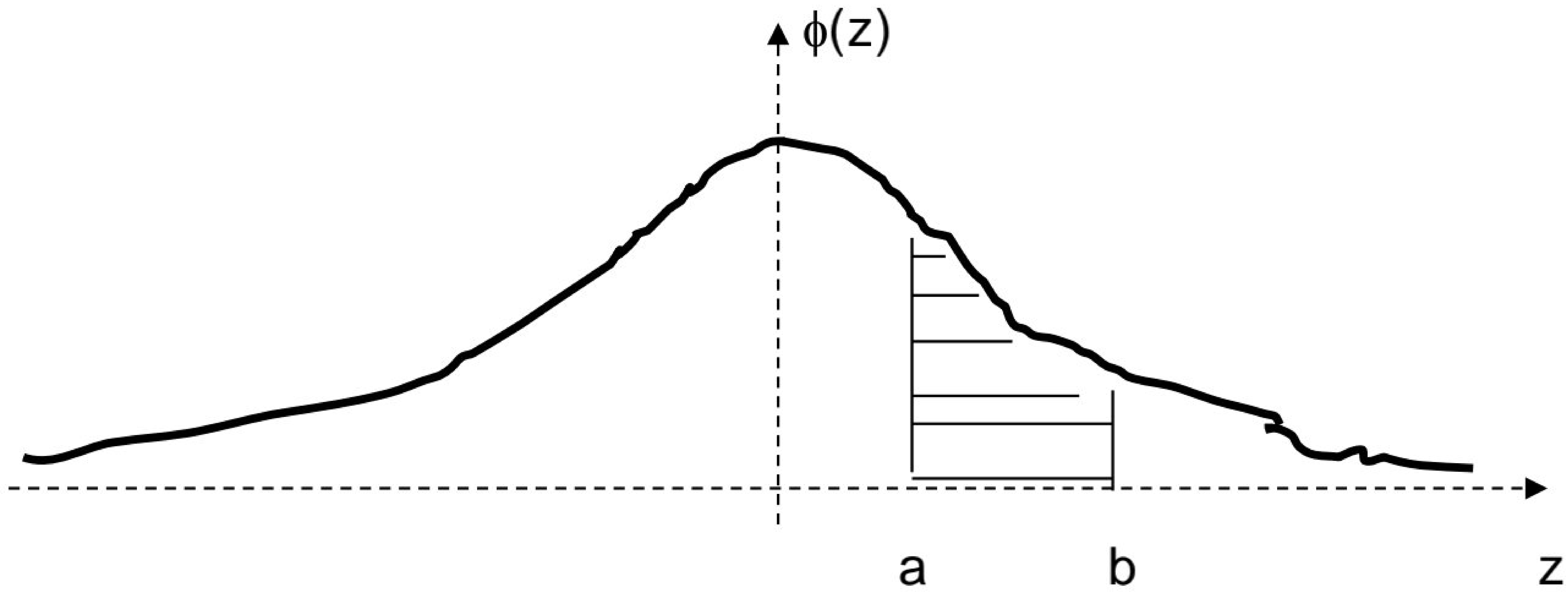
If X is a continuous random variable, then we find its probability density function f(x) which is constructed from observed data as frequency curve. A continuous random variable which has the density function as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is called normal random variable.

In such case, we find $P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x)dx$.

Due to complexity of the above integral, we use a substitution $Z = \frac{X-\mu}{\sigma}$.

Clearly, $\mathbf{P(\alpha \le X \le \beta) = P(a \le Z \le b)} = \int_{a}^{b} \phi(z)dz$ where $\frac{\alpha-\mu}{\sigma} = a$ and $\frac{\beta-\mu}{\sigma} = b$ and $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

Also, $P(-\infty \le Z \le \infty) = 1 \Rightarrow \text{Total area under the curve} = 1$



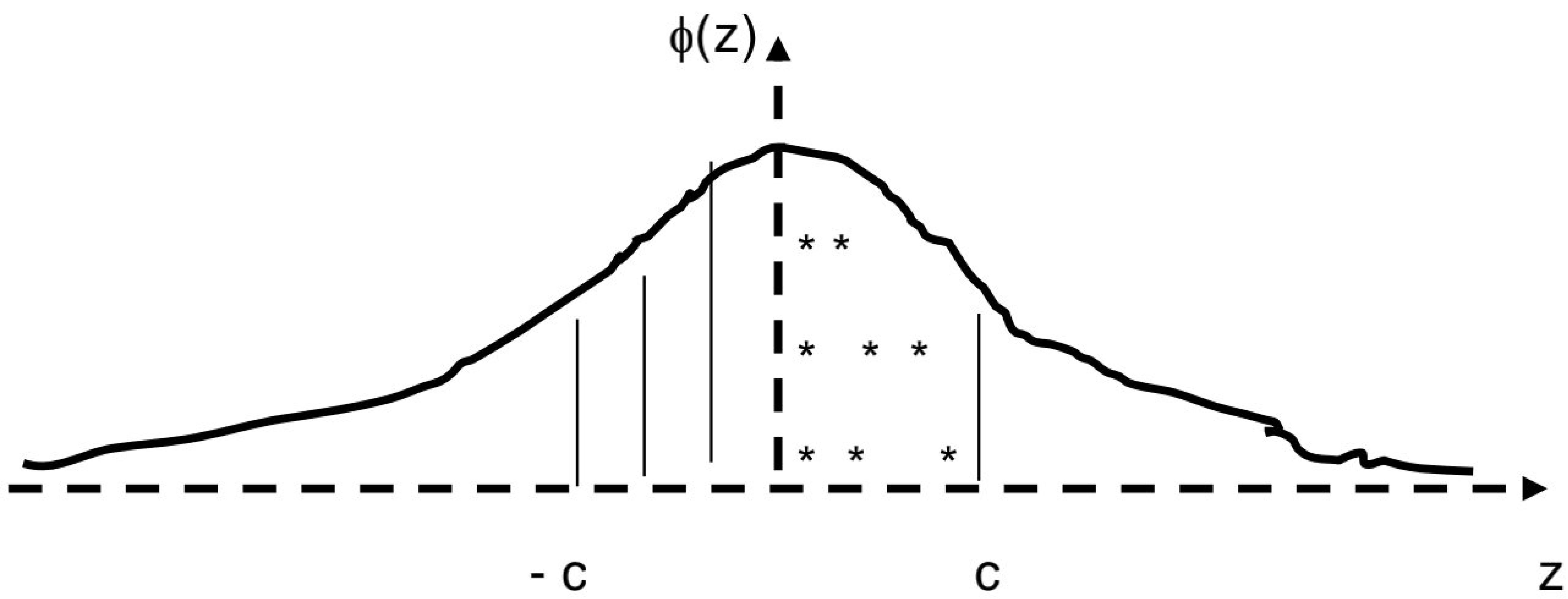
The shaded region in the figure stands for area indicated as

$P(a \le Z \le b) = \int_{a}^{b} \phi(z)dz$ = area bounded by the curve, the z-axis, between a and b.

Also, the above curve is symmetric about the Y-axis, so half of the total area which is 1 i.e., 0.5 lies in both the sides.

To know the value of $P(a \le Z \le b)$, we take help of a table called as Normal table in which we can find the area from 0 to c (must be positive).

So, $P(-c \le Z \le 0) = P(0 \le Z \le c)$.



How to proceed?

First read the question carefully then decide what to find. If to find $P(\alpha \le X \le \beta)$ then find a and b from the relation $a = \dfrac{\alpha - \mu}{\sigma}$ & $b = \dfrac{\beta - \mu}{\sigma}$ so that

$P(\alpha \le X \le \beta) = P(a \le Z \le b)$

If it is asked to find $P(X \le \beta)$ then $P(X \le \beta) = P(-\infty \le X \le \beta)$.

Similarly, for $P(\alpha \le X) = P(\alpha \le X \le \infty)$

Interpret P(a ≤ Z ≤ b) as area under the normal curve from a to b. Using normal table calculate the probability.

Functions of random variables; mathematical expectations:
If Y = f(X) then P(Y = y) = P(X = x) where Y and X are random variables.

Mathematical Expectation or Mean: Suppose X is a random variable whose possible values are $x_1,\ x_2,\ x_3, \ldots.\ x_n$ then it is the mean and denoted as E(X) and defined as

$$E(x) = \sum_{j=1}^{n} x_j P(X = x_j)$$

For continuous case,

$$E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$Variance = \sigma^2 = E(X^2) - \mu^2$$

Definition and classification of random processes, discrete-time Markov chains, Poisson process:

Stochastic Process or Random Process

Introduction

Suppose a die is thrown infinite no. of times. X_n is the outcome of the n – th throw. So the values of X_n can be 1, 2, …., 6. But one neither can surely say that at n – th throw the outcome will be 5 as am example nor say that it will not be 5. It may be or may not be nothing is certain. Some chance or probability is there of happening or not happening of this event. Therefore, X_n randomly takes any one value out of six (1, 2, …, 6) values and X_n is called random variable. The family $\{X_n, n \geq 1\}$ is called a ***stochastic process***. X_n is function of n. If we assume the throwing of die is at every minute then n is the time of throwing and n = 1, 2, ... is discrete time. The set of values of X_n i.e., {1, 2, 3, 4, 5, 6} is called state space. The above example is a case of discrete time and discrete state space stochastic process.

Another example: Suppose X(t) is the no. of incoming telephone at a switch board in the duration (0, t). Here, $\{X(t), t \in T\}$ is a continuous time discrete space stochastic process.

Few definitions:

III. <u>Density Function</u>**:** The density function is constructed from the data with variable and corresponding frequencies (known as the frequency curve). It is denoted by f(x).

If X is a discrete random variable, then

$P(X = x_k) = f(x_k)$

But if X is a continuous random variable then

$P(a \le X \le b) = \int_a^b f(x)dx$ = area bounded by the curve, the x-axis, between a and b.

Clearly, $f(x) \ge 0 \;\; and \;\; \sum_x f(x) = 1, \;\; \int_{-\infty}^{\infty} f(x)dx = 1$

IV. <u>Distribution function:</u>

It is defined as $F(x) = P(X \le x) = P(-\infty \le X \le x)$

$F(-\infty) = 0$, $F(\infty) = 1$

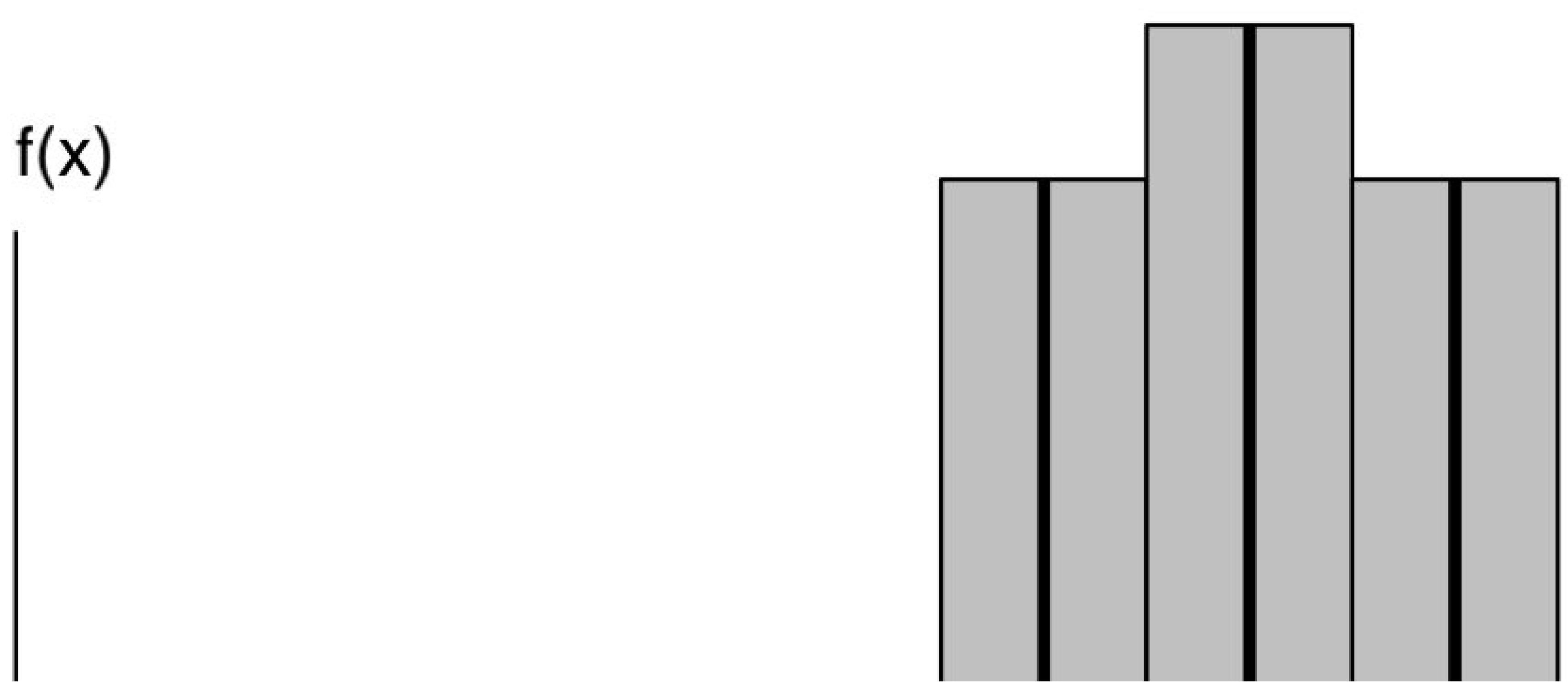
<u>Relation between them</u>: In continuous case: $f(x) = \frac{d}{dx}\{F(x)\}$ & $F(x) = \int_{-\infty}^{x} f(x)dx$

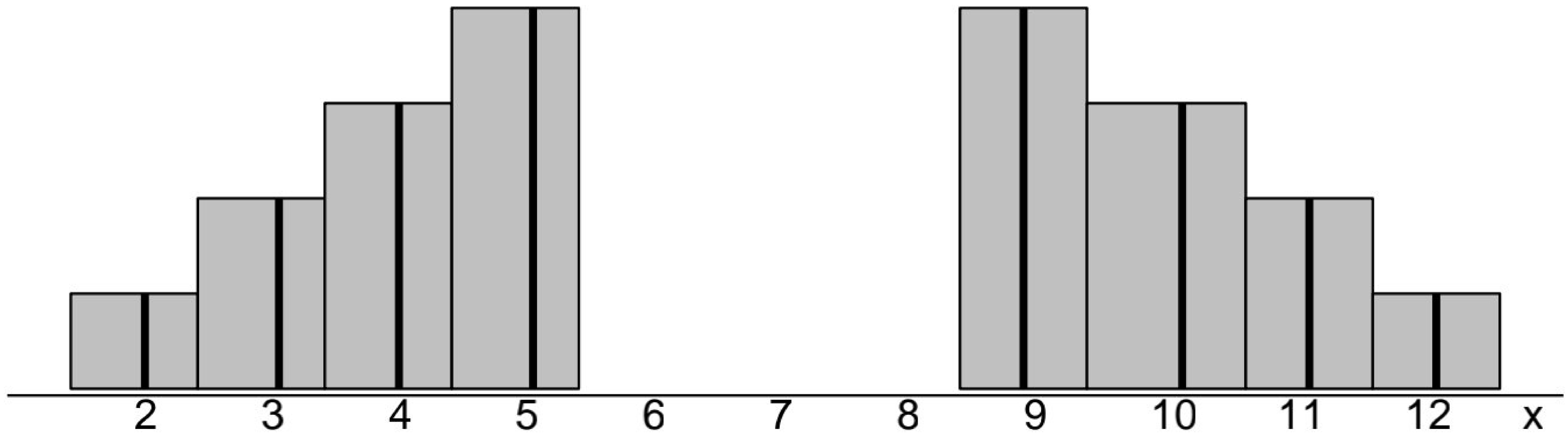
& $\int_{-\infty}^{\infty} f(x)dx = 1$ as $\int_{-\infty}^{\infty} f(x)dx = P(-\infty \le X \le \infty) = 1$

Ex.: Suppose a pair of dice are tossed and let the random variable X is the sum of the points.

In this case, the p. d. f. or probability function is

x	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36





The Distribution function F(x) is

$$F(x)=\begin{cases}0, & -\infty \le x < 2\\ 1/36, & 2 \le x < 3\\ 2/36, & 3 \le x < 4\\ 3/36, & 4 \le x < 5\\ --- & \\ 1, & 12 \le x < \infty\end{cases}$$

<u>Mathematical Expectation</u>: Suppose X is a random variable whose possible values are $x_1,\ x_2,\ x_3, \ldots.\ x_n$ then it is the mean and denoted as E(X) and defined as

$$E(x)=\sum_{j=1}^{n} x_j P(X = x_j)$$

For continuous case, $E(x)=\int_{-\infty}^{\infty} xf(x)dx$

The Markov Process

If $\{X(t),\ t \in T\}$ is a stochastic process such that, given the value X(s), the values of X(t), t > s, do not depend on the values X(u), u < s, then the process is said to be a Markov process.

In other words, if $t_1 < t_2 < t_3 < \ldots\ldots\ldots t_n < t$

$$P\left(\frac{a \le X(t) \le b}{X(t_1)=x_1,\ldots.,X(t_n)=x_n}\right)=P\left(\frac{a \le X(t) \le b}{X(t_n)=x_n}\right)$$

Markov process with discrete state space

Discrete time Markov Process is denoted as $\{X_n,\ n \ge 0\}$ and let the values of X_n are 0, 1, 2, 3, …… then the set {0, 1, 2, …..} is called discrete state space of $\{X_n,\ n \ge 0\}$ The discrete time Markov Process with discrete state space is called *Markov Chain*.

$X_n = i$ means the process is in the state i at the time n.

Hence, a random process $\{X_n, n \geq 0\}$ is called Markov Chain iff

$$P\left(\frac{X_{n+1} = j}{X_0 = i_0, X_1 = i_1 ...., X_n = i}\right) = P\left(\frac{X_{n+1} = j}{X_n = i}\right)$$

Markov Chain:

Let S is any system which can be randomly in any state at any time. The possible states are S_1, S_2, S_3, ……S_n(Discrete state space). Let at any time t_i the system is in the state S_i and at the next time t_{i+1} the system is in the state S_j. If it is found that next state S_j depends only on the proceeding state S_i not on previous to the proceeding state then it is called as Markov Chain.

Ex.: Suppose a simple coin is tossed for n number of times. The possible outcomes at each toss are two: head with probability p and tail with probability q. A random variable X_n which is the outcome of the n-th toss and defined as $X_n = \begin{cases} 1, \text{ if the outcome is head} \\ 0, \text{ if the outcome is tail} \end{cases}$ and $S_n = X_1 + X_2 + X_3 + ... + X_n$

Thus the possible values of S_n are 0, 1, 2, ……, n. In this case, $S_{n+1} = S_n + X_{n+1}$. Given that S_n= j, then S_{n+1} can have two possible values only j or j + 1 with probability q and p respectively.

$$P\left(\frac{S_{n+1} = j+1}{S_n = j}\right) = p \quad \& \quad P\left(\frac{S_{n+1} = j}{S_n = j}\right) = q$$

These probabilities are not effected by the values of S_1, S_2, S_3, …., S_n. Therefore, the outcome of (n + 1)-th toss depends only on the value of n-th toss.

The Transition Matrix:

Let $p_{ij} = P\left(\frac{X_{n+1} = j}{X_n = i}\right)$= probability that the system or process changes from i-th state to the j-th state in one step.

Then, the matrix $P = [p_{ij}]_{n \times n}$ is called as the one step transition matrix of the Markov chain $\{X_n, n \geq 0\}$ is as given below:

$S_1 \quad S_2 \quad .. \quad S_n$

$$\begin{array}{c} S_1 \\ S_2 \\ .. \\ S_n \end{array}\begin{bmatrix} p_{11} & p_{12} & .. & p_{1n} \\ p_{21} & p_{22} & .. & p_{23} \\ .. & .. & .. & .. \\ p_{n1} & p_{n2} & .. & p_{nn} \end{bmatrix}$$

where $p_{ij} \geq 0 \ \& \ \sum_{j=1}^{n} p_{ij} = 1 \ for \ i = 1,2,...,n$

Ex.: A particle performs a random walk with absorbing barrier at 0 and 4. Whenever, it is at any position r (0< r <4), it moves to (r+1) with probability p or to (r -1) with probability q such that p + q = 1. But whenever it reaches 0 or 4 it remains there(absorbing). Find the transition matrix.

Sol.: Let X_n be the position of the particle at n-th step or time and the states of this random movement are the positions as the particle occupy time to time. Clearly, this random movement of the particle is a Markov Chain as the n – th time position depends only on the position occupied by the particle at (n - 1) –th time. The one step state transition matrix is as given below:

$$\textbf{\textit{States of }} X_n$$

$$\textbf{\textit{States of }} X_{n-1} \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex.: Suppose a simple coin is tossed for n number of times. The possible outcomes at each toss are two: head with probability p and tail with probability q. A random variable X_n which is the outcome of the n-th toss and defined as

$X_n = \begin{cases} 1, \ if \ the \ outcome \ is \ head \\ 0, \ if \ the \ outcome \ is \ tail \end{cases}$ and $S_n = X_1 + X_2 + X_3 + ... + X_n$

Show that $\{S_n, \ n \geq 0\}$ is a Markov Chain and hence find the transition matrix.

Sol.: The possible values (states) of S_n are 0, 1, 2, ……, n. In this case, $S_{n+1} = S_n + X_{n+1}$. Given that S_n = j, then S_{n+1} can have two possible values only j or j + 1 with probability q and p respectively.

$$P\left(\frac{S_{n+1} = j+1}{S_n = j}\right) = p \ \ \& \ P\left(\frac{S_{n+1} = j}{S_n = j}\right) = q$$

These probabilities are not effected by the values of S_1, S_2, S_3, …., S_n. Therefore, the outcome of (n + 1)-th toss depends only on the value of n-th toss. Hence $\{S_n,\ n \geq 0\}$ is a Markov Chain.

The state transition matrix is as given below:

$$\textbf{\textit{States of }} S_{n-1}\quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ .. \end{array}\begin{array}{c} \textbf{\textit{States of }} S_n \\ \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & .. \end{array} \\ \begin{bmatrix} q & p & 0 & 0 & 0\,.. \\ 0 & q & p & 0 & 0\,.. \\ 0 & 0 & q & p & 0\,.. \\ 0 & 0 & 0 & q & p\,.. \\ .. & .. & .. & .. & ....\end{bmatrix}\end{array}$$

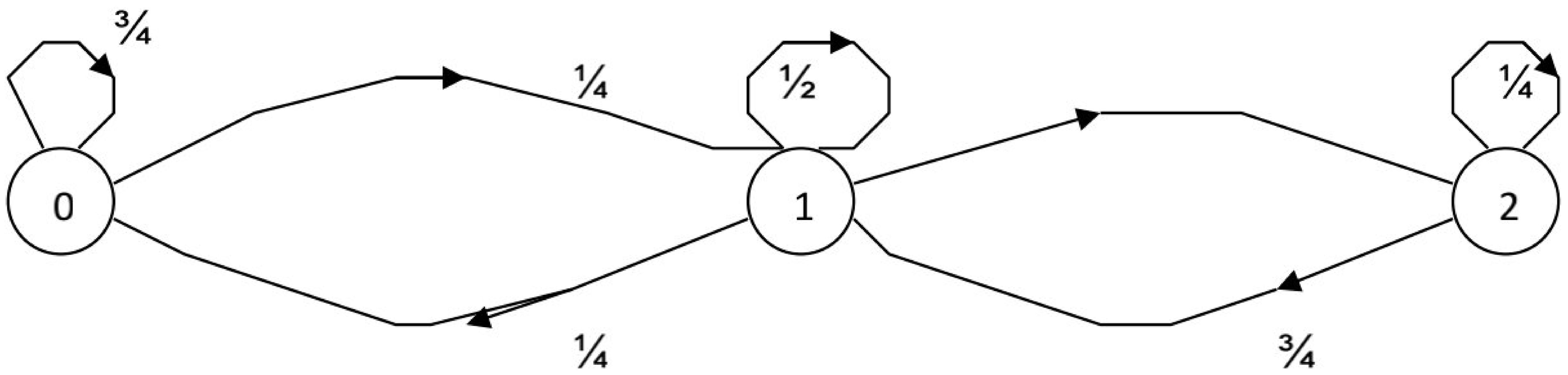
The initial probability distribution vector:

Let us define $p_i^{(0)}$ as the probability that the system or the process was in the i-th state initially. Then $\vec{p}^{(0)} = \left\{p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, ..., p_n^{(0)}\right\}$ is called as the initial probability distribution vector.

Markov Chain as Graphs:

Ex.: Let $\{X_n,\ n \geq 0\}$ is a Markov Chain with transition matrix as $\begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$

It can be displayed graphically as



Higher Transition probabilities:

Let us define the m step transition probability

$p_{ij}^{(m)} = P\left(\dfrac{X_{n+m} = j}{X_n = i}\right)$. It is the probability that the process changes from the i- th state to the j – th state in m steps.

Let $P^{(m)}$ is the m step transition matrix then $P^{(m)} = [p_{ij}^{(m)}]$

States of X_{n+m}

$0 \quad 1 \quad 2 \quad 3 \quad 4\ ..$

i.e., $P^{(m)}$ = ***States of*** X_n $\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ .. \end{array} \begin{bmatrix} p_{00}^{(m)} & p_{01}^{(m)} & p_{02}^{(m)} & p_{03}^{(m)} & p_{04}^{(m)} & .. \\ p_{10}^{(m)} & p_{11}^{(m)} & p_{12}^{(m)} & p_{13}^{(m)} & p_{14}^{(m)} & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix}$

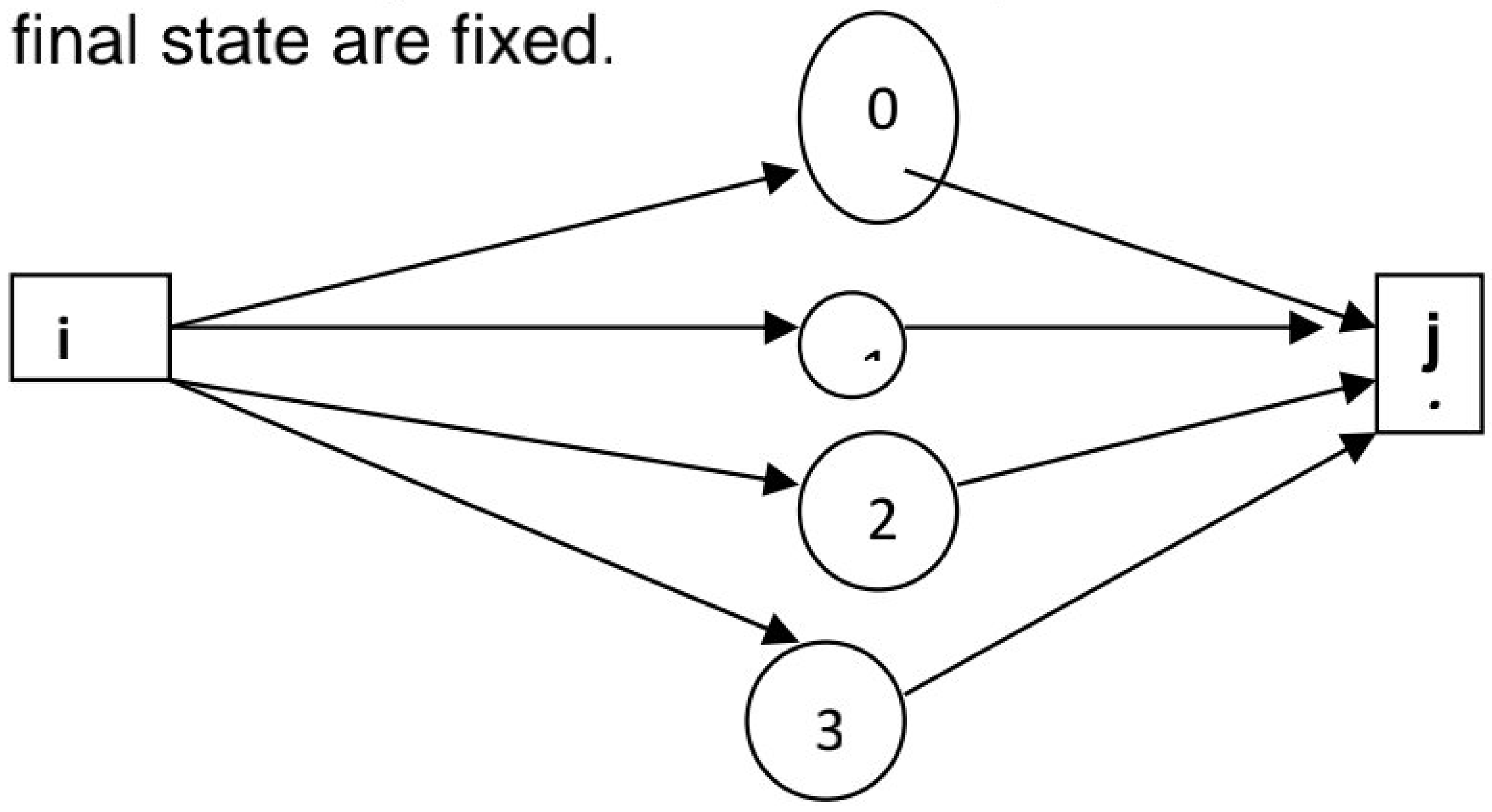
Theorem: If P is the transition matrix (one step) and $P^{(m)}$ is the m-th step transition matrix then $P^{(m)} = P^m$

Proof: Let us prove the result by mathematical induction.

Clearly, it is true for n = 1 as $P^{(1)} = P$

Let us prove the result for n = 2 i.e., to show $P^{(2)} = P^2$. For that let us compute $p_{ij}^{(2)}$ which is the probability that the system moves from the i-th state to the j-th state in 2 steps.

Let the system moves from i-th state to the k-th state in one step and then from k-th state to the j-th state in one step where k = 0, 1, 2, ….. as the starting state and the final state are fixed.



$$p_{ij}^{(2)} = \sum_k p_{ik}^{(1)} p_{kj}^{(1)} = \sum_k p_{ik} p_{kj}$$

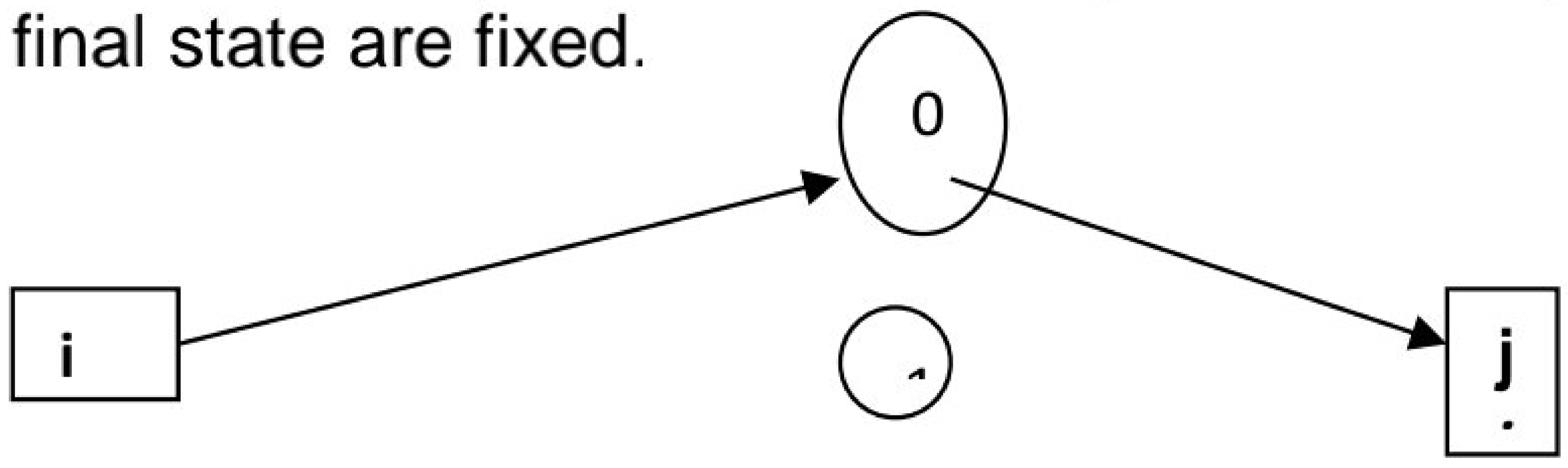
As an example, $p_{12}^{(2)} = \sum_k p_{1k} p_{k2} = p_{10}p_{02} + p_{11}p_{12} + p_{12}p_{22} + \ldots$ which is the (1,2) element of the matrix P^2.

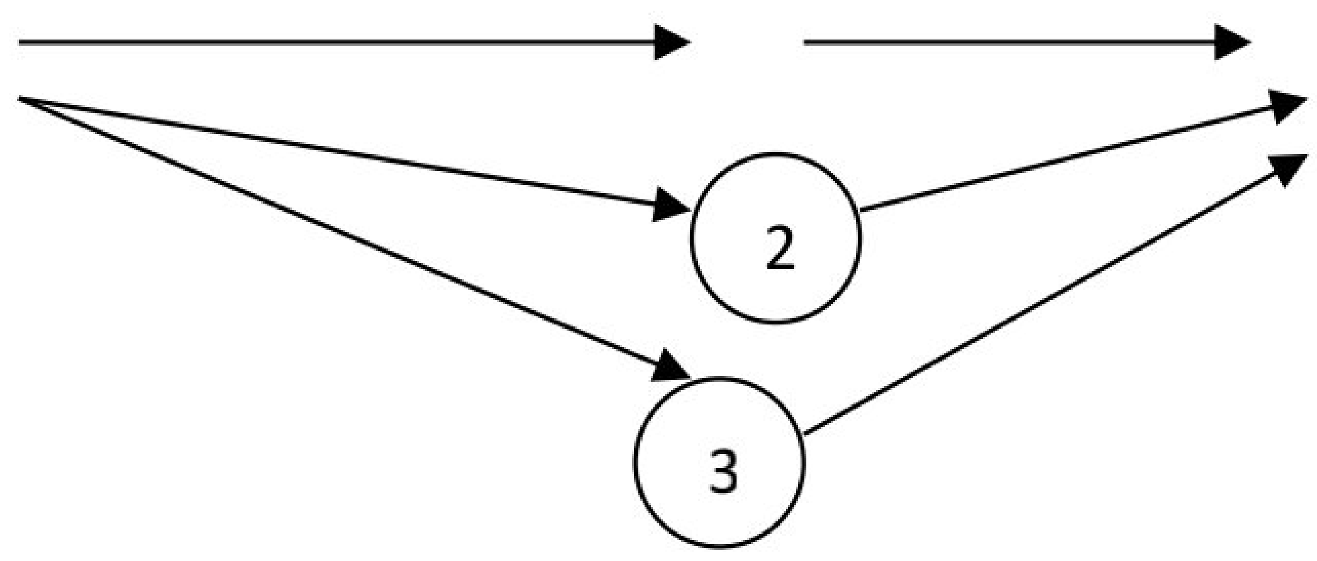
Hence, $P^{(2)} = P^2$

Let $P^{(m-1)} = P^{m-1}$ then to prove $P^{(m)} = P^m$

For that let us compute $p_{ij}^{(m)}$ which is the probability that the system moves from the i-th state to the j-th state in m steps.

Let the system moves from i-th state to the k-th state in (m-1) steps and then from k-th state to the j-th state in one step where k = 0, 1, 2, ….. as the starting state and the final state are fixed.





$$p_{ij}^{(m)} = \sum_k p_{ik}^{(m-1)} p_{kj}^{(1)} = \sum_k p_{ik}^{(m-1)} p_{kj}$$

As an example, $p_{12}^{(m)} = \sum_k p_{1k}^{(m-1)} p_{k2} = p_{10}^{(m-1)} p_{02} + p_{11}^{(m-1)} p_{12} + p_{12}^{(m-1)} p_{22} + \ldots$

which is the (1,2) element of the matrix $P^{(m-1)}P = P^{m-1}P = P^m$.

Hence, $P^{(m)} = P^m$

The n – steps state probability vector:

$\vec{p}^{(n)} = \left(p_0^{(n)}, p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, \ldots\ldots\ldots\right)$ where $p_i^{(n)}$ is the probability that the system is in the i-th state after n steps.

Theorem: $\vec{p}^{(n)} = \vec{p}^{(0)} P^n$ where P is the transition matrix.

Accessible state: The two states i and j are said accessible if they can communicate each other i.e., $p_{ij}^{(n)} > 0$

Absorbing state: The state i is called absorbing state iff $p_{ii} = 1$

Stationary distribution or fix point:

A vector $\vec{t} = (t_0, t_1, t_2, \ldots\ldots t_n)$ is called stationary distribution or fix point of the t. p. m. P iff

(i) $t_i \geq 0$

(ii) $\sum_{i=1}^{n} t_i = 1$ and

(iii) $\vec{t}P = \vec{t}$

Regular Markov Chain: A Markov chain with t. p. m. as P is called regular iff there exist a positive integer m such that $P^m = Q = [q_{ij}]$ *such that all* $q_{ij} > 0$

Theorem: For a regular Markov Chain, $\underset{n\to\infty}{Lt}\ \vec{p}^{(n)} = \vec{t}$ & $\underset{n\to\infty}{Lt}\ P^{(n)} = T$ where every row of T is $\vec{t}$.

Ex.: If a Markov Chain with state space {0, 1, 2} has the t. p. m. as given below $\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$ and the initial probability distribution $\vec{p} = (0,\ 1,\ 0)$ then find the following:

(i) probability that process will change from state 0 to state 2 in two steps.

(ii) Probability that the system will be found in the state 2 after 2 steps.
(iii) Probability that the system will be in the state 0 after a long run.

Sol.: (i) Find $p_{02}^{(2)}$ which is an element of $P^{(2)} = P^2$

(ii) find $p_2^{(2)}$ which is the third element of $\vec{p}^{(2)} = \vec{p}^{(0)} P^2$

(iii) find the first element of $\vec{t}$

Poisson process:

Let us consider an event E such as (i) incoming telephone calls (ii) arrivals of customer at a service counter (iii) occurrences of accidents.
Let N(t) is the total no. of occurrences of the event E in an interval of duration t and $p_n(t)$ = P{N(t) = n}
Let us make following assumptions:

(i) N(t) is independent of the no. of occurrences of the event in an interval prior to the interval (0, t).
(ii) $p_n(t)$ depends only on the length of the interval and is independent of where this interval is situated.
(iii) In an interval of very small length h, the probability of exactly one occurrence is $\lambda h + o(h)$ and that of more than one is o(h).

Under the above assumptions let us find out $p_n(t+h)$ *for* $n \geq 0$.
Occurrence of n events by the time (t + h) can happen in the following mutually exclusive ways $A_1, A_2, A_3, \ldots\ldots$

For n $\geq$1:
A_1: n events by the time t and no event between t and t + h:

$$P(A_1) = P\{N(t) = n\}\, P\left\{\frac{N(h)=0}{N(t)=n}\right\} = p_n(t)p_0(h) = p_n(t)\left(1-\lambda h + o(h)\right)$$

A_2: (n – 1) events by the time t and one event between t and t + h:

$$P(A_2) = P\{N(t) = n - 1\}\, P\left\{\frac{N(h)=1}{N(t)=n-1}\right\} = p_{n-1}(t)p_1(h) = p_n(t)(\lambda h) + o(h)$$

A_3: (n – 2) events by the time t and two events between t and t + h:

$$P(A_3) = P\{N(t) = n - 2\}\, P\left\{\frac{N(h)=2}{N(t)=n-2}\right\} = p_{n-2}(t)p_2(h) = o(h)$$

Similarly, $P(A_4)$, $P(A_5)$, …..

