

Problem 1

Prove that the subset relation " \subseteq " on all subsets of Z is a partial order but not a total order.

Proof: The subset relation is reflexive

Let A be a subset of Z . Every element in A also exists in A because they are the same. Therefore, A must be a subset of itself. There will be no value in A that is not also in A .

Proof: The subset relation is antisymmetric

Let A and B be subsets of Z such that A is a subset of B and B is a subset of A .

So, $A \subseteq B$ and $B \subseteq A$.

Let a be some element in A .

Because $A \subseteq B$, a exists in A and B for all values in A .

Because $B \subseteq A$, b exists in B and A for all values in B .

Every value in A is in B , and every value in B is in A .

Therefore, for every value in a , there is a value b such that $a = b$ and vice-versa.

Therefore $A = B$.

Proof: The subset relation is transitive

Let A , B , and C be subsets of Z .

Let $A \subseteq B$.

Let $B \subseteq C$.

Let a be some value that exists in A .

Because $A \subseteq B$, a exists in B .

Because $B \subseteq C$, a exists in C .

Therefore, because every value of A is in C , $A \subseteq C$.

Proof: The subset relation is not a total relation

Let $A = \{0, 1, 2\}$ and $B = \{3, 4, 5\}$.

Both A and B are subsets of Z , but no values of A are in B , so A is not a subset of B .

And no values of B are in A , so B is not a subset of A .

Therefore for any two subsets of Z , it is not the case that either $A \subseteq B$ or $B \subseteq A$.

Problem 2

Give examples of relations that are:

a. reflexive and symmetric, but not transitive

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$ on the set $\{1,2,3\}$

b. reflexive and transitive, but not symmetric

The \leq relation is reflexive and transitive, but not symmetric.

c. symmetric and transitive but not reflexive

$R = \{(1,2), (2,1), (1,1), (2,3), (3,2), (2,2), (3,3), (1,3), (3,1)\}$ on the set $\{1,2,3,4\}$. Note that the pair $\{4,4\}$ never appears, and thus, R is not reflexive.

Problem 3

In how many ways can we choose three distinct numbers from the set $\{1, 2, \dots, 99\}$ so that their sum is even?

Two possibilities:

- two odds and an even
- three evens

Number of ways to get three evens = $50C3$

>> 19600

Number of ways to get two odds and an even = $50C2 * 50C1$

>> $1225 * 50 = 61250$

$19600 + 61250 = 80850$

Problem 4

Suppose we shuffle a deck of 10 cards, each bearing a distinct number from 1 to 10, to mix the cards thoroughly. We then remove three cards, one at a time, from the deck. What is the probability that we select the three cards in sorted (increasing) order?

Number of permutations of 3 cards = $3! = 6$

Only one permutation of drawing 3 cards will be in increasing order, therefore, probability = $1/6$

Problem 5

Suppose we roll two ordinary, 6-sided dice. What is the expectation of the sum of the two values showing?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$\begin{aligned} & 2*(1/36) + 3*(2/36) + 4*(3/36) + 5*(4/36) + 6*(5/36) + 7*(6/36) \\ & + 8*(5/36) + 9*(4/36) + 10*(3/36) + 11*(2/36) + 12*(1/36) \\ & = (2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)/36 \\ & = 252/36 \\ & = 7 \end{aligned}$$

Whats is the expectation of the maximum of the two values showing?

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

$$\begin{aligned} & (1*1 + 2*3 + 3*5 + 4*7 + 5*9 + 6*11)/36 \\ & = (1 + 6 + 15 + 28 + 45 + 66)/36 \\ & = 161/36 \approx 4.472 \end{aligned}$$

Problem 6

Verify axiom 2 of the probability axioms for the geometric distribution.

Need to prove that the probability distribution of the sample space is 1.

Proof:

sum of the probabilities from 1 to infinity is equal to:

$$q^{(0)} * p + q^{(1)} * p + q^{(2)} * p + \dots \\ = p * [q^{(0)} + q^{(1)} + \dots]$$

$$[q^{(0)} + q^{(1)} + \dots] = 1/(1-q)$$

$$1-q = p$$

$$[q^{(0)} + q^{(1)} + \dots] = 1/p$$

$$p * [q^{(0)} + q^{(1)} + \dots] = p * (1/p) = 1$$

Problem 7

How many times on average must we flip 6 fair coins before we obtain 3 heads and 3 tails?

Probability of 3 heads on 6 flips

$$= (6C3) \cdot (0.5)^3 (0.5)^3$$

$$= 20 \cdot (0.125) \cdot (0.125)$$

$$= 0.3125$$

$$E(x) = 1/p$$

$$p = 0.3125$$

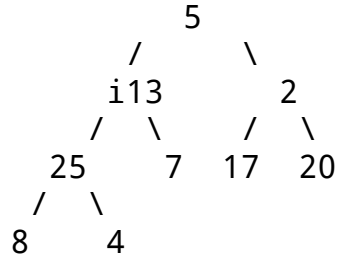
$$E(x) = 3.2 \text{ times}$$

Problem 8

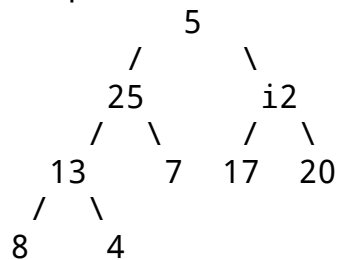
Using Figure 6.4 as a model, illustrate the operation of Heapsort on the array $A=[5,13,2,25,7,17,20,8,4]$. Sort from lowest to highest—so use a max-heap. Also, show the heap being built (as in Figure 6.3), as well as being taken apart.

