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CS 324
Homework 2
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PROBLEM 1

Prove that $x^3 + 1000x^2 + 50x + 2000 = \Theta(x^3)$. Prove this directly from the definition of Θ on page 44 of CLRS, by finding specific constants c_1, c_2, and n_0, and showing that all relevant inequalities hold.

Definition of Θ: $\Theta(g(n)) = \{ f(n)$: there exists positive constants c_1, c_2, and n_0 such that $0 \le c_1 * g(n) \le f(n) \le c_2 * g(n)$ for all $n \ge n_0$. $\}$

Need to prove: $0 \le c_1 * x^3 \le x^3 + 1000(x)^3 + 50(x) + 2000 \le c_2 * x^3$ for all $x \ge n_0$ with some c_1 and c_2.

$0 \le c_1 * x^3 \le x^3 + 1000(x)^3 + 50(x) + 2000 \le c_2 * x^3$
$c_1 \le 1 + 1000(1/x) + 50(1/x^2) + 2000(1/x^3) \le c_2$

As x increases to infinity, the terms $1000(1/x)$, $50(1/x^2)$, and $2000(1/x^3)$ will approach 0.

This means that the sum $1 + 1000(1/x) + 50(1/x^2) + 2000(1/x^3)$ will converge to 1 but never go below it.

Let $c_1 = ½$, $c_2 = 15$, and $n_0 = 100$.

$1 + 1000(1/100) + 50(1/10000) + 2000(1000000)$
$= 1 + 10 + 1/200 + 1/500$
$= 11.007$

$0 \le 1/2 \le 11.007 \le 15$

PROBLEM 2

Show that for any real constants *a* and *b*, where *b* > 0, $(n + a)^b = \Theta(n^b)$. Note that *a* and *b* might not be integers.

Definition of Θ: $\Theta(g(n)) = \{ f(n)$: there exists positive constants c_1, c_2, and n_0 such that $0 \le c_1 * g(n) \le f(n) \le c_2 * g(n)$ for all $n \ge n_0$. $\}$

Need to prove: $0 \le c_1 * n^b \le (n + a)^b \le c_2 * n^b$ for all $n \ge n_0$

$0 \le c_1 * n^b \le (n + a)^b \le c_2 * n^b$

Let $c_1 = 0.5$ and $c_2 = 2$

$0.5 * n^b \le (n + a)^b \le 2 * n^b$

Want to show the following for some n_0:
$0.5 \le (n + a)^b / n^b \le 2$

Let $n_0 = 4a$

$(n_0 + a)^b / n_0^b = (5a)^b / (4a)^b$
$= 5/4 = 1.25$
$0.5 \le 1.25 \le 2$

As n increases infinitely, the term '$(n + a)^b / n^b$' will converge to 1.

Therefore, $(n + a)^b = \Theta(n^b)$.

<u>PROBLEM 3</u>

Let *f*(*n*) and *g*(*n*) be asymptotically positive functions. Prove or disprove each of the following conjectures.

a. $f(n) = O(g(n))$ implies $g(n) = O(f(n))$

False.

Let f(n) = n and g(n) = n^2

n = $O(n^2)$ because $0 \le n \le c_1 * n^2$ when n becomes sufficiently large.

But $n^2 \ne O(n)$ because $c_1 * n < n^2$ when n becomes sufficiently large.

h. $f(n) + o(f(n)) = \Theta(f(n))$

Let g(n) = o(f(n)).

Then $0 \le g(n) < c * f(n)$ for some constant c for all $n \ge n_0$.

So $f(n) \le f(n) + g(n) < c * f(n)$.

Therefore, $f(n) + g(n) \ne \Theta(f(n))$ by definition of Θ.

PROBLEM 4

Prove or disprove: $f(x) = x^2 - 2x + 1$ is monotonically increasing for real values of $x > 1$.

Want to show: $x^2 - 2x + 1 < y^2 - 2y + 1$ when $x < y$

Let $1 < x < y$.

$x - 1 < y - 1$

Let $a = (x - 1)$ and $b = (y - 1)$.
Because x and y are both greater than 1, then a and b must also both be greater than 1.

Therefore, $a^2 < b^2$.
So $(x-1)^2 < (y-1)^2$,
then $x^2 - 2x + 1 < y^2 - 2x + 1$.

<u>PROBLEM 5</u>

Find a simple formula for

$$\sum_{k=1}^{n}(2k-1)$$

= (2*1 + 2*2 + … + 2*n) - n
= [2 * (1 + 2 + … + n)] - n
= 2 * [(n * (n + 1)) / 2] - n
= (n * (n + 1)) - n
= n^2

PROBLEM 6

Give asymptotically tight bounds on the following summations. Assume that r ≥ 0 and s ≥ 0 are constants. Show your derivation using approximation by integrals. (Note: "find asymptotically tight bounds" means "find the THETA".)

$$\sum_{k=1}^{n} k^r$$

$$\int_0^n x^r\,dx \;\le\; \sum_{k=1}^{n} k^r \;\le\; \int_1^{n+1} x^r\,dx$$

$$(x^{r+1}/r+1)|0,n \;\le\; \sum_{k=1}^{n} k^r \;\le\; (x^{r+1}/r+1)|1,(n+1)$$

$$(n^{r+1}/r+1) \;\le\; \sum_{k=1}^{n} k^r \;\le\; ((n+1)^{r+1}/r+1) - (1/r+1)$$

$$(n^{r+1}/r+1) \;\le\; \sum_{k=1}^{n} k^r \;\le\; [(n+1)^{r+1}-1] / (r+1)$$

$$\sum_{k=1}^{n} k^r = \Theta(n^{r+1})$$

PROBLEM 7

Show that the solution of $T(n) = T([n/2]) + 1$ is $O(\lg n)$.

Prove $T(n) = O(\lg n)$ (i.e. $T(n) \le c_1 * \lg(n)$

IH: $T(k) \le c_1 * \lg(k)$ (for some positive c_1, for all $k < n$)

$T(n) = T([n/2]) + 1$
$T([n/2]) \le c_1 * \lg([n/2])$
$T([n/2]) + 1 \le c_1 * \lg([n/2]) + 1$

$c_1 * \lg([n/2]) + 1$
$= c_1 * (\lg(n) - \lg(2)) + 1$
$= c_1 * \lg(n) - c_1 * \lg(2) + 1$

Let $c_1 = (2/\lg(2))$

$c_1 * \lg(n) - c_1 * \lg(2) + 1$
$= c_1 * \lg(n) - (2/\lg(2)) * \lg(2) + 1$
$= c_1 * \lg(n) - 2 + 1$
$= c_1 * \lg(n) - 1 < c_1 * \lg(n)$

Therefore,
$T(n) < c_1 * \lg(n)$ for some positive c_1.

PROBLEM 8

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

c. $T(n) = 2T(n/2) + n^4$.

Proof using the master theorem:

$a = 2$, $b = 2$, $f(n) = n^4$

$f(n) = n^4 = n^{(\log_2(16))}$

Check that $a*f(n/b) \le c*f(n)$ for some $c < 1$:
$2*(n/2)^4 = n/8 \le (1/2)(n^4)$

Therefore, $T(n) = \Theta(n^4)$

f. $T(n) = 2T(n/4) + \sqrt{(n)}$.

Proof using the master theorem:

$a = 2$, $b = 4$, $f(n) = n^{1/2}$

$f(n) = n^{1/2} = n^{(\log_4(2))} = n^{(\log_b(a))}$

Therefore, $T(n) = \Theta(n^{1/2}\lg(n))$

g. $T(n) = T(n-2) + n^2$.

IH: $T(k) \ge c_1k^2\lg(k)$

$T(n-2) \ge c_1(n-2)^2$
$T(n-2) + n^2 \ge c_1(n-2)^2 + n^2$
$c_1(n-2)^2 + n^2$
$= c_1(n^2) - c_1(4n) + c_1(4) + n^2$
$= (c_1+1)(n^2) - c_1(4n) + c_1(4)$
$= (c_1+1)(n^2) - (4*c_1)(n+1) \ge n^2$

So, $T(n) \ge n^2$
$T(n) = \Omega(n^2)$

IH: $T(k) \le c_1k^2$

$T(n-2) \le c_1(n-2)^2$
$T(n) \le c_1(n-2)^2 + n^2$

$c_1(n-2)^2 + n^2$
$= c_1(n^2) - c_1(4n) + c_1(4) + n^2$