So, $p_n(t+h) = p_n(t)(1-\lambda h) + p_{n-1}(t)\lambda h + o(h)$ for n $\geq$1

$$\frac{p_n(t+h)-p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(h)}{h}$$

$$\Rightarrow p_n^{/}(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \text{ for n} \geq 1 \qquad (1)$$

For n = 0:

$$p_0(t+h) = p_0(t)p_0(h) = p_0(t)\{1-\lambda h\} + o(h)$$

$$\Rightarrow p_0^{/}(t) = -\lambda\, p_0(t) \;\; for \;\; n=0 \qquad (2)$$

Initial conditions:

$$p_0(0) = 1 \;\; \& \;\; p_n(0) = 0 \;\; for \;\; n \ge 1 \qquad (3)$$

(1) is called differential difference equation.
(1), (2) and (3) describe the event.

Let us find out the solution of (1) by Laplace transform:

$$sp_n(s) - p_n(0) = -\lambda p_n(s) + \lambda p_{n-1}(s)$$

$$\Rightarrow (s+\lambda)p_n(s) = \lambda p_{n-1}(s) \quad \because p_n(0) = 0 \;\; for \;\; n \ge 1$$

$$\Rightarrow p_n(s) = \frac{\lambda}{s+\lambda} p_{n-1}(s)$$

$$\Rightarrow \left(E - \frac{\lambda}{s+\lambda}\right) p_{n-1}(s) = 0$$

$$\Rightarrow p_{n-1}(s) = c\left(\frac{\lambda}{s+\lambda}\right)^{n-1}$$

$$\Rightarrow p_n(s) = c\left(\frac{\lambda}{s+\lambda}\right)^{n}, n \ge 0 \qquad (4)$$

Again from (2),

$$sp_0(s) - p_0(0) = -\lambda p_0(s)$$

$$sp_0(s) = -\lambda p_0(s) \quad \because p_0(0) = 0$$

$$\Rightarrow p_0(s) = \frac{1}{s+\lambda}$$

Using it in (4) we get,

$$c = \frac{\lambda}{s+\lambda}$$

$$\Rightarrow p_n(s) = \frac{\lambda^n}{(s+\lambda)^{n+1}}$$

$$\Rightarrow p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \;\; n = 0, 1, 2, \; \ldots.$$

Correlation and Regression; Expectation and Variance:

Correlation and Regression:

Let the least square line which approximates the data $(x_1, y_1), (x_2, y_2), \; \ldots\ldots., (x_n, y_n)$ has the equation

$$y = mx + c \qquad (1)$$

Then the normal equations are

$$\sum y = m\sum x + nc \qquad (2)$$

$$\sum xy = m\sum x^2 + c\sum x \qquad (3)$$

The values of m and c so obtained from (2) and (3) are

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2} \quad and \quad c = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n\sum x^2 - (\sum x)^2}$$

But $m = \dfrac{\{n\sum xy - (\sum x)(\sum y)\}/n^2}{\{n\sum x^2 - (\sum x)^2\}/n^2} = \dfrac{\frac{\sum xy}{n} - \bar{x}\,\bar{y}}{\frac{\sum xy}{n} - \bar{x}^2} = \dfrac{\frac{\sum (x-\bar{x})(y-\bar{y})}{n}}{\frac{\sum (x-\bar{x})^2}{n}} = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} \qquad (4)$

Also, dividing (2) by n, we get,

$$\bar{y} = m\bar{x} + c \qquad (5)$$

Subtracting (5) from (1), we have,

$$y - \bar{y} = m(x - \bar{x})$$

$$\Rightarrow y - \bar{y} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}(x - \bar{x}) \qquad (5)$$

The above line is called the **regression line of y on x.**

Similarly, the **regression line of x on y** has the equation

$$\Rightarrow x - \bar{x} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}(y - \bar{y}) \qquad (6)$$

In general, lines (5) and (6) are different.

The sample variances and covariance of x and y are given by

$\sigma_x^2 = \dfrac{\sum (x-\bar{x})^2}{n}$, $\quad \sigma_y^2 = \dfrac{\sum (y-\bar{y})^2}{n}$ & $\sigma_{xy}^2 = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{n}$ respectively.

Therefore, the regression lines of y on x and x on y are as given below:

$$y - \bar{y} = \frac{\sigma_{xy}^2}{\sigma_x^2}(x - \bar{x}) \;\;\&\;\; x - \bar{x} = \frac{\sigma_{xy}^2}{\sigma_y^2}(y - \bar{y}) \qquad (7)$$

Hence, the correlation coefficient is defined as

$$r = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y}$$

The lines of (7) can also be written as

$$y-\bar{y}=\frac{r\sigma_y}{\sigma_x}(x-\bar{x}) \quad \& \quad x-\bar{x}=\frac{r\sigma_x}{\sigma_y}(y-\bar{y})$$

Thus if the regression lines of y on x and x on y are given as
y = mx + c and x = ny + d then
$r^2 = mn$ where the equations are to so expressed that m and n both are less than 1.

The significance of the correlation coefficient:

r measures how well the least square regression lines fit with the sample data. if r^2 = 1 i.e $r=\pm 1$, we say that there is a perfect linear correlation between the variables.

Mathematical Expectation or Mean: Suppose X is a random variable whose possible values are $x_1,\ x_2,\ x_3, \ldots.\ x_n$ then it is the mean and denoted as E(X) and defined as

$$E(x)=\sum_{j=1}^{n} x_j P(X=x_j)$$

For continuous case,

$$E(x)=\mu=\int_{-\infty}^{\infty} xf(x)dx$$

$$Variance=\sigma^2=E(X^2)-\mu^2$$

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