$A=[5,13,2,25,7,17,20,8,4]$

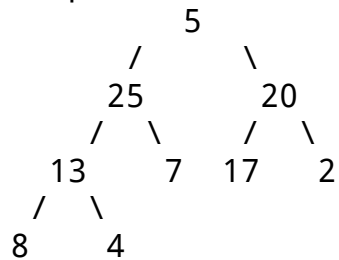
step 1:



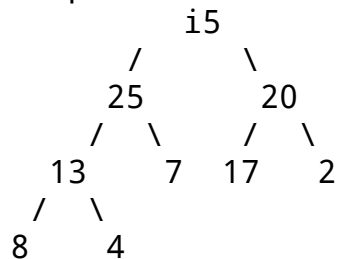
step 2:



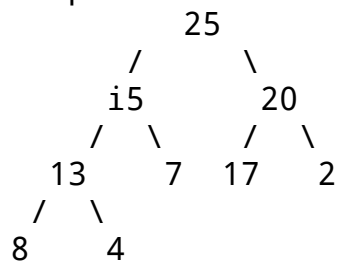
step 3:



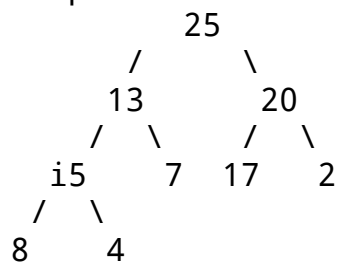
step 4:



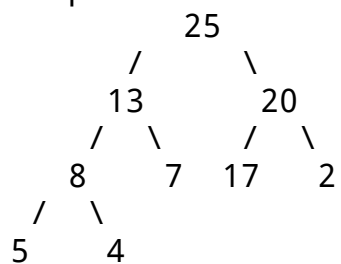
step 5:



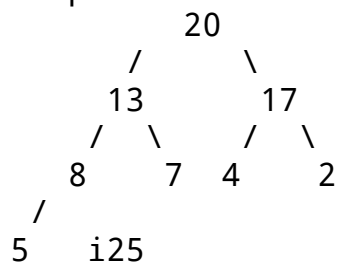
step 6:



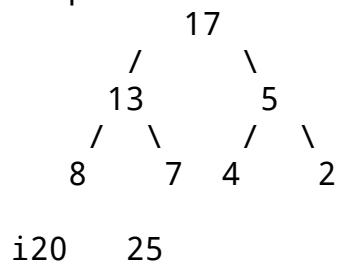
step 7:



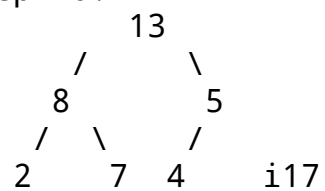
step 8:



step 9:

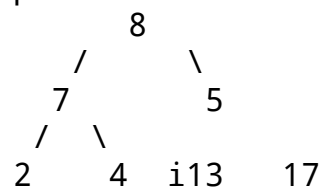


step 10:



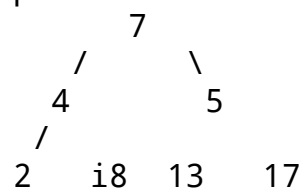
20 25

step 11:



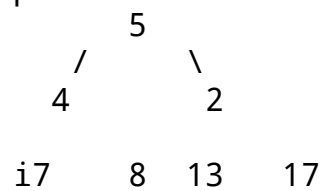
20 25

step 12:



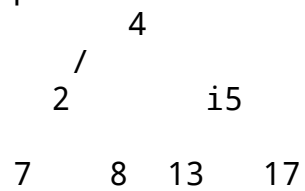
20 25

step 13:



20 25

step 14:



20 25

step 15:

2

i4

5

7

8

13

17

20

25

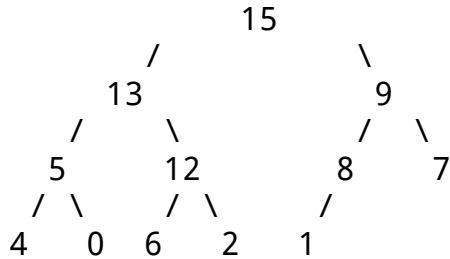
step 16:

A = [2,4,5,7,8,13,17,20,25]

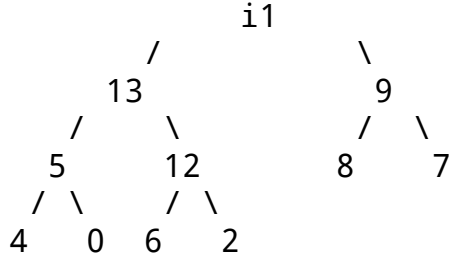
Problem 9

Illustrate the operation of Heap-Extract-Max on the heap A = [15,13,9,5,12,8,7,4,0,6,2,1]. Only illustrate a single call (i.e. removal of one element), but for that call, show each step (i.e. at each level), not just the beginning and ending configurations of the heap.

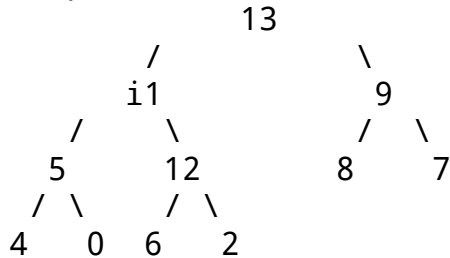
step 1:



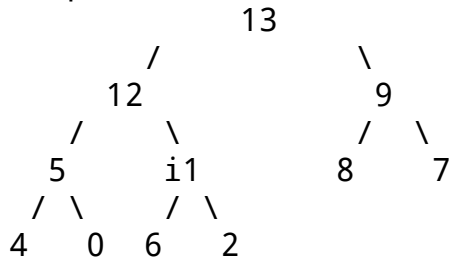
step 2:



step 3:



step 4:



step 5:

